**Using Diagonalization to Solve Recurrence Relations**

If $\mathbf{A}$ is a square matrix, recall that the $n$-th power of $\mathbf{A}$ can be calculated as follows: Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $\mathbf{A}$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be eigenvectors corresponding to these eigenvalues, respectively. Then if $\mathbf{P}$ is defined to be the matrix whose columns are $v_1, v_2, \ldots, v_n$, respectively; namely if

$$\mathbf{P} = \left[ \begin{array}{c|c|c|c} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{array} \right],$$

then it is true that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$ where

$$\mathbf{D} = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

Furthermore, the relation $\mathbf{P}^{-1}\mathbf{A}^n\mathbf{P} = \mathbf{D}^n$ holds. This procedure is known as "finding powers of a matrix via diagonalization".

Diagonalization can be used to calculate the general term of linear recurrence relations. For instance, consider the Fibonacci sequence given by

$$x_0 = 0, x_1 = 1,\ x_{n+2} = x_{n+1} + x_n \text{ for } n \geq 0.$$

So the first few terms are $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$ Terms of the Fibonacci sequence clearly satisfy

$$\begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}$$

for any $n \geq 0$. Let us denote the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ by $\mathbf{A}$. You can observe that $\begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = \mathbf{A}^n \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$ for any $n \geq 0$. Therefore we can easily obtain an explicit formula for $x_n$ once we calculate $\mathbf{A}^n$. The eigenvalues of $\mathbf{A}$ are $\lambda_1 = \dfrac{1-\sqrt{5}}{2}$ and $\lambda_2 = \dfrac{1+\sqrt{5}}{2}$. Two eigenvectors corresponding to these eigenvalues are $\mathbf{v}_1 = \begin{bmatrix} -\lambda_2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -\lambda_1 \\ 1 \end{bmatrix}$, respectively. Hence we have

$$\mathbf{P} = \begin{bmatrix} -\lambda_2 & -\lambda_1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -\dfrac{1+\sqrt{5}}{2} & \dfrac{\sqrt{5}-1}{2} \\ 1 & 1 \end{bmatrix} \text{ and } \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1-\sqrt{5}}{2} & 0 \\ 0 & \dfrac{1+\sqrt{5}}{2} \end{bmatrix}.$$

We can calculate the inverse of $\mathbf{P}$ as $\mathbf{P}^{-1} = \dfrac{1}{\sqrt{5}} \begin{bmatrix} -1 & \dfrac{\sqrt{5}-1}{2} \\ 1 & \dfrac{\sqrt{5}+1}{2} \end{bmatrix}$. Then from $\mathbf{P}^{-1}\mathbf{A}^n\mathbf{P} = \mathbf{D}^n$ can write $\mathbf{A}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$. So $\mathbf{A}^n$ can be calculated and simplified as

$$\begin{aligned} \mathbf{A}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1} &= \begin{bmatrix} -\dfrac{1+\sqrt{5}}{2} & \dfrac{\sqrt{5}-1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(\dfrac{1-\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\dfrac{1+\sqrt{5}}{2}\right)^n \end{bmatrix} \begin{bmatrix} -1 & \dfrac{\sqrt{5}-1}{2} \\ 1 & \dfrac{\sqrt{5}+1}{2} \end{bmatrix} \cdot \frac{1}{\sqrt{5}} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \dfrac{(1+\sqrt{5})^{n-1} - (1-\sqrt{5})^{n-1}}{2^{n-1}} & \dfrac{(1+\sqrt{5})^{n} - (1-\sqrt{5})^{n}}{2^{n}} \\ \dfrac{(1+\sqrt{5})^{n} - (1-\sqrt{5})^{n}}{2^{n}} & \dfrac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}} \end{bmatrix} \end{aligned}$$

so that we have

$$\begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = \mathbf{A}^n \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \mathbf{A}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \dfrac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n} \\ \dfrac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}} \end{bmatrix}.$$

As a result, the general term of Fibonacci sequence is

$$x_n = \frac{1}{\sqrt{5}} \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n}.$$

You might be tempted to think that every term of the Fibonacci sequence is an integer and so the above formula must be incorrect. However, the above formula gives an integer for every natural number $n$. You may convince yourself of this by trying it with small values of $n$. (For large $n$ the computer may not calculate integer results due to round-off error.)