MAT 222 Linear Algebra
Week 7
Lecture Notes

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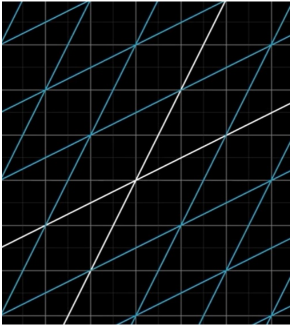
A linear transformation between vector spaces

Let V and W be vector spaces. A function $T : V \to W$ is called a **linear transformation** from V to W if it satisfies the following two conditions:

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u} \in V$ and $c \in \mathbb{R}$.

- We will mainly focus on linear transformations of Euclidean spaces.



- In the figure on the left, lines that are parallel to coordinate axes are transformed to the blue lines, which are also parallel. In addition, the coordinate axes are transformed to the white lines. Origin is left fixed.
- This is a common characteristic of linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$. They (1) keep the origin fixed, (2) map lines to lines, (3) map parallel lines to parallel lines.

Linear Transformations: Example

- Suppose we have a transformation T_1 from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that the **image** of $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ is $\begin{pmatrix} 3x + 2y \\ -2x + y \end{pmatrix}$.
- Is this transformation linear?
- To check, let us take $\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$. Also take $c \in \mathbb{R}$.
- Firstly, we have

$$T(c\mathbf{v}_1) = T\begin{pmatrix} cx_1 \\ cy_1 \end{pmatrix} = \begin{pmatrix} 3cx_1 + 2cy_1 \\ -2cx_1 + cy_1 \end{pmatrix} = c\begin{pmatrix} 3x_1 \\ 2y_1 \end{pmatrix} + c\begin{pmatrix} -2x_1 \\ y_1 \end{pmatrix} = cT(\mathbf{v}_1)$$

- Secondly, we have

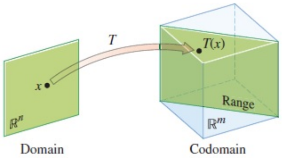
$$\begin{aligned} T(\mathbf{v}_1 + \mathbf{v}_2) = T\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} &= \begin{pmatrix} 3(x_1 + x_2) + 2(y_1 + y_2) \\ -2(x_1 + x_2) + (y_1 + y_2) \end{pmatrix} \\ &= \begin{pmatrix} 3x_1 + 2y_1 \\ -2x_1 + y_1 \end{pmatrix} + \begin{pmatrix} 3x_2 + 2y_2 \\ -2x_2 + y_2 \end{pmatrix} = T(\mathbf{v}_1) + T(\mathbf{v}_2) \end{aligned}$$

- So T is linear.

Exercise: Is the transformation $S_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + 2y + 1 \\ -2x + y \end{pmatrix}$ linear?

Linear Transformations: Example

- Now let us consider the transformation T_2 from $\mathbb{R}^2$ to $\mathbb{R}^3$ defined by
$$T_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-3y \\ 3x+5y \\ -x+7y \end{pmatrix}.$$
- Note that this time the **domain** of T_2 is $\mathbb{R}^2$ and the **codomain** of T_2 is $\mathbb{R}^3$. They need not be equal in general.
- Since $\dim(\mathbb{R}^2) < \dim(\mathbb{R}^3)$, there are elements of $\mathbb{R}^3$ such that they are not the **image** of any element of $\mathbb{R}^2$ under T_2.



- The elements of $\mathbb{R}^3$ that are the image of some element of $\mathbb{R}^2$ under T_2 constitute the **range** of T_2. More explicitly, it is
$$\{\mathbf{y} \in \mathbb{R}^3 : T_2(\mathbf{x}) = \mathbf{y} \text{ for some } x \in \mathbb{R}^2\}$$
- You can see the general picture on the left.
- How can you describe the range of T_2?

