OBJECT ORIENTED ANALYSIS & DESIGN
DATA STRUCTURES & ALGORITHMS

GRAPHS

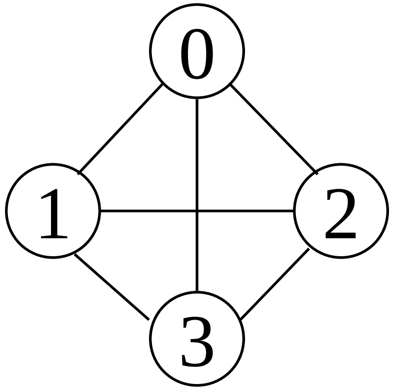
Definition

- A graph G consists of two sets
  - a finite, nonempty set of vertices V(G)
  - a finite, possible empty set of edges E(G)
  - G(V,E) represents a graph

- An undirected graph is one in which the pair of vertices in a edge is unordered, (v0, v1) = (v1,v0)

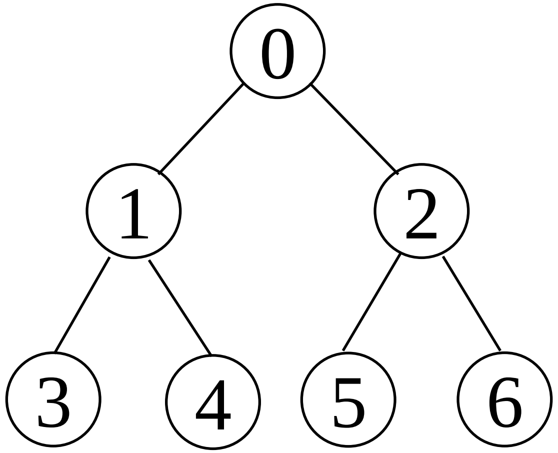
- A directed graph is one in which each edge is a directed pair of vertices, <v0, v1> != <v1,v0>

Examples for Graph



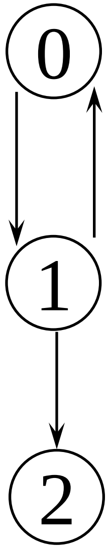
G_1

complete graph



G_2

incomplete graph



G_3

V(G1)={0,1,2,3}
V(G2)={0,1,2,3,4,5,6}
V(G3)={0,1,2}

E(G1)={(0,1),(0,2),(0,3),(1,2),(1,3),(2,3)}
E(G2)={(0,1),(0,2),(1,3),(1,4),(2,5),(2,6)}
E(G3)={<0,1>,<1,0>,<1,2>}

complete undirected graph: n(n-1)/2 edges
complete directed graph: n(n-1) edges

Complete Graph

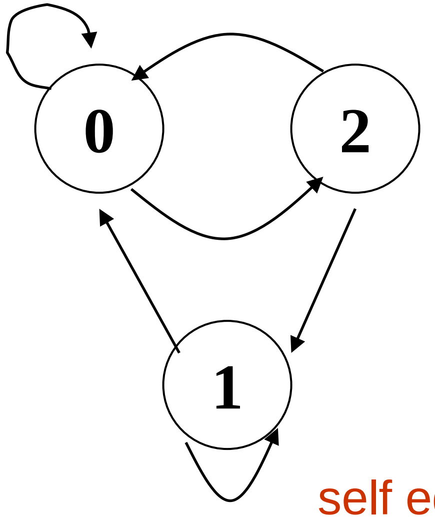
- A complete graph is a graph that has the maximum number of edges
  - for undirected graph with n vertices, the maximum number of edges is $n(n-1)/2$
  - for directed graph with n vertices, the maximum number of edges is $n(n-1)$
  - example: G1 is a complete graph

Adjacent and Incident

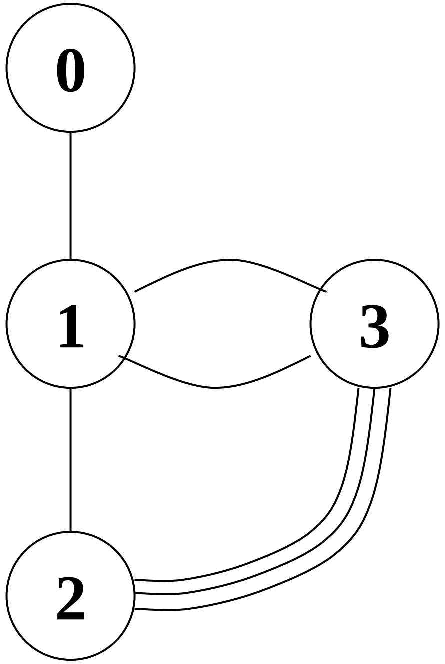
- If (v0, v1) is an edge in an undirected graph,
  - v0 and v1 are adjacent
  - The edge (v0, v1) is incident on vertices v0 and v1

- If <v0, v1> is an edge in a directed graph
  - v0 is adjacent to v1, and v1 is adjacent from v0
  - The edge <v0, v1> is incident on v0 and v1

Example of a graph with feedback loops and a multigraph



(a)



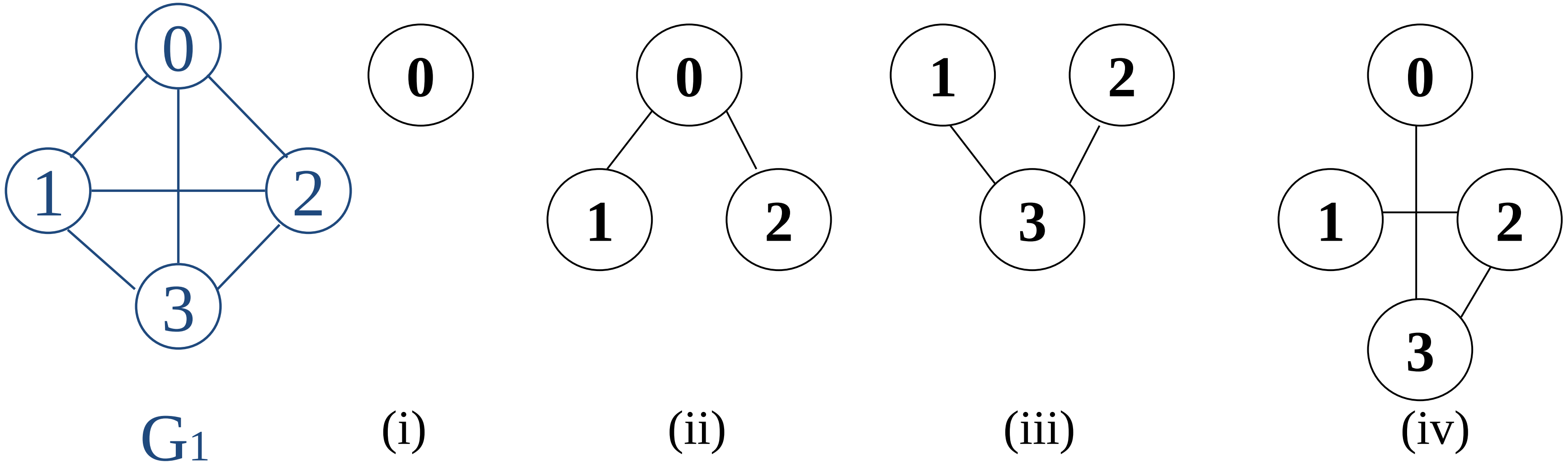
(b)

multigraph:
multiple occurrences
of the same edge

Subgraph and Path

- A subgraph of G is a graph G’ such that V(G’) is a subset of V(G) and E(G’) is a subset of E(G)
- A path from vertex vp to vertex vq in a graph G, is a sequence of vertices, vp, vi1, vi2, ..., vin, vq, such that (vp, vi1), (vi1, vi2), ..., (vin, vq) are edges in an undirected graph
- The length of a path is the number of edges on it

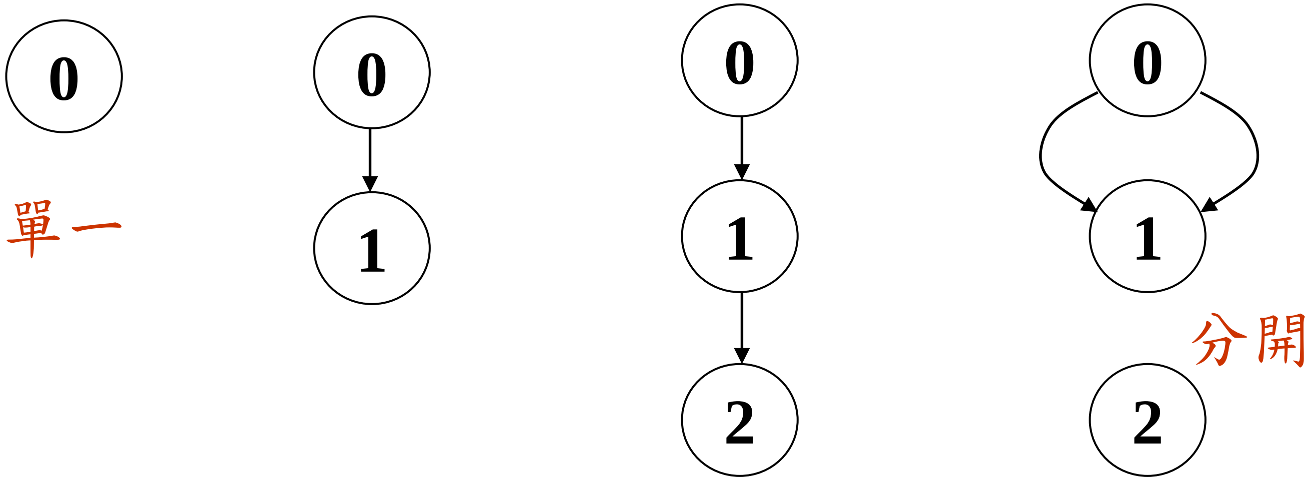
subgraphs of G1 and G3



(a) Some of the subgraph of G_1

subgraphs of G1 and G3 (contd.)

G3



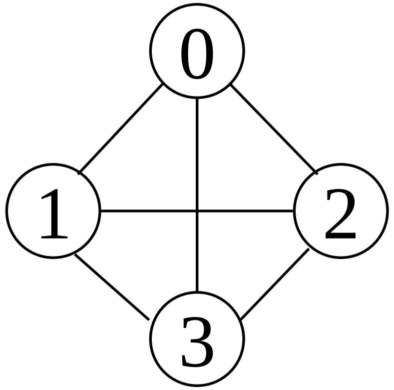
(i) (ii) (iii) (iv)

(b) Some of the subgraph of G_3

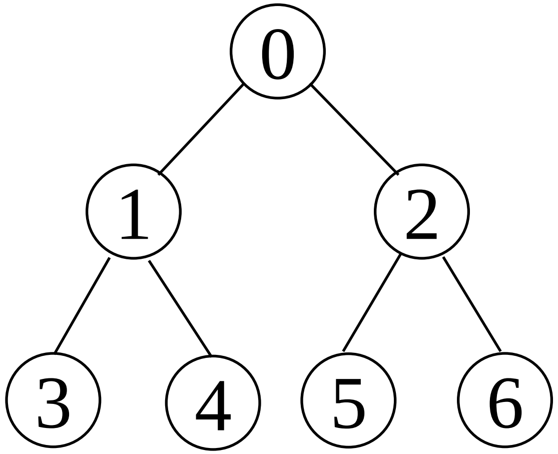
Simple Path and Style

- A simple path is a path in which all vertices, except possibly the first and the last, are distinct
- A cycle is a simple path in which the first and the last vertices are the same
- In an undirected graph G, two vertices, v0 and v1, are connected if there is a path in G from v0 to v1
- An undirected graph is connected if, for every pair of distinct vertices vi, vj, there is a path from vi to vj

