MAT 222 Linear Algebra and Numerical Methods
Week 6
Lecture Notes 3

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Extending a Given Set to a Basis

- Recall this theorem about spanning sets: If $\mathbf{u} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, then $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p, \mathbf{u}\}$.
- This theorem has a "dual", stated as follows: If $\dim(V) = n$ and $\mathcal{B}_1$ is a linearly independent subset of V with less than $p < n$ elements, then we can find a subset $\mathcal{B}_2 = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{n-p}\}$ of V such that $\mathcal{B}_1 \bigcup \mathcal{B}_2$ is a basis for V.
- This process is known as "extension to a basis".
- As an example, let us complete the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$ to a basis of $\mathbb{R}^3$.
- One method of doing this relies on the following fact: Given an $m \times n$ matrix A, if α is a basis for $R(A)$ and β is a basis for $N(A)$, then $\alpha \cup \beta$ is a linearly independent set. (We will prove this fact later.)
- This shows that $\alpha \cup \beta$ is a basis for $\mathbb{R}^n$ since its number of elements is n due to $\text{rank}(A) + \text{nullity}(A) = n$.
- Thus, we begin by forming the matrix A whose rows are the vectors of the given set.

Extending a Given Set to a Basis: Method 1

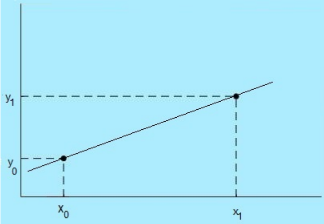
- So we form the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.
- We need only one vector to complete the given set to a basis of $\mathbb{R}^3$. This vector will come from the null space of A.
- Consider the system $A\mathbf{x} = \mathbf{0}$. An echelon form is $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \end{bmatrix}$, leading to the system $\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ -x_2 + x_3 &= 0 \end{aligned}$.
- The solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}$.
- So the set $\left\{ \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $N(A)$.
- Thus, the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

Extending a Given Set to a Basis: Method 2

- A second method is based on the following obvious fact: Given a vector space V and a basis $\mathcal{B}$ of V, any set containing $\mathcal{B}$ spans V.
- This implies that, given a linearly independent subset $\{\mathbf{v}_1, \mathbf{v}_2 \dots, \mathbf{v}_p\}$ of $\mathbb{R}^n$ with $p < n$, the set $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2 \dots, \mathbf{v}_p, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ spans $\mathbb{R}^n$.
- Thus, if we form the matrix A whose columns are the vectors in $\mathcal{S}$, then $C(A) = \mathbb{R}^n$. Hence a basis for $C(A)$ is a basis for $\mathbb{R}^n$.
- In order to extend the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$ to a basis of $\mathbb{R}^3$, we first form the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 4 & 0 & 0 & 1 \end{bmatrix}$.
- The reduced echelon form of A is $\begin{bmatrix} 1 & 0 & 0 & 4/5 & -1/5 \\ 0 & 1 & 0 & -3/5 & 2/5 \\ 0 & 0 & 1 & -1/5 & -1/5 \end{bmatrix}$. The pivot columns are the first 3 columns.
- Thus, a basis for $\mathbb{R}^3$ containing the given set is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

An Application to Polynomial Interpolation

- We have seen that the set $\mathbb{P}_n$ of polynomials with degree not exceeding n is an $(n+1)$-dimensional vector space.
- In mathematics and particularly in approximation theory, the problem of finding a polynomial passing through given points is important.
- Given $(n+1)$ points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ with different first coordinates, there is a unique polynomial $P \in \mathbb{P}_n$ passing through all of these $n+1$ points.
- This is an extension of the well-known fact that only one line can pass through two given lines. (See the figure)



- The equation of this line can be written in two ways:

$$(1) \quad y = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$

$$(2) \quad y = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}$$

An Application to Polynomial Interpolation

- Given $n+1$ points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, we want to find a polynomial P with $\deg(P) \leq n$ such that $P(x_0) = y_0, P(x_1) = y_1, \ldots, P(x_n) = y_n$.
- Let $P(x) = c_0 + c_1 x + \ldots + c_n x^n$. Our aim is to find the coefficients $c_0, c_1, \ldots, c_n$. The problem leads to the following equations.

