MAT 222 Linear Algebra and Numerical Methods
Week 4
Lecture Notes 1

Murat Karaçayır

Akdeniz University
Department of Mathematics

4th March 2025

Summary of Gauss-Jordan Elimination

- Below is a schematic summary of Gauss-Jordan Elimination performed on a full rank $n \times n$ matrix A.

$$\left[\, A \,\middle|\, \mathbf{I} \,\right] \longrightarrow \left[\, \mathbf{I} \,\middle|\, A^{-1} \,\right]$$

- The inverse A^{-1} can be used to obtain the unique solution of $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{b}$, but cost of computing it significantly high.
- It is useful when we want to solve a number of systems of type

$$A\mathbf{x} = \mathbf{b}_1,\ A\mathbf{x} = \mathbf{b}_2,\ \ldots, A\mathbf{x} = \mathbf{b}_k$$

  all having the same coefficient matrix.
- Still it would be better if we could find a faster method to solve multiple systems of this type.

Row Operations and Matrix Multiplication

- We had seen that performing Gauss-Jordan elimination on a matrix has the effect of multiplying the matrix by another matrix.
- In one of our examples, we had reduced the matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 10 \\ -1 & 2 & -2 \end{bmatrix}$ to its reduced echelon form, which was $\mathbf{I}$.
- It turned out that the result was just CA, where $C = \begin{bmatrix} 10 & -2 & 5 \\ 6 & -1 & 4 \\ 1 & 0 & 1 \end{bmatrix}$. (Actually it is the inverse of A.)
- Indeed, we will see that any single row operation amounts to multiplication by a matrix.

Elementary Matrices

- As an example, let us consider $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$.
- Let us apply Gauss elimination to A.
- $$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix} \xrightarrow{1R_1+R_3}$$
$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix} \xrightarrow{1R_2+R_3} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$
- The resulting echelon matrix is in upper triangular form. Let us call it U.
- The row operations are:
  1: Add -2 times Row 1 to Row 2
  2: Add 1 times Row 1 to Row 3
  3: Add 1 times Row 2 to Row 3
- Each of these row operations is equivalent to a matrix multiplication.

Elementary Matrices

- The row operation "Add -2 times Row 1 to Row 2" corresponds to multiplication (from left) by the matrix $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
- We have
$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}$$
- This can be quickly observed by using matrix multiplication in terms of row-vector products: The first row of $E_1 A$ is the first row of A, the second row of $E_1 A$ is -2 times first row of A plus second row of A, the third row of $E_1 A$ is third row of A.
- Note that E_1 can be obtained from $\mathbf{I}$ by the same row operation "Add -2 times Row 1 to Row 2".

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_1 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Similarly, the operation "Add 1 times Row 1 to Row 3" is equivalent to multiplication by $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and the operation "Add 1 times Row 2 to Row 3" is equivalent to multiplication by $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.
- So we have

$$E_2(E_1A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix} \text{ and}$$

$$E_3(E_2E_1A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

- In short, $A \xrightarrow{-2R_1+R_2} E_1A \xrightarrow{1R_1+R_3} E_2E_1A \xrightarrow{1R_2+R_3} E_3E_2E_1A = U$

Elementary matrices

An elementary matrix is a matrix which is obtained from the identity matrix $\mathbf{I}$ by a single elementary row operation.

- E_1, E_2 and E_3 are elementary matrices. They are obtained from $\mathbf{I}$ by a row operation of type (3) (Add multiple of one row to another).
- The other two row operations also correspond to elementary matrices. We have for example $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-5R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$.
- $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$ are also elementary matrices.
- The former is known as a special type of permutation matrices. It can be obtained from the identity by a row swap.

Inverses of Elementary Matrices

- Note that the inverse of an elementary matrix is an elementary matrix of the same type.
- In order to invert "Add -2 times Row 1 to Row 2", you perform "Add 2 times Row 1 to Row 2".
- So the inverse of $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (Please check it)
- Note that the inverse of a permutation-type elementary matrix is itself.

  $$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- What is the inverse of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$?

