MAT 222 Linear Algebra and Numerical Methods
Week 4
Lecture Notes 2

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- Determinant of a square matrix is a real number which conveys some information about the matrix.
- There are several ways to define determinants. We define it in terms of its basic properties.

Determinant of a square matrix

Let $M_{n\times n}(\mathbb{R})$ denote the set of all $n \times n$ matrices with real entries. Consider a function $d : M_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$. Suppose d has the following three properties:

(i) $d(\mathbf{I}) = 1$

(ii) If A and B are two elements of $M_{n\times n}(\mathbb{R})$ such that they can be obtained from each other by a single row exchange, then $d(A) = -d(B)$.

(iii) d is linear on any of the rows. In other words, let us represent A using its rows as $A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$. Then let us construct the matrices B and C such that
$B = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \rho_k, \ldots, \mathbf{a}_n)$, $C = (\mathbf{a}_1, \mathbf{a}_2, \ldots, t\cdot \mathbf{a}_k + \rho_k, \ldots, \mathbf{a}_n)$,
where ρ_k is a row vector of size n (Note that B and C differ from A only in the k-th row). Then it is true that $d(C) = t\cdot d(A) + d(B)$.

Then the function d is called a determinant function on $M_{n\times n}(\mathbb{R})$.

Determinant of a Matrix

- The determinant of A is denoted by $\det(A)$ or $|A|$.
- Determinant of nonsquare matrices are not defined.
- Let us demonstrate the properties (i)-(iii) using 2×2 matrices.

Examples

- $\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1$
- $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = -\det\left(\begin{bmatrix} c & d \\ a & b \end{bmatrix}\right)$
- Property (iii) implies $\det\left(\begin{bmatrix} t\cdot a & t\cdot b \\ c & d \end{bmatrix}\right) = t\cdot\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ and

  $\det\left(\begin{bmatrix} a+a' & b+b' \\ c & d \end{bmatrix}\right) = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + \det\left(\begin{bmatrix} a' & b' \\ c & d \end{bmatrix}\right)$.

  These two can be expressed in a single equality as follows:

  $\det\left(\begin{bmatrix} t\cdot a+a' & t\cdot b+b' \\ c & d \end{bmatrix}\right) = t\cdot\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + \det\left(\begin{bmatrix} a' & b' \\ c & d \end{bmatrix}\right)$

Other Properties of Determinants

- The properties (i)-(iii) define the determinant function uniquely. (Though this fact is not apparent at once.)
- A number of other properties can be inferred from these properties.
- Let us again use 2×2 matrices to demonstrate them.

(P4) If two rows of A are the same, $\det(A) = 0$.

By property (ii), $\begin{vmatrix} a & b \\ a & b \end{vmatrix} = -\begin{vmatrix} a & b \\ a & b \end{vmatrix}$. So we have $\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0$.

(P5) Adding a multiple of one row to another doesn't change the determinant. This follows from property (ii).

$$\begin{vmatrix} a & b \\ c+ka & d+kb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ ka & kb \end{vmatrix}$$
$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + k\begin{vmatrix} a & b \\ a & b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + 0 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

(P6) If A has a zero row, then $\det(A) = 0$.

(iii) and (P4) gives $0 = \begin{vmatrix} a & b \\ a & b \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ a & b \end{vmatrix} + \begin{vmatrix} a & b \\ a & b \end{vmatrix}$. So $\begin{vmatrix} 0 & 0 \\ a & b \end{vmatrix} = 0$.

Other Properties of Determinants

(P7) If A is triangular, $\det(A)$ is equal to the product of the diagonal entries.

To see this, first suppose that the diagonal entries $a_{1,1}, a_{2,2}, \ldots, a_{n,n}$ of A are all nonzero. Then $\det(A) = \det(diag(a_{1,1}, a_{2,2}, \ldots, a_{n,n}))$. (Why?) But this last determinant is equal to
$a_{1,1} \cdot a_{2,2} \cdot \ldots \cdot a_{n,n} \cdot \, |\mathbf{I}| = a_{1,1} \cdot a_{2,2} \cdot \ldots \cdot a_{n,n}$.
On the contrary, if one of $a_{1,1}, a_{2,2}, \ldots, a_{n,n}$ is 0, then the row containing that entry can be made a zero row by row operations of type (3), which don't change the determinant. So the determinant is equal to 0, which is $a_{1,1} \cdot a_{2,2} \cdot \ldots \cdot a_{n,n}$.

(P8) If A is noninvertible (singular), then $\det(A) = 0$.

