MAT 222 Linear Algebra and Numerical Methods
Week 2
Lecture Notes 1

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Every Linear System Can be Represented by a Matrix-Vector Product

- Let us consider the system
$$\begin{aligned} x+y+z&=9\\ 2x+3y-3z&=1.\\ -3x+2y-5z&=0 \end{aligned}$$

- Its column picture is

$$x\begin{bmatrix}1\\2\\-3\end{bmatrix}+y\begin{bmatrix}1\\3\\2\end{bmatrix}+z\begin{bmatrix}1\\-3\\-5\end{bmatrix}=\begin{bmatrix}1&1&1\\2&3&-3\\-3&2&-5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}9\\1\\0\end{bmatrix}$$

- The matrix $\begin{bmatrix}1&1&1\\2&3&-3\\-3&2&-5\end{bmatrix}$ is known as the coefficient matrix of the system and $\mathbf{b}=\begin{bmatrix}9\\1\\0\end{bmatrix}$ is the vector on the right-hand side. $\mathbf{x}=\begin{bmatrix}x\\y\\z\end{bmatrix}$ is the vector of unknowns. So the system can be written as

$$A\mathbf{x}=\mathbf{b}$$

Augmented Matrix of A Linear System

- Given a linear system $A\mathbf{x} = \mathbf{b}$, it is customary to refer to its size by the size of its coefficient matrix $\mathbf{A}$. Therefore the size of the system

$$\begin{aligned} x + y + z &= 9 \\ 2x + 3y - 3z &= 1 \\ -3x + 2y - 5z &= 0 \end{aligned}$$

  is 3×3.

- There is an even more concise way to refer to this system. Note that the system is completely determined by the coefficint matrix A and the right-hand side vector $\mathbf{b}$.

- Thus, the matrix $\left[A|\mathbf{b}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & 3 & -3 & 1 \\ -3 & 2 & -5 & 0 \end{array}\right]$ can be used to represent the system $A\mathbf{x} = \mathbf{b}$. It is called the augmented matrix of the system.

- Every linear system is determined by its augmented matrix.

Examples

(1) $$\begin{aligned} 2a - b &= 4 \\ -a + 3b &= 3 \end{aligned}$$ Augmented matrix is $\left[\begin{array}{rr|r} 2 & -1 & 4 \\ -1 & 3 & 3 \end{array}\right]$.

(2) $$\begin{aligned} 2x_1 - x_2 - x_3 &= 3 \\ 2x_2 + 3x_3 &= 5 \end{aligned}$$ Augmented matrix is $\left[\begin{array}{rrr|r} 2 & -1 & -1 & 3 \\ 0 & 2 & 3 & 5 \end{array}\right]$.

(3) $$\begin{aligned} 2x + y &= 5 \\ -x + 4y &= \sqrt{3} \\ 2x - 3y &= 0 \end{aligned}$$ Augmented matrix is $\left[\begin{array}{rr|r} 2 & 1 & 5 \\ -1 & 4 & \sqrt{3} \\ 2 & -3 & 0 \end{array}\right]$.

- Note that the vertical bar can be omitted: $\begin{bmatrix} 2 & -1 & 4 \\ -1 & 3 & 3 \end{bmatrix}$ etc.
- **Exercise:** Write the augmented matrix of the system
$$\begin{aligned} 3x_1 + x_2 - x_4 &= 1 \\ -2x_1 - x_3 + x_4 &= 0 \\ x_2 - 3x_1 &= 4 \end{aligned} \quad .$$

Matrix Representation of a Linear System

Matrix representation of a general linear system

Consider the $m \times n$ linear system

$$
\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n &= b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n &= b_2 \\
\vdots &= \vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n &= b_m \, .
\end{aligned}
$$

It can be represented by the matrix-vector product $A\mathbf{x} = \mathbf{b}$, where

$$
A = \begin{bmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m,1} & a_{m,2} & \ldots & a_{m,n} \end{bmatrix}, \; \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \; \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},
$$

or by its augmented matrix $\left[A|\mathbf{b}\right] = \left[\begin{array}{cccc|c} a_{1,1} & a_{1,2} & \ldots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} & b_2 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \ldots & a_{m,n} & b_m \end{array}\right]$.

Elementary Row Operations

- Matrix representation of a linear system will make it easier to solve it in a systematic way.
- First, we have several observations.
- Given a linear system of equations,

(O1) multiplying one of the equations by a nonzero scalar does not change its solution set.

(O2) switching the order of two equations does not change its solution set.

(O3) adding a scalar multiple of one equation to another does not change its solution set.

- These three operations are knowns as elementary row operations. They form the building blocks of many algorithms in linear algebra.

Elementary Row Operations

Example

- Consider the system $\begin{aligned} x+y+z&=9\\ 2x+3y-3z&=1\\ -3x+2y-5z&=0 \end{aligned}$.

