Ch. 9 Homework cover page

Problems Done :

9.1 : 3, 7, 9, 12, 16, 17, 21, 24, 28, 32 (ALL)

9.2 : 4, 11, 16, 20, 24, 31, 35, 41, 45 (ALL)

9.3 : 4, 11, 15, 20, 25, 26, 31, 35, 40, 43 (ALL)

9.4 : 5, 9, 12, 17, 20, 23, 29, 32, 36, 38 (ALL)

Problems Not Done : NONE

ch. 9 Sect. 1 Prob. 3

Use the sample space given in Ex. 9.1.1. write the event as a set, and compute its probability.

The event that the chosen card is red and not a face card.

(A deck has 52 cards : the reds are diamonds and hearts and the blacks are clubs and spades. Each group has 13 cards : 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, and A. The J, K, and Q are face cards.)

$\rightarrow$ $E = \{ 2\diamondsuit, 3\diamondsuit, \ldots, 10\diamondsuit, A\diamondsuit, 2\heartsuit, 3\heartsuit, \ldots, 10\heartsuit, A\heartsuit \}$

E contains 20 cards out of the total 52.

Thus, $P(E) = 20/52 \cong 38.5\%$

Ch. 9 Sect. 1 Prob. 7

Use the sample space given in Ex. 9.1.2. Write the event as a set and compute its probability.

The event that the sum of the numbers showing face up is 8.

(we have a pair of dice and each dice has 6 sides with 1-6 dots.)

→ $E = \{26, 35, 44, 53, 62\}$

There's a total of 6 possibilities per one dice, and since we have 2, that's $6 \times 6 = 36$ total outcomes.

Thus, $P(E) = 5/36 \approx 13.9\%$

Ch. 9 Sect. 1 Prob. 9

Same instructions as #7 on previous page.

The event that the sum of the numbers showing face up is at most 6. (≤ 6)

$E = \{11, 12, 13, 14, 15, 21, 22, 23, 24, 31, 32, 33, 41, 42, 51\}$

→ 36 possible outcomes

15 elements in E.

$P(E) = \frac{15}{36} \cong 42\%$

Ch. 9 Sect. 1 Prob. 12

Suppose that each child born is equally likely to be a boy or a girl. Consider a family with exactly 3 children. Let BBG indicate that the first 2 children born are boys and the third child is a girl, let GBG indicate that the 1st and 3rd are girls and the 2nd is a boy, and so forth.

a. List the 8 elements in the sample space whose outcomes are all possible genders of the 3 children.

{ BBB, BBG, BGB, BGG, GGG, GGB, GBG, GBB }

b. Write each as sets and find their probability:

(i) The event that exactly 1 child is a girl.

E = { BBG, BGB, GBB }, P(E) = 3/8.

(ii) The event that atleast 2 are girls. (2 ≤)

E = { BGG, GGG, GGB, GBG }, P(E) = 4/8 = 1/2.

(iii) The event that no child is a girl.

E = { BBB }, P(E) = 1/8.

Ch. 9 Sect. 1 Prob. 16

2-faces of a 6-sided die are painted red, 2 are painted blue, and 2 are painted yellow. The die is rolled 3 times, and the colors that appear face up on the 1st, 2nd, and 3rd rolls are recorded.

a. Let BBR denote the outcome where blue appears on the 1st 2 rolls and red appears on the 3rd roll. Because there are as many faces of 1 color as of any other, the outcomes of this experiment are equally likely. List all 27 outcomes.

{RRR, RRB, RRY, RBR, RBB, RBY, RYR, RYB, RYY,
BRR, BRB, BRY, BBR, BBB, BBY, BYR, BYB, BYY,
YRR, YRB, YRY, YBR, YBB, YBY, YYR, YYB, YYY}

b. Consider the event that all 3 rolls produce different colors. List all outcomes in this event. What is the probability?

E = {RBY, RYB, BRY, BYR, YRB, YBR}

$P(E) = 6/27 \cong 22.2\%$

c. Consider the event that 2 of the colors that appear face up are the same. List all outcomes in this event. What is the probability?

E = {RRB, RRY, RBR, RBB, RYR, RYY, BRR, BRB,
BBR, BBY, BYB, BYY, YRR, YRY, YBB, YBY,
YYR, YYB}

$P(E) = 18/27 = 6/9 = 2/3 \cong 66.7\%$

Ch. 9 Sect. 1 Prob. 17

Consider the situation described in # 16 on the previous page.

a. Find the probability of the event that exactly 1 of the colors that appears face up is red.

E = { RBB, RBY, RYB, RYY, BRB, BRY, BBR, BYR, YRB, YRY, YBR, YYR }

P(E) = 12/27 = 4/9.

b. Find the probability of the event that atleast 1 of the colors that appears face up is red.

