MAT 222 Linear Algebra and Numerical Methods
Week 2
Lecture Notes 2

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Echelon Matrices

- Observe that Gaussian elimination stops only when the augmented matrix of the system assumes a certain form. This exact form is defined in the following.

Echelon matrix

In any matrix, we call an entry the leading entry of its row if it is the leftmost nonzero entry in the row. We say that a matrix is in echelon form or it is an echelon matrix if the following conditions hold:

(1) Each leading entry is positioned to the right of the leading entry of the above row (if it is not in the first row).

(2) In each column containing a leading entry, all the entries below the leading entry are zero.

(3) The rows consisting of all-zeros (if there are any) are located below the nonzero rows.

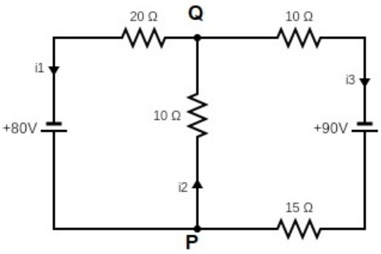
- Thus we can reformulate the goal of Gaussian elimination as follows: It is to reduce the coefficient matrix of the system to an echelon matrix.
- In our previous example systems, we have not yet come across with an all-zero row mentioned in condition (3).

Echelon Matrices

Examples

- The matrix $\begin{bmatrix} 1 & 4 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ is in echelon form.
- The matrix $\begin{bmatrix} 2 & 0 & -4 & -3 \\ 0 & 0 & 5 & 4 \\ 0 & 3 & 0 & 5 \end{bmatrix}$ is not in echelon form since the leading entry in the third row (which is in $(3, 2)$ position) is to the right of the leading entry of the second row (which is in the $(2, 3)$ position). It can be brought to echelon form by a row swap.
- The matrix $\begin{bmatrix} 2 & 4 & -3 \\ 0 & 0 & 0 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ is not in echelon form because the second row is a zero row and there are nonzero rows below it. It can be made echelon by two row swaps.

An Electrical Network



According to the electrical network in the figure, determine the intensity of the currents i_1, i_2 and i_3.

- In order to form the required equations, we need to use Kirchoff's Current Law and Kirchoff's Voltage Law.
- **Kirchoff's Current Law:** At any point in the circuit, sum of inflowing currents equals that of outflowing currents.
- **Kirchoff's Voltage Law:** Inside any given loop, sum of all voltage drops equals that of all voltage rises.

Gaussian Elimination: Example 4

- KCL for point P: $i_1 + i_3 = i_2$
  KCL for point Q: $i_2 = i_1 + i_3$
- KVL for left loop: $20i_1 + 10i_2 = 80$
  KVL for rgiht loop: $10i_2 + 25i_3 = 90$
- So we have the following 4×3 system:

$$\begin{aligned} i_1 - i_2 + i_3 &= 0 \\ -i_1 + i_2 - i_3 &= 0 \\ 10i_2 + 25i_3 &= 90 \\ 20i_1 + 10i_2 &= 80 \end{aligned}$$

- Its augmented matrix is

$$\left[\begin{array}{lll|l} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right].$$

Gaussian Elimination: Example 4

- $$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right] \xrightarrow{1R_1+R_2,\,-20R_1+R_4} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right]$$
- Now, the second row has become a zero row, so let us move it to the last row by a row swap. Also it is better to have the third row as the second row because that would make the arithmetic cleaner. So we make two successive row swaps.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right] \xrightarrow{R_2\leftrightarrow R_3,\,R_3\leftrightarrow R_4} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

- Then we can just proceed as usual ignoring the zero row.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-3R_2+R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Gaussian Elimination: Example 4

- The equation form of the reduced system is

$$\begin{aligned} i_1 - i_2 + i_3 &= 0 \\ 10i_2 + 25i_3 &= 90 \\ -95i_3 &= -190 \\ 0 &= 0 \end{aligned}$$

- Backward substitution gives the solution $i_1 = 1, i_2 = 4, i_3 = 2$.
- Clearly the zero row has no effect on the solution.
- We could have anticipated the forming of the zero row by inspecting the original system. Indeed the equations resulting from Kirchoff's Current Law were identical. (In other words, the planes corresponding to them were coincident.)
- This is an example of when an over-determined system has a unique solution.