Matrix for a Linear Transformation

- In order to address such questions, it may be useful to see linear transformations under a new perspective.
- The linear transformation T_2 can be written as

$$T_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-3y \\ 3x+5y \\ -x+7y \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

- Thus, applying T_2 to a vector in $\mathbb{R}_2$ has the effect of multiplying it by the above 3×2 matrix. Let us call it A.
- Then, many questions about T_2 can be stated in terms of our terminology about matrices.
- For example, the range of T_2 can be restated as follows:

$$\{\mathbf{y} \in \mathbb{R}^3 : \mathbf{y} = A\mathbf{x} \text{ for some } x \in \mathbb{R}^2\}$$

- But this is just the column space of A, which is $\text{Span}\{(1,3,-1)^T, (-3,5,7)^T\}$.
- We can also say that the range of T_2 is the plane $13x + 5y + 2z = 0$ in $\mathbb{R}^3$. (Exercise)

Matrix for a Linear Transformation

- Let us now investigate if T_2 is one-to-one.
- Saying T_2 is one-to-one is equivalent to saying that the system of equations $A\mathbf{x} = \mathbf{b}$ does not have infinitely many solutions for any $\mathbf{b}$.
- Recall that if linear system has infinitely many solutions, then its solution set can be written as $\{\mathbf{p} + \mathbf{v} : \mathbf{v} \in N(A)\}$ for some $\mathbf{p}$.
- Thus, it suffices to check the nullspace of A.
- $N(A) = \{\mathbf{0}\}$ since the homogeneous system

$$\begin{aligned} x - 3y &= 0 \\ 3x + 5y &= 0 \\ -x + 7y &= 0 \end{aligned}$$

  has only the trivial solution. $N(A)$ is called the **kernel** of T_2 and is denoted by $\ker(T_2)$.
- Thus, $\ker(T_2) = \{\mathbf{0}\}$, which shows T_2 is one-to-one.

Exercise: Find an element of $\mathbb{R}^3$ not in the range of T_2.
Exercise: Determine the range of T_1. Is T_1 one-to-one?

Linear Transformations: Example

- Consider the matrix $A = \begin{pmatrix} 1 & -1 & 2 \\ 4 & 1 & 3 \end{pmatrix}$. Let us define a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$.
- Note that range of T is $\mathbb{R}^2$ since the system $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^2$, in other words since the column space of A is $\mathbb{R}^2$.
- Let us find the kernel of T. Since it is $N(A)$, consider the system
$$\begin{aligned} x - y + 2z &= 0 \\ 4x + y + 3z &= 0 \end{aligned},$$
which has the solution set $\{t(-1, 1, 1)^T : t \in \mathbb{R}\}$. This set is $\ker(T)$.
- Geometrically, the line $x = -t, y = z = t$ is transformed to the origin.
- Let us consider any vector $\mathbf{b}$ from $\mathbb{R}^2$, for example $\mathbf{b} = \begin{pmatrix} 1 \\ 9 \end{pmatrix}$. Let us find the **inverse image** of $\mathbf{b}$.
- The system $\begin{aligned} x - y + 2z &= 1 \\ 4x + y + 3z &= 9 \end{aligned}$ has the solution set $\{(1, 2, 1)^T + t(-1, 1, 1)^T : t \in \mathbb{R}\}$. Thus, the inverse image of $\begin{pmatrix} 1 \\ 9 \end{pmatrix}$ is the kernel of T translated by $(1, 2, 1)^T$.

Matrix Representation of a Linear Transformation

- Does every linear transformation have the form $A\mathbf{x}$ where A is a matrix?
- As far as linear transformations of the Euclidean spaces are concerned, the answer is "Yes".
- To see this, let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation.
- Take an arbitrary element $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$ be the standard basis of $\mathbf{R}^m$. Then $\mathbf{x}$ can be expressed as

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \ldots + x_m\mathbf{e}_m.$$

- Then, the image of $\mathbf{x}$ under T is

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \ldots + x_m\mathbf{e}_m) = x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) + \ldots + x_m T(\mathbf{e}_m).$$

- But this is just equal to $A\mathbf{x}$, where A is the matrix whose i-th column is $T(\mathbf{e}_i)$, the image of $\mathbf{e}_i$.
- The $n \times m$ matrix $A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \ldots & T(\mathbf{e}_m) \end{bmatrix}$ is called the **standard matrix** for the linear transformation T.
- Thus, we reach the following conclusion: A linear transformation is completely determined by the images of the basis vectors.

