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Analysis of multimode interferometers

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Abstract: A general analytical formula describing the transfer function between the input and output of an arbitrary rectangular multimode interference (MMI) coupler has been derived using the elliptic theta function $\vartheta(x', z')$. This formula provides the positions, amplitudes and relative phases of all the self-images of a given source. It is shown how the transfer function can be used as a propagation matrix for any rectangular NxM MMI. Specific simplified solutions for NxN, symmetric and paired MMIs are also derived from the general formula.

OCIS codes: (130.0130) Integrated optics; (130.2790) Guided waves; (130.3120) Integrated optics devices.

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1. Introduction

Multimode Interference devices (MMIs) have their origin in the Talbot effect, a phenomenon first observed by H.F. Talbot [1] in 1890. Talbot observed that if a monochromatic optical plane wave is incident upon a periodic diffraction grating, then the image of grating will repeat itself at a fixed distance L_T away from the diffraction grating, as verified by Rayleigh [2]. A more rigorous proof based on Fresnel imaging is given in [3], correctly predicting "secondary" imaging [4] at $\frac{p}{q}L_T$ also. Winthrop and Worthington extended the Talbot effect to two dimensional images [5]. In doing so, they emphasised that the output image could be related to the input source via a convolution, and that the complex phase-amplitude of these images could be expressed as the sum of quadratic complex exponentials. The Talbot effect is studied in a diverse range of topics today [6,7].

Rivlin, and later Bryngdahl, suggested that by using total internal reflection in an optical fibre to replicate a periodic grating, it should be possible to induce a confined Talbot effect [8,9]. This is the fundamental idea behind the MMI, which was first experimentally tested by Ulrich [10]. MMIs are used extensively in photonic integrated circuits (PICs) due to their compact size, low loss, phase dependence and predictable performance [11].

In one of the earliest attempts to derive the re-imaging properties of MMIs, Chang and Keuster derived a Green's function for an MMI in terms of Jacobi's elliptic theta function $\vartheta(x', z')$ [12]. From this it was shown that the phase-amplitudes of the MMI images may be expressed in terms of the Gaussian sums $c_n(p,q)$. Berry also recognised the significance of $\vartheta(x', z)$ and $c_n(p,q)$ in relation the Talbot effect and quantum mechanics [6, 13, 14]. We shall closely follow this approach. Since then, there have been other distinct derivations of the image phase-amplitudes for MMIs [11,15–17], as well as for periodic diffraction gratings . In this paper, we have consolidated these different approaches into one clear framework [18, 19].

In this work a general analytical expression is fully derived, which describes the position, phase and amplitude relationships between the input and output of an arbitrary MMI. This is expressed in a single equation containing Dirac delta functions and complex phase amplitudes. Section II includes the general solution for an arbitrary MMI. Section III includes the simplified solutions for the special cases of restricted (symmetric and paired) interference. Section IV demonstrates the use of the general solution as a transfer function matrix as applied to more complex MMI structures. Section V provides a conclusion.

2. A general MMI transfer function equation

The multimode interference (MMI) device is a wide rectangular waveguide. To solve for the electric field in the waveguide, we assume that the field is TE polarised $\overrightarrow{E(x,y,z,t)} = E_y(x)e^{i(\omega t-\beta z)}\hat{y}$ and substitute into Laplace's equation $\nabla^2\vec{E} + \left(n^2/c^2\right)\vec{E} = 0$. This yields the one dimensional Helmholtz equation:

$$\frac{d^2E_y(x)}{dx^2} + \left(k^2n(x)^2 - \beta^2\right)E_y(x) = 0 \tag{1}$$

where k is the free space wavenumber. The solutions to this equation are the eigenmodes $\psi_m(x)$ with (possibly complex) propagation constants β_m. The initial field $E_y(x, z = 0) = E_y(x)$ may

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represented as an eigenmode expansion:

$$E_y(x) = \sum_{m=0}^{\infty} a_m \psi_m(x) \tag{2}$$

where eigenmode coefficients may be calculated as $a_m = \int E_y(x)\psi_m(x)\,dx / \int \psi_m(x)\psi_m(x)\,dx$. We will be considering step index slab waveguides such that $n(x) = n_r$ for $0 < x < W$ and $n(x) = n_c$ otherwise, with $n_r > n_c$. n_r and n_c are usually effective indices of more elaborate waveguides computed using the effective index method. For wide waveguides with a large number of modes $\left(V = \frac{W\pi}{\lambda}\sqrt{n_r^2 - n_c^2} \gg 1\right)$, we may approximate the eigenmodes to be entirely contained within an effective width $W_e = W + \frac{\lambda}{\pi\sqrt{n_r^2 - n_c^2}}$ [20]. Then the eigenmodes simplify to $\psi_m(x) = \sin\left(\frac{m\pi x}{W_e}\right)$ for $0 < x < W_e$, where $x = 0$ now corresponds to the edge of the effective waveguide. In this limit, the propagation constants simplify to

$$\beta_m = kn_r\sqrt{1 - \left(\frac{m\pi}{W_e k n_r}\right)^2} \tag{3}$$

As n_r and n_c are typically on the order of magnitude of 1, the condition $V \gg 1$ is equivalent to $W \gg \lambda \Leftrightarrow Wk \gg 1$. Thus for m small, we may expand β_m as

$$\beta_m \approx kn_r - \frac{1}{2kn_r}\left(\frac{m\pi}{W_e}\right)^2 \tag{4}$$

For TM modes, the effective width is $W_e = W + \left(\frac{n_c}{n_r}\right)^2 \frac{\lambda}{\pi\sqrt{n_r^2 - n_c^2}}$ [20]. As $V \gg 1$ the width of the waveguide is generally much larger than $\lambda/\pi\sqrt{n_r^2 - n_c^2}$. For both TE and TM polarizations, the effective width is very close to the actual width of the waveguide. However, the TM solutions can be calculated exactly by using the alternate effective width calculation.

We further make the assumption that in the eigenmode expansion of $E_y(x)$, the excitation of the higher order modes is negligible. This allows us to replace the higher order modes and propagation constants with something more manageable. For this reason, we assume that Eq. (4) is true for all integers m. To compute the electric field at an arbitrary distance z down the waveguide, we compute the effect of propagating each mode and then sum the result.

$$E_y(x,z) = \sum_{m=0}^{\infty} a_m \sin\left(\frac{m\pi x}{W_e}\right) e^{-i\beta_m z} \approx e^{-ikn_r z} \sum_{m=0}^{m=\infty} a_m \sin\left(\frac{m\pi x}{W_e}\right) e^{\frac{i}{2kn_r}\left(\frac{m\pi}{W_e}\right)^2 z} \tag{5}$$

The z-evolution in $E_y(x,z)$ is completely equivalent to the time evolution of a quantum particle in a 1-dimensional box [6, 13, 14]. Thus, all results derived here apply equally well to a wavefunction in an infinite potential well.

To facilitate the use of Fourier analysis, we follow the technique used in [15] and extend the electric field $E_y(x)$ periodically to an antisymmetric function as in Fig. 1. Then $F(x,z)$ is defined as a $2W_e$ periodic function with

$$F(x,z) = \begin{cases} E_y(x,z) & 0 \le x \le W_e \\ -E_y(2W_e - x, z) & W_e \le x \le 2W_e \end{cases} \tag{6}$$

This is analogous to the method of images in electrostatics. For mathematical calculations, $F(x,z)$ has a real part ($0 \le x \le W_e$) and a fictitious antisymmetric part ($W_e \le x \le 2W_e$). For

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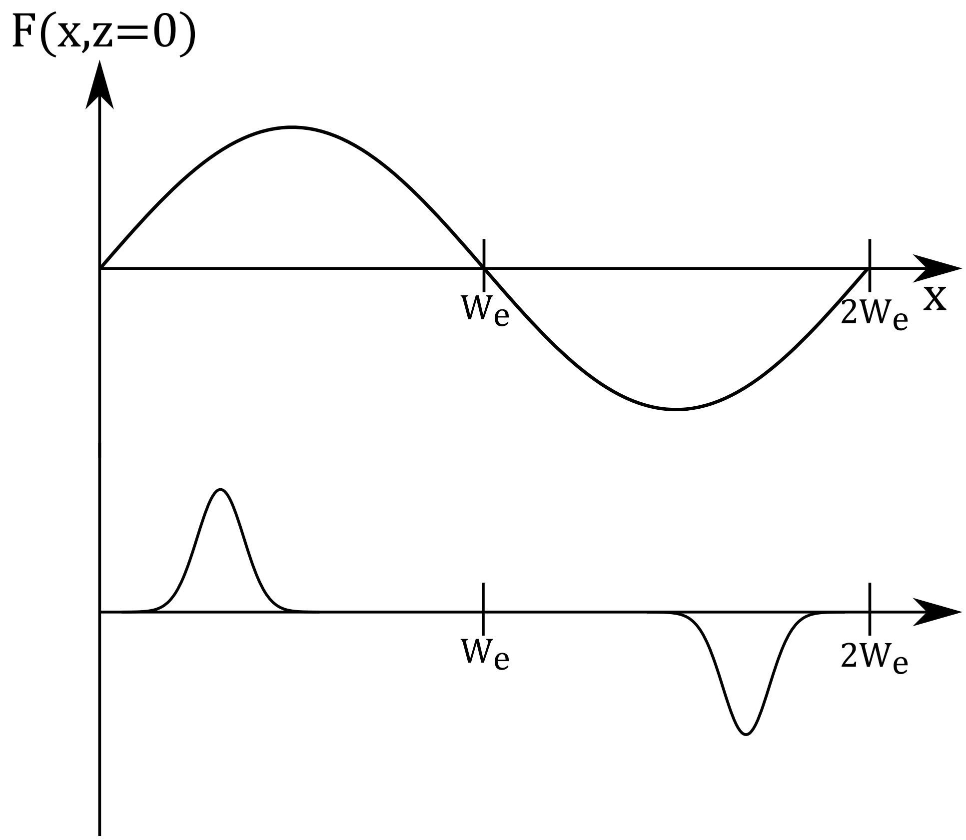

Fig. 1. Construction of the extended field.

the subsequent analysis, these will be referred to as the real and fictitious regions. The region $0 \le x \le 2W_e$ is referred to as the extended region, and $F(x,z)$ as the extended field.

Expanding $F(x,z)$ in the eigenbasis $\psi_m(x)$ is the same as taking the Fourier series of $F(x,z)$. Using complex exponentials for the modes instead of sinusoids:

$$F(x,z) = \sum_{m=-\infty}^{\infty} a_m e^{i\frac{m\pi x}{W_e}} e^{-i\beta_m z} \approx e^{-ikn_r z} \sum_{m=0}^{\infty} a_m e^{i\frac{m\pi x}{W_e}} e^{\frac{i}{2kn_r}\left(\frac{m\pi}{W_e}\right)^2 z} \tag{7}$$

with $a_m = \frac{1}{2W_e}\int_0^{2W_e} F(x,0)e^{-i\frac{m\pi x}{W_e}}\,dx$. Given that $F(x,0)$ is periodic, Eq. (7) describes the propagation in a homogeneous medium with refractive index n_r in the Fresnel approximation. Thus, $F(x,z)$ is actually the superposition of two Talbot gratings which are out of phase with each other.

We note that the effect of propagating down the waveguide a distance z is equivalent to multiplying each of the Fourier series coefficients of $F(x,0)$ by $e^{\frac{i}{2kn_r}\left(\frac{m\pi}{W_e}\right)^2 z} = e^{im^2\pi\frac{\pi}{2kn_r W_e^2}z}$. To simplify (7), we define the MMI length $L = 2kn_r W_e^2/\pi = 4n_r W_e^2/\lambda$. The corresponding time in the quantum mechanical case is $T = 4mW^2/h$, which is half the revival time of a particle in a box [21].

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We introduce the normalised variables $x' = x/2W_e$ and $z' = z/L$, which simplifies Eq. (7) to

$$F(x', z') = e^{-ikn_r z' L} \sum_{m=-\infty}^{\infty} a_m e^{im2\pi x'} e^{im^2 \pi z'} \tag{8}$$

which is now a periodic function of x' with period 1.

