Ch. 4 HW Cover Page: Aamir Khan

Problems Done:

4.1: 4, 9, 11, 12, 15, 20, 25 (ALL)

4.2: 4, 7, 10, 14, 15, 20, 24 (ALL)

4.3: 4, 9, 10, 13, 15, 22, 23, 27 (ALL)

4.4: 5, 8, 11, 16, 20, 22, 26 (ALL)

4.6: 4, 5, 10, 12, 15, 21, 22 (ALL)

4.7: 4, 5, 10, 12, 15, 21, 22, 27 (ALL)

Problems Not Done: NONE

Ch. 4 Sect. 1 Prob. 4

Prove the statement.

There are integers m and n such that $m>1$ and $n>1$ and $1/m + 1/n$ is an integer.

(This is an existential statement, so we only need to find one case for which this statement is true to prove its truth.)

For example, let $m=2$ and $n=2$.

then $1/m + 1/n = 1/2 + 1/2 = 1$, which is an integer.

$\therefore$ $m>1$ and $n>1$ and $1/m + 1/n$ is an integer.

Ch. 4 Sect. 1 Prob. 5

Prove the statement.

There are distinct integers m and n such that $1/m + 1/n$ is an integer.

For example, let $m = 4$ and $n = -4$, then

$1/m + 1/n = 1/4 + (1/-4) = 1/4 - 1/4 = 0$, which is an integer.
(m and $n \neq 0$ because we can't have division by 0.)

$\therefore$ $m \neq n$ (distinct) and $1/m + 1/n$ is an integer.

Ch. 4 Sect. 1 Prob. 11

Disprove the statement by giving a counter example.

For all real numbers a and b, if $a < b$ then $a^2 < b^2$.

(This is a universal conditional. To disprove it, we need one example such that:

$\forall x,\ P(x) \wedge \sim Q(x)$ or that $a<b$ and $a^2 \not< b^2$.)

For example, let $a = -3$ and $b = 1$, then

$a<b \rightarrow -3<1$, which is true. And...

$a^2<b^2 \rightarrow 9<1$, which is false.

$\therefore$ the negation is true which means that the statement has been disproved using a counterexample.

Ch. 4 Sect. 1 Prob. 12

Disprove by giving a counter example.

For all integers n, if n is odd then $n-1/2$ is odd.

(we need to prove the negation in order to disprove the statement, we find a counter example to do this.)

For example, let $n=5$, then

n is odd AND $n-1/2 = 5-1/2 = 4/2 = 2$, which is even.

$\therefore$ the statement has been disproved by proving the negation.

$\sim(\forall x, P(x) \rightarrow Q(x)) \equiv \exists x \mid P(x) \wedge \sim Q(x))$

Ch. 4 Sect. 1 Prob. 15

Determine whether the property is true for all integers, true for no integers, or true for some integers and false for other integers.

$-a^n = (-a)^n$.

① For odd numbers n, $-a^n = (-a)^n$.

For example, let $n = 1$, then

$-a^n = -a^1 = -a = (-a)^1$.

② For even numbers n, $-a^n \neq (-a)^n$.

For example, let $n = 2$, then

$-a^n = -(a^2) = -a^2 \neq (-a)^2$ since $(-a)^2 = a^2$ and $-(a^2) = -a^2$.

$\therefore$ this property is true for some integers and false for other integers.

Ch. 4 Sect. 1 Prob. 20

Rewrite the statement with the quantification implicit as If _, then _. Write the 1st sentence of the proof and the last sentence of the proof.

For all integers m, if $m > 1$ then $0 < 1/m < 1$.

Answer:

If an integer is greater than 1, then its reciprocal lies between 0 and 1.

1st sentence of proof: Suppose m is any integer where $m > 1$.

Last sentence of proof: Therefore, $0 < 1/m < 1$.

Ch. 4 Sect. 1 Prob. 25

Prove the statement. Use only the definitions of the terms and the assumptions listed on page 146, not any previously established properties of odd and even integers.

The difference of any even integer minus any odd integer is odd.

Proof:

Suppose m is any even integer such that $m = 2r$ and n is any odd integer such that $n = 2s + 1$ for some integers r and s. Then,

$m - n = (2r) - (2s + 1) = 2r - 2s - 1$.

