FACULTY OF ENGINEERING
DEPARTMENT OF COMPUTER ENGINEERING
MAT 222 LINEAR ALGEBRA
FINAL EXAM SOLUTIONS

1. Consider the matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. The eigenvalues of $\mathbf{A}$ are given to be 2 and -1. Find the entry in the $(3,1)$ position of the matrix $\mathbf{A}^{2025}$. (Hint: You can use that $\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = 3\mathbf{I}$, where $\mathbf{I}$ is the 3×3 identity matrix.)

Solution: The eigenvectors corresponding to 2 are the nonzero solutions of the system

$$(\mathbf{A} - 2\mathbf{I})\mathbf{x} = \mathbf{0} \longrightarrow \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The solutions are given by $x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ so we can take $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ as the first eigenvector.

The eigenvectors corresponding to -1 are the nonzero solutions of the system

$$(\mathbf{A} - (-1)\mathbf{I})\mathbf{x} = \mathbf{0} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The solutions are given by $x_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, which shows that we can take $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ as the independent eigenvectors corresponding to -1.

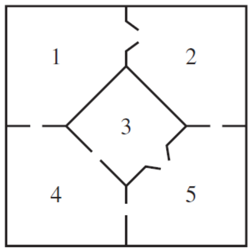
To this end, we have $\mathbf{P} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Since $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ we have $\mathbf{A}^{2025} = \mathbf{P}\mathbf{D}^{2025}\mathbf{P}^{-1}$. From the hint we can see that $\mathbf{P}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$. Then, carrying out the required matrix multiplications gives

$$\mathbf{A}^{2025} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2^{2025} & 0 & 0 \\ 0 & (-1)^{2025} & 0 \\ 0 & 0 & (-1)^{2025} \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2^{2025}-2}{3} & \frac{2^{2025}+1}{3} & \frac{2^{2025}+1}{3} \\ \frac{2^{2025}+1}{3} & \frac{2^{2025}-2}{3} & \frac{2^{2025}+1}{3} \\ \frac{2^{2025}+1}{3} & \frac{2^{2025}+1}{3} & \frac{2^{2025}-2}{3} \end{pmatrix}.$$

Therefore, the entry in the $(3,1)$ position of $\mathbf{A}^{2025}$ is $\frac{2^{2025}+1}{3}$. (Note that one can only calculate this entry without calculating the remaining of the matrix.)

2. Consider a maze where a mouse wanders between five distinct rooms through the available doorways. Note that some doors are one-way; the mouse can move from room 3 to room 5 and from room 1 to room

2 but it cannot move the other way around. Also assume that at each time step the mouse has to move to one of the available neighbouring rooms with equal probability; it cannot stay where it is.



According to this scenario, let us suppose that 1 million (10^6) time steps have passed and the mouse has somehow managed to survive. At how many time instants approximately would you expect the mouse to be in room 5?

Solution: The stochastic matrix of this Markov process is

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 & 1/3 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/3 & 0 \\ 1/2 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1 & 1/2 & 1/3 & 0 \end{pmatrix}.$$

Since $\mathbf{P}$ is a stochastic matrix, we know that 1 is its eigenvalue. The eigenvectors corresponding to 1 will determine the frequency of time instants that the mouse will spend in each room. Let us find it.

$$(\mathbf{P} - 1\mathbf{I})\mathbf{x} = \mathbf{0} \longrightarrow \begin{pmatrix} -1 & 0 & 0 & 1/3 & 0 \\ 1/2 & -1 & 0 & 0 & 1/2 \\ 0 & 0 & -1 & 1/3 & 0 \\ 1/2 & 0 & 1/2 & -1 & 1/2 \\ 0 & 1 & 1/2 & 1/3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

One can use Gaussian elimination or any other method to obtain the solutions as $x_1 \begin{pmatrix} 1/4 \\ 5/8 \\ 1/4 \\ 3/4 \\ 1 \end{pmatrix}$. So, the eigenvector which is also a probability vector (whose entries add up to 1) is $\begin{pmatrix} 2/23 \\ 5/23 \\ 2/23 \\ 6/23 \\ 8/23 \end{pmatrix}$. This means that in the long run, at a random time instant, the probability that the mouse is in room 5 is 8/23. Thus, out of the first 1 million time instants, the number of times that it is in room 5 is approximately $\dfrac{8}{23} \cdot 10^6 \approx 347826$.