Jacobi's elliptic theta function is defined as

$$\vartheta\,(x', z') = \sum_{m=-\infty}^{\infty} e^{im2\pi x'} e^{im^2 \pi z'} \tag{9}$$

Note that the exponentials of the theta function match those of $F(x', z')$. Using the convolution theorem for Fourier series:

$$\begin{aligned} F(x', 0) * \vartheta\,(x', z') &= \sum_{n=-\infty}^{\infty} a_n e^{in2\pi x'} * \sum_{m=-\infty}^{\infty} e^{im2\pi x'} e^{im^2\pi z'} \\ &= \sum_{n,m} a_n e^{im^2\pi z'} \left(e^{in2\pi x'} * e^{im2\pi x'}\right) = \sum_{m=-\infty}^{\infty} a_m e^{im2\pi x'} e^{im^2\pi z'} \end{aligned} \tag{10}$$

Thus

$$F(x', z') = e^{-ikn_r z' L} F(x', 0) * \vartheta(x', z') \tag{11}$$

A similar equation has been derived in [12] using Green's functions. It should be pointed out that the propagation effects of $F(x', z')$ are contained entirely in $\vartheta(x', z')$, and the re-imaging properties of the MMI are now evident. For instance,

$$\vartheta(x', z' + 2) = \sum_{m=-\infty}^{\infty} e^{im2\pi x'} e^{im^2\pi(z'+2)} = \sum_{m=-\infty}^{\infty} e^{im2\pi x'} e^{im^2\pi z'} = \vartheta(x', z') \tag{12}$$

Therefore $F(x', z' + 2) = e^{-ikn_r(z'+2)L} F(x', 0) * \vartheta(x', z' + 2) = F(x', z') e^{-ikn_r 2L}$. Thus both $F(x', z')$ and $E_y(x, z')$ are periodic in z' with period 2, up to a phase factor.

Note that

$$\vartheta(x', 0) = \sum_{m=-\infty}^{\infty} e^{im2\pi x'} = \sum_{m=-\infty}^{\infty} \delta(x' - m) = \delta_1(x') \tag{13}$$

where $\delta_1(x')$ represents a Dirac comb with period 1. As we have already imposed periodicity in x' with period 1, we may further identify $\delta_1(x') = \delta(x')$.

For the general case of $\vartheta(x', z')$ we investigate $z' = 1/q$. This leads to our generic propagation equation, which is fully derived in appendix A:

$$\vartheta\left(x', \frac{1}{q}\right) = \frac{1}{\sqrt{q}} \sum_{\substack{t=0 \\ t \equiv q (mod\, 2)}}^{2q-1} e^{\frac{i\pi}{4}\left(1 - \frac{t^2}{q}\right)} \delta\left(x' - \frac{t}{2q}\right) \tag{14}$$

Recall that $f(x') * \delta(x' - a) = f(x' - a)$, which corresponds to a shift of $f(x')$ in x' by a. Thus, $F(x', 1/q)$ is a sum of q evenly spaced copies of $F(x', 0)$, each multiplied by a phase term.

$$F\left(x', \frac{1}{q}\right) = \frac{e^{-ikn_r \frac{L}{q}}}{\sqrt{q}} \sum_{\substack{t=0 \\ t \equiv q (mod\, 2)}}^{2q-1} e^{\frac{i\pi}{4}\left(1 - \frac{t^2}{q}\right)} F\left(x' - \frac{t}{2q}, 0\right) \tag{15}$$

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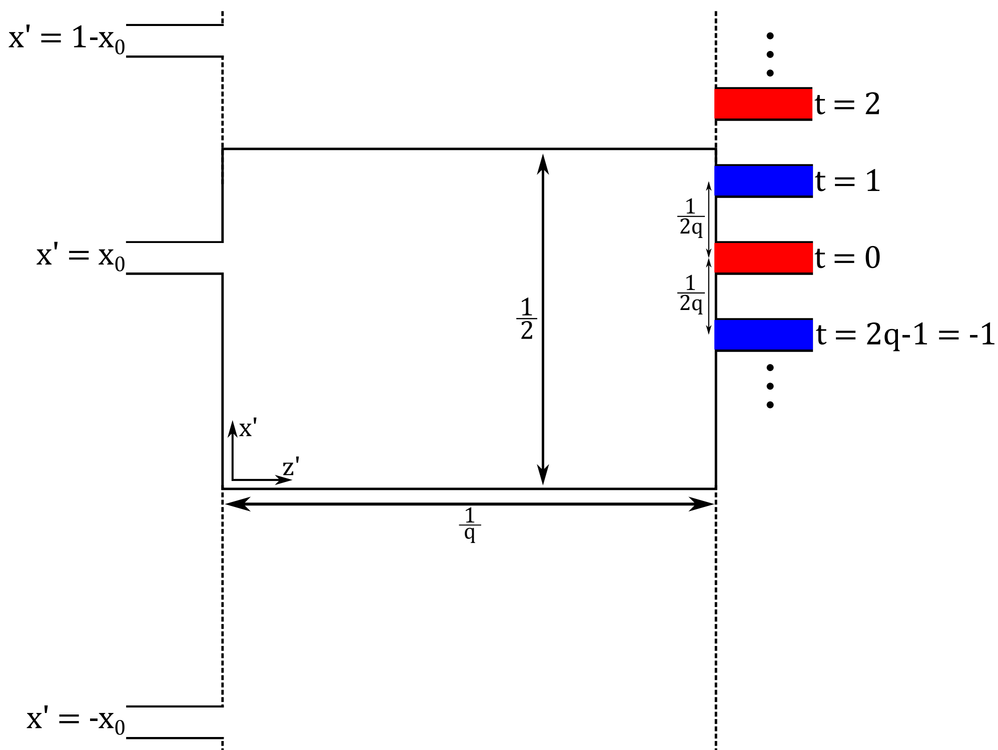

Fig. 2. The images of a single source located at x_0.

Note that $F(x' - t/2q, z') = F(x' - (t/2q) - 1, z') = F(x' - (t + 2q)/2q), z')$ and $e^{\frac{i\pi}{4}\left(1-\frac{t^2}{q}\right)} = e^{\frac{i\pi}{4}\left(1-\frac{(t+2q)^2}{q}\right)}$. Thus, in Eq. (15), t is a periodic variable with period $2q$. We have a certain freedom in setting the limits of summation in (15); summing t from 1 to $2q$ or from $-q$ to $q-1$ are both valid reinterpretations of (15). Thus, a more general form of Eq. (15) is

$$F\left(x', \frac{1}{q}\right) = \frac{e^{-ikn_r\frac{L}{q}}}{\sqrt{q}} \sum_{\substack{t\in\mathbb{Z}_{2q} \\ t\equiv q(mod\,2)}} e^{\frac{i\pi}{4}\left(1-\frac{t^2}{q}\right)} F\left(x' - \frac{t}{2q}, 0\right) \tag{16}$$

To derive the analogous result of (15) for $E_y(x', z')$, we just include the fictitious term:

$$E_y\left(x', \frac{1}{q}\right) = \frac{e^{-ikn_r\frac{L}{q}}}{\sqrt{q}} \sum_{\substack{t=0 \\ t\equiv q(mod\,2)}}^{2q-1} \left[e^{\frac{i\pi}{4}\left(1-\frac{t^2}{q}\right)} E_y\left(x' - \frac{t}{2q}, 0\right) - e^{\frac{i\pi}{4}\left(1-\frac{t^2}{q}\right)} E_y\left(-x' - \frac{t}{2q}, 0\right)\right] \tag{17}$$

for $0 < x' < 1/2$. Each entry in the above summand represents an image of the original field $E_y(x', 0)$.

Equation (17) is displayed graphically in Fig. 2, where both input and output images are represented as step index optical waveguides. The $t = 0$ images always corresponds to the image directly ahead of the input. However if q is odd, this image vanishes. Only q of these $2q$ images are non-vanishing. If q is even, then the q non vanishing images will be at the red waveguides. Conversely if q is odd, the images will be at the blue waveguides. We have neglected to display

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the outputs of the fictitious input. However note that as x' is periodic, we may use a fictitious source located at either $x' = -x'_0$ or $x' = 1 - x'_0$; they are equivalent.

In the application of MMIs, the input electric field $E_y(x, 0)$ will be generated from an array of optical waveguides with widths much smaller than the MMI. Each waveguide will carry its own electrical signal, but often these signals will be identical up to a change in phase and amplitude. To this end, suppose the signals all have a common shape $E_0(x)$. Then any such $E_y(x)$ may represented as a sum of sources, where each sources represents an input optical waveguide. Each source is associated with a complex phase amplitude A_j and location x_j:

$$E_y(x,0) = \sum_{j=1}^{N} A_j E_0\left(x - x_j\right) = \sum_{j=1}^{N} A_j E_0(x) * \delta\left(x - x_j\right) = E_0(x) * \left(\sum_{j=1}^{N} A_j \delta\left(x - x_j\right)\right) \quad (18)$$

with $0 \leq x'_j \leq \frac{1}{2}$. In the extended domain:

$$F\left(x',0\right) = \sum_{j=1}^{N} A_j\left(E_0\left(x' - x'_j\right) - E_0\left(-x' + x'_j\right)\right) = \sum_{j=1}^{N} A_j\left(E_0\left(x'\right) * \delta\left(x' - x'_j\right) - E_0\left(-x'\right) * \delta\left(x' + x'_j\right)\right) \quad (19)$$

For most cases, $E_0(x)$ will represent the lowest order mode of the optical input waveguide. These modes are even, and thus $E_0(x) = E_0(-x)$:

$$F\left(x',0\right) = \sum_{j=1}^{N} A_j\left(E_0\left(x'\right) * \delta\left(x' - x'_j\right) - E_0\left(x'\right) * \delta\left(x' + x'_j\right)\right) = E_0\left(x',0\right) * \left(\sum_{j=1}^{N} A_j\left(\delta\left(x' - x'_j\right) - \delta\left(x' + x'_j\right)\right)\right) \quad (20)$$

In the case of an odd mode, where $E_0(-x) = -E_0(x)$, then analogously

$$F\left(x',0\right) = \sum_{j=1}^{N} A_j\left(E_0\left(x'\right) * \delta\left(x' - x'_j\right) + E_0\left(x'\right) * \delta\left(x' + x'_j\right)\right) = E_0\left(x',0\right) * \left(\sum_{j=1}^{N} A_j\left(\delta\left(x' - x'_j\right) + \delta\left(x' + x'_j\right)\right)\right) \quad (21)$$

Unless otherwise stated, we will always assume that $E_0(x)$ is even. We can now apply Eq. (15):

$$F\left(x', \frac{1}{q}\right) = \left(E_0\left(x',0\right) * \left(\sum_{j=1}^{N} A_j\left(\delta\left(x' - x'_j\right) - \delta\left(x' + x'_j\right)\right)\right)\right) * \left(\frac{e^{-ikn_r\frac{L}{q}}}{\sqrt{q}} \sum_{\substack{t=0 \\ t\equiv q(mod\,2)}}^{2q-1} e^{\frac{i\pi}{4}\left(1-\frac{t^2}{q}\right)} \delta\left(x' - \frac{t}{2q}\right)\right) \quad (22)$$

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Recall that the convolution operator is associative, so that $(f(x) * g(x)) * h(x) = f(x) * (g(x) * h(x))$. Thus,

$$\begin{aligned} F\left(x', \frac{1}{q}\right) &= \frac{e^{-ikn_r \frac{L}{q}}}{\sqrt{q}} E_0\left(x'\right) * \\ &\left(\left(\sum_{j=1}^{N} A_j \left(\delta\left(x' - x_j'\right) - \delta\left(x' + x_j'\right)\right)\right) * \left(\sum_{\substack{t=0 \\ t \equiv q (mod\, 2)}}^{2q-1} e^{\frac{i\pi}{4}\left(1 - \frac{t^2}{q}\right)} \delta\left(x' - \frac{t}{2q}\right)\right)\right) \\ &= \frac{e^{-ikn_r \frac{L}{q}}}{\sqrt{q}} E_0\left(x'\right) * \left(\sum_{j=1}^{N} \sum_{\substack{t=0 \\ t \equiv q (mod\, 2)}}^{2q-1} A_j e^{\frac{i\pi}{4}\left(1 - \frac{t^2}{q}\right)} \left(\delta\left(x' - x_j'\right) - \delta\left(x' + x_j'\right)\right) * \delta\left(x' - \frac{t}{2q}\right)\right) \\ &= \frac{e^{-ikn_r \frac{L}{q}}}{\sqrt{q}} E_0\left(x'\right) * \left(\sum_{j=1}^{N} \sum_{\substack{t=0 \\ t \equiv q (mod\, 2)}}^{2q-1} A_j e^{\frac{i\pi}{4}\left(1 - \frac{t^2}{q}\right)} \left(\delta\left(x' - x_j' - \frac{t}{2q}\right) - \delta\left(x' + x_j' - \frac{t}{2q}\right)\right)\right) \end{aligned} \tag{23}$$

Finally, in the simplified case where there is a single input source $j = N = 1$ and $A_j = 1$, Eq. (23) reduces to

$$F\left(x', \frac{1}{q}\right) = \frac{e^{-ikn_r \frac{L}{q}}}{\sqrt{q}} E_0(x') * \left(\sum_{\substack{t=0 \\ t \equiv q (mod\, 2)}}^{2q-1} e^{\frac{i\pi}{4}\left(1 - \frac{t^2}{q}\right)} \left(\delta\left(x' - x_1' - \frac{t}{2q}\right) - \delta\left(x' + x_1' - \frac{t}{2q}\right)\right)\right) \tag{24}$$

To compute the relevant $E_0\left(x', z'\right)$, we simply restrict x' to be between 0 and $1/2$. One may suggest that in restricting x' to $[0, 1/2]$, we may disregard the Dirac deltas $\delta\left(x' - a\right)$ for which $a \in (1/2, 1)$, as theses images are in the fictitious domain. This is false, as although $\delta\left(x' - a\right) = 0$ for any $x' \in [0, 1/2]$, $E_0\left(x'\right) * \delta\left(x' - a\right) = E_0\left(x' - a\right)$ may be non-zero for $x' \in [0, 1/2]$. However in practice, $E_0\left(x'\right)$ will be narrow. We will say that $E_0\left(x'\right)$ has width ϵ if $E_0\left(x'\right)$ is negligible for $|x'| > \epsilon/2$. Thus we may ignore those images for which $a \in (1/2 + \epsilon/2, 1 - \epsilon/2)$. We will makes this argument more precise in the forthcoming examples.

