MAT 222 Linear Algebra
Week 9
Lecture Notes

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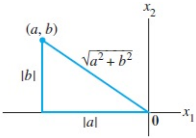
- The length or magnitude or norm of a vector v is defined as:

$$\|v\| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

  where $v_1, v_2, \ldots, v_n$ are the components of vector v.
- The length of a vector in $\mathbb{R}^n$ can be thought of as the hypotenuse of a right-angled triangle with sides parallel to the coordinate axes.
- This can be thought of a generalization of the Pythagorean Theorem. [1]



Example: Consider the vector $v = (1, -2, 3)$ in $\mathbb{R}^3$. The length of $\mathbf{v}$ is:

$$\|\mathbf{v}\| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}.$$

[1] Taken from Lay, Lay & Mcdonald, Linear Algebra, 5th Ed., page 333.

Inner Product of Vectors

- The inner product or dot product of two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is defined as

$$\mathbf{u} \cdot \mathbf{v} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n,$$

  where $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are the components of vectors u and v, respectively. Note that, if $\mathbf{u}$ and $\mathbf{v}$ are given as column vectors, then $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$.

Example: Consider the vectors $\mathbf{u} = (1, 2, -3)$ and $\mathbf{v} = (4, -5, 6)$ in $\mathbb{R}^3$. The dot product of $\mathbf{u}$ and $\mathbf{v}$ is: $\mathbf{u} \cdot \mathbf{v} = 1 \cdot 4 + 2 \cdot (-5) + (-3) \cdot 6 = -24$.

- Inner product is clearly commutative, i.e. we have $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
- The length of a vector $\mathbf{v}$ can also be expressed in terms of its inner product with itself, i.e., $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v} \longrightarrow \|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$.
- For any scalar c, we have $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$. (Why?)

Example: Consider the vector $\mathbf{v} = (1, 2, -2)$ in $\mathbb{R}^3$. The length of $\mathbf{v}$ is:

$$\|\mathbf{v}\| = \sqrt{v \cdot v} = \sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{9} = 3.$$

Exercise: Compute $\| - 5\mathbf{v}\|$.

Unit Vectors

- A unit vector is a vector of length one. A vector $\mathbf{u}$ in $\mathbb{R}^n$ is a unit vector if $\|\mathbf{u}\| = 1$.
- **Example 1:** Consider the vector $\mathbf{u} = \left(\frac{3}{5}, \frac{4}{5}\right) \in \mathbb{R}^2$. $\mathbf{u}$ is a unit vector since $\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = 1$.
- **Example 2:** Consider the vector $\mathbf{v} = \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right) \in \mathbb{R}^3$. Verify that $\mathbf{u}$ is a unit vector.
- Given a vector $\mathbf{u}$, we can find a unit vector having the same direction as $\mathbf{u}$. This can be done by dividing $\mathbf{u}$ by its length: The new vector $\dfrac{\mathbf{u}}{\|\mathbf{u}\|}$ is a unit vector since

  $\left\| \dfrac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = \dfrac{1}{\|\mathbf{u}\|} \cdot \|\mathbf{u}\| = 1$. This process is called normalizing a vector.

Example: Consider the vector $\mathbf{u} = (1, -2, 2)$ in $\mathbb{R}^3$. The length of $\mathbf{u}$ is $\|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + 2^2} = 3$. To normalize $\mathbf{u}$, we divide it by its length:

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{3}(1, -2, 2) = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right).$$

The vector $\mathbf{v}$ is a unit vector in the same direction as $\mathbf{u}$.

Distance Between Two Vectors

- The distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is the norm of the vector $\mathbf{u}-\mathbf{v}$ (or $\mathbf{v}-\mathbf{u}$). More explicitly, it is given by

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|.$$

Example: Let $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$. Then, the distance between $\mathbf{u}$ and $\mathbf{v}$ is

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \left\| \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} \right\|$$

$$= \sqrt{3^2+2^2+(-4)^2} = \sqrt{29}.$$

- Therefore, the distance between $\mathbf{u}$ and $\mathbf{v}$ is $\sqrt{29}$.

