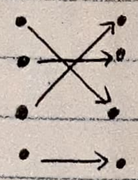
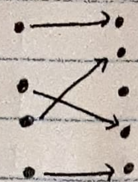
CSC-7 Final (6-8-2022): Aamir K.

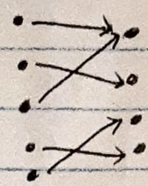
①



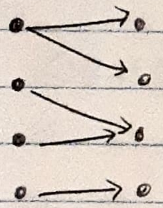
Bijective
Every pigeon is in its own pigeonhole.



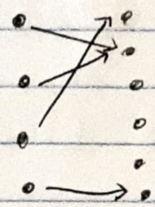
Injective
Every pigeon is in its own hole, but not all holes are occupied.



Surjective
Every hole is occupied, but atleast 1 has 2 pigeons.



NOT A Function
A pigeon is in 2 holes at the same time.



General Function
Pigeons can occupy the same hole, but no pigeon occupies 2 holes at once. Doesn't need to be injective or surjective.

(c)

	1	2	3	4	5	6
	a	b	c	~c	a ∧ ~c	b ∨ c
	1	1	1	0	0	1
	1	1	0	1	1	1
	1	0	1	0	0	1
	1	0	0	1	1	0
	0	1	1	0	0	1
	0	1	0	1	0	1
	0	0	1	0	0	1
	0	0	0	1	0	0

		b → 5	a → 6	b → c	
1		0	1	1	
2	*	1	1	0	← INVALID
3	*	1	1	1	
4		1	0	1	
5		0	1	1	
6		0	1	0	
7	*	1	1	1	
8	*	1	1	1	
		Premises		Conclusion	

(crit. rows) * - rows that have true premises

Row 2 has true premises but a false conclusion, so this form of argument is invalid. All critical rows would need to have a true conclusion. Since row 2 doesn't, this form of argument is invalid. Therefore, since all critical rows don't have a true conclusion, this argument is invalid.

							conclusion	premise	premise	premise
		1	2	3	4	5	6	7	8	9
(3)		p	q	r	$p \wedge q$	$\sim q$	$\sim r$	$p \vee \sim q$	$\sim q \to p$	$4 \to \sim r$
1		1	1	1	1	0	0	1	1	0
2	*	1	1	0	1	0	1	1	1	1
3	*	1	0	1	0	1	→ 0 ←	1	1	1
4	*	1	0	0	0	1	1	1	1	1
5		0	1	1	0	0	0	0	1	1
6		0	1	0	0	0	1	0	1	1
7		0	0	1	0	1	0	1	0	1
8		0	0	0	0	1	1	1	0	1

* - critical rows (All true premises)

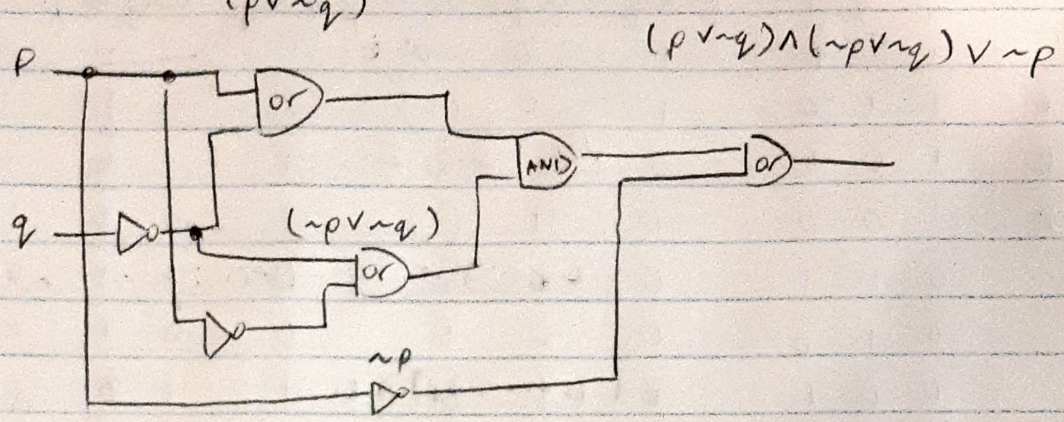
Row 3 has true premises but a false conclusion. This means that this form of argument is invalid. For an argument to be valid, all critical rows need to have true conclusions. Here, row 3 is critical but it has a false conclusion. Therefore, this argument is invalid.

