Ch. 5 Homework Cover Page: Aamir Khan

Problems Done:

5.1: 1, 8, 18, 27, 35, 44, 51, 53, 61, 75, 88 (ALL)

5.2: 3, 8, 14, 15, 20, 23, 28, 30, 33, 37 (ALL)

5.6: 3, 7, 13, 17, 20, 24, 27, 33, 39, 41 (ALL)

5.7: 5, 10, 14, 19, 25, 30, 35, 39, 45, 50 (ALL)

Problems Not Done: NONE

Ch. 5 Sect. 1 Prob. 1

write the first 4 terms.

$$a_k = \frac{k}{10+k}, \text{ for all integers } k \geq 1.$$

$$a_1 = \frac{1}{10+1} = \frac{1}{11}.$$

$$a_2 = \frac{2}{10+2} = \frac{2}{12} = \frac{1}{6}.$$

$$a_3 = \frac{3}{10+3} = \frac{3}{13}.$$

$$a_4 = \frac{4}{10+4} = \frac{4}{14} = \frac{2}{7}.$$

Finally, $a_1 = \frac{1}{11}$, $a_2 = \frac{1}{6}$, $a_3 = \frac{3}{13}$, and $a_4 = \frac{2}{7}$.

Ch. 5 Sect. 1 Prob. 8

Compute the 1st 15 terms, and describe the general behavior in words.

$g_n = \lfloor \log_2 n \rfloor$ for all integers $n \geq 1$.

$g_1 = \lfloor \log_2 1 \rfloor = 0$

$g_2 = \lfloor \log_2 2 \rfloor = 1$

$g_3 = \lfloor \log_2 3 \rfloor = 1$

$g_4 = \lfloor \log_2 4 \rfloor = 2$

$g_5 = \lfloor \log_2 5 \rfloor = 2$

$g_6 = \lfloor \log_2 6 \rfloor = 2$

$g_7 = \lfloor \log_2 7 \rfloor = 2$

$g_8 = \lfloor \log_2 8 \rfloor = 3$

$g_9 = \lfloor \log_2 9 \rfloor = 3$

$g_{10} = \lfloor \log_2 10 \rfloor = 3$

$g_{11} = \lfloor \log_2 11 \rfloor = 3$

$g_{12} = \lfloor \log_2 12 \rfloor = 3$

$g_{13} = \lfloor \log_2 13 \rfloor = 3$

$g_{14} = \lfloor \log_2 14 \rfloor = 3$

$g_{15} = \lfloor \log_2 15 \rfloor = 3$

If n is a power of 2, then g_n is the exponent that 2 is raised to. For example, let $n = 4 = 2^2$, then $g_4 = 2$. Every term in the sequence between the current power of 2 and the next power of 2 is the same as g_n for the current power of 2. For example, let $n = 4 = 2^2$, then the next power of 2 is $2^3 = 8$. This means that g_n for terms between 4 and 8 are the same as $g_4 = 2$. Likewise, if $n = 8 = 2^3$, then the next power of 2 is $2^4 = 16$, so, all g_n between g_8 and $g_{16} = g_8 = 3$.

Ch. 9 Sect. 1 Prob. 18

Let $a_0 = 2$, $a_1 = 3$, $a_2 = -2$, $a_3 = 1$, $a_4 = 0$, $a_5 = -1$, & $a_6 = -2$.

Compute each of the following:

a. $\sum_{i=0}^{6} a_i = a_0 + a_1 + \dots + a_6 = 2 + 3 - 2 + 1 + 0 - 1 - 2 = 5 - 4 = 1.$

b. $\sum_{i=0}^{0} a_i = a_0 = 2.$

c. $\sum_{j=1}^{3} a_{2j} = a_2 + a_4 + a_6 = -2 + 0 - 2 = -4.$

d. $\prod_{k=0}^{6} a_k = a_0 \cdot a_1 \cdot a_2 \cdots a_6 = 0$ since one of our terms is 0 and anything times $0 = 0$.

e. $\prod_{k=2}^{2} a_k = a_2 = -2.$

Ch. 5 Sect. 1 Prob. 27

Compute the summation.

$$\sum_{n=1}^{10} \left(\frac{1}{n} - \frac{1}{n+1} \right) =$$

$$(1-\tfrac{1}{2}) + (\tfrac{1}{2}-\tfrac{1}{3}) + (\tfrac{1}{3}-\tfrac{1}{4}) + (\tfrac{1}{4}-\tfrac{1}{5}) + (\tfrac{1}{5}-\tfrac{1}{6}) + (\tfrac{1}{6}-\tfrac{1}{7}) + (\tfrac{1}{7}-\tfrac{1}{8}) + (\tfrac{1}{8}-\tfrac{1}{9}) + (\tfrac{1}{9}-\tfrac{1}{10}) + (\tfrac{1}{10}-\tfrac{1}{11}).$$

$$\sum_{n=1}^{10} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{11}$$

$$= \frac{11}{11} - \frac{1}{11}$$

$$= \frac{10}{11}.$$

Finally,

$$\sum_{n=1}^{10} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 10/11.$$

Ch. 5 Sect. 1 Prob. 35

Evaluate the product for the indicated value of the variable.

$$\left(\frac{1}{1+1}\right)\left(\frac{2}{2+1}\right)\left(\frac{3}{3+1}\right)\cdots\left(\frac{k}{k+1}\right); \quad k = 3$$

For $k = 3$,

$$\prod_{k=1}^{3}\left(\frac{k}{k+1}\right) = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)$$

$$= \frac{1}{4}$$

Ch. 5 Sect. 1 Prob. 44

Write the following using summation notation.

$(1^3-1)-(2^3-1)+(3^3-1)-(4^3-1)+(5^3-1)$

- we have alternating signs, so we'll need an alternator, $(-1)^{n+1}$ $\forall n \geq 1$.

- we cube each n and subtract by 1.

