MAT 222 Linear Algebra
Week 10
Lecture Notes 1

Murat Karaçayır

Akdeniz University
Department of Mathematics

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Orthogonal Set of Vectors

- Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \in \mathbb{R}^n$. If these vectors are mutually orthogonal, that is if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $i \neq j$, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is an orthogonal set of vectors.
- Let $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$. Then the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set. (Please check)
- An orthogonal set of nonzero vectors is always linearly independent. To see this, write, $0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p$ and take inner product with $\mathbf{v}_k$ for $k = 1, 2, \ldots, k$.
- This implies the following: If the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is orthogonal, then it is a basis, called an orthogonal basis, for Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.
- So, for example, if $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are as given above, the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$.

Orthogonal Basis

- Working with an orthogonal basis is extremely easy.
- To see this, assume that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is an orthogonal basis and we want to express the vector $\mathbf{u}$ with respect to this basis. So we have

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p$$

- Taking inner product with $\mathbf{v}_1$ gives

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v}_1 &= (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p) \cdot \mathbf{v}_1 \\
&= c_1\mathbf{v}_1 \cdot \mathbf{v}_1 + c_2\mathbf{v}_2 \cdot \mathbf{v}_1 + \ldots + c_p\mathbf{v}_p \cdot \mathbf{v}_1 \\
&= c_1\|\mathbf{v}_1\|^2 \longrightarrow c_1 = \frac{\mathbf{u} \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2}
\end{aligned}$$

- Similarly, $c_k = \dfrac{\mathbf{u} \cdot \mathbf{v}_k}{\|\mathbf{v}_k\|^2}$ for all $k = 1, 2, \ldots, p$.
- This shows the following: the coordinates of any vector with respect to an orthogonal basis can easily be computed just by taking inner products.

Orthogonal Basis: Example

- As an example, consider the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for $\mathbf{R}^3$, where $\mathbf{u}_1 = (3, 1, 1), \mathbf{u}_2 = (-1, 2, 1), \mathbf{u}_3 = (-1, -4, 7)$. This is an orthogonal basis. Let us express the vector $\mathbf{v} = (1, 2, 3)$ with respect to this basis.
- Let $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$. c_1 is given by

$$c_1 = \frac{\mathbf{v} \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2} = \frac{8}{11}.$$

- Similarly, for c_2 and c_3 we have

$$c_2 = \frac{\mathbf{v} \cdot \mathbf{u}_2}{\|\mathbf{u}_2\|^2} = \frac{6}{6} = 1$$

and

$$c_3 = \frac{\mathbf{v} \cdot \mathbf{u}_3}{\|\mathbf{u}_3\|^2} = \frac{12}{66}$$

.

- So we have $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{8}{11} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + \frac{12}{66} \begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix}$.

Orthonormal Basis

- If an orthogonal basis has the additional property that every vector in it is a unit vector, then it is called an orthonormal basis.
- Standard basis is the simplest orthonormal basis.

Example: Consider the vectors

$$\mathbf{v}_1 = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right), \ \mathbf{v}_2 = \left(-\frac{1}{\sqrt{18}}, \frac{4}{\sqrt{18}}, -\frac{1}{\sqrt{18}}\right), \ \mathbf{v}_3 = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right).$$

These three vectors are unit vectors and they are mutually orthogonal (Please check). So the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis for $\mathbf{R}^3$.

- Given an orthogonal basis, normalizing every vector in the basis gives an orthonormal basis. More explicitly, if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is an orthogonal basis, then $\left\{\frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}, \ldots, \frac{\mathbf{v}_p}{\|\mathbf{v}_p\|}\right\}$ is an orthonormal basis.
- Coordinates with respect to an orthonormal basis is particularly easy.