E = { RRR, RRB, RRY, RBR, RBB, RBY, RYR, RYB, RYY, BRR, BRB, BRY, BBR, BYR, YRR, YRB, YRY, YBR, YYR }

P(E) = 19/27.

Another way to solve for P(E):

$P(E) = 1 - P(\tilde{E})$

complement: probability that none of the colors are red.

$\tilde{E}$ = { BBB, BBY, BYB, BYY, YBB, YBY, YYB, YYY }

$P(\tilde{E}) = 8/27$.

THEN:

$P(E) = 1 - 8/27 = 19/27$.

Ch. 9 Sect. 1 Prob. 21

a. How many positive 2-digit integers are multiples of 3?

$$\underset{}{10}, 11, \underset{\substack{\downarrow \\ 4\cdot 3}}{12}, 13, 14, \underset{\substack{\downarrow \\ 5\cdot 3}}{15}, 16, 17, \underset{\substack{\downarrow \\ 6\cdot 3}}{18}, \dots, \underset{\substack{\downarrow \\ 33\cdot 3}}{99}$$

we have as many 2-digits integers that are multiples of 3 that are between 4-33 inclusive.

so, $33 - 4 + 1 = 30$ integers. (upper - lower + 1)

b. what is the probability that a randomly chosen positive 2-digit integer is a multiple of 3?

$P(E) = \dfrac{30}{90}$ (# of ints that are multiples of 3) / (total # of 2-digit integers)

$= 1/3$. $(90 = 99 - 10 + 1)$

c. what is the probability that a randomly chosen positive 2-digit integer is a multiple of 4?

$$10, 11, \underset{\substack{\downarrow \\ 3\cdot 4}}{12}, 13, 14, 15, \underset{\substack{\downarrow \\ 4\cdot 4}}{16}, \dots, \underset{\substack{\downarrow \\ 24\cdot 4}}{96}, 97, 98, 99$$

$\rightarrow (24 - 3 + 1) = 22$ integers that are multiples of 4.

$P(E) = 22/90 = 11/45$.

Ch. 9 Sect. 1 Prob. 24

Suppose $A[1], A[2], \ldots, A[n]$ is a 1-D array and $n \geq 2$. Consider the subarray $A[1], A[2], \ldots, A[\lfloor n/2 \rfloor]$.

a. How many elements are in the subarray if n is even and if n is odd?

If n is even, then we have $\lfloor n/2 \rfloor - \cancel{1} + \cancel{1}$
$= n/2$. $(\lfloor n/2 \rfloor = n/2$ if n is even$)$

If n is odd, then $\lfloor n/2 \rfloor - \cancel{1} + \cancel{1}$
$= n-1/2$ elements
$(\lfloor n/2 \rfloor = n-1/2$ if n is odd$)$

b. What is the probability that a randomly chosen array element is in the subarray if n is even and if n is odd?

If n is even, $p = \dfrac{n/2}{n}$ - # of elements in subarray / - total # of elements
$= n/2n = 1/2$.

If n is odd, $p = \dfrac{n-1/2}{n}$
$= n-1/2n$
$= n/2n - 1/2n$
$= 1/2 - 1/2n$.

Ch. 9 Sect. 1 Prob. 28

If the largest of 56 consecutive integers is 279, what is the smallest?

* If m and n are integers and $m \le n$, then there are $n-m+1$ integers from m to n inclusive.

Here, $n = 279$ and $n-m+1 = 56$.

Then,

$$\begin{aligned} m &= n - 56 + 1 \\ &= 279 - 56 + 1 \\ &= 224. \end{aligned}$$

Ch. 9 Sect. 1 Prob. 32

A certain nonleap year has 365 days, and January 1 occurs on a Monday.

a. How many Sundays are in a year?

M	T	W	Th	F	S	Sun	...	Sun	...	Sun	Mon
1	2	3	4	5	6	7	...	14	...	364	365
						↓		↓		↓	
						$7 \cdot 1$		$7 \cdot 2$		$7 \cdot 52$	

- we have as many Sundays as we have #'s between 1 - 52, so we have

(52 - 1 + 1 = 52 Sundays).

b. How many Mondays are in a year?

M	T	W	Th.	F	Sat.	Sun.	M	...	M
1	2	3	4	5	6	7	8	...	365
↓							↓		↓
$7 \cdot 0 + 1$							$7 \cdot 1 + 1$		$7 \cdot 52 + 1$

(There are 7 days in a week)

→ we have as many Mondays as we have #'s between 0 - 52 =

(52 - 0 + 1 = 53 Mondays)

Ch. 9 sect. 2 prob. 4

The 1st team to win 4 games wins the series.

- we have 2 teams: team A and B.

How many ways can a world series be played if no team wins 2 games in a row?

A - B - A - B - A - B - A

AND

B - A - B - A - B - A - B

→ There are only 2 ways.

