Chs. 9-11 Homework Cover Page:

Problems Done:

9.7: 5, 10, 15, 25, 31 (ALL)

9.8: 8, 16 (ALL)

9.9: 5, 10, 15, 25, 31 (ALL)

10.1: 8, 16, 24, 32, 40 (ALL)

10.5: 5, 10, 15, 25, 31 (ALL)

11.1: 5, 10, 15, 20, 25 (ALL)

11.2: 10, 20, 31, 40, 50 (ALL)

Problems Not Done: NONE

ch. 9 section 7 prob. 9

Prove algebraically that $\binom{n}{r} = \binom{n}{n-r}$, for integers n and r w/ $0 \le r \le n$.

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

Since $n-(n-r) = r$,

$r! = (n-(n-r))!$ and then,

$$\frac{n!}{(n-r)!\,r!} = \frac{n!}{(n-r)!\,(n-(n-r))!}$$

$$= \binom{n}{n-r}$$

Ch. 9 Sech 7 Prob. 10

Use Pascal's triangle to compute

$\binom{6}{2}, \binom{6}{3}, \binom{6}{4},$ and $\binom{6}{5}$.

$\binom{6}{2} = \binom{5}{1} + \binom{5}{2} = 10 + 5 = 15$

$\binom{6}{3} = \binom{5}{2} + \binom{5}{3} = 10 + 10 = 20$

$\binom{6}{4} = \binom{5}{3} + \binom{5}{4} = 10 + 5 = 15$

$\binom{6}{5} = \binom{5}{4} + \binom{5}{5} = 5 + 1 = 6$

Compute $\binom{7}{3}, \binom{7}{4},$ and $\binom{7}{5}$:

$\binom{7}{3} = \binom{6}{2} + \binom{6}{3} = 15 + 20 = 35$

$\binom{7}{4} = \binom{6}{3} + \binom{6}{4} = 20 + 15 = 35$

$\binom{7}{5} = \binom{6}{4} + \binom{6}{5} = 15 + 6 = 21$

For $n = 7$, the values of the triangle are:

1 7 21 35 35 21 7 1

Ch. 9 Sect. 7 Prob. 15

Prove: Let r be a fixed nonnegative integer. For all integers n w/ $r \leq n$,

$$\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}.$$

Proof: Let r be a fixed nonnegative integer and n be an integer w/ $r \leq n$.

$$\sum_{i=r}^{n} \binom{i}{r} = \binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \dots + \binom{r+k}{r},$$

where $r+k = n$. Since $\binom{r}{r} = \binom{r+1}{r+1} = 1$, we can replace $\binom{r}{r}$ w/ $\binom{r+1}{r+1}$. Then,

$$\left[\binom{r+1}{r+1} + \binom{r+1}{r}\right] + \binom{r+2}{r} + \dots + \binom{r+k}{r}$$

$$\left[\binom{r+2}{r+1} + \binom{r+2}{r}\right] + \dots + \binom{r+k}{r}$$

$$\left[\binom{r+3}{r+1} + \dots + \binom{r+k}{r}\right]$$

$$\dots \binom{r+k}{r+1} + \binom{r+k}{r}$$

$$\binom{r+k+1}{r+1} = \binom{n+1}{r+1}$$

Ch. 9 Sect. 7 Prob. 25

Use the binomial theorem to expand.

$\left(x + \frac{1}{x}\right)^5$. $a = x$

$b = 1/x$.

$\rightarrow \sum_{i=0}^{5} \binom{5}{i} a^{5-i} b^{i}$

$= \binom{5}{0} a^5 + \binom{5}{1} a^4 b^1 + \binom{5}{2} a^3 b^2$

$+ \binom{5}{3} a^2 b^3 + \binom{5}{4} a^1 b^4 + \binom{5}{5} b^5$.

$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

$= x^5 + 5(x^4)(1/x) + 10(x^3)(1/x)^2$
$+ 10(x^2)(1/x)^3 + 5(x)(1/x)^4 + (1/x)^5$

$= x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5}$.

Ch. 9 Sect 7 Prob. 31

Find the coefficient of the given term when the expression is expanded by the binomial theorem.