At $z' = 1/q$, each sources has $2q$ images in the extended domain. We say that each source *re-images* $2q$ times in the extended domain ($0 \leq x' < 1$), and q times in the real domain ($0 \leq x' < 1/2$). However in general these images may overlap or even destructively interfere, which we will exploit.

Example 2.1 (Single image: $z = L, q = 1$)**.** This simple case is evaluated using Eq. (23) with $q = 1$:

$$F\left(x', z' = \frac{1}{1}\right) = e^{-ikn_r L} E_0(x') * \left(\delta\left(x' - x_1' - \frac{1}{2}\right) - \delta\left(x' + x_1' - \frac{1}{2}\right)\right) \tag{25}$$

With a real source $\left(0 \leq x_1' \leq 1/2\right)$ and $E_0\left(x'\right)$ having a width less than $\min\left(\frac{x_1'}{2}, \left(\frac{1}{2} - x_1'\right)/2\right)$, then the real field at $z' = 1$ may be expressed as

$$E_y(x', z' = 1) = -e^{-ikn_r L} E_0\left(x'\right) * \delta\left(x' + x_1' - \frac{1}{2}\right) \tag{26}$$

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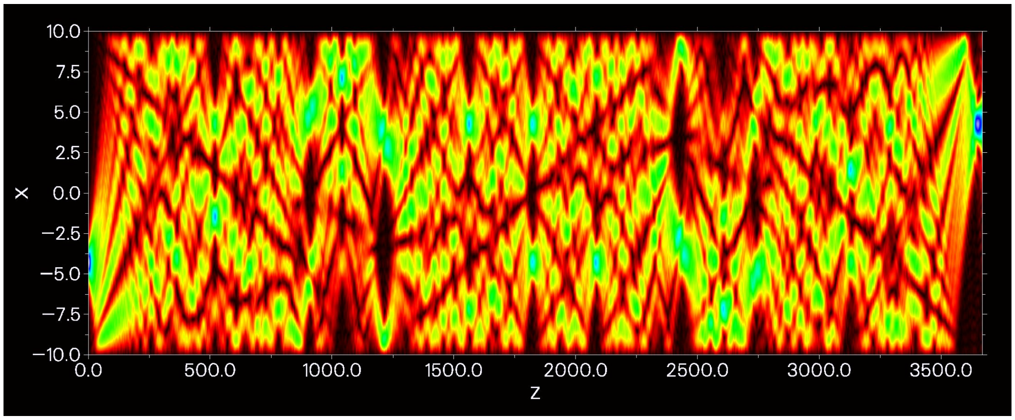

Fig. 3. A single image.

This solution describes a source which upon propagating a distance $z = L$, has shifted laterally by $x' = 1/2$ in the extended domain. Taking into account the fictitious source at $x' = 1 - x_1' \simeq -x_1'$, the real image is then located at $x' = -x_1 + 1/2$. The average of these locations is $x' = 1/4$ which is the centre of the waveguide. Thus, the overall effect is that the source and the image are located symmetrically about the centre of the waveguide.

One such possible optical field $E(x,z)$ is plotted in Fig. 3. This is a numerical solution using $n_r = 3.5$, $n_c = 1.0$ and $\lambda = 1.55\mu m$. Therefore $L \approx 3.6mm$ and $z' \approx z/3600\mu m$. These parameters are used for all the numerical simulations in this article. Such a pattern is referred to as a Talbot carpet [22]. In the limiting case when the waveguide supports infinitely many modes, Talbot carpets yield a fascinating fractal structure [14].

This solution is presented graphically in Fig. 4. Note that from a ray optics point of view, the real image propagates until it hits the boundary of the waveguide, whereupon it bounces via total internal reflection and yields the single real image. However from an extended domain point of view, there is no reflection. Thus, any image created from the fictitious domain may be thought of as a "reflection" of the original image. Total internal reflection replicates these fictitious images, which further induces a periodic grating. Thus, total internal reflection is the driving mechanism behind the Talbot effect in dielectric waveguides.

Example 2.2 (Double image: $z = L/2, q = 2$)**.**

$$F\left(x', \frac{1}{2}\right) = \frac{e^{-ikn_r\frac{L}{2}}}{\sqrt{2}} E_0\left(x'\right) * \left(\sum_{\substack{t=0 \\ t\equiv 2(\mathrm{mod}\,2)}}^{3} e^{\frac{i\pi}{4}\left(1-\frac{t^2}{2}\right)} \left(\delta\left(x' - x_1' - \frac{t}{4}\right) - \delta\left(x' + x_1' - \frac{t}{4}\right)\right)\right)$$

$$= \frac{e^{-ikn_r\frac{L}{2}}}{\sqrt{2}} E_0\left(x'\right) * \left(e^{\frac{i\pi}{4}}\left(\delta\left(x' - x_1'\right) - \delta\left(x' + x_1'\right)\right) + e^{-\frac{i\pi}{4}}\left(\delta\left(x' - x_1' - \frac{1}{2}\right) - \delta\left(x' + x_1' - \frac{1}{2}\right)\right)\right)$$

With a real source $\left(0 \le x_1' \le 1/2\right)$ and $E_0\left(x'\right)$ again having a width less than $\min\left(\frac{x_1'}{2}, \left(\frac{1}{2} - x_1'\right)/2\right)$, the real images are described by

$$E_y\left(x', \frac{1}{2}\right) = \frac{e^{-ikn_r\frac{L}{2}}}{\sqrt{2}} E_0\left(x'\right) * \left(e^{\frac{i\pi}{4}}\delta\left(x' - x_1'\right) - e^{-\frac{i\pi}{4}}\delta\left(x' + x_1' - \frac{1}{2}\right)\right)$$

These two images are located at x_1' and $\frac{1}{2} - x_1'$ which are symmetrically located about the center

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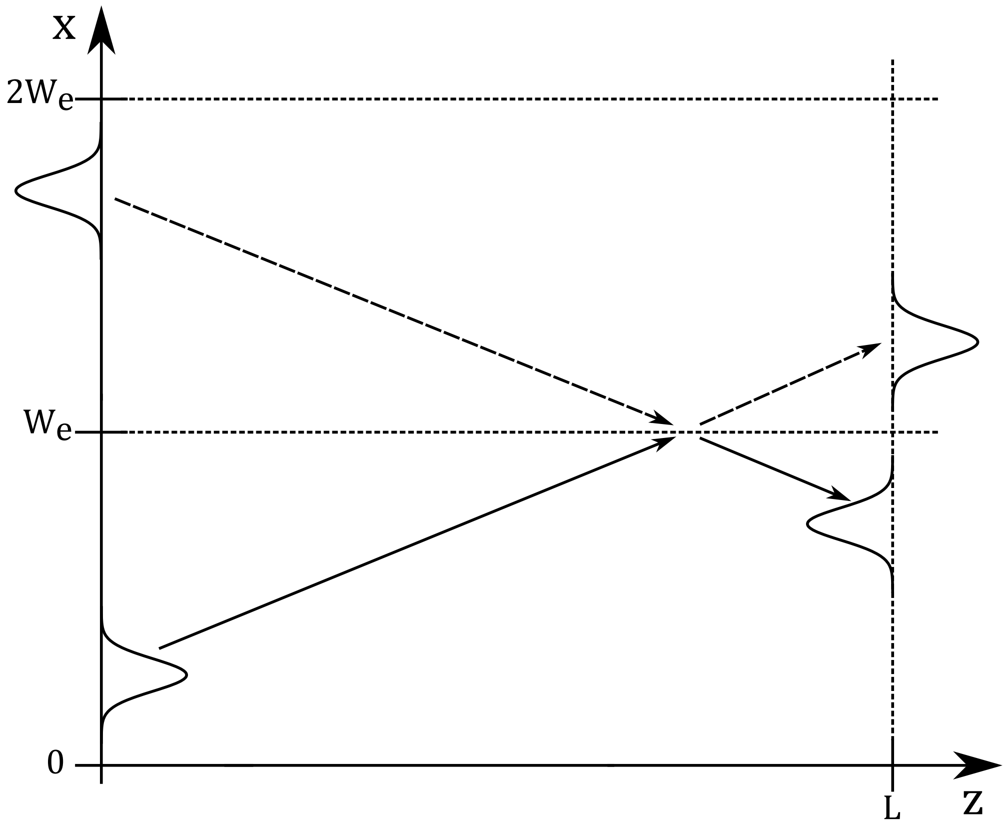

Fig. 4. The two interpretations of the single image at $z = L$.

of the MMI. The two images have a phase difference of $\frac{\pi}{2}$ and are equal in amplitude, as can be seen in Fig. 5.

Example 2.3 (Double image: $(z = L/4, q = 4)$)**.** This is a special case for a double image, when the input source is located at $x_1' = \frac{1}{8}$.

$$F\left(x', \frac{1}{4}\right) = \frac{e^{-ikn_r \frac{L}{4}}}{2} E_0\left(x'\right) * \left(\sum_{\substack{t=0 \\ t \equiv 4(\mathrm{mod}\,2)}}^{7} e^{\frac{i\pi}{4}\left(1-\frac{t^2}{4}\right)} \left(\delta\left(x' - \frac{1}{8} - \frac{t}{8}\right) - \delta\left(x' + \frac{1}{8} - \frac{t}{8}\right)\right)\right)$$

$$= \frac{e^{-ikn_r \frac{L}{4}}}{2} E_0\left(x'\right) * \left(e^{\frac{i\pi}{4}} \left(\delta\left(x' - \frac{1}{8}\right) - \delta\left(x' + \frac{1'}{8}\right)\right) + \left(\delta\left(x' - \frac{3}{8}\right) - \delta\left(x' - \frac{1}{8}\right)\right)\right.$$
$$\left. + e^{-\frac{i3\pi}{4}} \left(\delta\left(x' - \frac{5}{8}\right) - \delta\left(x' - \frac{3}{8}\right)\right) + \left(\delta\left(x' - \frac{7}{8}\right) - \delta\left(x' - \frac{5}{8}\right)\right)\right)$$

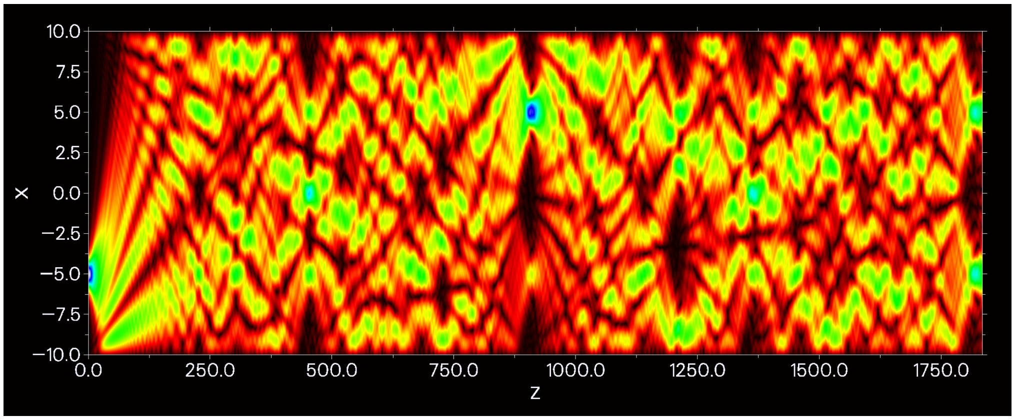

Fig. 5. Two images at $z' = 1/2$.