Matrix Representation of a Linear Transformation

- The linear transformation T_1 we considered before has the standard matrix $\begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}$.
- Note that the first column is the image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the second column is the image of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
- Similarly, columns of the standard matrix $\begin{pmatrix} 1 & -1 & 2 \\ 4 & 1 & 3 \end{pmatrix}$ of T_2 are the images of the standard basis vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, respectively.

Matrix Rep. of a Linear Transformation: Example

Example: Consider a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1,1,1) = (1,1)$, $T(1,2,3) = (1,2)$ and $T(1,2,4) = (1,4)$. Find the image of an arbitrary vector in $\mathbb{R}^3$.

- Let the standard matrix of T be A. Then the information we have about T can be written as follows:

$$AB = A\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}.$$

- Since the vectors $(1,1,1)^T, (1,2,3)^T, (1,2,4)^T$ are linearly independent, the matrix B is invertible. Its inverse is
$$B^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix}.$$

- Thus, the standard matrix is equal to
$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & 2 \end{pmatrix}$. The image of $\mathbf{x} \in \mathbb{R}^3$ is $A\mathbf{x}$.

- This example shows that a linear transformation is uniquely determined by the images of all vectors in an arbitrary basis, not only the standard basis.

Matrix Representation of a Linear Transformation

- Although working with the standard matrix of a linear transformation is easy, sometimes it may be useful to consider other bases in the domain and codomain.
- Suppose $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Let $\mathcal{B}_1 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ and $\mathcal{B}_2 = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ be bases of $\mathbb{R}^m$ and $\mathbb{R}^n$.
- Let us first consider the images of the vectors in $\mathcal{B}_1$. $T(\mathbf{u}_1)$ has a representation with respect to $\mathcal{B}_2$ such as $a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \ldots + a_{n1}\mathbf{w}_n$.
- Similarly, the other k values $T(\mathbf{u}_k)$ can be represented by

  $$T(\mathbf{u}_k) = a_{1k}\mathbf{w}_1 + a_{2k}\mathbf{w}_2 + \ldots + a_{nk}\mathbf{w}_n$$

  in terms of elements of $\mathcal{B}_2$.
- This forms an $n \times m$ matrix $A = [a_{ij}]$.
- Let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^m$. There are scalars $c_1, c_2, \ldots, c_m$ such that $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_m\mathbf{u}_m$. In other words, $c_1, c_2, \ldots, c_m$ are the coordinates of $\mathbf{x}$ with respect to $\mathcal{B}_1$. We write $[\mathbf{x}]_{\mathcal{B}_1} = (c_1, c_2, \ldots, c_m)^T$.

Matrix Representation of a Linear Transformation

- Let us consider $T(\mathbf{x})$.

$$T(\mathbf{x}) = T(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_m\mathbf{u}_m) = c_1T(\mathbf{u}_1) + c_2T(\mathbf{u}_2) + \ldots + c_mT(\mathbf{u}_m)$$
$$= c_1(a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \ldots + a_{n1}\mathbf{w}_n) + \ldots + c_m(a_{1m}\mathbf{w}_1 + a_{2m}\mathbf{w}_2 + \ldots + a_{nm}\mathbf{w}_n)$$
$$= (a_{11}c_1 + a_{12}c_2 + \ldots + a_{1m}c_m)\mathbf{w}_1 + \ldots + (a_{n1}c_1 + a_{n2}c_2 + \ldots + a_{nm}c_m)\mathbf{w}_n$$

- But this shows that the coordinates of $T(\mathbf{x})$ with respect to the basis $\mathcal{B}_2$ of $\mathbb{R}^n$ is just $A[\mathbf{x}]_{\mathcal{B}_1}$. We can state this as follows:

$$A[\mathbf{x}]_{\mathcal{B}_1} = [T(\mathbf{x})]_{\mathcal{B}_2}.$$

- Verbally, the matrix A transforms the coordinates with respect to $\mathcal{B}_1$ to coordinates with respect to $\mathcal{B}_2$.

