MAT 222 Linear Algebra and Numerical Methods
Week 3
Lecture Notes 1

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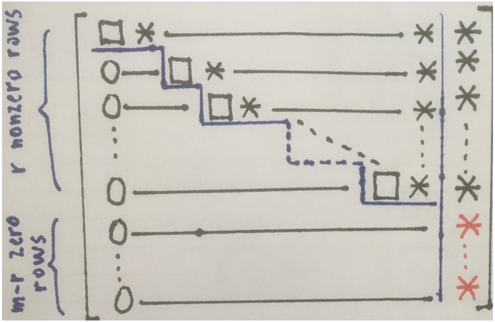
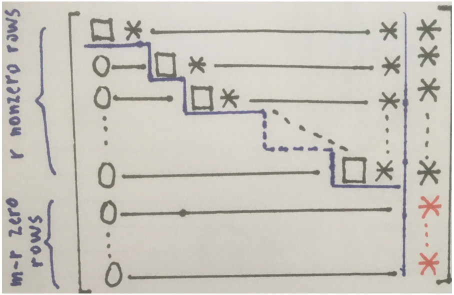
Summary of Gaussian Elimination



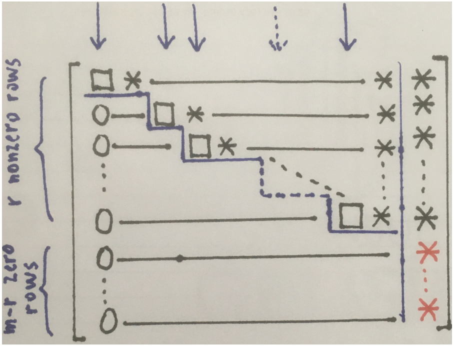
- Above is a typical echelon form of an $m \times n$ linear system obtained after Gaussian elimination.
- Let r denote the number of nonzero rows in the echelon form. This means we have r leading entries, indicated in squared positions.
- Entries in black asterisk are allowed to have any value. On the other hand, every entry in red asterisk has to be zero; otherwise the system is inconsistent.

Rank of a Matrix



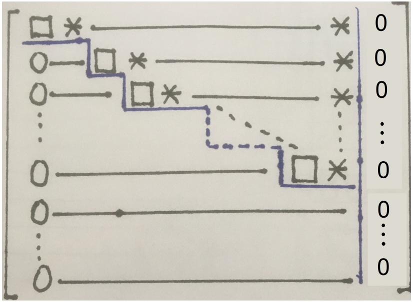
- Given a system $A\mathbf{x} = \mathbf{b}$, number of nonzero rows in echelon form of $\mathbf{A}$ $= r$ is called the rank of $\mathbf{A}$.
- Note that if one of the red asterisks is nonzero, rank of the augmented matrix $[A|\mathbf{b}]$ becomes more than the rank of A.
- This gives us a consistency condition for a general linear system $A\mathbf{x} = \mathbf{b}$: The system is consistent if and only if $\text{rank}(A) = \text{rank}([A|\mathbf{b}])$.

Leading Variables and Free Variables



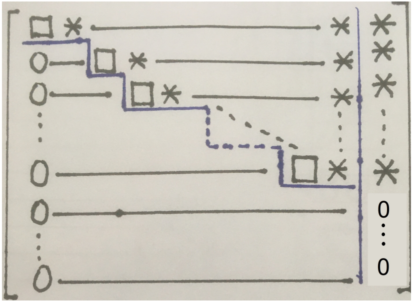
- If the echelon form of the augmented matrix contains r nonzero rows ($\mathrm{rank}(A) = r$), there are r leading entries (one for each nonzero row).
- The columns containing this leading entries are called pivot columns and the unknowns corresponding to these columns are called leading variables.
- The remaining $n - r := k$ unknowns are called free variables.
- All leading variables can be expressed in terms of free variables.