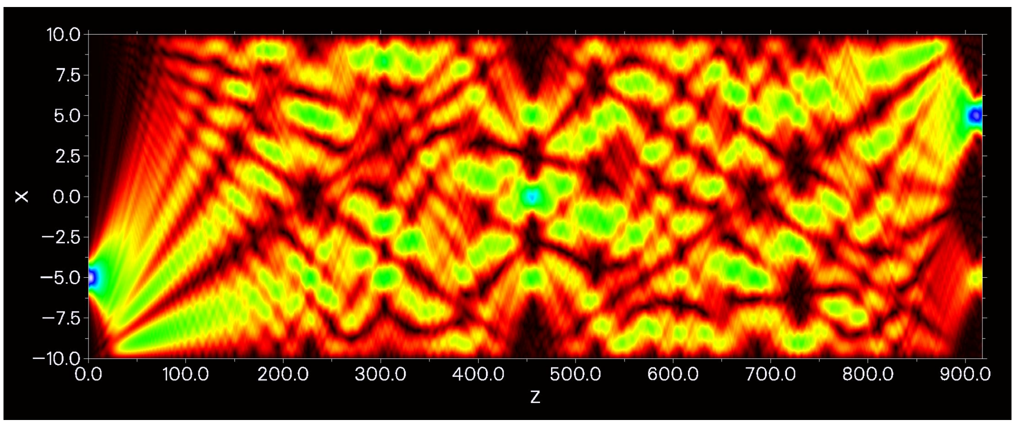

Fig. 6. Two of the four images destructively interfere.

When $E_0(x')$ has a width less than $1/16$, the real images are described by:

$$E_y\left(x', \frac{1}{4}\right) = \frac{e^{-ikn_r\frac{L}{4}}}{2} E_0(x') * \left(\left(e^{\frac{i\pi}{4}} - 1\right)\delta\left(x' - \frac{1}{8}\right) + \left(1 - e^{-\frac{i3\pi}{4}}\right)\delta\left(x' - \frac{3}{8}\right)\right)$$

$$= e^{-ikn_r\frac{L}{4}} E_0(x') * \left(\sqrt{\frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)} e^{\frac{i5\pi}{8}}\delta\left(x' - \frac{1}{8}\right) + \sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)} e^{\frac{i\pi}{8}}\delta\left(x' - \frac{3}{8}\right)\right)$$

These two images can be seen in Fig. 6, where the asymmetry in the intensities is also clearly visible. Each of these numerical calculations validates our analytical results, as both the amplitudes and locations of the images exactly match. What's more, as the vanishing of certain images is due to destructive interference we also indirectly verify our phase calculations. This will become especially evident in section 3.

From Fig. 6 note that there is also a triple image at $z' = 1/8$ ($z \approx 450\mu m$). We will return to both of these examples in section 4.

2.1. Multimode interference in 2 dimensions

The above analysis may be generalised to more than one dimension. Suppose we have a rectangular waveguide with effective widths W_x and W_y in the x and y directions and infinite in

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the z direction. We perform the standard seperation of variables to find solutions of the form $\psi_{m,n}(x,y)e^{-i\beta_{m,n}z}$. Then the lower order modes are given by

$$\psi_{m,n}(x,y) = \sin\left(\frac{m\pi x}{W_x}\right)\sin\left(\frac{n\pi y}{W_y}\right) \tag{27}$$

The propagation constants may be expanded as

$$\begin{aligned}\beta_{m,n} &= \sqrt{n_r^2k^2 - \left(\frac{m\pi}{W_x}\right)^2 - \left(\frac{n\pi}{W_y}\right)^2} \\ &= n_r k\sqrt{1 - \left(\frac{m\pi}{n_r k W_x}\right)^2 - \left(\frac{n\pi}{n_r k W_y}\right)^2} \\ &\approx n_r k\left(1 - \frac{1}{2}\left(\frac{m\pi}{n_r k W_x}\right)^2 - \frac{1}{2}\left(\frac{n\pi}{n_r k W_y}\right)^2\right) \\ &= n_r k - \frac{m^2\pi}{L_x} - \frac{n^2\pi}{L_y}\end{aligned} \tag{28}$$

where L_x and L_y are the x and y MMI lengths. Thus for an arbitrary input field $E(x,y,0)$, the field $E(x,y,z)$ may be expressed to a good approximation as

$$\begin{aligned}E(x,y,z) &= \sum_{m,n} a_{m,n}\psi_{m,n}(x,y)e^{-i\beta_{m,n}z} \\ &\approx e^{-ikn_r z}\sum_{m,n} a_{m,n}\sin\left(\frac{m\pi x}{W_x}\right)\sin\left(\frac{n\pi y}{W_y}\right)e^{im^2\pi\frac{z}{L_x}+in^2\pi\frac{z}{L_y}}\end{aligned} \tag{29}$$

where $a_{m,n} = \frac{4}{W_xW_y}\iint E(x,y,0)\sin\left(\frac{m\pi x}{W_x}\right)\sin\left(\frac{n\pi y}{W_y}\right)dx\,dy$. If the input $E(x,y,0)$ originates from a smaller waveguide, then often $E(x,y,0)$ may be expressed as a product of fields in x and y, say $E(x,y,0) = E_x(x)E_y(y)$. Thus $a_{m,n} = b_m c_n$, where $b_m = \frac{2}{W_x}\int E_x(x)\sin(m\pi x/W_x)\,dx$ and $c_n = \frac{2}{W_y}\int E_y(y)\sin(n\pi y/W_y)\,dy$. Further, $E(x,y,x)$ may be expressed as a product

$$E(x,y,z) = e^{-ikn_r z}\left(\sum_m b_m\sin\left(\frac{m\pi x}{W_x}\right)e^{im^2\pi\frac{z}{L_x}}\right)\left(\sum_n c_n\sin\left(\frac{n\pi y}{W_y}\right)e^{im^2\pi\frac{z}{L_y}}\right) \tag{30}$$

Each of these factors is equivalent to MMI propagation in one dimension, which we have already analysed. Thus we may break down the analysis of 2 dimensional MMIs into an analysis of two 1 dimensional MMIs.

More explicitly, suppose that the input $E(x,y,0) = E_x(x)E_y(y)$ is centered about (x_0,y_0). For the x direction consider the 1 dimensional MMI of width W_x and length z with an input signal $E_x(x)$ centered at x_0, and similarly for the y direction. If in the x MMI there are images located at x_i with phase amplitudes u_i and in the y MMI there are images located at y_j with phase amplitudes v_j, then in the 2D MMI there will be images located at (x_i,y_j) with phase amplitudes u_iv_j. If there is more than one source, these will have to be dealt with seperately and the results summed.

As will be derived in the next section, if a symmetric input is launched through the centre of an MMI, then there will be 2 images located at $z = L/8$ and 3 images located at $z = L/12$. Thus, if we design a 2 dimensional MMI such that $L_x/12 = L_y/8$, then at this common distance there will be six images arranged in a 3×2 array as in Fig. 7. As $L \sim W_e^2$, this implies that $W_x^2/12 = W_y^2/8 \Leftrightarrow W_x = \sqrt{3/2}W_y$.

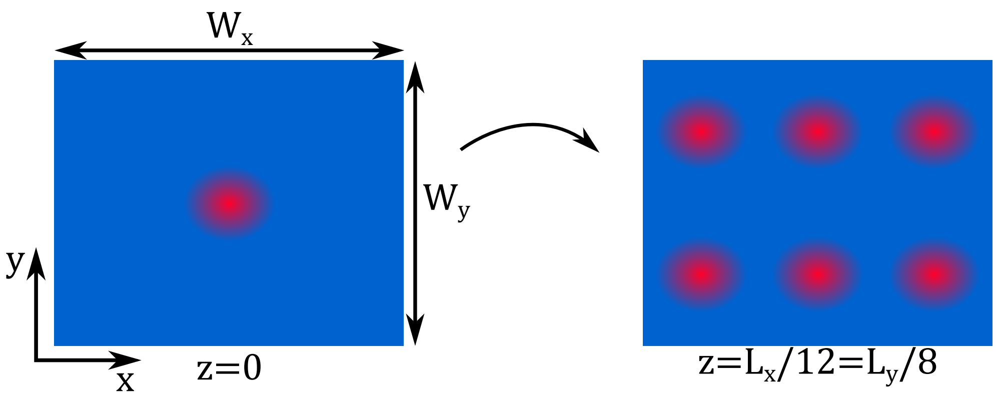

Fig. 7. Propagation in a 2D MMI with $W_x = \sqrt{3/2}W_y$.

3. Restricted interference

The analysis so far has been carried out for general interference; no assumptions on the eigenmode coefficients a_m have been placed. However by careful selection of the input field, certain modes will not be excited [11]. This in turn allows for the extended field $F(x', z')$ to re-image at distances smaller than L. The locations and phases of these images were first derived by Bachmann et al. [16].

3.1. Symmetric interference

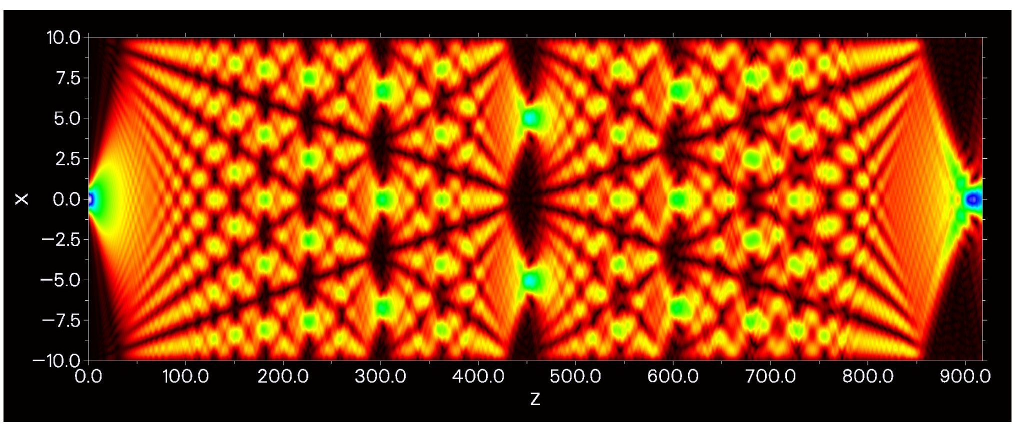

Fig. 8. A single image at $z = L/4$ due to symmetric interference.

A very important class of MMIs come from the observation that if a signal with a symmetric profile enters the centre of the MMI at $x' = 1/4$, $(E_y(x', 0) = E_0(x') * \delta(x' - 1/4)$ where $E_0(x')$ is even), then in the Fourier representation of the incoming signal

$$F(x', 0) = \sum_{m=-\infty}^{m=\infty} a_m e^{im2\pi x'} \tag{31}$$

all of the even modes will vanish. In other words, $a_m = 0$ for m even. This is known as symmetric

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interference. Then

$$F(x',0) = \sum_{m\text{ odd}} a_m e^{im2\pi x'} \tag{32}$$

As before:

$$F(x',z') = e^{-ikn_r z' L} \sum_{m\text{ odd}} a_m e^{im2\pi x'} e^{im^2\pi z'} = e^{-ikn_r z' L} F(x',0) * \vartheta_{\text{odd}}(x',z') \tag{33}$$

where

$$\vartheta_{\text{odd}}(x',z') = \sum_{m\text{ odd}} e^{im2\pi x'} e^{im^2\pi z'} \tag{34}$$

m odd implies that $m^2 \equiv 1 \pmod 4$ for all m, implying that $\vartheta_{\text{odd}}(x', 1/4q)$ may simplify more than $\vartheta(x', 1/4q)$. This is supported by Fig. 8, which shows a single image at $z' = 1/4$. Thus for a symmetric source entering the MMI at $x' = 1/4$ we examine the $q = 4q'$ non vanishing images in the extended domain that exist at $z' = 1/q = 1/4q'$. These are given by

$$F\left(x', \frac{1}{q}\right) = F\left(x', \frac{1}{4q'}\right)$$
$$\frac{e^{-ikn_r \frac{L}{q}}}{\sqrt{4q'}} E_0(x') * \left(\sum_{\substack{t=0 \\ t\equiv 4q' \pmod 2}}^{8q'-1} e^{\frac{i\pi}{4}\left(1-\frac{t^2}{4q'}\right)} \left(\delta\left(x' - \frac{1}{4} - \frac{t}{8q'}\right) - \delta\left(x' + \frac{1}{4} - \frac{t}{8q'}\right)\right)\right) \tag{35}$$