	1	2	3	4	5	6
(4)	p	q	$p \to q$	$q \to p$	$3 \wedge 4$	$p \leftrightarrow q$
	1	1	1	1	1	1
	1	0	0	1	0	0
	0	1	1	0	0	0
	0	0	1	1	1	1
					↑ $(p\to q)\wedge(q\to p)$	↑ $p\leftrightarrow q$

Since 5 and 6 match in values, they are logically equivalent. In other words

$$(p\to q)\wedge(q\to p) \equiv p\leftrightarrow q.$$

⑤



⑥ Simplify $((p \vee \sim q) \wedge (\sim p \vee \sim q)) \vee \sim p$

$\rightarrow (\sim q \vee (p \wedge \sim p)) \vee \sim p$

$\rightarrow (\sim q \vee c) \vee \sim p$

$\rightarrow \sim q \vee \sim p$ (simplified)

⑦ Rewrite the above using sheffer strokes.

$\sim q \vee \sim p \equiv \sim (q \wedge p)$

$\equiv q \mid p$ or $p \mid q$

By def. of $\mid$, $p \mid q \equiv \sim (p \wedge q)$.

(8) Negate

(1) $\forall x \in R,\ x > 4 \rightarrow x^2 > 16$.

$\rightarrow \exists x \in R \mid x > 4 \wedge x^2 \leq 16$.

$(\sim(\forall x,\ P(x) \rightarrow Q(x)) = \exists x,\ P(x) \wedge \sim Q(x))$

(2) $\forall a,b,c \in Z,\ ((a-b)\ \%\ 2 = 1) \wedge ((b-c)\ \%\ 2 = 1) \rightarrow$
$((a-c)\ \%\ 2 = 1)$.

$\rightarrow \exists a,b,c \in Z \mid ((a-b)\ \%\ 2 = 1) \wedge ((b-c)\ \%\ 2 = 1) \wedge$
$((a-c)\ \%\ 2 \neq 1)$.

$(\sim(\forall x,\ (P(x) \wedge Q(x)) \rightarrow R(x)) = \exists x \mid (P(x) \wedge Q(x)) \wedge \sim R(x))$.

(9) Convert BE_{16} to binary, octal, decimal.

$BE_{16} = 1011\ 1110_2$

$= 2\ \ 7\ \ 6_8$

$= (11 \times 16^1) + (14 \times 16^0)$
$= 176 + 14 = 190_{10}$

(10) Derive 2's comp of 67_{10}. Add this complement to BE_{16}. Display the results.

$67_{10} = 64 + 2 + 1 = 01000011$

Then $-67_{10} =$

$$\begin{array}{r} 10111100 \\ +\quad 1 \\ \hline 10111101_2 \end{array}$$

Then, $BE_{16} + (-67_{10}) =$

$$\begin{array}{r} 10111110_2 \\ +\ 10111101_2 \\ \hline \cancel{1}01111011_2 \end{array}$$

Truncated →

→ $BE_{16} + (-67_{10}) = 01111011_2$

$= 1\ 7\ 3_8$

$= 7B_{16}$.

$= 123_{10}$.

(11) Day of the week = (Current + Days Passed) % 7

= (Thursday + 662) % 7

= (3 + 662) % 7

= 665 % 7

= 0

0	1	2	3	4	5	6
M	T	W	TH	F	S	Sun

If today is Thursday, then 662 days from today is a Monday.

(12) Prove $\sum_{i=0}^{n} c^i = \dfrac{1-c^{n+1}}{1-c}$

proof: Let $P(n)$ be the equation given above for any integers.

① $P(0) = \sum_{i=0}^{0} c^i = c^0 = 1$ and $\dfrac{1-c^{0+1}}{1-c^1} = 1$

so, $P(1)$ is true.

② Suppose $P(k)$ is true for some integer k.

Then $P(k+1) = \sum_{i=0}^{k+1} c^i = \underbrace{\sum_{i=0}^{k} c^i}_{P(k)} + c^{k+1}$

$$\rightarrow \frac{1-c^{k+1}}{1-c} + \underbrace{c^{k+1}}_{\frac{(1-c)c^{k+1}}{1-c}}$$

$$\rightarrow \frac{1-c^{k+1} + (1-c)c^{k+1}}{1-c} = \underbrace{\frac{1-c^{k+1} + c^{k+1} - c^{k+2}}{1-c}}_{P(k+1)}$$

$P(k+1) = \dfrac{1-c^{k+2}}{1-c}$ so we've shown this.

Hence, $\sum_{i=0}^{n} c^i = \dfrac{1-c^{n+1}}{1-c}$ for all n.

