MAT 222 Linear Algebra and Numerical Methods
Week 6
Lecture Notes 2

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Basis and Dimension: Summary

- Consider a vector space V and vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$.
- For any vector $\mathbf{u} \in V$, if $\mathbf{u} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, then $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p, \mathbf{u}\}$. (Adding $\mathbf{u}$ to the set does not enlarge the span.)
- Conversely, if any of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$, say $\mathbf{v}_p$, is a linear combination of the others, then $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{p-1}\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$. (Removing $\mathbf{v}_p$ does not shrink the span.)
- On the contrary, if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a linearly independent set, then it is a basis for $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.
- Hence, to check that if a given set S is a basis for V one must check if
  (1) S spans V. (i.e. $V = \text{Span}(S)$).
  (2) S is linearly independent.

  If both (1) and (2) are satisfied, S is a basis for V. Otherwise it is not.
- This gives an alternative definition for the dimension of V. $\dim(V)$ is the size of the largest linearly independent subset of V.
- Hence, if $\dim(V) = n$, we have
  (1) Any linearly independent n-element subset of V is a basis of V.
  (2) Any set of n elements that span V is linearly independent.

Not Every Vector Space is Finite Dimensional

- Of course, not every vector space is finite dimensional.
- Consider for example the set of all sequences with real elements.
- For obvious reasons, it is generally denoted by $\mathbb{R}^\infty$ (sometimes by $\mathbb{R}^\mathbb{N}$).
- Its elements are of the form

$$(a_n)_{n=0}^{\infty} = (a_0, a_1, a_2, \ldots) \text{ where each } a_n \text{ is a real number.}$$

- $\mathbb{R}^\infty$ is infinite dimensional. (Can you see why?)

- Another example is $\mathcal{F}(\mathbb{R})$, which is the set of all real-valued functions $f : \mathbb{R} \to \mathbb{R}$.

- Considering real-valued functions as vectors, $\mathcal{F}(\mathbb{R})$ is obviously closed under the operations

  (1) $(f+g)(x) = f(x) + g(x)$, where $f, g \in \mathcal{F}(\mathbb{R})$.
  (2) $(cf)(x) = c \cdot f(x)$, where $c \in \mathbb{R}$.

  So $\mathcal{F}(\mathbb{R})$ is a vector space.

- Although they are both infinite dimensional, it is true that $\dim(\mathcal{F}(\mathbb{R})) > \dim(\mathbb{R}^\infty)$. (Can you see why?)

The Space of Polynomials over $\mathbb{R}$

- Consider the set of all polynomials with real coefficients, which is denoted by $\mathbb{R}[x]$.
- Clearly $\mathbb{R}[x]$ is a vector space under the operations
  (1) $(P+Q)(x) = f(x) + g(x)$, where $P, Q \in \mathbb{R}[x]$,
  (2) $(cP)(x) = c \cdot P(x)$, where $c \in \mathbb{R}$.
- It can be considered as a subspace of $\mathcal{F}(\mathbb{R})$.
- For every nonnegative integer n, consider the set

$$\mathbb{P}_n = \{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n : a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}\}$$

  of all polynomials having degree at most n.
- Obviously $\mathbb{P}_n$ is a subspace of $\mathbb{R}[x]$.
- The set $\{1, x, x^2, \ldots, x^n\}$ obviously spans $\mathbb{P}_n$. In addition, it is linearly independent. Hence it is a basis (the standard basis) of $\mathbb{P}_n$.

The Space of Polynomials over $\mathbb{R}$

- As an example, let us take $n = 3$ and consider $\mathbb{P}_3$, the set of polynomials of degree 3 or less.
- The standard basis for $\mathbb{P}_3$ is $\{1, x, x^2, x^3\}$.
- Consider the function $f : \mathbb{P}_3 \to \mathbb{R}^4$ defined by

