Problem 5: Error orders.

① Order of error w/ respect to Sine approximation:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Approximation: $\sin(1/n) \approx 1/n$.

Finding the error: (Approx. − Actual)

$$[1/n] - [1/n - 1/n^3 3! + 1/n^5 5! - 1/n^7 7! + \dots]$$

$$= 1/n^3 3! - 1/n^5 5! + 1/n^7 7! - \dots$$

As $n \to \infty$, the denominators get larger and larger. Thus, the terms get smaller and become less significant.

$1/n^3 3! > 1/n^5 5! > 1/n^7 7! > \dots$ (Terms get smaller)

we can observe that the greatest term is $1/n^3 3!$, given that it has the smallest denominator. Therefore, we can conclude that the greatest error comes from this term. This means that all other errors are bounded above by $1/n^3 3!$.

$\therefore$ Error of $\sin(1/n)$ = $O(1/n^3)$, for $n > 1$.

Next up: $\cos(x)$.

② order of error w/ respect to the cosine approx.:

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Approximation: $\cos(1/n) \approx 1 - 1/2n^2$.

Finding the error: (Actual − Approx.)

$$\left[1 - 1/n^2 2! + 1/n^4 4! - 1/n^6 6! + \cdots\right] - \left[1 - 1/n^2 2!\right]$$

$$= 1/n^4 4! - 1/n^6 6! + 1/n^8 8! - \cdots$$

As $n \to \infty$, the denominators get larger and larger. Thus, the terms get smaller and become less significant.

$$1/n^4 4! > 1/n^6 6! > 1/n^8 8! > \cdots \text{ (Terms get smaller.)}$$

we can observe that the greatest term is $1/n^4 4!$, given that it has the smallest denominator. Therefore, we can conclude that the greatest error comes from this term. This means that all other errors are bounded above by $1/n^4 4!$.

$\therefore$ Error of $\cos(1/n)$ = $O(1/n^4)$, for $n > 1$.

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