Objectives: After finishing this chapter you will be able to:

- Define probability, sample space, and event.
- Distinguish between subjective and objective probability.
- Describe the complement of an event, the intersection, and the union of two events.
- Compute probabilities of various types of events.
- Explain the concept of conditional probability and how to compute it.
- Explain Bayes' theorem and its applications.

Chapter 3.1 Events, Sample Spaces and Probability

Consider the population of the families with two children.

The following **questions** are **equivalent**:

- **What proportion** of all such families has two girls?
- **How likely** is it to select randomly a family with two girls?
- **What is the probability** that a randomly selected family with two kids has only girls?

Key Concepts:

Random experiment: the outcome cannot be predicted
A sample point: the most basic outcome of an experiment
Sample space (S): the set of all possible individual outcomes of an experiment (the collection of all sample points)
Probability: a branch of mathematics that deals with modeling of *random phenomena or experiments*, which is, experiments whose outcomes may vary when repeated

Examples:

Experiment	Sample Space
• Toss a Coin, Note Face	{Head, Tail}
• Toss 2 Coins, Note Faces	{HH, HT, TH, TT}
• Select 1 Card, Note Kind	{2♥, 2♠, ..., A♦} (total 52)
• Select 1 Card, Note Color	{Red, Black}
• Play a Football Game	{Win, Lose, Tie}
• Inspect a Part, Note Quality	{Defective, Good}
• Observe Gender	{Male, Female}

Example of an experiment that is **not random**: count the money in your wallet three times. Each time the result is quite predictable and won't change.)

A random experiment is described by the following:

1. Sample Space
2. Events
3. Probability

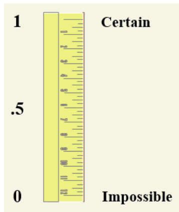
Simple event: (e_1, e_2,...) a **sample point;** individual outcome of an experiment
Example: experiment roll a die. Simple event: {2}.
Other events (compound events) (denoted as A, B, C, ...or E_1, E_2...): the sets of simple events; the subsets of sample space

Example: experiment - roll a die. Event: {even number was rolled}.

Probability: a number measuring the likelihood of an outcome of an experiment

- Probability is a function. Its domain is a set of all sample points from given sample space and the range is a set of real numbers from interval [0,1].
- Every event A in given sample space is assigned a corresponding number between 0 and 1, where 0 means "impossible" and 1 means "certain". The number is called probability of an event A, and is denoted P(A)
- Probabilities of all possible simple outcomes in a finite sample space add up to 1; that is, P(S)= $\sum_{i=1}^{n} p_i$=1
- Equally likely events all have the same probability

Probability can often be determined by considering what fraction of the population has the desired property

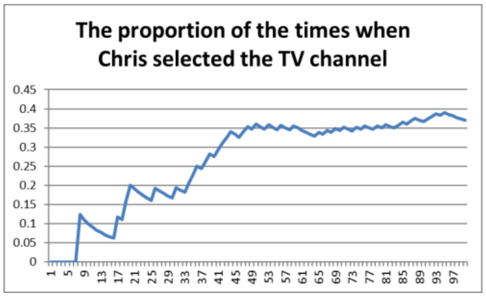


Example: Two interpretations of probability

1) 50% of newborn children are boys.

Probability that the next newborn child is a boy is 50% or 0.50.

The Law of Large Numbers
As the number of repetitions of a probability experiment increases, the proportion with which a certain outcome is observed gets closer to the probability of the outcome.



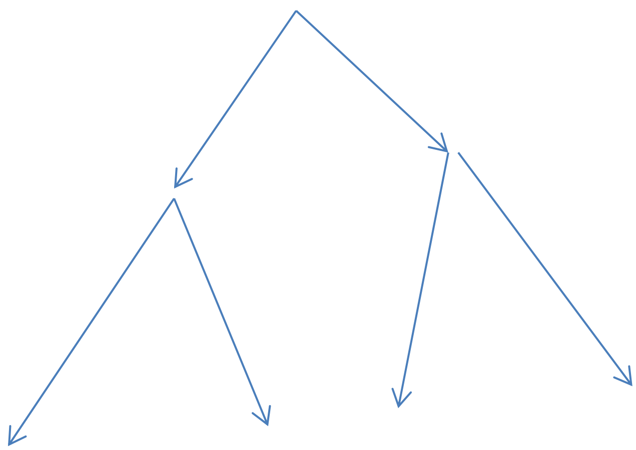
$$\frac{\#of\ times\ A\ has\ occured}{\text{Total}\ \#of\ repeted\ observations} \to P(A)$$