Ch. 9 Sect. 2 Prob. 11

a. A bit string is a finite sequence of 0's and 1's. How many bit strings have length 8?

$$\underset{0-1}{\downarrow}\ \underset{0-1}{\downarrow}\ \underset{0-1}{\downarrow}\ \underset{0-1}{\downarrow}\ \underset{0-1}{\downarrow}\ \underset{0-1}{\downarrow}\ \underset{0-1}{\downarrow}\ \underset{0-1}{\downarrow}$$

we have 2 options per place, so we have $\underbrace{2 \cdot 2 \cdots 2}_{8 \text{ terms}}$ possible strings.

$\rightarrow 2^8 = 256$.

b. How many bit strings of length 8 begin with 3 0's?

$$0\ 0\ 0\ \underset{0-1}{\downarrow}\ \underset{0-1}{\downarrow}\ \underset{0-1}{\downarrow}\ \underset{0-1}{\downarrow}\ \underset{0-1}{\downarrow}$$

Since the 1st 3 need to be 0's, we can have 2 options for the last 5.

$\rightarrow$ $2^5 = 32$ strings that begin w/ 3 0's.

c. How many strings of length 8 begin and end w/ a 1?

$$1\ \underset{0-1}{\downarrow}\ \underset{0-1}{\downarrow}\ \underset{0-1}{\downarrow}\ \underset{0-1}{\downarrow}\ \underset{0-1}{\downarrow}\ \underset{0-1}{\downarrow}\ 1$$

$\rightarrow$ we have 2 options for the 6 bits inbetween, so

$2^6 = 64$ strings.

Ch. 9 Sect. 2 Prob. 16

a. How many integers are there from 10 - 99?

$\underset{1-9}{\downarrow}$ $\underset{0-9}{\downarrow}$

9 possible choices for digit 1

10 possible choices for digit 2.

$= 9 \cdot 10 = 90$ integers.

b. How many odd integers are there from 10 - 99?

For digit 1, we can have
1-9 (9 choices)

* we just need to make sure that the 2nd digit is a 1, 3, 5, 7, or 9.
(5 choices)

→ $9 \cdot 5 = 45$ odd integers

c. How many integers from 10-99 have distinct digits?

$\underset{1-9}{\downarrow}$ $\underset{1-9}{\downarrow}$ $= 9 \cdot 9 = 81$

d. How many odd integers from 10-99 have distinct digits?

$\underset{\text{8 choices}}{\downarrow}$ $\underset{\text{5 choices}}{\downarrow}$ $= 8 \cdot 5 = 40$

e. Probability that a randomly chosen 2-digit integer has distinct digits AND has distinct digits and is odd?

$P(E_1) = 81/90 = 9/10$.

$P(E_2) = 40/90 = 4/9$.

Ch. 9 Sect 2 Prob. 20

A PIN mustn't begin w/ the letters A-M and must end w/ a digit. Continue to assume that no symbol may be used more than once and that the total # of PINs is to be determined.

a. Find the error in the following: "Constructing a PIN is a 4-step process.

1. choose the leftmost symbol.
2. Choose the 2nd from the left.
3. choose the 3rd from the left.
4. choose the rightmost.

Because none of the 13 letters A-M can be chosen for ①, there are 36 - 13 = 23 ways to perform ①. There are 35 ways to do ② and 34 for ③. Since for ④, it must be a previously unused digit, there are 10 - 3 = 7 ways for ④. Thus, there are 23 · 35 · 34 · 7 = 191,540 different PINs that satisfy the given conditions."

→ The # of ways to perform ④ isn't constant. If 3 letters are chosen for ①-③, we'd have 10 options for ④.

b. Reorder ① → ④ as follows:

① is now ④
② is now ①
③ is now ②
④ is now ③

Find the # of PINs that satisfy the given conditions.

→ ①: we can choose 0-9 (10 choices)

②: we can choose N-Z, but we can't reuse the digit from ①, so we have 23-1 = 22.

③: we can choose anything except what we've used in ① and ②, so 34 choices

④: Same idea w/ ③, so 33 choices.

→ 10 · 22 · 34 · 33 = 246,840 PINs.

Ch. 9 Sect. 2 prob. 24

Determine how many times the inner most loop will be iterated.

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for i := 1 to 30
    for j := 1 to 15
    next j
next i
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we have $15 \cdot 30 = 450$ iterations.

Ch. 9 Sect. 2 Prob. 31

a. If p is a prime # and a is a positive integer, how many distinct positive divisors does p^a have?

(Any # is divisible by a prime #, Here

$\underbrace{p^a = p \cdot p \cdot p \cdot p \cdots p}_{a \text{ times}}$, THEN, p^a is divisible by $1, p, p^2, \ldots, p^a$.

→ It has $a+1$ divisors.

(For ex: $2^3 = 2 \cdot 2 \cdot 2 = 8$, so it has $1, 2, 4, 8$ or $3+1$.)

b. If p and q are distinct prime #s and a and b are positive integers, how many distinct positive divisors does $p^a q^b$ have?