Since $4q'$ is even, we let $2t' = t$ which gives

$$F\left(x', \frac{1}{q}\right) = F\left(x', \frac{1}{4q'}\right) =$$
$$\frac{e^{-ikn_r \frac{L}{q}}}{\sqrt{4q'}} E_0(x') * \left(\sum_{t'=0}^{4q'-1} e^{\frac{i\pi}{4}\left(1-\frac{t'^2}{q'}\right)} \left(\delta\left(x' - \frac{1}{4} - \frac{t'}{4q'}\right) - \delta\left(x' + \frac{1}{4} - \frac{t'}{4q'}\right)\right)\right) \tag{36}$$

Note that t' is a periodic variable with period $4q'$. Thus, in order to simplify Eq. (36), we will shift the images due to the fictitious source so as to overlap with those of the real source. This is achieved by taking $t' \to t' + 2q'$, as $\delta\left(x' + \frac{1}{4} - \frac{(t'+2q')}{4q'}\right) = \delta\left(x' + \frac{1}{4} - \frac{1}{2} - \frac{t'}{4q'}\right) = \delta\left(x' - \frac{1}{4} - \frac{t'}{4q'}\right)$.

$$F\left(x', \frac{1}{4q'}\right) = \frac{e^{-ikn_r \frac{L}{q}}}{\sqrt{4q'}} E_0(x') *$$
$$\left(\sum_{t'=0}^{4q'-1} e^{\frac{i\pi}{4}\left(1-\frac{t'^2}{q'}\right)} \delta\left(x' - \frac{1}{4} - \frac{t'}{4q'}\right) - e^{\frac{i\pi}{4}\left(1-\frac{(t'+2q')^2}{q'}\right)} \delta\left(x' + \frac{1}{4} - \frac{t'+2q'}{4q'}\right)\right)$$
$$= \frac{e^{-ikn_r \frac{L}{q}}}{\sqrt{4q'}} E_0(x') * \left(\sum_{t'=0}^{4q'-1} \left(e^{\frac{i\pi}{4}\left(1-\frac{t'^2}{q'}\right)} - e^{\frac{i\pi}{4}\left(1-\frac{(t'+2q')^2}{q'}\right)}\right) \delta\left(x' - \frac{1}{4} - \frac{t'}{4q'}\right)\right) \tag{37}$$

But

$$e^{\frac{i\pi}{4}\left(1-\frac{t'^2}{q'}\right)} - e^{\frac{i\pi}{4}\left(1-\frac{(t'+2q')^2}{q'}\right)} = e^{\frac{i\pi}{4}\left(1-\frac{t'^2}{q'}\right)} - e^{\frac{i\pi}{4}\left(1-\frac{t'^2+4t'q'+4q'^2}{q'}\right)} = e^{\frac{i\pi}{4}\left(1-\frac{t'^2}{q'}\right)}\left(1 - e^{-i\pi(t'+q')}\right)$$

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Note that $1 - e^{-i\pi(t'+q')}$ equals 0 when $t' + q'$ is even, which is equivalent to $t' \equiv q' \pmod 2$. Otherwise, $1 - e^{-i\pi(t'+q')} = 2$. Thus Eq. (37) may be simplified to

$$F\left(x', \frac{1}{4q'}\right) = \frac{e^{-ikn_r\frac{L}{q}}}{\sqrt{q'}} E_0(x') * \left(\sum_{\substack{t'=0 \\ t' \not\equiv q' \pmod 2}}^{4q'-1} e^{\frac{i\pi}{4}\left(1-\frac{t'^2}{q'}\right)} \delta\left(x' - \frac{1}{4} - \frac{t'}{4q'}\right) \right) \tag{38}$$

In the extended domain, there are $2q'$ images, so in the real domain there are q' images. Recalling that t' is periodic with period $4q'$, we will sum over t' from $-2q'$ to $2q' - 1$ for the sake of symmetry.

$$F\left(x', \frac{1}{4q'}\right) = \frac{e^{-ikn_r\frac{L}{q}}}{\sqrt{q'}} E_0(x') * \left(\sum_{\substack{t'=-2q' \\ t' \not\equiv q' \pmod 2}}^{2q'-1} e^{\frac{i\pi}{4}\left(1-\frac{t'^2}{q'}\right)} \delta\left(x' - \frac{1}{4} - \frac{t'}{4q'}\right) \right) \tag{39}$$

The $2q'$ images are centred at $1/4 + t'/4q'$. For an image to be in the real domain, we require that $0 \le 1/4 + t'/4q' \le 1/2 \leftrightarrow -1/4 \le t'/4q' \le 1/4 \leftrightarrow -q' \le t' \le q'$. However when $t' = q'$ or $-q'$, then $t' \equiv q' \pmod 2$. Therefore, if the width of $E_0(x')$ is less than $1/2q'$, we obtain the following expression for the real field.

$$E_y\left(x', \frac{1}{4q'}\right) = \frac{e^{-ikn_r\frac{L}{q}}}{\sqrt{q'}} E_0(x') * \left(\sum_{\substack{t'=-q'+1 \\ t' \not\equiv q' \pmod 2}}^{q'-1} e^{\frac{i\pi}{4}\left(1-\frac{t'^2}{q'}\right)} \delta\left(x' - \frac{1}{4} - \frac{t'}{4q'}\right) \right) \tag{40}$$

Thus $z' = L/4q'$, we can reduce the number of real images from $4q'$ to q' by sending the input signal symmetrically into the MMI. Furthermore, these images are evenly spaced along the MMI. This works by two mechanisms; the $2q'$ real images from the real and fictitious source perfectly overlap, and of these $2q'$ real images q' vanish via destructive interference.

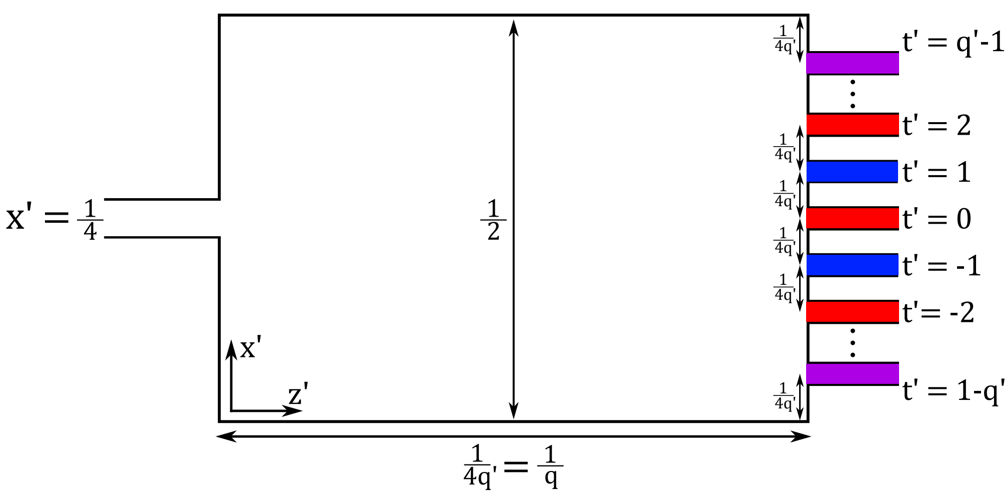

Fig. 9. A schematic of symmetric interference.

Equation (40) is further explained in Fig. 9. If q is even, the images will be located at the blue waveguides. If q is odd, the images will be located at the red waveguides instead. As

$q-1 \equiv 1-q \not\equiv q' \pmod 2$, there will always be images at the purple waveguides next to the edge of the MMI.

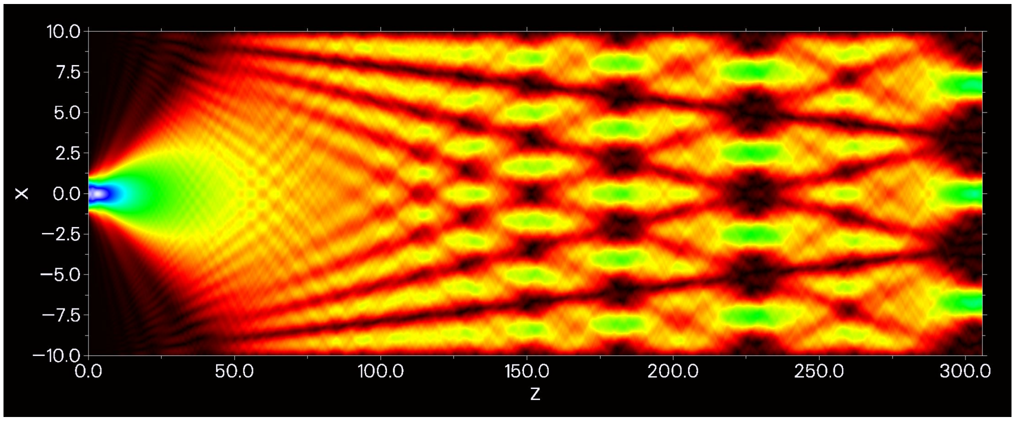

Fig. 10. Symmetric interference producing 3 images.

Example 3.1 (1×3 MMI: $z = L/12$, $q' = 3$, $q = 12$)**.** This example describes a typical MMI splitter; in this case a 1×3 splitter as in Fig. 10. Using Eq. (40), we write:

$$E_y\left(x', \frac{1}{12}\right) = \frac{e^{-ikn_r \frac{L}{12}}}{\sqrt{3}} E_0\left(x', 0\right) * \left(\sum_{\substack{t'=-2 \\ t' \text{ even}}}^{2} e^{\frac{i\pi}{4}\left(1-\frac{t'^2}{3}\right)} \delta\left(x' - \frac{1}{4} - \frac{t'}{12}\right) \right)$$

$$= \frac{e^{-ikn_r \frac{L}{12}}}{\sqrt{3}} E_0\left(x', 0\right) * \left(e^{\frac{i\pi}{4}\left(1-\frac{4}{3}\right)} \delta\left(x' - \frac{1}{4} + \frac{1}{6}\right) + e^{\frac{i\pi}{4}} \delta\left(x' - \frac{1}{4}\right) + e^{\frac{i\pi}{4}\left(1-\frac{4}{3}\right)} \delta\left(x' - \frac{1}{4} - \frac{1}{6}\right) \right) \tag{41}$$

So, the real images can then be described by:

$$E_y\left(x', \frac{1}{12}\right) = \frac{e^{-ikn_r \frac{L}{12}} e^{-\frac{i\pi}{4}}}{\sqrt{3}} E_0\left(x', 0\right) * \left(e^{-\frac{i\pi}{3}} \delta\left(x' - \frac{1}{12}\right) + \delta\left(x' - \frac{1}{4}\right) + e^{-\frac{i\pi}{3}} \delta\left(x' - \frac{5}{12}\right) \right) \tag{42}$$

In this basic splitter structure, the magnitude of all outputs is the same, while there is a phase difference of $-\pi/3$ that is symmetric around the centre of the structure.

This example can also be used to create a 3×1 combiner. As $\vartheta\left(x', -z'\right) = \overline{\vartheta\left(x', z'\right)}$, it follows that $E_y\left(x', z'\right) = \overline{E_y\left(x', -z'\right)}$. In other words, *backwards propagation is equivalent to conjugation*. Thus, if

$$E_y\left(x', -\frac{1}{12}\right) = \frac{e^{ikn_r \frac{L}{12}} e^{\frac{i\pi}{4}}}{\sqrt{3}} E_0\left(x', 0\right) * \left(e^{+\frac{i\pi}{3}} \delta\left(x' - \frac{1}{12}\right) + \delta\left(x' - \frac{1}{4}\right) + e^{+\frac{i\pi}{3}} \delta\left(x' - \frac{5}{12}\right) \right) \tag{43}$$

then $E_y\left(x', 0\right) = E_0\left(x'\right) * \delta\left(x' - 1/4\right)$. This principle may be applied to invert any MMI structure. Note that if the phases of the inputs in (43) are altered, then there will be more than one image. Therefore this combiner does not function as a 3×1 MMI for arbitrary inputs. The impossibility of such combining MMIs will be discussed in appendix B.