$a^5 b^7$ in $(a-2b)^{12}$ $\quad (a=a, \; b=-2b)$

$(a-2b)^{12} =$

$$\binom{12}{0}a^{12}b^0 + \binom{12}{1}a^{11}b^1 + \binom{12}{2}a^{10}b^2 + \binom{12}{3}a^9b^3$$

$$+ \binom{12}{4}a^8b^4 + \binom{12}{5}a^7b^5 + \binom{12}{6}a^6b^6 + \binom{12}{7}a^5b^7$$

$$+ \binom{12}{8}a^4b^8 + \binom{12}{9}a^3b^9 + \binom{12}{10}a^2b^{10} + \binom{12}{11}a^1b^{11}$$

$$+ \binom{12}{12}a^0b^{12}.$$

Coefficient of $a^5b^7 = \binom{12}{7}a^5(-2b)^7$

$= 792a^5 \cdot (-128b^7)$

$= -101,376\, a^5b^7$

So, the coefficient is ...

$-101,376$.

ch. 9 sect. 8 prob. 8

Suppose a sample space S consists of 3 outcomes: 0, 1, and 2. Let $A = \{0\}$, $B = \{1\}$, and $C = \{2\}$, and suppose $P(A) = 0.5$ and $P(B) = 0.4$. Find the following:

a. $P(A \cup B)$ b. $P(C)$ c. $P(A \cup C)$

d. $P(A^c)$ e. $P(A^c \cap B^c)$ f. $P(A^c \cup B^c)$

a. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.5 + 0.4 - 0 \quad = 0.9$

b. $P(C) \quad = 1 - 0.5 - 0.4 = 0.1$

c. $P(A \cup C) = P(A) + P(C) - P(A \cap C)$

$= 0.5 + 0.1 - 0 \quad = 0.6$

d. $P(A^c) = 1 - P(A) = 1 - 0.5 = 0.5$

e. $P(A^c \cap B^c) = P((A \cup B)^c)$

$= 1 - P(A \cup B)$

$= 1 - 0.9 = 0.1$

f. $P(A^c \cup B^c) = P((A \cap B)^c)$

$= 1 - P(A \cap B)$

$= 1 - 0 \quad = 1.$

Ch. 9 Sect 8 Prob. 16

An urn contains 4 balls numbered 2, 2, 5, and 6. If a person selects a set of 2 balls at random, what is the expected value of the sum of the numbers on the balls?

All different ways to take 2 balls:

$\{2_1, 2_2\}$, sum = 4

$\{2_1, 5\}$ and $\{2_2, 5\}$, sum = 7

$\{2_1, 6\}$ and $\{2_2, 6\}$, sum = 8

$\{5, 6\}$, sum = 11.

There are a total of 6 sums w/ the 6 different ways to take 2 balls.

For a sum of 4, that is 1/6 of the sums.

For a sum of 7, that is 2/6 of the sums.

For a sum of 8, that is 2/6 of the sums.

For a sum of 11, that is 1/6 of the sums.

→ Expected value:

$4 \cdot 1/6 + 7 \cdot 2/6 + 8 \cdot 2/6 + 11 \cdot 1/6$

$= 7.5$.

Ch. 9 Sect. 9 Prob. 5

Suppose that A and B are events in a sample space S and that $P(A)$, $P(B)$, and $P(A|B)$ are known. Derive a formula for $P(A|B^c)$.

$$P(A|B^c) = \frac{P(A \cap B^c)}{P(B^c)}.$$

Since $A = (A \cap B) \cup (A \cap B^c)$, then

$$P(A) = P((A \cap B) \cup (A \cap B^c))$$
$$= P(A \cap B) + P(A \cap B^c).$$ Then,

$$P(A|B^c) = \frac{P(A) - P(A \cap B)}{P(B^c)}$$

$$= \frac{P(A) - P(A \cap B)}{1 - P(B)},$$ since $P(B^c) = 1 - P(B)$.

Moreover, $P(A|B) = \frac{P(A \cap B)}{P(B)}$, so

$$P(A \cap B) = P(A|B) \cdot P(B).$$

Finally,

$$P(A|B^c) = \frac{P(A) - P(A|B)P(B)}{1 - P(B)}.$$

Ch. 9 Sect. 9 Prob. 10

Prove Bayes' Theorem.

Proof. Suppose $A_1, A_2, \ldots, A_n$ are mutually disjoint events in a sample space S, and suppose that all A_i have a nonzero probability. Suppose B is an event in S w/ a nonzero probability. (S is the union of the disjoint events).

Then, $P(B) = P(B \cap S)$

↳ This makes sense because B is an event in S.