If A is noninvertible, then row operations of type (3) leads to a zero row in its echelon form. So $\det(A) = 0$.

(P9) $\det(AB) = \det(A)\det(B)$ (In particular $\det(A^{-1}) = \dfrac{1}{\det(A)}$)

Let us define a function $f(A) = \dfrac{\det(AB)}{\det(B)}$. One can verify that f has all three defining properties (i)-(iii) of the determinant function (**Exercise**). So f should be the determinant function.

Other Properties of Determinants

(P10) The transpose of a matrix has the same determinant as the matrix. $\det(A^T) = \det(A)$. (Note that A^T is the matrix whose columns are the rows of A in that order.)

The proof is omitted.

- Note that the converse of (P8) is also true: If $\det(A) = 0$, then A is noninvertible. So we have reached this particularly important conclusion: A **is invertible if and only if** $\det(A) \neq 0$**.**
- These properties can be used to compute determinants. Let us start with the $n = 2$ case.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \left(\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} \right) + \left(\begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} \right)$$

$$= (0 + ad) + (-bc + 0) = ad - bc$$

- In a similar manner, the derived properties of the determinant can be used to compute the determinant of bigger matrices.

- Let us apply the same procedure to an arbitrary 3×3 matrix.

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = \begin{vmatrix} a_{1,1} & 0 & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} + \begin{vmatrix} 0 & a_{1,2} & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}$$

- For the first term on the right we have

$$\begin{vmatrix} a_{1,1} & 0 & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = \begin{vmatrix} a_{1,1} & 0 & 0 \\ a_{2,1} & 0 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} + \begin{vmatrix} a_{1,1} & 0 & 0 \\ 0 & a_{2,2} & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} + \begin{vmatrix} a_{1,1} & 0 & 0 \\ 0 & 0 & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}$$

- The first term on the right is equal to 0. For the other two we have

$$\begin{vmatrix} a_{1,1} & 0 & 0 \\ 0 & a_{2,2} & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = \begin{vmatrix} a_{1,1} & 0 & 0 \\ 0 & a_{2,2} & 0 \\ a_{3,1} & 0 & 0 \end{vmatrix} + \begin{vmatrix} a_{1,1} & 0 & 0 \\ 0 & a_{2,2} & 0 \\ 0 & a_{3,2} & 0 \end{vmatrix} + \begin{vmatrix} a_{1,1} & 0 & 0 \\ 0 & a_{2,2} & 0 \\ 0 & 0 & a_{3,3} \end{vmatrix}$$

and

$$\begin{vmatrix} a_{1,1} & 0 & 0 \\ 0 & 0 & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = \begin{vmatrix} a_{1,1} & 0 & 0 \\ 0 & 0 & a_{2,3} \\ a_{3,1} & 0 & 0 \end{vmatrix} + \begin{vmatrix} a_{1,1} & 0 & 0 \\ 0 & 0 & a_{2,3} \\ 0 & a_{3,2} & 0 \end{vmatrix} + \begin{vmatrix} a_{1,1} & 0 & 0 \\ 0 & 0 & a_{2,3} \\ 0 & 0 & a_{3,3} \end{vmatrix}$$

Determinant of a 3 × 3 Matrix

- In each equality, two of the determinants are clearly equal to 0. So we have
$$\begin{vmatrix} a_{1,1} & 0 & 0 \\ 0 & a_{2,2} & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = \begin{vmatrix} a_{1,1} & 0 & 0 \\ 0 & a_{2,2} & 0 \\ 0 & 0 & a_{3,3} \end{vmatrix} \text{ and } \begin{vmatrix} a_{1,1} & 0 & 0 \\ 0 & 0 & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = \begin{vmatrix} a_{1,1} & 0 & 0 \\ 0 & 0 & a_{2,3} \\ 0 & a_{3,2} & 0 \end{vmatrix}.$$
- Combining these two we have
$$\begin{vmatrix} a_{1,1} & 0 & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = \begin{vmatrix} a_{1,1} & 0 & 0 \\ 0 & a_{2,2} & 0 \\ 0 & 0 & a_{3,3} \end{vmatrix} + \begin{vmatrix} a_{1,1} & 0 & 0 \\ 0 & 0 & a_{2,3} \\ 0 & a_{3,2} & 0 \end{vmatrix}.$$
- A similar reasoning gives
$$\begin{vmatrix} 0 & a_{1,2} & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = \begin{vmatrix} 0 & a_{1,2} & 0 \\ a_{2,1} & 0 & 0 \\ 0 & 0 & a_{3,3} \end{vmatrix} + \begin{vmatrix} 0 & a_{1,2} & 0 \\ 0 & 0 & a_{2,3} \\ a_{3,1} & 0 & 0 \end{vmatrix}$$
and
$$\begin{vmatrix} 0 & 0 & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = \begin{vmatrix} 0 & 0 & a_{1,3} \\ a_{2,1} & 0 & 0 \\ 0 & a_{3,2} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{1,3} \\ 0 & a_{2,2} & 0 \\ a_{3,1} & 0 & 0 \end{vmatrix}.$$
- Thus $\det(A)$ can be expressed as the sum of 6 determinants in each of which the nonzero entries are all in different rows and columns.