That means: probability can often be determined by computing the fraction of times when an event of interest has occurred, out of all repetitions of an experiment (empirical method)

Equally Likely Outcomes: if a sample space contains n outcomes and if each outcome (sample point) is equally likely, then
P(sample point)=1/n

Experiment: Select at random a family with two children. Note Gender.
Sample Space: …………………….. (a list of possible outcomes)

Exercise: Use Tree Diagram to find all possible simple outcomes:



Other Events:	**Outcomes**
• 1 Boy & 1 Girl	{BG, GB}
• First was a Girl	…
• At Least 1 Girl	…

Example: find the probability of the event "First Girl, then Boy" that is, P(GB)

Let x=number of outcomes (sample points) in the event A
Let n=total number of sample points in sample space
If the event A contains x points from sample space, then P(A)=x/n

Notation: Probability of an event A={one girl}; P(A)=…

Example: find the probability of the event B="at least one Girl in the family with two children".

P(B)=P({GB}or {BG} or {GG})=…

Exercise: What is the probability that in such family the children are of different genders?

…………………………

Example: Two unbalanced coins are tossed. Probabilities of simple events are as follow:
P(HH)=4/9
P(HT)=2/9
P(TH)=2/9
P(TT)=1/9

Find the probability of A, P(A), if A={at least one head turns up} and P(B), B={no heads}

Exercise:

3.1 An experiment results in one of the following sample points: E_1, E_2, E_3, E_4, or E_5.

a. Find $P(E_3)$ if $P(E_1) = .1, P(E_2) = .2, P(E_4) = .1$, and $P(E_5) = .1$.

Problem:
How many simple points are in an experiment "tossing 3 coins"? "Tossing 10 coins"?

<u>Combinations Rule</u>

Definition: <u>n-factorial</u> $n! = n(n-1)\cdot ... \cdot 3\cdot 2\cdot 1$

$3! = 3\cdot 2\cdot 1; \quad 2! = 2\cdot 1; \quad 1! = 1; \quad 0! = 1$

Suppose that a group of objects (e.g. population) consists of ***N*** distinct objects. Suppose that we select ***r*** objects from that group. Such selection is called a **combination** of size ***r*** out of ***N*** objects.

The rule of combination

The number of all combinations is denoted by nCr or $\binom{N}{r}$, reads as "*N* choose *r*".

$$nCr = \binom{N}{r} = \frac{N!}{r!(N-r)!}$$

Example: A "poker hand" consists of 5 cards selected from a standard deck of 52 cards , order does not matter. The number of different "poker hands" is

$$\binom{52}{5} = \frac{52!}{5!47!} = {}_{52}C_5 = 2,598,960$$

Check: TI-83/84 type 52 then select MATH – PRB –3:nCr – ENTER – 5 – ENTER

Example: In how many ways can you select the numbers in **Pick6** lotto? (In that game you are to select any 6 out of 51 numbers)

(Will your answer change if the order of selection matters? How?)

Graphical representations of the events in a finite sample space

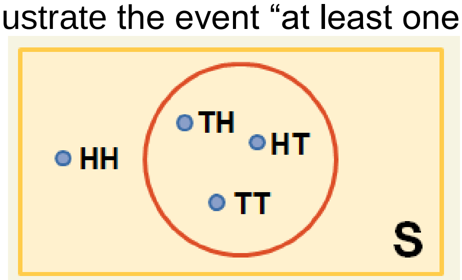
a) Venn Diagram

b) Probability tree

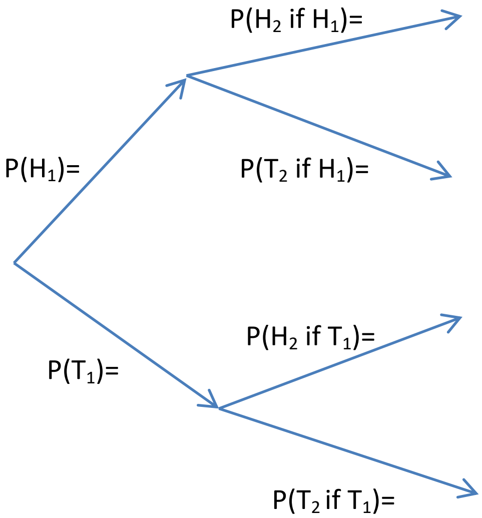
Example: Experiment: tossing two coins (or one coin twice)

a) What is the sample space (a list of sample points) for this experiment?