$$\begin{aligned} P(x_0) &= c_0 + c_1 x_0 + c_2 x_0^2 + \ldots + c_n x_0^n &= y_0 \\ P(x_1) &= c_0 + c_1 x_1 + c_2 x_1^2 + \ldots + c_n x_1^n &= y_1 \\ \vdots &= \qquad \vdots &= \vdots \\ P(x_n) &= c_0 + c_1 x_n + c_2 x_n^2 + \ldots + c_n x_n^n &= y_n \end{aligned}$$

- In matrix form, we have $\underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}.$
- The determinant $\det(A)$, known as the **Vandermonde determinant**, is known to be nonzero, so the above system has a unique solution.

An Application to Polynomial Interpolation

- Solving the system requires $\mathcal{O}(n^3)$ multiplications, which may be too much for large systems.
- We have another method which requires $\mathcal{O}(n^2)$ multiplications.
- This method proceeds as follows: For each $k = 0, 1, \ldots, n$, construct a polynomial $L_k(x)$ such that $L_k(x_k) = 1$ and $L_k(x_j) = 0$ for $j \neq k$.
- These polynomials are

$$\begin{aligned} L_k(x) &= \frac{(x - x_0)(x - x_1)\ldots(x - x_{k-1})(x - x_{k+1})\ldots(x - x_n)}{(x_k - x_0)(x_k - x_1)\ldots(x_k - x_{k-1})(x_k - x_{k+1})\ldots(x_k - x_n)} \\ &= \prod_{j \neq k} \frac{x - x_j}{x_k - x_j} \end{aligned},$$

  where $k = 0, 1, \ldots, n$.
- We claim that the set $\mathcal{L} = \{L_0(x), L_1(x), \ldots, L_n(x)\}$ is a basis for $\mathbb{P}_n$.
- This basis is called the Lagrange basis for $\mathbb{P}_n$ and the polynomials $L_k(x)$ are called Lagrange basis polynomials.

An Application to Polynomial Interpolation

- To see that the set $\mathcal{L}$ is a basis for $\mathbb{P}_n$, it suffices to show that $\text{Span}(\mathcal{L}) = \mathbb{P}_n$ since $\mathcal{L}$ has $\dim(\mathbb{P}_n) = n+1$ elements.
- To show $\text{Span}(\mathcal{L}) = \mathbb{P}_n$, let us take any polynomial $q \in \mathbb{P}_n$.
- Consider the values of q at the $n+1$ points $x_0, x_1, \ldots, x_n$. Let $q(x_0) = b_0, q(x_1) = b_1, \ldots, q(x_n) = b_n$.
- Then the polynomial $s(x) = b_0L_0(x) + b_1L_1(x) + \ldots + b_nL_n(x)$ has the same values as $p(x)$ at the $n+1$ points $x_0, x_1, \ldots, x_n$.
- This implies that the polynomial $(s-q)(x) = 0$ at these $n+1$ points.
- Since $\deg(s-q) \leq n$, this shows that $q(x) = s(x)$ since a polynomial with degree at most n cannot have $n+1$ roots.
- As a result, any polynomial in $\mathbb{P}_n$ is in $\text{Span}(\mathcal{L})$, which shows that $\mathcal{L}$ is a basis for $\mathbb{P}_n$.
- This procedure also solves the interpolation problem. The Lagrange interpolating polynomial is the polynomial $P(x) = y_0L_0(x) + y_1L_1(x) + \ldots + y_nL_n(x)$.