Connected



G1



G2

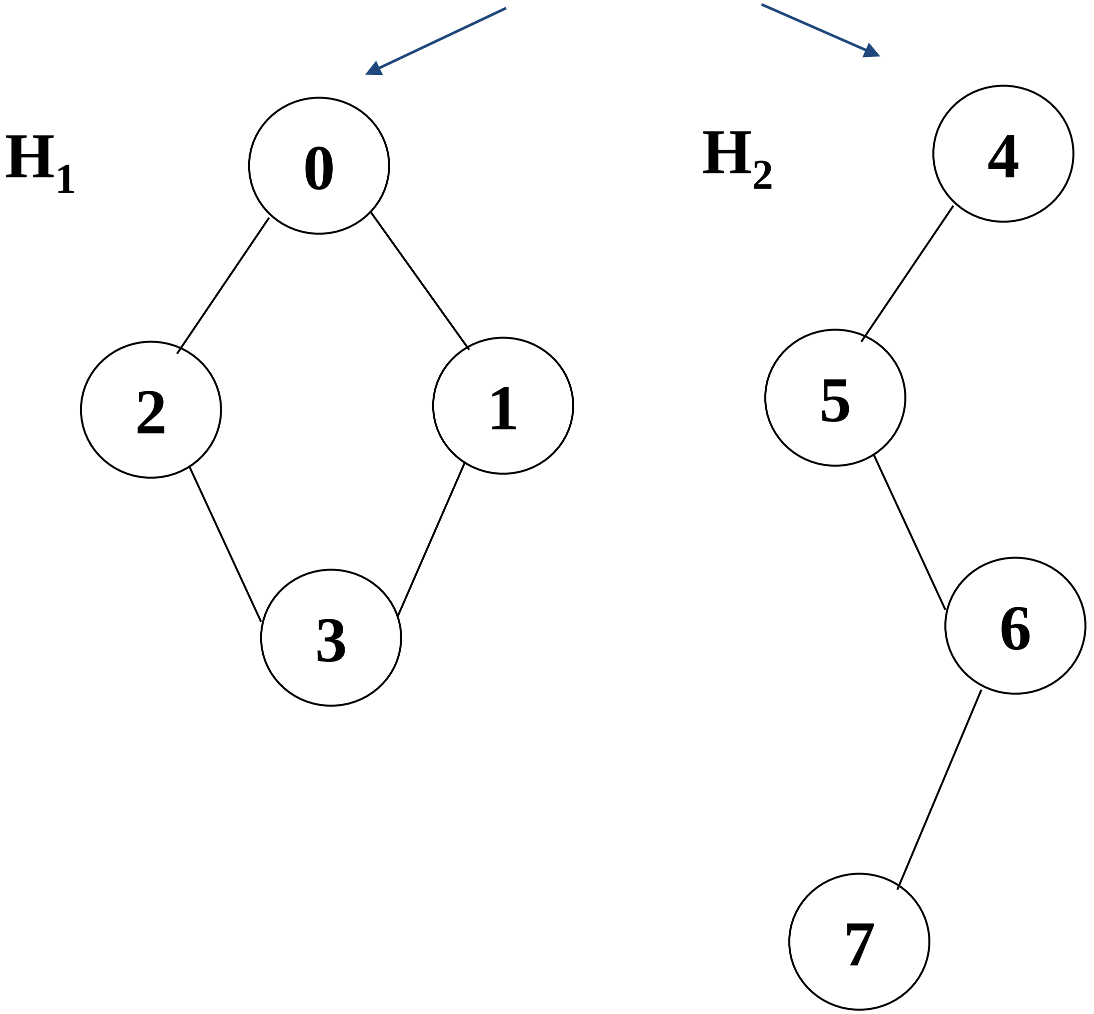
tree (acyclic graph)

Connected Component

- A connected component of an undirected graph is a maximal connected subgraph.
- A tree is a graph that is connected and acyclic.
- A directed graph is strongly connected if there is a directed path from vi to vj and also from vj to vi.
- A strongly connected component is a maximal subgraph that is strongly connected.

A graph with two connected components

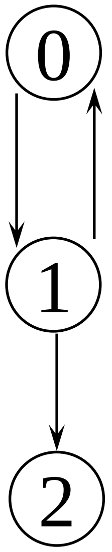
connected component (maximal connected subgraph)



G_4 (not connected)

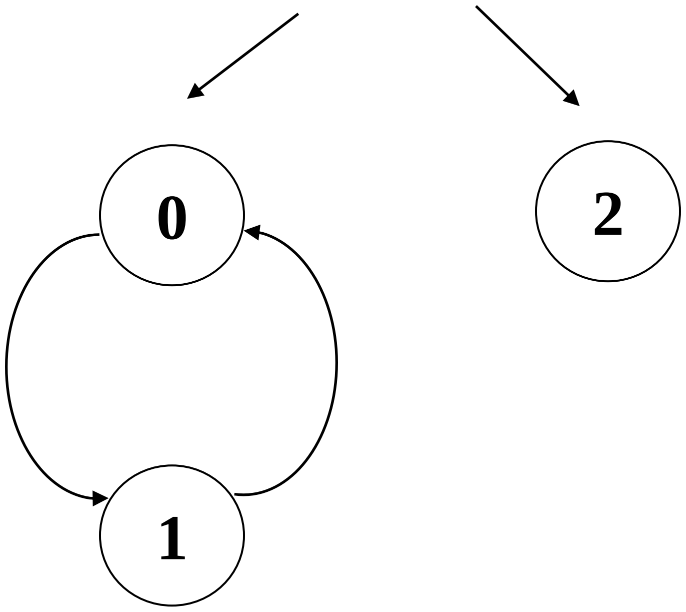
Strongly connected components of G3

not strongly connected



G_3

strongly connected component
(maximal strongly connected subgraph)

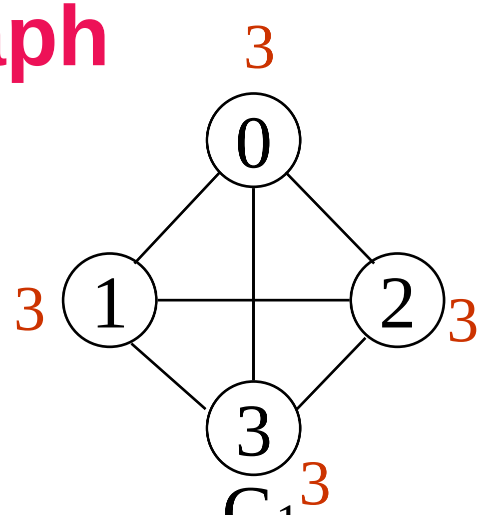


Degree

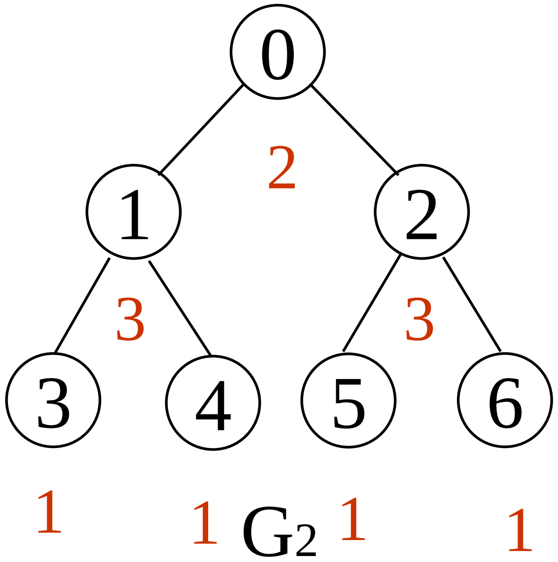
- The degree of a vertex is the number of edges incident to that vertex
- For directed graph,
  - the in-degree of a vertex *v* is the number of edges that have *v* as the head
  - the out-degree of a vertex *v* is the number of edges that have *v* as the tail
  - if *di* is the degree of a vertex *i* in a graph *G* with *n* vertices and *e* edges, the number of edges is

$$e = \left(\sum_{0}^{n-1} d_i\right) / 2$$

undirected graph



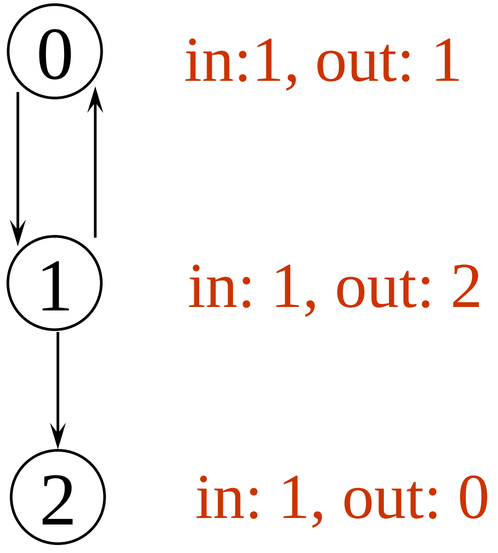
G1



G2

directed graph

in-degree
out-degree



G3

ADT for Graph

structure Graph is

objects: a nonempty set of vertices and a set of undirected edges, where each edge is a pair of vertices

functions: for all *graph* $\in$ *Graph*, *v*, v_1 and v_2 $\in$ *Vertices*

Graph Create()::=return an empty graph

Graph InsertVertex(*graph*, *v*)::= return a graph with *v* inserted. *v* has no incident edge.

Graph InsertEdge(*graph*, *v1*,*v2*)::= return a graph with new edge between *v1* and *v2*

Graph DeleteVertex(*graph*, *v*)::= return a graph in which *v* and all edges incident to it are removed

Graph DeleteEdge(*graph*, *v1*, *v2*)::=return a graph in which the edge (*v1*, *v2*) is removed

Boolean IsEmpty(*graph*)::= if (*graph==empty graph*) return TRUE else return FALSE

List Adjacent(*graph*,*v*)::= return a list of all vertices that are adjacent to *v*

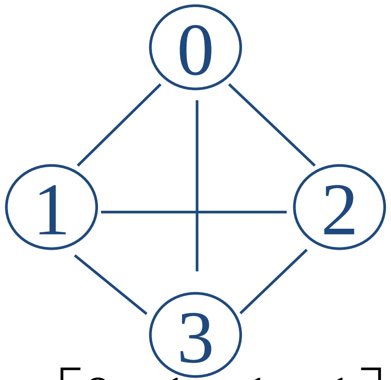
Graph Representations

- Adjacency Matrix
- Adjacency Lists
- Adjacency Multilists

Adjacency Matrix

- Let G=(V,E) be a graph with n vertices.
- The adjacency matrix of G is a two-dimensional n by n array, say adj_mat
- If the edge (vi, vj) is in E(G), adj_mat[i][j]=1
- If there is no such edge in E(G), adj_mat[i][j]=0
- The adjacency matrix for an undirected graph is symmetric; the adjacency matrix for a digraph need not be symmetric

Examples for Adjacency Matrix

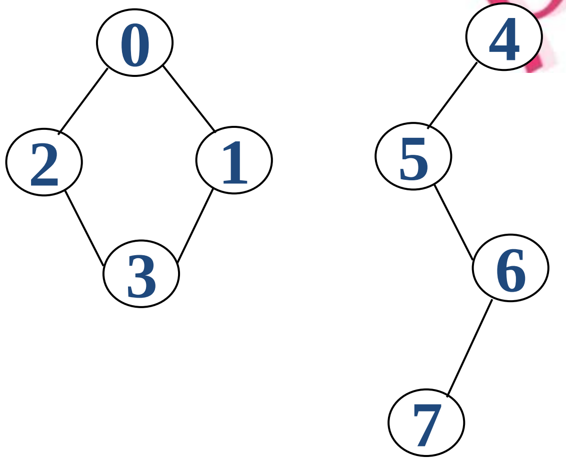


$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

G_1

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

G_2



$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

G_4

symmetric

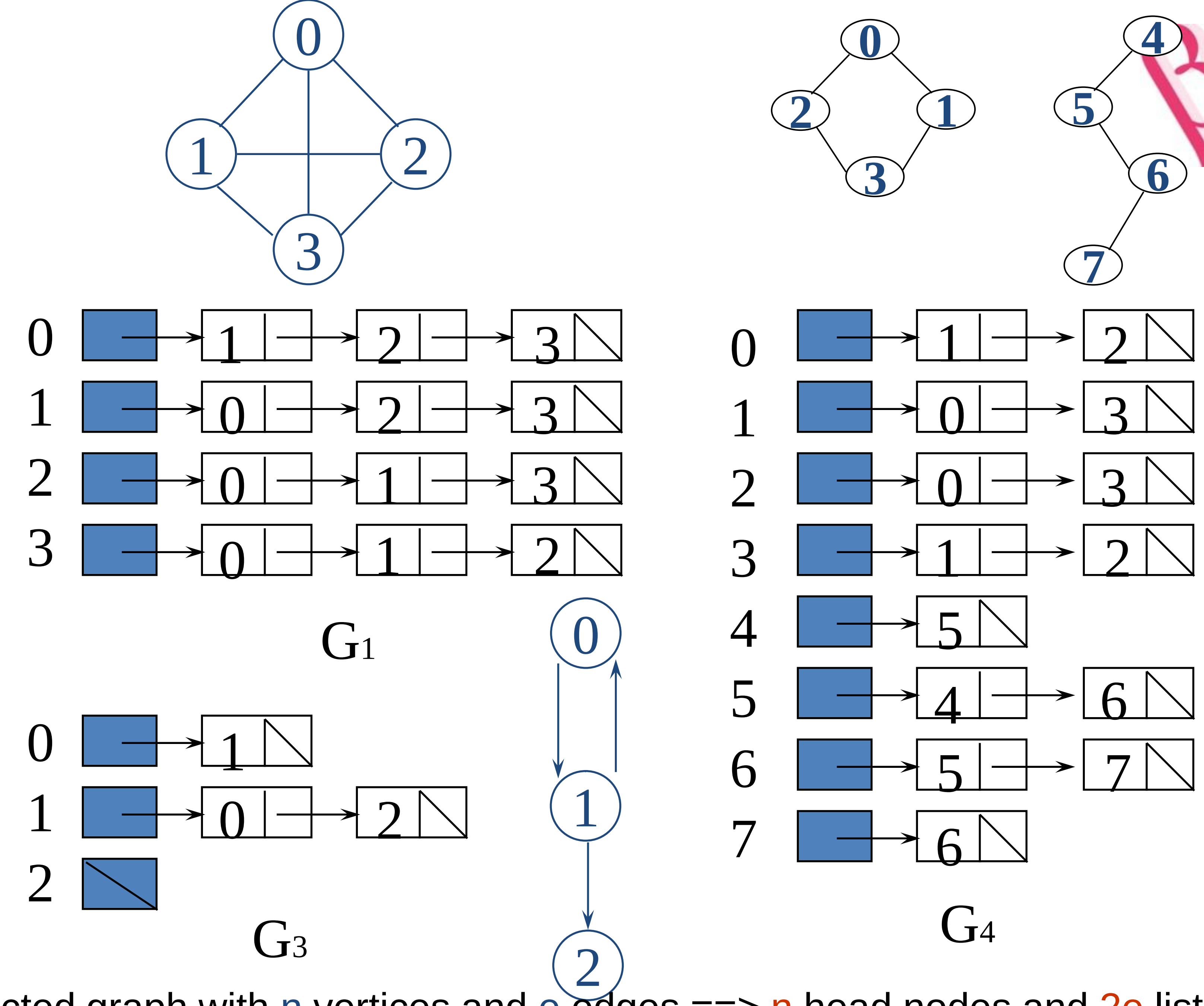
undirected: $n^2/2$
directed: n^2

Merits of Adjacency Matrix

- From the adjacency matrix, to determine the connection of vertices is easy
- The degree of a vertex is
-
- For a digraph, the row sum is the out_degree, while the column sum is the in_degree

$$\sum_{j=0}^{n-1} adj\_mat[i][j]$$

$$ind(vi) = \sum_{j=0}^{n-1} A[j,i] \quad outd(vi) = \sum_{j=0}^{n-1} A[i,j]$$



An undirected graph with n vertices and e edges ==> n head nodes and 2e list nodes