(13)

$33 + 34 + \cdots + 134$

$$= \sum_{i=1}^{134} i - \sum_{i=1}^{32} i$$

$$= \frac{(134)(135)}{2} - \frac{(32)(33)}{2}$$

$$= (135)(67) - (33)(16)$$

$$= 8517.$$

(14)

Sum $\{85, 95, 105, \ldots, 775, 785\}$

$785 - 85 + 1 = 701$ ( )

$a_n = 85 + (n-1)10$

$85 + 10n - 10$

$a_n = 75 + 10n$

$785 - 75 = 10n$, $n = 710/10 = 71$ terms.

$$\sum_{i=0}^{70} 85 + 10i$$

$$= \sum_{i=0}^{70} 85 + \sum_{i=0}^{70} 10i$$

$$(71)(85) + 10 \cdot \frac{(70)(71)}{2}$$

$$6035 + 24850$$

$$= 30,885.$$

(a) $t_1 = 0$, $t_2 = 1$

$t_n = 3t_{n-1} - 2t_{n-2}$, and $t(n) = f(n)$.

$t_3 = 3t_2 - 2t_1 = 3 - 0 = 3 = 4 - 1$

$t_4 = 3t_3 - 2t_2 = 9 - 2 = 7 = 8 - 1$

$t_5 = 3t_4 - 2t_3 = 21 - 6 = 15 = 16 - 1$

$t_6 = 3t_5 - 2t_4 = 45 - 14 = 31 = 32 - 1$

$\vdots$

$\uparrow$ powers of 2

$t_n = 2^{n-1} - 1$.

(b) $f(n) = 2^{n-1} - 1$, develop a recursive function for c_i.

$f(n-1) = 2^{n-2} - 1$ $\qquad 2^n \cdot 2^{-2} - 1$

$f(n) = f(n-1) \cdot 2$

so $c_i = f(n) = 2 \cdot f(n-1)$.

(17) $1.1\underline{3}$, $\quad .1\underline{3} = \frac{1}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots$

$$= \frac{1}{10} + \frac{3}{100}\Big(\underbrace{1 + \frac{1}{10} + \frac{1}{100} + \cdots}_{1/1-c \,=\, 10/9}\Big)$$

$1.1\underline{3} =$

$1 + 1/10 + (3/100 \cdot 10/9)$

$= 1 + 1/10 + 1/30$

$= 1\ 2/15$ or $17/15$.

$1.4\underline{25}$ $\quad .4\underline{25} = \frac{4}{10} + \frac{25}{1000} + \frac{25}{10000} + \cdots$

$$= \frac{4}{10} + \frac{25}{1000}\Big(\underbrace{1 + \frac{1}{10} + \frac{1}{100} + \cdots}_{10/9}\Big)$$

$1.4\underline{25} = 1 + 4/10 + (25/1000 \cdot 10/9)$

$= 1 + 4/10 + 1/36 = \frac{257}{180}$ or $1\ 77/180$.

(18)

Reminders will be between 0 and B-1.
This means that you can have atmost
B remainders between 0 - B-1. If you
have more, the pigeonhole principle
states that you'll have repetition from
there onwards.

So the max length of the repeated
decimal can't exceed B digits.

(19) $p = 0.9$ and $q = 0.1$, find when $n = 4$.

$$\sum_{i=0}^{n-1} \binom{n}{i} p^i q^{n-i}$$

$$= \sum_{i=0}^{3} \binom{4}{i} p^i q^{4-i}$$

$$= \binom{4}{0} p^0 q^4 + \binom{4}{1} p^1 q^3 + \binom{4}{2} p^2 q^2 + \binom{4}{3} p^3 q^1$$

$$= q^4 + 4pq^3 + 6p^2q^2 + 4p^3q^1.$$

(20) $\sum_{i=0}^{n} \binom{n}{i} 0.5^i = \sum_{i=0}^{n} \binom{n}{i} 0.5^i \underbrace{1^{n-i}}_{1}$

$= (1+0.5)^n = 1.5^n.$

(21)

$$f(n) = \sum_{j=1}^{n} j^{0.5} .$$

power is 0.5, so

$$\Omega(g(n)) = \Omega(x^{0.5})$$

$$O(g(n)) = O(x^{0.5})$$

22

$P((D_1 = 2) \wedge (D_2 = 3))$

$1/5 \quad \wedge \quad 1/5 \quad = \quad 1/25.$

$P((D_1 = 2) \vee (D_2 = 3))$

$= P(A) + P(B) - P(A \wedge B)$

$= 1/5 + 1/5 - 1/25 \quad = \quad 9/25.$

(23) edges = 13 (Tree)

Then it has 14 vertices

(n vertices, n-1 edges)

$Deg(T) = 2 \cdot 13 = 26$