($a+1$ for p and $b+1$ for q = $(a+1)(b+1)$ divisors)

c. If p, q, r are distinct prime #s and a, b, c are positive integers, how many distinct positive divisors does $p^a q^b r^c$ have?

→ $(a+1)(b+1)(c+1)$ divisors.

d. If $p_1, p_2, \ldots, p_m$ are distinct prime #s and $a_1, a_2, \ldots, a_m$ are positive integers, how many distinct positive divisors does $p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ have?

→ $(a_1+1)(a_2+1) \cdots (a_m+1)$ divisors.

e. What is the smallest positive integer w/ exactly 12 divisors?

we could have $2^{11} = 2048$

or we could have $2^5 \cdot 3^1 = 32 \cdot 3 = 96$

or we could have $2^2 \cdot 3^3 = 4 \cdot 27 = 108$

or we could have $2^2 \cdot 3^1 \cdot 5^1 = 4 \cdot 15 = 60$

→ The smallest is 60.

Ch. 9 Sect 2 Prob. 35

Write all the 2 permutations of $\{w, x, y, z\}$

(Elements can't be the same)

= wx, wy, wz, xw, xy, xz,
yw, yx, yz, zw, zx, zy.

→ 12 permutations.

$$= {}_nP_m = {}_4P_2 = \frac{n!}{(n-m)!} = \frac{4!}{(4-2)!} = \frac{4!}{2!}$$

$$\frac{4!}{2!} = \frac{4 \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}{\cancel{2} \cdot \cancel{1}} = 12.$$

Ch. 9 Sect. 2 Prob. 41

Prove that for all integers $n \geq 2$,

$P(n+1, 2) - P(n, 2) = 2P(n, 1)$

$(P(n,r) = {}_nP_r = n!/(n-r)!)$

Proof: Let an integer $n \geq 2$. Then:

$$P(n+1,2) - P(n,2) = \frac{(n+1)!}{(n+1-2)!} - \frac{n!}{(n-2)!}$$

$$= \frac{(n+1)!}{(n-1)!} - \frac{n!}{(n-2)!}$$

$$= \frac{(n+1)(n)\cancel{(n-1)!}}{\cancel{(n-1)!}} - \frac{n \cdot (n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}}$$

$$= (n+1)(n) - n(n-1)$$

$$= n^2 + n - n^2 + n = 2n$$

$$= 2 \cdot \frac{n(n-1)!}{(n-1)!}$$

$$= 2 \cdot \frac{n!}{(n-1)!}$$

$$= 2P(n,1).$$

Ch. 9 Sect. 2 prob. 45

Prove that for any integer n with $n \geq 1$, the number of permutations of a set of n elements is $n!$ by mathematical induction.

Proof: Let n be an integer w/ $n \geq 1$ and suppose $n!$ is the # of permutations of a set of n elements.

① when $n=1$, we only have 1 element, so the # of permutations is simply $1 = 1!$. TRUE.

② suppose there exist $k!$ permutations of a set of k elements. Let X be a set w/ $k+1$ elements, then $n = k+1$ and there are

$(k+1)\underbrace{(k)(k-1)(k-2)\cdots(2)(1)}$ ways to choose elements in X.

$(k+1) \cdot k!$

$= (k+1)!$.

Hence, the property is true for all $n \geq 1$.

Ch. 9 Sect. 3 Prob. 4

How many arrangements in a row of no more than 3 letters can be formed using the letters of the word NETWORK (w/ no repetitions allowed)?

Arrangements of no more than 3 letters =
Arrangements of no more than 1 letter
+ Arrangements of no more than 2 letters
+ Arrangements of no more than 3 letters.

$\rightarrow P(7,1) + P(7,2) + P(7,3)$

$\downarrow$ # of letters in our word

$$= \frac{7!}{(7-1)!} + \frac{7!}{(7-2)!} + \frac{7!}{(7-3)!}$$

$$= \frac{7!}{6!} + \frac{7!}{5!} + \frac{7!}{4!}$$

$$= \frac{7 \cdot \cancel{6!}}{\cancel{6!}} + \frac{7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}} + \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4!}}{\cancel{4!}}$$

$$= 7 + (7 \cdot 6) + (7 \cdot 6 \cdot 5)$$

$$= 7 + 42 + 210$$

$= 259$ arrangements.

Ch. 9 Sect. 3 Prob. 11

a. How many ways can the letters of the word QUICK be arranged in a row?

→ we have 5 letters taken 5 at a time, so

$5! = 120$ total permutations

b. How many ways can the letters of the word QUICK be arranged in a row if the Q and U must remain next to each other in the order QU?

→ we have to consider QU as a single element. Then, we have QU, I, C, K as elements. So, the # of permutations becomes

$4! = 24$. (4 elements)

c. Same as b., but the order can be QU or UQ?