Finally =
$$\sum_{n=1}^{5} (-1)^{n+1} \cdot (n^3-1) =$$

$(1^3-1)-(2^3-1)+(3^3-1)-(4^3-1)+(5^3-1)$.

Ch. 5 Sect. 1 Prob. 51

Write the following using summation notation.

$n + (n-1) + (n-2) + \dots + 1$

- I choose i as my index variable.
- If we start at $i = 0$, then each individual term turns out to be $n - i$.

For ex:

$(n-0) + (n-1) + (n-2) + \dots + (n-(n-1))$

Finally,

$$\sum_{i=0}^{n-1} n-i = n + (n-1) + (n-2) + \dots + 1 \ .$$

- If we went till $i = n$, then our last term would be $n - n = 0$, but we need it to be 1. In this case, we go till $i = n-1$. That way, our last term is $n - (n-1) = 1$.

Ch. 5 Sect. 1 Prob. 53

Make the change of variable, $i = k+1$.

$$\sum_{k=0}^{5} k(k-1).$$ Since $i = k+1$, $k = i - 1$.

If we subsitute $i-1$ for k, we get

$k=0$ becomes $i-1=0$, so $i = 1$. (Lower bound)

$k=5$ becomes $i-1 = 5$, so $i = 6$. (Upper bound)

Finally, $k(k-1)$ becomes $(i-1)(i-1-1)$

we end up with:

$$\sum_{i=1}^{6} (i-1)(i-2)$$

by making the change of variable $i = k+1$.

Ch. 5 Sect. 1 Prob. 61

Write as a single product.

$$\left(\prod_{k=1}^{n} \frac{k}{k+1}\right) \cdot \left(\prod_{k=1}^{n} \frac{k+1}{k+2}\right)$$

$$\rightarrow \prod_{k=1}^{n} \left(\frac{k}{\cancel{k+1}}\right)\left(\frac{\cancel{k+1}}{k+2}\right)$$

$$= \prod_{k=1}^{n} \frac{k}{k+2}$$

$$* \quad \prod a_k \cdot \prod b_k = \prod (a_k \cdot b_k)$$

Ch. 5 Sect. 1 Prob. 75

Compute. Assume the values of the variables are restricted so that the expression is defined.

$$\binom{n}{n-1}$$

$$* \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Then, $\binom{n}{n-1} = \dfrac{n!}{(n-1)!(n-(n-1))!}$

- $n! = n \cdot (n-1)!$
- $(n-(n-1))! = 1! = 1$

Finally, $\binom{n}{n-1} = \dfrac{n \cdot \cancel{(n-1)!}}{\cancel{(n-1)!} \cdot 1} = n$

$$\binom{n}{n-1} = n.$$

Ch. 5 Sect. 1 Prob. 88

Convert to hexadecimal notation.

287_{10}

$\frac{287}{16} = 17 \text{ r } 15$ $(15 = F_{16})$

$\frac{17}{16} = 1 \text{ r } 1$

$\frac{1}{16} = 0 \text{ r } 1$

Finally, $287_{10} = 11F_{16}$.

(Our answer is made up of our remainders in reverse order.)

- we divide our number by our base, set aside the remainders, and keep dividing the quotient until we end up with a quotient of 0.

$$11F_{16} = (15 \times 16^0) + (1 \times 16^1) + (1 \times 16^2)$$
$$= 15 + 16 + 256$$
$$= 287_{10}$$

Ch. 5 Sect. 2 Prob. 3

For each positive integer n, let $P(n)$ be the formula

$$1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

a. write $P(1)$. Is $P(1)$ true?

$$P(1) = \frac{(1)(1+1)(2(1)+1)}{6} = \frac{6}{6} = 1.$$

$P(1)$ is true because $P(1) = 1^2 = 1$ which is what we ended up with.

b. write $P(k)$.

$$P(k) = \frac{k(k+1)(2k+1)}{6}.$$

c. write $P(k+1)$.

$$P(k+1) = \frac{(k+1)(k+2)(2k+3)}{6}$$

d. In a proof by mathematical induction that the formula holds for all integers $n \geq 1$, what must be shown in the inductive step?

we'd need to show that for some integer $k \geq 1$,

if $P(k) = \frac{k(k+1)(2k+1)}{6}$, then

$$P(k+1) = \frac{(k+1)(k+2)(2k+3)}{6}.$$

Ch. 5 Sect. 2 Prob. 8

Prove using mathematical induction.

For all integers $n \geq 0$, $1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$.

Proof: The property $P(n)$ is the equation

$$1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1.$$

① Show $P(0)$ is true:

$P(0) = 1 = 2^{0+1} - 1 = 2 - 1 = 1.$

Thus, $P(0) = 1$ on both sides and is true.

② Show that if $P(k)$ is true for all integers $k \geq 0$, then $P(k+1)$ is true:

Let k be any integer ≥ 0, and suppose $P(k)$ is true.

Then, $P(k) = 1 + 2 + 2^2 + \dots + 2^k = 2^{k+1} - 1$

Also, $P(k+1) = 1 + 2 + 2^2 + \dots + 2^{k+1} = 2^{k+2} - 1$

→ The left-hand side of $P(k+1)$ is

$$\underbrace{1 + 2 + 2^2 + \dots + 2^k} + 2^{k+1}$$

$$= (2^{k+1} - 1) + 2^{k+1}$$

$$= (2^{k+1} + 2^{k+1}) - 1$$

$$= (2 \cdot 2^{k+1}) - 1$$

$$= 2^{k+2} - 1,$$

so $P(k+1)$ is true if $P(k)$ is true.

$\therefore$ $P(n)$ is true for all integers $n \geq 0$.

