MAT 222 Linear Algebra
Week 12
Lecture Notes 2

Murat Karaçayır

Akdeniz University
Department of Mathematics

22nd May 2025

Linear Programming

- Suppose that an agricultural company produces two different type of bags.
- Bag Type I (Green) contains 3 kg of Seed A, 1 kg of Seed B, 1 kg of grass. Bag Type II (Blue) contains 2 kg of Seed A, 2 kg of Seed B, 1 kg of grass.
- The company has a total of 1200 kg of Seed A, 800 kg of Seed B, 450 kg of grass in its warehouse.
- It is known that the company makes 3 dollar profit/ unit sold of Type I bag and 2 dollar profit/unit sold of Type II bag.
- In order to maximize the profit, how many units of each type of bag must the company produce?
- If x_1 = Units of Type I and x_2 = Units of Type II, mathematical formulation of the problem is as follows: "Maximize the function $f(x_1, x_2) = 3x_1 + 2x_2$ subject to the condition that $3x_1 + 2x_2 \leq 1200, x_1 + 2x_2 \leq 800, x_1 + x_2 \leq 450$".
- Here the linear function $f(x_1, x_2) = 3x_1 + 2x_2$ is called the **objective function** and the above conditions are **inequality constraints**.
- This type of constrained optimization problems is known as **linear programming**.

Canonical Linear Programming Problem

Canonical form of linear programming

Given $\mathbf{b} = (b_1, b_2, \ldots, b_m)^T \in \mathbb{R}^m$, $\mathbf{c} = (c_1, c_2, \ldots, c_n)^T \in \mathbb{R}^n$ and an $m \times n$ matrix $A = [a_{ij}]$, the **canonical linear programming problem** is defined as follows:
Find a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ that maximizes the function

$$f(x_1, x_2, \ldots, x_n) = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots a_{mn}x_n &\leq b_m \end{aligned}$$

and

$$x_j \geq 0 \ \text{ for } j = 1, 2, \ldots, n.$$

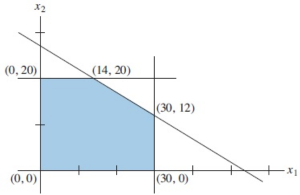
In matrix form:
Maximize $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$
subject to the constraints $A\mathbf{x} \leq \mathbf{b}$
and $\mathbf{x} \geq \mathbf{0}$

Linear Programming Problem: Example

Example: Maximize $f(x_1, x_2) = 2x_1 + 3x_2$ subject to

$$x_1 \leq 30, \; x_2 \leq 20, \; x_1 + 2x_2 \leq 54$$
$$\text{and } x_1 \geq 0, \; x_2 \geq 0$$

- Each of the inequality constraints determine a half plane in $\mathbb{R}^2$. The borders of these regions are the lines formed by replacing "$\leq$" or "$\geq$" by "$=$".
- The intersection of these half planes determines a convex region. In the figure it is the pentagon colored in blue.



  - This region is called the **feasible region**. Denoting it by $\mathcal{F}$, the problem can be stated as follows: "Maximize $f(\mathbf{x})$ for $\mathbf{x} \in \mathcal{F}$".
  - The vertices of the feasible set are called the **extreme points** of $\mathcal{F}$.
  - If the feasible set $\mathcal{F}$ is not empty and the objective function $f(\mathbf{x})$ is bounded on $\mathcal{F}$, the canonical linear programming problem always has a solution. In addition, at least one of the solutions is at the extreme points of $\mathcal{F}$.

Linear Programming Problem: Example

- Thus, in order to solve a linear programming problem in canonical form, it suffices to check the extreme points.
- For this example, there are five extreme points, each of which is the intersection of the border lines of two constraints.
- The below table lists the values of $f(x_1, x_2) = 2x_1 + 3x_2$ at the extreme points.

(x_1, x_2)	$2x_1 + 3x_2$
(0,0)	0
(30,0)	60
(30,12)	96
(14,20)	88
(0,20)	60

- According to the table, it is seen that the maximum value of f is 96 and it occurs at the point $(30, 12)$.

The Simplex Method

- In general, visualizing the feasible region and the extreme points gets complicated as the number of variables and the constraints gets bigger.
- The **Simplex Method** gives a well-defined way of switching between the extreme points until a maximum is reached.
- The outline of the method is as follows.
  1. Select an extreme point $\mathbf{x}$ of $\mathcal{F}$.
  2. Consider all edges of $\mathbf{x}$ whose one vertex is $\mathbf{x}$. If $f(\mathbf{x})$ cannot be increased by moving along any of these edges, then $\mathbf{x}$ is the optimal point.
  3. If not, then move along the edge that results in the largest increase in $f(\mathbf{x})$, thus moving to a new edge.
  4. Repeat until a situation described in Step 2 arises.
- Since the value of f increases at each point, the algorithm won't go through the same extreme point twice. Since there are finitely many extreme points, it will give a solution after a finite number of steps.

