MAT 222 Linear Algebra
Week 6
Lecture Notes 1

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Linear Independence of Vectors

- Previously we considered the 4×4 matrix $A = \begin{bmatrix} -2 & -5 & 8 & 0 \\ 1 & 3 & -5 & 1 \\ 3 & 11 & -19 & 7 \\ 1 & 7 & -13 & 5 \end{bmatrix}$ and showed that its row space can actually be spanned by three vectors $\mathbf{b}_1 = (1, 3, -5, 1), \mathbf{b}_2 = (0, 1, -2, 2)$ and $\mathbf{b}_3 = (0, 0, 0, 4)$.
- What does this imply regarding the rows of A?
- It implies that each row of A is a linear combination of $\mathbf{b}_1, \mathbf{b}_2$ and $\mathbf{b}_3$.
- So there are real numbers $a_i, b_i, c_i, d_i,\ i = 1, 2, 3,$ such that

$$\begin{aligned} \mathbf{a}_1 &= (-2, -5, 8, 0) = a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + a_3\mathbf{b}_3 \\ \mathbf{a}_2 &= (1, 3, -5, 1) = b_1\mathbf{b}_1 + b_2\mathbf{b}_2 + b_3\mathbf{b}_3 \\ \mathbf{a}_3 &= (3, 11, -19, 7) = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 \\ \mathbf{a}_4 &= (1, 7, -13, 5) = d_1\mathbf{b}_1 + d_2\mathbf{b}_2 + d_3\mathbf{b}_3 \end{aligned}$$

- Now consider the homogeneous system $k_1\mathbf{a}_1 + k_2\mathbf{a}_2 + k_3\mathbf{a}_3 + k_4\mathbf{a}_4 = \mathbf{0}$.
- After some manipulations, we can write it as $(a_1k_1 + b_1k_2 + c_1k_3 + d_1k_4)\mathbf{b}_1 + (a_2k_1 + b_2k_2 + c_2k_3 + d_2k_4)\mathbf{b}_2$ $+ (a_3k_1 + b_3k_2 + c_3k_3 + d_3k_4)\mathbf{b}_3 = \mathbf{0}$.

Linear Independence of Vectors

- This last equation is satisfied if the system of equations

$$a_1k_1 + b_1k_2 + c_1k_3 + d_1k_4 = 0$$
$$a_2k_1 + b_2k_2 + c_2k_3 + d_2k_4 = 0$$
$$a_3k_1 + b_3k_2 + c_3k_3 + d_3k_4 = 0$$

- But this is an under-determined (3×4) homogeneous system in the unknowns k_1, k_2, k_3, so it should have a nontrivial solution.
- So there is a nonzero vector (k_1, k_2, k_3, k_4) such that $k_1\mathbf{a}_1 + k_2\mathbf{a}_2 + k_3\mathbf{a}_3 + k_4\mathbf{a}_4 = \mathbf{0}$.
- The following gives a new name to this important concept

Linear independence

Let V be a vector space and let $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_m$ be elements of V. If there are scalars $c_1, c_2, \ldots, c_m$, not all zero, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots c_m\mathbf{v}_m = \mathbf{0}$, then the set $\{\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_m\}$ is said to be a linearly dependent set and the vectors $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_m$ are said to be linearly dependent vectors. Otherwise the set $\{\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_m\}$ (and the vectors in it) are called linearly independent.

- In the preceding example, the rows $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ of A are **NOT** linearly independent. They are linearly dependent.

Linear Independence of Vectors: Example

- Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$.
- Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent?
- If the system $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ has a nontrivial solution, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. If the only solution is $(c_1, c_2, c_3) = (0, 0, 0)$, then it is linearly independent.
- This system can be written as

$$\begin{aligned} 3c_1 - 4c_2 - 2c_3 &= 0 \\ c_2 + c_3 &= 0 \\ -6c_1 + 7c_2 + 5c_3 &= 0 \end{aligned} \quad \text{or equivalently} \quad \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

- The reduced echelon form of the coefficient matrix is the identity, so the only solution to this system is $c_1 = c_2 = c_3 = 0$.
- Thus, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.
- **Exercise:** Considering this example, decide if the following expression is true or false: "Columns of A is linearly independent if and only if $A\mathbf{x} = \mathbf{0}$ has only one solution."