$$a_0 + a_1x + a_2x^2 + a_3x^3 \longrightarrow (a_0, a_1, a_2, a_3).$$

- For example $f(2 - x^2 + 4x^3) = (2, 0, -1, 4)$.
- The function f "respects" both addition and scalar multiplication. In other words we have
  - (i) $f(p + q) = f(p) + f(q)$ where $p, q \in \mathbb{P}_3$,
  - (ii) $f(cp) = cf(p)$ where $c \in \mathbb{R}$.
- In addition f is one-to-one. We call f an isomorphism from $\mathbb{P}_3$ to $\mathbb{R}^4$. The spaces $\mathbb{P}_3$ and $\mathbb{R}^4$ are called isomorphic.
- This means that the spaces $\mathbb{P}_3$ and $\mathbb{R}^4$ are like copies of each other. The difference is only "cosmetic".
- **Exercise:** Show that the set $\{1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3\}$ is a basis for $\mathbb{P}_3$. Then express the polynomial $a(x) = -2 + x + x^2 + x^3$ with respect to this basis.

Example: A Basis of $\mathbb{P}_2$

- Given the set $S = \{1+x, 2+3x, 1+2x+3x^2\}$, let us determine if S is a basis for $\mathbb{P}_2$.
- Let us make use of the mapping in the previous page. We have

$$1+x \to (1,1,0),\ 2+3x \to (2,3,0), 1+2x+3x^2 \to (1,2,3).$$

- Then S is a basis for $\mathbb{P}_2$ if and only if the set $\{(1,1,0),(2,3,0),(1,2,3)\}$ is a basis for $\mathbb{R}^3$.
- To check this, one way is to form the 3×3 matrix whose rows (or columns) are $(1,1,0),(2,3,0),(1,2,3)$.
- This matrix is $\begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix}$, let us call it A.
- $\det(A) = 3 \neq 0$, so A is invertible.
- So the set $B = \{(1,1,0),(2,3,0),(1,2,3)\}$ is a basis for $\mathbb{R}^3$. This shows that S is a basis for $\mathbb{P}_2$.

- Now that we have a basis S of $\mathbb{P}_2$, we can use it to express any element of $\mathbb{P}_2$.
- Let us express the polynomial $Q(x) = 1 + x + x^2$ in terms of S.
- We will find c_1, c_2, c_3 such that

$$1 + x + x^2 = c_1(1 + x) + c_2(2 + 3x) + c_3(1 + 2x + 3x^2).$$

- One way to do this is to think in terms of the elements of $\mathbb{R}^3$. So we have $(1, 1, 1) = c_1(1, 1, 0) + c_2(2, 3, 0) + c_3(1, 2, 3)$.
- This leads to the system

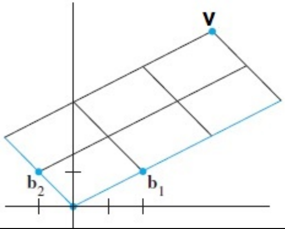
$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

- The solution is $c_1 = 4/3, c_2 = -1/3, c_3 = 1/3$.
- We have $(1, 1, 1) = \frac{4}{3}(1, 1, 0) - \frac{1}{3}(2, 3, 0) + \frac{1}{3}(1, 2, 3)$. Equivalently, $1 + x + x^2 = \frac{4}{3}(1 + x) - \frac{1}{3}(2 + 3x) + \frac{1}{3}(1 + 2x + 3x^2)$.

Coordinates of a vector relative to a basis

Let V be a vector space with a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and let $\mathbf{v} \in V$. Then, if $\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \ldots + c_n\mathbf{b}_n$, the unique numbers $c_1, c_2, \ldots, c_n$ are called the coordinates of $\mathbf{v}$ relative to the basis $\mathcal{B}$. It is denoted by $[\mathbf{v}]_{\mathcal{B}} = (c_1, c_2, \ldots, c_n)^T$. (Note that the elements of the basis $\mathcal{B}$ must be in a fixed order.)

- So we have $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_B = [1 + x + x^2]_S = \begin{bmatrix} 4/3 \\ -1/3 \\ 1/3 \end{bmatrix}.$
- Below you see the same vector $\mathbf{v}$ in $\mathbb{R}^2$ in two different coordinate systems.