$= P(B \cap \bigcup_{i=1}^{n} A_i)$ (P of a union is the sum of the probabilities)

$= \sum_{i=1}^{n} P(B \cap A_i)$

($P(B|A) = P(A \cap B) / P(A)$, so $P(A \cap B) = P(B|A)P(A)$)

$\rightarrow \sum_{i=1}^{n} P(B|A_i)P(A_i)$. Then, for

$$P(A_k|B) = \frac{P(A_k \cap B)}{P(B)}$$

$$= \frac{P(B|A_k)P(A_k)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$$

∘ $P(B|A_k) = \frac{P(A_k \cap B)}{P(A_k)}$, so $P(A_k \cap B) = P(B|A_k)P(A_k)$.

Ch. 9 Sect 9 Prob. 15

2 different factories both produce a certain automobile part. The probability that a component from the 1st factory is defective is 2%, and the probability that a component from the 2nd factory is defective is 5%. In a supply of 180 of the parts, 100 were from the 1st and 80 were from the 2nd.

a. what is the prob. that a part chosen from the 180 is from the 1st factory? $100/180 = 5/9$.

b. what is the prob. that a part chosen from the 180 is from the 2nd factory? $80/180 = 4/9$.

c. what is the prob. that a part chosen at random from the 180 is defective?

$$P(\text{defective}) = \frac{100}{180} \cdot 0.02 + \frac{80}{180} \cdot 0.05$$

$$= \frac{5}{9} \cdot \frac{2}{100} + \frac{4}{9} \cdot \frac{5}{100} = \frac{30}{900} = \frac{1}{30}.$$

d. If the chosen part is defective, what is the prob. that it came from the 1st factory?

$$P(1^{st} \text{ factory} \mid \text{defective}) = \frac{P(\text{defective} \mid 1^{st} \text{ factory}) \cdot P(1^{st} \text{ factory})}{P(\text{defective} \mid 1^{st} \text{ factory})P(1^{st}) + P(\text{defective} \mid 2^{nd})P(2^{nd})}$$

$$= \frac{0.02 \cdot 100/180}{((0.02 \cdot 100/180) + (0.05 \cdot 80/180))}$$

$$= \frac{1/90}{1/30} = \frac{30}{90} = \frac{1}{3}.$$

$$* \; P(B_k \mid A) = \frac{P(A \mid B_k)P(B_k)}{P(A \mid B_1)P(B_1) + \cdots + P(A \mid B_n)P(B_n)}$$

ch. 9 Sect. 9 Prob. 25

A coin is loaded so that the prob. of heads is 0.7 and the prob. of tails is 0.3. Suppose that the coin is tossed twice and that the results of the tosses are independent.

a. what is the prob. of obtaining exactly 2 heads?

$P(H_1 \cap H_2) = 0.7 \cdot 0.7$

2 heads for 2 tosses $= 0.49$ or 49%.

b. what is the prob. of obtaining exactly 1 head?

$P((H_1 \cap T_2) \cup (T_1 \cap H_2)) = P(H_1 \cap T_2) + P(T_1 \cap H_2)$

$= P(H_1)P(T_2) + P(T_1)P(H_2)$

$= (0.7)(0.3) + (0.3)(0.7)$

$= 0.21 \cdot 2 = 0.42$ or 42%.

c. what is the probability of no heads?

$= 1 - P(1 \text{ head}) - P(2 \text{ heads})$

$= 1 - 0.42 - 0.49 = 0.09$ or 9%. OR...

$P(T_1 \cap T_2) = P(T_1)P(T_2) = (0.3)(0.3) = 0.09$.

No heads

d. what is the prob. of at least 1 head?

$P(1 \text{ head}) + P(2 \text{ heads}) =$

$0.42 + 0.49 = 0.91$ or 91%.

Ch. 9 Sect. 9 Prob. 31

Empirical data indicate that approximately 103 out of every 200 children born are male. Hence, the prob. of a newborn being male is about 51.5%.

Suppose that a family has 6 children and suppose that the genders of all children are mutually independent.

a, Prob. that none are male? (All female).

$P(\text{female}) = 1 - P(\text{male}) = 1 - 0.515 = 0.485$.

$P(\text{All female}) = (0.485)^6 \cong 1.3\%$

b. Prob. that at least 1 is male?

$P(\text{at least 1 male}) = 1 - P(\text{none are male})$

$= 1 - 0.013 = 0.987$.

c, Prob. that exactly 5 are male?