Interesting Operations

- degree of a vertex in an undirected graph

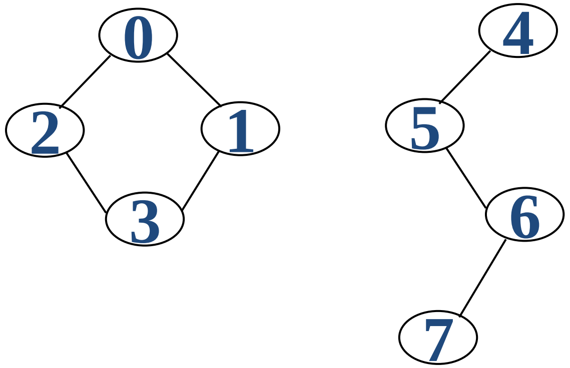
  # of nodes in adjacency list
- # of edges in a graph

  determined in O(n+e)
- out-degree of a vertex in a directed graph

  # of nodes in its adjacency list
- in-degree of a vertex in a directed graph

  traverse the whole data structure

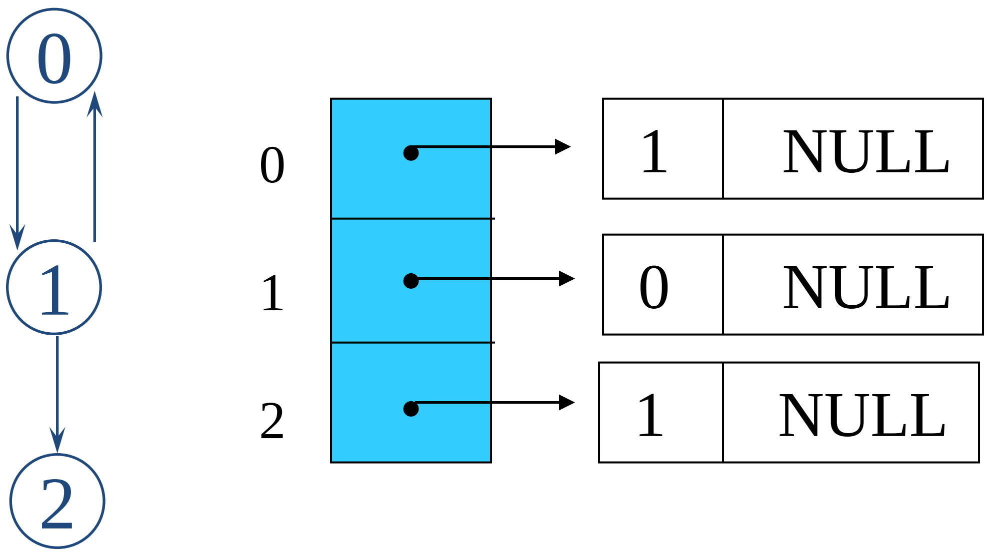
Compact Representation



node[0] … node[n-1]: starting point for vertices
node[n]: n+2e+1
node[n+1] … node[n+2e]: head node of edge

[0]	9		[8]	23		[16]	2
[1]	11	0	[9]	1	4	[17]	5
[2]	13		[10]	2	5	[18]	4
[3]	15	1	[11]	0		[19]	6
[4]	17		[12]	3	6	[20]	5
[5]	18	2	[13]	0		[21]	7
[6]	20		[14]	3	7	[22]	6
[7]	22	3	[15]	1			

Inverse adjacency list for G3

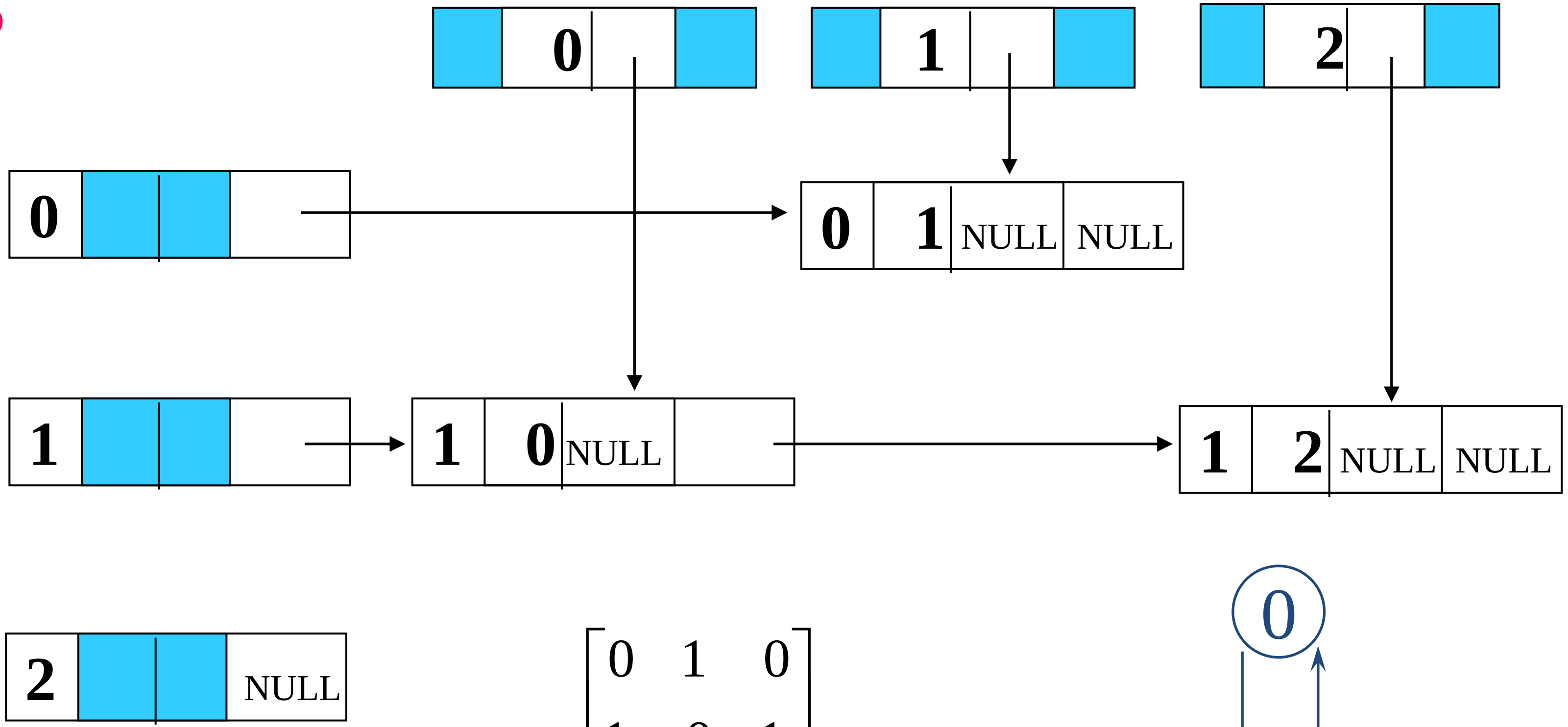


Determine in-degree of a vertex in a fast way.

Alternate node structure for adjacency lists

tail	head	column link for head	row link for tail

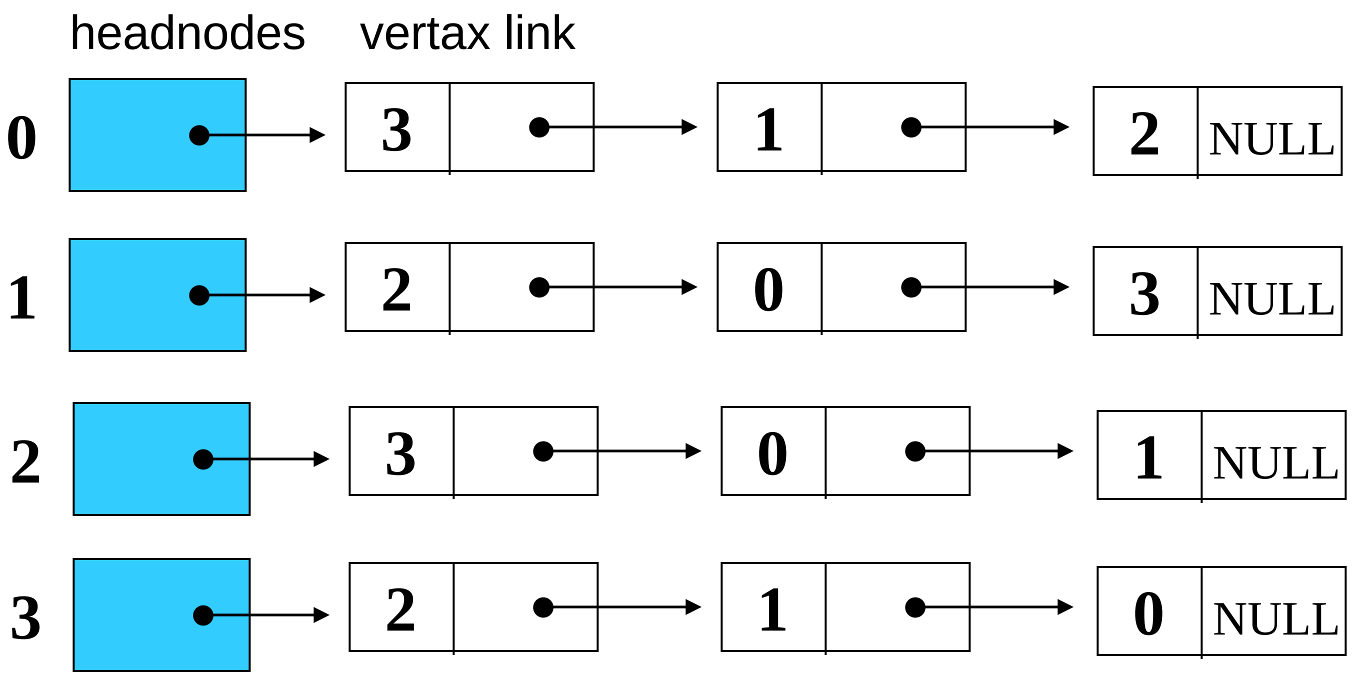
Orthogonal representation for graph G3

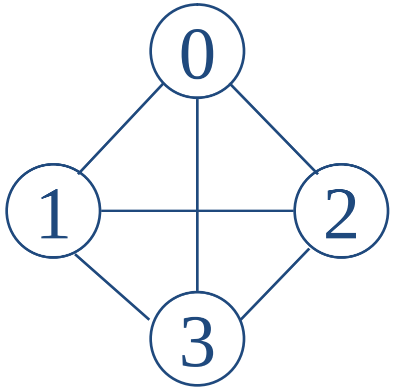


$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Alternate order adjacency list for G1

Order is of no significance.





Adjacency Multilists

- An edge in an undirected graph is represented by two nodes in adjacency list representation.