An Application to Polynomial Interpolation: Example

- The 4 points $(x_0, y_0) = (1, 10), (x_1, y_1) = (2, 9), (x_2, y_2) = (2,5, 9.25), (x_3, y_3) = (3, 10)$ are given. Let us find the polynomial $P \in \mathbb{P}_3$ passing through these four points.
- The Lagrange basis polynomials are

$$L_0(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} = -\frac{1}{3}(x - 2)(x - 2.5)(x - 3)$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} = 2(x - 1)(x - 2.5)(x - 3)$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} = -\frac{8}{3}(x - 1)(x - 2)(x - 3)$$

$$L_3(x) = \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} = (x - 1)(x - 2)(x - 2.5)$$

- Thus, the polynomial we are looking for is

$$\begin{aligned} P(x) &= y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x) + y_3 L_3(x) \\ &= 10L_0(x) + 9L_1(x) + 9.25L_2(x) + 10L_3(x) \\ &= -\frac{10}{3}(x - 2)(x - 2.5)(x - 3) + 18(x - 1)(x - 2.5)(x - 3) \\ &- \frac{74}{3}(x - 1)(x - 2)(x - 3) + 10(x - 1)(x - 2)(x - 2.5) \end{aligned}$$

An Application to Polynomial Interpolation: Example

- After some simplifications, the Lagrange interpolating polynomial is computed to be $P(x) = x^2 - 4x + 13$.
- This shows that the four given points are on the same parabola.
- **Exercise:** Find the Lagrange interpolating polynomial $P \in \mathbb{P}_2$ passing through the points $(x_0, y_0) = (1, 3.5), (x_1, y_1) = (3, -1), (x_2, y_2) = (4, 2)$.
- **Exercise:** Extend the set $\left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ to a basis of $\mathbb{R}^4$ using both methods.

An Application to Nutrition Plans

- Recall the following problem from the first week.

Setting up a diet

Rabbits in a lab are given a strict diet consisting of three food types labeled Mix A, Mix B, Mix C. Total micro contents of these mixtures are given below.

	Carbohydrate (g)	Protein (g)	Fat (g)
Mix A	3	1	2
Mix B	1	5	6
Mix C	12	4	0.5

If each rabbit is required to take 35 grams of carbohydrates, 28 grams of proteins and 27 grams of fats daily, set up the system of equations required to calculate how many units of each food type should be given to the rabbits daily to meet the mentioned dietary need.

- It turns out that the correct combination is 2.5 units of Mix A, 3.5 units of Mix B, 2 units of Mix C.
- In vector form we have $\begin{bmatrix} 35 \\ 28 \\ 27 \end{bmatrix} = 2.5 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + 3.5 \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} + 2 \begin{bmatrix} 12 \\ 4 \\ 0.5 \end{bmatrix}$.

An Application to Nutrition Plans

- We also posed the following problem: Is it possible to satisfy every possible dietary need using only these three food types?
- This is roughly equal to asking: Do the set of foods $\{$Mix A, Mix B, Mix C$\}$ form a basis for the 3-dimensional "nutrition space"?
- In vector notation, this is equivalent to the following question "Is the set $\left\{ \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 12 \\ 4 \\ 0.5 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?"
- There is a diference of course: In $\mathbb{R}^3$ negative coordinates is allowed, but in the "nutrition space" negative coordinates is nonsense.
- For instance, the nutrition goal "38 grams of carbohydrate, 8 grams of protein, 5 grams of fat" is attained by 5 units of Mix A, -1 units of Mix B, 2 units of Mix C, which is a little disturbing.
- Ignoring this fact, let us pose the following problem. Suppose Mix A contains 120 mg of calcium, Mix B contains 90 mg of calcium and Mix C contains no calcium. Invent a 4th food mixture such that the 4 mixtures together span the 4-dimensional nutrition space. (Your invented mixture must have nonnegative components, of course.) (**Exercise**)

Linear Algebra in a Nutshell

- Let A be an $n \times n$ matrix. Then the following are equivalent.

(1) A is invertible.

(2) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(3) $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in M_{n\times 1}(\mathbb{R})$.

(4) Rows of A are linearly independent.

(5) Columns of A are linearly independent.

(6) Column space of A is $\mathbb{R}^n$.

(7) Row space of A is $\mathbb{R}^n$.

(8) Columns of A constitute a basis for $\mathbb{R}^n$.

(9) Rows of A constitute a basis for $\mathbb{R}^n$.

(10) $\text{rank}(A) = n$.

(11) A has no zero rows in the echelon form.

(12) Null space of A is the trivial subspace.

(13) $\det(A) \neq 0$

(14) The reduced echelon form of A is I.