MAT 222 – Exercise Set 2

1. In each of the following, determine if the given set is a subspace of the given vector space V.

(a) $\{\begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{R}\},\ V = \mathbb{R}^2$

(b) $\{\begin{bmatrix} x \\ x+1 \end{bmatrix} : x \in \mathbb{R}\},\ V = \mathbb{R}^2$

(c) $\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + 5y - z = 0\},\ V = \mathbb{R}^3$

(d) $\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 1\},\ V = \mathbb{R}^3$

(e) $\{\begin{bmatrix} a+b \\ 2a-3b \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R}\},\ V = \mathbb{R}^4$

(f) $\{\begin{bmatrix} a+b+1 \\ 2a-3b \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R}\},\ V = \mathbb{R}^4$

(g) $\{a_1 x + a_2 x^2 : a_1, a_2 \in \mathbb{R}\},\ V = \mathcal{P}_2 =$ The set of all polynomials of degree 2 or less

(h) $\{a_0 + a_1 x + a_2 x^2 : a_0 + a_1 + a_2 = 5\},\ V = \mathcal{P}_2 =$ The set of all polynomials of degree 2 or less

(i) $\{f : [a,b] \to \mathbb{R} : f''(x) - 2f'(x) - 3f(x) = 0\},\ V = \mathcal{F}([a,b]) =$ The set of all real-valued functions on $[a,b]$

2. In each of the following, determine if $\mathbf{v}$ is in the span of the given set.

(a) $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\operatorname{span}\{\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\}$

(b) $\mathbf{v} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$, $\operatorname{span}\{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\}$

(c) $\mathbf{v} = x - x^3$, $\operatorname{span}\{x^2, 2x + x^2, x + x^3\}$

(d) $\mathbf{v} = \begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix}$, $\operatorname{span}\{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix}\}$

3. Write two elements of $\mathbb{R}^2$ which do not span $\mathbb{R}^2$. Also write two elements of $\mathbb{R}^2$ which span $\mathbb{R}^2$. Then, write three elements of $\mathbb{R}^3$ which do not span $\mathbb{R}^3$? Can two elements of $\mathbb{R}^3$ span $\mathbb{R}^3$? Can three elements of $\mathbb{R}^2$ span $\mathbb{R}^3$? Why/Why not?

4. For each of the following, determine if the given set of vectors spans $\mathbb{R}^3$. In case it spans, decide if it is a basis or not.

(a) $\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}\}$

(b) $\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\}$

(c) $\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\}$

(d) $\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\}$

(e) $\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}\}$

5. Suppose that $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ and $\mathbf{v_4}$ are vectors such that $\mathbf{v_4} \in \operatorname{span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$. Show that $\operatorname{span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\} = \operatorname{span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$.

6. Find a basis for each vector space given below.

(a) The xz-plane in $\mathbb{R}^3$

(b) $\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = 2y = 3z\}$

(c) $\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : 3x + 2y + z = 0\}$

(d) $\{\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} : x + 2y + t = 0 \,\text{ve}\, y + 2z = 0\}$

(e) The set of degree 2 or less polynomials P such that $P(7) = 0$

7. In problem (1), find a basis for each set that is also a vector space.

8. Consider the following matrices. Find a basis for the nullspace of each matrix. Then find a basis for the column space of the matrix.

(a) $\mathbf{A} = \begin{bmatrix} -1 & 3 \\ 2 & -6 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 1 & -1 & -1 & 3 \\ 2 & -2 & 0 & 4 \end{bmatrix}$

(c) $\mathbf{C} = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 6 & 3 \end{bmatrix}$

(d) $\mathbf{D} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & -1 & 3 \\ 1 & 3 & -11 \end{bmatrix}$

(e) $\mathbf{E} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$

(f) $\mathbf{F} = \begin{bmatrix} 1 & 2 & 0 \\ 4 & -1 & 7 \\ 3 & -5 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

9. Let $\mathbf{A} = \begin{bmatrix} 4 & -5 & -2 \\ 5 & -6 & -2 \\ -8 & 9 & 3 \end{bmatrix}$. Find a vector from the column space of $\mathbf{A}$. Then find a vector that is not in the column space of $\mathbf{A}$. If you take a random vector from $\mathbb{R}^3$, which probability is greater: That it is in $C(\mathbf{A})$ or that it is not in $C(\mathbf{A})$?

10. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{bmatrix}$. Also let $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 9 \\ -17 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 3 \\ 15 \\ 2 \end{bmatrix}$. One of the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ is in the column space of $\mathbf{A}$. Determine which one it is and write it as a linear combination of columns of $\mathbf{A}$.

11. Let the matrix $\mathbf{A}$ be as in Problem 10 and let $\mathbf{v}_1 = \begin{bmatrix} 8 \\ -8 \\ 4 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 0 \\ -7 \end{bmatrix}$ be given. Determine which one of these two vectors is in the nullspace of $\mathbf{A}$.

12. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 4 & -1 & 7 \\ 3 & -5 & 2 \\ 0 & 1 & 3 \end{bmatrix}$. If possible, find nonzero vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ such that

(i) The system $\mathbf{Ax} = \mathbf{b}_1$ has no solution.

(ii) The system $\mathbf{Ax} = \mathbf{b}_2$ has a unique solution.

(iii) The system $\mathbf{Ax} = \mathbf{b}_3$ has infinitely many solutions.

For each part, in case it is not possible, state briefly why it is not.

13. Same as in Problem 12 but this time $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 4 & -1 & 7 & 4 \\ 3 & -5 & 2 & 0 \end{bmatrix}$.

14. In each of the following, complete the given set to a basis of the given vector space V. In other words, add one or more elements to the set so as to form a basis for V.

(a) $\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$, $V = \mathbb{R}^2$ (b) $\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\}$, $V = \mathbb{R}^3$ (c) $\{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} -7 \\ -8 \\ -3 \\ 2 \end{bmatrix}\}$, $V = \mathbb{R}^4$

(d) $\{1 + x^2, x + x^2\}$, $V = \mathcal{P}_2 =$ The set of all polynomials of degree 2 or less

(e) $\{1 + x^2, x + x^2\}$, $V = \mathcal{P}_3 =$ The set of all polynomials of degree 3 or less

15. Let V be a vector space and $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m \in V$. Let v be a vector in V such that v can be written as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ in more than one way. Then show that the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ is linearly dependent.

16. **Column rank** of a matrix is defined to be the maximum number of independent columns of a matrix. Determine the column rank of each matrix in Problem 1. Does anything draw your attention?

17. In each of the following, determine if the given set of vectors is linearly independent or not.

(a) $\{(1, 2), (3, -5)\}$ (b) $\{(1, 1), (1/2, -1), (4, 7)\}$ (c) $\{(3, 0, 1, 2), (6, 1, 0, 0), (12, 1, 2, 4)\}$

(d) $\{(2, 6, 1), (-3, 0, 5), (5, 4, -7)\}$ (e) $\{(3, 6, 12), (0, 1, 1), (1, 0, 2), (2, 0, 4)\}$

18. Show that the rank of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & -1 & 3 & 5 \\ -4 & 3 & -5 & -13 \\ 1 & 1 & 0 & 3 \\ 4 & 1 & 3 & 11 \end{bmatrix}$ is equal to 3. Find 3 rows of $\mathbf{A}$ that are linearly independent.

19. Suppose we have n vectors each having m components. If $n > m$ prove that these n vectors are linearly dependent. (Hint: What is the situation if we have three 2×1 vectors?)

20. Let us consider the vectors $\mathbf{v}_1 = (1, 2, 3), \mathbf{v}_2 = (-2, 3, 5)$ and $\mathbf{v}_3 = (4, 1, 1)$. $\mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2$, so these three vectors are linearly dependent. Suppose we want to find a vector $\mathbf{w}$ such that the three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}$ become linearly independent. For this purpose we try changing the last coordinate of $\mathbf{v}_3$ from 1 to 2 and setting it as the new vector, so we have $\mathbf{w} = (4, 1, 2)$. Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}$ are indeed linearly independent. Does this method work in general? In other words, if the vectors $(a_1, b_1, c_1), (a_2, b_2, c_2)$ and (a_3, b_3, c_3) are linearly dependent, is it true that $(a_1, b_1, c_1), (a_2, b_2, c_2)$ and (a_3, b_3, c_4) are certainly independent as long as $c_4 \neq c_3$? If yes, prove it; if no, give a counterexample.

21. If $\mathbf{A}$ is a nonsquare matrix, show that either the rows of $\mathbf{A}$ or the columns of $\mathbf{A}$ are linearly dependent. (Hint: You can use Problem 19.)