3. Consider the data below.

x_i	1	2	4	5
y_i	-1	0.5	3	4

(a) Let $y = \beta_0 + \beta_1 x$ be the least squares regression line corresponding to the above data. Explain how the problem of finding β_0 and β_1 can be expressed as a least squares problem. (In other words, find a matrix

$\mathbf{A}$ and a vector $\mathbf{b}$ such that $\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$ is the least squares solution to $\mathbf{Ax} = \mathbf{b}$. You don't need to calculate β_0 and β_1.)

(b) Explain how β_0 and β_1 can be approximately computed using the gradient descent method. You don't need to make any computations.

Solution: **(a)** For each i let $\hat{y}_i = \beta_0 + \beta_1 x_i$. If all the data were linear, then we would have $y_i = \hat{y}_i$ for all i. In this case, the relation between x_i and y_i would be given by

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0.5 \\ 3 \\ 4 \end{pmatrix}.$$

Since the relation between x_i and y_i is not linear, this system cannot be satisfied exactly; it can only be satisfied in a least squares sense. Hence, $\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$ is the least squares solution to $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 0.5 \\ 3 \\ 4 \end{pmatrix}$.

(b) Define the loss function $f(\beta_0, \beta_1) = \sum_{i=1}^{4} (\hat{y}_i - y_i)^2 = \sum_{i=1}^{4} (\beta_0 + \beta_1 x_i - y_i)^2$. Then, the method of gradient descent computes the vector $\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$ approximately. Namely, we choose a step length $\alpha > 0$ and a stopping tolerance $\varepsilon > 0$. Then, starting from any initial guess $\mathbf{x}_0$, we compute the iterations $\mathbf{x}_{k+1} = \mathbf{x}_k - \alpha \nabla f(\mathbf{x}_k)$ for $k = 0, 1, 2, \ldots$ When $\|\nabla f(\mathbf{x}_k)\| < \varepsilon$, the current approximation $\mathbf{x}_k$ gives the approximate value for the vector $\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$.

4. Let $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 1 & -1 \\ 1 & -3 \end{pmatrix}$. Consider the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Find the vector in the column space of $\mathbf{A}$ that is closest to $\mathbf{b}$. (Hint: You can find the least squares solution to the system $\mathbf{Ax} = \mathbf{b}$.)

Solution: The asked vector is the orthogonal projection of $\mathbf{b}$ onto $V = C(\mathbf{A})$. To find it, let us first obtain an orthogonal basis of $C(\mathbf{A})$. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the columns of $\mathbf{A}$ respectively. $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for V but it is not orthogonal. In order to make it orthogonal, we can make use of the projection of $\mathbf{v}_2$ onto $\mathbf{v}_1$. Define the vector $\mathbf{u}$ as follows:

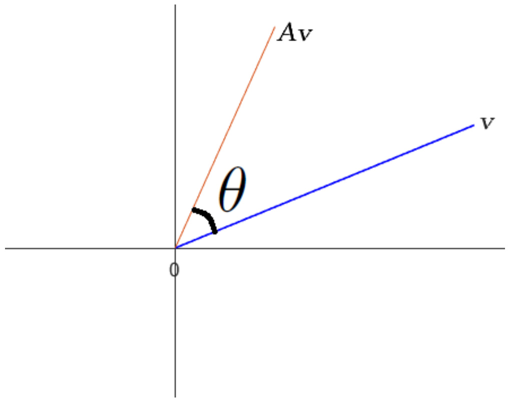
$$\mathbf{u} = \mathbf{v}_2 - \mathrm{proj}_{\mathbf{v}_1} \mathbf{v}_2 = (5, -1, -3) - \frac{(5, -1, -3) \cdot (2, 1, 1)}{(2, 1, 1) \cdot (2, 1, 1)} (2, 1, 1) = (5, -1, -3) - (2, 1, 1) = (3, -2, -4)$$

Then, $\{\mathbf{v}_1, \mathbf{u}\}$ is an orthogonal basis for V. Let us denote the requested vector by $\hat{\mathbf{b}}$. Then, it is found by

$$\begin{aligned} \hat{\mathbf{b}} = \mathrm{proj}_V \mathbf{b} = \mathrm{proj}_{\mathbf{v}_1} \mathbf{b} + \mathrm{proj}_{\mathbf{u}} \mathbf{b} &= \frac{\mathbf{b} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 + \frac{\mathbf{b} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} \\ &= \frac{(1, 0, 1) \cdot (2, 1, 1)}{(2, 1, 1) \cdot (2, 1, 1)} (2, 1, 1) + \frac{(1, 0, 1) \cdot (3, -2, -4)}{(3, -2, -4) \cdot (3, -2, -4)} (3, -2, -4) = \frac{1}{2} (2, 1, 1) + \frac{-1}{29} (3, -2, -4) \\ &= \left(\frac{26}{29}, \frac{33}{58}, \frac{37}{58} \right). \end{aligned}$$

As a result, the vector in $C(\mathbf{A})$ that is closest to $\mathbf{b}$ is $\left(\frac{26}{29}, \frac{33}{58}, \frac{37}{58}\right)$.

5. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & a \\ -1 & 1 \end{bmatrix}$. Suppose that a particle moves freely on the unit circle in $\mathbb{R}^2$. Let us denote the varying position vector of this particle by $\mathbf{v}$. Suppose also that a second particle is tracing the curve indicated by $\mathbf{Av}$. Suppose further that, if θ is the angle between $\mathbf{v}$ and $\mathbf{Av}$, for all time instants we have $0 < \theta < \pi/2$. See the figure below. Find a value for a such that this scenario is possible. (There may be different answers.)



Solution: Consider the inner product of $\mathbf{v}$ and $\mathbf{Av}$. We have

$$\mathbf{v} \cdot \mathbf{Av} = \|\mathbf{v}\| \|\mathbf{Av}\| \cos\theta.$$

Since $0 < \theta < \pi/2$, we have $\cos\theta > 0$, which shows that $\mathbf{v} \cdot \mathbf{Av}$ is positive (also since the norm of a nonzero vector is always positive). Now denote the coordinates of $\mathbf{v}$ by x and y, i.e. let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then, using the other definition of inner product we can write

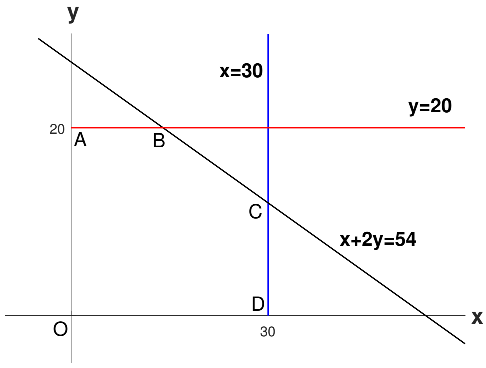
$$\mathbf{v} \cdot \mathbf{Av} = \begin{pmatrix} x \\ y \end{pmatrix} \cdot \left(\begin{pmatrix} 1 & a \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x + ay \\ -x + y \end{pmatrix} = x^2 + y^2 + (a-1)xy$$

Since $\mathbf{v} \cdot \mathbf{Av}$ must be positive, we want $x^2 + y^2 + (a-1)xy > 0$ for all choices of x, y (except $(x, y) = (0, 0)$). The choice $a = 3$ ensures this since $\mathbf{v} \cdot \mathbf{Av} = (x + y)^2 > 0$ in this case.

All that remains is to check that θ cannot be 0. But $\theta = 0$ would mean that $\mathbf{A}$ has a real eigenvalue, which is not possible since its characteristic polynomial $(1 - \lambda)^2 + a$ does not have a real root for $a = 3$. As a result, $a = 3$ satisfies the requirements of the problem.

6. A food store sells two kinds of mixtures. In one unit of Mix A, 1 kg of walnuts are mixed with 1 kg of Antep peanuts. In one unit of Mix B, 1 kg of Karadeniz nuts is mixed with 2 kg of Antep peanuts. The store has available 30 kg of walnuts, 20 kg of Karadeniz nuts and 54 kg of Antep peanuts. Profit of Mix A is 20 TL/unit while profit of Mix B is 30 TL/unit. If the store can sell all of the mixtures, what are the numbers of each mixture that provides the maximum possible profit? (You can use any method except trial and error.)

Solution: Let x denote the number of Mix A to be sold and let y denote the number of Mix B to be sold. Then, we will maximize the function $f(x, y) = 20x + 30y$ under the conditions $0 \le x \le 30, 0 \le y \le 20$ and $x + 2y \le 54$. The feasible region of this linear programming problem is the inside (including the borders) of the pentagon OABCD in the below figure.



We know that the optimal solution of the problem occurs in the vertices of the pentagon OABCD. The coordinates of the vertices can be computed as $O(0,0), A(0,20), B(14,20), C(30,12), D(30,0)$. The profit corresponding to these points are

$$f(0,0) = 0,\ f(0,20) = 600,\ f(14,20) = 880,\ f(30,12) = 960,\ f(30,0) = 600.$$

Since 960 is the greatest of these, this is the optimal solution of the problem. The store must sell 30 units of Mix A and 12 units of Mix B in order to maximize its profit. The maximum possible profit is 960 liras.