- Adjacency Multilists

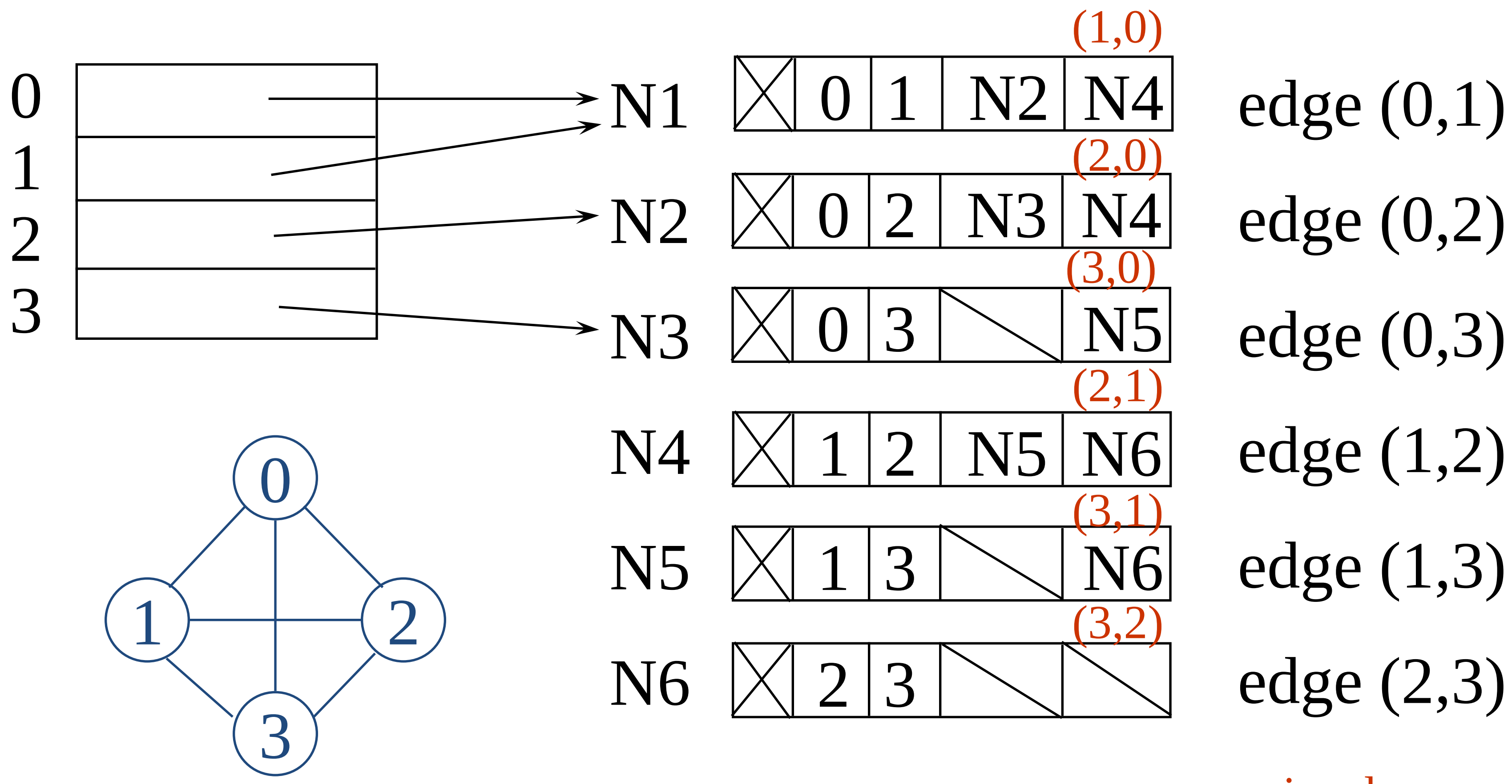
  lists in which nodes may be shared among several lists.

  (an edge is shared by two different paths)

marked	vertex1	vertex2	path1	path2

Example for Adjacency Multlists

Lists: vertex 0: M1->M2->M3, vertex 1: M1->M4->M5
vertex 2: M2->M4->M6, vertex 3: M3->M5->M6



six edges

Some Graph Operations

- Traversal

Given G=(V,E) and vertex v, find all $w \in V$,
such that w connects v.

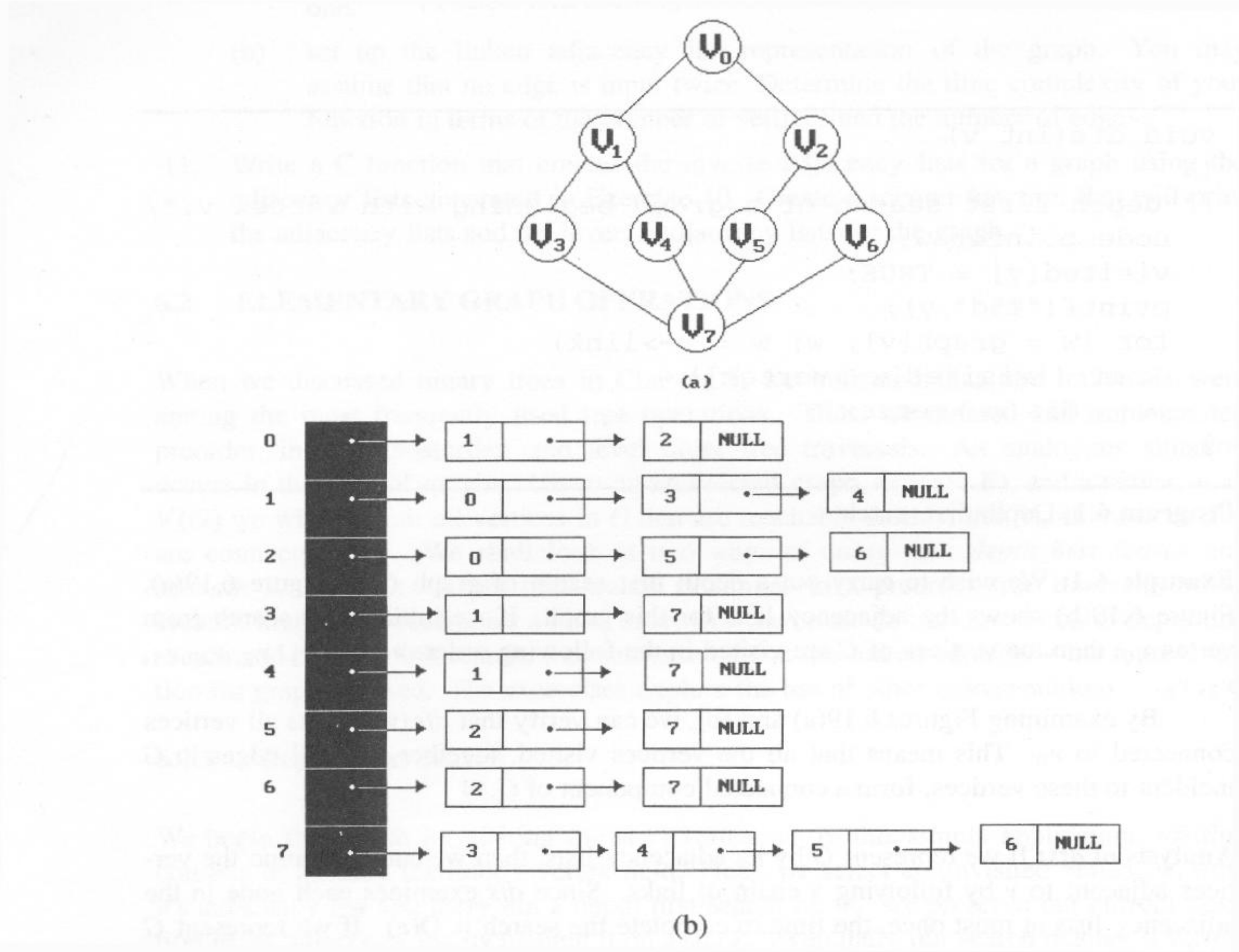
Depth First Search (DFS)
preorder tree traversal
Breadth First Search (BFS)
level order tree traversal

- Connected Components

- Spanning Trees

Graph G and its adjacency lists

depth first search: v0, v1, v3, v7, v4, v5, v2, v6



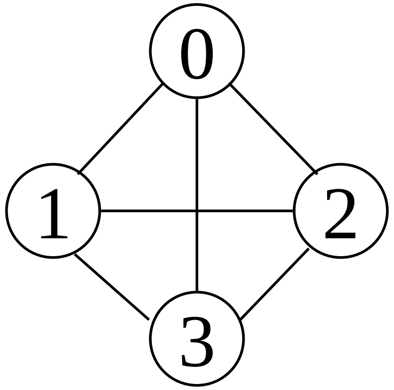
breadth first search: v0, v1, v2, v3, v4, v5, v6, v7

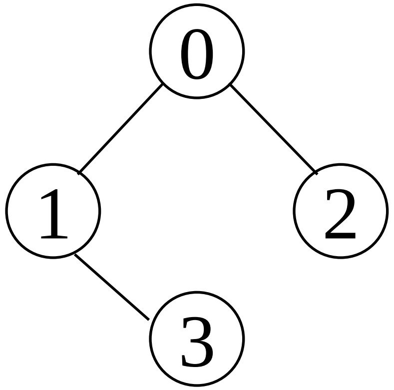
Spanning Trees

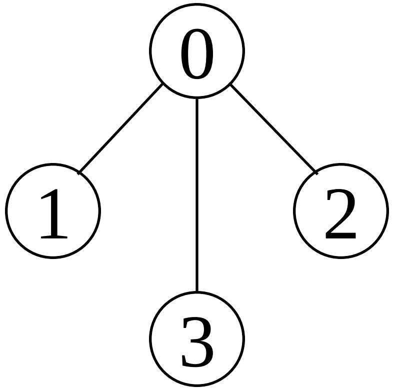
- When graph G is connected, a depth first or breadth first search starting at any vertex will visit all vertices in G
- A spanning tree is any tree that consists solely of edges in G and that includes all the vertices
- E(G): T (tree edges) + N (nontree edges)

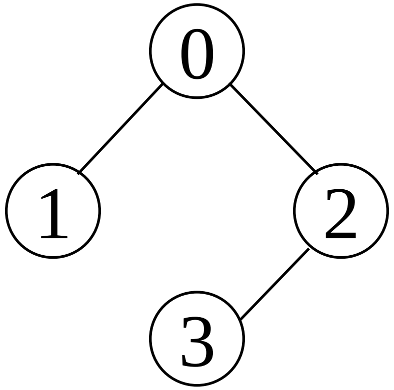
where T: set of edges used during search
N: set of remaining edges

Examples of Spanning Tree









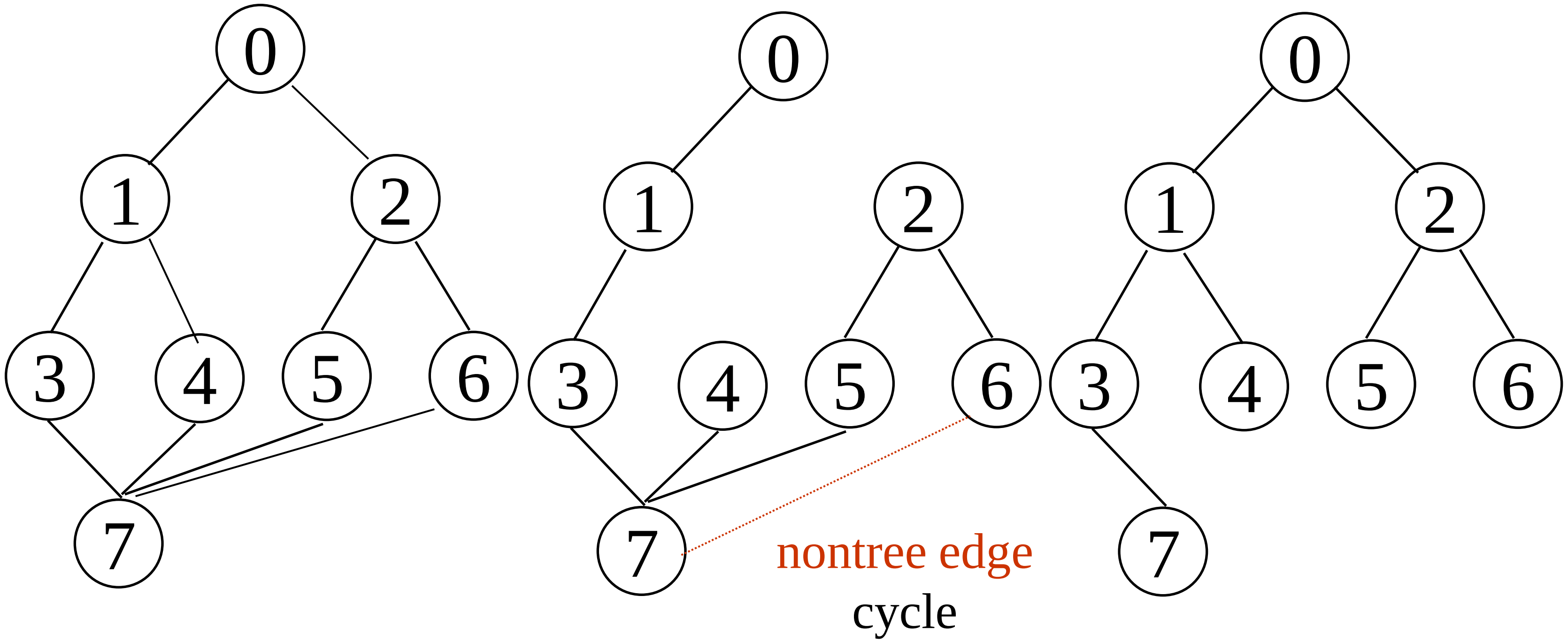
G_1

Possible spanning trees

Spanning Trees

- Either dfs or bfs can be used to create a spanning tree
  - When dfs is used, the resulting spanning tree is known as a depth first spanning tree
  - When bfs is used, the resulting spanning tree is known as a breadth first spanning tree
- While adding a nontree edge into any spanning tree, this will create a cycle

DFS VS BFS Spanning Tree



DFS Spanning

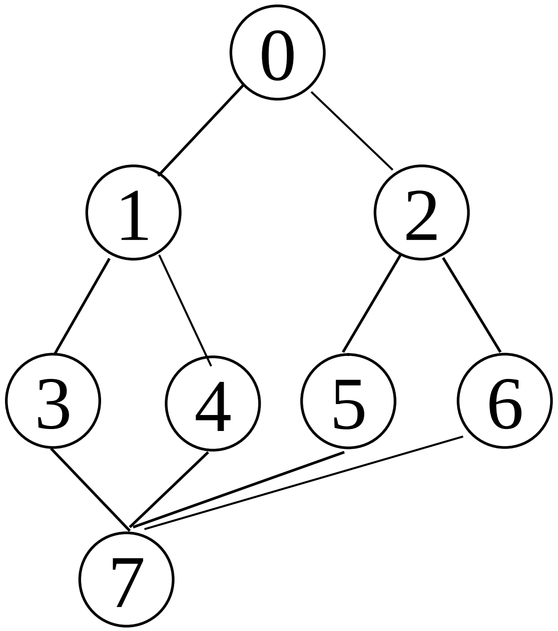
BFS Spanning

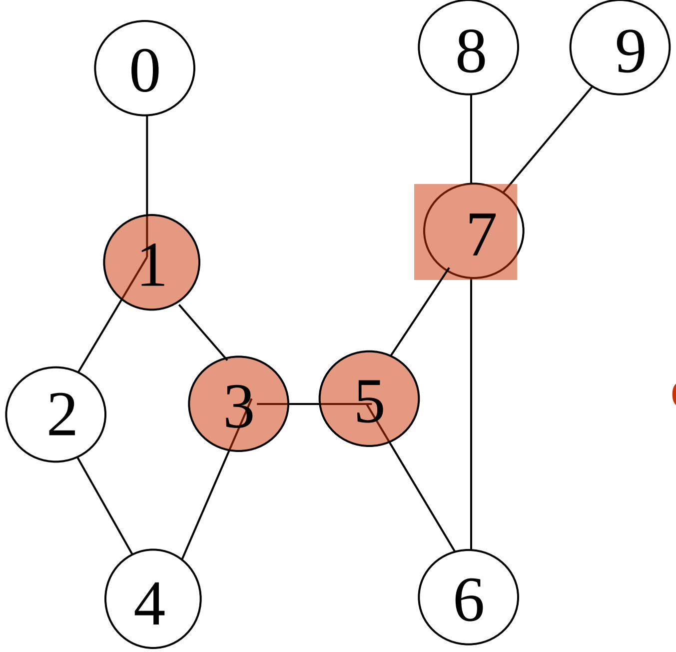
A spanning tree is a minimal subgraph, G’, of G such that V(G’)=V(G) and G’ is connected.