→ w/ the same idea in B,

① QU, I, C, K → $4!$

② UQ, I, C, K → $4!$

Finally, the total is ① + ②

$= 4! + 4! = 24 + 24$

$= 48$.

Ch. 9 Sect. 3 Prob. 15

Identifiers in a certain database language must begin w/ a letter, and then the letter may be followed by other characters, which can be letters, digits, or underscores. However, 82 keywords (all consisting of 15 or fewer characters) are reserved and can't be used. How many identifiers w/ 30 or fewer characters are possible?

→ we have 26 choices for the 1^{st} character (A-Z).

→ we have (A-Z) or (0-9) or (_) as choices for the 2^{nd} character; so there are 37 choices.

⋮

For identifiers of 1 char, we have 26 options

For identifiers of 2 chars, we have $26 \cdot 37$ options

For identifiers of 3 chars, we have $26 \cdot 37^2$ options.

⋮

For identifiers of 30 chars, we have $26 \cdot 37^{29}$ options.

The total # of options is their sum =

$$26 \sum_{i=0}^{29} 37^i \qquad * \sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$$

$$= 26 \cdot \left(\frac{37^{30}-1}{37-1} \right)$$

$$= 26 \cdot \left(\frac{37^{30}-1}{36} \right) = \frac{13}{18} \left(37^{30}-1 \right)$$

FINALLY, 82 of these can't be used, so we have =

$$\frac{13}{18} \left(37^{30}-1 \right) - 82$$

ch. 9 Sect 3 Prob. 20

a. How many integers from 1 - 100,000 contain the digit 6 exactly once?

→ At most, we can have a 5 digit # w/ a 6 as 1 of our digits. _ _ _ _ _
1 2 3 4 5

1 of these can be our 6, THEN:

① we have 5 ways to choose which digit will be our 6.

② we have 9 ways to choose the rest of our digits for each position (0-9 excluding 6)

By the multiplication rule, we have

$5 \cdot 9 \cdot 9 \cdot 9 \cdot 9 = 32,805$ integers.

b. How many integers from 1-100,000 contain 6 atleast once?

→ # of digits that contain atleast 1 6 = total options − # of digits that contain no 6

* To find the # of digits w/ no 6's:

_ _ _ _ _ Each can be 0-9 except
1 2 3 4 5 for 6, so we have 9 choices
for each. → $9 \cdot 9 \cdot 9 \cdot 9 \cdot 9$ digits w/ no 6.

→ we can have 1-99,999 as options, so

$99,999 - 9^5 = 40950$ w/ atleast 1 6.

c. Probability that a # 1-100,000 contains 2 or more occurrences of 6?

→ # of integers w/ atleast 1 6 −
# of integers w/ exactly 1 6
$= 40,950 - 32,805 = 8145$

$P(E) = \frac{8145}{100,000}$.

Ch. 9 Sect 3 Prob. 25

a. Make a list of all bit strings of lengths 0, 1, 2, 3, and 4 that don't contain the bit pattern 111.

- 0: 0

1: 0, 1

2: 00, 01, 10, 11

3: 000, 001, 010, 011, 100, 101, 110

4: 0000, 0001, 0010, 0011, 0100, 0101, 0110, 1000, 1001, 1010, 1011, 1100, 1101

b. For each integer $n \geq 0$, let d_n = # of bit strings of length n that don't contain 111.

Find d_0, d_1, d_2, d_3, and d_4.

$d_0 = 1$, $d_1 = 2$, $d_2 = 4$,

$d_3 = 7$, and $d_4 = 13$

c. Find a recurrence relation for $d_0, d_1, d_2, \ldots$

Let k be an integer w/ $k \geq 3$. Bit strings of length k start w/ a 1 or 0. If 0, it can be followed by a string of $k-1$ bits that don't contain 111. There are d_{k-1} of these. If it starts w/ a 1, then the 1st 2 bits are 10 or 11. If 10, it can be followed by a string of $k-2$ bits that don't contain 111. There are d_{k-2} of these. If 11, the 3rd bit must be 0, and it can be followed by a string of $k-3$ bits that don't contain 111. There are d_{k-3} of these.

$\therefore$ For all $k \geq 3$, $d_k = d_{k-1} + d_{k-2} + d_{k-3}$,

$d_0 = 1$, $d_1 = 2$, and $d_2 = 4$.

d. Find the # of bit strings of length 5 that don't contain 111.

$\rightarrow d_5 = d_4 + d_3 + d_2$

$= 13 + 7 + 4$

$=$ 24 strings.

Ch. 9 Sect 3 Prob. 26

Consider the set of all strings of a's, b's, and c's.

a. Make a list of all of these of lengths 0 - 3 that don't contain the pattern aa.