(we want $m-n$ to be twice some integer $+1$).

$m - n = 2r + 2s - 1 \rightarrow m - n = 2r + 2s \underbrace{- 2 + 1}_{-1}$.

$m - n = 2r + 2s - 2 + 1 = 2(r + s - 1) + 1$.

Let $q = r + s - 1$, then q is an integer since the sums and differences of integers are integers.

$\therefore$ $m - n = 2q + 1$ for some integer q and by definition of odd, $m-n$ is odd.

Ch. 4 Sect. 2 Prob. 4

The number is rational. Write it as a ratio of 2 integers.

$0.37373737\ldots$

Let $x = 0.37373737\ldots$

If we multiply by 100, we get $100x = 37.37373737\ldots$

Subtracting 100x by x will allow us to get rid of everything after the decimal point.

$$100x - x = 37.37373737\ldots - 0.37373737\ldots = 37$$

$$99x = 37$$

$x = \frac{37}{99}$ which is a ratio of 2 integers.

Ch. 4 Sect. 2 Prob. 7

The number is rational. Write it as a ratio of 2 integers.

52.4672167216721

Let $x = 52.46721\,6721\,6721\ldots$

If we multiply x by 100,000, we get:

$5246721.672167216721\ldots$

If we multiply x by 10, we get:

$524.672167216721\ldots$

By subtracting $100{,}000x$ by $10x$, we can get rid of everything after the decimal.

$$100{,}000x - 10x = 5246721.672167216721\ldots - 524.672167216721\ldots$$

$$99990x = 5246197$$

$$x = \frac{5246197}{99990},$$ which is a ratio of 2 integers.

Ch. 4 Sect. 2 Prob. 10

Assume that m and n are both integers and that $n \neq 0$. Explain why $(5m + 12n)/(4n)$ must be a rational number.

① The numerator is an integer as the sum and scalar multiples of integers are integers.

② The denominator is also an integer as scalar multiples of integers are integers.

By the zero-product property $4n \neq 0$ since neither of the integers are 0.

∴ we have a ratio of 2 integers and by definition of rational, $(5m + 12n)/(4n)$ is a rational number.

Ch. 4 Sect. 2 Prob. 14

Consider the statement: The square of any rational number is a rational number.

a) Write formally using a quantifier and a variable.

b) Determine whether true or false.

a) $\forall r$, if r is rational then r^2 is rational.

b) Let r be a rational number. Then there exists 2 integers s and t where $t \neq 0$ such that $r = s/t$.

Then, $r^2 = \left(\frac{s}{t}\right)^2 = \frac{s^2}{t^2}$.

s^2 and t^2 are both integers since s and t are integers and the scalar multiples of integers are integers.

$\therefore$ r^2 is a ratio of 2 integers where $t^2 \neq 0$, so r^2 is a rational number and the statement is true.

Ch. 4 Sect. 2 Prob. 15

In case the statement is false, determine whether a small change would make it true. If so, make the change and prove the new statement.

The product of any 2 rational numbers is a rational number.

This statement is true.

Proof:

Suppose r and s are rational numbers. Then $r = \frac{a}{b}$ and $s = \frac{c}{d}$ for some integers a, b, c, d and $b \neq 0$ and $d \neq 0$.

$$rs = \left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \left(\frac{ac}{bd}\right).$$

Since $a, b, c,$ and d are integers, ac and bd are integers. Since neither b nor d is 0, $bd \neq 0$ by the zero-product property.

$rs = \left(\frac{ac}{bd}\right)$ is a ratio of 2 integers and $bd \neq 0$.

$\therefore$ rs is rational by definition of a rational number and the product of any 2 rational numbers is a rational number.

Ch. 4 Sect. 2 Prob. 20

Same instructions as # 13 on the previous page.

Given any 2 rational numbers r and s with $r < s$, there is another rational number between r and s.

This statement is true.

Proof:

Let r and s be rational numbers with $r < s$.

Suppose $r < s$, then (If $a < b$, then $a + a < a + b$) From Appendix B.

$r + r < r + s$ (Added r to both sides)

$2r < r + s$

$r < \frac{r+s}{2} < s$.

$r = \frac{a}{b}$ and $s = \frac{c}{d}$ for some integers $a, b, c,$ and d. $b \neq 0$ and $d \neq 0$.

