MAT 222 Linear Algebra Week 8 Lecture Notes

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Eigenvalues and Eigenvectors

- Let us consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ and the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
- Let us consider the matrix-vector product $A\mathbf{v}$.

$$A\mathbf{v} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

- Notice that the resulting vector, which is $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$, is just 3 times $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
- In other words, for the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, multiplication by the matrix A is the same as scalar multiplication by 3. In other words we have $A\mathbf{v} = 3\mathbf{v}$.
- Is there any other vector $\mathbf{u}$ such that $A\mathbf{u} = 3\mathbf{u}$?
- Is there any other scalar c such that $A\mathbf{w} = c\mathbf{w}$ for some vector $\mathbf{w}$?

Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix. If there exist a scalar (real or complex) c and a nonzero $n \times 1$ vector $\mathbf{v}$ such that $A\mathbf{v} = c\mathbf{v}$, then c is called an eigenvalue of A and $\mathbf{v}$ is called an eigenvector of A corresponding to c.

- Note that the zero vector $\mathbf{0}$ always satisfies $A\mathbf{0} = c\mathbf{0}$ for any scalar c. This is the reason why it is excluded from the above definition.
- For the matrix $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$, 3 is an eigenvalue and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of A corresponding to 3.
- Notice that any nonzero scalar multiple of $\mathbf{v}$ is also an eigenvector of A. You can easily verify that $3\mathbf{v} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ and $-\frac{1}{2}\mathbf{v} = \begin{bmatrix} -1/2 \\ -1 \end{bmatrix}$ are also eigenvectors corresponding to 3.

Eigenvalues and Eigenvectors

- Does A have any other eigenvalue?
- To tackle this problem, we form the system $A\mathbf{x} = c\mathbf{x}$.
- If this system has a nontrivial solution for some c, then c is an eigenvalue of A and any nontrivial solution $\mathbf{x}$ is an eigenvector of A corresponding to the eigenvalue c.
- Let us rearrange this system as

$$A\mathbf{x} = c\mathbf{I}\mathbf{x} \longrightarrow A\mathbf{x} - c\mathbf{I}\mathbf{x} = \mathbf{0} \longrightarrow (A - c\mathbf{I})\mathbf{x} = \mathbf{0}.$$

- This system has a nontrivial solution if and only if the matrix $A - c\mathbf{I}$ is singular (noninvertible).
- Thus, c is an eigenvalue of A if and only if $\det(A - c\mathbf{I}) = 0$.
- For $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$, we have $|A - c\mathbf{I}| = \begin{vmatrix} 1-c & 1 \\ 4 & 1-c \end{vmatrix} = (1-c)^2 - 4$
- If $(1-c)^2 - 4 = 0$ then $c = 3$ or $c = -1$. These are the eigenvalues of A.

Characteristic polynomial of a matrix

Let A be an $n \times n$ matrix. Then the n-th degree polynomial $\det(A - \lambda \mathbf{I})$ is called the characteristic polynomial of A and the equation $\det(A - \lambda \mathbf{I}) = 0$ is called the characteristic equation of A.

- Thus, the eigenvalues of a matrix are exactly the roots of its characteristic equation.
- Since an n-th degree polynomial has at most n zeros (roots), an $n \times n$ matrix can have at most n eigenvalues.
- The characteristic polynomial of $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ is $(1 - c)^2 - 4$, which has two distinct real roots. So A has two real eigenvalues, which are 3 and -1.

Eigenvalues and Eigenvectors: Example

- Let us now see how to find the eigenvectors corresponding to each eigenvalue.
- Consider again $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$. The eigenvalues are $c_1 = 3$ and $c_2 = -1$.
- For $c_1 = 3$, we have the system $A\mathbf{x} = 3\mathbf{x}$, which is the same as $(A - 3\mathbf{I})\mathbf{x} = \mathbf{0}$.

$$(A - 3\mathbf{I})\mathbf{x} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow -2x_1 + x_2 = 0 \longrightarrow \mathbf{x} = \begin{bmatrix} t \\ 2t \end{bmatrix}.$$

- Thus, every nonzero scalar multiple of $(1, 2)^T$ is an eigenvector corresponding to 3.
- For the other eigenvalue $c_2 = -1$, we solve the system $(A - (-1)\mathbf{I})\mathbf{x} = \mathbf{0}$.

$$(A - (-1)\mathbf{I})\mathbf{x} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow 2x_1 + x_2 = 0 \longrightarrow \mathbf{x} = \begin{bmatrix} t \\ -2t \end{bmatrix}.$$

- Thus, every scalar nonzero multiple of $(1, -2)^T$ is an eigenvector corresponding to -1.

Eigenvalues and Eigenvectors: Example

- As a second example, let us consider $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.
- The characteristic polynomial is
$$\det(B - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 3\lambda + 2.$$
- One can write $-\lambda^3 + 3\lambda + 2 = (\lambda + 1)(\lambda^2 - \lambda - 2) = (\lambda + 1)^2(\lambda - 2)$. Its roots are $\lambda_1 = 2$ and $\lambda_2 = -1$ (double root). These are the eigenvalues of B.
- For the eigenvalue $\lambda_1 = 2$, we have the system
$$(A - 2\mathbf{I})\mathbf{x} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
- This homogeneous system must be solved. The reduced echelon form is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, which is equivalent to $\begin{matrix} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{matrix}$.

Eigenvalues and Eigenvectors: Example

- So we have $x_1 = x_2 = x_3$, or equivalently $\mathbf{x} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$.
- Thus, every nonzero multiple of $(1, 1, 1)^T$ is an eigenvector corresponding to 2.
- For the other eigenvalue $\lambda_2 = -1$, the system is

$$(A - (-1)\mathbf{I})\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow x_1 + x_2 + x_3 = 0.$$

- Since $x_1 = -x_2 - x_3$, we have $\mathbf{x} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.
- Thus, every nonzero vector in Span $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is an eigenvector corresponding to -1.

Eigenspaces of a Matrix

- Let A be an $n \times n$ matrix and let λ be an eigenvalue of A.
- We have seen that the eigenvectors of A corresponding to λ are just the nontrivial solutions of $(A - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$.
- Recall that the solution set of $(A - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ (including the zero vector) is $N(A - \lambda\mathbf{I})$, the nullspace of $A - \lambda\mathbf{I}$.
- It is also called the eigenspace of A corresponding to λ. We can denote this subspace of $\mathbb{R}^n$ by $E_\lambda(A)$.
- For the matrix $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, we had two eigenvalues, 2 and -1.
- The eigenspace corresponding to 2 is $E_2(A) = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$. We have $\dim(E_2(A)) = 1$.
- The eigenspace corresponding to -1 is $E_{-1}(A) = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$. We have $\dim(E_{-1}(A)) = 2$.

Eigenspaces of a Matrix: Example

- Consider the 3×3 identity $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
- For every $\mathbf{x} \in \mathbb{R}^3$ we have $\mathbf{I}\mathbf{x} = \mathbf{x}$, so 1 is the only eigenvalue of $\mathbf{I}$ and every nonzero vector is an eigenvector corresponding to 1.
- In other words, $E_1(\mathbf{I}) = \mathbb{R}^3$.
- **Remark:** Note that eigenspace is only defined in connection with a single eigenvalue. For example, an eigenvector of the matrix $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ corresponding to 2 is the vector $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$, while an eigenvector corresponding to -1 is $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$. But any linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ is not an eigenvector of A. (For instance the vector $\mathbf{v}_1 + 2\mathbf{v}_2 = \begin{bmatrix} -1 \\ 5 \\ 5 \end{bmatrix}$ is not an eigenvector.)