Solution of Homogeneous Systems



- If every entry on the augmented part (the right-most column) is zero, we have a homogeneous system.
- A homogeneous system has always the trivial solution $\mathbf{x} = \mathbf{0}_{n\times 1}$.
- If $\text{rank}(A) = r < n$, then there are infinitely many solutions.
- In this case, $k = n - r$ of the unknowns are free and every solution can be expressed in the form $\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \ldots + t_k\mathbf{v}_k$.
- Here the parameters $t_1, t_2, \ldots, t_k$ indicate the values of the free variables and the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are solutions of the system satisfying a certain additional condition (will be described later).

General=Particular+Homogeneous



- The solution set of a general linear system $A\mathbf{x} = \mathbf{b}$ turns out to be closely related to the solution set of its homogeneous part.
- Namely, if the column vector $\mathbf{p}$ is any solution of $A\mathbf{x} = \mathbf{b}$, then every solution of $A\mathbf{x} = \mathbf{b}$ can be expressed as $\mathbf{x} = \mathbf{p} + \mathbf{x}_h$.
- Here $\mathbf{x}_h$ is the general solution of the homogeneous sytem $A\mathbf{x} = \mathbf{0}$. In other words, $\mathbf{x}_h = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \ldots + t_k\mathbf{v}_k$ as in the previous page.
- Each solution of $A\mathbf{x} = \mathbf{b}$ can be written $\mathbf{x} = \mathbf{p} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \ldots + t_k\mathbf{v}_k$.
- The solution set of $A\mathbf{x} = \mathbf{b}$ is a translated version of the solution set of $A\mathbf{x} = \mathbf{0}$.

General=Particular+Homogeneous: Example

- Consider one of last week's examples: $\begin{aligned} 3x_1 + 2x_2 + 2x_3 - 5x_4 &= 8 \\ 0.6x_1 + 1.5x_2 + 1.5x_3 - 5.4x_4 &= 2.7 \\ 1.2x_1 - 0.3x_2 - 0.3x_3 + 2.4x_4 &= 2.1 \end{aligned}$.
- We expressed the general solution as $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 4 \\ 0 \\ 1 \end{bmatrix}$.
- Here, the vector $\mathbf{p} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is a solution of the system. It corresponds to the choice of the values $t_1 = t_2 = 0$ for the free variables.
- The choice of values $t_1 = 2$, $t_2 = 1$ gives another solution: $\mathbf{q} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}$.
- **Exercise:** Is it also true that every solution can be expressed as $\mathbf{x} = \mathbf{q} + t_1 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 4 \\ 0 \\ 1 \end{bmatrix}$?

Observations on the Echelon Forms of a Matrix

- Note that the echelon form of a system is not unique: Multiplying a nonzero row of a matrix by a nonzero scalar gives another echelon matrix.
- On the other hand, for a given system $A\mathbf{x} = \mathbf{b}$, the echelon form has a determined number of leading entries in determined positions.
- In other words, which unknowns are leading variables and which ones are free can be inferred from the system in only one way.
- Furthermore, once we reach an echelon form for a given system, we can proceed row reduction in the opposite direction in order to obtain an even more simplified version.

A Simpler Echelon Form

- For example, the system
$$\begin{aligned} 3x_1 + 2x_2 + 2x_3 - 5x_4 &= 8 \\ 0.6x_1 + 1.5x_2 + 1.5x_3 - 5.4x_4 &= 2.7 \\ 1.2x_1 - 0.3x_2 - 0.3x_3 + 2.4x_4 &= 2.1 \end{aligned}$$
has an echelon form $\left[\begin{array}{cccc|c} 2 & 5 & 5 & -18 & 9 \\ 0 & 1 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.
- Without performing backward-substitution at once, we can make further simplification by
$$\left[\begin{array}{cccc|c} 2 & 5 & 5 & -18 & 9 \\ 0 & 1 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-5R_2+R_1} \left[\begin{array}{cccc|c} 2 & 0 & 0 & 2 & 4 \\ 0 & 1 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
- The resulting echelon form can no longer be simplified and it corresponds to the following system:
$$\begin{aligned} x_1 + x_4 &= 2 \\ x_2 + x_3 - 4x_4 &= 1 \end{aligned}$$
- The first equation is simpler than the previous version, which was $2x_1 + 5x_2 + 5x_3 - 18x_4 = 9$. Solution is obtained almost without effort.
- The effort of backward-substitution has been expended on doing two more row operations.

