MAT 222 Linear Algebra and Numerical Methods
Week 5
Lecture Notes 2

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Vector Spaces

- From now on, we deal with sets of vectors rather than individual vectors.
- This takes us to the concept of vector spaces.
- Loosely speaking, a vector space is a set of objects such that a linear combination of these objects is also an element of this set.
- These objects are called vectors.
- More explicitly, a set V of vectors is called a vector space if

  (i) addition of two vectors in V
  (ii) multiplication of a vector in V by a scalar

  also produces a vector in V.
- In other words, a vector space is a set which is closed under linear combination of its elements.
- The properties (i) and (ii) imply a number of other mathematical properties.

Vector Space Axioms

- Suppose V is a nonempty set. Suppose further that the following hold for every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and every real numbers c and d:
  (1) $\mathbf{u} + \mathbf{v} \in V$
  (2) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  (3) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
  (4) There is a zero vector $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
  (5) For each $\mathbf{u} \in V$ there is a vector $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
  (6) The scalar multiple $c\mathbf{u}$ is in V.
  (7) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
  (8) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
  (9) $c(d\mathbf{u}) = (cd)\mathbf{u}$
  (10) $1\mathbf{u} = \mathbf{u}$
- Then V is called a real vector space.
- Unless otherwise stated, we will deal with real vector spaces.
- The properties (1)-(10) are called vector space axioms.

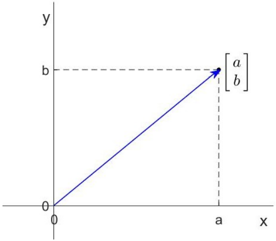
Euclidean Space

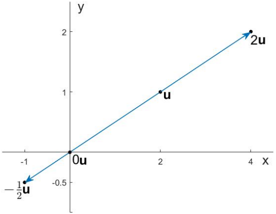
- Among all vector spaces, the Euclidean space $\mathbb{R}^n$ (sometimes denoted by E^n) is the most important.
- It is the space of all ordered tuples of n real numbers.
- $\mathbb{R}^n = \{(x_1, x_2 \ldots, x_n) : \ x_1, x_2, \ldots, x_n \in \mathbb{R}\}$.
- Previously we represented its elements by column vectors $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.
- Let $\mathbf{u} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{v} = (b_1, b_2, \ldots, b_n)$ are two elements of $\mathbb{R}^n$ and $c_1, c_2 \in \mathbb{R}$. Then the linear combination

$$c_1\mathbf{u} + c_2\mathbf{v} = (c_1 a_1 + c_2 b_1, c_1 a_2 + c_2 b_2, \ldots, c_1 a_n + c_2 b_n)$$

  is clearly in $\mathbb{R}^n$. So $\mathbb{R}^n$ is closed under linear combination.
- Other vector space axioms can easily be verified for $\mathbb{R}^n$.

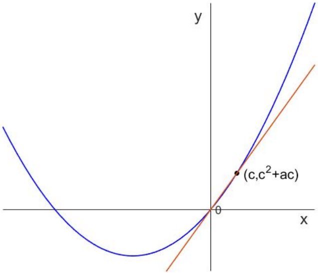
The Euclidean Space $\mathbb{R}^2$





- Every point (x, y) (or $\begin{bmatrix} x \\ y \end{bmatrix}$) in the analytic plane can be considered as a vector.
- Is any subset of $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ a vector space?
- Clearly, the set $\{(0, 0)\}$ consisting of only the zero vector is a vector space. It is called the trivial subspace.
- Consider a line passing through the origin. It has an equation of the form $y = mx$ where $m \in \mathbb{R}$.
- Such a line is the set $L = \{t\mathbf{u} : t \in \mathbb{R}\}$, where $\mathbf{u} = (1, m)$.
- If c is any real number and $\mathbf{v} = (t, tm)$ is any vector in L, then $c\mathbf{v} = (ct, ctm) = ct(1, m)$ is a vector in L.
- In addition, if $\mathbf{u}_1 = (t_1, t_1 m)$ and $\mathbf{u}_2 = (t_2, t_2 m)$ are two vectors in L, then $\mathbf{u}_1 + \mathbf{u}_2 = (t_1 + t_2)(1, m)$ is also in L.
- Thus, any line passing through the origin is a vector space.

The Euclidean Space $\mathbb{R}^2$



- Note that any vector space should contain the zero vector.
- In $\mathbb{R}^2$ the zero vector is the origin $(0,0)$ so lines (and other curves) not passing from the origin are not vector spaces.
- Is a parabola passing through the origin a vector space?
- Such a parabola is a set of the form $P = \{(x, y) \in \mathbb{R}^2 : y = ax^2 + bx\}$. Parametrically, it is $P = \{(t, at^2 + bt) : t \in \mathbb{R}\}$.
- Take any point **q** from P. Is every scalar multiple of **q** is in P?
- The set of all scalar multiples of **q** is the line connecting the origin and **q**.
- Since this line is not contained in the parabola P, the answer is "No".