Linear Independence and Column Space

- Consider $A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 3 & 7 \\ 1 & 4 & 9 \end{bmatrix}$. Is the vector $\mathbf{b} = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}$ in $C(A)$?
- In other words, are there real numbers x, y, z such that

$$x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + z \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 3 & 7 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} ?$$

- This system has infinitely many solutions. For instance, you can check that $\mathbf{b} = 3\mathbf{v}_1 + 1\mathbf{v}_2 - 1\mathbf{v}_3$, where $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are columns of A, respectively. (Can you find other solutions than $(3, 1, -1)$?)
- So $\mathbf{b} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
- But it is true that $\mathbf{v}_3 = 1\mathbf{v}_1 + 2\mathbf{v}_2$.
- For $\mathbf{b}$ this implies the following:

$$\mathbf{b} = 3\mathbf{v}_1 + 1\mathbf{v}_2 - 1\mathbf{v}_3 = 3\mathbf{v}_1 + 1\mathbf{v}_2 - 1(1\mathbf{v}_1 + 2\mathbf{v}_2) = 2\mathbf{v}_1 - 1\mathbf{v}_2.$$

- This means that actually $\mathbf{b} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.
- It is also true that $C(A) = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

Dimension of a Vector Space

- Thus the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ has the property that
  - (i) It spans $C(A)$.
  - (ii) No smaller set can span $C(A)$
- An immediate result is this: 2 is the smallest number of vectors that span $C(A)$.
- We call it the dimension of $C(A)$.

Dimension of a vector space

Suppose that V is a vector space and there are vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ in V such that $V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$. Suppose further that V cannot be spanned by $m - 1$ elements. Than the number m is called the dimension of V and it is denoted by $\dim(V) = m$.

- If a vector space cannot be spanned by a finite number of vectors, it is called infinite dimensional. Otherwise it is finitely generated.

Dimension and Linear Independence

- In the example with $A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 3 & 7 \\ 1 & 4 & 9 \end{bmatrix}$, we started with $C(A) = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
- Then we realized that $\mathbf{v}_3 = 1\mathbf{v}_1 + 2\mathbf{v}_2$.
- This shows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent since $1\mathbf{v}_1 + 2\mathbf{v}_2 - 1\mathbf{v}_3 = 0$.
- This also showed that we can remove $\mathbf{v}_3$ from the spanning set, so that $C(A) = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.
- The set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is clearly linearly independent, so we cannot make any further removal.
- The following is true in general: If a vector space V is the span of n vectors that are linearly dependent, than it can be written as the span of a less number number of elements. So $\dim(V) < n$.
- This gives an alternative definition of dimension: If a vector space is spanned by a set of m linearly independent vectors, then $\dim(V) = m$.

Dimension and Basis

- The preceding discussion shows the following: If $\dim(V) = m$, then any set of $m + 1$ (or more) elements in V is linearly dependent.
- So we have still another definition of dimension. It is the maximum possible number of elements in a linearly independent subset of V.
- In order to determine the dimension of V, we need a subset S such that
  - (i) S spans V.
  - (ii) S is a linearly independent subset.
- Such a set S deserves a special name.

Basis of a vector space

Suppose V is a vector space and that S is a linearly independent subset of S such that $V = \text{Span}(S)$. Then S is called a basis of V.

- Basis of a vector space can be thought of as the "seed" that generates the vector space. It is sufficient to generate the whole space and it does not contain any unnecessary elements. It is perfect in this sense.
- In the case of $A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 3 & 7 \\ 1 & 4 & 9 \end{bmatrix}$, what is a basis of $C(A)$?