$P(5 \text{ males}) = 6 \cdot (0.515)^5 \cdot (0.485)$

$\approx 10.54\%$

If we have 6 children, and we want to take 5 to be male, there are a total of $6C5$ ways to do this.

$6C5 = 6$ combinations.

Then, $(0.515)^5$ is the prob. of 5 males

and (0.485) is the prob. of 1 female.

Ch. 10 Sect. 1 Prob. 8

i. Find all edges incident on v_1.

e_1, e_2, and e_3

ii. Find all vertices adjacent to v_3.

v_1, v_2, and v_3

iii. Find all edges adjacent to e_1.

e_2, e_3, e_8, and e_9

iv. Find all loops.

e_6 and e_7

v. Find all parallel edges.

e_8, e_9, e_4, and e_5

vi. Find all isolated vertices.

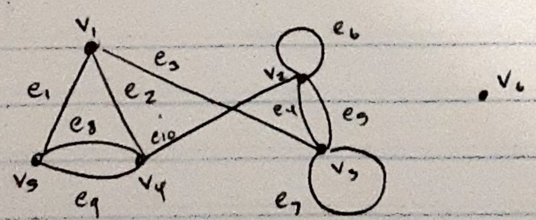
v_6

vii. Find the degree of v_3.

5

viii. Total degree of graph?

$20 = 2 \cdot 10$ edges



Ch. 10 sect 1 Prob. 16.

A graph has vertices of degrees 1, 1, 4, 4, and 6.
How many edges does the graph have?

Total degree $= 1+1+4+4+6 = 16$

$16 = 2 \cdot \#$ of edges

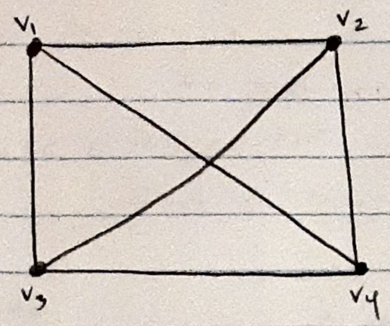
So, $\#$ of edges $= 16/2$
$= 8.$

$\therefore$ the graph has 8 edges.

Ch. 10 Sect. 1 Prob. 24

Either draw the graph or explain why no such graph exists.

simple graph w/ 6 edges and all vertices of degree 3.



Ch. 10 sect. 1 prob. 32

Deduce that for any positive integer n, if there is a sum of n odd integers that is even, then n is even.

Proof:

Suppose n is odd, and suppose that the sum of n odd integers is even.

The sum of an odd number of odd integers is odd. This contradicts the fact that the sum is even.

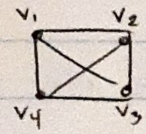
$\therefore$ n is even.

Ch. 10 Sect. 1 Prob. 40

a. Find the complement of the graph K_4, the complete graph on 4 vertices.

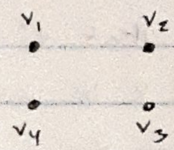
* If G is a simple graph, the complement of G is obtained as follows: The vertex set of G' is identical to the vertex set of G. However, 2 distinct vertices v and w of G' are connected by an edge iff v and w are not connected by an edge in G.

$K_4 =$



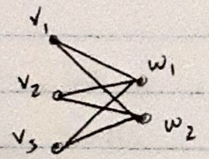
so

$K_4' =$



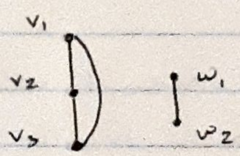
b. Find the complement of the graph $K_{3,2}$.

$K_{3,2} =$



so,

$K_{3,2}' =$



Ch. 10 sect. 5 prob. 9

Show that a tree w/ more than 1 vertex has at least 2 vertices of degree 1.

Proof: Let T be a tree w/ more than 1 vertex.

Step 1: Choose a vertex v of T and let e be an edge incident on v. Let v_0 and e_0 be the 1st chosen.

Step 2: while $\deg(v) > 1$, repeat 2a, 2b, and 2c.

2a: choose e' to be an edge incident on v such that $e' \neq e$.

2b: Let v' be the vertex at the other end of e' from v.

2c: $e = e'$ and $v = v'$.

