MAT 222 – Exercise Set I

1. Let the following matrices be given.

$$\mathbf{A} = \begin{bmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ -2 & 0 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 3 & -5 & 6 \\ 4 & 1 & 12 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 1 & -2 & 0 \end{bmatrix}, \mathbf{E} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}.$$

Some pairs of the above matrices can be multiplied and some of them cannot. Multiply any two that can be multiplied. Which matrix or matrices can be multiplied by itself?

2. For each of the following statements, determine whether it is true or false. If false, give a counterexample. (In other words, find two matrices $\mathbf{A}, \mathbf{B}$ that show the statement is false.)

(a) If the first and the second columns of $\mathbf{B}$ are the same, then so are the first and the second columns of $\mathbf{AB}$.

(b) If the first and the second rows of $\mathbf{B}$ are the same, then so are the first and the second rows of $\mathbf{AB}$.

(c) If the first and the second rows of $\mathbf{A}$ are the same, then so are the first and the second rows of $\mathbf{AB}$.

(d) $(\mathbf{AB})^2 = \mathbf{A}^2\mathbf{B}^2$.

3. Let the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be given as

$$\mathbf{A} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \mathbf{C} = \mathbf{AB}.$$

Calculate the matrices $\mathbf{A}^2, \mathbf{A}^3, \mathbf{B}^2, \mathbf{B}^3$ and $\mathbf{C}^2, \mathbf{C}^3$. Try to find a general rule for the matrices $\mathbf{A}^n, \mathbf{B}^n$ and $\mathbf{C}^n$ for $n > 3$.

4. If a matrix $\mathbf{A}$ satisfies $\mathbf{A}^2 = \mathbf{A}$, $\mathbf{A}$ is called idempotent. If a nonzero matrix $\mathbf{B}$ satisfies $\mathbf{B}^m =$ zero matrix for some $m > 0$, $\mathbf{B}$ is called nillpotent.

a) Find a 2×2 nonzero idempotent matrix.

b) Find a 2×2 nillpotent matrix.

5. Suppose that $\mathbf{U}_1, \mathbf{U}_2$ are upper triangular and $\mathbf{L}_1, \mathbf{L}_2$ are lower triangular matrices. Suppose also that these four are square matrices of the same size. Which of the following matrices are upper triangular or lower triangular?

$$\mathbf{U}_1 + \mathbf{U}_2 \qquad \mathbf{U}_1\mathbf{U}_2 \qquad \mathbf{L}_1 + \mathbf{L}_2 \qquad \mathbf{L}_1\mathbf{L}_2 \qquad \mathbf{U}_1 + \mathbf{L}_1 \qquad \mathbf{U}_1\mathbf{L}_1$$

6. In the below system, find a value for b such that the system (a) doesn't have any solutions, (b) has infinitely many solutions.

$$3x + 2y = 10$$
$$6x + 4y = b$$

7. The following system is given.

$$2x + by = 16$$
$$4x + 8y = c$$

(a) For which value(s) of b does this system have a unique solution? Does your answer depend on the value of c?

(b) For which value(s) of b can the system not have a unique solution? In this case, find value(s) of c that makes the system inconsistent(not having any solutions). Find value(s) of c that makes the system to have infinitely many solutions.

8. Solve the following systems using Gaussian elimination. There may be no solutions or infinitely many solutions. In case there are infinitely many solutions, express the solution set in a suitable way by using one or more free parameters.

(a)
$$\begin{aligned} 2x + y + z &= 5 \\ 4x - 6y &= -2 \\ -2x + 7y + 2z &= 9 \end{aligned}$$

(b)
$$\begin{aligned} x + y + z &= -1 \\ 2x + 2y + 5z &= 0 \\ 4x + 6y + 8z &= 5 \end{aligned}$$

(c)
$$\begin{aligned} 4y + z &= 20 \\ 2x - 2y + z &= 0 \\ x + z &= 5 \\ x + y - z &= 10 \end{aligned}$$

(d)
$$\begin{aligned} x_1 - 3x_2 + x_3 &= 4 \\ -x_1 + 2x_2 - 5x_3 &= 3 \\ 5x_1 - 13x_2 + 13x_3 &= 8 \end{aligned}$$

(e)
$$\begin{aligned} x - 3y + z &= 1 \\ x + y + 2z &= 14 \end{aligned}$$

(f)
$$\begin{aligned} 2x + z + w &= 5 \\ y - w &= -1 \\ 3x - z - w &= 0 \\ 4x + y + 2z + w &= 9 \end{aligned}$$

(g)
$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 6 \\ -x_1 + 2x_2 - 5x_3 + x_4 &= 2 \\ x_1 + 4x_2 - 3x_3 + 3x_4 &= 14 \\ 3x_1 + 6x_2 - x_3 + 5x_4 &= 26 \end{aligned}$$

(h)
$$\begin{aligned} x + 3y + z + t &= 1 \\ -4x - 9y + 2z - t &= -1 \\ -y - 2z - t &= -1 \\ y + 2z + t &= 1 \end{aligned}$$

9. We are given the following 2×3 system.

$$\begin{aligned} 3x - 2y - z &= 9 \\ x + y + 3z &= 5 \end{aligned}$$

(a) Write down a third equation in x, y, z such that the resulting 3×3 system has a unique solution.

