MAT 222 Linear Algebra 1st Assignment

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Problem 1: Find the general term of the sequence whose reccurrence and first two terms are given by the following.

$x_0=1$, $x_1=2$ $\qquad x_{n+2}=-2x_{n+1}+8x_n$

$$\begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix} = \begin{bmatrix} -2 & 8 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} \qquad A=\begin{bmatrix} -2 & 8 \\ 1 & 0 \end{bmatrix} \qquad |A-\lambda I|=0$$

Eigen values: $-4, 2$

A also $\begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix}$

$$\begin{vmatrix} -2-\lambda & 8 \\ 1 & -\lambda \end{vmatrix} = 0$$

$(-2-\lambda)(-\lambda)-8.1=\lambda^2+2\lambda-8$

$(\lambda+4)(\lambda-2)=0$

$\lambda=-4, 2$

for $\lambda=-4$

$(A+4I)\vec{v}=0$

$$\begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 4 & 1 & 0 \\ 8 & 2 & 0 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|c} 4 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

x_2 free variable

$x_2=a$

$4x_1+a=0$

$x_1=-a/4$

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a\begin{bmatrix} -1/4 \\ 1 \end{bmatrix}}_{\text{eigen vector}}$$

for $\lambda=2$

$(A-2I)\vec{v}=0$

$$\begin{bmatrix} -2 & 1 \\ 8 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 8 & -4 & 0 \end{array}\right] \xrightarrow{4R_1+R_2} \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

x_2 free variable

$x_2=a$

$-2x_1+x_2=0$

$x_1=a/2$

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}}_{\text{eigen vector}}$$

$$D=\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix}, \quad P=\begin{bmatrix} -1/4 & 1/2 \\ 1 & 1 \end{bmatrix}, \quad P^{-1}=\frac{1}{\frac{-1}{4}-\frac{1}{2}}\begin{bmatrix} 1 & -1/2 \\ -1 & -1/4 \end{bmatrix} = -\frac{4}{3}\begin{bmatrix} 1 & -1/2 \\ -1 & -1/4 \end{bmatrix}$$

$$P^{-1}A^nP=D^n \Rightarrow A^n=PD^nP^{-1} \Rightarrow \begin{bmatrix} -1/4 & 1/2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (-4)^n & 0 \\ 0 & 2^n \end{bmatrix}\begin{bmatrix} 1 & -1/2 \\ -1 & -1/4 \end{bmatrix}\cdot -\frac{4}{3}$$

$$= -\frac{4}{3}\begin{bmatrix} \frac{-(-4^n)}{4}+\frac{-2^n}{2} & \frac{(-4)^n}{4.2}+\frac{-2^n}{4.2} \\ (-4)^n-2^n & \frac{-(-4)^n}{2}-\frac{2^n}{4} \end{bmatrix} = -\frac{4}{3}\begin{bmatrix} -2^{n-1}-(-4)^{n-1} & \frac{(-4)^{n-1}-2^{n-2}}{2} \\ (-4)^n-2^n & \frac{-(-4)^n-2^{n-1}}{2} \end{bmatrix}$$

$$\begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = A^n\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = A^n\begin{bmatrix} 1 \\ 2 \end{bmatrix} = -\frac{4}{3}\begin{bmatrix} -2^{n-1}-(-4)^{n-1}+(-4)^{n-1}-2^{n-2} \\ (-4)^n-2^n-(-4)^n-2^{n-1} \end{bmatrix} = -\frac{4}{3}\begin{bmatrix} -2^{n-1}-2^{n-2} \\ -2^n-2^{n-1} \end{bmatrix}$$

$$x_n = -\frac{4}{3}(-2^{n-1}-2^{n-2}) = \frac{2^{n+1}+2^n}{3} = \boxed{\frac{2^n.3}{3}=2^n}$$

Problem 2: Consider a discrete dynamical model about the population of two animal species in the same ecological system, where x_k and y_k denote the predator and prey population at the end of k-th month, respectively. The model is of the form

$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = A \begin{bmatrix} x_k \\ y_k \end{bmatrix}$, where A is a 2×2 matrix.

- What is $\lim_{k\to\infty} x_k$? What is $\lim_{k\to\infty} y_k$?
- What is $\lim_{k\to\infty} \frac{x_k}{y_k}$?
- A comment on the long term behaviour.

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$A = \begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix}$, $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 30 \\ 40 \end{bmatrix}$

characteristic polynomial $|A - \lambda I| = 0$

$$\begin{vmatrix} 0.86-\lambda & 0.08 \\ -0.12 & 1.14-\lambda \end{vmatrix} = 0$$

$$= (0.86-\lambda)(1.14-\lambda) - (0.08)(-0.12) = 0$$

$$0,9804 - 0.86\lambda - 1,14\lambda + \lambda^2 + 0,0096 = 0$$

$$\lambda^2 - 2\lambda + 0,99 = 0$$

$$\lambda_1 = 1,1 \quad , \lambda_2 = 0,9$$

Eigenvalues : 1,1 , 0,9

For $\lambda = 1,1$

$(A - 1,1I)\vec{v} = 0$

$$\left[\begin{array}{cc|c} -0,24 & 0,08 & 0 \\ -0,12 & 0,04 & 0 \end{array}\right] \xrightarrow{-1/2 R_1 + R_2} \left[\begin{array}{cc|c} -0,24 & 0,08 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