  Multiplying the second row by -2 results in the system

  $$\begin{aligned} x+y+z&=9\\ -4x-6y+6z&=-2\\ -3x+2y-5z&=0 \end{aligned}.$$

  Clearly, it has the same solution set as the original system.
- This time, let us add 3 times the first equation to the third equation, which gives the system $\begin{aligned} x+y+z&=9\\ 2x+3y-3z&=1\\ 5y-2z&=27 \end{aligned}$.

  This also has the same solution set as the original system. (Why?)
- The fact that switching the order of equations does not change the solution set is trivial.

Working with the Augmented Matrix

- When performing row operations, note that one can use the augmented matrix of the system instead of the equations themselves.
- Consider the row operation $3R_1 + R_3$ (add 3 times row 1 to row 3) applied to the system in the previous page.

$$\begin{aligned} x + y + z &= 9 \\ 2x + 3y - 3z &= 1 \\ -3x + 2y - 5z &= 0 \end{aligned} \xrightarrow{3R_1+R_3} \begin{aligned} x + y + z &= 9 \\ 2x + 3y - 3z &= 1 \\ 5y - 2z &= 27 \end{aligned}$$

The resulting system has the augmented matrix $\left[\begin{array}{rrr|r} 1 & 1 & 1 & 9 \\ 2 & 3 & -3 & 1 \\ 0 & 5 & -2 & 27 \end{array}\right]$.

- Alternatively, let us apply the same operation to the augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 9 \\ 2 & 3 & -3 & 1 \\ -3 & 2 & -5 & 0 \end{array}\right] \xrightarrow{3R_1+R_3} \left[\begin{array}{rrr|r} 1 & 1 & 1 & 9 \\ 2 & 3 & -3 & 1 \\ 0 & 5 & -2 & 27 \end{array}\right].$$

- Two results are the same. This shows that we can work with the augmented matrix. The result will not change.

Using Row Operations to Simplify a Linear System

- **Question:** What are elementary row operations for?
- **Answer:** Simple. In order to transform a linear system into a more easily tractable form.
- Consider the systems $\begin{aligned} x+y+z&=9 \\ 2x+4y-3z&=1 \\ 3x+6y-5z&=0 \end{aligned}$ and $\begin{aligned} x+y+z&=9 \\ 2y-5z&=-17 \\ -z&=-3 \end{aligned}$.
- The second system is just a transformed version of the first system by means of three elementary row operations. Thus they have the same solutions. They are called equivalent systems.
- Note that the second equation is in a much simpler form: The last equation contains only the unknown z and the second equation contains y and z.
- This special structure of the second system enables us to obtain the values of x, y, z almost without effort.
- We will see that any linear system can be converted to an eqivalent system in a similar simple form.

Gaussian Elimination

- Let us now examine the last system in more detail.

$$\begin{aligned} x + y + z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

- An obvious way to proceed is to use the first equation to eliminate the terms containing the unknown x from the other two equations.
- This can be achieved using elementary row operations of type 3: Add a scalar multiple of a row to another.
- Thus, we begin by asking ourselves the following question: "What multiple of the first equation should I add to the second equation so that the unknown x vanishes from the second equation?"
- The answer is -2.
- Can you answer the corresponding question for the third equation?

Gaussian Elimination

- The method begins as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 2 & -5 & -17 \\ 3 & 6 & -5 & 0 \end{array}\right] \xrightarrow{-3R_1+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 2 & -5 & -17 \\ 0 & 3 & -8 & -27 \end{array}\right]$$

- In fact, both row operations may be performed together:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right] \xrightarrow[]{\substack{-2R_1+R_2 \\ -3R_1+R_3}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 2 & -5 & -17 \\ 0 & 3 & -8 & -27 \end{array}\right]$$

- We have used the entry in the $(1,1)$ position of the augmented matrix (which is equal to 1) to make the other entries in the first column equal to 0. In other words, we have eliminated all the entries in the first column except the first entry.

- Stated in terms of equations, we now have $\begin{aligned} x+y+z&=9 \\ 2y-5z&=-17 \\ 3y-8z&=-27 \end{aligned}$.

Gaussian Elimination

- How to proceed to the final solution?
- An obvious way is to use the entry in the $(2,2)$ position to eliminate the entry in the $(2,3)$ position, which holds the term containing y.
- The required row operation is Add $-\frac{3}{2}$ multiple of second row to third row. The result is

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 9 \\ 0 & 2 & -5 & -17 \\ 0 & 3 & -8 & -27 \end{array}\right] \xrightarrow{-3/2R_2+R_3} \left[\begin{array}{rrr|r} 1 & 1 & 1 & 9 \\ 0 & 2 & -5 & -17 \\ 0 & 0 & -1/2 & -3/2 \end{array}\right].$$

- Note that we could not have eliminated the entry in the $(3,2)$ position using the first row. (Why not?)
- The resulting system is

$$\begin{aligned} x+y+z &= 9 \\ 2y-5z &= -17 \\ -\frac{1}{2}z &= -\frac{3}{2} \end{aligned}$$

Backward Substitution

- At this point, there is no need for further simplification.
- The solution can be obtained by the process of backward substitution.
- Namely, the last equation $-\frac{1}{2}z = -\frac{3}{2}$ gives $z = 3$.
- Substituting this value in the second equation $2y - 5z = -17$, we find $y = -1$.
- Lastly, substituting $y = -1$ and $z = 3$ in the first equation $x + y + z = 9$, we obtain $x = 7$.
- Thus, the solution of the system is $x = 7, y = -1, z = 3$. It is the only solution.
- The entire method is called the method of elimination or Gaussian elimination.
- The goal of Gaussian elimination is to convert, or to reduce, the given linear system into a form that can be solved by backward substitution. So it is also known as row reduction.
- Note that only row operation of type 3 was enough for this system.