The Subspaces of $\mathbb{R}^2$

- We have seen that the subsets of $\mathbb{R}^2$ that are subspaces are the following:
  (1) The trivial vector space $\{(0,0)\}$
  (2) Lines passing through origin
  (3) The space $\mathbb{R}^2$ itself
- These are called the subspaces of $\mathbb{R}^2$.

Subspace of a vector space

Suppose that V is a vector space and U is a subset of V. If U is also a vector space, it is called a subspace of V.

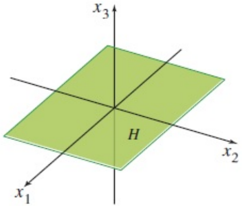
- Given a vector space V and a subset U of V, it suffices to check the following to see if U is a subspace:
  (1) U is closed under vector addition.
  (2) U is closed under scalar multiplication.
- If (1) and (2) is satisfied, then U is a subspace of V. Otherwise it is not.
- **Exercise:** Show that the first quadrant $\{(x,y) : x \geq 0, y \geq 0\}$ is not a subspace of $\mathbb{R}^2$.

Subspaces of $\mathbb{R}^3$: An Example

- Consider the subset $Q = \{(x, y, z) : 2x + y - z = 0\}$ of $\mathbb{R}^3$. Is Q a subspace of $\mathbb{R}^3$?
- Let $\mathbf{u}_1 = (a_1, b_1, c_1)$ and $\mathbf{u}_2 = (a_2, b_2, c_2)$ be two elements of Q.
- Both $\mathbf{u}_1$ and $\mathbf{u}_2$ are solutions of $2x + y - z = 0$. So we have $2a_1 + b_1 - c_1 = 0$ and $2a_2 + b_2 - c_2 = 0$.
- Is $\mathbf{u}_1 + \mathbf{u}_2$ an element of Q?
- We should check if $\mathbf{u}_1 + \mathbf{u}_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$ satisfies the equation $2x + y - z = 0$.
- We have
  $2(a_1 + a_2) + (b_1 + b_2) - (c_1 + c_2) = (2a_1 + b_1 - c_1) + (2a_2 + b_2 - c_2) = 0 + 0 = 0.$
- So $\mathbf{u}_1 + \mathbf{u}_2$ satisfies the equation, which means $\mathbf{u}_1 + \mathbf{u}_2 \in Q$.
- Secondly, we should check if $k\mathbf{u_1} = (ka_1, kb_1, kc_1) \in Q$.
- Is $k\mathbf{u_1} = (ka_1, kb_1, kc_1)$ a solution of $2x + y - z = 0$?
- Since $2(ka_1) + kb_1 - kc_1 = k(2a_1 + b_1 - c_1) = k \cdot 0 = 0$, we have $\mathbf{u}_1 \in Q$.
- Thus, Q is a subspace of $\mathbb{R}^3$.
- **Exercise:** Is the set $T = \{(x, y, z) : 2x + y - z = 1\}$ a subspace of $\mathbb{R}^3$?

Subspaces of $\mathbb{R}^3$

- The set $Q = \{(x, y, z) : 2x + y - z = 0\}$ is a plane through origin $(0, 0, 0)$. We have shown that it is a subspace of $\mathbb{R}^3$.
- **Exercise:** Show that every plane through origin is a subspace of $\mathbb{R}^3$. (Hint: A plane through origin has an equation $ax + by + cz = 0$.
- Similarly one can show lines through origin are also subspaces of $\mathbb{R}^3$.
- Thus, the subspaces of $\mathbb{R}$ are
  1. $\{(0, 0, 0)\}$, the trivial subspace
  2. Lines through origin
  3. Planes through origin
  4. $\mathbb{R}^3$
- Subspaces in the above list gets "bigger" as we move downward.



- Each plane through origin is like a copy of $\mathbb{R}^2$ inside $\mathbb{R}^3$.
- In particular the $x_1 x_2$-plane $\{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{R}\}$ is like an "embedded" version of $\mathbb{R}^2$ in $\mathbb{R}^3$. (See the figure)
- Note that $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$. (Why?)

Span of a Set of Vectors

- Let us again consider the subspace $Q = \{(x, y, z) : 2x + y - z = 0\}$ of $\mathbb{R}^3$.
- A typical element of Q can be written

$$\begin{bmatrix} x \\ y \\ 2x + y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

- So Q is the set of all linear combinations of the vectors $(1, 0, 2)$ and $(0, 1, 1)$. (Do you remember this structure?)

Span of a set of vectors

Let V be a vector space and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ be a subset of V. Then the set

$$\{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m : c_1, c_2, \ldots, c_m \in \mathbb{R}\}$$

is called the span of S and is denoted by $\mathrm{Span}(S)$.

- In the language of this new definition, the plane Q is the span of the set $\{(1, 0, 2), (0, 1, 1)\}$.