$\rightarrow$ $\frac{r+s}{2}$ is a rational number since it is the ratio of 2 integers and $2 \neq 0$.

$\rightarrow$ $\frac{r+s}{2} = \frac{a/b + c/d}{2} = \frac{ad+bc/bd}{2} = \frac{ad+bc}{2bd}$.

Let $f = ad + bc$ and $g = 2bd$. Then f and g are integers as the sums and products of integers are integers. Moreover, $2bd \neq 0$.

$\frac{r+s}{2} = \frac{f}{g}$ where $g \neq 0$.

$\therefore$ $\frac{r+s}{2}$ is a rational number by definition of rational number that lies between r and s.

Ch. 4 Sect. 2 Prob. 24

Derive the statement as a corollary of theorems 4.2.1 and 4.2.2, and the results of exercises 12-15 and 17.

For any rational numbers r and s, $2r+3s$ is rational.

Proof:

Suppose r and s are any 2 rational numbers.
By theorem 4.2.1, any integer is a rational number, so 2 and 3 are both rational numbers.
By exercise 15, the product of any 2 rational numbers is a rational number. This means that $2r$ and $3s$ are both rational numbers.
By theorem 4.2.2, the sum of any 2 rational numbers is a rational number.

$\therefore$ $2r+3s$ is a rational number.

Ch. 4 Sec. 3 Ex. 4

Give a reason for your answer. Assume all variables are integers.

Does 3 divide $(3k+1)(3k+2)(3k+3)$?

If n and d are integers, then d divides n iff $n = dk$ for some integer k.

Here, $n = (3k+1)(3k+2)(3k+3)$ and $d = 3$.

$$\begin{aligned} (3k+1)(3k+2)(3k+3) &= (3k+1)(3k+2)\cdot 3(k+1) \\ &= 3\,[\underbrace{(3k+1)(3k+2)(k+1)}_{k}] \\ n &= \underset{d}{3}\,k \end{aligned}$$

And $(3k+1)(3k+2)(k+1)$ is an integer because sums and products of integers are integers.

$\therefore$ 3 divides $(3k+1)(3k+2)(3k+3)$.

Ch. 4 Sect. 3 Prob. 9

Give a reason for your answer. Assume all variables are integers.

Is 4 a factor of $2a \cdot 34b$?

If n and d are integers, then d divides n iff $n = dk$ for some integer k.

Here, $n = 2a \cdot 34b$ and $d = 4$.

$$2a \cdot 34b = \underbrace{68ab}_{n} = \underbrace{4}_{d}\underbrace{(17ab)}_{k}$$

And $17ab$ is an integer since the product of integers is an integer.

$\therefore$ 4 is a factor of $2a \cdot 34b$.

Ch. 4 Sect. 3 Prob. 10

Give a reason for your answer.

Does $7 \mid 34$?

If n and d are integers, then $d \mid n$ iff

$n = dk$ for some integer k.

Here, $n = 34$ and $d = 7$.

There is no such integer k such that

$7k = 34$, so $7 \nmid 34$.

- k would need to be $34/7$ which is not

an integer.

$\therefore$ $7 \nmid 34$.

Ch. 4 Sect. 3 Prob. 13

Give a reason for your answer. Assume all variables are integers.

If $n = 4k+3$, does 8 divide n^2-1?

$$
\begin{aligned}
n^2-1 = (4k+3)^2 - 1 &= (16k^2 + 24k + 9) - 1 \\
&= 16k^2 + 24k + 8 \\
&= \underbrace{8}_{d}\underbrace{(2k^2 + 3k + 1)}_{k}
\end{aligned}
$$

n^2-1 is divisible by 8 iff $n^2-1 = 8k$ for some integer k.

$2k^2 + 3k + 1$ is an integer because sums and products of integers are integers.

$\therefore$ 8 divides n^2-1.

Ch. 4 Sect. 3 Prob. 15

Prove the statement directly from the def. of divisibility.

For all integers a, b, and c, if $a \mid b$ and $a \mid c$
then $a \mid (b+c)$.

Proof:

Suppose a, b, and c are integers such that $a \mid b$ and $a \mid c$.
Then, $b = ar$ and $c = ak$ for some integers r and k.