3.2. Paired interference

If a symmetric input field is sent into an MMI at either $x' = 1/6$ or $x' = 2/6$, then no modes with a mode number divisible by 3 will be excited. Thus in the sum $\sum_{m=-\infty}^{\infty} e^{im2\pi x'} e^{im^2\pi z'}$,

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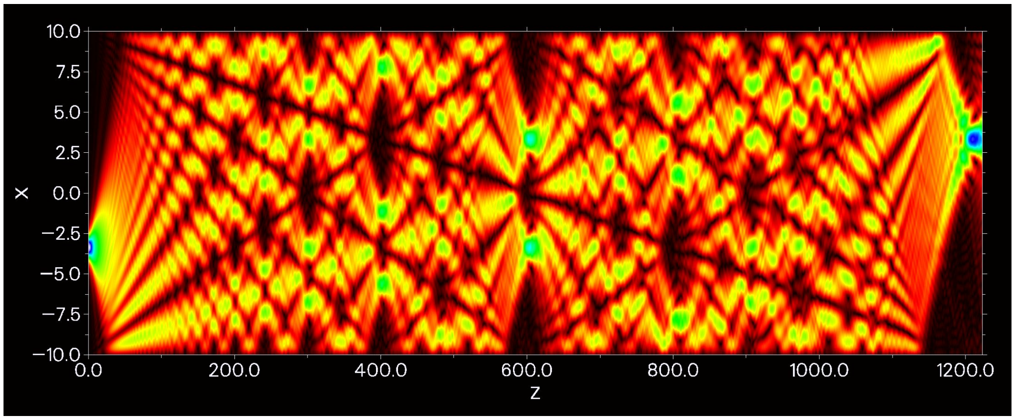

Fig. 11. Paired interference producing a single image at $z = L/3$.

$m \equiv 1, -1 \pmod 3$ and therefore $m^2 \equiv 1 \pmod 3$. Further, observe that for such an input in Fig. 11, there is a single image at $z' = 1/3$. Thus we will consider the image of a symmetric input at $x' = 1/6$ at a distance $z' = 1/3q$. As before, we apply Eq. (23)

$$F\left(x', \frac{1}{3q}\right) = \frac{e^{-ikn_r\frac{L}{3q}}}{\sqrt{3q}} E_0\left(x'\right) * \left(\sum_{\substack{t=0 \\ t\equiv 3q \pmod 2}}^{6q-1} e^{\frac{i\pi}{4}\left(1-\frac{t^2}{3q}\right)} \left(\delta\left(x' - \frac{1}{6} - \frac{t}{6q}\right) - \delta\left(x' + \frac{1}{6} - \frac{t}{6q}\right)\right)\right) \tag{44}$$

Note that as 3 is odd, $t \equiv 3q \pmod 2$ is equivalent to $t \equiv q \pmod 2$. Proceeding as before, we will shift the indexing of t for the images generated by fictitious source, $t \to t + 2q$, so that we may superimpose the images from both the real and fictitious sources. Then $\delta\left(x' + \frac{1}{6} - \frac{t+2q}{6q}\right) = \delta\left(x' + \frac{1}{6} - \frac{1}{3} - \frac{t}{6q}\right) = \delta\left(x' - \frac{1}{6} - \frac{t}{6q}\right)$ and

$$F\left(x', \frac{1}{3q}\right) = \frac{e^{-ikn_r\frac{L}{3q}}}{\sqrt{3q}} E_0\left(x'\right) * \left(\sum_{\substack{t=0 \\ t\equiv q \pmod 2}}^{6q-1} \left(e^{\frac{i\pi}{4}\left(1-\frac{t^2}{3q}\right)} - e^{\frac{i\pi}{4}\left(1-\frac{(t+2q)^2}{3q}\right)}\right) \delta\left(x' - \frac{1}{6} - \frac{t}{6q}\right)\right) \tag{45}$$

Then

$$e^{\frac{i\pi}{4}\left(1-\frac{t^2}{3q}\right)} - e^{\frac{i\pi}{4}\left(1-\frac{(t+2q)^2}{3q}\right)} = e^{\frac{i\pi}{4}\left(1-\frac{t^2}{3q}\right)} - e^{\frac{i\pi}{4}\left(1-\frac{t^2+4tq+4q^2}{3q}\right)} = e^{\frac{i\pi}{4}\left(1-\frac{t^2}{3q}\right)}\left(1 - e^{-\frac{i\pi}{3}(t+q)}\right) \tag{46}$$

We then use the identity $e^{i\alpha} - e^{i\beta} = 2i\sin\left(\frac{\alpha-\beta}{2}\right)e^{i\left(\frac{\alpha+\beta}{2}\right)}$: $1 - e^{-\frac{i\pi}{3}(t+q)} = 2i\sin\left(\frac{\pi}{6}(t+q)\right)e^{-i\frac{\pi}{6}(t+q)}$. Enforcing t and q to have the same parity is equivalent to forcing $t + q$ to be even. Further, if $3|t + q$ then $\sin\left(\frac{\pi}{6}(t+q)\right) = 0$. Thus in Eq. (45) there are only $2q$ non-vanishing images in the extended domain.

Currently, $t = 0$ corresponds to the image at $x' = 1/6$. Let $t' = (t + q)/2$, so that $\delta\left(x' - \frac{1}{6} - \frac{t}{6q}\right) = \delta\left(x' - \frac{1}{6} - \frac{2't-q}{6q}\right) = \delta\left(x' - \frac{t'}{3q}\right)$. $t' = 0$ corresponds to $t = -q$, or the image at $x' = 0$. As $t + q$ is even, t' is an integer and $2i\sin\left(\frac{\pi}{6}(t+q)\right)e^{-i\frac{\pi}{6}(t+q)} = 2i\sin\left(\frac{\pi t'}{3}\right)e^{-\frac{i\pi}{3}t'}$.

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To account for $\sin\left(\frac{\pi t'}{3}\right)$, let $\tau\left(t'\right) = \frac{2}{\sqrt{3}}\sin\left(\frac{\pi t'}{3}\right)$:

$$\tau\left(t'\right) = \begin{cases} 0 & t' \equiv 0 \pmod 3 \\ 1 & t' \equiv 1,2 \pmod 3 \\ -1 & t' \equiv 1,2 \pmod 3 \end{cases} \tag{47}$$

so $2i\sin\left(\frac{\pi t'}{3}\right)e^{-\frac{i\pi}{3}t'} = sqrt3i\tau\left(t'\right)e^{-\frac{i\pi}{3}t'}$. Next, the phase term in (46) is expanded in t' and q:

$$e^{\frac{i\pi}{4}\left(1-\frac{t^2}{3q}\right)} = e^{\frac{i\pi}{4}\left(1-\frac{(2t'-q)^2}{3q}\right)} = e^{\frac{i\pi}{4}\left(1-\frac{4t'^2-4t'q+q^2}{3q}\right)} = e^{\frac{i\pi}{4}}e^{-\frac{i\pi}{3q}t'^2}e^{\frac{i\pi}{3}t'}e^{-\frac{i\pi}{12}q} \tag{48}$$

Thus in all:

$$e^{\frac{i\pi}{4}\left(1-\frac{t^2}{3q}\right)}\left(1-e^{-\frac{i\pi}{3}(t+q)}\right) = \frac{\tau\left(t'\right)i}{\sqrt{3}}e^{-\frac{i\pi}{3}t'}e^{\frac{i\pi}{4}}e^{-\frac{i\pi}{3q}t'^2}e^{\frac{i\pi}{3}t'}e^{-\frac{i\pi}{12}q} = \frac{\tau\left(t'\right)}{\sqrt{3}}e^{\frac{i3\pi}{4}}e^{-\frac{i\pi}{3q}t'^2}e^{-\frac{i\pi}{12}q} \tag{49}$$

Therefore:

$$F\left(x', \frac{1}{3q}\right) = \frac{e^{-ikn_r\frac{L}{3q}}e^{i\pi\left(\frac{9-q}{12}\right)}}{\sqrt{q}}E_0\left(x'\right) * \left(\sum_{t'=0}^{3q-1}\tau\left(t'\right)e^{-i\frac{\pi}{3q}t'^2}\delta\left(x' - \frac{t'}{3q}\right)\right) \tag{50}$$

To determine which images are real, we require that $0 \le \frac{t'}{3q} \le \frac{1}{2} \Leftrightarrow 0 \le t' \le \frac{3q}{2} \Leftrightarrow 0 \le t' \le \left\lfloor\frac{3q}{2}\right\rfloor$. Thus, provided that the width of $E_0\left(x'\right)$ is less than $1/3q'$, then the real field is given by

$$E_y\left(x', \frac{1}{3q}\right) = \frac{e^{-ikn_r\frac{L}{3q}}e^{i\pi\left(\frac{9-q}{12}\right)}}{\sqrt{q}}E_0\left(x'\right) * \left(\sum_{t'=0}^{\left\lfloor\frac{3q}{2}\right\rfloor}\tau\left(t'\right)e^{-i\frac{\pi}{3q}t'^2}\delta\left(x' - \frac{t'}{3q}\right)\right) \tag{51}$$

Of the $3q$ Dirac deltas being summed in the extended domain, only $2q$ are non-vanishing. Thus there are q images in the real domain.

Fig. 12. Graphical representation of equation (51)

Equation (51) is displayed schematically in Fig. 12. The red output waveguides correspond to the non-vanishing images, and the blue waveguides to the vanishing images. Notice how the images come in pairs. If q is odd, there will be an isolated image at $x' = 1/2 - 1/6q$.

For the input at $x' = 2/6$, the images are described by

$$F\left(x', \frac{1}{3q}\right) = \frac{e^{-ikn_r\frac{L}{3q}}}{\sqrt{3q}}E_0\left(x'\right) * \left(\sum_{\substack{t=0 \\ t\equiv q \pmod 2}}^{6q-1} e^{\frac{i\pi}{4}\left(1-\frac{t^2}{3q}\right)}\left(\delta\left(x' - \frac{2}{6} - \frac{t}{6q}\right) - \delta\left(x' + \frac{2}{6} - \frac{t}{6q}\right)\right)\right) \tag{52}$$

For the images of the fictitious source, we shift $t \to t - 2q$. Then $\delta\left(x' + \frac{2}{6} - \frac{t-2q}{6q}\right) = \delta\left(x' + \frac{4}{6} - \frac{t}{6q}\right) = \delta\left(x' - \frac{2}{6} - \frac{t}{6q}\right)$ as $\frac{4}{6} \equiv -\frac{2}{6} \pmod 1$. Then

$$F\left(x', \frac{1}{3q}\right) = \frac{e^{-ikn_r\frac{L}{3q}}}{\sqrt{3q}}E_0\left(x'\right) * \left(\sum_{\substack{t=0 \\ t\equiv q \pmod 2}}^{6q-1}\left(e^{\frac{i\pi}{4}\left(1-\frac{t^2}{3q}\right)} - e^{\frac{i\pi}{4}\left(1-\frac{(t-2q)^2}{3q}\right)}\right)\delta\left(x' - \frac{2}{6} - \frac{t}{6q}\right)\right) \tag{53}$$

As before:

$$e^{\frac{i\pi}{4}\left(1-\frac{t^2}{3q}\right)} - e^{\frac{i\pi}{4}\left(1-\frac{(t-2q)^2}{3q}\right)} = e^{\frac{i\pi}{4}\left(1-\frac{t^2}{3q}\right)} - e^{\frac{i\pi}{4}\left(1-\frac{t^2-4tq+4q^2}{3q}\right)} = e^{\frac{i\pi}{4}\left(1-\frac{t^2}{3q}\right)}\left(1-e^{-\frac{i\pi}{3}(q-t)}\right) \tag{54}$$

$$\Rightarrow F\left(x', \frac{1}{3q}\right) = \frac{e^{-ikn_r\frac{L}{3q}}}{\sqrt{3q}} E_0\left(x'\right) * \left(\sum_{\substack{t=0 \\ t\equiv q\ (\mathrm{mod}\ 2)}}^{6q-1} \left(e^{\frac{i\pi}{4}\left(1-\frac{t^2}{3q}\right)}\left(1-e^{-\frac{i\pi}{3}(q-t)}\right)\right)\delta\left(x' - \frac{2}{6} - \frac{t}{6q}\right)\right) \tag{55}$$