Basis: Example of $\mathbb{R}^2$

- Let us first consider $\mathbb{R}^2$.
- It is clear that $\mathbb{R}^2$ cannot be generated by a single vector, because if $\mathbf{u} \in \mathbb{R}^2$ then Span$\{\mathbf{u}\}$ is a line.
- Take an arbitrary element $(a, b) \in \mathbb{R}^2$. Then observe that
$$(a, b) = a(1, 0) + b(0, 1)$$
- This shows that any element of $\mathbb{R}^2$ can be expressed as a linear combination of $(1, 0)$ and $(0, 1)$.
- This means that the set $\{(1, 0), (0, 1)\}$ is a basis of $\mathbb{R}^2$. It is called the standard basis or the canonical basis. (This also shows $\dim(\mathbb{R}^2) = 2$.)
- It is customary to denote $(1, 0)$ and $(0, 1)$ by $\mathbf{e}_1$ and $\mathbf{e}_2$.
- Of course it is not the only basis of $\mathbb{R}^2$.
- Any two vectors which are not multiples of each other (which are not on the same line) constitute a basis for $\mathbb{R}^2$.
- For example if $\mathbf{v}_1 = (1, -1)$ and $\mathbf{v}_2 = (3, 5)$ then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^2$.
- **Exercise:** Express $(1, 0)$ and $(0, 1)$ as linear combinations $\mathbf{v}_1$ and $\mathbf{v}_2$. Then use this to express any elements (a, b) as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Basis: Example of $\mathbb{R}^3$

- Similarly, $\mathbb{R}^3$ cannot be generated by a single vector or two vectors. These generate a line and a plane, respectively.
- The standard basis for $\mathbb{R}^3$ is $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ where $\mathbf{e}_1 = (1, 0, 0), \mathbf{e}_2 = (0, 1, 0), \mathbf{e}_3 = (0, 0, 1)$.
- In addition, any three vectors which are not on the same plane consitute a basis for $\mathbb{R}^3$.
- For example, consider the matrix $A = \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix}$. We had shown that A is invertible. The set of its columns (written horizontally) $\{(3, 0, -6), (-4, 1, 7), (-2, 1, 5)\}$ is a basis for $\mathbb{R}^3$. Of course, the set of its rows is also a basis.
- In order to express any element $\mathbf{u} = (a, b, c)$ in terms of the columns of A, just solve the system $A\mathbf{x} = \mathbf{u}^T$. The entries of the solution will give you the correct combination.
- Of course, all these discussions extend to Euclidean spaces $\mathbb{R}^n$ in general. Define $\mathbf{e}_i$ to be the vector whose i-th entry is 1 and the others are 0. Then $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is the standar basis for $\mathbb{R}^n$.
- **Exercise:** Express $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ as linear combinations of the rows of A.

Basis and Dimension: Summary

- Consider a vector space V and vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$.
- For any vector $\mathbf{u} \in V$, if $\mathbf{u} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, then $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p, \mathbf{u}\}$. (Adding $\mathbf{u}$ to the set does not enlarge the span.)
- Conversely, if any of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$, say $\mathbf{v}_p$, is a linear combination of the others, then $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{p-1}\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$. (Removing $\mathbf{v}_p$ does not shrink the span.)
- On the contrary, if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a linearly independent set, then it is a basis for $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.
- Hence, to check that if a given set S is a basis for V one must check if
  (1) S spans V. (i.e. $V = \text{Span}(S)$).
  (2) S is linearly independent.

  If both (1) and (2) are satisfied, S is a basis for V. Otherwise it is not.
- This gives an alternative definition for the dimension of V. $\dim(V)$ is the size of the largest linearly independent subset of V.
- **Exercise:** Write down a 4×4 triangular matrix with nonzero diagonal entries. Prove that the set of its rows constitute a basis for $\mathbb{R}^4$. Then express the last 4 digit of your student id with respect to this basis.