$b + c = ar + ak = a(r+k)$

Let $q = r + k$. q is an integer as the sum of integers is an integer.

$\Rightarrow$ $b + c = aq$ where $q = r + k$.

$\therefore$ $a \mid (b+c)$ by definition of divisibility.

Ch. 4 Sect. 3 Prob. 22

Determine whether the statement is true or false. Prove the statement directly from the definitions if it is true, and give a counterexample if it is false.

A necessary condition for an integer to be divisible by 6 is that it be divisible by 2.

The statement is true.

Proof:

Suppose n is any integer such that n is divisible by 6. Then, $n = 6k$ for some integer k.

$$n = 6k = 2 \cdot 3k$$

$$n = \underbrace{d}\ \underbrace{\text{some integer}}$$

Let $r = 3k$. Then r is an integer since the product of integers is an integer.

$\therefore$ by definition of divisibility, n is divisible by 2. Any integer n that is divisible by 6 is divisible by 2.

Ch. 4 Sect. 3 Prob. 23

Same instructions as # 22 on the previous page.

A sufficient condition for an integer to be divisible by 8 is that it be divisible by 16.

The statement is true.

Proof:

Suppose n is any integer such n is divisible by 16.
Then, $n = 16k$ for some integer k.

$$n = 16k = \underbrace{8}_{d} \cdot \underbrace{2k}_{\text{some integer}}$$

Let $q = 2k$. Then q is an integer since the product of integers is an integer.
By def. of divisibility, 8 divides n.

∴ Any integer n that is divisible by 16 is also divisible by 8.

Ch. 4 Sect. 3 Prob. 27

Same instructions as # 22 (2 pages back).

For all integers a, b, and c, if $a \mid (b+c)$ then $a \mid b$ or $a \mid c$.

This statement is false. To disprove it, we simply need to prove its negation with a counterexample.

Negation: $\exists$ integers a, b, and c $\mid$ $a \mid (b+c)$ and $a \nmid b \wedge a \nmid c$.

For example, let $a = 2$, $b = 9$, and $c = 5$.

Then $a \mid (b+c) = 2 \mid (9+5) = 2 \mid 14$ which is true since $14 = 2 \cdot 7$.

$a \mid b = 2 \mid 9$ which is false since $9 \neq 2 \cdot$ some integer.

So, $a \nmid b$.

$a \mid c = 2 \mid 5$ which is also false since $5 \neq 2 \cdot$ some integer.

So, $a \nmid c$.

$\rightarrow a \mid (b+c)$ and $a \nmid b$ and $a \nmid c$.

$\therefore$ by proving the negation, we've disproved the statement using a counterexample.

Ch. 4 Sect. 4 Prob. 5

For the values of n and d given, find integers q and r such that $n = dq + r$ and $0 \le r < d$.

$n = -45$, $d = 11$

$$-45 = 11(-5) + 10$$

$$\begin{array}{ccccc} \downarrow & & \downarrow & \downarrow & \downarrow \\ n & = & d & q & r \end{array}$$

$q = -5$ and $r = 10$

$\downarrow$ $\downarrow$

quotient remainder

Ch. 4 Sect. 4 Prob. 8

Evaluate the expressions

a. 50 div 7

b. 50 mod 7

a) 50 div 7 = the quotient when 50 is divided by 7.

b) 50 mod 7 = the remainder when 50 is divided by 7.

$$50 \div 7 = 7 + 1$$

$$\underset{n}{\underset{\downarrow}{50}} = \underset{d}{\underset{\downarrow}{7}}(\underset{q}{\underset{\downarrow}{7}}) \underset{+}{+} \underset{r}{\underset{\downarrow}{1}}$$

$q = 50 \text{ div } 7 = 7$

$r = 50 \text{ mod } 7 = 1$

Ch. 4 Sect. 4 Prob. 11

Check the correctness of formula (4.4.1) given in example 4.4.3 for the following values of DayT and N.

a. DayT = 6 (Saturday) and N = 15.

b. DayT = 0 (Sunday) and N = 7.

c. DayT = 4 (Thursday) and N = 12.

Formula: DayN = (DayT + N) mod 7

a. DayN = (6 + 15) mod 7

= 21 mod 7

$= \underset{q}{3} + \underset{r}{0}$ $\quad 21 = 7(3) + 0$

So, DayN = 0 (Sunday).