Gaussian Elimination: Example 5

- Consider the system $\begin{array}{r} 3x+2y+z=3 \\ 2x+y+z=0 \\ 6x+2y+4z=6 \end{array}$.

- $$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right] \xrightarrow{-2/3R_1+R_2,\,-2R_1+R_3} \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & -2 & 2 & 0 \end{array}\right]$$
$$\xrightarrow{-3R_2} \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & 1 & -1 & 6 \\ 0 & -2 & 2 & 0 \end{array}\right] \xrightarrow{2R_2+R_3} \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 0 & 12 \end{array}\right]$$

- So we have the system $\begin{array}{r} 3x+2y+z=3 \\ y-z=6 \\ 0=12 \end{array}$, which is clearly a contradiction.
- Therefore the system has no solutions.
- This example reveals the following fact: Applying row reduction to an inconsistent system produces a row of the form $\begin{bmatrix} 0 & 0 & \ldots & 0 & c \end{bmatrix}$, where c is nonzero.

Homogeneous Systems

Homogeneous system

An $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$ is called homogeneous if the vector $\mathbf{b}$ is the all-zero column vector of size m.

- An homogeneous sytem has the following "nice" property: It always has at least one solution, because $\mathbf{x} = \mathbf{0}_{n\times 1}$ is always a solution. (All unknowns equal zero certainly satisfies the system)
- Another way to see this is that, since the right-hand sides of every equation is zero in an homogeneous system, they will always remain zero during row reduction, hence a row of the form $\begin{bmatrix} 0 & 0 & \dots & 0 & c \end{bmatrix}$, $c \neq 0$, cannot be produced.
- Futhermore, the solution set of a homogeneous system possesses a "desirable" structure, as the following example shows.

- Consider the 3×4 system $\begin{aligned} x + 2y &= 0 \\ 2x + z &= 0 \\ 3x + 2y + z - w &= 0 \end{aligned}$.
- **Exercise:** Can you guess how many solutions this system has without solving it?
- $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 2 & 1 & -1 \end{bmatrix} \xrightarrow{-2R_1+R_2,-3R_1+R_3} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & -4 & 1 & -1 \end{bmatrix}$
- Note that there is no need to write the right-hand side all-zero vector since row reduction does not change 0's.
- $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & -4 & 1 & -1 \end{bmatrix} \xrightarrow{-1R_2+R_3} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$
- The echelon form is equivalent to $\begin{aligned} x + 2y &= 0 \\ -4y + z &= 0 \\ -w &= 0 \end{aligned}$.
- Note that y cannot be obtained uniquely, it can only be obtained in terms of z. Namely, $y = \frac{1}{4}z$.
- Writing this in the first equation gives $x + \frac{1}{2}z = 0$ and hence $x = -\frac{1}{2}z$.

Gaussian Elimination: Example 6

- So the solution can be expressed by $x = -\frac{1}{2}z,\ y = \frac{1}{4}z,\ z = z,\ w = 0.$
- As you can see there is no condition on z: It can take any value. We call z to be a free variable.
- It is customary to denote free variables by other letters, let us use "t" in this example.
- So the solution is $x = -\frac{1}{2}t,\ y = \frac{1}{4}t,\ z = t,\ w = 0.$
- We can also write this as

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}t \\ \frac{1}{4}t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{4} \\ 1 \\ 0 \end{bmatrix}.$$

So the solution set consists of all scalar multiples of the vector

$$\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{4} \\ 1 \\ 0 \end{bmatrix}.$$

Gaussian Elimination: Example 7

- Consider the 3×4 homogeneous system

$$\begin{aligned} 3x_1 + 2x_2 + 2x_3 - 5x_4 &= 0 \\ 0.6x_1 + 1.5x_2 + 1.5x_3 - 5.4x_4 &= 0\,. \\ 1.2x_1 - 0.3x_2 - 0.3x_3 + 2.4x_4 &= 0 \end{aligned}$$