Simplex Method: Slack Variables

- Step 3 of the Simplex Algorithm is the critical step. In order to decide on the next extreme point, Simplex Method begins by converting each inequality constraint to an equality.
- This is done by introducing **slack variables**.
- For example, the inequality constraint $5x_1 + 7x_2 \leq 80$ can be replaced by the equality $5x_1 + 7x_2 + x_3 = 80$, where $x_3 \geq 0$.
- x_3 is a slack variable here.
- After doing this for all the inequality constraints, the system of inequalities $A\mathbf{x} \leq \mathbf{b}$ is converted to a system of m equations in $m + n$ unknowns(Let us denote it by $\tilde{A}\tilde{\mathbf{x}} = \mathbf{b}$).
- The extra m slack variables are not part of the original problem but are used to switch between extreme points of the feasible set.
- Each extreme point corresponds to a **basic feasible solution** of the system $A\mathbf{x} \leq \mathbf{b}$. A basic feasible solution is a solution of $\tilde{A}\tilde{\mathbf{x}} = \mathbf{b}$ such that at most m of the variables are positive and the remaining are zero.

Basic Feasible Solution: Example

- As an example, let us consider the system of inequalities

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &\leq 60 \\ 3x_1 + x_2 + 5x_3 &\leq 46 \, . \\ x_1 + 2x_2 + x_3 &\leq 50 \end{aligned}$$

- To convert it to a system of equations, introduce the variables x_4, x_5, x_6:

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 + x_4 &= 60 \\ 3x_1 + x_2 + 5x_3 + x_5 &= 46 \, . \\ x_1 + 2x_2 + x_3 + x_6 &= 50 \end{aligned} \tag{1}$$

- Setting $x_4 = 60$, $x_5 = 46$, $x_6 = 50$ gives a simple basic feasible solution. We will call it "the basic feasible solution associated with (1)". It corresponds to the extreme point $(0, 0, 0)$ of the feasible set.
- The varibles x_4, x_5, x_6 are said to be **in the solution**, while we say that the other variables x_1, x_2, x_3 are **out of the solution** since they assume the value of 0.
- Note that, in the matrix of the system (1), the columns corresponding to the variables x_4, x_5, x_6 (which are "in" the solution) are the columns of the identity matrix.

Updating the Basic Feasible Solution

- Suppose the current state of the system is as given in (1) and we want to bring the variable x_2 in to the solution. This will correspond to moving from the extreme point $(0, 0, 0)$ to an extreme point of the form $(0, c, 0)$.
- For this purpose, we must convert the column of x_2 in the augmented matrix to a column of the identity matrix.
- We have three choices to do this: Using the x_2 term in the first equation, in the second equation, and in the third equation.
- Since negative values are not allowed in feasible solutions, the values on the right-hand side should be nonnegative in the updated system.
- We can ensure this by calculating the ratios $\dfrac{b_i}{a_{i2}}$ in each row and select the row resulting in the smallest ratio (among the positive ratios, of course).
- For the first row this ratio is $\dfrac{60}{3} = 20$, for the second it is $\dfrac{46}{1} = 46$ and for the third $\dfrac{50}{2} = 25$. So we use the first equation.
- As a preparation, we divide the first equation by 3 and write

$$\frac{2}{3}x_1 + x_2 + \frac{4}{3}x_3 + \frac{1}{4}x_4 = 20.$$

Updating the Basic Feasible Solutions

- Using x_2 as the pivot in the first equation, we obtain

$$\left[\begin{array}{cccccc|c} \frac{2}{3} & 1 & \frac{4}{3} & \frac{1}{3} & 0 & 0 & 20 \\ 3 & 1 & 5 & 0 & 1 & 0 & 46 \\ 1 & 2 & 1 & 0 & 0 & 1 & 50 \end{array}\right] \xrightarrow{-1R_1+R_2,-2R_1+R_3} \left[\begin{array}{cccccc|c} \frac{2}{3} & 1 & \frac{4}{3} & \frac{1}{3} & 0 & 0 & 20 \\ \frac{7}{3} & 0 & \frac{11}{3} & -\frac{1}{3} & 1 & 0 & 26 \\ -\frac{1}{3} & 0 & -\frac{5}{3} & -\frac{2}{3} & 0 & 1 & 10 \end{array}\right]$$

- In the resulting system, basic variables are x_2, x_5, x_6 and the basic feasible solution is $(x_2, x_5, x_6) = (20, 26, 10)$ or $\mathbf{x} = (0, 20, 0, 0, 26, 10)$. It corresponds to the extreme point (0,20,0) of the feasible set.
- As a result, the variable x_4 has been moved out of the solution and x_2 has been moved in to the solution
- It only remains to outline the procedure about how to decide which variable to move in to the solution.

