MAT 222 Linear Algebra and Numerical Methods
Week 9
Lecture Notes

Murat Karaçayır

Akdeniz University
Department of Mathematics

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Eigenvalues and Eigenvectors (Summary)

- Let A be an $n \times n$ matrix. If there exist a scalar (real or complex) c and a nonzero $n \times 1$ vector $\mathbf{v}$ such that $A\mathbf{v} = c\mathbf{v}$, then c is called an eigenvalue of A and $\mathbf{v}$ is called an eigenvector of A corresponding to c.
- Eigenvalues of A are the roots (real or complex) of the polynomial $\det(A - \lambda\mathbf{I})$, called the charactersitic polynomial of A.
- A can have at most n eigenvalues.
- Let c be an eigenvalue of A. The set of all eigenvectors of A corresponding to c, union the zero vector, is a vector space, called the eigenspace corresponding to c.

Complex Eigenvalues

- Consider the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
- Its characteristic polynomial is $\lambda^2 + 1$, giving the eigenvalues i and $-i$.
- For the eigenvalue $\lambda = i$, we have the system

$$(A - i\mathbf{I})\mathbf{x} = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow -(ix_1 + x_2) = 0 \rightarrow x_1 = ix_2.$$

- So a basis for the eigenspace corresponding to i is $\begin{bmatrix} i \\ 1 \end{bmatrix}$.
- Similarly, a basis for the eigenspace corresponding to $-i$ is $\begin{bmatrix} 1 \\ i \end{bmatrix}$.
- Note that the two eigenvalues i and $-i$ are conjugates of each other.
- This is always the case: If an eigenvalue of A (real matrix) is $a + bi$, then $a - bi$ is also an eigenvalue.
- A has the following property: If $\mathbf{x}$ is a real 2×1 vector, then $A\mathbf{x}$ rotates $\mathbf{x}$ counter-clockwise by 90 degrees. (This implies there is no real eigenvalue. Can you see why?)

Complex Eigenvalues: A 3×3 Example

- Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$.
- The characteristic polynomial is $(1-\lambda)(\lambda^2 - 2\lambda + 5)$. (Please check)
- The eigenvalues are $1, 1+2i, 1-2i$.
- An eigenvector corresponding to 1 is $(2, -2, 3)^T$. (Please check)
- For $\lambda = 1 + 2i$, we have the system

$$(\mathbf{A} - (1+2i)\mathbf{I})\mathbf{x} = \begin{bmatrix} -2i & 0 & 0 \\ 3 & -2i & -2 \\ 2 & 2 & -2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{aligned} -2ix_1 &= 0 \\ 3x_1 - 2ix_2 - 2x_3 &= 0 \\ 2x_1 + 2x_2 - 2ix_3 &= 0 \end{aligned} \longrightarrow \begin{aligned} -2ix_2 - 2x_3 &= 0 \\ 2x_2 - 2ix_3 &= 0 \end{aligned} \longrightarrow x_1 = 0, x_2 = ix_3$$

- So an eigenvector corresponding to $1 + 2i$ is $(0, i, 1)^T$.
- **Exercise:** Find an eigenvector corresponding to $\lambda = 1 - 2i$.

Singular Matrices Have Zero as Eigenvalue

- Suppose 0 is an eigenvalue of a matrix A. What does this imply about A?
- This implies we have $A\mathbf{v} = \mathbf{0}$ for some nonzero vector $\mathbf{v}$.
- But this is only true if A is noninvertible.
- So we have the following. A is noninvertible if and only if 0 is an eigenvalue of A.
- For example, let us consider the matrix $B = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 1 \\ 2 & 4 & -3 \end{bmatrix}$.
- The characteristic polynomial of B is $-\lambda^3 + 9\lambda$ (please check), which means that 0 is an eigenvalue.
- **Exercise:** Verify that B is noninvertible.

Similar Matrices

Similar matrices

Suppose A and B are $n \times n$ matrices such that there is an invertible matrix P satisfying $A = PBP^{-1}$ (or $A = P^{-1}BP$). Then A and B are called similar matrices.

- An immediate observation that follows is the following: Similar matrices have the same eigenvalues.
- To see this, we start with $A = PBP^{-1}$ and noting that $\mathbf{I} = PP^{-1}$ we have

$$A - \lambda\mathbf{I} = PBP^{-1} - \lambda PP^{-1} = P(BP^{-1} - \lambda P^{-1}) = P(B - \lambda\mathbf{I})P^{-1}$$

- Then, recalling that multiplication preserves determinant we have $\det(A - \lambda\mathbf{I}) = \det(P)\det(B - \lambda\mathbf{I})\det(P^{-1})$.
- This shows that A and B have the same characteristic polynomial. Thus, they have the same eigenvalues (with the same multiplicites).
- Note that the converse is not true: Matrices having the same eigenvalues may not be similar. For instance, the matrices $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ have the same eigenvalues but are not similar.