(Day of the week in N days.)

b. DayN = (0 + 7) mod 7

= 7 mod 7

$= \underset{q}{1} + \underset{r}{0}$ $\quad 7 = 7(1) + 0$

So, DayN = 0 (Sunday).

c. DayN = (4 + 12) mod 7

= 16 mod 7

$= \underset{q}{2} + \underset{r}{2}$ $\quad 16 = 7(2) + 2$

So, DayN = 2 (Tuesday).

Ch. 4 Sect. 4 Prob. 16

Suppose d is a positive integer and n is any integer. If $d|n$, what is the remainder obtained when the quotient-remainder theorem is applied to n w/ divisor d?

If $d|n$, then $n = dk$ for some integer k.

If we apply the quotient-remainder theorem, then

$n = dk + 0$ for some integer k.

So, the remainder would be 0.

(If $d|n$, then n is evenly divisible by d. In other words, the remainder is 0.)

Ch. 4 Sect. 4 Prob. 20

Suppose a is an integer. If $a \bmod 7 = 4$, what is $5a \bmod 7$? In other words, if division of a by 7 gives a remainder of 4, what is the remainder when $5a$ is divided by 7.

$a \bmod 7 = 4$. Then

$a = 7k + 4$ for some integer k.

Then, $5a \bmod 7 =$

$= 5(a \bmod 7)$ (we want $5a \bmod 7$ to be in the form $n = dq + r$, so we can determine r.)

$= 5(7k + 4)$

$= 35k + 20$

$= 35k + 14 + 6$ $- \; 0 \leq r < d$

$= 7(5k + 2) + 6$

$n = d \; q + r$

$\therefore$ $5a \bmod 7 = 6$.

Ch. 4 Sect. 4 Prob. 22

Suppose c is an integer. If $c \bmod 15 = 3$, what is $10c \bmod 15$?

$c \bmod 15 = 3$, so

$c = 15k + 3$ for some integer k.

Then, $10c \bmod 15 =$

$= 10(c \bmod 15)$

$= 10(15k + 3)$

$= 150k + 30$

$= 15(10k + 2) + 0$

$\downarrow \qquad \downarrow$

$d \quad q \quad + \quad r$ where $0 \le r < d$.

$\therefore \; 10c \bmod 15 = 0$.

Ch 4. Sect 4. Prob. 26

Prove that a necessary and sufficient condition for a non-negative integer n to be divisible by a positive integer d is that $n \bmod d = 0$.

Proof.

① Suppose n is any non-negative integer and d is any positive integer such that n is divisible by d.

Then,

$$n = dk + 0 \quad \text{for some integer } k.$$

Then, $n \bmod d = 0$ since the remainder is 0.

② Suppose $n \bmod d = 0$. Then,

$$n = dq + 0$$
$$= dq \quad \text{for some integer } q.$$

Then, n is divisible by d.

∴ a necessary and sufficient condition for a non-negative integer n to be divisible by a positive integer d is that $n \bmod d = 0$.

$$d \mid n \Leftrightarrow n \bmod d = 0.$$

(Because this is an iff statement, we need to prove it in both directions.)

Ch. 4 Sect. 6 Prob. 4

Use proof by contradiction to show that for all integers m, $7m+4$ is not divisible by 7.

1. Suppose the negation is true.
2. Show that this leads to a contradiction.
3. Conclude the statement is therefore true.

Proof:

Suppose there exists an integer m such that $7m+4$ is divisible by 7. Then,

$7m+4 = 7k$ for some integer k.

$4 = 7k - 7m$

$4 = 7(k-m)$.

Let $q = k-m$, then q is an integer as the difference of integers is an integer. We then have,

$4 = 7q$ for some integer q.

This means 4 is divisible by 7 or $7 \mid 4$.

But 4 is not divisible by 7 or $7 \nmid 4$.

* Theorem 4.3.1: $\forall$ integers a and b, if a and b are positive and $a \mid b$, then $a \leq b$.

→ $7 \nmid 4$ because $7 \nleq 4$ $(7 > 4)$.

∴ $7 \mid 4$ and $7 \nmid 4$ which is a contradiction, so $\forall$ integers m, $7m+4$ is not divisible by 7.