Reversing Gaussian Elimination

- For our example matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$, we have $E_3 E_2 E_1 A = U$, where $U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ is the echelon form of A.
- How can we get from U to A?
- The row operations corresponding to E_1, E_2 and E_3 (in that order) took us from A to U.
- $A \xrightarrow{-2R_1+R_2} E_1 A \xrightarrow{1R_1+R_3} E_2 E_1 A \xrightarrow{1R_2+R_3} E_3 E_2 E_1 A = U$
- To reverse the procedure, we start with the last row operation and apply its inverse. We do the same thing with the other operations until we obtain back A.

  $$U \xrightarrow{-1R_2+R_3} E_3^{-1} U \xrightarrow{-1R_1+R_3} E_2^{-1} E_3^{-1} U \xrightarrow{2R_1+R_2} E_1^{-1} E_2^{-1} E_3^{-1} U = A$$
- Therefore we have $A = E_1^{-1} E_2^{-1} E_3^{-1} U$.

Reversing Gaussian Elimination

- The inverse matrices $E_1^{-1}, E_2^{-1}, E_3^{-1}$ can easily be inferred from E_1, E_2, E_3. No elimination is required.
- Recall that
$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$
- Their inverses are computed just by multiplying the only nondiagonal nonzero entry by -1.
- $$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$
- Forming the product is also easy: Just put together the nondiagonal nonzero entries in a single matrix.
- Thus $E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$. (Note that the order is important.)
- This is lower triangular with diagonal entries equal to 1. Let us denote it by L.

LU Decomposition

- In summary, we have
$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$
- Thus we have factorized A into two matrices, one of which is lower triangular and the other upper triangular.

LU decomposition

Suppose A is an $n \times n$ invertible matrix and it requires no row swaps in Gaussian elimination. Then we can factorize A into two matrices such that

$$A = LU.$$

Here U is an echelon form of A and L is a lower triangular matrix with diagonal entries equal to 1. This is called an LU decomposition of A, LU (lower-upper) factorization of A or triangular factorization of A. LU decomposition of a matrix, if it exists, is unique.

Making Use of $A = LU$

- Suppose that we have a system $A\mathbf{x} = \mathbf{b}$ and we are given the LU-decomposition of A, so that $A = LU$. How can we use this?
- Write $A\mathbf{x} = \mathbf{b}$ as $LU\mathbf{x} = \mathbf{b} \to L(U\mathbf{x}) = \mathbf{b}$.
- $U\mathbf{x}$ is a vector of unknowns, call it $\mathbf{c}$.
- So we have $L\mathbf{c} = \mathbf{b}$, which can be solved by forward substitution.
- After finding $\mathbf{c}$, solve $U\mathbf{x} = \mathbf{c}$ by backward substitution.
- Hence, the factorization $A = LU$ enables us to divide the problem into two simpler parts.

Making Use of $A = LU$: Example

- As an example, consider $\begin{aligned} 2x + y + z &= 5 \\ 4x - 6y &= -2 \\ -2x + 7y + 2z &= 9 \end{aligned}$.
- We have seen that the coefficient matrix is $A = LU$, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix},\ L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix},\ U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

- Setting $U\mathbf{x} = \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, first we solve $L\mathbf{c} = \mathbf{b}$. This is the system

$$\begin{aligned} c_1 &= 5 \\ 2c_1 + c_2 &= -2 \\ -c_1 - c_2 + c_3 &= 9 \end{aligned}.$$

- We can solve it by forward substitution. So $c_1 = 5, c_2 = -12, c_3 = 2$.
- Then we solve $U\mathbf{x} = \mathbf{c}$, which is $\begin{aligned} 2x + y + z &= 5 \\ -8y - 2z &= -12 \\ z &= 2 \end{aligned}$.
- Backward substitution gives $x = 1, y = 1, z = 2$.

Cost of LU-Decomposition

- This scheme looks fine, but is it faster than the previous methods?
- An upper triangular system such as $L\mathbf{c} = \mathbf{b}$ in the example requires 3 multiplications and 3 additions.
- A lower triangular system such as $U\mathbf{x} = \mathbf{c}$ in the example requires 3 divisions, 3 multiplications and 3 additions.
- In total, 9 multiplications/divisions and 6 additions.
- For a general $n \times n$ system, the figures are approximately $\frac{3n^2}{2}$ multiplications/divisions and n^2 additions.
- Note we should add to these numbers the cost of finding the LU-decomposition itself. But it is the same as the cost of Gaussian elimination, which is approximately $\frac{n^3}{3}$ multiplications/divisions and the same number of additions/subtractions for large n.
- Therefore, apart from finding L and U, the cost of solving a system by this method is $\mathcal{O}(n^2)$. Close to the cost of backward substitution.
- Furthermore, it is advantageous over inverse matrix method when we want to solve many systems with the same coefficient matrix.