Matrix Rep. of a Linear Transformation: Example

Example: Consider $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1 - x_2, x_1, x_2)^T$. Let the basis $\mathcal{B}_1 = \{(1,1), (-1,1)\}$ and $\mathcal{B}_2 = \{(1,0,1), (0,1,1), (1,1,0)\}$ be given for $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Find the matrix representation of T with respect to these bases.

- Let $\mathbf{u}_1 = (1,1)$ and $\mathbf{u}_2 = (-1,1)$. Also let $\mathbf{w}_1 = (1,0,1)$, $\mathbf{w}_2 = (0,1,1)$ and $\mathbf{w}_3 = (1,1,0)$. We know the columns of A are the coordinate vectors of $T(\mathbf{u}_1)$ and $T(\mathbf{u}_2)$ with respect to $\mathcal{B}_2$.
- Let us start with $\mathbf{u}_1$. $T(\mathbf{u}_1) = T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, which is equal to $0\mathbf{w}_1 + 1\mathbf{w}_2 + 0\mathbf{w}_3$. So we have $[T(\mathbf{u}_1)]_{\mathcal{B}_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.
- Similarly, $T(\mathbf{u}_2) = T\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$. This is equal to $0\mathbf{w}_1 + 1\mathbf{w}_2 - 2\mathbf{w}_3$ (please check), so its coordinates with respect to $\mathcal{B}_2$ is $[T(\mathbf{u}_2)]_{\mathcal{B}_2} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$.
- Thus, the matrix representation of T with respect to $\mathcal{B}_1$ and $\mathcal{B}_2$ is $A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & -2 \end{pmatrix}$.

Change-of-Coordinates Matrix

- The matrix representation of a linear transformation with respect to nonstandard bases can be used to switch between different bases within the same vector space as follows.
- Let $\mathcal{B}_1 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ and $\mathcal{B}_2 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be different bases for $\mathbb{R}^n$.
- Consider the identity transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. Its matrix representation with respect to $\mathcal{B}_1$ and $\mathcal{B}_2$ is the matrix A whose k-th column is the coordinate vector of $\mathbf{u}_k$ with respect to $\mathcal{B}_2$. More explicitly, we have

$$A = \begin{bmatrix} [\mathbf{u}_1]_{\mathcal{B}_2} & [\mathbf{u}_2]_{\mathcal{B}_2} & \ldots & [\mathbf{u}_n]_{\mathcal{B}_2} \end{bmatrix}.$$

- Given any element $\mathbf{x} \in \mathbb{R}^n$, we know that the following relation holds: $A[\mathbf{x}]_{\mathcal{B}_1} = [\mathbf{x}]_{\mathcal{B}_2}$.
- Thus, A converts the coordinates of $\mathbf{x}$ with respect to $\mathcal{B}_1$ to the coordinates with respect to $\mathcal{B}_2$. It is called the **change-of-coordinates matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$**. We can denote it by ${}_{\mathcal{B}_1}A_{\mathcal{B}_2}$.

Example: Let $\mathbf{b}_1 = \begin{pmatrix} -9 \\ 1 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} -5 \\ -1 \end{pmatrix}, \mathbf{c}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}, \mathbf{c}_2 = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$. Consider the bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ of $\mathbb{R}^2$. Let us find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

- Let us first find the coordinates of $\mathbf{b}_1$ with respect to $\mathcal{C}$. Let $\mathbf{b}_1 = x_1\mathbf{c}_1 + y_1\mathbf{c}_2$. This gives rise to the system
$$\begin{pmatrix} 1 & 3 \\ -4 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -9 \\ 1 \end{pmatrix},$$
which has the solution $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \end{pmatrix} = [\mathbf{b}_1]_{\mathcal{C}}$.
- Similarly, one can show that $[\mathbf{b}_2]_{\mathcal{C}} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$. (Exercise)
- The change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ is ${}_{\mathcal{B}}A_{\mathcal{C}} = \begin{pmatrix} 6 & 4 \\ -5 & -3 \end{pmatrix}$.