Ch. 4 Sect. 6 Prob. 5

Carefully negate the statement. Then prove it by contradiction.

There is no greatest even integer.

Negation: There is a greatest even integer.

Proof.

Suppose there is a greatest even integer N. Then N is even and $N \geq n$ for all even integers n. Let $M = N + 2$. Then M is even since it is the sum of even integers and $M > N$ since $M = N + 2$. This contradicts our supposition that $N \geq n$ for all even integers n because $M > N$.

$\therefore$ the supposition is false and there is no greatest even integer.

Ch. 4 Sect. 6 Prob. 10

Prove by contradiction.

The square root of any irrational number is irrational.

Proof:

Suppose there is an irrational number x such that the square root of x is rational. Then by definition of rational,

$\sqrt{x} = a/b$ for some integers a and b where $b \neq 0$.

Then, $(\sqrt{x})^2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$. (solving for x.)

$\frac{a^2}{b^2}$ is rational because a^2 and b^2 are both integers (products of integers are integers) and $b^2 \neq 0$ by the zero-product property.

→ x is irrational and x is rational.

∴ the supposition is false and the square root of any irrational number is irrational.

Ch. 4 Sect. 6 Prob. 12

Proof by contradiction.

If a and b are rational numbers, $b \neq 0$, and r is an irrational number, then $a + br$ is irrational.

Proof:

Suppose a and b are rational numbers, $b \neq 0$, r is an irrational number and $a + br$ is rational.

Then,

$a = p/q$ for some integers p and q w/ $q \neq 0$.

$b = k/s$ for some integers k and s w/ $s \neq 0$.

$a + br = m/n$ for some integers m and n w/ $n \neq 0$.

$a + br = \left(\frac{p}{q}\right) + \left(\frac{k}{s}\right)r$. Solving for r, we get,

$$r = \left(\frac{m}{n}\right) - \left(\frac{p}{q}\right) / \left(\frac{k}{s}\right) = \left(\frac{mq - pn}{nq}\right) \cdot \left(\frac{s}{k}\right)$$

$r = \frac{mqs - pns}{nqk}$. Let $f = mqs - pns$ and $g = nqk$. f and g are both integers since differences and products of integers are integers.

Then, $r = \frac{f}{g}$ where $g \neq 0$ by zero-product property.

$\rightarrow$ r is irrational and r is rational.

$\therefore$ the supposition is false and if a and b are rational numbers, $b \neq 0$, and r is irrational, then $a + br$ is irrational.

Ch. 4 Sect. 6 Prob. 15

Prove by contradiction.

If a, b, and c are integers and $a^2+b^2=c^2$, then atleast one of a and b is even.

Proof:

Suppose a, b, and c are integers and $a^2+b^2=c^2$ and both a and b are odd.

case 1: consider when c is odd. Then c^2 is odd and a^2+b^2 would need to be odd since $a^2+b^2=c^2$. But a and b are both odd, so a^2+b^2 is even since the sum of two odd integers is even.

→ a^2+b^2 is odd and a^2+b^2 is even for case 1.

case 2: consider when c is even. Then c^2 is even and a^2+b^2 would need to be even since $a^2+b^2=c^2$. Since both a and b are odd, a^2+b^2 is even, so this case is satisfied when both a and b are odd.

→ a^2+b^2 is even when both a and b are odd.

∴ the supposition is false because case 1 says that a^2+b^2 is both odd and even (either a or b is even would mean a^2+b^2 is odd and both a and b are odd would mean a^2+b^2 is even.) So, if a, b, and c are integers and $a^2+b^2=c^2$, then atleast one of a and b is even.

* Case 1 says that atleast one of a and b is even BUT it also says both a and b are odd which is a contradiction.

Ch. 4 sect. 6 Prob. 21

Consider the statement, "For all integers n, if n^2 is odd, then n is odd."

a. Write what you would suppose and what you would to show to prove by contradiction.

b. Write what you would suppose and what you would need to show to prove by contraposition.

a) Suppose there is an integer n such that n^2 is odd and n is even. We'd need to show that this supposition leads logically to a contradiction.

b) Suppose that n is any integer such that n is not odd. We'd need to show that n^2 is also not odd.