Ch. 5 Sect. 2 Prob. 14

Prove by mathematical induction.

$$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2, \text{ for all integers } n \geq 0.$$

① Show $P(0)$ is true:

$$P(0) = \sum_{i=1}^{0+1} i \cdot 2^i = 1 \cdot 2^1 = 0 \cdot 2^{0+2} + 2 = 2.$$

since both sides of $P(0) = 2$,

it is true.

② Show that if $P(k)$ is true, then $P(k+1)$ is true:

Let k be any integer ≥ 0, and suppose $P(k)$ is true.

Then, $P(k) = \sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{k+2} + 2$.

Also, $P(k+1) = \sum_{i=1}^{k+2} i \cdot 2^i = (k+1) \cdot 2^{k+3} + 2$.

The left-hand side of $P(k+1)$ is:

$$\left(\sum_{i=1}^{k+1} i \cdot 2^i \right) + (k+2) \cdot 2^{k+2}$$

$$= (k \cdot 2^{k+2} + 2) + (k+2 \cdot (2^{k+2}))$$

$$= 2^{k+2} (k + k + 2) + 2$$

$$= 2^{k+2} (2k + 2) + 2$$

$$= 2^{k+2} (2(k+1)) + 2$$

$$= (k+1) \cdot 2^{k+3} + 2$$ which is the right-hand side of $P(k+1)$.

$\therefore$ $P(n)$ is true for all integers $n \geq 0$.

Ch. 5 Sect. 2 Prob. 15

Prove by mathematical induction,

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1, \text{ for all integers } n \geq 1.$$

Proof. Let $P(n) = \sum_{i=1}^{n} i(i!) = (n+1)! - 1$

① Show $P(1)$ is true:

$P(1) = \sum_{i=1}^{1} i(i!) = 1 \cdot 1! = (2)! - 1 = 2 - 1 = 1.$

Since both sides of $P(1) = 1$,

$P(1)$ is true.

② Show that if $P(k)$ is true, then $P(k+1)$ is true:

Let k be any integer ≥ 1, and suppose $P(k)$ is true.

Then $P(k) = \sum_{i=1}^{k} i(i!) = (k+1)! - 1$.

Also, $P(k+1) = \sum_{i=1}^{k+1} i(i!) = (k+2)! - 1$.

$$= \underbrace{\sum_{i=1}^{k} i(i!)} + (k+1)(k+1!)$$

$$= (k+1)! - 1 + (k+1)!(k+1)$$

$$= (k+1)! [1 + (k+1)] - 1$$

$$= \underbrace{(k+1)! [k+2]} - 1$$

$$= (k+2)! - 1$$ which is the right-hand side of $P(k+1)$.

$\therefore$ $P(n)$ is true for all integers $n \geq 1$.

Ch. 5 Sect. 2 Prob. 20

Use the formula for the sum of the first n integers and/or the formula for the sum of a geometric sequence to evaluate the sum.

$4 + 8 + 12 + 16 + \cdots + 200$

$= 4(1 + 2 + 3 + 4 \cdots + 50)$

* Sum of the first n integers $= \frac{n(n+1)}{2}$

Then,

$$4 \cdot \frac{50(51)}{2} = 2 \cdot 50(51) = 2(2550)$$

$$= 5100.$$

Ch. 5 Sect. 2 Prob. 23

Same instructions as # 20 on previous page.

$$7 + 8 + 9 + 10 + \cdots + 600$$

$$= (1 + 2 + 3 + \cdots + 600) - (1 + 2 + 3 + 4 + 5 + 6)$$

$$\frac{600(601)}{2} - 21$$

$$= 300(601) - 21$$

$$= 180,279.$$

$$\begin{array}{r} 601 \\ \times\ 300 \\ \hline 000 \\ 0000 \\ 180300 \\ -\ 21 \\ \hline 180279 \end{array}$$

Ch. 5 Sect 2 Prob. 28

Same instructions as # 20 (2 pages back).

$1 + 1/2 + 1/2^2 + 1/2^3 + \dots + 1/2^n$, where n is a positive integer.

* Sum of a Geometric sequence:

For any real # r except 1 and any integer $n \geq 0$,

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$

we have $$\sum_{i=0}^{n} \frac{1}{2}^i = \frac{1/2^{n+1} - 1}{1/2 - 1} = \frac{1/2^{n+1} - 1}{-1/2}$$

$$= -2 \cdot (1/2^{n+1} - 1)$$

$$= 2 - 2/2^{n+1}$$

$$= 2 - 1/2^n.$$

- 1 raised to any power is 1, so $1^{n+1} = 1$.
- $2/2^{n+1} = 1/2^{n+1-1}$ or $1/2^n$.

Ch. 5 Sect 2 Prob. 30

Find a formula in n, a, m, and d for the sum

$(a+md)+(a+(m+1)d)+(a+(m+2)d)+\cdots+(a+(m+n)d)$,

where m and n are integers, $n \geq 0$, and a and d are real numbers.

Hint: $c+(c+d)+(c+2d)+\cdots+(c+nd)$
$= (n+1)c + d \cdot \frac{n(n+1)}{2}$.

$(a+md)+(a+(m+1)d)+\cdots+(a+(m+n)d)$

* Since we have a in every term and we have $n+1$ terms which is where we get $(n+1)a$.
* we also have md in every term and we also have $(n+1)$ of it so we have $(n+1)md$.
* Finally, we $d+2d+\cdots+nd$ or $d(1+2+\cdots+n)$ so we have $d \cdot \frac{n(n+1)}{2}$. $\quad \frac{n(n+1)}{2}$

→ we have a common factor of $n+1$, so we get:
$(n+1)[a+md+dn/2]$.

$(a+md)+(a+md+2d)+(a+md+3d)+\cdots+(a+md+nd)$
we have a $n+1$ times, md $n+1$ times, and
$d(1+2+\cdots+n) = d \cdot n(n+1)/2$.

Ch. 5 sect. 2 Prob. 33

Find the mistakes.