Gaussian Elimination: Example 2

- Let us now solve the following system: $\begin{array}{r} 2x - 4y - z = 4 \\ x - 2y + 2z = -1 \\ 4x + z = 1/2 \end{array}$.
- We first eliminate the second and third entries of the first column.
$$\left[\begin{array}{ccc|c} 2 & -4 & -1 & 4 \\ 1 & -2 & 2 & -1 \\ 4 & 0 & 1 & 1/2 \end{array}\right] \xrightarrow{-1/2R_1+R_2,\,-2R_1+R_3} \left[\begin{array}{ccc|c} 2 & -4 & -1 & 4 \\ 0 & 0 & 5/2 & -3 \\ 0 & 8 & 3 & -15/2 \end{array}\right]$$
- This time we have a zero in the $(2,2)$ position, so we cannot use it to eliminate the entry in the $(3,2)$ position.
- We can now swap the last two rows to get
$$\left[\begin{array}{ccc|c} 2 & -4 & -1 & 4 \\ 0 & 0 & 5/2 & -3 \\ 0 & 8 & 3 & -15/2 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 2 & -4 & -1 & 4 \\ 0 & 8 & 3 & -15/2 \\ 0 & 0 & 5/2 & -3 \end{array}\right],$$
which has the corresponding equation form $\begin{array}{r} 2x - 4y - z = 4 \\ 8y + 3z = -15/2. \\ 5z/2 = -3 \end{array}$
- Backward substitution gives $x = 17/40$, $y = -39/80$, $z = -6/5$.

Gaussian Elimination: Example 3

- Consider the 4×4 system

$$\begin{aligned} x - 3y + z &= -2 \\ 2x - 6y - z + 2t &= 6 \\ -x + y + 3z + 4t &= -8 \\ 4y - z - 4t &= 1 \end{aligned}$$

- Gauss elimination begins as follows:

$$\left[\begin{array}{cccc|c} 1 & -3 & 1 & 0 & -2 \\ 2 & -6 & -1 & 2 & 6 \\ -1 & 1 & 3 & 4 & -8 \\ 0 & 4 & -1 & -4 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2,\,1R_1+R_3} \left[\begin{array}{cccc|c} 1 & -3 & 1 & 0 & -2 \\ 0 & 0 & -3 & 2 & 10 \\ 0 & -2 & 4 & 4 & -10 \\ 0 & 4 & -1 & -4 & 1 \end{array}\right]$$

- Now, the position $(2, 2)$ contains a zero, so we perform a row swap.

$$\left[\begin{array}{cccc|c} 1 & -3 & 1 & 0 & -2 \\ 0 & 0 & -3 & 2 & 10 \\ 0 & -2 & 4 & 4 & -10 \\ 0 & 4 & -1 & -4 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & -3 & 1 & 0 & -2 \\ 0 & -2 & 4 & 4 & -10 \\ 0 & 0 & -3 & 2 & 10 \\ 0 & 4 & -1 & -4 & 1 \end{array}\right]$$

Gaussian Elimination: Example 3

- $$\left[\begin{array}{cccc|c} 1 & -3 & 1 & 0 & -2 \\ 0 & -2 & 4 & 4 & -10 \\ 0 & 0 & -3 & 2 & 10 \\ 0 & 4 & -1 & -4 & 1 \end{array}\right] \xrightarrow{2R_2+R_4} \left[\begin{array}{cccc|c} 1 & -3 & 1 & 0 & -2 \\ 0 & -2 & 4 & 4 & -10 \\ 0 & 0 & -3 & 2 & 10 \\ 0 & 0 & 7 & 4 & -19 \end{array}\right]$$
$$\xrightarrow{7/3R_3+R_4} \left[\begin{array}{cccc|c} 1 & -3 & 1 & 0 & -2 \\ 0 & -2 & 4 & 4 & -10 \\ 0 & 0 & -3 & 2 & 10 \\ 0 & 0 & 0 & 26/3 & 13/3 \end{array}\right]$$
- So we have obtained

$$\begin{aligned} x - 3y + z &= -2 \\ -2y + 4z + 4t &= -10 \\ -3z + 2t &= 10 \\ 26t/3 &= 13/3 \end{aligned}$$

- Backward substitution gives $x = 1, y = 0, z = -3, t = \frac{1}{2}$.
- Note that we still have not used any row operation of type 1 (multiply an equation by a nonzero scalar). Indeed it is not necessary in Gaussian elimination.