FACULTY OF ENGINEERING
DEPARTMENT OF COMPUTER ENGINEERING
MAT 222 LINEAR ALGEBRA
MIDTERM EXAM SOLUTIONS

1. Three truck drivers, named Haydar, Recep and Vahit, enter a restaurant in a way station. Haydar orders 1 portion of meat, 2 portions of rice and 3 portions of salad for 680 TL. Recep orders 1 portion of meat, 3 portions of rice and 5 portions of salad for 840 TL. Vahit, who has only 600 TL, wants to have exactly 1 portion of meat with 2 portions of salad and spend all his remaining money on rice. Determine at most how many portions of rice he can take in addition to meat and salad. (You can use any method except trial and error. The answer need not be an integer.)

Solution: First, we will make the assumption that rice is not free. Secondly, let x denote the number of portions of rice that Vahit can take with his remaining money. Let a, b, c denote the portion prices (in TL) of meat, salad and rice, respectively. Then we have the following system of equations:

$$\begin{aligned} a + 3b + 2c &= 680 \\ a + 5b + 3c &= 840 \\ a + 2b + xc &= 600 \end{aligned}$$

Let us apply row reduction to this linear system.

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 680 \\ 1 & 5 & 3 & 840 \\ 1 & 2 & x & 600 \end{array}\right] \xrightarrow{-1R_1+R_2,-1R_1+R_3} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 680 \\ 0 & 2 & 1 & 160 \\ 0 & -1 & x-2 & -80 \end{array}\right] \xrightarrow{\frac{1}{2}R_2+R_3} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 680 \\ 0 & 2 & 1 & 160 \\ 0 & 0 & x-3/2 & 0 \end{array}\right]$$

The third row corresponds to the equation $\left(x - \dfrac{3}{2}\right) c = 0$. Since rice is not free, we have $c > 0$. This means that $x = \frac{3}{2}$; in other words, Vahit can take at most 1.5 portions of rice.

2. Consider the following system of equations.

$$\begin{aligned} x_1 + 3x_2 + x_3 + x_4 &= 1 \\ -4x_1 - 9x_2 + 2x_3 - x_4 &= -1 \\ 2x_2 + 4x_3 + 2x_4 &= 2 \\ x_1 + 5x_2 + 5x_3 + 3x_4 &= 3 \end{aligned}$$

(a) Find the solution set of this system using Gaussian elimination.

Solution: The steps of Gaussian elimination are given below.

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 1 \\ -4 & -9 & 2 & -1 & -1 \\ 0 & 2 & 4 & 2 & 2 \\ 1 & 5 & 5 & 3 & 3 \end{array}\right] \xrightarrow{4R_1+R_2,-1R_1+R_4} \left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 1 \\ 0 & 3 & 6 & 3 & 3 \\ 0 & 2 & 4 & 2 & 2 \\ 0 & 2 & 4 & 2 & 2 \end{array}\right] \xrightarrow{-\frac{2}{3}R_2+R_3,-\frac{2}{3}R_2+R_4} \left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 1 \\ 0 & 3 & 6 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system is now in echelon form and the corresponding equations are given below:

$$\begin{aligned} x_1 + 3x_2 + x_3 + x_4 &= 1 \\ 3x_2 + 6x_3 + 3x_4 &= 3 \end{aligned}$$

Taking x_3 and x_4 as free variables, from the second equation we can write $x_2 = 1 - 2x_3 - x_4$. Using this in the first equation and simplifying gives $x_1 = 1 + 5x_3 + 2x_4$. Thus, we can write down the general solution as follows:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 + 5x_3 + 2x_4 \\ 1 - 2x_3 - x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 5 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

(b) In the solution set you found in part (a), how many solutions satisfy the equation $x_1 = x_2 + x_3 + x_4$? Briefly state your reasoning.