Example: Let us express the vector $\mathbf{w} = (1, 2, 3)$ with respect to the above orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. If $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$, then

$$c_1 = \mathbf{w} \cdot \mathbf{v}_1 = \frac{10}{3}, \ c_2 = \mathbf{w} \cdot \mathbf{v}_2 = \frac{2\sqrt{2}}{3}, \ c_3 = \mathbf{w} \cdot \mathbf{v}_3 = \sqrt{2}.$$

So we have $\mathbf{w} = \frac{10}{3}\mathbf{v}_1 + \frac{2\sqrt{2}}{3}\mathbf{v}_2 + \sqrt{2}\mathbf{v}_3$. (Please check)

Computing the Projection onto a Subspace

- Now we are ready to tackle the problem of computing the orthogonal projection of a vector onto a subspace.
- Let $V = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, where $\mathbf{u}_1 = (3, 1, 1), \mathbf{u}_2 = (-1, 2, 1)$. Let $\mathbf{y} = (1, 1, 1)$. Observe that $\mathbf{y} \notin V$.
- We had shown that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for V.
- To find the orthogonal projection of $\mathbf{y}$ onto V, we compute the projections of $\mathbf{y}$ onto both $\mathbf{u}_1$ and $\mathbf{u}_2$.
- We have

$$\text{proj}_{\mathbf{u}_1}\mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = \left(\frac{15}{11}, \frac{5}{11}, \frac{5}{11}\right) \text{ and}$$

$$\text{proj}_{\mathbf{u}_2}\mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = \left(-\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right).$$

- Then we add these projections. $\hat{\mathbf{y}} = \text{proj}_{\mathbf{u}_1}\mathbf{y} + \text{proj}_{\mathbf{u}_2}\mathbf{y} = \left(\frac{34}{33}, \frac{37}{33}, \frac{26}{33}\right)$.
- $\mathbf{y} - \hat{\mathbf{y}} = \left(-\frac{1}{33}, -\frac{4}{33}, \frac{7}{33}\right)$ is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$ (please verify), which shows $\mathbf{y} - \hat{\mathbf{y}} \in V^{\perp}$.
- But this shows that $\hat{\mathbf{y}} = \text{proj}_{\mathbf{u}_1}\mathbf{y} + \text{proj}_{\mathbf{u}_2}\mathbf{y} = \text{proj}_V\mathbf{y}$.
- We can use this process whenever we have an orthogonal basis for V.

- Since orthogonal bases are so easy to work with, it is important to be able to construct them.

Example

Let $\mathbf{v}_1 = (1, 3, -1), \mathbf{v}_2 = (2, 1, 0), \mathbf{v}_3 = (3, 4, 1)$. The set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a non-orthogonal basis for $\mathbf{R}^3$.

In order to construct an orthogonal basis from S, we will proceed as follows:

- First, we will use $\mathbf{v}_2$ to construct a vector $\mathbf{w}_2$ that is orthogonal to $\mathbf{v}_1$.
- Then we will use $\mathbf{w}_3$ to construct a vector $\mathbf{w}_3$ that is orthogonal to both $\mathbf{v}_1$ and $\mathbf{w}_2$.
- The set $\{\mathbf{v}_1, \mathbf{w}_2, \mathbf{w}_3\}$ will be an orthogonal basis for $\mathbb{R}^3$.

Step 1: $\mathbf{w}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{v}_1}\mathbf{v}_2 = (2, 1, 0) - \left(\frac{5}{11}, \frac{15}{11}, -\frac{5}{11}\right) = \left(\frac{17}{11}, -\frac{4}{11}, \frac{5}{11}\right)$

Step 2: $\mathbf{w}_3 = \mathbf{v}_3 - \text{proj}_{\mathbf{v}_1}\mathbf{v}_3 - \text{proj}_{\mathbf{w}_2}\mathbf{v}_3$

$= (3, 4, 1) - \left(\frac{14}{11}, \frac{42}{11}, -\frac{14}{11}\right) - \left(\frac{68}{33}, -\frac{16}{33}, \frac{20}{33}\right) = \left(-\frac{1}{3}, \frac{2}{3}, \frac{5}{3}\right)$

Step 3: The set $\{\mathbf{v}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthogonal basis for $\mathbb{R}^3$. (Please check)

Gram-Schmidt Orthogonalization

- The above mentioned method is known as Gram-Schmidt (Orthogonalization) Process and can be generalized as follows:

Gram-Schmidt Process

Given a set of linearly independent vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$, we can construct an orthonormal basis for the subspace spanned by these vectors as follows:

1. Set $\mathbf{w}_1 = \mathbf{v}_1$.
2. For $k = 2, 3, \ldots, p$, define

$$\mathbf{w}_k = \mathbf{v}_k - \sum_{j=1}^{k-1} \mathrm{proj}_{\mathbf{w}_j}(\mathbf{v}_k)$$

The resulting set $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_p\}$ is an orthonormal basis for the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.