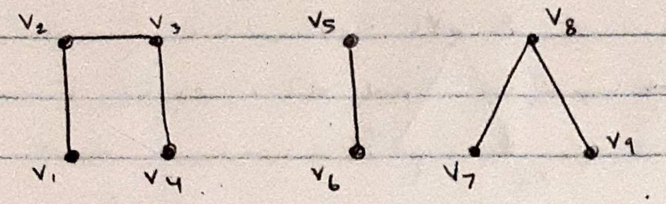
Step 3: Once 1 vertex of degree 1 is found, go back to v_0 and search for another vertex of degree 1 by moving outward of e_0. (Repeat step 2 w/ v_0 and e_0)

DONE.

Ch. 10 sect. 5 Prob. 10

Draw a graph or explain why no such graph exists.

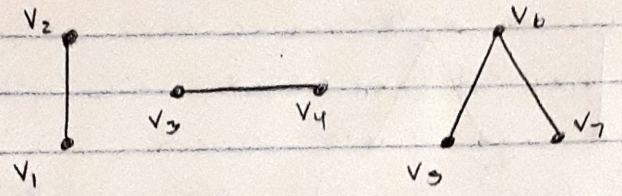
Graph, circuit-free, 9 vertices, 6 edges.



Ch. 10 Sect. 5 Prob. 13

Draw a graph or explain why no such graph exists.

Graph, circuit-free, 7 vertices, 4 edges.



ch. 10 Sect. 5 prob. 25

A graph has 8 vertices and 6 edges. Is it connected?

No, if this were a tree w/ 8 vertices, then it would need to have 7 edges. It has 6, so there is no such connected graph w/ 8 vertices and 6 edges.

Ch. 10 Sect. 5 Prob. 31

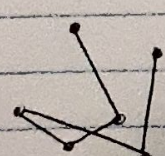
a. Prove that the following is an invariant for graph isomorphism: A vertex of degree i is adjacent to a vertex of degree j.

Proof: Let G be a graph w/ more than 1 vertex and let H be a graph that is isomorphic to G. Then,

By the fact that H is isomorphic to G, H and G have the same degrees on their vertices. This means that a vertex of degree i that is adjacent to a vertex of degree j in G is also true for vertices in H. Thus, this is an invariant of graph isomorphism.

b. Find all nonisomorphic trees w/ 6 vertices.
(A tree w/ 6 vertices has 5 edges).
(The total degree would have to be 10).
(A tree w/ more than 1 vertex has atleast 2 vertizes of degree 1).

① 1,1,2,2,2,2.
② 1,1,1,1,2,4.
③ 1,1,1,1,1,5
④ 1,1,1,1,3,3
⑤ 1,1,1,2,2,3
⑥ 1,1,1,3,2,2

①

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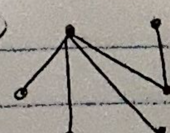
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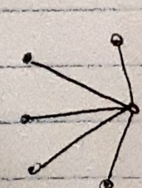

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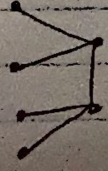
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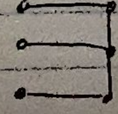
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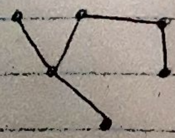
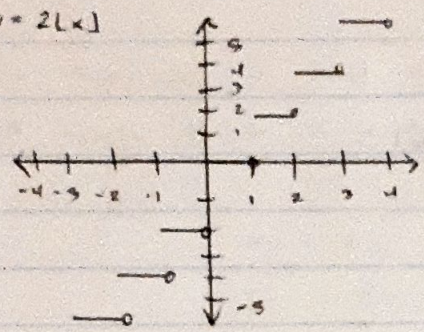


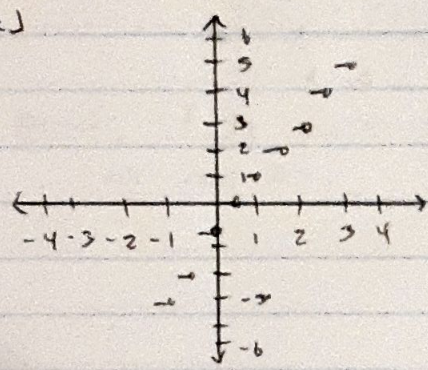
ch. 11 Sect. 1 Prob. 5

Draw the graphs of $y = 2\lfloor x \rfloor$ and $y = \lfloor 2x \rfloor$ for all real #s x. What can you conclude?