Theorem: For any integer $n \geq 1$,

$1^2 + 2^2 + \dots + n^2 = n(n+1)(2n+1)/6$.

"Proof by mathematical induction: Certainly the theorem is true for $n=1$ because $1^2 = 1$ and $1(1+1)(2\cdot1+1)/6 = 1$. So the basis step is true. For the inductive step, suppose that for some integer $k \geq 1$, $k^2 = k(k+1)(2k+1)/6$. We must show that $(k+1)^2 = (k+1)(k+1+1)(2(k+1)+1)/6$."

① The inductive step should be $1^2 + 2^2 + \dots + k^2 = k(k+1)(2k+1)/6$,
NOT $k^2 = k(k+1)(2k+1)/6$. (Forgot all other terms on the lefthand side.)

② We'd need to show that $1^2 + 2^2 + \dots + (k+1)^2 = (k+1)(k+2)(2k+3)/6$,
but it says above that we'd need to show that
$(k+1)^2 = (k+1)(k+2)(2k+3)/6$ which is wrong.

Ch. 5 Sect. 2 Prob. 37

Prove that if p is any prime number with $p \geq 5$, then the sum of squares of any p consecutive integers is divisible by p.

Proof: Let p be any prime number with $p \geq 5$ and suppose $n+1, n+2, \dots, n+p$ are p consecutive integers. Then, the sum of the squares is

$$n^2 + (n+1)^2 + (n+2)^2 + \dots + (n+(p-1))^2$$

$$= n^2 + (n^2 + 2n + 1^2) + (n^2 + 4n + 2^2) + \dots + (n^2 + 2(np-n) + (p-1)^2)$$

$$= n^2 + (n^2 + 2n + 1^2) + (n^2 + 2(2n) + 2^2) + \dots + (n^2 + 2n(p-1) + (p-1)^2)$$

$$= pn^2 + [2n + 4n + 6n + \dots + 2n(p-1)] + [1 + 4 + 6 + \dots + (p-1)^2]$$

→ we have p terms and all terms have n^2, so we end up w/ p n^2 terms or pn^2.

→ we have multiples of $2n$ in each term so we take the sum of all of them.

→ we have the sum of all squares starting w/ $1^2, 2^2, \dots, (p-1)^2$, so we add that sum to our result.

Then, we get

$$[pn^2 + 2n(1 + 2 + \dots + (p-1))] + (1^2 + 2^2 + 3^2 + \dots + (p-1)^2)$$

Since we have pn^2, we know that pn^2 is divisible by p because we have the common factor p.

Then, simply adding $2n(1+2+\dots+(p-1)) + (1^2 + 2^2 + \dots + (p-1)^2)$ will produce a new number, but we still will have the common factor p since $p \mid n^2$ which means $p \mid n$, so $2n(1+2+\dots+(p-1))$ is divisible by p. Similarly, p divides $(1^2 + 2^2 + \dots + (p-1)^2)$ because we'll have a p^2 term at the end, and p^2 is divisible by p.

∴ the sum of squares of any p consecutive integers is divisible by p if p is a prime ≥ 5.

Ch. 5 Sect. 6 Prob. 3

Find the 1st 4 terms.

$c_k = k(c_{k-1})^2$, for all integers $k \geq 1$.

$c_0 = 1$

① Term $c_1 = 1(c_{1-1})^2$

$= 1\, c_0^2$

$= 1(1) = 1$.

② Term $c_2 = 2(c_{2-1})^2$

$= 2\, c_1^2$

$= 2(1) = 2$.

③ Term $c_3 = 3(c_{3-1})^2$

$= 3\, c_2^2$

$= 3(2)^2 = 12$.

Finally, the 1st 4 terms are:

$c_0 = 1$

$c_1 = 1$

$c_2 = 2$

$c_3 = 12$

Ch. 5 Sect. 6 Prob. 7

Find the 1st 4 terms.

$U_k = kU_{k-1} - U_{k-2}$, for all integers $k \geq 3$.

$U_1 = 1$ and $U_2 = 1$

$U_3 = 3(U_{3-1}) - U_{3-2}$

$= 3U_2 - U_1$

$= 3(1) - 1 = 2.$

$U_4 = 4(U_{4-1}) - U_{4-2}$

$= 4U_3 - U_2$

$= 4(2) - 1 = 7.$

Finally, the 1st 4 terms are:

$U_1 = 1$

$U_2 = 1$

$U_3 = 2$

$U_4 = 7$

ch. 9 sect. 6 prob. 13

Let $t_0, t_1, t_2, \ldots$ be defined by the formula $t_n = 2+n$ for all integers $n \geq 0$. show that this sequences satisfies the recurrence relation $t_k = 2t_{k-1} - t_{k-2}$.

① $t_{k-2} = 2+(k-2) = k$.

② $t_{k-1} = 2+(k-1) = k+1$.

③ $t_k = 2+k = k+2$.

Then, for all integers $k \geq 2$,

$$\begin{aligned} t_k &= 2t_{k-1} - t_{k-2} \\ &= 2(k+1) - k \\ &= 2k + 2 - k \\ &= k + 2 \\ &= t_k \text{ defined by } t_n = 2+n. \end{aligned}$$

Ch. 5 Sect. 6 Prob. 17

Tower of Hanoi w/ Adjacency Requirement: suppose that in addition to the requirement that they never move a larger disk on top of a smaller disk, the priests who move the disks of the Tower of Hanoi are also allowed only to move disks one by one from one pole to an adjacent pole. Assume poles A and C are at 2 ends of the row and pole B is in the middle. Let

$$a_n = \left[\begin{array}{l} \text{the minimum number of moves} \\ \text{needed to transfer a tower of } n \\ \text{disks from pole A to pole C} \end{array}\right].$$

a) Find a_1, a_2, and a_3.

$a_1 = 2$ because we'd go from $A \to B$ then $B \to C$.