Exercise: Let $\mathbf{v}_1 = (1, 1, 1, 1), \mathbf{v}_2 = (0, 1, 1, 1), \mathbf{v_3} = (0, 0, 1, 1)$. Let $V = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Construct an orthogonal basis for V. Then extend it to an orthogonal basis for $\mathbf{R}^4$.

Finding the Distance to a Subspace

- Let $\mathbf{u}_1 = (2, 5, -1)$ and $\mathbf{u}_2 = (2, 11, -1)$. Find the distance of the vector $\mathbf{y} = (1, 2, 3)$ to the subspace $V = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$.
- To compute the distance of $\mathbf{y}$ to V, we must find the vector in V that is closest to $\mathbf{y}$. But this vector is just $\text{proj}_V \mathbf{y}$.
- In order to calculate $\text{proj}_V \mathbf{y}$, we must first find an orthogonal basis of V. Observe that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is not orthogonal.
- To construct an orthogonal basis of V, we start with the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ and apply Gram-Schmidt process to it.

(1) Set $\mathbf{w}_1 = \mathbf{u}_1 = (2, 5, -1)$.

(2) Compute $\mathbf{w}_2 = \mathbf{u}_2 - \text{proj}_{\mathbf{w}_1} \mathbf{u}_2 = (-2, 1, 1)$. (Please check)

- Then $\{\mathbf{w}_1, \mathbf{w}_2\} = \{(2, 5, -1), (-2, 1, 1)\}$ is an orthogonal basis for V.
- The projection of $\mathbf{y}$ onto V is computed by

$$\text{proj}_V \mathbf{y} = \text{proj}_{\mathbf{w}_1} \mathbf{y} + \text{proj}_{\mathbf{w}_2} \mathbf{y} = \left(\frac{3}{5}, \frac{3}{2}, -\frac{3}{10}\right) + \left(-1, \frac{1}{2}, \frac{1}{2}\right) = \left(-\frac{2}{5}, 2, \frac{1}{5}\right).$$

- Thus, the point in V that is closest to $\mathbf{y}$ is $\text{proj}_V \mathbf{y} = \left(-\frac{2}{5}, 2, \frac{1}{5}\right)$.
- The distance of $\mathbf{y}$ to V is $\|\mathbf{y} - \text{proj}_V \mathbf{y}\| = \left\| \left(\frac{7}{5}, 0, \frac{14}{5}\right) \right\| = \frac{7\sqrt{5}}{5}$.

Inner Product in Function Spaces

- In a similar way that we can define an inner product between vectors in $\mathbf{R}^n$, we can define an inner product between functions in a function space.
- Let's consider the vector space of real-valued functions defined on the interval $[a, b]$. This space is denoted by $\mathcal{F}([a, b])$.
- The inner product of two functions $f, g \in \mathcal{F}[a, b]$ is defined as $f \cdot g = \int_a^b f(x)g(x)dx$

Example: Let's consider the functions $f(x) = x$ and $g(x) = \sin(x)$ defined on the interval $[0, \pi]$. Then their inner product is

$$f \cdot g = \int_0^\pi x \sin(x)\, dx = [\sin(x) - x\cos(x)]_0^\pi = \pi$$

- Just like in Euclidean spaces, two functions f and g are said to be orthogonal to each other if $f \cdot g = 0$.
- For example, the functions $f(x) = x - \dfrac{\pi}{2}$ and $g(x) = \sin(x)$ are orthogonal on the interval $[0, \pi]$. (Note that the interval matters; they are not orthogonal e.g. on $[0, 2\pi]$.)