Exercise: Let the coordinates of $\mathbf{v}$ with respect to $\mathcal{B}$ be $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Without finding $\mathbf{v}$, find its coordinates with respect to $\mathcal{C}$.

Composition of Linear Transformations

Composition of two linear transformations

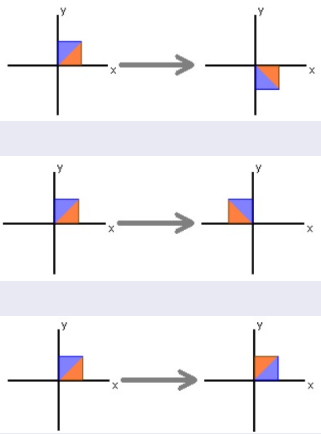
Let $T : \mathbb{R}^m \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^r$ be linear transformations with standard matrices A and B, respectively. Then the composite transformation $(S \circ T) : \mathbb{R}^m \to \mathbb{R}^r$ is a linear transformation with standard matrix BA.

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ -x_2 \\ -x_1 + 3x_2 \end{pmatrix}$, and let $S : \mathbb{R}^3 \to \mathbb{R}^2$ such that $S\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_3 \\ -x_2 \end{pmatrix}$. Describe the composite transformation $S \circ T$.

- $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ -1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ are the standard matrices of T and S.
- Thus the standard matrix of $S \circ T$ is $BA = \begin{pmatrix} 0 & 5 \\ 0 & 1 \end{pmatrix}$.
- So $(S \circ T) : \mathbb{R}^2 \to \mathbb{R}^2$ and $(S \circ T)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 5 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5x_2 \\ x_2 \end{pmatrix}$.

Some Linear Transformations of $\mathbb{R}^2$

Reflections in $\mathbb{R}^2$

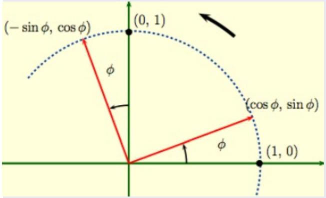


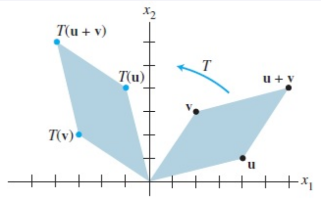
- The matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ represents reflection about the x-axis.
- Given $\mathbf{x} \in \mathbb{R}^2$, $A\mathbf{x}$ is the reflection of $\mathbf{x}$ around the x-axis. Note that the standard basis vector $(1, 0)$ stays fixed and $(0, 1)$ is moved to $(0, -1)$.
- Similarly, the matrix for reflection around the y-axis is $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Note its effect on the standard basis vectors.
- Given an arbitrary vector $\begin{pmatrix} a \\ b \end{pmatrix}$, its reflection around the line $y = x$ is the vector $\begin{pmatrix} b \\ a \end{pmatrix}$. So the matrix $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the corresponding matrix.

Exercise: Find the matrix that corresponds to reflection about the line $y = -x$.

Some Linear Transformations of $\mathbb{R}^2$

Rotations in $\mathbb{R}^2$



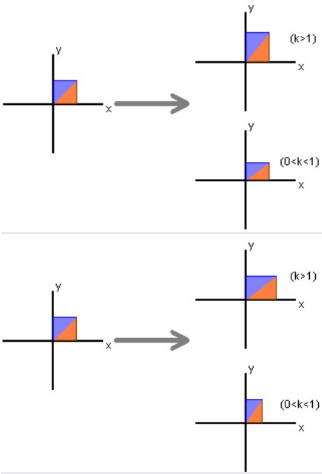


- If you rotate the standard basis vectors $(1, 0)$ and $(0, 1)$ around the origin by an angle of θ in the positive direction, then $(1, 0)$ is moved to $(\cos\theta, \sin\theta)$ and $(0, 1)$ is moved to $(-\sin\theta, \cos\theta)$.
- Thus, the matrix $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ represents rotation by an angle of θ around the origin.
- In particular the matrix corresponding to rotation by 90 degress is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
- The parallelogram on the left is transformed to another parallelogram of the same size after rotated by 90 degrees.