Any connected graph with n vertices must have at least n-1 edges.

A biconnected graph is a connected graph that has no articulation points.

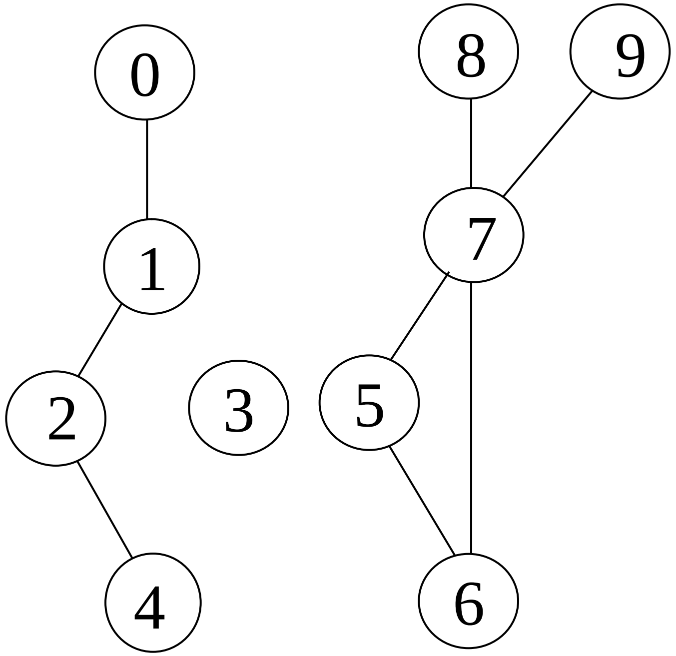
biconnected graph



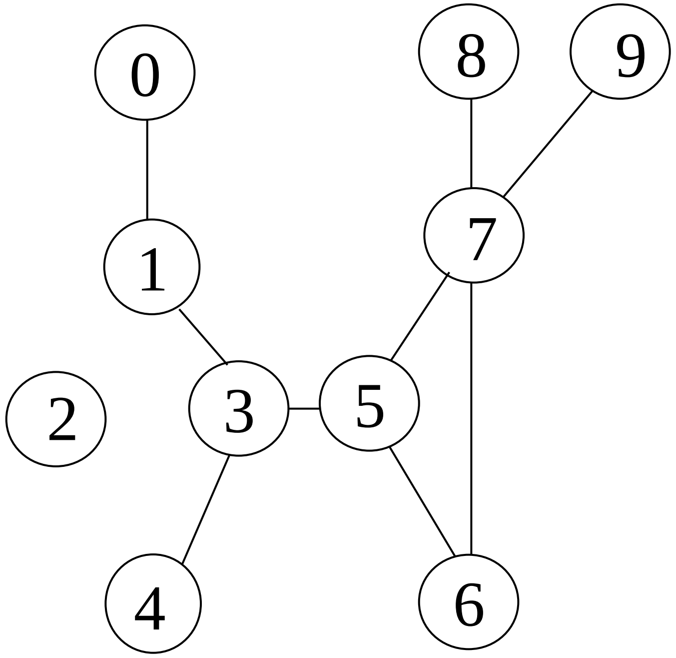


connected graph

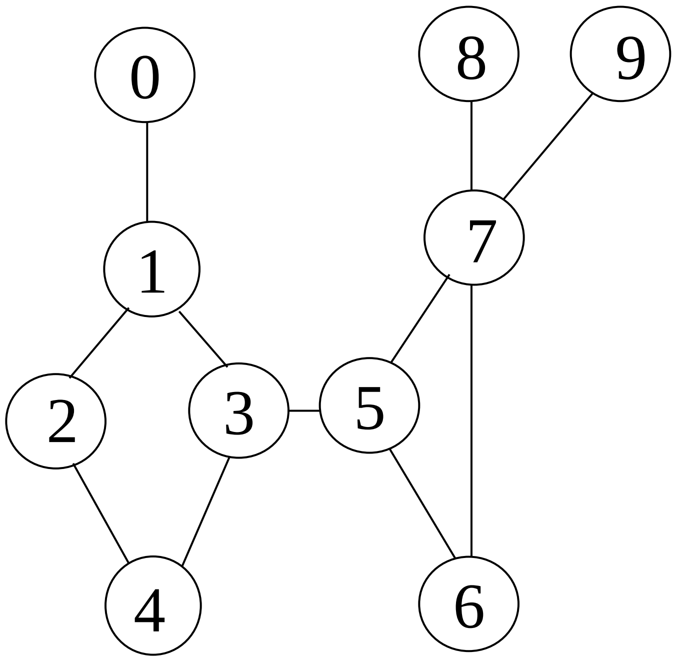
two connected components

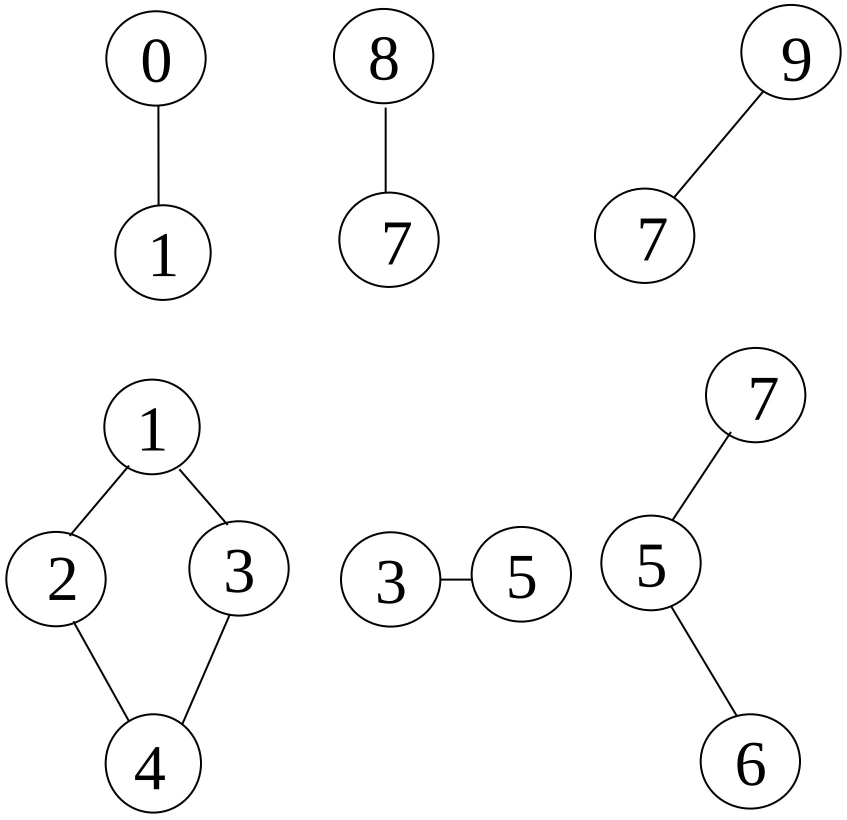


one connected graph



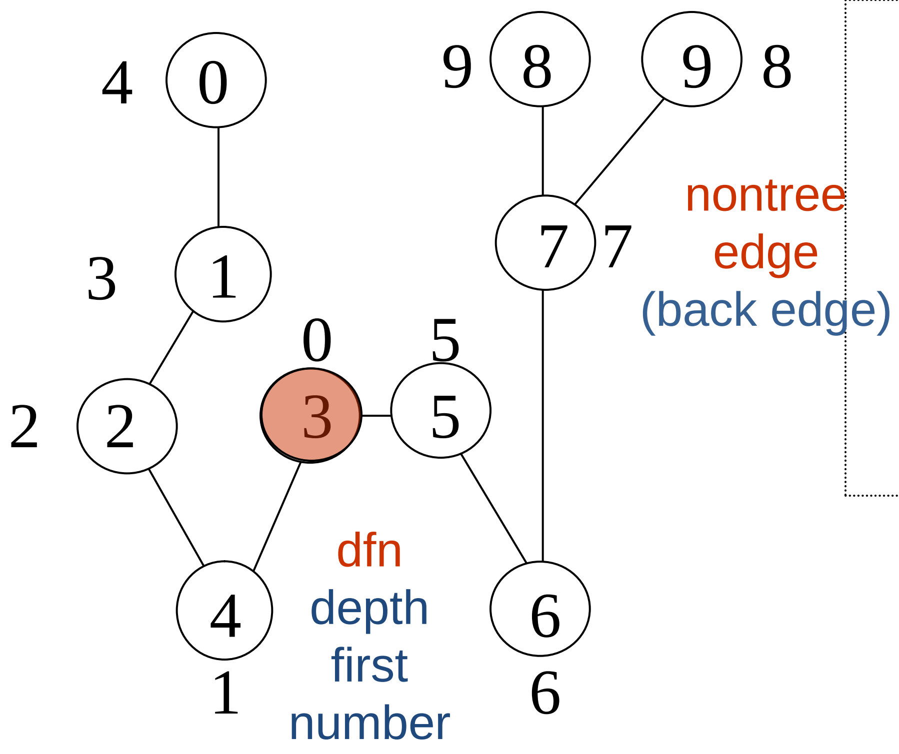
Biconnected component: a maximal connected subgraph H (no subgraph that is both biconnected and properly contains H)



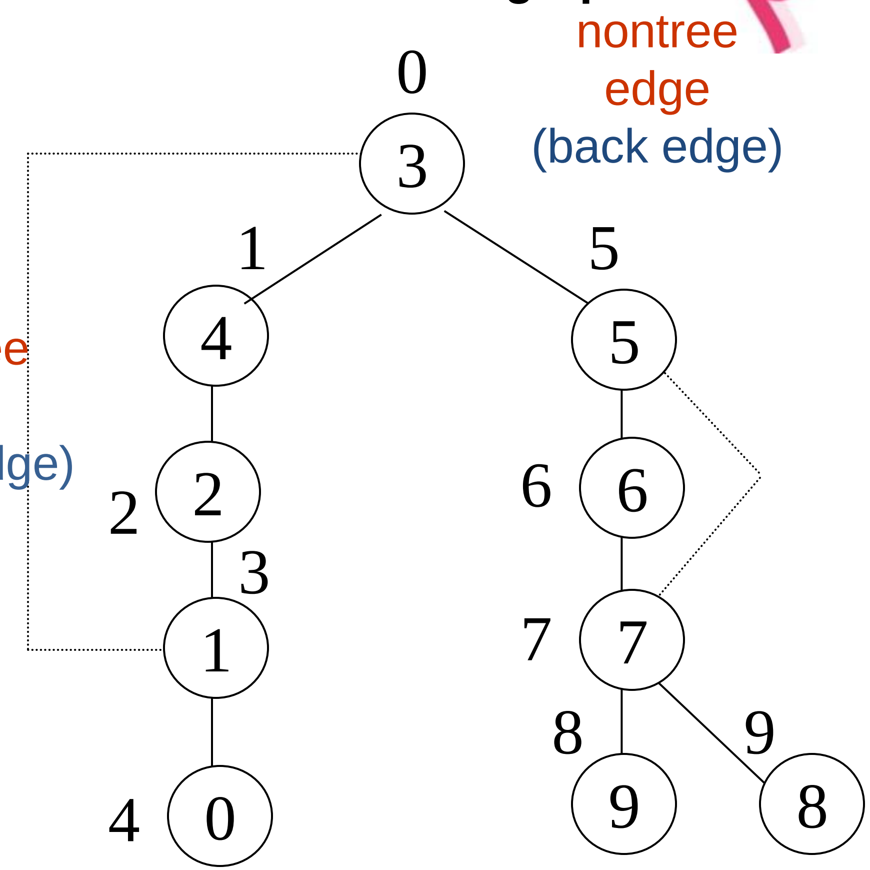


biconnected components

Find biconnected component of a connected undirected graph by depth first spanning tree