$y = 2\lfloor x \rfloor$



$y = \lfloor 2x \rfloor$



we can conclude that

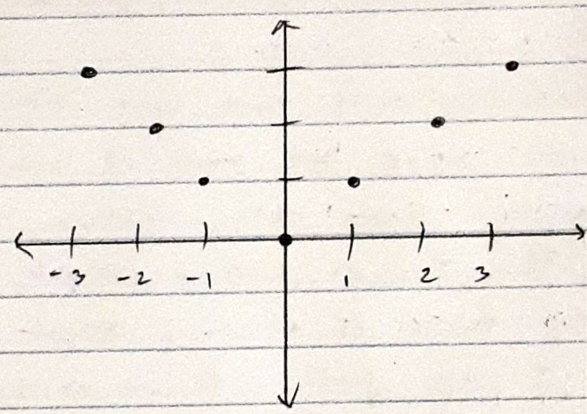
$2\lfloor x \rfloor \neq \lfloor 2x \rfloor$ for many x.

Ch.11 Sect. 1 Prob. 10

Graph: (Defined on a set of integers)

$f(n) = |n|$ for each integer n.

$f(n) = |n|$.



Ch. 11 Sect. 1 Prob. 15

Show that the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x - 3$ is increasing on the set of real #s.

Proof. Suppose x_1 and x_2 are any real #s such that $x_1 < x_2$. Then,

$x_1 < x_2$ then

$2x_1 < 2x_2$ and

$2x_1 - 3 < 2x_2 - 3$. So

$f(x_1) < f(x_2)$.

Hence, f is increasing on the set of all real #s.

Ch. 11 Sect. 1 Prob. 20

Given real-valued functions f and g w/ the same domain D, the sum of f and g, denoted $f+g$, is defined as:

For all real #s x, $(f+g)(x) = f(x) + g(x)$.

Show that if f and g are both increasing on a set S, then $f+g$ is also increasing on S.

Proof. Suppose x_1 and x_2 are real #s such that $x_1 < x_2$. Let f and g be increasing on a set S. Then $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$.

Then,

$$x_1 < x_2$$
$$f(x_1) < f(x_2)$$
$$f(x_1) + g(x_1) < f(x_1) + g(x_1)$$
$$(f+g)(x_1) < (f+g)(x_2)$$

Hence, $f+g$ is also increasing.

Ch. 11 Sect. 1 Prob. 25

Let f be a real-valued function of a real variable. Show that if f is increasing on a set S and if M is any negative real #, then Mf is decreasing on S.

Proof: Let x_1 and x_2 be real #s such that $x_1 < x_2$ and let M be any negative real #. Also, let f be increasing on a set S. Then,

$$x_1 < x_2$$
$$f(x_1) < f(x_2) \quad \text{for } f$$

$$x_1 < x_2$$
$$Mx_1 > Mx_2 \quad (M \text{ is negative})$$
$$Mf(x_1) > Mf(x_2)$$

Hence, Mf is decreasing on S.

Ch. 11 Sect. 2 Prob. 10

Prove that if $g(x)$ is $O(f(x))$, then $f(x)$ is $\Omega(g(x))$. Assume f and g are real-valued functions defined on the same set of nonnegative real #s.

Proof: Suppose f and g are r.v. functions defined on the same set of nonnegative real #s, and suppose $g(x)$ is $O(f(x))$.

By def. of O-notation, there exist positive real #s b and B such that

$|g(x)| \leq B|f(x)| \quad \forall x > b$. Then...

$1/B\, |g(x)| \leq |f(x)|$. Let $A = 1/B$ and $a = b$.

Then, $A|g(x)| \leq |f(x)| \quad \forall x > a$.

Thus, by def. of Ω-notation,

$f(x)$ is $\Omega(g(x))$.

ch. 11 Sect. 2 prob. 20

a. show that for any real # x, if $x > 1$, then
$|x^2| \leq |\lceil x^2 \rceil|$.

By def. of ceiling, $\forall x \in \mathbb{R}$, $\lceil x^2 \rceil$ is the integer n such that $n-1 < x^2 \leq n$. So, by substitution, $x^2 \leq \lceil x^2 \rceil$. Since $x > 1$, both sides of the inequality are positive. Thus, $|x^2| \leq |\lceil x^2 \rceil|$.

b. Show that for any real # x, if $x > 1$ then
$\frac{1}{2} |\lceil x^2 \rceil| \leq |x^2|$. ($\lceil x^2 \rceil$ is n where $n-1 < x^2 \leq n$).