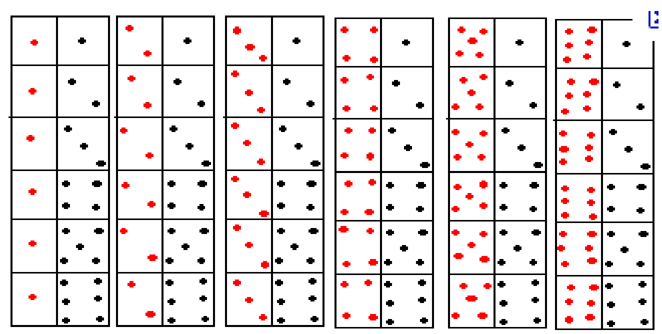
b) Illustrate the event "at least one tail" on Venn diagram.



c) Illustrate the experiment on tree diagram:



Experiment: two dice roll



Sample space - a list of sample points:

(1,1) (1,2) (1,3) (1,4) (1,5) (1,6)
(2,1) (2,2) (2,3) (2,4) (2,5) (2,6)
(3,1) (3,2) (3,3) (3,4) (3,5) (3,6)
(4,1) (4,2) (4,3) (4,4) (4,5) (4,6)
(5,1) (5,2) (5,3) (5,4) (5,5) (5,6)
(6,1) (6,2) (6,3) (6,4) (6,5) (6,6)

All are equally likely.

Other events (examples):
A={rolling even numbers}
B={rolling the sum of 5}
C={rolling both the same}

Exercise: Find the probabilities P(A)=? P(B) =? P(C)=?

Exercise: simulation. Problem 3.8

It is often HARD to find the probability of a given event! But we can approximate it quite well by simulating the event over and over again.

3.7 Two marbles are drawn at random and without replacement from a box containing two blue marbles and three red marbles.

a. List the sample points for this experiment.
b. Assign probabilities to the sample points.
c. Determine the probability of observing each of the following events:

A: {Two blue marbles are drawn.}B: {A red and a blue marble are drawn.}C: {Two red marbles are drawn.}

3.8 Simulate the experiment described in Exercise 3.7 using any five identically shaped objects, two of which are one color and three another color. Mix the objects, draw two, record the results, and then replace the objects. Repeat the experiment a large number of times (at least 100). Calculate the proportion of time events *A, B*, and *C* occur. How do these proportions compare with the probabilities you calculated in Exercise 3.7? Should these proportions equal the probabilities? Explain.

Solution: Use the calculator and RANDINT function to simulate the selections. Assign numbers 1 and 2 for a blue ball, and 3, 4, and 5 for red ball. Function: randInt(1,5,2). Each click simulates a selection of two balls.
Repeat the total 10 times. Record the results. Then combine your results with the other students to obtain the total of 100 repetitions.
Find empirical probability of selecting two blue, two red, and two of different colors.

(Color1, color2)	Tally	Number of repetitions (your)	Total number of repetitions (ten students)
Both blue			
Both red			
One of each			
TOTAL		10	100

Compute basing on the last column:

P(BB)=

P(RR)=

P(BR or RB)=

At home, after finishing homework #3.7, compare the results.
Exercise #3.10 and 14 p.140

3.10 Workers' unscheduled absence survey. Each year CCH, Inc., a firm that provides human resources and employment law information, conducts a survey on absenteeism in the workplace. The latest *CCH Unscheduled Absence Survey* found that of all unscheduled work absences, 34% are due to "personal illness," 22% for "family issues," 18% for "personal needs," 13% for "entitlement mentality," and 13% due to "stress." Consider a randomly selected employee who has an unscheduled work absence.

a. List the sample points for this experiment.

b. Assign reasonable probabilities to the sample points.

c. What is the probability that the absence is due to something other than "personal illness"?

3.14 Working on summer vacation. Is summer vacation a break from work? Not according to an *Adweek/Harris* (July 2011) poll of 3,304 U.S. adults. The poll found that 46% of the respondents work during their summer vacation, 35% do not work at all while on vacation, and 19% were unemployed. Consider the work status during summer vacation of a randomly selected poll respondent.

a. List the sample points for this experiment.

b. Assign reasonable probabilities to the sample points.

c. What is the probability that a randomly selected poll respondent will not work while on summer vacation or is unemployed?