(b) Write down a third equation in x, y, z such that the resulting 3×3 system has infinitely many solutions.

(c) Write down a third equation in x, y, z such that the resulting 3×3 system has no solutions.

10. Consider the following system.

$$\begin{aligned} 2\sin\alpha - \cos\beta + 3\tan\gamma &= -1 \\ 4\sin\alpha + 2\cos\beta - 2\tan\gamma &= 2 \\ 6\sin\alpha - 3\cos\beta + \tan\gamma &= -3 \end{aligned}$$

Is this sytem linear? Can you find α, β, γ uniquely? If not, what extra condition must be imposed on them so that you can determine α, β, γ uniquely?

11. Consider an $m \times n$ linear system. If $m < n$ is it possible that the system has a unique solution? Infinitely many solutions? No solutions? Answer the same three questions in the cases $m = n$ and $m > n$.

12. The following system is known to have a unique solution.

$$\begin{aligned} 2t - u + 3v + 4w &= 9 \\ t - 2v + 7w &= 11 \\ 3t - 3u + v + 5w &= 8 \\ 2t + u + 4v + 4w &= 10 \end{aligned}$$

(a) Change one number in the system so that the resulting system becomes inconsistent (doesn't have a solution).

(b) Change one number in the system so that the resulting system has infinitely many solutions.

(c) Change one number in the system so that the resulting system also has a unique solution. (Hint: This can be answered in one second.)

13. Explain why a linear system cannot have **exactly** two solutions. (Hint: Consider a 3×3 system and assume that (a_1, b_1, c_1) and (a_2, b_2, c_2) satisfy the system. Using these two solutions, how can you find new ones. Generalize your reasoning to any $m \times n$ system.)

14. Consider an $m \times n$ system $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{b}$ is not equal to the zero vector. Suppose that the system has infinitely many solutions. Suppose further that the vector $\mathbf{p} = (a_1, a_2, \ldots, a_n)$ is a solution of the system. If $\mathbf{u}$ is any other solution of the system, show that the vector $\mathbf{u} - \mathbf{p}$ is a solution of the corresponding homogeneous system $\mathbf{Ax} = \mathbf{0}$. (Hint: How do you represent the infinitely many solutions of the original system in terms of the free vairables?)

15. Calculate the ranks of the following matrices.

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \\ -2 & -4 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 3 & -5 & 6 \\ 4 & 1 & 12 \\ 1 & -17 & -6 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 0 & 8 & 4 & 2 \\ 1 & 2 & 4 & 8 \\ 2 & 4 & 8 & 1 \end{bmatrix},$$

$$\mathbf{E} = \begin{bmatrix} 1 & 5 & 2 & 3 & 5 \\ -2 & -10 & -3 & -3 & -6 \\ 7 & 35 & 12 & 16 & 29 \\ 8 & 40 & 13 & 18 & 34 \end{bmatrix}, \mathbf{F} = \begin{bmatrix} 2 & 1 & 3 & -1 \\ 3 & -1 & 2 & 0 \\ 1 & 3 & 4 & -2 \\ 4 & -3 & 1 & 1 \end{bmatrix}$$

16. Two of the following systems are consistent (have at least one solution) and two af them are not. Determine which two of them are consistent. Then, calculate the ranks of the coefficient matrix and the augmented matrix in each system. Does anything draw your attention?

(a) $$\begin{aligned} 2x + y + z &= 5 \\ 4x - 6y &= 2 \\ -2x + 7y + 2z &= 9 \end{aligned}$$

(b) $$\begin{aligned} x + 3y &= 1 \\ 2x + y &= -3 \\ 2x + 2y &= 3 \end{aligned}$$

(c)
$$\begin{aligned} x - y &= 0 \\ 2x - 2y + z + 2w &= 4 \\ y + w &= 0 \\ 2z + w &= 5 \end{aligned}$$

(d)
$$\begin{aligned} 3x - 2y - z &= 9 \\ x + y + 3z &= 5 \\ -x + 4y + 7z &= -2 \end{aligned}$$

17. For each of the below matrices, find the LU factorization.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 8 \\ 1 & 1 & 3 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} -1 & -2 & 3 & 0 \\ 2 & 4 & -6 & 5 \\ 1 & 1 & -1 & 3 \\ 2 & 5 & -10 & 1 \end{bmatrix}$$

18. The matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 8 \\ 1 & 1 & 3 \end{bmatrix}$ and the vectors $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$ be given. Use the LU factorization of $\mathbf{A}$ to solve the three systems $\mathbf{Ax} = \mathbf{b}_1, \mathbf{Ax} = \mathbf{b}_2$ and $\mathbf{Ax} = \mathbf{b}_3$.