x_2 free variables

$x_2 = a$

$-0,24x_1 + 0,08a = 0$

$x_1 = 3a$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ } eigen vector

for $\lambda = 0,9$

$(A - 0,9I)\vec{v} = 0$

$$\left[\begin{array}{cc|c} -0,04 & 0,08 & 0 \\ -0,12 & 0,24 & 0 \end{array}\right] \xrightarrow{-3R_1 + R_2} \left[\begin{array}{cc|c} -0,04 & 0,08 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

x_2 free variables

$x_2 = a$

$-0,04x_1 + 0,08a = 0$

$x_1 = 2a$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$D = \begin{bmatrix} 1,1 & 0 \\ 0 & 0,9 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, $P^{-1} = \frac{1}{1-6}\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} \Rightarrow P^{-1}A^kP = D^k \Rightarrow A^k = PD^kP^{-1} \Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} (1,1)^k & 0 \\ 0 & (0,9)^k \end{bmatrix} \cdot \left(-\frac{1}{5}\right) \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$

$\begin{bmatrix} (1,1)^k & -2(1,1)^k \\ -3(0,9)^k & (0,9)^k \end{bmatrix}$

$$= -\frac{1}{5}\begin{bmatrix} (1,1)^k - 6(0,9)^k & 2(0,9)^k - 2\cdot(1,1)^k \\ 3(1,1)^k - 3(0,9)^k & (0,9)^k - 6(1,1)^k \end{bmatrix}$$

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} = A^k\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_k \\ y_k \end{bmatrix} = A^k\begin{bmatrix} 30 \\ 40 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -50(1,1)^k - 100(0,9)^k \\ -150(1,1)^k - 50(0.9)^k \end{bmatrix}$$

$$x_k = 10(1,1)^k + 20(0.9)^k \qquad y_k = 30(1,1)^k + 10(0,9)^k$$

$$\lim_{k\to\infty} x_k = 10 \cdot \lim_{k\to\infty} \overbrace{(1,1)^k}^{\infty} + 20 \cdot \lim_{k\to\infty} \overbrace{(0.9)^k}^{0} = \infty$$

$$\lim_{k\to\infty} y_k = 30 \lim_{k\to\infty} \overbrace{(1,1)^k}^{\infty} + 10 \cdot \lim_{k\to\infty} \overbrace{(0,9)^k}^{0} = +\infty$$

$$\lim_{k\to\infty} \frac{x_k}{y_k} = \left(\frac{10 + 20\overbrace{\frac{(0,9)^k}{(1,1)^k}}^{0}}{30 + 10\underbrace{\frac{(0,9)^k}{(1,1)^k}}_{0}} \right) = \frac{10}{30} = \frac{1}{3}$$

- Both populations grow indefinetely they do not perish nor approach a specific finite limit.

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Problem 3: Find the unique polynomial of degree less than or equal to 3 that passes through the given points.

$(-3,-51)$, $(-1,-7)$, $(1,-3)$, $(2,14)$

$(x_0,y_0)=(-3,-51)$, $(x_1,y_1)=(-1,-7)$, $(x_2,y_2)=(1,-3)$

$(x_3,y_3)=(2,14)$

- The Lagrange basis polynomials are

$$L_0(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}=-\frac{1}{40}(x+1)(x-1)(x-2)$$
$$(-3+1)\ (-3-1)\ (-3-2)$$

$$L_1(x)=\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}=\frac{1}{12}(x+3)(x-1)(x-2)$$
$$(-1+3)\ (-1-1)\ (-1-2)$$

$$L_2(x)=\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}=-\frac{1}{8}(x+3)(x+1)(x-2)$$
$$(1+3)\ (1+1)\ (1-2)$$

$$L_3(x)=\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}=\frac{1}{15}(x+3)(x+1)(x-1)$$
$$(2+3)\ (2+1)\ (2-1)$$

- So, the polynomial we are looking for is

$$P(x)=y_0L_0(x)+y_1L_1(x)+y_2L_2(x)+y_3L_3(x) \Rightarrow P(x)=-51L_0(x)-7L_1(x)-3L_2(x)+14L_3(x)$$

$$P(x)=\frac{51}{40}(x+1)(x-1)(x-2)-\frac{7}{12}(x+3)(x-1)(x-2)+\frac{3}{8}(x+3)(x+1)(x-2)+\frac{14}{15}(x+3)(x+1)(x-1)$$

$$P(x)=2x^3+x^2-6$$

Let's check whether the polynomial we found passes through the given points.

$(-3,-51)\Rightarrow P(-3)=2.(-3)^3+(-3)^2-6=-51$ (true)

$(-1,-7)\Rightarrow P(-1)=2.(-1)^3+(-1)^2-6=-7$ (true)

$(1,-3)\Rightarrow P(1)=2.1^3+1^2-6=-3$ (true)

$(2,14)\Rightarrow P(2)=2.(2)^3+2^2-6=14$ (true)

A unique polynomial that passes through given points and has degree less than or equal to 3:

$$\boxed{P(x)=2x^3+x^2-6}$$