0: ϵ (empty string)

1: a, b, c

2: ab, ac, ba, bb, bc, ca, cb, cc

3: aba, abb, abc, aca, acb, acc, bab, bac, bba, bbb, bbc, bca, bcb, bcc, cab, cac, cba, cbb, cbc, cca, ccb, ccc

b. For each integer $n \geq 0$, let s_n = # of strings of a's, b's, and c's of length n that don't contain aa. Find s_0, s_1, s_2, and s_3.

$s_0 = 1$ $\quad$ $s_2 = 8$

$s_1 = 3$ $\quad$ $s_3 = 22$

c. Find a recurrence relation.

Let k be an integer w/ $k \geq 2$. Strings of length k can begin w/ a, b, or c. If "a", the 2nd must be b or c. If it's a b, "a" can be followed w/ $k-1$ characters. The same applies to if c comes after "a". We then have $2s_{k-1}$ (for b and c).
If it begins w/ b or c, then the 2nd can be a, b, or c. If "a", the 3rd has to be a b or c. The "a" can be followed by $k-2$ characters, so we have $2s_{k-2}$ (for b and c after "a").

$\rightarrow$ $s_k = 2s_{k-1} + 2s_{k-2}$. ($s_0 = 1$ and $s_1 = 3$)

d. How many of length 4 don't contain aa?

$s_4 = 2s_3 + 2s_2 = 2(22) + 2(8) = 44 + 16 = 60$

e. Find an explicit formula.

For all integers $n \geq 0$,

$$s_n = \frac{\sqrt{3}+2}{2\sqrt{3}}(1+\sqrt{3})^n + \frac{\sqrt{3}-2}{2\sqrt{3}}(1-\sqrt{3})^n$$

Ch. 9 Sect. 3 Prob. 31

Assume that B-days are equally likely to occur in any 1 of the 12 months of the year.

a. Given a group of 4 people, A-D, what is the total # of ways in which birth months could be associated w/ A-D?

→ $_{12}P_4^R$ w/ repetition allowed = $12^4 = 20{,}736$.

b. How many ways could birth months be associated w/ A-D if no 2 people would share the same month?

→ $_{12}P_4$ w/o rep: $12!/(12-4)! = 12 \cdot 11 \cdot 10 \cdot 9 = 11{,}880$.

c. How many ways could birth months be associated w/ A-D so that atleast 2 people would share the same month?

→ (# of total ways) − (# of ways w/ no duplicates)

= (# of ways w/ duplicates)

= (20,736) − (11,880) = 8856 ways.

d. What is the probability that atleast 2 people would share the same birth month?

→ $P(E) = \dfrac{8856}{20{,}736} \approx 42.7\%$

e. How large must n be so that in any group of n people, the probability that 2 or more share the same birth month is atleast 50%?

when $n=5$, $P(E) = \dfrac{_{12}P_5^R - {_{12}P_5}}{_{12}P_5^R} \approx 61.8\%$

* P(E) = probability that 2 or more share the same month.

- $_{12}P_5^R$ (Total options)
- $_{12}P_5$ (No duplicates)

$\}$ $12^5 - {_{12}P_5}$ = Dups Allowed.

when $n=5$, $P(E) = \dfrac{12^5 - 12!/(12-5)!}{12^5} \approx 61.8\%$

ch. 9 sect. 3 Prob. 35

An interesting use of the inclusion/exclusion rule is to check survey #s for consistency. For example, suppose a public opinion polltaker reports that out of a national sample of 1200 adults, 675 are married, 682 are from 20-30 years old, 684 are female, 195 are married and are from 20-30 years old, 467 are married females, 318 are females from 20-30 years old, and 165 are married females from 20-30 years old. Are the polltaker's figures consistent? Could they have occurred as a result of an actual sample survey?

$$* \; N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C)$$

Let M = set of married people in the sample.
Y = set of people from 20-30 years old.
F = set of females in the sample.

→ $1200 \geq N(M \cup Y \cup F)$ because we don't account for everyone in the sample. For ex, you could be an unmarried male that isn't 20-30 years old.

Then, $N(M \cup Y \cup F) = N(M) + N(Y) + N(F) - N(M \cap Y) - N(M \cap F) - N(Y \cap F) + N(M \cap Y \cap F) =$

$675 + 682 + 684 - 195 - 467 - 318 + 165$

$= 1226$ which is > 1200, so

$1200 \not\geq 1226$ (#s are inconsistent)

They could NOT have occurred as a result of an actual survey sample.

Ch. 9 Sect. 3 Prob. 40

For each integer $n \geq 1$, $\phi(n)$ = # of positive integers $\leq n$ that have no common factors w/ n except ± 1. (# of positive integers that are coprime w/ n). Use the inclusion/exclusion rule to prove the following:

If $n = pq$, where p and q are distinct prime #s, then $\phi(n) = (p-1)(q-1)$.