Similar Matrices: Example

- Consider $A = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & -3 \\ 6 & 5 \end{bmatrix}$. Also consider $P = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$.
- You can verify that $P^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$.
- So we have $P^{-1}AP = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -5 & -3 \\ 6 & 5 \end{bmatrix} = B$. (Please check)
- This shows that A and B are similar.
- But given two matrices A and B, how to determine if they are similar? Also, how to find P such that $P^{-1}AP = B$ (or $PAP^{-1} = B$) if they are similar?

Using Eigenbases to Construct a Similar Matrix

- Let us now give a partial answer to this question.
- As an example, let us consider $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$.
- We had find its eigenvalues as 3 and -1. We had also shown that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ are eigenvectors corresponding to them, respectively.
- Let us consider the matrix formed by writing these eigenvectors in columns; i.e. consider $\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$. Let us call it S.
- Consider the product AS. The columns of these product are $A\mathbf{v}_1 = 3\mathbf{v}_1$ and $A\mathbf{v}_2 = -1\mathbf{v}_2$. Thus, we have
$$AS = \left[\ 3\mathbf{v}_1 \mid -1\mathbf{v}_2\ \right] = \begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = SD,$$
where we have defined $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$.
- Note that $AS = SD$ can be written as $A = SDS^{-1}$. So A is similar to D, the diagonal matrix having the eigenvalues of A in the diagonal.

- This process is called diagonalization and can be generalized as follows:

Diagonalization of a matrix

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues (including possible repetitions) of **A** and assume that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent eigenvectors corresponding to these eigenvalues, respectively. Then if **P** is defined to be the matrix whose columns are $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, respectively; namely if

$$\mathbf{P} = \left[\begin{array}{c|c|c|c} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{array}\right],$$

then it is true that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$ where

$$\mathbf{D} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

In this case, A is called a diagonalizable matrix.

Diagonalization

- In general, an $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors.
- Notice that not every matrix is diagonalizable. For instance the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has only 1 as an eigenvalue and the corresponding eigenspace has dimension 1 since $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is a basis. So the matrix does not have 2 linearly independent eigenvectors, thus it is not diagonalizable.
- This example shows that there is not any relation between invertibility and diagonalizability. An invertible matrix may be nondiagonalizable.
- Another conclusion is the following: If A has n distinct eigenvalues, than A is diagonalizable.
- The converse is not true: $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ has only two distinct eigenvalues but has three linearly independent eigenvectors. The matrix $\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ diagonalizes B. (Please check)

Using Diagonalization to Compute Powers of Matrices

- Diagonalization has a direct application: computing high powers.
- To begin with, observe that taking a positive integer power of a diagonal matrix is straightforward. If $C = \text{diag}(c_1, c_2, \ldots, c_n)$ and $k \in \mathbb{Z}^+$, then $C^k = \text{diag}(c_1^k, c_2^k, \ldots, c_n^k)$.
- For instance, we have
$$\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}^2 = \begin{bmatrix} 3^2 & 0 \\ 0 & (-1)^2 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}^3 = \begin{bmatrix} 3^3 & 0 \\ 0 & (-1)^3 \end{bmatrix} \text{ etc.}$$
- The rest is simple: If A is diagonalizable and $P^{-1}AP = D$ where D is a diagonal matrix with the eigenvalues of A in the diagonal, then we have $D^k = (P^{-1}AP)^k = \underbrace{(P^{-1}AP)(P^{-1}AP)\ldots(P^{-1}AP)}_{k \text{ times}} = P^{-1}A^kP$.
- Thus the k-th power of A is $A^k = PD^kP^{-1}$.

Computing Powers Using Diagonalization: Example

- As an example, consider the matrix $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. A is diagonalizable and we have $A = PDP^{-1}$, where
$D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$. (Please check)
- We can compute A^{10} as follows:
- Compute D^{10} as $D^{10} = \begin{bmatrix} 3^{10} & 0 \\ 0 & 5^{10} \end{bmatrix}$
- Compute P^{-1} as $P^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$
- Compute A^{10} as $A^{10} = (PDP^{-1})^{10} = PD^{10}P^{-1}$
- Substitute the values of D^{10} and P^{-1} to get
$A^{10} = \begin{bmatrix} 2\cdot 5^{10} - 3^{10} & 5^{10} - 3^{10} \\ -2\cdot 5^{10} + 2\cdot 3^{10} & -5^{10} + 2\cdot 3^{10} \end{bmatrix}$.

