Quiz 4b : (4-6-2022) Aamir K.

① $\forall a,b \in \mathbb{R}, (a<b) \to (a < a+b/2 < b)$

Proof : suppose a and b are any real numbers such that $a<b$.

case 1 : $a < a+b/2$. Then

$2a < a+b$ and since $a<b$, $a+b > a+a$,

so $2a < a+b$. Hence, this equality is true and $a < a+b/2$.

case 2 : $\frac{a+b}{2} < b$. Then,

$a+b < 2b$ and since

$a<b$, $a+b < b+b$, so $a+b < 2b$.

Hence, this equality is also true and $\frac{a+b}{2} < b$.

$\therefore$ since $a < \frac{a+b}{2}$ and $\frac{a+b}{2} < b$,

$a < \frac{a+b}{2} < b$ if $a<b$.

② Prove, $\forall n \in \mathbb{Z}, (n \bmod 5 = 3) \to (n^2 \bmod 5 = 4)$

Proof. Suppose n is any integer such that $n \bmod 5 = 3$. Then, that would imply $n = 5q + 3$ for some integer q.

Then, $n^2 = (5q+3)^2 = (5q+3)(5q+3) = 25q^2 + 30q + 9$

$$n^2 = 25q^2 + 30q + 9 = 25q^2 + 30q + 5 + 4$$
$$= 5(5q^2 + 6q + 1) + 4$$

Let $z = 5q^2 + 6q + 1$. Then z is an integer as products and sums of integers are integers.

$n^2 = 5z + 4$. We can see that we have a remainder of 4, so $n^2 \bmod 5 = 4$.

$\therefore$ $n^2 \bmod 5 = 4$ if $n \bmod 5 = 3$.

③ Prove by contradiction. $\forall r, s: ((r \in \mathbb{Q}) \wedge (s \notin \mathbb{Q})) \to ((r+s) \notin \mathbb{Q})$

Proof: Suppose r is any rational number and s is any irrational number such that $r+s$ is rational. Then,

$r = a/b$ for some integers a and b, $b \neq 0$.

$r+s = c/d$ for some integers c and d, $d \neq 0$.

$r+s = \frac{a}{b} + s = \frac{c}{d}$. Then, $s = \frac{c}{d} - \frac{a}{b}$

$$s = \frac{c}{d} - \frac{a}{b} = \frac{bc}{bd} - \frac{ad}{bd} = \frac{bc - ad}{bd}.$$

Since the numerator and denominator consist of integers, they are closed under mult. and subtraction.

Thus, $s = \frac{e}{f}$ for some integers e and f, $f \neq 0$.

s is both irrational and rational which is a contradiction.

$\therefore$ $r+s \notin \mathbb{Q}$ if $r \in \mathbb{Q}$ and $s \notin \mathbb{Q}$.