Proof: Let A and B be the sets of all positive integers that are divisible by p and q, respectively.

Then: The # of positive integers $\leq n$ that are coprime w/ n = (the total # of integers between $1-n$) − (# of integers that are divisible by p or q). This is because the integers that are divisible by p or q are not coprime w/ n since they'd have p or q as a factor and since $n = pq$, they have a common factor besides ± 1.

$$\rightarrow \phi(n) = n - N(A \cup B)$$

The only integer that's divisible by both p and q = n (so $N(A \cap B) = 1$)

$$= n - [\underbrace{N(A)}_{p} + \underbrace{N(B)}_{q} - \underbrace{N(A \cap B)}_{1}]$$

$$= n - p - q + 1$$

$$= pq - p - q + 1$$

$$= (p-1)(q-1).$$

Hence, $\phi(n) = (p-1)(q-1)$ if $n = pq$.

ch. 9 Sect. 3 Prob. 43

A derangement of the set $\{1, 2, \dots, n\}$ is a permutation that moves every element of the set away from its "natural" position. Thus 21 is a derangement of $\{1, 2\}$, and 231 and 312 are derangements of $\{1, 2, 3\}$. For each positive integer n, let d_n = # of derangements of the set $\{1, 2, \dots, n\}$.

a. Find d_1, d_2, and d_3.

$\{1\} = \varphi$, so $d_1 = 0$ (You can't derange 1 element)

$\{1, 2\} = 21$, so $d_2 = 1$

$\{1, 2, 3\} = 231, 321$, so $d_3 = 2$.

b. Find d_4.

$\{1, 2, 3, 4\} = 2143, 2413, 2341, 3142, 3421, 3412,$
$4321, 4213, 4123$, so $d_4 = 9$.

c. Find a recurrence relation:

If we divide the set of all derangements into 2 subsets: ① consists of all derangements in which the # 1 changes places w/ another #, AND ② consists of all derangements in which the # 1 goes to a position $i \neq 1$ but i doesn't go to the first position.

→ For ①, if 1 changed places w/ another #, it could be repeated $k-1$ times until the remaining $k-1$ #s were out of order. Then we'd have $(k-1)d_{k-1}$ of these.

→ For ②, if 1 changed places w/ a # at position i, then the # i would need to be sent somewhere besides position 1. We'd need to do this $k-1$ times for the remaining $k-1$ #s, but each operation would have an additional step of sending i elsewhere, so we'd have 1 less # to deal w/ once i is sent somewhere. Then, we'd have $(k-1)d_{k-2}$ of these.

$$d_k = (k-1)d_{k-1} + (k-1)d_{k-2}. \quad (k \geq 2)$$

ch. 9 Sect. 4 Prob. 5

a. Given any set of 4 integers, must there be 2 that have the same remainder when divided by 3?

→ we have 4 pigeons : our set of 4 integers.

Any # divided by 3 can have only 1 of these 3 remainders : 0, 1, or 2.

(once it hits 3, the remainder goes back to 0)

Thus, by the pigeonhole principle, atleast 2 will have the same remainder.

b. Given any set of 3 integers, must there be 2 that have the same remainder when divided by 3?

→ 3 pigeons : our 3 integers

3 holes : 0, 1, or 2 (3 possible remainders)

The pigeonhole principle can't be applied because we don't have more pigeons than pidgeonholes. Thus, there doesn't need to be 2 that have the same remainder.

ch. 9 Sect. 4 Prob. 9

a. If 7 integers are chosen from between 1 and 12 inclusive, must atleast one of them be odd?

7 pigeons: our 7 integers

→ From 1 to 12 inclusive, we have 6 odd numbers and 6 even numbers.

Thus, by the pigeonhole principle, atleast one needs to be odd. (7 pigeons and 6 holes)

b. If 10 integers are chosen from 1 to 20 inclusive, must atleast one of them be even?

10 pigeons: our 10 integers

→ From 1-20 inclusive, we have 10 even and 10 odd #s.

Since the principle can't be applied, it doesn't guarantee an even #.

Ch. 9 Sect. 4 prob. 12

How many cards must you pick from a standard 52-card deck to be sure of getting atleast 1 red card?

of these 52,

26 are black AND

26 are red.

If we want atleast 1 red, we'd have to choose 27 because then we'd have:

27 pigeons (cards chosen)

AND

26 holes (red / black).

Then, by the p.h. principle, atleast 1 would be red.

Ch. 9 Sect. 4 Prob. 17

How many integers must you pick in order to be sure that atleast 2 of them have the same remainder when divided by 7?

when divided by 7, a # can have:

0, 1, 2, 3, 4, 5, 6 as remainders.

(7 pigeonholes)

Then, we'd need to have 8 pigeons or 8 integers.

Ch. 9 Sect 4 Prob. 20

a. If repeated divisions by 20,483 are performed, how many distinct remainders can be obtained?