$a_2 =$ 2 (moves to move the top disk from A to C)

+ 1 (move to move the bottom disk from A to B)

+ 2 (moves to move the top disk from C to A)

+ 1 (move to move the bottom disk from B to C)

+ 2 (moves to move the top disk from A to C)

= 8 moves.

$a_3 = 8+1+8+1+8 = 26$.

b) Find a_4. $a_4 = 26+1+26+1+26 = 3(26)+2 = 80$.

c) Find a recurrence relation for $a_1, a_2, a_3, \ldots$

→ Notice that for each a_n, we have 3 times the previous term and we add 2. For ex.

$a_2 = 2+1+2+1+2 = 3(2)+2$ where $a_1 = 2$.

$a_3 = 8+1+8+1+8 = 3(8)+2$ where $a_2 = 8$.

Then, $a_k = 3a_{k-1} + 2$, for all integers $k \geq 2$.

$a_1 = 2$.

Ch. 5 Sect. 6 Prob. 20

Tower of Hanoi Poles in a Circle: suppose that instead of being lined up in a row, the 3 poles are placed in a circle. The monks move the disks one by one from one pole to another, but they may only move disks one over in a clockwise direction and they may never move a larger disk on top of a smaller one. Let c_n be the minimum # of moves needed to transfer a pile of n disks from one pole to the next adjacent pole in the clockwise direction.

a) Justify the inequality $c_k \leq 4c_{k-1} + 1$ $\forall$ integers $k \geq 2$.

Consider $a_1 = 1$ (moved from $A \to B$ clockwise)

$a_2 = 2$ (Moves to move top dish from $A \to C$)

$+ 1$ (Move to move bottom dish from $A \to B$)

$+ 2$ (Moves to move top dish from $C \to B$)

$= 5$ steps.

$a_3 = 2$ (Moves to move top disk from $A \to C$)

$+ 1$ (Move to move middle dish from $A \to B$)

$+ 1$ (move to move top dish from $C \to A$)

$+ 1$ (move to move middle dish from $B \to C$)

$+ 2$ (moves to move top disk from $A \to C$).

- In a similar way, a modified version of the steps above can be used to move the bottom disk around. Then, we end up with $7 + 1 + 7 = 15$ moves.

we see $a_2 = 5 \leq 4a_1 + 1 = 4(1) + 1 = 5$ True

$a_3 = 15 \leq 4a_2 + 1 = 4(5) + 1 = 21$ True

→ This can be repeated to show it is true $\forall$ integers $k \geq 2$.

b) Explain why $c_3 \neq 4c_2 + 1$

$c_2 = 5$

$c_3 = 15 \neq 4(5) + 1 = 21$

$15 \neq 21$,

so $c_3 < 4c_2 + 1$.

Ch. 5 Sect. 6 Prob. 24

Use the recurrence relation and values for $F_0, F_1, F_2, \ldots$ to compute F_{13} and F_{14}. (Fibonacci Sequence)

$F_0 = 1$ $\qquad F_n = F_{n-1} + F_{n-2}$

$F_1 = 1$

$F_2 = 2 \quad = F_0 + F_1$

$F_3 = 3 \quad = F_1 + F_2$

$F_4 = 5 \quad = F_2 + F_3$

$F_5 = 8 \quad = F_3 + F_4$

$F_6 = 13 \quad = F_4 + F_5$

$F_7 = 21 \quad = F_5 + F_6$

$F_8 = 34 \quad = F_6 + F_7$

$F_9 = 55 \quad = F_7 + F_8$

$F_{10} = 89 \quad = F_8 + F_9$

$F_{11} = 144 \quad = F_9 + F_{10}$

$F_{12} = 233 \quad = F_{10} + F_{11}$

Finally,

$$F_{13} = F_{11} + F_{12}$$
$$= 144 + 233 = 377$$

$$F_{14} = F_{12} + F_{13}$$
$$= 233 + 377 = 610$$

Ch. 5 Sect. 6 Prob. 27

Prove that $F_k^2 - F_{k-1}^2 = F_k F_{k+1} - F_{k+1} F_{k-1}$,

for all integers $k \geq 1$.

(Use the def. of Fibonacci sequence)

Lefthand-side:

$$F_k^2 - F_{k-1}^2$$

$$= (F_k - F_{k-1})(F_k + F_{k-1})$$

F_{k+1} because

$$F_{k+1} = F_{k+1-1} + F_{k+1-2}$$

$$= F_k + F_{k-1}$$

$$= (F_k - F_{k-1}) F_{k+1}$$

$$= F_k F_{k+1} - F_{k-1} F_{k+1}$$

Ch. 5 Sect. 6 Prob. 33

It turns out that the Fibonacci sequence satisfies the following explicit formula: For all integers $F_n \geq 0$,

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

Verify that the sequence defined by this formula satisfies the recurrence relation $F_k = F_{k-1} + F_{k-2}$ $\forall$ integers $k \geq 2$.

① $$F_{k-2} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right]$$

② $$F_{k-1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\right]$$

③ $$F_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$$

$$F_{k-1} + F_{k-2} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k} + \left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\left(\frac{1-\sqrt{5}}{2}\right)^{k} + \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right] = F_k$$ as defined by the formula.

$\therefore$ $F_k = F_{k-1} + F_{k-2}$ $\forall$ integers $k \geq 2$.

Ch. 9 Sect. 6 Prob. 39

W/ each step you take when climbing a staircase, you can move up either one or two stairs. As a result, you can climb the entire staircase taking one stair at a time, taking two at a time, or taking a combination of one and two stair increments. For each integer $n \geq 1$, if the staircase consists of n stairs, let c_n be the # of different ways to climb the staircase. Find a recurrence relation for $c_1, c_2, c_3, \ldots$

The last step is either one or two stairs taken.

- If the last step is a single stair, the # of different ways to climb n stairs is c_{n-1}.
- If it's two stairs, the # of different ways to climb n stairs is c_{n-2}.