- In the figure, black axes correspond to the standard basis $\mathcal{B}_1 = \{\mathbf{e}_1, \mathbf{e}_2\}$. Blue axes correspond to the basis $\mathcal{B}_2 = \{\mathbf{b}_1, \mathbf{b}_2\}$, where $\mathbf{b}_1 = (2, 1), \mathbf{b}_2 = (-1, 1)$.
- The coordinates of the vector $\mathbf{v} = (4, 5)$ relative to the basis $\mathcal{B}_2$ is $[\mathbf{v}]_{\mathcal{B}_2} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Null Space of a Matrix

- After the column space and row space, a third vector space associated with a matrix is its null space.
- Given an $m \times n$ matrix A, the null space of A is defined as the solution set of $A\mathbf{x} = \mathbf{0}$. It is denoted by $N(A)$ or $\mathrm{Null}(A)$.
- We have seen that this set is either the trivial subspace or can be expressed as the span of some vectors in $\mathbb{R}^n$. In any case it is a subspace of $\mathbb{R}^n$.
- As an example let us consider $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix}$.
- To solve $A\mathbf{x} = \mathbf{0}$, we reduce A to echelon form as usual. An echelon form of A is $\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (Actually it is the reduced echelon form of A). This leads to the system

$$\begin{aligned} x_1 + 3x_3 + 2x_4 &= 0 \\ x_2 - 2x_3 - x_4 &= 0 \end{aligned}$$

Null Space of a Matrix

- The solution is $x_1 = -3x_3 - 2x_4$, $x_2 = 2x_3 + x_4$, where x_3 and x_4 are free variables.
- In vector form, the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_3 - 2x_4 \\ 2x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

- So the solution set is $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. (These are the **fundamental** solutions.)
- Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is actually a basis for $N(A)$. Thus $\dim(N(A)) = 2$, which is equal to the number of free variables.
- **Exercise:** Find a basis for the null space of $B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 2 & 1 & -1 \end{bmatrix}$.

Finding a Basis for the Column Space

- Let us now consider the problem of finding a basis for the column space.
- Consider again $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix}$.
- In the problem of finding $N(A)$, we solved the system $A\mathbf{x} = \mathbf{0}$. This can be written as

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

- Now substituting the first fundamental solution $\mathbf{x} = (-3, 2, 1, 0)^T$ gives

$$-3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}.$$

- Similarly, the second fundamental solution $\mathbf{x} = (-2, 1, 0, 1)^T$ gives

$$-2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}.$$

Finding a Basis for the Column Space

- Thus, we have the following: $\begin{bmatrix}1\\4\\1\end{bmatrix}$ and $\begin{bmatrix}1\\3\\2\end{bmatrix}$, which are the columns of A corresponding to the free variables x_3 and x_4, respectively, are both linear combinations of the first two columns of A.
- This shows $\begin{bmatrix}1\\4\\1\end{bmatrix} \in \text{Span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\1\\4\end{bmatrix}\right\}$ and $\begin{bmatrix}1\\3\\2\end{bmatrix} \in \text{Span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\1\\4\end{bmatrix}\right\}$.
- This means that the columns $\begin{bmatrix}1\\4\\1\end{bmatrix}$ and $\begin{bmatrix}1\\3\\2\end{bmatrix}$ can be removed from the spanning set of $C(A)$.
- As a result, we have $C(A) = \text{Span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\1\\4\end{bmatrix}\right\}$.
- Furthermore, the set $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\1\\4\end{bmatrix}\right\}$ is linearly independent so it is a basis for $C(A)$. Hence $\dim(C(A)) = 2$.

- This procedure can be generalized as follows: The pivot columns of A constitute a basis for $C(A)$.
- On one hand, the number of pivot columns of A (which is equal to the number of the leading variables) is equal to the number of nonzero rows in the echelon form of A, which is rank(A). So $\dim(C(A)) = \text{rank}(A)$.
- The number of the free variables, on the other hand, is equal to $\dim(N(A))$, which is also called the nullity of A.
- Since there are n unknowns in total, this shows the following: rank(A) + nullity$(A) = n$.
- **Exercise:** Find a basis for the column space of $B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 2 & 1 & -1 \end{bmatrix}$ and verify the theorem rank(B) + nullity$(B) = n$ for B.