Have you heard of the Three Doors game? Check below and try yourself.

http://math.ucsd.edu/~crypto/Monty/monty.html

Chapter 3.2-3,4 Unions and Intersections, Complements, Addition Rule and Mutually Exclusive Events

Operations on the sets of events:

Given events A and B:

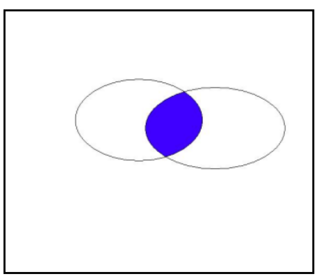
1. **Intersection AB** (means: A and B occur, denoted: $A \cap B$), represents the event that ***both A and B occur*** and consists of all elementary outcomes which are **in both events A and B.**

2. **Union A or B** (denoted: A∪B), represents the event that ***either A or B or both occur*** and consist of all elementary outcomes which are **either in A or in B or in both**

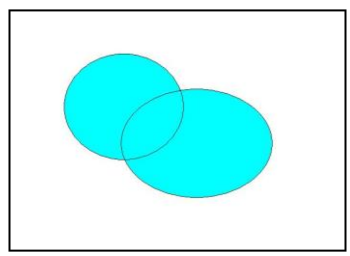
3. **Complement** A^C, represents the event that ***A does not occur*** and consists of all elementary outcomes which **are not in A**

Venn diagrams

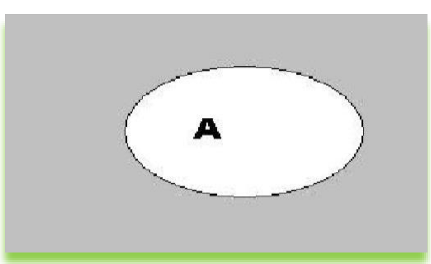
1. Intersection of A **and** B ($A \cap B$)



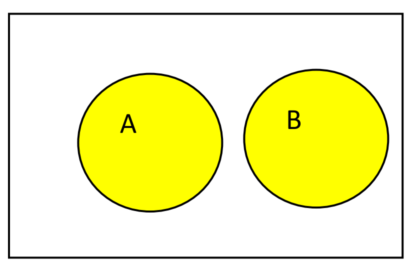
2. Union of A **or** B (A∪B)



3. Complement of A (A^C)



4. Mutually exclusive events ($A \cap B$=%)



Disjoin (mutually exclusive) events: two events are disjoint if they do not share any sample points.

If A and B are disjoint, then P(A or B)=P(A)+P(B) (Additive Rule for disjoint events)

Complementary events: A and A^C are complementary if A^C contains all single points that are not in A.

If A and A^C are complementary, then $P(A)+P(A^C)=1$

In particular, $P(A) = 1- P(A^C)$ and $P(A^C) = 1- P(A)$

Class Exercise: A die is rolled once.

Let A={odd number} B={less than 5}.

Use Venn diagram to illustrate and find the probabilities of the following events

$P(A)$

$P(B)$

$P(A\cap B)$

$P(A\cup B)$

$P(A^C)$

$P(B^C)$

Probability – the Rules

Range of Values for P(A): $0 \le P(A) \le 1$

Rule of Complements: $P(A^C) + P(A) = 1$; so, $P(A^C) = 1 - P(A)$

Rule of Complements for "at least one" or "none": P(at least one)=1-P(none)

Additive Rule $P(A \cup B) = P(A) + P(B)$ **(mutually exclusive events):**

Additive Rule $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (for any A, B)

Intersection - Probability of both A *and* B $P(A \cap B) = \frac{n(A \cap B)}{n(S)}$

Mutually exclusive (or disjoint) events: Events A and C are called mutually exclusive) if $P(A \cap C) = 0$

Agreement about rounding the probabilities:

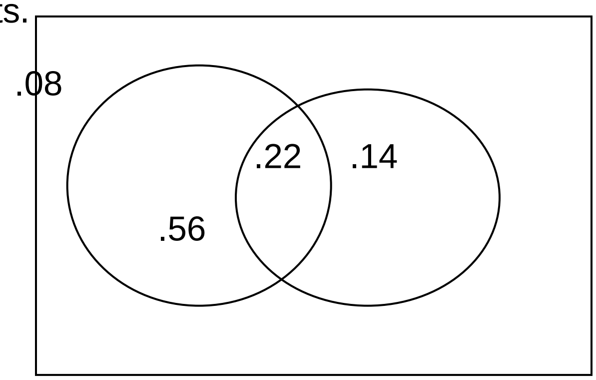
If the result is a simple fraction, like ¾ or 5/9, you may leave them in this form. If the fraction is more complicated, write it as a decimal rounded to **three significant digits** or as percent with the **accuracy 0.1** %.