But $1 - e^{-\frac{i\pi}{3}(q-t)} = 2i\sin\left(\frac{\pi(q-t)}{6}\right)e^{-\frac{i\pi(q-t)}{6}}$. The condition $t \equiv q \pmod 2$ is equivalent to $q-t$ being even. Thus let $q - t/2 = t'$, so that $x' - \frac{2}{6} - \frac{t}{6q} = x' - \frac{2}{6} - \frac{q-2t'}{6q} = x' - \frac{1}{2} + \frac{t'}{3q}$. Whereas $t = 0$ corresponds to the $x' = 2/6$ image, $t' = 0$ corresponds to the $x' = 1/2$ image. Thus, $2i\sin\left(\frac{\pi(q-t)}{6}\right)e^{-\frac{i\pi(q-t)}{6}} = 2i\sin\left(\frac{\pi t'}{3}\right)e^{-\frac{i\pi t'}{3}} = \sqrt{3}i\tau\left(t'\right)e^{-\frac{i\pi}{3}t'}$ and

$$e^{\frac{i\pi}{4}\left(1-\frac{t^2}{3q}\right)} = e^{\frac{i\pi}{4}\left(1-\frac{(q-2t')^2}{3q}\right)} = e^{\frac{i\pi}{4}\left(1-\frac{(2t'-q)^2}{3q}\right)} \tag{56}$$

which is identical to the expression in Eq. (48). $\sqrt{3}i\tau\left(t'\right)e^{-\frac{i\pi}{3}t'}$ has already been evaluated in Eq. (49), thus

$$F\left(x', \frac{1}{3q}\right) = \frac{e^{-ikn_r\frac{L}{3q}}e^{i\pi\left(\frac{9-q}{12}\right)}}{\sqrt{q}} E_0\left(x'\right) * \left(\sum_{t'=0}^{3q-1} \tau\left(t'\right)e^{-i\frac{\pi}{3q}t'^2}\delta\left(x' - \frac{1}{2} + \frac{t'}{3q}\right)\right) \tag{57}$$

Unsurprisingly, Eq. (57) is identical to Eq. (50) under the transformation $x' \rightarrow \frac{1}{2} - x'$. This is because the second case is the mirror image of the first about $x' = 1/4$. For the real field:

$$E_y\left(x', \frac{1}{3q}\right) = \frac{e^{-ikn_r\frac{L}{3q}}e^{i\pi\left(\frac{9-q}{12}\right)}}{\sqrt{q}} E_0\left(x'\right) * \left(\sum_{t'=0}^{\left\lfloor\frac{3q}{2}\right\rfloor} \tau\left(t'\right)e^{-i\frac{\pi}{3q}t'^2}\delta\left(x' - \frac{1}{2} + \frac{t'}{3q}\right)\right) \tag{58}$$

provided $E_0\left(x'\right)$ has a width less than $1/3q$.

It is natural to ask when do the images from the $x' = 1/6$ and $x' = 2/6$ sources overlap. If this were the case, then for some integers a and b,

$$\frac{1}{2} - \frac{a}{3q} = \frac{b}{3q} \Leftrightarrow a + b = \frac{3q}{2} \tag{59}$$

This equation can only be satisfied when q is even. Thus when q is odd, the images from the two sources are completely disjoint. Conversely, if q is even, then given some integer a Eq. (59) may be solved for some b and vice versa. This implies that for q even, the images of the two sources completely overlap.

Example 3.2 (Paired 2×2 MMI)**.** We use the case of paired interference with $3q = 6$ to construct a paired 2×2 MMI. The output of the $x' = 1/6$ input is shown in Fig. 13. As q is even, the images of both sources overlap. Thus, the real images are located at xâĂš = 1/6 and xâĂš = 2/6 only. Note that while there are images at the edges of the MMI (x' = âĹŠ0, 1/2), in both cases $\tau = 0$ and so these vanish. This is true for all paired MMIs.

Suppose $E_y\left(x', 0\right) = E\left(x', 0\right) * \left(A_1\delta\left(x' - \frac{1}{6}\right) + A_2\delta\left(x' - \frac{2}{6}\right)\right)$, with $E_0\left(x'\right)$ even and having

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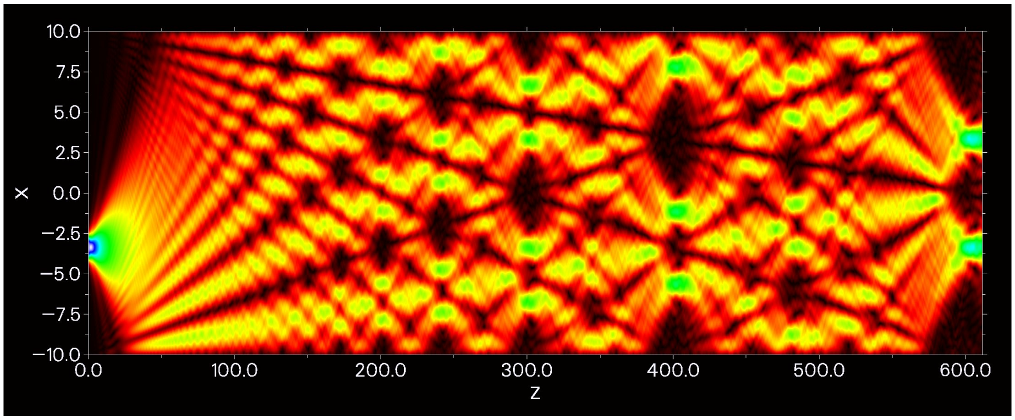

Fig. 13. The two outputs of one input for a paired 2×2 MMI

a width less than $\frac{1}{6}$. Then by Eqs. (51) and (58)

$$
\begin{aligned}
E_y\left(x', z' = \frac{1}{6}\right) &= \\
&\frac{e^{-ikn_r\frac{L}{6}}e^{i\pi\left(\frac{9-2}{12}\right)}}{\sqrt{2}}E_0\left(x'\right) * \left(A_1\left(\tau(1)e^{-i\frac{\pi}{6}}\delta\left(x' - \frac{1}{6}\right) + \tau(2)e^{-i\frac{\pi 2^2}{6}}\delta\left(x' - \frac{2}{6}\right)\right)\right. \\
&\qquad \left. +A_2\left(\tau(1)e^{-i\frac{\pi}{6}}\delta\left(x' - \frac{2}{6}\right) + \tau(2)e^{-i\frac{\pi 2^2}{6}}\delta\left(x' - \frac{1}{6}\right)\right)\right) \\
= \frac{e^{-ikn_r\frac{L}{6}}e^{i\pi\left(\frac{7}{12}\right)}}{\sqrt{2}}E_0\left(x'\right) * &\left(\left(A_1e^{-i\frac{\pi}{6}} + A_2e^{-i\frac{2\pi}{3}}\right)\delta\left(x' - \frac{1}{6}\right) + \left(A_1e^{-i\frac{2\pi}{3}} + A_2e^{-i\frac{\pi}{6}}\right)\delta\left(x' - \frac{1}{3}\right)\right)
\end{aligned} \tag{60}
$$

Letting $E_y\left(x', z' = 1/6\right) = E_0\left(x'\right) * \left(B_1\delta\left(x' - \frac{1}{6}\right) + B_2\delta\left(x' - \frac{2}{6}\right)\right)$, Eq. (60) implies that

$$
\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \frac{e^{-ikn_r\frac{L}{6}}e^{i\pi\left(\frac{5}{12}\right)}}{\sqrt{2}}M\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \tag{61}
$$

where M is the 2×2 matrix

$$
M = \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \tag{62}
$$

Example 3.3 (Paired 2×4 MMI)**.** Let $E_y\left(x', 0\right) = E_0\left(x'\right) * \left(A_1\delta\left(x' - \frac{1}{6}\right) + A_2\delta\left(x' - \frac{2}{6}\right)\right)$ and $E_y\left(x', z' = 1/12\right) = E_0\left(x'\right) * \left(B_1\delta\left(x' - \frac{1}{12}\right) + B_2\delta\left(x' - \frac{1}{6}\right) + B_3\delta\left(x' - \frac{1}{3}\right) + B_4\delta\left(x' - \frac{5}{12}\right)\right)$. Then using Eqs. (51) and (58), we may relate $E_y\left(x', z' = 1/12\right)$ to $E_y\left(x', z' = 0\right)$ via the matrix equation:

$$
\begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} = \frac{e^{-ikn_r\frac{L}{12}}}{2}M\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \tag{63}
$$

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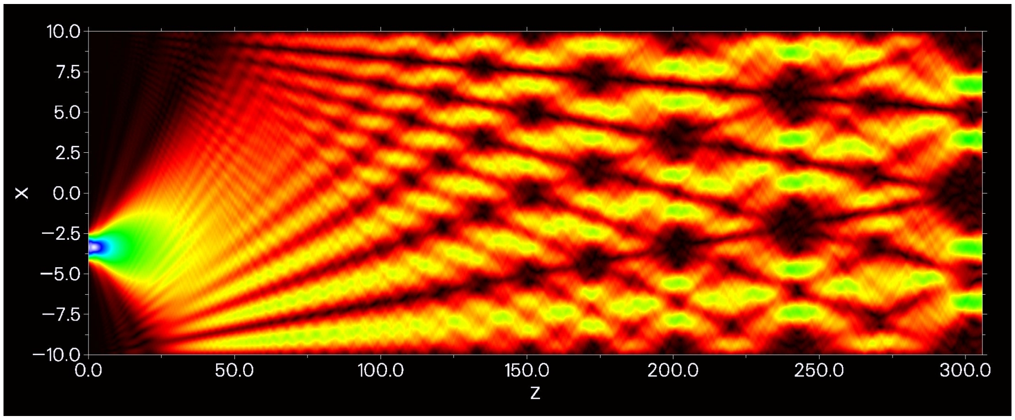

Fig. 14. The four outputs of one input for a paired 2×4 MMI.

with

$$M = \begin{pmatrix} e^{-\frac{i\pi}{3}} & e^{-\frac{i2\pi}{3}} \\ e^{-\frac{i3\pi}{4}} & e^{-\frac{i3\pi}{4}} \\ e^{-\frac{i3\pi}{4}} & e^{-\frac{i3\pi}{4}} \\ e^{-\frac{i2\pi}{3}} & e^{-\frac{i\pi}{3}} \end{pmatrix} \tag{64}$$

The output of the $x' = 1/6$ input is shown in Fig. 14.

4. $N \times N$ MMIs and a general MMI transfer function

$N \times N$ MMIs are an important class of MMIs, capable of both splitting and combining signals. By simply not using some input ports, an $N \times N$ MMI may be used as an $M \times N$ MMI for $M \le N$. Thus for any given positive integers M and N, it is always possible to construct an $M \times N$ MMI provided that $M \le N$. It is shown in appendix B that it is impossible to construct an $M \times N$ with $N \le M$. In this sense, $N \times N$ MMIs are very general.

The N inputs and N outputs are spaced out evenly along either end of the MMI. There are two ways in which this can be achieved, which we denote as regular and restricted. Both of these cases have been considered by Bachmann et al. [15, 16] for $L = 1/q$. We will extend both of these classes of MMIs to the case $L = 1/2q$ also, which we denote as the half integer MMIs. Thus for each N, we provide analytic formulae for phase amplitude relations between inputs and outputs for 4 distinct $N \times N$ MMIs. Although only a small extension, half integer MMIs may be useful as a more compact alternative to previously existing MMIs.

4.1. Regular $N \times N$ MMIs

For a given integer q, suppose that the inputs for the MMI are spaced evenly as in Fig. 15. Thus, the input field has the form:

$$E_y\left(x', 0\right) = \sum_{s=1}^{q} E_0\left(x', 0\right) * \left(A_s(0)\delta\left(x' + \frac{1}{4q} - \frac{s}{2q}\right)\right) \tag{65}$$

with $E_0\left(x'\right)$ even and having a width less than or equal to $1/2q$. The complex numbers $A_s(0)$ represent the complex phase-amplitude of each input. We claim that

$$E_y\left(x', z' = \frac{p}{q}\right) = \sum_{t=1}^{q} E_0\left(x', 0\right) * \left(A_t(p)\delta\left(x' + \frac{1}{4q} - \frac{t}{2q}\right)\right) \tag{66}$$

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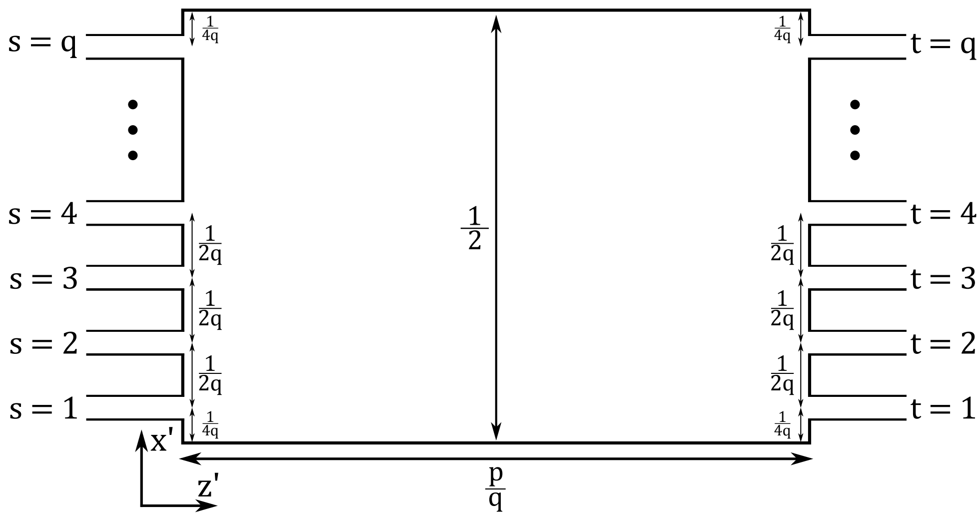

Fig. 15. Schematic of a general $N \times N$ MMI.

where p is some half-integer, as in Fig. 15.