Ch. 4 Sect. 6 Prob. 22

Consider the statement, "For all real numbers r, if r^2 is irrational then r is irrational."

a. Write what you would suppose and what you would need to show to prove by contradiction.

b. Write what you would suppose and what you would need to show to prove by contraposition.

a) Suppose there is a real number r such that r^2 is irrational and r is rational. Show this supposition leads logically to a contradiction.

b) Suppose r is any real number such that r is not irrational. Show that r^2 is not irrational.

Ch. 4 Sect. 7 Prob. 4

Determine whether true or false. Prove if true, disprove if false.

$3\sqrt{2} - 7$ is irrational. TRUE.

Proof:

Suppose $3\sqrt{2} - 7$ is rational. Then,

$3\sqrt{2} - 7 = m/n$ for some integers m and n w/ $n \neq 0$.

$$3\sqrt{2} - 7 = m/n$$

$$3\sqrt{2} = m/n + 7$$

$$= \frac{m+7n}{n}$$

$$\sqrt{2} = \frac{m+7n}{n} / 3$$

$$= \frac{m+7n}{3n}.$$

Since $m, n, 7,$ and 3 are integers, $m+7n$ and $3n$ are integers since sums and products of integers are integers. Then $\sqrt{2} = \frac{m+7n}{3n}$ is a rational number w/ $3n \neq 0$.

$\rightarrow \sqrt{2}$ is irrational and $\sqrt{2}$ is rational.

$\therefore$ the supposition is false and $3\sqrt{2} - 7$ is irrational.

Ch. 4 Sect. 7 Prob. 5

Determine whether true or false. Prove if true, disprove if false.

$\sqrt{4}$ is irrational. FALSE.

$\sqrt{4} = 2 = 2/1$ which is rational.

$\therefore$ $\sqrt{4}$ is not irrational, so the statement is false.

Ch. 4 Sect. 7 Prob. 10

Determine whether true or false. Prove if true, disprove if false.

If r is any rational number and s is any irrational number, then r/s is irrational. FALSE.

Disproof by counterexample:

we need to find one example where r/s is rational to disprove by counterexample.

For example, let $r = 0$ and $s = \sqrt{2}$, then

$\frac{r}{s} = \frac{0}{\sqrt{2}} = 0$, and 0 is rational since $0 = \frac{0}{1}$.

∴ the statement is false.

Ch. 4 Sect. 7 Prob. 12

Determine whether true or false. Prove if true, disprove if false.

The product of any 2 irrational number is irrational. FALSE.

Disproof by counter example:

For example, let $x = \sqrt{2}$ and $y = 3\sqrt{2}$, then

$xy = (\sqrt{2})(3\sqrt{2}) = 3 \cdot 2 = 6$ which is rational since $6 = 6/1$.

$\therefore$ the statement is false.

Ch. 4 Sect. 7 Prob. 15

a. Prove that for all integers a, if a^3 is even then a is even.

b. Prove that $\sqrt[3]{2}$ is irrational.

a) Proof: (By contraposition)

Suppose a is odd. Then,

$a = 2k+1$ for some integer k.

$$a^3 = (2k+1)^3 = (2k+1)(2k+1)(2k+1)$$
$$= (4k^2+4k+1)(2k+1)$$
$$= (8k^3 + 4k^2 + 8k^2 + 4k + 2k + 1)$$
$$= 8k^3 + 12k^2 + 6k + 1.$$
$$= 2(4k^3 + 6k^2 + 3k) + 1.$$

Let $s = 4k^3 + 6k^2 + 3k$. Then s is an integer because sums and products of integers are integers.

$a^3 = 2s+1$, so a^3 is odd.

∴ For all integers a, if a^3 is even then a is even.

b) Proof:

Suppose $\sqrt[3]{2}$ is rational. Then,

$\sqrt[3]{2} = m/n$ for some integers m and n w/ no common factors.

$(\sqrt[3]{2})^3 = \left(\frac{m}{n}\right)^3 = \frac{m^3}{n^3}$ which is rational since m and n are integers w/ $n^3 \neq 0$.

Then, $2 = \frac{m^3}{n^3}$, so $m^3 = 2n^3$, m^3 is even, so m is even by part a.