(a) depth first spanning tree



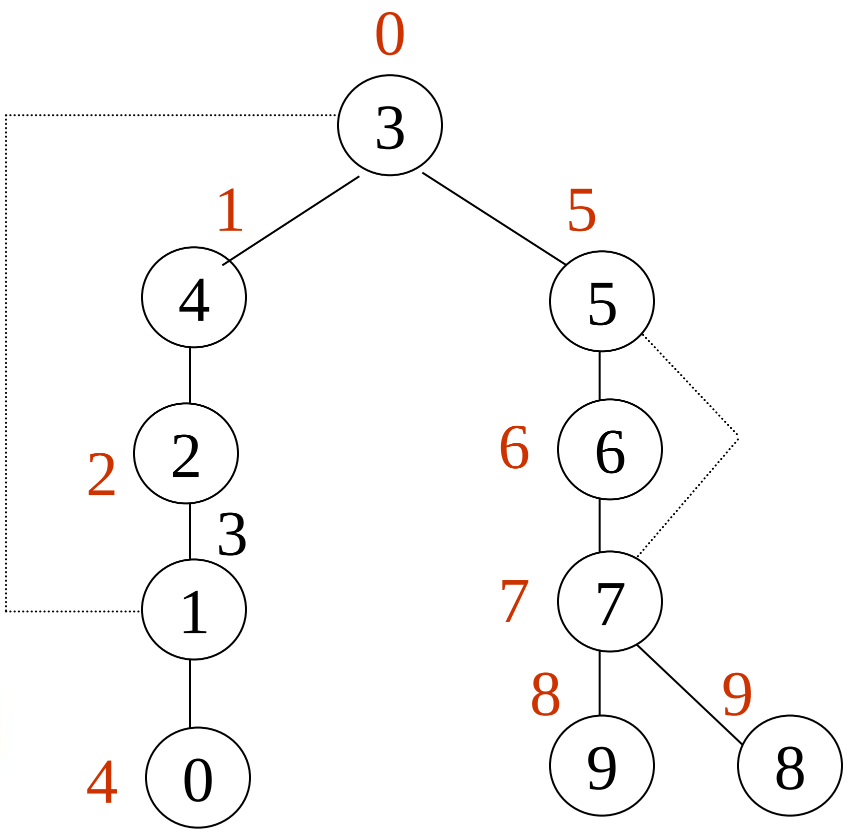
Why is cross edge impossible?

(b)

If u is an ancestor of v then dfn(u) < dfn(v).

dfn and *low* values for *dfs* spanning tree with *root* =3

Vertax	0	1	2	3	4	5	6	7	8	9
dfn	4	3	2	0	1	5	6	7	9	8
low	4	0	0	0	0	5	5	5	9	8



- The root of a depth first spanning tree is an articulation point iff it has at least two children.

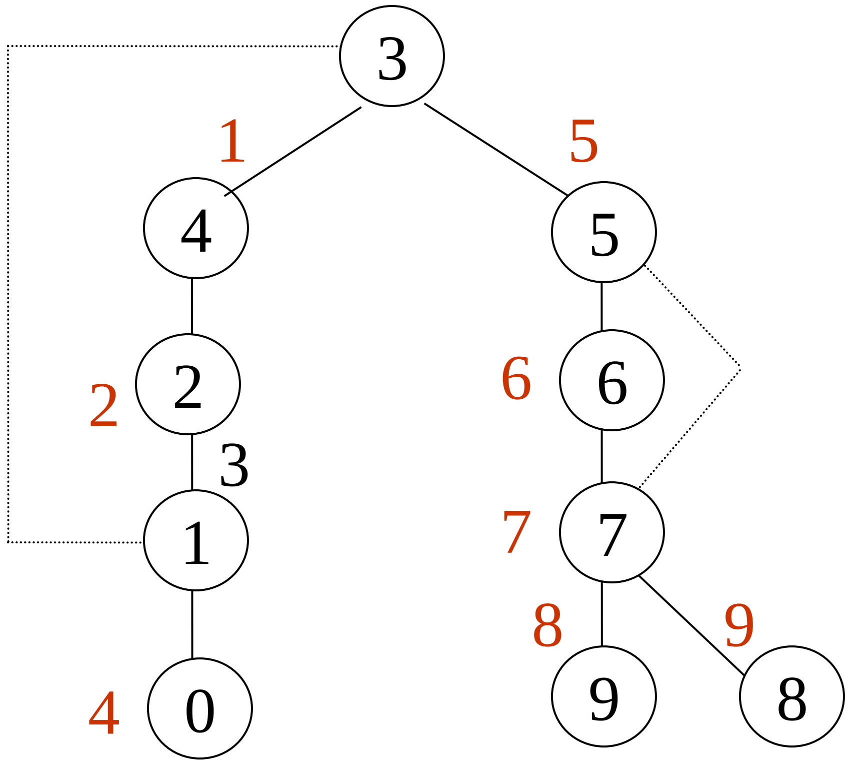
- Any other vertex u is an articulation point iff it has at least one child w such that we cannot reach an ancestor of u using a path that consists of

(1) only w (2) descendants of w (3) single back edge.

low(u)=min{dfn(u),
min{low(w)|w is a child of u},
min{dfn(w)|(u,w) is a back edge}

u: articulation point
low(child) $\geq$ dfn(u)

vertex	dfn	low	child	low_child	low:dfn
0	4	4 (4,n,n)	null	null	null:4
1	3	0 (3,4,0)	0	4	$4 \geq 3$ •
2	2	0 (2,0,n)	1	0	$0 < 2$
3	0	0 (0,0,n)	4,5	0,5	$0,5 \geq 0$ •
4	1	0 (1,0,n)	2	0	$0 < 1$
5	5	5 (5,5,n)	6	5	$5 \geq 5$ •
6	6	5 (6,5,n)	7	5	$5 < 6$
7	7	5 (7,8,5)	8,9	9,8	$9,8 \geq 7$ •
8	9	9 (9,n,n)	null	null	null, 9
9	8	8 (8,n,n)	null	null	null, 8


3
1
5
4
5
2
6
2
6
3
1
7
7
8
9
4
0
9
8

Minimum Cost Spanning Tree

- The cost of a spanning tree of a weighted undirected graph is the sum of the costs of the edges in the spanning tree
- A minimum cost spanning tree is a spanning tree of least cost
- Three different algorithms can be used
  - Kruskal
  - Prim
  - Sollin

Select n-1 edges from a weighted graph of n vertices with minimum cost.

Greedy Strategy

- An optimal solution is constructed in stages
- At each stage, the best decision is made at this time
- Since this decision cannot be changed later, we make sure that the decision will result in a feasible solution
- Typically, the selection of an item at each stage is based on a least cost or a highest profit criterion

Kruskal’s Idea

- Build a minimum cost spanning tree T by adding edges to T one at a time
- Select the edges for inclusion in T in nondecreasing order of the cost
- An edge is added to T if it does not form a cycle
- Since G is connected and has n > 0 vertices, exactly n-1 edges will be selected