Exercise: Find the matrix corresponding to reflection about the origin.
Exercise: Find the matrix corresponding to rotation by an angle of θ in clockwise direction.

Some Linear Transformations of $\mathbb{R}^2$

Dilations and Contractions in $\mathbb{R}^2$

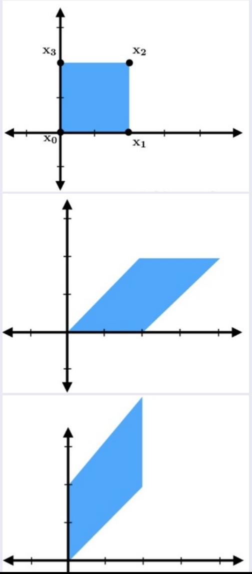


- Diagonal matrices also have a particular geometric meaning.
- The matrix $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ corresponds to a dilation (expansion) along the x-axis if $k > 1$ and a contraction (compression) along the x-axis if $0 < k < 1$.
- Similarly, $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$ corresponds to a dilation (expansion) along the y-axis if $k > 1$ and a contraction (compression) along the y-axis if $0 < k < 1$.
- Taking composition of a transformation of one type with another can combine these effects in a single matrix.

Exercise: Find a matrix corresponding to dilation along the x-axis and contraction along the y-axis.

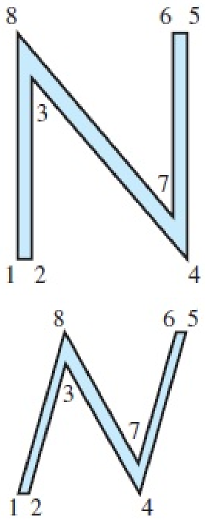
Some Linear Transformations of $\mathbb{R}^2$

Shear Transformation



- Another important transformation is the so called **shear transformation**. It holds one coordinate fixed while changing the other.
- A shear transformation transforms a rectangle to a parallelogram in general
- A shear in x keeps the y-coordinate fixed while changing the x-coordinate. It has a matrix of the form $\begin{pmatrix} 1 & k \\ 0 & 1. \end{pmatrix}$
- Similarly, a shear in y keeps the x-coordinate fixed while changing the y-coordinate. It has a matrix of the form $\begin{pmatrix} 1 & 0 \\ k & 1. \end{pmatrix}$
- It is widely used in computer graphics and image processing.

An Application of Shear Transformations



- Consider the captial letter "N" with normal font on the left.
- The coordinates of its vertices are given in the below table.

x_i	0	0.5	0.5	6	6	5.5	5.5	0
y_i	0	0	6.42	0	8	8	1.58	8

- Thus, this "normal N" can be represented by the matrix
$$D = \begin{pmatrix} 0 & 0.5 & 0.5 & 6 & 6 & 5.5 & 5.5 & 0 \\ 0 & 0 & 6.42 & 0 & 8 & 8 & 1.58 & 8 \end{pmatrix}.$$
- Given $A = \begin{pmatrix} 1 & 0.25 \\ 0 & 1 \end{pmatrix}$, let us consider the shear transformation $T : \mathbf{x} \mapsto A\mathbf{x}$.
- T transforms the coordinates of the vertices of this "normal" N to the points given by
$$AD = \begin{pmatrix} 0 & 0.5 & 2.105 & 6 & 8 & 7.5 & 5.895 & 2 \\ 0 & 0 & 6.42 & 0 & 8 & 8 & 1.58 & 8 \end{pmatrix}.$$
- This is the "slanted N" on the left.