Let $m = 2k$ for some integer k. Then,

$(\sqrt[3]{2})^3 = \frac{m^3}{n^3}$ → $m^3 = 2n^3$ → $8k^3 = 2n^3$, so $n^3 = 2(2k^3)$.

Therefore, n^3 is even and n is even by part a.

Since both m and n are even, they share a common factor of 2. But this contradicts our supposition that m and n have no common factors.

∴ our supposition is false and $\sqrt[3]{2}$ is irrational.

Ch. 4 Sect. 7 Prob. 21

An alternative proof of the irrationality of $\sqrt{2}$ counts the number of 2's on the 2 sides of the equation $2n^2 = m^2$ and uses the unique factorization of integers theorem to deduce a contradiction. Write a proof that uses this approach.

Proof:

Suppose $\sqrt{2}$ is rational, Then,

$\sqrt{2} = a/b$ for some integers a and b w/ $b \neq 0$.

$(\sqrt{2})^2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$, so $a^2 = 2b^2$. Now, consider the prime factorizations for a^2 and $2b^2$.

By the unique factorization of integers theorem, these prime factorizations are unique except for the order in which the factors are written. Since every prime factor of a appears twice in the prime factorization of a^2, the factorization contains an even number of 2's (If 2 is a factor of a, then we'll have an even number of 2's in the factorization for a^2). However, since every prime factor of b appears twice in the prime factorization of b^2, the factorization contains an odd number of 2's (we have an additional 2 on the right-hand side of the equation which makes the number of 2's we have odd).

∴ the equation $a^2 = 2b^2$ is false because the number of 2's is not even on both sides of the equation, so $\sqrt{2}$ is irrational.

Ch. 4 Sect. 7 Prob. 22

Prove that if n is any integer that is not a perfect square, then $\sqrt{n}$ is irrational.

Proof:

Suppose $\sqrt{n}$ is rational. Then, $\sqrt{n} = a/b$ for some integers a and b w/ $b \neq 0$.

$(\sqrt{n})^2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$, so $a^2 = nb^2$. Consider the prime factorizations of a^2 and nb^2.

Since every prime factor of a appears twice in the prime factorization of a^2, the number of 2's in the prime factorization of a^2 is even. On the other hand, since the prime factors of b appear twice in the prime factorization of b^2, the prime factorization contains an odd number of 2's.

$\therefore$ the equation $a^2 = nb^2$ is false since the number of 2's is not even on both sides, so $\sqrt{n}$ is irrational if n is any integer that is not a perfect square.

Ch. 4 sect. 7 Prob. 27

Let $p_1, p_2, p_3, \ldots$ be a list of prime numbers in ascending order. Here is a table of the first 6:

p_1	p_2	p_3	p_4	p_5	p_6
2	3	5	7	11	13

a. For each $i = 1, 2, 3, 4, 5, 6$, let $N_i = p_1 p_2 \cdots p_i + 1$. Calculate $N_1 - N_6$.

b. For each $i = 1, 2, 3, 4, 5, 6$, find the smallest prime number q_i such that q_i divides N_i.

* Test for Primality:
Given an integer $n > 1$, to test whether n is prime, check to see if it is divisible by a prime number less than or equal to its square root. If it is not divisible by any of these numbers, then it is prime.

a) $N_1 = 2 + 1 = 3$, $N_2 = 2 \cdot 3 + 1 = 7$

$N_3 = 2 \cdot 3 \cdot 5 + 1 = 31$, $N_4 = 2 \cdot 3 \cdot 5 \cdot 7 + 1 = 211$

$N_5 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 + 1 = 2311$, $N_6 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 30{,}031$

b) To find q_1 for N_1, we check to see if N_1 is divisible by a prime number $\leq \sqrt{N_1}$. Since there are no primes that are $\leq \sqrt{3}$, q_1 for $N_1 = 3$ or N_1 itself.

$q_2 = 7$ since 7 is not divisible by 2.

$q_3 = 31$ since 31 is not divisible by 2, 3, or 5.

$q_4 = 211$ since 211 is not divisible by 2, 3, 5, 7, 11, or 13.

$q_5 = 2311$ since 2311 is not divisible by 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, or 47.

$q_6 = 59$ since 30,031 is divisible by 59.

$30{,}031 = 59 \cdot 509$.

Prime numbers between 2 and $\sqrt{N_6}$ = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, and 173.