→ we could have anything between 0 and 20,483 − 1 as remainders.

So, we could have 20,483 distinct remainders.

b. When 5/20,483 is written as a decimal, what is the maximum length of the repeating section of the representation?

When 2 remainders are equal, our digits will repeat from then on. If we have 20,483 pigeonholes, the max # of pigeons can be 20,483. If we have more, some pigeons will occupy the same hole. Thus, the max length of the repeating section is 20,483.

ch. 9 Sect. 4 Prob. 23

Is 56.5566555666555566666... (where the strings of 5's and 6's becomes longer in each repetition) rational or irrational?

Since the decimal expansion doesn't terminate, nor does it repeat, this number is an irrational number.

- Goes to infinity

- Each repetition is different
(Different string of 5's and 6's)

Ch. 9 Sect. 4 Prob. 29

A certain college class has 40 students. All the students in the class are known to be from 17-34 years of age. You want to make a bet that the class contains atleast x students of the same age. How large can you make x and yet be sure to win your bet?

40 pigeons - 40 students

18 pigeonholes - years 17-34

$\frac{40}{18} = 2\,2/9$. $(40 > 18 \cdot 2$ But $40 < 18 \cdot 3)$

→ If we had 19 students w/ 18 years, then atleast 2 would be the same age.

However, we have 40 students $(40 > 18 \cdot 2$ but $40 < 18 \cdot 3)$ so there could be atleast 3, but no more than 3. Hence, $x = 3$. $(40 > 19 \cdot 2)$.

Ch 9 Sect 4 Prob. 32

Let A be a set of 6 positive integers each of which is less than 13. Show that there must be 2 distinct subsets of A whose elements when added up give the same sum. (For ex., if $A = \{5, 12, 10, 1, 3, 4\}$, then the elements of the subsets $S_1 = \{1, 4, 10\}$ and $S_2 = \{5, 10\}$ both add up to 15.)

- The # of subsets of 6 integers is $2^6 = 64$. Since each is < 13, the largest possible sum is 57.

→ Largest Possible subset sum:

$S_1 = \{12, 11, 10, 9, 8, 7\}$, sum = 57

Smallest Possible Subset sum:

$S_2 = \{1, 2, 3, 4, 5, 6\}$, Sum = 21

(sums range from 21-57, so there are 36 sums)

→ 64 pigeons : 64 subsets

36 pigeonholes : 36 different sums

Therefore, by the p.h. principle, atleast 2 subsets of 6 integers will have the same sum.

Ch. 9 Sect. 4 Prob. 36

Show that if 101 integers are chosen from 1 - 200 inclusive, there must be 2 w/ the property that one is divisible by the other.

- we can represent each of the 101 integers x_i as $a_i 2^{k_i}$ where a_i is odd and $k_i \geq 0$. Since $1 \leq x_i \leq 200$, then $1 \leq a_i \leq 199$ for all i. There are only 100 odd integers from 1 - 199 inclusive.

→ 101 pigeons : our 101 integers
100 pigeonholes : our 100 possible odd a_i

By the p.h. principle, atleast 2 share the same odd number a_i in their product.

Since they share the same odd number a_i in their product, then one is divisible by the other.

For ex.

Let $x_i = a \cdot 2^{k_i}$ and $x_j = a \cdot 2^{k_j}$ where $i \neq j$. If

① $i > j$ then x_i is divisible by x_j $= 2^{k_i - k_j}$.

② $i < j$ then x_j is divisible by x_i $= 2^{k_j - k_i}$.

Ch. 9 Sect. 4 Prob. 38

Observe that the sequence 12, 15, 8, 13, 7, 18, 19, 11, 14, 10
has 3 increasing subsequences of length 4:

① 12, 15, 18, 19

② 12, 13, 18, 19

③ 8, 13, 18, 19.

It also has one decreasing subsequence of length 4:

① 15, 13, 11, 10.

Show that in any sequence of n^2+1 distinct real #s, there must be a sequence of length $n+1$ that is either strictly increasing or strictly decreasing.

Let $a_1, a_2, \dots, a_{n^2+1}$ be any sequence of n^2+1 distinct real #s. Suppose this doesn't contain a strictly increasing / decreasing subsequence of length $n+1$. Let S be the set of all ordered pairs of integers (i, d), where $1 \le i \le n$ and $1 \le d \le n$. For each a_k, let $F(a_k) = (i_k, d_k)$, where i_k is the length of the longest increasing sequence starting at a_k, and d_k is the length of the longest decreasing sequence starting at a_k. Suppose F is 1-to-1.

If both i and $d < n+1$, then i and $d \le n$. Then the total # of ordered pairs is $\le n^2$, but we have n^2+1 terms, so it should be a total of n^2+1 ordered pairs. Thus, we have a contradiction.

∴ we have a subsequence of length $n+1$ that is either strictly increasing / decreasing.