Examples:

Police report that 78% of drivers stopped on suspicion of drunk driving are given a breath test, 36% a blood test, and 22% both tests. What is the probability that a randomly selected DUI suspect is given

1. A test? Ans: 0.78 + 0.36 – 0.22 = 0.92
2. A blood test or a breath test, but not both? Ans: 0.70 (see diagram)
3. Neither test? Ans: 0.08 (see diagram)

Solution: 78% of the drivers who took breath test include the 22% who took both tests.



<u>Example:</u> Suppose that the probability that it rains on Friday P(F)=0.4, and that it rains on Saturday P(S)=0.8, and the probability that it rains both days $P(F \cap S)$= 0.3.

Then the probability of rain on Friday or Saturday is ....?

<u>More exercises:</u>

Two dice are rolled.

Sample space:

(1,1) (1,2) (1,3) (1,4) (1,5) (1,6)

(2,1) (2,2) (2,3) (2,4) (2,5) (2,6)

(3,1) (3,2) (3,3) (3,4) (3,5) (3,6)

(4,1) (4,2) (4,3) (4,4) (4,5) (4,6)

(5,1) (5,2) (5,3) (5,4) (5,5) (5,6)

(6,1) (6,2) (6,3) (6,4) (6,5) (6,6)

We define events E1, E2, E3 and E4 as follows
E1: Getting a sum equal to 8
E2: Getting a double
E3: Getting a sum less than 5
E4: Getting a sum less to 8

a) Are events E1 and E2 mutually exclusive? Find the probability of a) union; b) Intersection

b) Are events E2 and E3 mutually exclusive? Find the probability of a) union; b) Intersection

c) Are events E3 and E4 mutually exclusive? Find the probability of a) union; b) Intersection

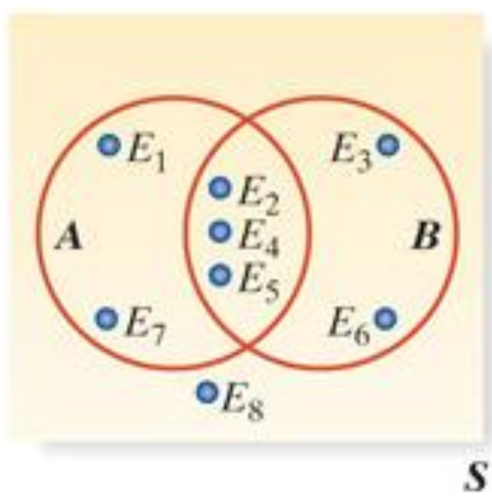
d) Are events E4 and E1 mutually exclusive? Find the probability of a) union; b) Intersection

Class exercises:

3.32 Consider the Venn diagram below, where

P(E1)=.10, P(E2)=.05, P(E3)=P(E4)=.2,P(E5)=.06, P(E6)=.3, P(E7)=.06, andP(E8)=.03.

Find the following probabilities:



a.$P(A^c)$

b.$P(B^c)$

c.$P(A^c \cap B)$

d.$P(A \cup B)$

e.$P(A \cap B)$

f.$P(A^c \cap B^c)$

g. Are events *A* and *B* mutually exclusive? Why?

(Exercises #36, 38 if time allows)

Problems at major companies. The *Organization Development Journal* (Summer 2006) reported on the results of a survey of human resource officers (HROs) at major employers located in a southeastern city. The focus of the study was employee behavior, namely absenteeism, promptness to work, and turnover. The study found that 55% of the HROs had problems with employee absenteeism; also, 41% had problems with turnover. Suppose that 22% of the HROs had problems with both absenteeism and turnover. Use this information to find the probability that an HRO selected from the group surveyed had problems with either employee absenteeism or employee turnover.

3.38 Scanning errors at Wal-Mart. The National Institute for Standards and Technology (NIST) mandates that for every 100 items scanned through the electronic checkout scanner at a retail store, no more than 2 should have an inaccurate price. A study of the accuracy of checkout scanners at Wal-Mart stores in California was conducted (*Tampa Tribune*, Nov. 22, 2005). Of the 60 Wal-Mart stores investigated, 52 violated the NIST scanner accuracy standard. If one of the 60 Wal-Mart stores is randomly selected, what is the probability that the store does not violate the NIST scanner accuracy standard?