Solution: In part (a), we obtained x_1 as $x_1 = 1 + 5x_3 + 2x_4$. Now we also want to have $x_1 = x_2 + x_3 + x_4$. As a result, it should be true that

$$1 + 5x_3 + 2x_4 = x_2 + x_3 + x_4 \longrightarrow x_2 = 1 + 4x_3 + x_4.$$

In part (a), we also obtained x_2 as $1 - 2x_3 - x_4$. Combining these two expressions for x_2, we get

$$1 - 2x_3 - x_4 = 1 + 4x_3 + x_4 \longrightarrow x_4 = -3x_3.$$

As a result, any solution of part (a) satisfying $x_4 = -3x_3$ will also satisfy the equation $x_1 = x_2 + x_3 + x_4$. This means that there are infinitely many such solutions. Indeed, the asked solutions can be expressed as follows:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 5 \\ -2 \\ 1 \\ 0 \end{pmatrix} + (-3x_3) \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \\ -3 \end{pmatrix}.$$

3. (a) Consider the polynomials $p_1(x) = 2 + x, p_2(x) = 3x + x^2$ and $p_3(x) = -4 + x + x^2$. Find a basis for $\text{Span}\{p_1(x), p_2(x), p_3(x)\}$.

Solution: Let us check whether the set $S = \{p_1(x), p_2(x), p_3(x)\}$ is linearly independent or not. To this end, we form the system $c_1p_1(x) + c_2p_2(x) + c_3p_3(x) = \mathbf{0}$, where c_1, c_2, c_3 are unknown real numbers. More explicitly, this system can be written as follows:

$$\begin{aligned} 2c_1 - 4c_3 &= 0 \\ c_1 + 3c_2 + c_3 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

Applying row reduction to this homogeneous system, one can get

$$\begin{pmatrix} 2 & 0 & -4 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \quad \text{(The details are omitted.)}$$

This shows that the system $c_1p_1(x) + c_2p_2(x) + c_3p_3(x) = \mathbf{0}$ has infinitely many solutions, so the set S is linearly dependent. This implies that $\dim(\text{Span}(S)) < 3$. On the other hand, the set $\{p_1(x), p_2(x)\}$ is obviously linearly independent. Thus, $\dim(\text{Span}(S)) = 2$ and $\{p_1(x), p_2(x)\}$ is a basis for $\text{Span}(S)$.

(b) Consider the matrix $\mathbf{A} = \begin{pmatrix} 2 & 0 & -4 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. If possible, find a vector $\mathbf{b} \in \mathbb{R}^3$ such that the system $\mathbf{Ax} = \mathbf{b}$ has infinitely many solutions. If not, briefly explain why such a vector $\mathbf{b}$ does not exist.

Solution: Our work in part (a) already showed that the columns of $\mathbf{A}$ are linearly dependent. $\mathbf{b}$ can be chosen as any vector from the column space of $\mathbf{A}$. For instance, let us choose $\mathbf{b}$ to be the last column of $\mathbf{A}$, which is $\mathbf{a}_3 = \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}$. Then, the system $\mathbf{Ax} = \mathbf{a}_3$ has infinitely many solutions. This is true because the columns of $\mathbf{A}$ are linearly dependent, which implies that $\mathbf{a}_3$ can be expressed as a linear combination of the columns of $\mathbf{A}$ in infinitely many different ways. Each of these gives rise to a solution of $\mathbf{Ax} = \mathbf{a}_3$. In fact, you can verify that the general solution of $\mathbf{Ax} = \mathbf{a}_3$ is

$$\mathbf{x} = \begin{pmatrix} -2+2t \\ 1-t \\ t \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

4. Let V be a vector space and let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p, \mathbf{v}\}$ be a subset of V. Suppose further that $(a_1, a_2, \ldots, a_p)$ and $(b_1, b_2, \ldots, b_p)$ are two distinct vectors from $\mathbb{R}^p$ such that

$$\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \ldots + a_p\mathbf{u}_p = b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + \ldots + b_p\mathbf{u}_p.$$

Show that the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is not a basis of $\mathrm{Span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$.

Solution: Let us focus on the equality $a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \ldots + a_p\mathbf{u}_p = b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + \ldots + b_p\mathbf{u}_p$. From this it follows that

$$(a_1 - b_1)\mathbf{u}_1 + (a_2 - b_2)\mathbf{u}_2 + \ldots + (a_p - b_p)\mathbf{u}_p = \mathbf{0}. \tag{1}$$

Since $(a_1, a_2, \ldots, a_p)$ and $(b_1, b_2, \ldots, b_p)$ are different vectors, it follows that at least one of the coefficients $a_i - b_i$ in the equality (1) is nonzero. This means that the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is linearly dependent. Therefore, it cannot be a basis.

5. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Let the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ be such that $T(\mathbf{v}_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $T(\mathbf{v}_2) = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$ and $T(\mathbf{v}_3) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Find a matrix $\mathbf{A}$ such that $T(\mathbf{x}) = \mathbf{Ax}$ for every $\mathbf{x} \in \mathbb{R}^3$.

Solution method 1: In order to find the requested matrix $\mathbf{A}$, one method is to compute the images of the standard basis vectors under T. More explicitly, if $\mathbf{e}_1 = (1,0,0)^T, \mathbf{e}_2 = (0,1,0)^T, \mathbf{e}_3 = (0,0,1)^T$, then $T(\mathbf{e}_1), T(\mathbf{e}_2)$ and $T(\mathbf{e}_3)$ will constitute the columns of $\mathbf{A}$, in the same order. To this end, we will first express $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ in terms of the given vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$. One can check that this is possible since the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. Then, we must find the numbers $c_{j,k}$ such that

$$\begin{aligned} \mathbf{e}_1 &= c_{1,1}\mathbf{v}_1 + c_{1,2}\mathbf{v}_2 + c_{1,3}\mathbf{v}_3 \\ \mathbf{e}_2 &= c_{2,1}\mathbf{v}_1 + c_{2,2}\mathbf{v}_2 + c_{2,3}\mathbf{v}_3 \\ \mathbf{e}_3 &= c_{3,1}\mathbf{v}_1 + c_{3,2}\mathbf{v}_2 + c_{3,3}\mathbf{v}_3 \end{aligned}$$

Here we have three linear systems all of which can be solved by Gaussian elimination. Indeed, since each has the same coefficient matrix, we can combine them in a single problem whose augmented matrix is

$$\left[\begin{array}{c|c} \mathbf{B} & \mathbf{I} \end{array}\right] = \left[\begin{array}{ccc|ccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & -2 & 0 & 0 & 0 & 1 \end{array}\right].$$

Therefore, the solutions of these systems, which are the coefficients $c_{j,k}$, will give the inverse of the matrix $\mathbf{B}$. Gauss-Jordan elimination (the details are left out of this solution for brevity) gives the inverse matrix

$$\mathbf{B}^{-1} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & -1 & -1 \\ 1 & -1 & 0 \end{bmatrix},$$

so that we have

$$\begin{aligned} \mathbf{e}_1 &= 0\mathbf{v}_1 + 0\mathbf{v}_2 + 1\mathbf{v}_3 \\ \mathbf{e}_2 &= 2\mathbf{v}_1 - 1\mathbf{v}_2 - 1\mathbf{v}_3 \\ \mathbf{e}_3 &= 1\mathbf{v}_1 - 1\mathbf{v}_2 + 0\mathbf{v}_3. \end{aligned}$$

Then, using the linearity of T we can write

$$\begin{aligned} T(\mathbf{e}_1) &= 0T(\mathbf{v}_1) + 0T(\mathbf{v}_2) + 1T(\mathbf{v}_3) = 1\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ T(\mathbf{e}_2) &= 2T(\mathbf{v}_1) - 1T(\mathbf{v}_2) - 1T(\mathbf{v}_3) = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 1\begin{pmatrix} 0 \\ -3 \end{pmatrix} - 1\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \\ T(\mathbf{e}_3) &= 1T(\mathbf{v}_1) - 1T(\mathbf{v}_2) + 0T(\mathbf{v}_3) = 1\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 1\begin{pmatrix} 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}. \end{aligned}$$

As a result, the requested matrix is

$$\mathbf{A} = \left[\; T(\mathbf{e}_1) \;\middle|\; T(\mathbf{e}_2) \;\middle|\; T(\mathbf{e}_3) \;\right] = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 6 & 4 \end{bmatrix}.$$

Solution method 2: There is a more concise method for this type of problem. It is described in the solution of the example problem in the 10th page of “Week 7 Lecture Notes”. You can access and read it via the “Lecture Notes” folder in the Teams group for this course.