Adding 1 to all sides:

$n-1+1 < x^2+1 \leq n+1$, so $\lceil x^2 \rceil \leq x^2+1$.

Then, $|\lceil x^2 \rceil| \leq \lceil x^2 \rceil$ and

$|\lceil x^2 \rceil| \leq x^2+1$

$\Rightarrow$ $|\lceil x^2 \rceil| < x^2 + x^2$ since $1 < x^2$

$\Rightarrow$ $|\lceil x^2 \rceil| \leq 2x^2$

$\Rightarrow$ $|\lceil x^2 \rceil| \leq 2|x^2|$.

c. Use O- and Ω- to express a and b.

a) Let $A = 1$ and $a = 1$, then
$|x^2| \leq A|\lceil x^2 \rceil| \ \forall x > a$, so
$\lceil x^2 \rceil$ is $\Omega(x^2)$.

b) Let $B = 2$ and $b = 1$, then
$|\lceil x^2 \rceil| \leq B|x^2| \ \forall x > b$, so
$\lceil x^2 \rceil$ is $O(x^2)$.

d. What can you deduce about the order of $\lceil x^2 \rceil$?

We can deduce that
$\lceil x^2 \rceil$ is $\Theta(x^2)$ by part c.

Ch. 11 Sect. 2 Prob. 31

Find an order for $7x^4 - 95x^3 + 3$ from among the set of power functions.

* Suppose $a_0, a_1, \ldots, a_n$ are real #s and $a_n \neq 0$.

$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0$ is $\Theta(x^n)$.

So, $7x^4 - 95x^3 + 3$ is $\Theta(x^4)$.

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Prove, assuming n is a variable that takes positive integer values.

$1^2 + 2^2 + 3^2 + \dots + n^2$ is $\Theta(n^3)$.

* The sum of the squares of the 1st n terms is: $\dfrac{n(n+1)(2n+1)}{6}$.

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Then, the highest power in the numerator is 3 and the highest power in the denominator is 0.

The order is $\Theta(n^{3-0}) = \Theta(n^3)$.

- For rational functions, the order is $\Theta(x^{\text{highest num. power} - \text{highest denom. power}})$.

Ch. 11 Sect. 2 Prob. 50

a. Let x be any positive real #. Prove that $\forall n \in \mathbb{Z} \geq 1$,

if $x \leq 1$ then $x^n \leq 1$.

Proof: Let $P(n)$ be the property $x^n \leq 1$.

Let $x \leq 1$ and n be any integer ≥ 1.

① $x^1 \leq 1$ and $P(1) = x^1 \leq 1$ is true, so

$P(1)$ is true.

② Suppose $P(k)$ is true. Then,

$P(k+1) = x^{k+1} = x^k \cdot x^1 \leq 1$

$\underset{\sim}{P(k)} \cdot x^1 \leq 1$

and since $x^1 \leq 1$, $P(k) \cdot x^1 \leq 1$, so

$P(k+1)$ is true.

b. Explain how it follows from part a that if x is any positive real #, then $\forall n \in \mathbb{Z} \geq 1$, if $x^n > 1$ then $x > 1$.

From part a, if $x \leq 1$ then $x^n \leq 1$.

For x^n to be > 1, $x \nleq 1$, so $x > 1$

for $x^n > 1$ to be true.

c. Explain how it follows from part b that if x is any positive real #, then $\forall n \in \mathbb{Z} \geq 1$, if $x^n > 1$ then $x^{1/n} > 1$.

If $x^n > 1$, then $x > 1$. Moreover,

$x^{1/n} = \sqrt[n]{x}$ and since $x > 1$,

the n^{th} root will always be > 1.

Hence, if $x^n > 1$, then $x^{1/n} > 1$.

d. Let p, q, and s be positive integers, let r be a nonneg. integer, and suppose $p/q > r/s$.

If $x > 1$, then $x^{p/q} > 1$ and $x^{r/s}$ is > 1.

Since, $p/q > r/s$, $ps - qr > 0$ and

$\frac{x^{p/q}}{x^{r/s}} = x^{(p/q - r/s)} = x^{(ps-qr)/qs}$. If $x > 1$, then

$x^{1/qs} > 1$ and since $ps - qr > 0$,

$x^{ps-qr} > 1$ so $x^{(ps-qr)/qs} > 1$.

Thus, $x^{p/q} > x^{r/s}$ since $p/q > r/s$.