Examples for Kruskal's Algorithm

0 —10— 5

2 —12— 3

1 —14— 6

1 —16— 2

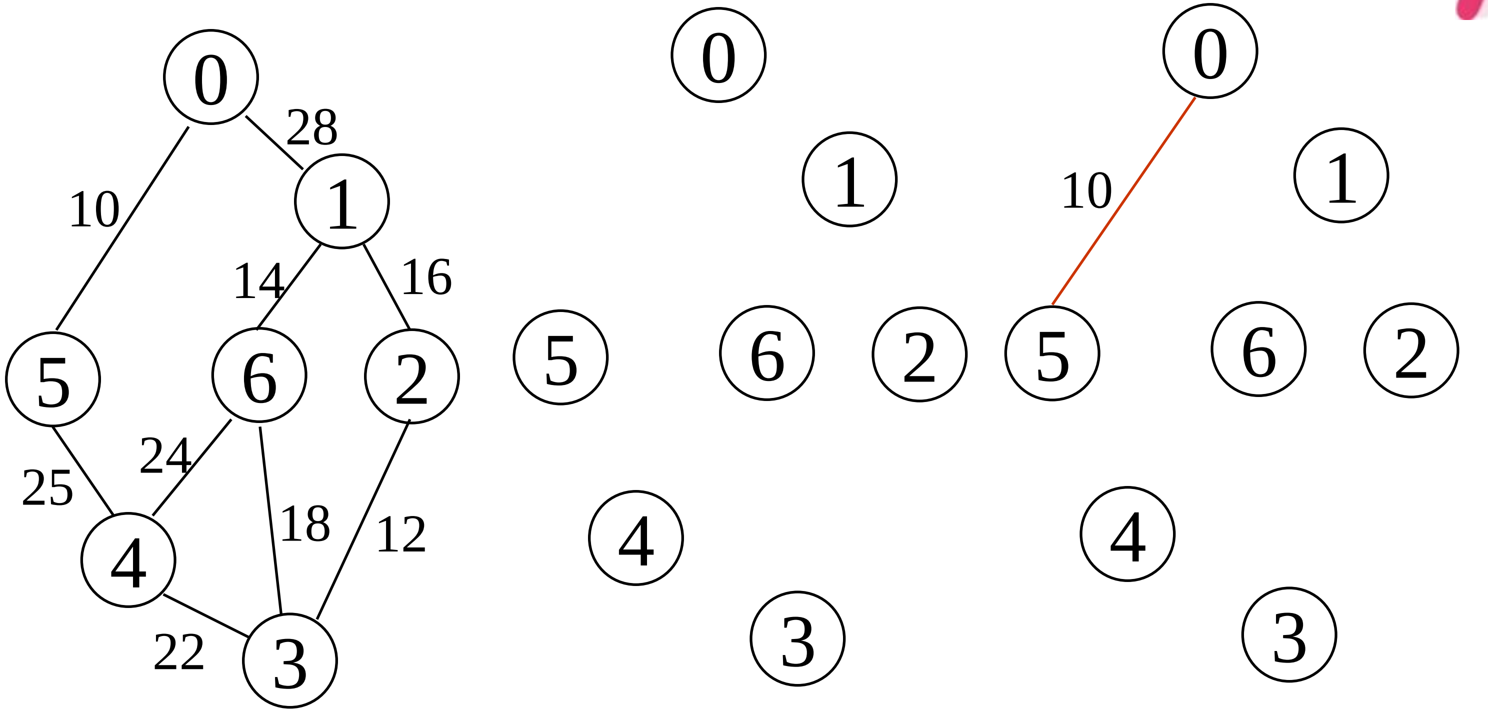
3 —18— 6

3 —22— 4

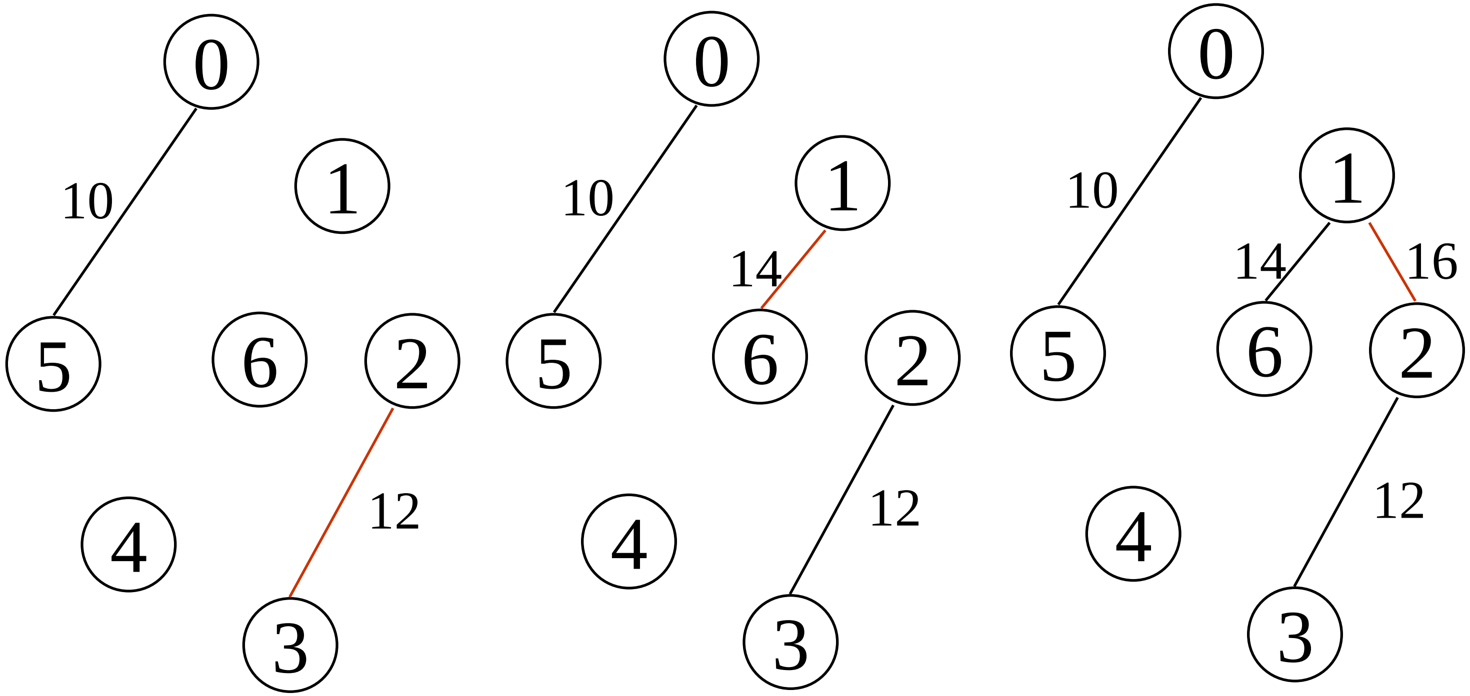
4 —24— 6

4 —25— 5

0 —28— 1



0	10	5
2	12	3
1	14	6
1	16	2
3	18	6
3	22	4
4	24	6
4	25	5
0	28	1



+ 3 — 6

cycle

0 —10— 5

2 —12— 3

1 —14— 6

1 —16— 2

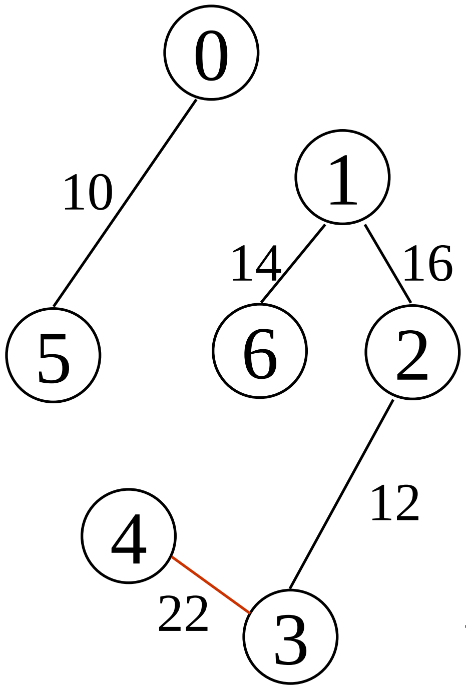
3 —18— 6

3 —22— 4

4 —24— 6

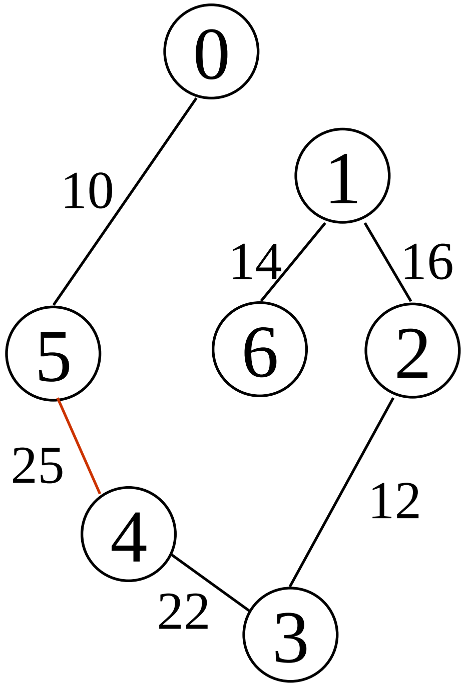
4 —25— 5

0 —28— 1



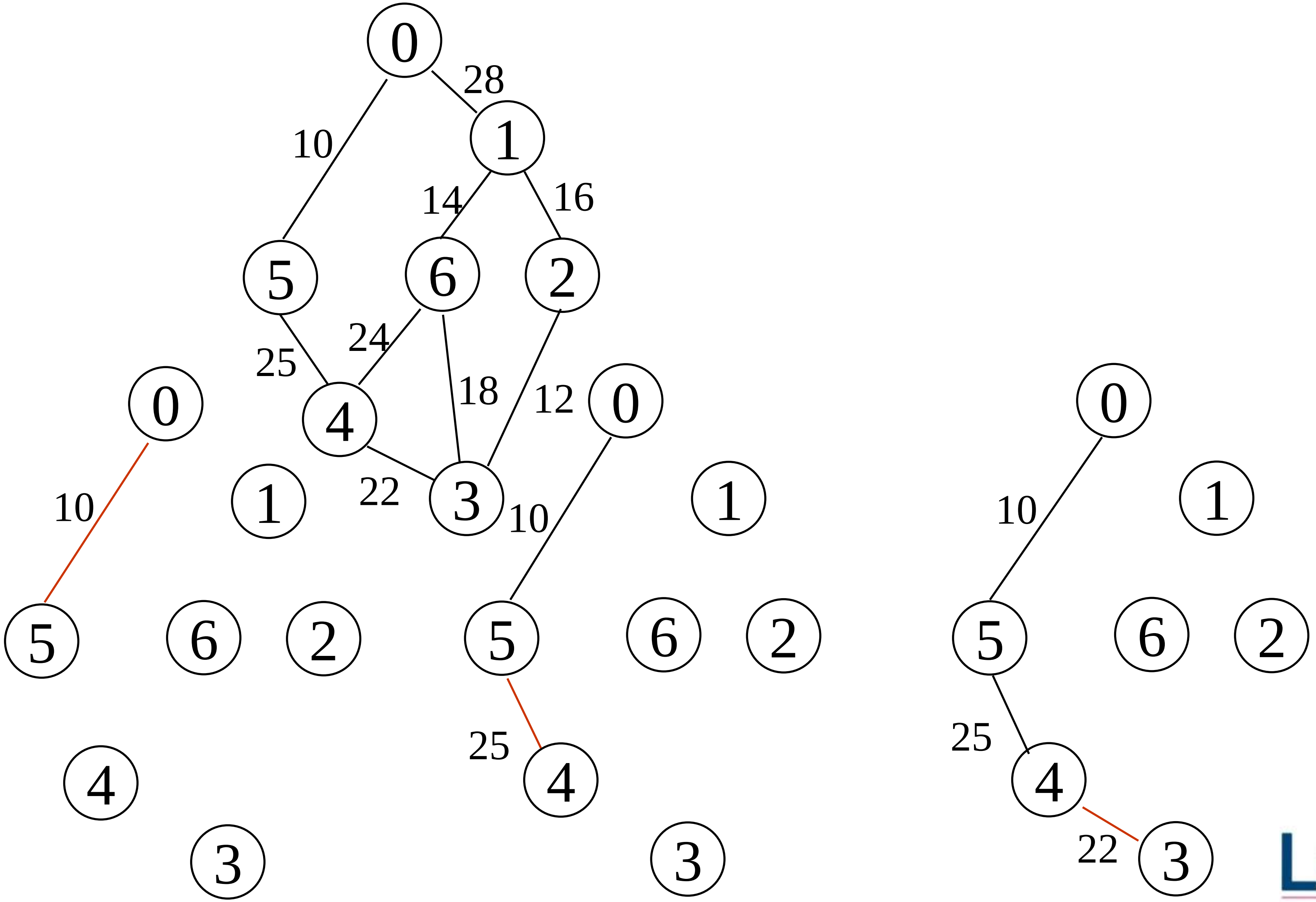
+ 4 —— 6

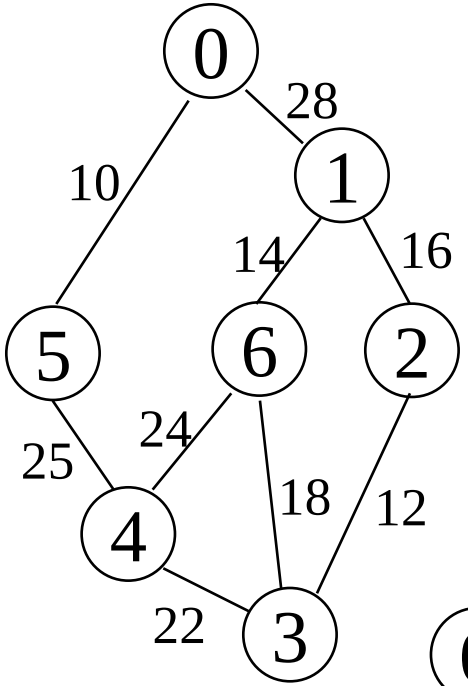
cycle

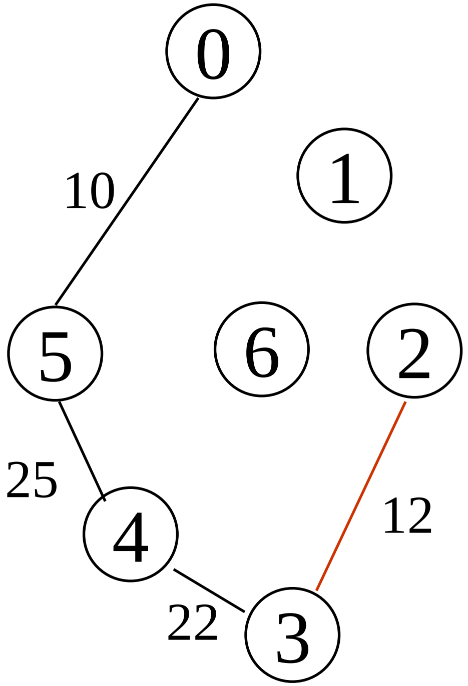


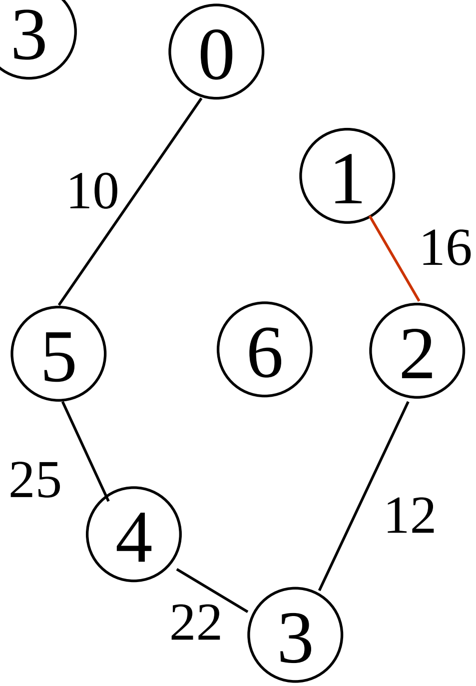
cost = 10 +25+22+12+16+14

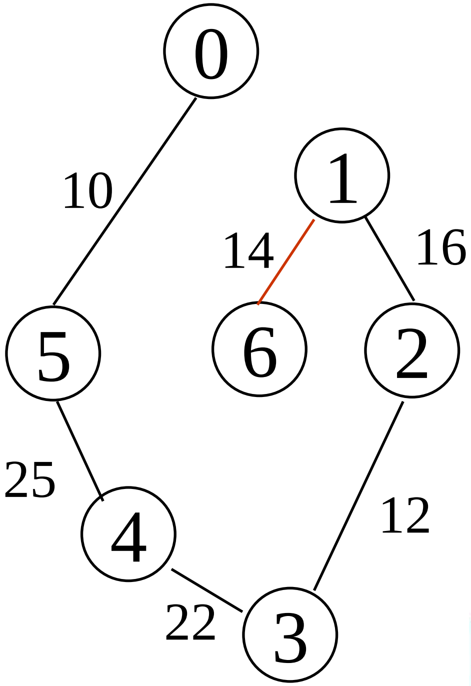
Examples for Prim’s Algorithm



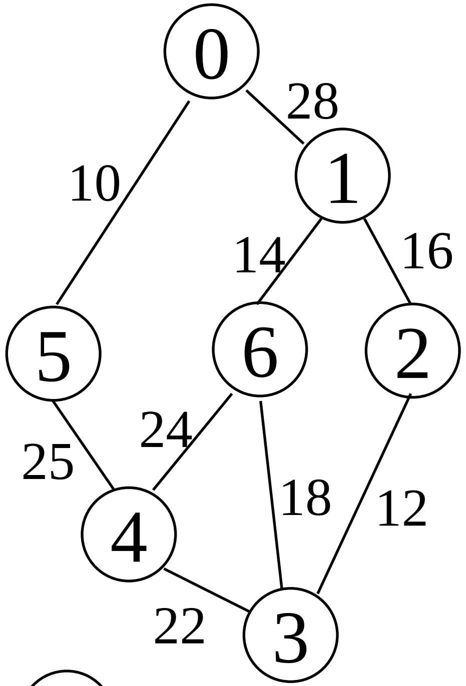

0
28
10
1
14
16
5
6
2
24
25
18
12
4
22
3


0
10
1
5
6
2
25
12
4
22
3

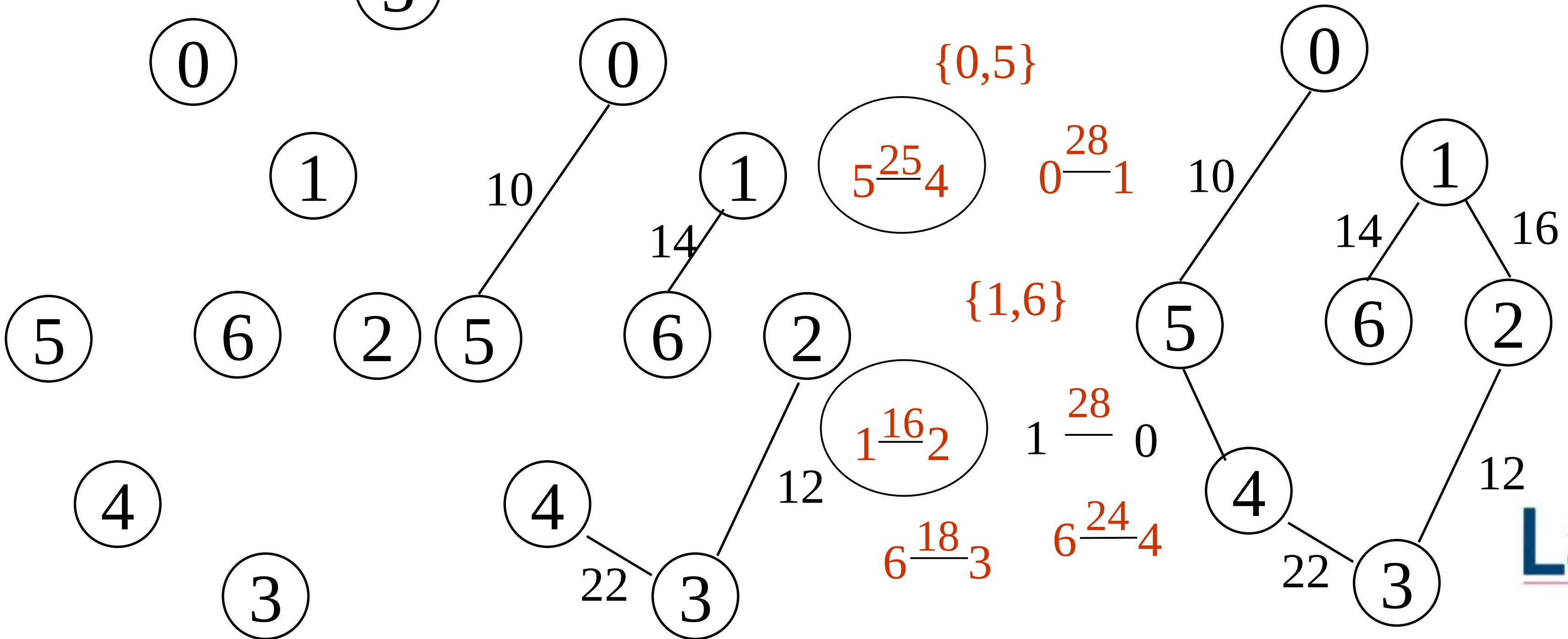

0
10
1
16
5
6
2
25
12
4
22
3


0
10
1
14
16
5
6
2
25
12
4
22
3

Sollin's Algorithm

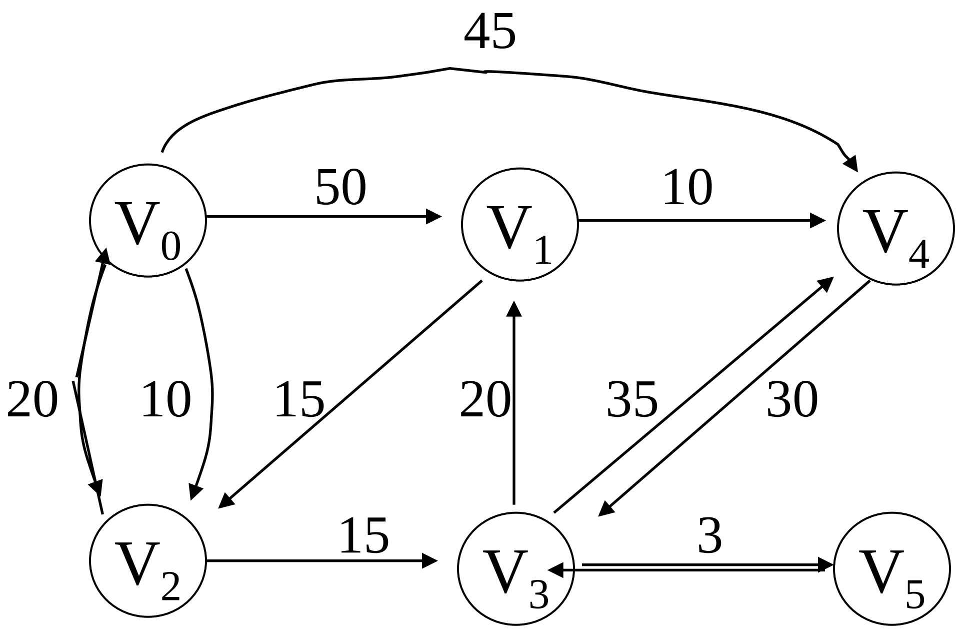


vertex	edge
0	0 -- 10 --> 5, 0 -- 28 --> 1
1	1 -- 14 --> 6, 1-- 16 --> 2, 1 -- 28 --> 0
2	2 -- 12 --> 3, 2 -- 16 --> 1
3	3 -- 12 --> 2, 3 -- 18 --> 6, 3 -- 22 --> 4
4	4 -- 22 --> 3, 4 -- 24 --> 6, 5 -- 25 --> 5
5	5 -- 10 --> 0, 5 -- 25 --> 4
6	6 -- 14 --> 1, 6 -- 18 --> 3, 6 -- 24 --> 4



Single Source All Destinations

Determine the shortest paths from v0 to all the remaining vertices.

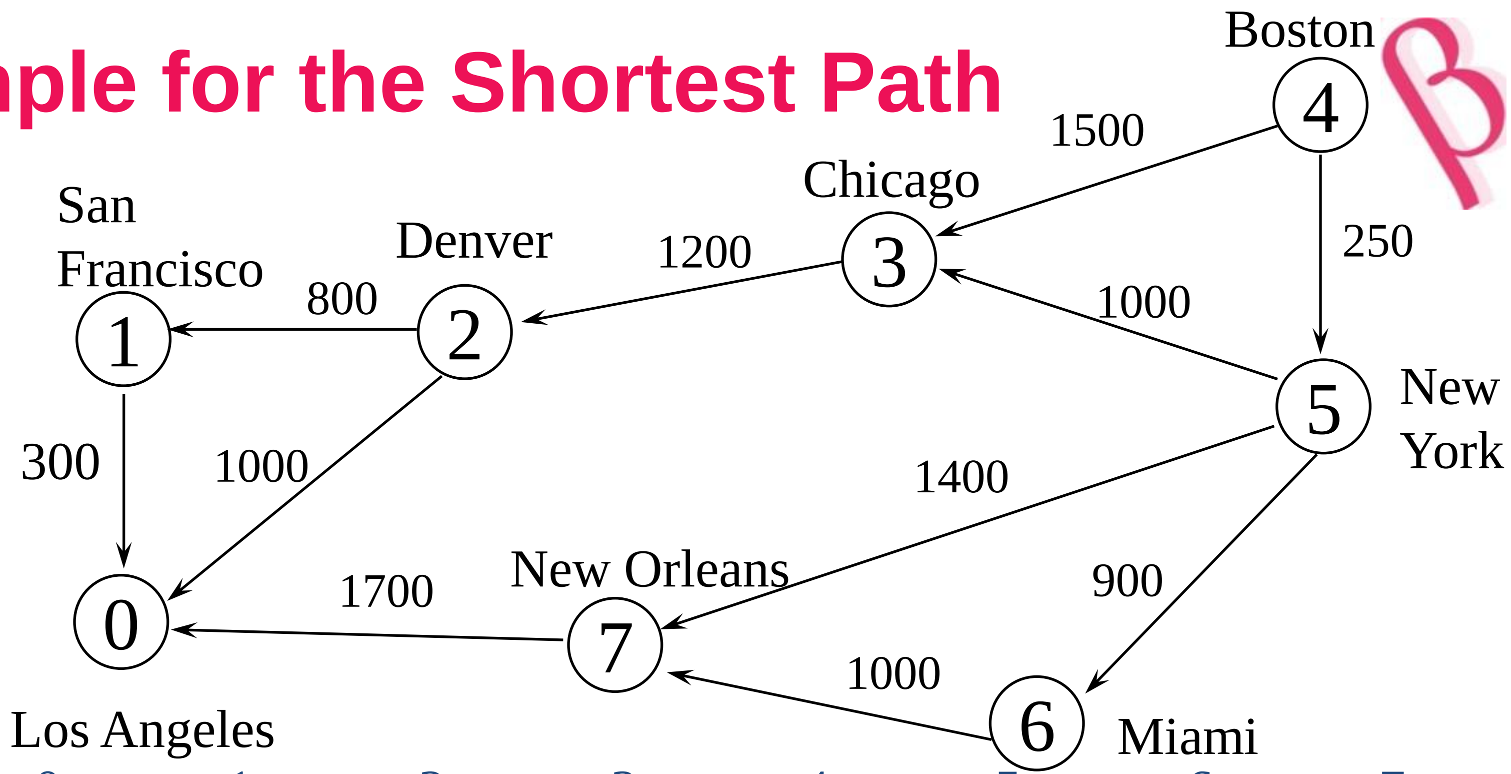


(a)

path	length
1) v0 v2	10
2) v0 v2 v3	25
3) v0 v2 v3 v1	45
4) v0 v4	45

(b)

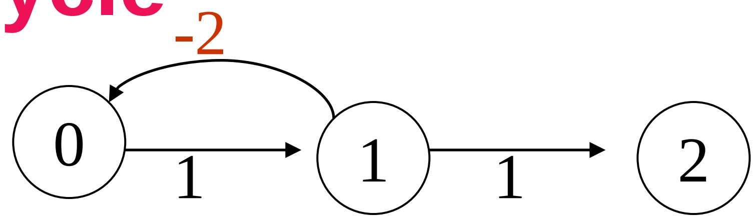
Example for the Shortest Path



	0	1	2	3	4	5	6	7
0	0							
1	300	0						
2	1000	800	0					
3			1200	0				
4				1500	0	250		
5				1000		0	900	1400