Row-reduced echelon matrix

Suppose A is an echelon matrix. Besides being echelon, if A satisfies the following two conditions, then it is said to be a row-reduced echelon matrix.

(1) Every leading entry of A is 1.

(2) Every leading entry of A is the only nonzero entry of its column.

- For example, the matrix $\begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 0 & 12 \end{bmatrix}$ is in echelon form but not in row-reduced echelon form. In order to reduce it, first make every leading entry 1 by scaling its rows.

$$\begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 0 & 12 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1,\, \frac{1}{12}R_3} \begin{bmatrix} 1 & 2/3 & 1/3 & 1 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- Then, use each leading entry to eliminate the entries above it.

$$\begin{bmatrix} 1 & 2/3 & 1/3 & 1 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-6R_3+R_2,\, -1R_3+R_1,\, -\frac{2}{3}R_2+R_1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Row-Reduced Echelon Form

- An advantage of working with row-reduced echelon matrices is that the row-reduced echelon form of every matrix is unique.
- This makes it possible to determine if two given matrices can be obtained from each other by a series of elementary row operations.
- Consider $A = \begin{bmatrix} 2 & -1 & 4 & 1 \\ 1 & 3 & 5 & 2 \\ -2 & -3 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -6 & 1 & 1 \\ 2 & 0 & 3 & 0 \\ 4 & 1 & 5 & -1 \end{bmatrix}$.
  Are the two matrices row-equivalent?
- Both matrices have the row-reduced echelon form $\begin{bmatrix} 1 & 0 & 0 & -\frac{15}{13} \\ 0 & 1 & 0 & -\frac{3}{13} \\ 0 & 0 & 1 & \frac{10}{13} \end{bmatrix}$, so they are row-equivalent.
- This means the systems $\begin{aligned} 2x - y + 4z &= 1 \\ x + 3y + 5z &= 2 \\ -2x - 3y &= 3 \end{aligned}$ and $\begin{aligned} x - 6y + z &= 1 \\ 2x + 3z &= 0 \\ 4x + y + 5z &= -1 \end{aligned}$
  have the same solution set.
- **Exercise:** Obtain B by performing elementary row operations on A.

Gauss-Jordan Elimination

- Computing the row-reduced echelon form of a system involves what is known as Gauss-Jordan elimination.

Gauss-Jordan elimination

Given a system $A\mathbf{x} = \mathbf{b}$, Gauss-Jordan elimination on this system is performed by the following steps:

(1) Reduce the augmented matrix $[A|\mathbf{b}]$ to echelon form using row reduction.

(2) Make each leading entry equal to 1 by scaling each nonzero row in the echelon form of $[A|\mathbf{b}]$.

(3) Starting from the last nonzero row, eliminate all the entries above each leading entry by applying row operations of type (3).

- In some sources, the above algorithm is known simply as Gaussian elimination. We have chosen to use this term for only step (1).
- After Gauss-Jordan elimination is over, there is no need for backward-substitution. We just read each leading variable from the reduced echelon form.

Gauss-Jordan Elimination: Example 1

- Consider the following system: $\begin{aligned} x-3y+z&=-2\\ 2x-6y-z+2t&=6\\ -x+y+3z+4t&=-8\\ 4y-z-4t&=1 \end{aligned}$.

- Last week we reduced it to $\left[\begin{array}{cccc|c} 1 & -3 & 1 & 0 & -2\\ 0 & -2 & 4 & 4 & -10\\ 0 & 0 & -3 & 2 & 10\\ 0 & 0 & 0 & 26/3 & 13/3 \end{array}\right]$, which is in echelon form.