As a result, the total # of different ways to climb n stairs is:

$$c_n = c_{n-1} + c_{n-2}$$

$$c_1 = 1 \text{ and } c_2 = 2.$$

($c_2 = 2$, since two stairs can be climbed one by one or all in one shot as a unit.)

Ch. 5 Sect. 6 Prob. 41

Use the recursive definition of summation, together w/ mathematical induction, to prove the generalized distributive law that for all positive integers n, if $a_1, a_2, \ldots, a_n$ and c are real numbers, then

$$\sum_{i=1}^{n} ca_i = c\left(\sum_{i=1}^{n} a_i\right).$$

Proof: Let $P(n)$ be the equation given above, where $a_1, a_2, \ldots, a_n$ and c are real numbers.

① Show that $P(1)$ is true.

$P(1) = \sum_{i=1}^{1} ca_i = ca_1$ and $\sum_{i=1}^{1} a_i = a_1$, so

$$c \cdot \sum_{i=1}^{1} a_i = c \cdot a_1 = ca_1, \text{ so } P(1) \text{ is true.}$$

② Show that if $P(k)$ is true, then $P(k+1)$ is true $\forall k \geq 1$.

Let k be any integer ≥ 1, and suppose $P(k)$ is true.

Then, $P(k) = \sum_{i=1}^{k} ca_i = c \cdot \sum_{i=1}^{k} a_i$.

$$P(k+1) = \sum_{i=1}^{k+1} ca_i = \sum_{i=1}^{k} ca_i + ca_{k+1}$$

$$= c \cdot \sum_{i=1}^{k} a_i + ca_{k+1}$$

$$= c\left(\sum_{i=1}^{k} a_i + a_{k+1}\right)$$

$$= c \cdot \sum_{i=1}^{k+1} a_i.$$

$\therefore$ the property is true for all positive integers n.

Ch. 9 Sect. 7 Prob. 5

The sequence is defined recursively. Use iteration to guess an explicit formula for the sequence.

$c_k = 3c_{k-1} + 1$, $\forall$ integers $k \geq 2$

$c_1 = 1$

$c_2 = 3c_1 + 1 = 3(1) + 1 = 3 + 1$

$c_3 = 3c_2 + 1 = 3(3+1) + 1 = 3^2 + 3 + 1$

$c_4 = 3c_3 + 1 = 3(3^2 + 3 + 1) + 1 = 3^3 + 3^2 + 3 + 1$

$\vdots$

$c_n = 3^{n-1} + 3^{n-2} + \dots + 3^2 + 3 + 1$

* $1 + r + r^2 + \dots + r^n = \dfrac{r^{n+1} - 1}{r - 1}$

Then, $c_n = 1 + 3 + 3^2 + \dots + 3^{n-2} + 3^{n-1}$, w/ $r = 3$.

$= \dfrac{3^{n+1-1} - 1}{3 - 1}$ (since we go till $n-1$ and not n, we account for that in our formula)

$= \dfrac{3^n - 1}{2}$

Ch. 5 Sect. 7 Prob. 10

Same instructions as #5 on previous page.

$h_k = 2^k - h_{k-1}, \ \forall$ integers $k \geq 1$

$h_0 = 1$

$h_1 = 2^1 - h_0 = 2 - 1$

$h_2 = 2^2 - h_1 = 2^2 - 2 + 1$

$h_3 = 2^3 - h_2 = 2^3 - 2^2 + 2 - 1$

$h_4 = 2^4 - h_3 = 2^4 - 2^3 + 2^2 - 2 + 1$

$\vdots$

$h_n = 2^n - 2^{n-1} + \cdots + (-1)^n \cdot 1$

$= (-1)^n [1 - 2 + 2^2 - \cdots + (-1)^n \cdot 2^n]$

$= (-1)^n [1 + (-2) + (-2)^2 + \cdots + (-2)^n]$

$= (-1)^n \left[\frac{(-2)^{n+1} - 1}{(-2) - 1}\right]$ w/ $r = -2$.

$= (-1)^n \left[\frac{(-2)^{n+1} - 1}{-3}\right]$. So we can make $(-1)^n \to (-1)^{n+1}$, but we need to multiply everything by $-1/-1$.

$= (-1)^{n+1} \left[\frac{(-2)^{n+1} - 1}{(-3) \cdot (-1)}\right]$

$= \frac{(-1)^{n+1} (-2)^{n+1} - (-1)^{n+1}}{3}$

$= \frac{2^{n+1} - (-1)^{n+1}}{3}$

Ch. 5 Sect. 7 Prob. 14

Same instructions as # 5 (2 pages back)

$x_k = 3x_{k-1} + k$, $\forall$ integers $k \geq 2$.

$x_1 = 1$

$$x_2 = 3x_1 + 2 = 3 + 2$$

$$x_3 = 3x_2 + 3 = 3(3+2) + 3 = 3^2 + 3 \cdot 2 + 3$$

$$x_4 = 3x_3 + 4 = 3(3^2 + 6 + 3) + 4 = 3^3 + 3 \cdot 3 \cdot 2 + 3^2 + 4.$$

$$x_5 = 3x_4 + 5 = 3(3^3 + 3^2 \cdot 2 + 3^2 + 4) + 5 = 3^4 + 3^3 \cdot 2 + 3^2 \cdot 3 + 3 \cdot 4 + 5$$

$$x_6 = 3x_5 + 6 = 3(3^4 + 3^3 \cdot 2 + 3^2 \cdot 3 + 3 \cdot 4 + 5) + 6$$

$$= 3^5 + 3^4 \cdot 2 + 3^3 \cdot 3 + 3^2 \cdot 4 + 3 \cdot 5 + 6$$

$\vdots$

$$x_n = 3^{n-1} + 3^{n-2} \cdot 2 + 3^{n-3} \cdot 3 + \dots + 3(n-1) + n$$

$$= 3^{n-1} + \underbrace{3^{n-2} + 3^{n-2}}_{2 \text{ times}} + \underbrace{3^{n-3} + 3^{n-3} + 3^{n-3}}_{3 \text{ times}} + \underbrace{3 + 3 + \dots + 3}_{n-1 \text{ times}} + \underbrace{1 + 1 + \dots + 1}_{n \text{ times}}$$