Orthogonality of Vectors

- Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are said to be **orthogonal** if their inner product is zero, that is, if $\mathbf{u} \cdot \mathbf{v} = 0$.
- **Remark:** The definition of orthogonality comes from the familiar notion of geometrical perpendicularity. Namely, if $\mathbf{u} \perp \mathbf{v}$, then $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{u}, -\mathbf{v})$. More explicitly we have

$$\begin{aligned}(d(\mathbf{u}, -\mathbf{v}))^2 &= \|\mathbf{u} - (-\mathbf{v})\|^2 = \|\mathbf{u} + \mathbf{v}\|^2 \\ &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u} \cdot \mathbf{v}\end{aligned}$$

  and

$$\begin{aligned}(d(\mathbf{u}, \mathbf{v}))^2 &= \|\mathbf{u} - \mathbf{v}\|^2 \\ &= (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v}\end{aligned}$$

  which are equal. Bu this can only be true if $\mathbf{u} \cdot \mathbf{v} = 0$.
- **Example:** Consider the vectors $\mathbf{u} = (1, 2, -1)$ and $\mathbf{v} = (-2, 3, -4)$ in $\mathbb{R}^3$. We have

$$\mathbf{u} \cdot \mathbf{v} = (1)(-2) + 2 \cdot 3 + (-1)(4) = 0.$$

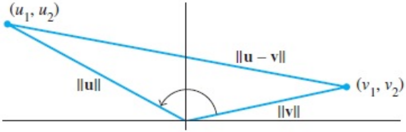
  So $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

The Angle Between Two Vectors

- Let **u** and **v** be nonzero vectors in $\mathbb{R}^n$. The cosine of the angle θ between **u** and **v** is defined by

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

- This comes from the cosine theorem. See the figure below:



- Assume **u** and **v** are in $\mathbb{R}^2$ and $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$. By the Law of Cosines, we have

$$\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

- Rearranging , we get:

$$\cos\theta = \frac{1}{2\|\mathbf{u}\|\|\mathbf{v}\|}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2)$$
$$= \frac{u_1^2 + u_2^2 + v_1^2 + v_2^2 - (u_1 - v_1)^2 - (u_2 - v_2)^2}{2\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{u_1v_1 + u_2v_2}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}.$$

- Although this proof applies only to $n = 2$, the definition is valid for all $\mathbb{R}^n$.

Example: Let $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix}$. Then we have:

$$\begin{aligned}
\cos\theta &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \\
&= \frac{(2)(0) + (1)(3) + (0)(-4)}{\sqrt{2^2 + 1^2 + 0^2}\sqrt{0^2 + 3^2 + (-4)^2}} \\
&= \frac{3}{5\sqrt{5}}
\end{aligned}$$

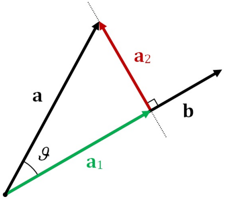
- Note that if vectors $\mathbf{a}$ and $\mathbf{b}$ are orhogonal, then $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta = 0$, which shows that $\cos\theta = 0$, hence they are perpendicular.

Orthogonal Projection

- Given vectors **u** and **v** in $\mathbb{R}^n$, we can decompose **u** into two components: $\mathbf{u}_1$, which is parallel to **v**, and $\mathbf{u}_2$, which is orthogonal to **v**.
- We can express this decomposition as $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, where

$$\mathbf{u}_1 = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} \quad \text{and} \quad \mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1.$$

- The vector $\mathbf{u}_1$ is called the orthogonal projection of **u** onto **v** and denoted by $\text{proj}_{\mathbf{v}}(\mathbf{u})$.
- To understand this formula, let us look at the following figure.