Simplex Method: Example

Example: Maximize $f(x_1, x_2, x_3) = 25x_1 + 33x_2 + 18x_3$ subject to

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &\leq 60 \\ 3x_1 + x_2 + 5x_3 &\leq 46 \\ x_1 + 2x_2 + x_3 &\leq 50 \end{aligned}$$

$$\text{and } x_1 \geq 0,\ x_2 \geq 0, x_3 \geq 0$$

- We will start by converting the objective function to an equality by introducing a dependent variable M as follows:

$$M = 25x_1 + 33x_2 + 18x_3.$$

- Then, the objective function is represented by the equality

$$-25x_1 - 33x_2 - 18x_3 + M = 0.$$

- We will use this equation as a control row to keep track of which variable to move in to the solution at the end of each step.

Simplex Method: Example

- Then, the state of the problem can be represented by the following augmented matrix:

$$\left[\begin{array}{ccccccc|c} 2 & 3 & 4 & 1 & 0 & 0 & 0 & 60 \\ 3 & 1 & 5 & 0 & 1 & 0 & 0 & 46 \\ 1 & 2 & 1 & 0 & 0 & 1 & 0 & 50 \\ \hline -25 & -33 & -18 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

- Note that the seventh column corresponds to M.
- This matrix is also known as the **initial simplex tableau**.
- This matrix correponds to the basic feasible solution $(x_4, x_5, x_6) = (60, 46, 50)$, as we have seen.
- In order to decide which vaiable to introduce to the solution, we look at the last row and solve for M:

$$M = 25x_1 + 33x_2 + 18x_3$$

- Since the coefficient of x_2 is the greatest, increasing x_2 will result in the maximum increase in M. So we'll bring x_2 into the solution.
- We have already moved x_2 in to the solution, but this time we will also update the control row by eliminating the x_2-term.

Simplex Method: Example

- The result of the pivot operation is

$$\left[\begin{array}{ccccccc|c} \frac{2}{3} & 1 & \frac{4}{3} & \frac{1}{3} & 0 & 0 & 0 & 20 \\ \frac{7}{3} & 0 & \frac{11}{3} & -\frac{1}{3} & 1 & 0 & 0 & 26 \\ -\frac{1}{3} & 0 & -\frac{5}{3} & -\frac{2}{3} & 0 & 1 & 0 & 10 \\ \hline -3 & 0 & 26 & 11 & 0 & 0 & 1 & 660 \end{array}\right].$$

- Now, the control row says $M = 660 + 3 + x_1 - 26x_3 - 11x_4$ so only bringing x_1 in to the solution will cause an increase in M.
- In order to decide which row to use as a pivot, calculate the ratios. For the first row it is $\dfrac{b_1}{a_{11}} = 30$ and for the second row $\dfrac{b_2}{a_{21}} = \dfrac{78}{7}$. The coefficient $-\dfrac{1}{3}$ is negative so there is no need to look at the third row.
- The smallest ratio comes from the second row, so we will use the x_1-term in the second row for pivoting.

Simplex Method: Example

- After pivoting, the updated tableau is as follows:

$$\left[\begin{array}{ccccccc|c} 0 & 1 & \frac{2}{7} & \frac{3}{7} & -\frac{2}{7} & 0 & 0 & \frac{88}{7} \\ 1 & 0 & \frac{11}{7} & -\frac{1}{7} & \frac{3}{7} & 0 & 0 & \frac{78}{7} \\ 0 & 0 & -\frac{8}{7} & -\frac{5}{7} & \frac{1}{7} & 1 & 0 & \frac{96}{7} \\ \hline 0 & 0 & \frac{215}{7} & \frac{74}{7} & \frac{9}{7} & 0 & 1 & \frac{4854}{7} \end{array}\right].$$

- The updated tableau corresponds to the basic feasible solution $(x_1, x_2, x_6) = \left(\frac{78}{7}, \frac{88}{7}, \frac{96}{7}\right)$
- In the updated tableau, we have $M = \dfrac{4854 - 215x_3 - 74x_4 - 9x_5}{7}$. So there is no need to increase M.
- As a result, the maximum value of the objective function $f(x_1, x_2, x_3) = 25x_1 + 33x_2 + 18x_3$ is $\dfrac{4854}{7}$ and it occurs at the point $(x_1, x_2, x_3) = \left(\frac{78}{7}, \frac{88}{7}, 0\right)$.

Simplex Method: Exercise

Exercise: A food store sells two kinds of mixtures. In one unit of Mix A, 1 kg of walnuts are mixed with 1 kg of Antep peanuts. In one unit of Mix B, 1 kg of Karadeniz nuts is mixed with 2 kg of Antep peanuts. The store has available 30 kg of walnuts, 20 kg of Karadeniz nuts and 54 kg of Antep peanuts. Profit of Mix A is 20 TL/unit while profit of Mix B is 30 TL/unit. If the store can sell all of the mixtures, what are the numbers of each mixture that provides the maximum possible profit?