Chapter 3.5-3.6 Conditional Probability, Independent Events and Multiplication Rule

Conditional probability: P(A occurs, given B has occurred) =

$P(A|B) = \frac{P(A \cap B)}{P(B)}$ (assuming P(B) ≠ 0); $P(B|A) = \frac{P(A \cap B)}{P(A)}$ (assuming P(A) ≠ 0)

Multiplicative Rule of Probability

$P(A \cap B) = P(B)\,P(A|B) = P(A)\,P(B|A)$

Two events are independent if $P(A|B) = P(A)$ or $P(A \text{ and } B) = P(A)P(B)$

Example: Consider group of students (in our class) who like pets like dogs or cats. Complete the table using relative frequencies (in percent).

In our class:	Prefer cats	Prefer dogs	TOTAL
Male			
Female			
TOTAL			

Find the probability that a randomly selected person:

a) Prefers cats; P(C)

b) Is a female; P(F)

c) Is a female who prefers cats; $P(C \cap F)$

d) Prefers cats, given a female is selected; P(C|F)

e) Is a female, given a person who prefers cats is selected P(F|C)

Examples and Exercises:

Ex. 3.18 (p. 159) A county welfare agency employs 10 welfare workers who interview prospective food stamp recipients. Periodically the supervisor selects, at random, the forms completed by 2 workers to audit for illegal deductions. Let

A ={First worker selected gives illegal deductions}

B ={Second worker selected gives illegal deductions}

Unknown to the supervisor, 3 of the workers have regularly been giving illegal deductions to applicants. What is the probability that both of the 2 workers chosen have been giving illegal deductions? That is, compute P(A ∩ B).

Solution: The tree can help see the options.

Example: Experiment – a coin is tossed three times.

A = "exactly two H in three tosses" ={HHT, HTH, THH}

B = "H in the first toss" ={HHH, HHT, HTH, HTT}

Are A and B independent?

CLASS EXERCISES:

3.52 An experiment results in one of three mutually exclusive events, A, B, or C. It is known that $P(A) = .30$, $P(B) = .55$, and $P(C) = .15$. Find each of the following probabilities:

a. $P(A \cup B)$
b. $P(A \cap C)$
c. $P(A|B)$
d. $P(B \cup C)$
e. Are B and C independent events? Explain.

3.60 Guilt in decision making. Refer to the *Journal of Behavioral Decision Making* (Jan. 2007) study of the effect of guilt emotion on how a decision maker focuses on a problem, Exercise 3.44 (p. 153). The results (number responding in each category) for the 171 study participants are reproduced in the table below. Suppose one of the 171 participants is selected at random.

(Hint:

Emotional State	Choose Stated Option	Do Not Choose Stated Option	Totals
Guilt	45	12	57
Anger	8	50	58
Neutral	7	49	56
Totals	60	111	171

Source: Based on Gangemi, A., & Mancini, F. "Guilt and focusing in decision-making." *Journal of Behavioral Decision Making*, Vol. 20, Jan. 2007 (Table 2).

a. Given that the respondent is assigned to the guilty state, what is the probability that the respondent chooses the stated option?

b. If the respondent does not choose to repair the car, what is the probability that the respondent is in the anger state?

c. Are the events {repair the car} and {guilty state} independent?

Independent evens: A and B are independent if and only if $P(A \cap B)=P(A)P(B)$, or $P(A)=P(A|B)$

Working mothers with children. The U.S. Census Bureau reports a decline in the percentage of mothers in the work-force who have infant children. The following table gives a breakdown of the marital status and working status of the 1.8 million mothers with infant children in the year 2010. (The numbers in the table are reported in thousands.) Consider the following events: A = {Mom with infant works}, B = {Mom with infant is married and living with husband}. Are A and B independent events?

	Working	Not Working
Married/living with husband	1,174	89
All other arrangements	416	121

NOTE:
Small Samples from Large Populations

If a sample size is no more than 5% of the size of the population, we treat the selections as being *independent* (even if the selections are made without replacement, so they are technically dependent).

Key concepts:

Events, sample points, sample space, probability, union and intersection of events, conditional probability, multiplicative rule, tree diagram, independent events