$$= (3^{n-1} + 3^{n-2} + \dots + 3^2 + 3 + 1) + (3^{n-2} + 3^{n-3} + 3^2 + 3 + 1) + \dots + (3^2 + 3 + 1) + (3+1) + 1$$

$$= \frac{3^n - 1}{2} + \frac{3^{n-1} - 1}{2} + \dots + \frac{3^3 - 1}{2} + \frac{3^2 - 1}{2} + \frac{3 - 1}{2}$$

$$= \frac{1}{2}\left[(3^n + 3^{n-1} + \dots + 3^2 + 3) - n\right]$$

(each term has $\frac{1}{2}(-1)$ and we have n terms, so we get $\frac{1}{2}(-1) \cdot n = -n/2$)

$$= \frac{1}{2}\left[3(3^{n-1} + 3^{n-2} + \dots + 3 + 1) - n\right]$$

$$= \frac{1}{2}\left[3\left(\frac{3^n - 1}{3 - 1}\right) - n\right] = \frac{1}{2}\left[3\left(\frac{3^n - 1}{2}\right) - n\right] = \frac{1}{2}\left[3\left(\frac{3^n - 1}{2}\right) - \frac{2n}{2}\right]$$

$$= \frac{1}{4}\left[3^{n+1} - 3 - 2n\right]$$

Ch. 5 Sect 7 Prob. 19

A worker is promised a bonus if he can increase his productivity by 2 units a day every day for a period of 30 days. If on day 0 he produces 170 units, how many units must he produce on day 30 to qualify for the bonus?

Let U_n = # of units produced on day n, then

$U_n = U_{n-1} + 2$, $\forall$ integers $n \geq 1$:

$U_0 = 170$.

we see that $U_0, U_1, U_2, \ldots$ is an arithmetic sequence w/ constant factor of 2.

$U_1 = 170 + 2$

$U_2 = 170 + 2 + 2$

$U_3 = 170 + 2 + 2 + 2$

$\vdots$

$U_n = 170 + 2 \cdot n$

Finally, if $n = 30$, then

$U_{30} = 170 + 2 \cdot 30$

$= 170 + 60$

$= 230$ units.

The worker must produce 230 units on day 30.

Ch. 5 Sect. 7 Prob. 25

A certain comp. algorithm executes twice as many operations when it's run w/ an input of size k as when it is run w/ an input size of k-1 (where k is an integer >1), when the algorithm is run w/ an input size of 1, it executes 7 operations. How many operations does it execute when it is run w/ an input of size 25?

Let a_n = # of operations on input size of k.

$a_k = 2a_{k-1}$, $\forall$ integers $k \geq 2$.

$a_1 = 7$.

$a_2 = 2a_1 = 2 \cdot 7$

$a_3 = 2a_2 = 2 \cdot 2 \cdot 7$

$\vdots$

$a_n = 2a_{n-1} = \underbrace{2 \cdot 2 \cdots \cdot 2}_{n-1 \text{ times}} \cdot 7 = 7 \cdot 2^{n-1}$

Finally, when $n = 25$

$a_{25} = 7 \cdot 2^{25-1}$

$= 7 \cdot 2^{24}$

$= 117,440,512$.

The computer executes 117,440,512 operations w/ an input of size 25.

Ch. 5 Sect. 7 Prob. 30

Use mathematical induction to verify the correctness of the formula.

$C_k = 3C_{k-1} + 1$, $\forall$ integers $k \geq 2$

$C_1 = 1$.

Proof: Let $P(n)$ be the formula $C_n = \frac{3^n - 1}{2}$, $\forall$ integers $n \geq 1$.

1. Show that $P(1)$ is true:

$P(1) = \frac{3^1 - 1}{2} = \frac{3-1}{2} = \frac{2}{2} = 1 = C_1$, so $P(1)$ is true.

2. Show that if $P(k)$ is true, then $P(k+1)$ is true:

Suppose $C_k = \frac{3^k - 1}{2}$ for some integer $k \geq 1$.

Lefthand side:

$$C_{k+1} = 3C_k + 1$$

$$= 3\left(\frac{3^k - 1}{2}\right) + 1$$

$$= \frac{3^1 \cdot 3^k - 3}{2} + \frac{2}{2}$$

$$= \frac{3^{k+1} - 3 + 2}{2}$$

$$= \frac{3^{k+1} - 1}{2} = P(k+1)$$

$\therefore$ This property is true for all integers $n \geq 1$.

Ch. 5 Sect. 7 Prob. 35

Same instructions as # 30 on previous page.

$h_k = 2^k - h_{k-1}$, for all integers $k \geq 1$

$h_0 = 1$.

we found that $h_n = \dfrac{2^{n+1} - (-1)^{n+1}}{3}$.

Proof: Let $P(n)$ be the formula given above for all integers $n \geq 1$

① Show that $P(1)$ is true:

$h_1 = 2^1 - h_0 = 2-1 = 1$ and $P(1) = \dfrac{2^{1+1} - (-1)^{1+1}}{3} = \dfrac{2^2 - 1}{3} = \dfrac{3}{3} = 1$.

So, $P(1)$ is true.