19. Let $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$ be given. Use the given factorization of $\mathbf{A}$ to solve $\mathbf{Ax} = \mathbf{b}$.

20. Find the inverse of each matrix, if it exists. If it does not exist, shortly state why it does not.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 5 & 9 & 1 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 5 & 1 \\ -3 & 13 & 3 \end{bmatrix},$$

$$\mathbf{E} = \begin{bmatrix} -1 & -2 & 3 & 0 \\ 2 & 4 & -6 & 5 \\ 1 & 1 & -1 & 3 \end{bmatrix}, \mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 0 & 1 & 4 \end{bmatrix}.$$

21. The inverse of the matrix $\mathbf{A} = \begin{bmatrix} 2 & 2 & -2 \\ 1 & 4 & 2 \\ -1 & 1 & 5 \end{bmatrix}$ is given to be the matrix $\begin{bmatrix} 3/2 & -1 & 1 \\ -7/12 & 2/3 & -1/2 \\ 5/12 & -1/3 & 1/2 \end{bmatrix}$. Using this information, solve the three systems $\mathbf{Ax} = \mathbf{b}_1, \mathbf{Ax} = \mathbf{b}_2$ and $\mathbf{Ax} = \mathbf{b}_3$, where the right-hand sides are $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$.

22. Solve the following systems by computing the inverse of the coefficient matrix.

(a) $$\begin{aligned} \frac{1}{2}x + y &= -1 \\ y + z &= 2 \\ x + 2z &= 3 \end{aligned}$$

(b) $$\begin{aligned} \frac{1}{2}x + y &= 1 \\ y + z &= -3 \\ x + 2z &= 2 \end{aligned}$$

(c) $$\begin{aligned} \frac{1}{2}x + y &= 3 \\ y + z &= -2 \\ x + 2z &= 1 \end{aligned}$$

23. Let $\mathbf{A} = \begin{bmatrix} 3 & 1 & 2 \\ 3 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 2 & 3 \\ 1 & -2 & -3 \\ 4 & -2 & -3 \end{bmatrix}$. Find a matrix $\mathbf{C}$ such that $\mathbf{AC} = \mathbf{B}$. (Hint: One method is the following: Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the first, second and third column of $\mathbf{B}$, respectively. Then solve the systems $\mathbf{Ax} = \mathbf{v}_1, \mathbf{Ax} = \mathbf{v}_2$ and $\mathbf{Ax} = \mathbf{v}_3$. Writing the solutions of these systems in columns will give you the matrix $\mathbf{C}$.)

24. Evaluate the determinants of the following matrices. You can use Gaussian elimination or cofactor expansion.

$$\mathbf{A} = \begin{bmatrix} -1 & 5 & 1 \\ 2 & -4 & -4 \\ 1 & 1 & -3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 2 & 7 & 1 \\ -1 & 0 & -4 \\ 1 & 0 & 3 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 5 & 3 & 1 \\ -2 & 5 & 7 \\ 1 & 9 & 4 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} -1 & -2 & 3 & 0 \\ 2 & 4 & -6 & 5 \\ 1 & 1 & -1 & 3 \\ 2 & 5 & -10 & 1 \end{bmatrix}$$

25. Evaluate the following determinant in terms of a and b.

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1-a & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1-b \end{vmatrix}$$

26. Let $\mathbf{A}$ be a 3×3 matrix whose determinant is 4. It is known that $\mathbf{A}(\mathbf{B} - 2\mathbf{A})$ is the 3×3 zero matrix. What is $\det(\mathbf{B})$?

27. Solve the following systems using Cramer's rule.

(a) $$\begin{aligned} \frac{1}{2}x + y &= -1 \\ 2x - 3y &= 2 \end{aligned}$$

(b) $$\begin{aligned} 2x + y + z &= 1 \\ 3x + z &= 4 \\ x - y - z &= 2 \end{aligned}$$

(c) $$\begin{aligned} -2x_1 + 3x_3 &= 13 \\ 2x_1 + 3x_2 - 2x_3 &= 0 \\ x_2 - 3x_4 &= 0 \\ -3x_3 + 2x_4 &= -15 \end{aligned}$$