6							0	1000
7	1700							0

Cost adjacency matrix

Graph with negative cycle



$$\begin{bmatrix} 0 & 1 & \infty \\ -2 & 0 & 1 \\ \infty & \infty & 0 \end{bmatrix}$$

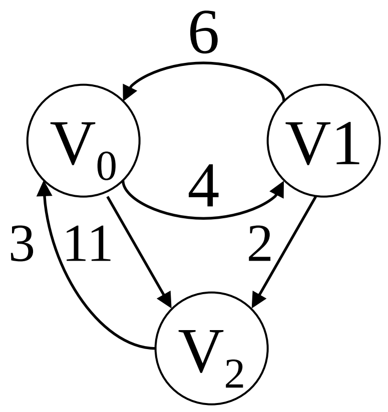
(a) Directed graph

(b) A^{-1}

The length of the shortest path from vertex 0 to vertex 2 is $-\propto$.

0, 1, 0, 1,0, 1, …, 0, 1, 2

Directed graph and its cost matrix

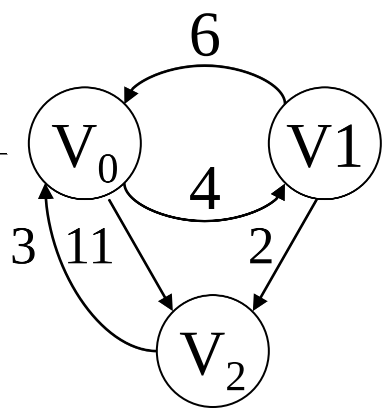


(a)Digraph G

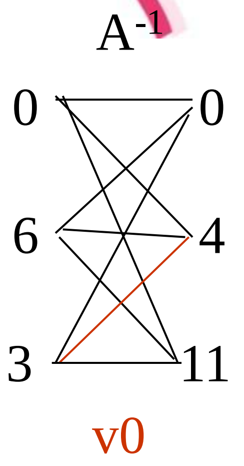
	0	1	2
0	0	4	11
1	6	0	2
2	3	∞	0

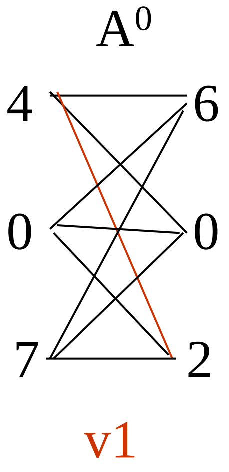
(b)Cost adjacency matrix for G

A^{-1}	0	1	2
0	0	4	11
1	6	0	2
2	3	∞	0



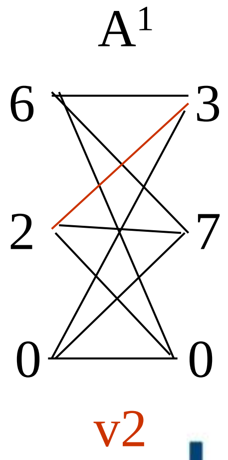
A^{0}	0	1	2
0	0	4	11
1	6	0	2
2	3	7	0





A^{1}	0	1	2
0	0	4	6
1	6	0	2
2	3	7	0

A^{2}	0	1	2
0	0	4	6
1	5	0	2
2	3	7	0

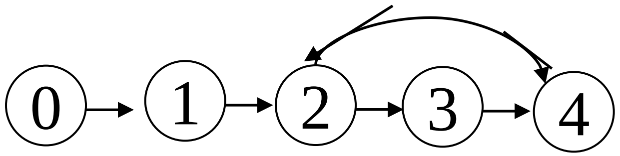


Transitive Closure

Goal: given a graph with unweighted edges, determine if there is a path from i to j for all i and j.

(1) Require positive path (> 0) lengths. transitive closure matrix

(2) Require nonnegative path (≥0) lengths. reflexive transitive closure matrix



(a) Digraph G

$$\begin{array}{c} 0\\1\\2\\3\\4 \end{array} \begin{bmatrix} 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(b) Adjacency matrix A for G

$$\begin{array}{c} 0\\1\\2\\3\\4 \end{array} \begin{bmatrix} 0 & 1 & 1 & 1 & 1\\ 0 & 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

cycle

(c) transitive closure matrix A+

There is a path of length > 0

$$\begin{array}{c} 0\\1\\2\\3\\4 \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 & 1\\ 0 & 1 & 1 & 1 & 1\\ 0 & 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

reflexive

(d) reflexive transitive closure matrix A*

There is a path of length ≥0

Thank You!