- Now, we first make every leading entry equal to 1 by scaling rows.

$$\xrightarrow{-\frac{1}{2}R_2,-\frac{1}{3}R_3,\frac{3}{26}R_4} \left[\begin{array}{cccc|c} 1 & -3 & 1 & 0 & -2\\ 0 & 1 & -2 & -2 & 5\\ 0 & 0 & 1 & -\frac{2}{3} & -\frac{10}{3}\\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

- Step 2 is complete. Let us now perform step 3.

$$\xrightarrow{\frac{2}{3}R_4+R_3,2R_4+R_2} \left[\begin{array}{cccc|c} 1 & -3 & 1 & 0 & -2\\ 0 & 1 & -2 & 0 & 6\\ 0 & 0 & 1 & 0 & -3\\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

- $$\left[\begin{array}{cccc|c} 1 & -3 & 1 & 0 & -2 \\ 0 & 1 & -2 & 0 & 6 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{array}\right] \xrightarrow{2R_3+R_2,\,-1R_3+R_1} \left[\begin{array}{cccc|c} 1 & -3 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$
  $$\xrightarrow{-3R_2+R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$
- The resulting matrix is in reduced echelon form, so Gauss-Jordan elimination is complete.
- One can read the values of unknowns from the matrix as $x = 1, y = 0, z = -3, t = \frac{1}{2}$. This is the same solution as we had obtained.

Gauss-Jordan Elimination: Example 2

- Let us now consider the following system:

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15 \end{aligned}$$

- The augmented matrix is $\left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right]$.
- An echelon form is $\left[\begin{array}{ccccc|c} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$. (Please check it)
- Now we start the backward phase of Gauss-Jordan elimination.
- $\xrightarrow{-2R_3+R_2,\,-6R_3+R_1} \left[\begin{array}{ccccc|c} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \xrightarrow{\frac{1}{3}R_1,\,\frac{1}{2}R_2}$

$$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \xrightarrow{3R_2+R_1} \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Gauss-Jordan Elimination: Example 2

- So we have obtained the row-reduced echelon form. The equations are

$$\begin{aligned} x_1 - 2x_3 + 3x_4 &= -24 \\ x_2 - 2x_3 + 2x_4 &= -7 \\ x_5 &= 4 \end{aligned} .$$

- This gives the solution $\begin{aligned} x_1 &= -24 + 2x_3 - 3x_4 \\ x_2 &= -7 + 2x_3 - 2x_4 \\ x_5 &= 4 \end{aligned}$, where x_3 and x_4 are free.

- In vector form,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -24 + 2t_1 - 3t_2 \\ -7 + 2t_1 - 2t_2 \\ t_1 \\ t_2 \\ 4 \end{bmatrix} = \begin{bmatrix} -24 \\ -7 \\ 0 \\ 0 \\ 4 \end{bmatrix} + t_1 \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$, where

we have replaced x_3 and x_4 by the free parameters t_1 and t_2.

- **Exercise:** Use G-J elimination to solve $\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\ -2x_1 + 4x_2 + 5x_3 - 5x_4 &= 3 \\ 3x_1 - 6x_2 - 6x_3 + 8x_4 &= 2 \end{aligned}$.

Operation Cost

- One might ask: Is Gauss-Jordan elimination advantageous over Gauss elimination?
- A good way to tackle this question is to count the arithmetic operations used in both methods.
- To simplify the discussion, let us assume that the system is $n \times n$.
- In this case, one can show that Gaussian elimination with backward substitution requires at most $\dfrac{n^3}{3} + n^2 - \dfrac{n}{3}$ multiplications/divisions and $\dfrac{n^3}{3} + \dfrac{n^2}{2} - \dfrac{5n}{6}$ additions/subtractions.
- For Gauss-Jordan elimination, these figures are $\dfrac{n^3}{2} + n^2 - \dfrac{n}{2}$ multiplications/divisions and $\dfrac{n^3}{2} - \dfrac{n}{2}$ additions/subtractions.
- For both methods operation count is $\mathcal{O}(n^3)$.
- Gauss-Jordan method is more costly because of the increased number of multiplications/divisions.
- If our goal is to solve a given system, Gauss-Jordan method brings an unnecessary extra cost.