- $$\begin{bmatrix} 3 & 2 & 2 & -5 \\ 0.6 & 1.5 & 1.5 & -5.4 \\ 1.2 & -0.3 & -0.3 & 2.4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0.6 & 1.5 & 1.5 & -5.4 \\ 3 & 2 & 2 & -5 \\ 1.2 & -0.3 & -0.3 & 2.4 \end{bmatrix}$$

$$\xrightarrow[-2R_1+R_3]{-5R_1+R_2} \begin{bmatrix} 0.6 & 1.5 & 1.5 & -5.4 \\ 0 & -5.5 & -5.5 & 22 \\ 0 & -3.3 & -3.3 & 13.2 \end{bmatrix} \xrightarrow{-3/5R_2+R_3} \begin{bmatrix} 0.6 & 1.5 & 1.5 & -5.4 \\ 0 & -5.5 & -5.5 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{10}{3}R_1,\,-\frac{1}{5.5}R_2} \begin{bmatrix} 2 & 5 & 5 & -18 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- So the final system is $\begin{aligned} 2x_1 + 5x_2 + 5x_3 - 18x_4 &= 0 \\ x_2 + x_3 - 4x_4 &= 0 \end{aligned}\,.$

Gaussian Elimination: Example 7

- This time we have 2 free variables: x_3 and x_4.
- x_1 and x_2 can be expressed in terms of the free variables as $x_1 = -x_4$ and $x_2 = -x_3 + 4x_4$.
- Setting $x_3 = t_1$ and $x_4 = t_2$, we can express the solution as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -t_2 \\ -t_1 + 4t_2 \\ t_1 \\ t_2 \end{bmatrix} = t_1 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 4 \\ 0 \\ 1 \end{bmatrix}.$$

- In other words, the solution set consists of all linear combinations of the vectors $\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 4 \\ 0 \\ 1 \end{bmatrix}$. (These two vectors are also solutions.)
- This structure holds for a general homogeneous system.

Solution set of a homogeneous system

Suppose that an $m \times n$ homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has r nonzero rows in its echelon form and $r + k = n$. Then k of the unknowns are free variables and the solution set is $\{t_1\beta_1 + t_2\beta_2 + \ldots t_k\beta_k \, : \, t_1, t_2, \ldots, t_k\}$, where the column vectors $\beta_1, \beta_2, \ldots, \beta_k$ of size n are solutions of the system satisfying a certain property (to be defined later).

Gaussian Elimination: Example 8

- Consider the 3×4 system $\begin{array}{r} 3x_1 + 2x_2 + 2x_3 - 5x_4 = 8 \\ 0.6x_1 + 1.5x_2 + 1.5x_3 - 5.4x_4 = 2.7 \\ 1.2x_1 - 0.3x_2 - 0.3x_3 + 2.4x_4 = 2.1 \end{array}$.
- This is the same as the system in Example 7 with the distinction that right-hand side is nonzero.
- The corresponding homogeneous system in Example 7 is known as the homogeneous part of this system.
- Applying the same row operation reduces the system to

$$\left[\begin{array}{cccc|c} 2 & 5 & 5 & -18 & 9 \\ 0 & 1 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

  which is equivalent to $\begin{array}{r} 2x_1 + 5x_2 + 5x_3 - 18x_4 = 9 \\ x_2 + x_3 - 4x_4 = 1 \end{array}$.
- So we have $x_2 = 1 - x_3 + 4x_4$ and $x_1 = 2 - x_4$.
- Setting $x_3 = t_1$ and $x_4 = t_2$, we can write in vector form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 - t_2 \\ 1 - t_1 + 4t_2 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 4 \\ 0 \\ 1 \end{bmatrix}.$$

- In other words, it is the solution set of its homogeneous part translated by the constant vector $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. (Note that this vector is a solution of the system)

Exercises

- **Exercise:** Solve the system
$$\begin{aligned} x + y &= 1 \\ 2x + z &= 2. \\ 3x + 2y + z - w &= 4 \end{aligned}$$
- **Exercise:** Solve the dietary plan (of rabbits) problem from the previous week.
- **Exercise:** Solve the traffic network problem from the previous week.