Multiple Systems with the Same Coefficient Matrix

- **Exercise:** Consider the following systems of equations:

$$\begin{aligned} 2x - 3y + z &= 2 \\ x + y - z &= -1 \\ -x + y - 3z &= 0 \end{aligned} \quad , \quad \begin{aligned} 2x - 3y + z &= 6 \\ x + y - z &= 4, \\ -x + y - 3z &= 5 \end{aligned}$$

$$\begin{aligned} 2x - 3y + z &= 0 \\ x + y - z &= 1 \\ -x + y - 3z &= -3 \end{aligned} \quad \text{and} \quad \begin{aligned} 2x - 3y + z &= -1 \\ x + y - z &= 0 \\ -x + y - 3z &= 0 \end{aligned} \quad .$$

Find the LU-decomposition of A and use it to solve these four systems.

Solving $Ax = b$ Using LU-Decomposition: Example 2

- Solve the system
$$\begin{aligned} 3x_1 - 7x_2 - 2x_3 + 2x_4 &= -9 \\ -3x_1 + 5x_2 + x_3 &= 5 \\ 6x_1 - 4x_2 - 5x_4 &= 7 \\ -9x_1 + 5x_2 - 5x_3 + 12x_4 &= 11 \end{aligned}$$
using LU decomposition.

- $$\begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} \xrightarrow{1R_1+R_2,-2R_1+R_3,3R_1+R_4} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 10 & 4 & -9 \\ 0 & -16 & -11 & 18 \end{bmatrix}$$
$$\xrightarrow{5R_2+R_3,-8R_2+R_4} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -3 & 2 \end{bmatrix} \xrightarrow{-3R_3+R_4} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

- So we have $U = \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$.

Solving $Ax = b$ Using LU-Decomposition: Example 2

- As for L, we list the row operations we performed during row reduction.
- **Step 1:** $1R_1 + R_2, -2R_1 + R_3, 3R_1 + R_4$
- The corresponding elementary matrices are $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}$.
- **Step 2:** $5R_2 + R_3, -8R_2 + R_4$
- The matrices are $E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -8 & 0 & 1 \end{bmatrix}$.
- **Step 3:** $-3R_3 + R_4$, which has the matrix $E_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix}$.

- The inverse elementary matrices are $E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix}, E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$E_5^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 8 & 0 & 1 \end{bmatrix}, E_6^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}.$$

- Their product is $E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1} = L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix}$.

- So $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ is the LU-decomposition of A.

Solving $Ax = b$ Using LU-Decomposition: Example 2

- Let us first solve $L\mathbf{y} = \mathbf{b}$. It is equivalent to

$$\begin{aligned} y_1 &= -9 \\ -y_1 + y_2 &= 5 \\ 2y_1 - 5y_2 + y_3 &= 7 \\ -3y_1 + 8y_2 + 3y_3 + y_4 &= 11 \end{aligned}.$$

- Forward substitution gives $y_1 = -9$, $y_2 = -4$, $y_3 = 5$, $y_4 = 1$.
- Then we solve $U\mathbf{x} = \mathbf{y}$, which is

$$\begin{aligned} 3x_1 - 7x_2 - 2x_3 + 2x_4 &= -9 \\ -2x_2 - x_3 + 2x_4 &= -4 \\ -x_3 + x_4 &= 5 \\ -x_4 &= 1 \end{aligned}.$$

- Backward substitution gives $x_1 = 3$, $x_2 = 4$, $x_3 = -6$, $x_4 = -1$.

LU-Decomposition for $m \times n$ Systems

- Consider the under-determined system $\begin{aligned} x_1 + 2x_2 - 3x_3 + x_4 &= 1 \\ -x_1 + 3x_2 + 2x_3 &= 8 \\ 2x_1 + 4x_2 + 7x_4 &= 8 \end{aligned}$.
- Gaussian elimination for A goes as follows.
- $\begin{bmatrix} 1 & 2 & -3 & 1 \\ -1 & 3 & 2 & 0 \\ 2 & 4 & 0 & 7 \end{bmatrix} \xrightarrow{1R_1+R_2,-2R_1+R_3} \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 5 & -1 & 1 \\ 0 & 0 & 6 & 5 \end{bmatrix}$
- The echelon form is $\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 5 & -1 & 1 \\ 0 & 0 & 6 & 5 \end{bmatrix}$, which is U.
- The performed row operations are **(1)** $1R_1 + R_2$ and **(2)** $-2R_1 + R_3$ with elementary matrices $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.
- The inverses are $E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$.