② Show that if $P(k)$ is true, then $P(k+1)$ is true:

Suppose $h_k = \dfrac{2^{k+1} - (-1)^{k+1}}{3}$ for some integer $k \geq 1$. Then,

$$h_{k+1} = 2^{k+1} - h_k = 2^{k+1} - \left(\frac{2^{k+1} - (-1)^{k+1}}{3}\right)$$

$$= \frac{3 \cdot 2^{k+1}}{3} - \left(\frac{2^{k+1} - (-1)^{k+1}}{3}\right)$$

$$= \frac{2^{k+1}(3-1) - (-1)^{k+1}}{3}$$

$$= \frac{2 \cdot 2^{k+1} + (-1)^{k+1}}{3} = \frac{2^{k+2} - (-1)^{k+2}}{3}.$$

$P(k+1) = \dfrac{2^{k+1+1} - (-1)^{k+1+1}}{3}$ which is what we got above.

∴ the property is true for all integers $n \geq 1$.

Ch. 5 Sect. 7 Prob. 39

Same instructions as # 30 (2 pages back).

$x_n = 3x_{n-1} + n$, $\forall$ integers $n \geq 2$.

$x_1 = 1$.

we found that $x_n = 1/4\,[3^{n+1} - 3 - 2n]$.

Proof: Let $P(n)$ be the formula given above $\forall$ integers $n \geq 1$.

① Show that $P(1)$ is true:

$P(1) = 1/4\,[3^{1+1} - 3 - 2(1)] = 1/4\,[3^2 - 5] = 4/4 = 1 = x_1$.

So, $P(1)$ is true.

② Show that if $P(k)$ is true, then $P(k+1)$ is true:

Suppose $x_k = 1/4\,[3^{k+1} - 3 - 2k]$ $\forall\, k \geq 1$.

$x_{k+1} = 3x_k + (k+1)$

$= 3\left(\frac{1}{4}\left[3^{k+1} - 3 - 2k\right]\right) + k+1 = \frac{1}{4}\left[3^{k+2} - 9 - 6k\right] + k+1$

$= \frac{1}{4}\left[3^{k+2} - 9 - 6k\right] + \frac{4(k+1)}{4} = \frac{1}{4}\left[3^{k+2} - 9 - 6k + 4k + 4\right]$

$= \frac{1}{4}\left[3^{k+2} - 2k - 5\right] = \frac{1}{4}\left[3^{k+1+1} - 2(k+1) - 3\right] = P(k+1)$.

$\therefore$ the property is true for all integers $n \geq 1$.

Ch. 5 Sect. 7 Prob. 45

The sequence is defined recursively. Use iteration to guess an explicit formula. Then, use mathematical induction to verify the formula.

$$v_k = v_{\lfloor k/2 \rfloor} + v_{\lfloor (k+1)/2 \rfloor} + 2, \ \forall \text{ integers } k \geq 2.$$
$$v_1 = 1.$$

$v_2 = v_{\lfloor 2/2 \rfloor} + v_{\lfloor (2+1)/2 \rfloor} + 2 = v_1 + v_1 + 2 = 1 + 1 + 2 = 4.$

$v_3 = v_{\lfloor 3/2 \rfloor} + v_{\lfloor (3+1)/2 \rfloor} + 2 = v_1 + v_2 + 2$
$= 1 + (1+1+2) + 2 = 3 + 2 \cdot 2$

$v_4 = v_{\lfloor 4/2 \rfloor} + v_{\lfloor (4+1)/2 \rfloor} + 2 = v_2 + v_2 + 2$
$= (1+1+2) + (1+1+2) + 2 = 4 + 2 \cdot 3$

⋮

$v_n = n + 2(n-1) = 3n - 2, \ \forall n \geq 1.$

Proof: Let $v_1, v_2, v_3, \ldots$ be the recursively defined sequence w/ $v_1 = 1$ and $v_k = v_{\lfloor k/2 \rfloor} + v_{\lfloor (k+1)/2 \rfloor} + 2, \ \forall k \geq 2.$

Let $P(n)$ be the formula $3n - 2, \ \forall n \geq 1.$

① Show that $P(1)$ is true:

$P(1) = 3(1) - 2 = 3 - 2 = 1 = v_1$, so $P(1)$ is true;

② Show that if $P(k)$ is true $\forall k \geq 1$, then $P(k+1)$ is true:

$v_{k+1} = v_{\lfloor (k+1)/2 \rfloor} + v_{\lfloor (k+1+1)/2 \rfloor} + 2$
$= (3\lfloor k+1/2 \rfloor - 2) + (3\lfloor k+2/2 \rfloor - 2) + 2$
$= 3(\lfloor k+1/2 \rfloor + \lfloor k+2/2 \rfloor) - 2$

$$= \begin{cases} 3\left(\frac{k}{2} + \frac{k+2}{2}\right) - 2 & \text{if } k \text{ is even.} \\ 3\left(\frac{k+1}{2} + \frac{k+1}{2}\right) - 2 & \text{if } k \text{ is odd.} \end{cases}$$

$= 3\left(\frac{2k+2}{2}\right) - 2 = 3(k+1) - 2 = 3k + 1.$

$P(k+1) = 3(k+1) - 2 = 3k + 1$, which we got above.

∴ this property is true for all integers $n \geq 1$.

Ch. 5 Sect. 7 Prob. 50

Determine whether the recursively defined sequence satisfies the formula $a_n = (n-1)^2$, $\forall n \geq 1$.

$a_k = 2a_{k-1} + k - 1$, $\forall$ integers $k \geq 2$.

$a_1 = 0$

$a_2 = 2a_1 + 2 - 1 = 2(0) + 1 = 1$ and $(2-1)^2 = 1$. True

$a_3 = 2a_2 + 3 - 1 = 2(1) + 2 = 4$ and $(3-1)^2 = 4$. True

$a_4 = 2a_3 + 4 - 1 = 2(4) + 3 = 11$ and $(4-1)^2 = 9$. False

We see that when $n = 4$, the recursively defined sequence doesn't satisfy the formula $a_n = (n-1)^2$.

$\therefore$ this recursively defined sequence does NOT satisfy the formula.