Exercise: Let $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Compute B^{30} by diagonalizing B.

Using Eigenvalues to Analyze Dynamical Systems

- Now we consider a simple dynamical system of the form $\mathbf{x}_{k+1} = A\mathbf{x}_k$.
- The state of the system in $k+1$-th instant is related to the state in the k-th instant through multiplication by A.
- It turns out that eigenvalues and eigenvectors of A can be used to analyze the long term behavior of such a system.
- As an example[1], consider the discrete predator-prey model about the populations of certain species of owl and mouse in California forests.
- Denoting owl population at time (month) k by O_k and rat population at time (month) k by R_k, let us consider the following model:

$$\mathbf{x}_{k+1} = \begin{bmatrix} O_{k+1} \\ R_{k+1} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.4 \\ -p & 1.1 \end{bmatrix} \begin{bmatrix} O_k \\ R_k \end{bmatrix} = A\mathbf{x}_k$$

- A short explanation is as follows:
  - Without any rat, half of the owl population will survive each month.
  - The coefficient 0.4 represents the contribution of the rat population to the owl population, while the parameter p represents the (negative) effect of the owl population over the rat population.
  - Without any owls, rat population grows by 10 percent each month.

[1] This example is taken from Lay, Lay & Mcdonald, Linear Algebra, 5th Ed., page 304.

Using Eigenvalues to Analyze Dynamical Systems

- Let us analyze this system using $p = 0.104$.
- In this case $A = \begin{bmatrix} 0.5 & 0.4 \\ -0.104 & 1.1 \end{bmatrix}$. The eigenvalues of A are 1.02 and 0.58. (Please check)
- An eigenvector of A corresponding to 1.02 is $\mathbf{v_1} = \begin{bmatrix} 10 \\ 13 \end{bmatrix}$ and an eigenvector corresponding to 0.58 is $\mathbf{v_2} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.
- Since $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent, together they form a basis for $\mathbb{R}^2$.
- In particular, the initial state $\mathbf{x}_0$ can be expressed as a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$.
- Let us assume $\mathbf{x}_0 = c_1\mathbf{v_1} + c_2\mathbf{v_2}$. Then observe that

$$\mathbf{x}_1 = A\mathbf{x}_0 = c_1A\mathbf{v_1} + c_2A\mathbf{v_2} = (1.02)c_1\mathbf{v_1} + (0.58)c_2\mathbf{v_2}.$$

- Similarly, we have

$$\mathbf{x}_2 = A\mathbf{x}_1 = c_1A((1.02)c_1\mathbf{v_1} + (0.58)c_2\mathbf{v_2}) = (1.02)^2c_1\mathbf{v_1} + (0.58)^2c_2\mathbf{v_2}.$$

- In general, it is true that $\mathbf{x}_k = (1.02)^kc_1\mathbf{v_1} + (0.58)^kc_2\mathbf{v_2}$.

Using Eigenvalues to Analyze Dynamical Systems

- So we have found a formula for the general term of the system: $\mathbf{x_k} = \begin{bmatrix} O_k \\ R_k \end{bmatrix} = (1.02)^k c_1 \begin{bmatrix} 10 \\ 13 \end{bmatrix} + (0.58)^k c_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.
- From this we can obtain the owl and rat populations at the k-th month:

$$O_k = 10(1.02)^k c_1 + 5(0.58)^k c_2$$
$$R_k = 13(1.02)^k c_1 + (0.58)^k c_2$$

- Recall that c_1 and c_2 depend on the initial owl and rat populations.
- As $k \to \infty$, $(0.58)^k$ will tend to zero, so in the long run we will have $O_k \approx 10(1.02)^k c_1$ and $R_k \approx 13(1.02)^k c_1$.
- Note that the ratio of the populations becomes $\dfrac{O_k}{R_k} = \dfrac{10}{13}$ in the long run. This ratio is determined by the eigenvector $\mathbf{v_1} = \begin{bmatrix} 10 \\ 13 \end{bmatrix}$.
- We can also comment that, in the long run populations of both species will increase by 2 percent each month.
- **Exercise:** If the predation parameter is $p = 0.2$, determine the long term behavior of both populations.