Fourier Series

- Now we will see a famous application of inner product in function spaces.
- A trigonometric polynomial of degree n on the interval $[a, b]$ is a function of the form

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx))$$
$$= \frac{a_0}{2} + a_1 \cos(x) + \ldots + a_n \cos(nx) + b_1 \sin(x) + \ldots + b_n \sin(nx)$$

  where $a_0, a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are real numbers.
- The folllowing result, due to Joseph Fourier, makes trigonometric polynomials extremely important in approximation theory: For any continuous function $f(x)$ on $[a, b]$, we can approximate f by a trigonometric polynomial as accurately as we desire.
- We are particularly interested function defined on the interval $[-\pi, \pi]$. Given $f \in C([a, b])$, there are real numbers $a_0, a_1, a_2 \ldots$ and $b_1, b_2, \ldots$, called the Fourier coefficients, such that

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx)) \text{ for all } x \in [-\pi, \pi].$$

- In other words, the infinite set $\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots\}$ is a basis for $C([-\pi, \pi])$. It is called the Fourier basis.

Calculating Fourier coefficients

- Given a function $f \in C([-\pi, \pi])$, how to calculate its Fourier coefficients?
- These calculations are made substantially easier by the following observation: A Fourier basis is orthogonal.
- To see this, just observe that $\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,dx = 0$ for integers m, n and also that $\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = 0$ for $m \neq n$. (Please check)
- Observe also that $\int_{-\pi}^{\pi} \cos(kx)\cos(kx)\,dx = \int_{-\pi}^{\pi} \sin(kx)\sin(kx)\,dx = \pi$.
- Then, calculating Fourier coefficients is straightforward:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx \text{ for } k = 0, 1, 2, \ldots$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(x)\,dx \text{ for } k = 1, 2, \ldots$$

Calculating Fourier coefficients: Example

- Let us calculate the Fourier coefficients of the function $f(t) = t^2$.
- First observe that f is an even function. Since $\sin(kt)$ is odd and $\cos(kt)$ is even, $f(t)\sin(kt)$ is odd and $f(t)\cos(kt)$ is even for all $k = 1, 2, \ldots$
- This shows that $\int_{-\pi}^{\pi} f(t)\cos(kt)\, dt = 2\int_{0}^{\pi} f(t)\cos(kt)\, dt$ and $\int_{-\pi}^{\pi} f(t)\sin(kt)\, dt = 0$. Thus all the Fourier coefficients corresponding to the sine terms are zero.
- For the constant term we have $a_0 = \frac{1}{\pi} \cdot 2\int_{0}^{\pi} t^2\, dt = \frac{2\pi^2}{3}$.
- For the remaining cosine terms we have

$$a_k = \frac{1}{\pi} \cdot 2\int_{0}^{\pi} t^2 \cos(kt)\, dt = \frac{2}{\pi}\frac{2\pi}{k^2}(-1)^k = \frac{4(-1)^k}{k^2} \text{ for } k = 1, 2, \ldots$$

(Please check)

- Thus, the Fourier series for f is

$$f(t) = t^2 = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2}\cos(kt) =$$

$$\frac{\pi^2}{3} + 4\left(-1\cos(t) + \frac{1}{2^2}\cos(2t) - \frac{1}{3^2}\cos(3t) + \ldots\right)$$

Calculating Fourier coefficients: Example

- We have computed the Fourier series for the function $f(t) = t^2$ defined on $[-\pi, \pi]$.
- Let us now evaluate this series at $t = \pi$. Since $f(\pi) = \pi^2$ we have

$$\pi^2 = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2} \cos(k\pi)$$

- Since $\cos(k\pi) = (-1)^k$ for every integer k, the above implies

$$\pi^2 = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2}(-1)^k = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{1}{k^2} \longrightarrow 4\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{2\pi^2}{3}$$

- From this follows the famous identity

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}$$

Exercise: Find the Fourier coefficients for the square wave defined by

$$g(t) = \begin{cases} -1, \text{ if } -\pi \le t < 0 \\ 1, \text{ if } 0 \le t < \pi \end{cases}.$$