- In the figure, the projection of **a** onto **b** is $\mathbf{a}_1$, whose length is $\|\mathbf{a}_1\| = \|\mathbf{a}\| \cos(\nu)$.
- Since $\mathbf{a}_1$ is in the same direction as **b**, it must be true that $\mathbf{a}_1 = k\mathbf{b}$.
- But $\mathbf{a}_1 = \|\mathbf{a}_1\|\mathbf{b}'$, where $\mathbf{b}' = \dfrac{\mathbf{b}}{\|\mathbf{b}\|}$ is the normalized form of **b**, so $k = \dfrac{\|\mathbf{a}_1\|}{\|\mathbf{b}\|} \longrightarrow \mathbf{a}_1 = \dfrac{\|\mathbf{a}\| \cos(\nu)}{\|\mathbf{b}\|}\mathbf{b}$.
- Using the definition $\cos(\nu) = \dfrac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}$ gives the result.

Orthogonal Projection: Example

Example: Let $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ be vectors in $\mathbb{R}^2$. The orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$ is

$$\mathbf{u}_1 = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{(2)(1) + (1)(2)}{(1)^2 + (2)^2} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4/5 \\ 8/5 \end{pmatrix}.$$

The component of $\mathbf{u}$ that is orthogonal to $\mathbf{v}$ is

$$\mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 4/5 \\ 8/5 \end{pmatrix} = \begin{pmatrix} 6/5 \\ -3/5 \end{pmatrix}.$$

- In essence, if we substract from $\mathbf{u}$ its projection onto $\mathbf{v}$, then what is left behind is the component of $\mathbf{u}$ that is orthogonal to $\mathbf{v}$.
- This process can be generalized to more than two vectors.

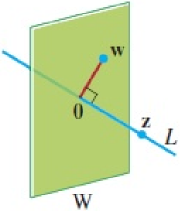
Orthogonal Complement of a Subspace

- Given a vector $\mathbf{x}$, just like we can compute its component orthogonal to some other vector $\mathbf{u}$, we can compute its component orthogonal to a set of vectors. To this end we have the following definition.

Orthogonal complement

Let V be a subspace of $\mathbb{R}^n$. The **orthogonal complement** of V, denoted by $V^\perp$, consists of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in V.

- An important property of orthogonal complement is that it is a subspace. (Exercise: Please prove it.)
- An obvious example occurs when V is any plane through the origin in $\mathbb{R}^3$. Then $V^\perp$ is the line through origin that is perpendicular to V.



Orthogonal Complement: Example

- Let us focus on planes in $\mathbb{R}^3$. Let $V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} < \mathbb{R}^3$.
- Then for any vector $\mathbf{x} \in V^\perp$ we have $\mathbf{x} \perp \mathbf{v}_1$ and $\mathbf{x} \perp \mathbf{v}_2$.
- Conversely, since any vector $\mathbf{w} \in V$ can be written as $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, $\mathbf{y} \perp \mathbf{v}_1$ and $\mathbf{y} \perp \mathbf{v}_2$ implies

$$\mathbf{y} \cdot \mathbf{w} = \mathbf{y} \cdot (c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1\mathbf{y} \cdot \mathbf{v}_1 + c_2\mathbf{y} \cdot \mathbf{v}_2 = 0 \longrightarrow \mathbf{y} \perp \mathbf{w}.$$

- Thus, we have proved the following: A vector is in $V^\perp$ if and only if it is orthogonal to every vector in a basis of V.

Example: Consider $\mathbf{v}_1 = (1, 1, -1), \mathbf{v}_2 = (1, 2, 3)$. Let $V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Find the orthogonal complement of V.