Determinant of a 3 × 3 Matrix

- For the 6 nonzero determinants we have
- $$\begin{vmatrix} a_{1,1} & 0 & 0 \\ 0 & a_{2,2} & 0 \\ 0 & 0 & a_{3,3} \end{vmatrix} = a_{1,1}a_{2,2}a_{3,3}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix},$$
$$\begin{vmatrix} a_{1,1} & 0 & 0 \\ 0 & 0 & a_{2,3} \\ 0 & a_{3,2} & 0 \end{vmatrix} = a_{1,1}a_{2,3}a_{3,2}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix},$$
$$\begin{vmatrix} 0 & a_{1,2} & 0 \\ a_{2,1} & 0 & 0 \\ 0 & 0 & a_{3,3} \end{vmatrix} = a_{1,2}a_{2,1}a_{3,3}\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix},$$
$$\begin{vmatrix} 0 & a_{1,2} & 0 \\ 0 & 0 & a_{2,3} \\ a_{3,1} & 0 & 0 \end{vmatrix} = a_{1,2}a_{2,3}a_{3,1}\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix},$$
$$\begin{vmatrix} 0 & 0 & a_{1,3} \\ a_{2,1} & 0 & 0 \\ 0 & a_{3,2} & 0 \end{vmatrix} = a_{1,3}a_{2,1}a_{3,2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix},$$
$$\begin{vmatrix} 0 & 0 & a_{1,3} \\ 0 & a_{2,2} & 0 \\ a_{3,1} & 0 & 0 \end{vmatrix} = a_{1,3}a_{2,2}a_{3,1}\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}.$$
- So computing the determinant is reduced to computing determinants of permutation matrices.

Determinant of Permutation Matrices

- Each permutation matrix has determinant 1 or -1, depending on how many row exchanges is required to obtain it from the identity.
- So $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$, $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1$, $\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$, $\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 1$, $\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1$, $\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -1$.
- Combining everything gives

$$\det(A) = \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} =$$

$$a_{1,1}a_{2,2}a_{3,3} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,3}a_{2,2}a_{3,1}.$$

- There are 6 terms in the determinant. This is no coincidence since there are $3! = 6$ matrices that can be obtained from the 3×3 identity matrix (including itself) by row exchanges.

Determinant and Permutations

- Another way of expressing the same thing is in terms of permutations. Each permutation matrix corresponds to a certain permutation. For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ corresponds to the permutation (132) (swap rows 2 and 3) and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ corresponds to (123) (do nothing).
- If a permutation can be obtained from the identity permutation (which does nothing) by an odd number of exchanges then it is called an odd permutation, otherwise it is called an even permutation.
- For example (132) is odd because it is obtained from (123) by exchanging 2 and 3.
- Then the determinant of a permutation matrix is just the parity of the corresponding permutation: It is 1 if the permutation is even, it is -1 if the permutation is odd.

Determinant and Permutations

- Then each of the terms in the formula $\det(A) = a_{1,1}a_{2,2}a_{3,3} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,3}a_{2,2}a_{3,1}$ corresponds to a certain permutation and the sign of that term is just the parity of that permutation.
- $a_{1,1}a_{2,2}a_{3,3}$ corresponds to (123), with parity 1. $a_{1,1}a_{2,3}a_{3,2}$ corresponds to (132), with parity -1. $a_{1,2}a_{2,1}a_{3,3}$ corresponds to (213), with parity -1, and so on.
- This gives an explicit formula for the determinant, generalized in the following:

Leibniz formula for the determinants

Let S_n denote the set of all permutations of the set $\{1, 2, \ldots, n\}$. Then the determinant of $A = [a_{i,j}]_{n\times n}$ can be calculated by

$$\det(A) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \ldots a_{n,\sigma(n)},$$

where $\operatorname{sgn}(\sigma)$ denotes the parity of σ.