Span of a Set of Vectors

- By definition, the span of a set of vectors is a vector space.
- The converse is also true for Euclidean spaces: Every subspace of $\mathbb{R}^n$ is the span of a set of vectors from $\mathbb{R}^n$.
- For example, in $\mathbb{R}^3$ every line through origin can be expressed as $\{t\mathbf{u} : t \in \mathbb{R}\} = \text{Span}\{\mathbf{u}\}$ for some $\mathbf{u} \in \mathbb{R}^3$.
- Similarly, every plane through origin can be expressed as $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ for some $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3$.
- We have seen that the plane $Q = \{(x, y, z) : 2x + y - z = 0\}$ is equal to the span of $\{(1, 0, 2), (0, 1, 1)\}$.
- Note that $(-1, 2, 0)$ is also in Q. So Q is also equal to the span of $\{(1, 0, 2), (0, 1, 1), (-1, 2, 0)\}$.
- This is true in general: If $U = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ and $\mathbf{v} \in U$, then we have $U = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m, \mathbf{v}\}$.
- **Exercise:** Is it true that $Q = \text{Span}\{(1, 0, 2), (-1, 2, 0)\}$?
- **Exercise:** Can you find a subset S of $\mathbb{R}^3$ such that $\mathbb{R}^3 = \text{Span}(S)$?

Span of a Set of Vectors: Example

- Let us once more consider the plane $Q = \{(x, y, z) : 2x + y - z = 0\}$.
- Using column notation, we have seen that $Q = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.
- Given a vector $\mathbf{v} = (a, b, c) \in \mathbb{R}^3$, we have two ways to determine if $\mathbf{v} \in Q$.
- The first way is to check if (a, b, c) satisfies $2x + y - z = 0$.
- The second way is to check if $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.
- To understand the second approach, let us consider $\mathbf{v} = (3, 4, -1)$. $\mathbf{v}$ is not in Q since $2 \cdot 3 + 4 - (-1) = 11 \neq 0$.
- To check if $\begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$ is in $\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$, we answer the following:

  Does there exist c_1 and c_2 such that $\begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$?

Span of a Set of Vectors: Example

- This is the same as looking for c_1 and c_2 that satisfy

$$\begin{aligned} c_1 &= 3 \\ c_2 &= 4 \\ 2c_1 + c_2 &= -1 \end{aligned} \quad \text{or equivalently} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}.$$

- This over-determined system clearly has no solutions so we conclude that $\begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$ is not in Span $\left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$. (which we already knew)
- In general, any vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in Span $\left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$ iff the system $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is consistent.
- The following definition gives a new way to express this fact.

Column space of a matrix

Let A be an $m \times n$ matrix whose columns are $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Then the set $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is called the column space of A and is denoted by $C(A)$.

Column Space of a Matrix

- If A is an $m \times n$ matrix, $C(A)$ is a subspace of $\mathbb{R}^m$.
- Consider the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}$ from the previous example.
- Any vector $\in \mathbb{R}^3$ is in $C(A)$ if and only if

$$\mathbf{b} = c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$ has at least one solution.

- This can be generalized as follows: $\mathbf{b} \in C(A)$ if and only if the system $A\mathbf{x} = \mathbf{b}$ is consistent.
- An important result is that the column space of an invertible matrix is the whole space.
- For example, the matrix $A = \begin{bmatrix} 10 & -2 & 5 \\ 6 & -1 & 4 \\ 1 & 0 & 1 \end{bmatrix}$ is invertible. So we have

$$\mathbb{R}^3 = \text{Span}\left(\begin{bmatrix} 10 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} \right).$$ (We'll return to this later)

Row Space of a Matrix

- Another vector space associated with a matrix is its row space.
- Row space of an $m \times n$ matrix A (denoted by $R(A)$) is the set spanned by its rows. More explicitly, if $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ are the rows of A, then $R(A) = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$. It is a subspace of $\mathbb{R}^n$.
- Note that the row space of A is equal to the column space of A^T. (Though one consists of row vectors while the other consists of column vectors.)
- Clearly, a row operation on A produces a matrix with the same row space as A.
- This implies the following: If B is an echelon form of A, then the nonzero rows of B span $R(A)$.
- More explicitly, if $\text{rank}(A) = r$ and the nonzero rows of B are $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r$, then we have $R(A) = \text{Span}\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r\}$.

Row Space of a Matrix: Example

- Let us consider $A = \begin{bmatrix} -2 & -5 & 8 & 0 \\ 1 & 3 & -5 & 1 \\ 3 & 11 & -19 & 7 \\ 1 & 7 & -13 & 5 \end{bmatrix}$.
- $R(A) = \text{Span}\{(-2,-5,8,0),(1,3,-5,1),(3,11,-19,7),(1,7,-13,5)\}$. Can we write $R(A)$ as the span of a less number of elements?
- To see this, let us reduce A to echelon form. An echelon form of A is the matrix $B = \begin{bmatrix} 1 & 3 & -5 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
- This shows that $\text{rank}(A) = 3$ and the 3 nonzero rows of B span A. So we have $R(A) = \text{Span}\{(1,3,-5,1),(0,1,-2,2),(0,0,0,-4)\}$.
- **Exercise:** Can this process be continued? In other words, are there $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^4$ such that $R(A) = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$? (We'll return to this later)
- **Exercise:** Does there exist 3 vectors in $\mathbb{R}^4$ whose span is equal to $C(A)$? (We'll return to this later)