- So we have $L = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$
- Then we first solve $L\mathbf{y} = \mathbf{b}$, which is $\begin{aligned} y_1 &= 1 \\ -y_1 + y_2 &= 8 \\ 2y_1 + y_3 &= 8 \end{aligned}$. The solution is $y_1 = 1, y_2 = 9, y_3 = 6$.
- Then we solve $U\mathbf{x} = \mathbf{y}$, which is $\begin{aligned} x_1 + 2x_2 - 3x_3 + x_4 &= 1 \\ 5x_2 - x_3 + x_4 &= 9 \\ 6x_3 + 5x_4 &= 6 \end{aligned}$.
- $x_4 = t$ is free. Backward substitution gives $x_3 = 1 - 5t/6, x_2 = 2 - 11t/30, x_1 = -83t/30$.
- In vector notation, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -83/30 \\ -11/30 \\ -5/6 \\ 1 \end{bmatrix}$$

Exercises

- **Exercise:** Solve the system
$$\begin{aligned} x + y &= 1 \\ 2x + z &= 2 \\ 3x + 2y + z - w &= 4 \end{aligned}$$
using LU-decomposition method.

- **Exercise:** It is known that if B and C are invertible matrices of the same size, then BC is invertible and $(BC)^{-1} = C^{-1}B^{-1}$. Use this information to find the inverse of $\begin{bmatrix} 3 & 1 & -1 \\ -1 & 2 & -2 \\ 1 & -5 & 1 \end{bmatrix}$.

LU-Decomposition with Row Exchanges

- Consider $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix}$. Let us row reduce A.
- $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \xrightarrow{-1R_1+R_2, -2R_1+R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix}$
- Normally we would expect a pivot (leading) entry in the $(2, 2)$ position but there is a zero there. Furthermore there is a 0 below it so we should perform a row swap. $\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} = U$
- It was not possible to obtain U without a row exchange. Thus we cannot find a lower triangular matrix L such that $A = LU$.
- Gaussian elimination for A went as follows:

$$A \xrightarrow{-1R_1+R_2} E_1A \xrightarrow{-2R_1+R_3} E_2E_1A \xrightarrow{R_2 \leftrightarrow R_3} PE_2E_1A = U,$$

where $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

LU-Decomposition with Row Exchanges

- These elementary matrices have the inverses
$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$
- Now since $(PE_2E_1)^{-1} = E_1^{-1}E_2^{-1}P^{-1}$, we have $A = (PE_2E_1)^{-1}U = E_1^{-1}E_2^{-1}P^{-1}U$.
- One can verify that $E_1^{-1}E_2^{-1}P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$, which is not lower triangular. Call this matrix L'.
- We have $A = L'U$, where L' can be made lower triangular by row exchanges.
- If we had known that Gaussian elimination would have required a row swap, we could have performed it as the first step to "pave the way".
$$A \xrightarrow{R_1 \leftrightarrow R_2} PA = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 8 \\ 1 & 1 & 3 \end{bmatrix}$$
- Then we could find a lower triangular matrix L such that $PA = LU$.

$PA = LU$

Suppose that A is an invertible matrix. Then there exists a permutation matrix P such that PA has a unique LU-decomposition, in other words $PA = LU$. We can call it a permuted LU-decomposition of A. (Note that a permutation matrix is a matrix which can be obtained from the identity by a series of row exchanges.)