Computing the Determinant

- Some sources use Leibniz's Formula as the definition of determinants.
- It is good for theoretical purposes, but not efficient for computation.
- We can try to find an easier way looking at the formula for $n = 3$.
- $\det(A) = a_{1,1}a_{2,2}a_{3,3} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,3}a_{2,2}a_{3,1} =$ $a_{1,1}(a_{2,2}a_{3,3} - a_{2,3}a_{3,2}) + a_{1,2}(a_{2,3}a_{3,1} - a_{2,1}a_{3,3}) + a_{1,3}(a_{2,1}a_{3,2} - a_{2,2}a_{3,1})$.
- But this can be written as

$$a_{1,1}\begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2}\begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3}\begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}$$

- Let us look at this formula more closely. The first term $a_{1,1}\begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix}$ is the multiplication of $a_{1,1}$ with the determinant of the matrix

$$\begin{bmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{bmatrix}.$$

- This matrix is the submatrix of A which excludes row 1 and column 1. Let us denote it by $A(1|1)$.

Cofactor Expansion

- In general, let us denote by $A(i|j)$ the $(n-1)\times(n-1)$ submatrix of A which is obtained by removing row i and column j of A.
- The formula $\det(A) = a_{1,1}\begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2}\begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3}\begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}$ can then be written as

$$\det(A) = a_{1,1}\det(A(1|1)) - a_{1,2}\det(A(1|2)) + a_{1,3}\det(A(1|3)).$$

- The number $(-1)^{i+j}\det(A(i|j))$ is called the cofactor of $a_{i,j}$ in A and denoted by $C_{i,j}$. Then the above formula becomes

$$\det(A) = a_{1,1}C_{1,1} + a_{1,2}C_{1,2} + a_{1,3}C_{1,3}.$$

- This is called the cofactor expansion of $\det(A)$ along row 1.
- $\det(A)$ can be computed along any row or column. For example, the cofactor expansion along column 2 is given by

$$\det(A) = a_{1,2}C_{1,2} + a_{2,2}C_{2,2} + a_{3,2}C_{3,2},$$

which gives the same result as any cofactor expansion, which is $\det(A)$.

Cofactor Expansion: Example

Cofactor expansion

Let A be an $n \times n$ matrix and let $k \in \{1, 2, \ldots, n\}$. Then the determinant of A is equal to

$$\sum_{i=1}^{n} a_{i,k} C_{i,k} = a_{1,k} C_{1,k} + a_{2,k} C_{2,k} + \ldots + a_{n,k} C_{n,k},$$

where $C_{i,k} = (-1)^{i+k} \det(A(i|j))$ is the cofactor of $a_{i,k}$ in A. The above formula for $\det(A)$, which is the cofactor expansion of the determinant along column k, can also be defined along any row in a straightforward manner.

- Let $A = \begin{bmatrix} 1 & 6 & 2 \\ 3 & 4 & 5 \\ 2 & -1 & -4 \end{bmatrix}$. Let us compute $\det(A)$ using cofactor expansion.
- Let us use the cofactor expansion along column 2. (Note that the result doesn't change)
- We begin by computing the cofactors of the entries in the 2nd column.
- $C_{1,2} = (-1)^{1+2} |A(1|2)| = (-1)^3 \begin{vmatrix} 3 & 5 \\ 2 & -4 \end{vmatrix} = (-1)(3 \cdot (-4) - 5 \cdot 2) = 22.$

Cofactor Expansion: Example

- Similarly,
$$C_{2,2} = (-1)^{2+2}|A(2|2)| = (-1)^4 \begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix} = 1(1\cdot(-4) - 2\cdot 2) = -8 \text{ and}$$
$$C_{3,2} = (-1)^{3+2}|A(3|2)| = (-1)^5 \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = (-1)(1\cdot 5 - 2\cdot 3) = 1.$$
- So we have
$\det(A) = a_{1,2}C_{1,2} + a_{2,2}C_{2,2} + a_{3,2}C_{3,2} = 6\cdot 22 + 4\cdot(-8) + (-1)\cdot 1 = 99.$
- **Exercise:** Let $A = \begin{bmatrix} -2 & 3 & 0 \\ 3 & 1/2 & 1 \\ 5 & -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 4 & -4 \\ -1 & 0 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{bmatrix}$.
Calculate $\det(A)$ and $\det(B)$ using cofactor expansion along any row or column.