- $\mathbf{x} = (a, b, c) \in V^\perp$ if and only if $\mathbf{x} \perp \mathbf{v}_1$ and $\mathbf{x} \perp \mathbf{v}_2$.
- But this means $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- But this is the same thing as saying that $\mathbf{x} \in N(A)$, where A is the matrix whose rows are $\mathbf{v}_1$ are $\mathbf{v}_2$. So $V^\perp = N(A)$.
- In general it holds that $(R(A))^\perp = N(A)$ and $(C(A))^\perp = N(A^T)$. (Exercise)

Orthogonal Projection onto a Subspace

- Given two vectors $\mathbf{y}$ and $\mathbf{u}$ in $\mathbb{R}^3$, recall that we can decompose $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$, where $\mathbf{y}_2 \perp \mathbf{u}$ and $\mathbf{y}_1$ is the projection of $\mathbf{y}$ onto $\mathbf{u}$.
- Now we can do the same in connection with any subspace V.

Orthogonal projection onto a subspace

Let V be a subspace of $\mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$. Then $\mathbf{y}$ can be decomposed as

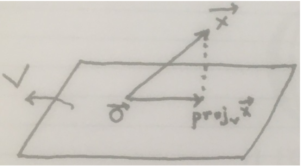
$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{w}$$

such that $\hat{\mathbf{y}} \in V$ and $\mathbf{w} \in V^{\perp}$. Here $\hat{\mathbf{y}}$ is known as the **projection of y onto** V, denoted by $\text{proj}_V\mathbf{y}$ and $\mathbf{w}$ is the component of $\mathbf{y}$ that is ortogonal to V. Furthermore, this decomposition is unique. Note that, if $\mathbf{y} \in V$, then $\text{proj}_V\mathbf{y} = \mathbf{y}$.

- **Example:** In the previous example $V = \text{Span}\{(1,1,-1),(1,2,3))\}$ and $V^{\perp} = \text{Span}\{(5,-4,1)\}$. (Please check)
- Let $\mathbf{y} = (2,0,3)$. Then $\mathbf{y} \notin V$. (Please check).
- So there is a unique real number c such that
$$\mathbf{y} = (2,0,3) = c(5,-4,1) + \text{proj}_V\mathbf{y}.$$
- We will soon see how to compute this real number c and hence $\text{proj}_V\mathbf{y}$.

Projection and the Closest Point

- An important conclusion is as follows: Orthogonal projection of a vector $\mathbf{x}$ onto a subspace is the vector in the subspace which is closest to $\mathbf{x}$.



- To see this, let us consider a subspace V of $\mathbb{R}^n$. Let us then take a vector $\mathbf{x}$ not in V and an arbitrary vector $\mathbf{u} \in V$.
- We claim that $\|\mathbf{x} - \text{proj}_V\mathbf{x}\| < \|\mathbf{x} - \mathbf{u}\|$.
- To see this, first define $\mathbf{a} = \mathbf{x} - \text{proj}_V\mathbf{x}$ and $\mathbf{b} = \text{proj}_V\mathbf{x} - \mathbf{u}$
- Then we have $\mathbf{x} - \mathbf{u} = \mathbf{a} + \mathbf{b}$. Now consider $\|\mathbf{x} - \mathbf{u}\|^2 = \|\mathbf{a} + \mathbf{b}\|^2$.
  $\|\mathbf{a}+\mathbf{b}\|^2 = (\mathbf{a}+\mathbf{b}) \cdot (\mathbf{a}+\mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} + 2\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\mathbf{a} \cdot \mathbf{b}$.
- But $\mathbf{a}$ is the component of $\mathbf{x}$ which is orthogonal to V, so it is orthogonal to every vector in V. In particular it is orthogonal to $\mathbf{b}$, so $\mathbf{a} \cdot \mathbf{b} = 0$.
- This implies $\|\mathbf{a} + \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 \geq \|\mathbf{a}\|^2 = \|\mathbf{x} - \text{proj}_V\mathbf{x}\|^2$
- So we have: The vector in V closest to $\mathbf{x}$ is the projection of $\mathbf{x}$ onto V and this distance is the norm of the component of $\mathbf{x}$ orthogonal to V.
- This result is known as the **Best Approximation Theorem**. This is because it shows the best approximation to $\mathbf{x}$ by elements of V is $\text{proj}_V\mathbf{x}$.