- The above fact cannot be used in the first try since it is not feasible to determine P before performing Gaussian elimination. But once it is found, it can be used to solve other systems with the same A.
- If one wants to solve a single system only, there is no real need to obtain a "perfect" LU-decomposition.
- In our previous example, we have

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 8 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

Permuted LU-Decomposition: Example

- As an example, consider the system $\begin{aligned} x_1 + 3x_2 + x_3 + 2x_4 &= 1 \\ 2x_1 + 6x_2 + 2x_3 - 3x_4 &= 2 \\ -2x_1 - 5x_2 - 2x_3 + x_4 &= 3 \\ x_1 + 2x_2 + 4x_3 + 3x_4 &= 4 \end{aligned}$.
- We will find a permuted LU-decomposition of $A = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 2 & -3 \\ -2 & -5 & -2 & 1 \\ 1 & 2 & 4 & 3 \end{bmatrix}$.
- $\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 2 & -3 \\ -2 & -5 & -2 & 1 \\ 1 & 2 & 4 & 3 \end{bmatrix} \xrightarrow{-2R_1+R_2,\,2R_1+R_3,\,-1R_1+R_4} \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 0 & -7 \\ 0 & 1 & 0 & 5 \\ 0 & -1 & 3 & 1 \end{bmatrix}$
- In order not to waste time, let us construct L in parallel. So far we have $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -2 & & 1 & 0 \\ 1 & & & 1 \end{bmatrix}$ (Note that the multipliers are multiplied by -1.)
- Now we need a row exchange.

Permuted LU-Decomposition: Example

- $\begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 0 & -7 \\ 0 & 1 & 0 & 5 \\ 0 & -1 & 3 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & -7 \\ 0 & -1 & 3 & 1 \end{bmatrix}$
- The corresponding **permutation** matrix is $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.
- Meanwhile, what happens to L? If P had been applied to A before, then the multipliers in step 1 would have been exchanged. $-2R_1 + R_2$ would have become $-2R_1 + R_3$ and $2R_1 + R_3$ would have become $2R_1 + R_2$.
- Therefore, the nonzero multipliers in the 2nd and 3rd rows of L should swap.

$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & & 1 & 0 \\ 1 & & & 1 \end{bmatrix}$ (Exchange only the multipliers, not the entire rows.)

- $\begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & -7 \\ 0 & -1 & 3 & 1 \end{bmatrix} \xrightarrow{1R_2 + R_4} \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & -7 \\ 0 & 0 & 3 & 6 \end{bmatrix}$

Permuted LU-Decomposition: Example

- L is updated as $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & -1 & & 1 \end{bmatrix}$ (Note the row operation $0R_2 + R_3$.)
- Then the last two rows should be swapped:
$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & -7 \\ 0 & 0 & 3 & 6 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & -7 \end{bmatrix}$$
. This is U.
- This updates L by swapping the multipliers in the 3rd and 4th rows. It also updates P by swapping its 3rd and 4th rows.
- $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. ($0R_3 + R_4$ is "hidden".)
- One can verify that $PA = LU$. (Please do it.)

Permuted LU-Decomposition: Example

- In order to solve $A\mathbf{x} = \mathbf{b}$, we apply P to both sides and write $PA\mathbf{x} = P\mathbf{b}$. We have

$$PA = \begin{bmatrix} 1 & 3 & 1 & 2 \\ -2 & -5 & -2 & 1 \\ 1 & 2 & 4 & 3 \\ 2 & 6 & 2 & -3 \end{bmatrix} \text{ and } P\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 2 \end{bmatrix}.$$

- So we have $LU\mathbf{x} = L(U\mathbf{x}) = P\mathbf{b}$. Setting $U\mathbf{x} = \mathbf{y}$ as usual, we solve $L\mathbf{y} = P\mathbf{b}$ first. That is
$$\begin{aligned} y_1 &= 1 \\ -2y_1 + y_2 &= 3 \\ y_1 - y_2 + y_3 &= 4 \\ 2y_1 + y_4 &= 2 \end{aligned}.$$

- Forward substitution gives $y_1 = 1, y_2 = 5, y_3 = 8, y_4 = 0$.

- Then we solve $U\mathbf{x} = \mathbf{y}$, which is
$$\begin{aligned} x_1 + 3x_2 + x_3 + 2x_4 &= 1 \\ x_2 + 5x_4 &= 5 \\ 3x_3 + 6x_4 &= 8 \\ -7x_4 &= 0 \end{aligned}.$$

- Backward substitution gives $x_1 = -50/3, x_2 = 5, x_3 = 8/3, x_4 = 0$.

Exercise

- **Exercise:** Solve
$$\begin{aligned} x + 2y - z &= 1 \\ 3x + 6y + 2z - w &= 0 \\ x + y - 7z + 2w &= 0 \\ x - y + 2z + w &= 3 \end{aligned}$$
by using LU (or permuted LU) decomposition.