Let $\overrightarrow{A(p)} = \left(A_1(p)\, A_2(p) \cdots A_q(p)\right)$. The complex vector $\overrightarrow{A(p)}$ may be thought of as a discrete representation of $E_y\left(x', \frac{p}{q}\right)$. We claim that $\overrightarrow{A(p)}$ is linearly related to $\overrightarrow{A(0)}$. To this end, let

$$\overrightarrow{A(p)} = e^{-ikn_r\frac{pL}{q}} \mathcal{M}(p,q)\, \overrightarrow{A(0)} \tag{67}$$

where $\mathcal{M}(p,q)$ is a $q \times q$ complex valued matrix. We call $\mathcal{M}(p,q)$ the *regular MMI matrices*, which may be thought of as a discrete representation of $\vartheta(x', z')$. We may thus derive many useful properties of $\mathcal{M}(p,q)$ from $\vartheta(x', z')$ without prior knowledge of the coefficients of $\mathcal{M}(p,q)$.

Table 1. Properties of the MMI matrices derived from $\vartheta(x', z')$.

	Properties	**Justification**
1)	$\mathcal{M}(0,q) = I$	$\vartheta(x', z') = \delta(x')$
2)	$\mathcal{M}(p_1,q)\,\mathcal{M}(p_2,q) = \mathcal{M}(p_1+p_2,q)$	$\vartheta\left(x', z_1'\right) * \vartheta\left(x', z_2'\right) = \vartheta\left(x', z_1' + z_2'\right)$
3)	$\mathcal{M}(-p,q) = \overline{\mathcal{M}(p,q)}$	$\vartheta(x' - z') = \overline{\vartheta(x', z')}$
4)	$\mathcal{M}(p,q)^{-1} = \overline{\mathcal{M}(p,q)}$	Follows from 1), 2) and 3)
5)	$\mathcal{M}(p,q)^{\dagger}\,\mathcal{M}(p,q) = I$	Energy conservation
6)	$\mathcal{M}(p,q)^{T} = \mathcal{M}(p,q)$	Follows from 4) and 5)
7)	$\mathcal{M}(p,q)_{q-s,q-t} = \mathcal{M}(p,q)_{s,t}$	Symmetry about $x' = 1/4$

Our aim is to show that Eqs. (66) and (67) hold and compute the coefficients of $\mathcal{M}(1,q)$ and

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$\mathcal{M}(1/2, q)$. By Eq. (15),

$$F\left(x', \frac{1}{q}\right) = \frac{e^{-ikn_r \frac{L}{q}}}{\sqrt{q}} E_0\left(x'\right) * \left(\sum_{s=1}^{q} \sum_{\substack{t=1 \\ t\equiv q\ (\mathrm{mod}\ 2)}}^{2q} e^{\frac{i\pi}{4}\left(1-\frac{t^2}{q}\right)} A_s\left(0\right)\left(\delta\left(x' + \frac{1}{4q} - \frac{s}{2q} - \frac{t}{2q}\right) - \delta\left(x' - \frac{1}{4q} + \frac{s}{2q} - \frac{t}{2q}\right)\right)\right) \tag{68}$$

In order to massage Eq. (68) into the form of (66), it is necessary to shift the summation index t for both the real and fictitious sources. For the real source, we shift t by $t \to t - s$. Then $\delta\left(x' + \frac{1}{4q} - \frac{s}{2q} - \frac{t-s}{2q}\right) = \delta\left(x' + \frac{1}{4q} - \frac{t}{2q}\right)$. For the fictitious source, we shift t by $t \to s + t - 1$. Then $\delta\left(x' - \frac{1}{4q} + \frac{s}{2q} - \frac{s+t-1}{2q}\right) = \delta\left(x' + \frac{1}{4q} - \frac{t}{2q}\right)$. Thus, being careful with the parity constraints and recalling that $s \equiv -s\ (\mathrm{mod}\ 2)$, Eq. (68) becomes

$$\begin{aligned} F\left(x', \frac{1}{q}\right) &= \frac{e^{-ikn_r \frac{L}{q}}}{\sqrt{q}} E_0\left(x'\right) * \left(\sum_{s=1}^{q} \sum_{\substack{t=1 \\ t-s\equiv q\ (\mathrm{mod}\ 2)}}^{2q} e^{\frac{i\pi}{4}\left(1-\frac{(t-s)^2}{q}\right)} A_s\left(0\right)\delta\left(x' + \frac{1}{4q} - \frac{t}{2q}\right) - \right. \\ &\qquad \left. \sum_{\substack{t=1 \\ t-s\equiv q+1\ (\mathrm{mod}\ 2)}}^{2q} e^{\frac{i\pi}{4}\left(1-\frac{(t+s-1)^2}{q}\right)} A_s\left(0\right)\delta\left(x' + \frac{1}{4q} - \frac{t}{2q}\right)\right) \\ &= e^{-ikn_r \frac{L}{q}} E_0\left(x'\right) * \left(\sum_{s=1}^{q} \sum_{t=1}^{2q} \alpha\left(s,t\right) A_s\left(0\right)\delta\left(x' + \frac{1}{4q} - \frac{t}{2q}\right)\right) \end{aligned} \tag{69}$$

with

$$\alpha(s,t) = \begin{cases} \frac{1}{\sqrt{q}} e^{\frac{i\pi}{4}\left(1-\frac{(t-s)^2}{q}\right)} & t - s \equiv q\ (\mathrm{mod}\ 2) \\ \frac{-1}{\sqrt{q}} e^{\frac{i\pi}{4}\left(1-\frac{(t+s-1)^2}{q}\right)} & t - s \equiv q + 1\ (\mathrm{mod}\ 2) \end{cases} \tag{70}$$

Notice that the images due to the real and fictitious sources are completely disjoint. To simplify $\alpha\left(s,t\right)$, we introduce the reflection parameter σ:

$$\sigma\left(t - s, q\right) = \begin{cases} 1 & t - s \equiv q\ (\mathrm{mod}\ 2) \\ 0 & t - s \equiv q + 1\ (\mathrm{mod}\ 2) \end{cases} \tag{71}$$

$\alpha(s,t)$ now takes on a more compact definition:

$$\alpha(s,t) = \frac{(-1)^{\sigma}}{\sqrt{q}} e^{\frac{i\pi}{4}\left(1-\frac{(t-(-1)^{\sigma}s-\sigma)^2}{q}\right)} \tag{72}$$

Thus for the real field:

$$\begin{aligned} E_y\left(x', 0\right) &= e^{-ikn_r \frac{L}{q}} E_0\left(x'\right) * \left(\sum_{s=1}^{q} \sum_{t=1}^{q} \alpha\left(s,t\right) A_s\left(0\right)\delta\left(x' + \frac{1}{4q} - \frac{t}{2q}\right)\right) \\ &= E_0\left(x'\right) * \left(\sum_{t=1}^{q} A_t\left(1\right)\delta\left(x' + \frac{1}{4q} - \frac{t}{2q}\right)\right) \end{aligned} \tag{73}$$

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As $\overrightarrow{A(1)} = e^{-ikn_r \frac{L}{q}} \mathcal{M}(1,q)\overrightarrow{A(0)}$, we have found the coefficients for MMI matrix $\mathcal{M}(1,q)$.

$$\mathcal{M}(1,q)_{s,t} = \alpha(s,t) = \frac{(-1)^{\sigma}}{\sqrt{q}} e^{\frac{i\pi}{4}\left(1-\frac{(t-(-1)^{\sigma}-\sigma)^2}{q}\right)} \tag{74}$$

For the case of $E_0(x')$ odd, we define the odd MMI matrix $\widetilde{\mathcal{M}}$ with coefficients

$$\widetilde{\mathcal{M}}(1,q)_{s,t} = \alpha(s,t) = \frac{1}{\sqrt{q}} e^{\frac{i\pi}{4}\left(1-\frac{(t-(-1)^{\sigma}-\sigma)^2}{q}\right)} \tag{75}$$

so that $\overrightarrow{A(1)} = e^{-ikn_r \frac{L}{q}} \widetilde{\mathcal{M}}(1,q)\overrightarrow{A(0)}$. The $(-1)^{\sigma}$ term has dropped due to $E_0(x') = -E_0(x')$. For future reference, tildes will be used to indicate the the relevant MMI matrix is for $E_0(x')$ odd.

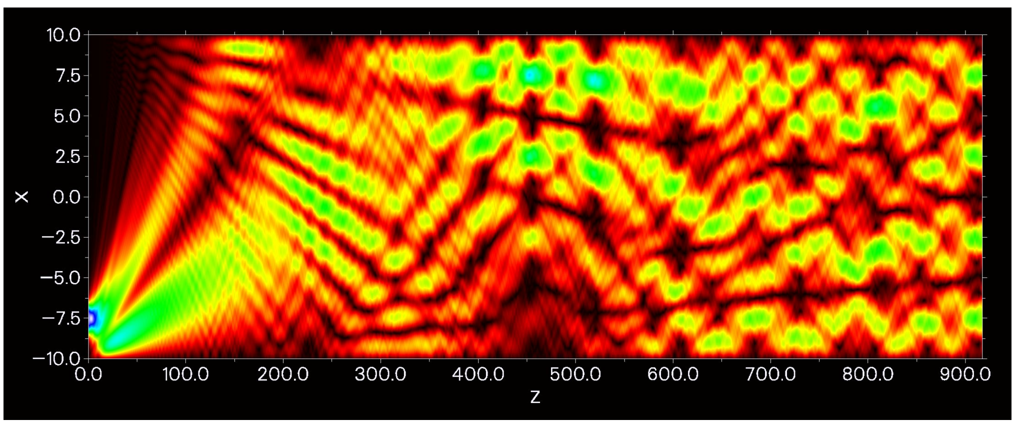

Fig. 16. $s = 1$ output for a regular 4×4 MMI.

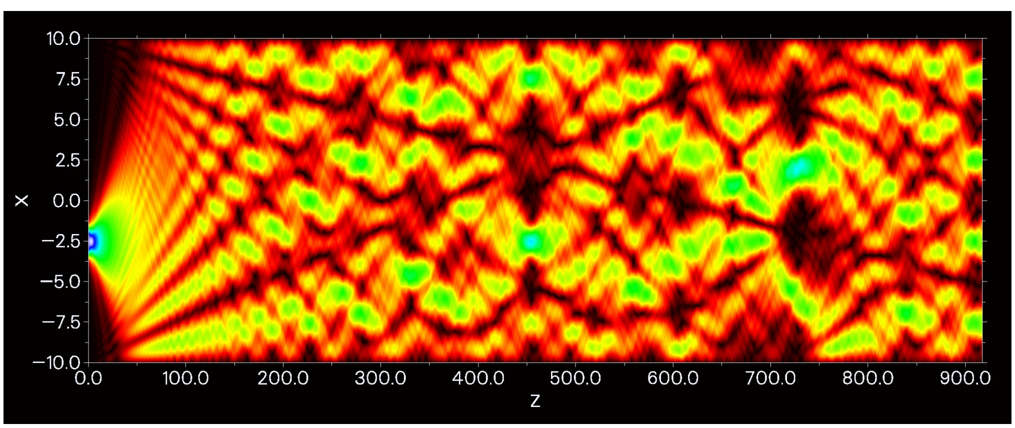

Fig. 17. $s = 2$ output for a regular 4×4 MMI.

Example 4.1 (Regular 4×4 MMI)**.** We will examine the case of $p = 1$, $q = 4$ to construct a

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4×4 MMI as in Figs. 16 and 17. The length of this MMI is $L/4$, and the MMI matrix is

$$\mathcal{M}(1,4) = \frac{1}{2}\begin{pmatrix} e^{\frac{i\pi}{4}} & -1 & 1 & e^{\frac{i\pi}{4}} \\ -1 & e^{\frac{i\pi}{4}} & e^{\frac{i\pi}{4}} & 1 \\ 1 & e^{\frac{i\pi}{4}} & e^{\frac{i\pi}{4}} & -1 \\ e^{\frac{i\pi}{4}} & 1 & -1 & e^{\frac{i\pi}{4}} \end{pmatrix} \tag{76}$$

This example illustrates one of the nicer properties of $\mathcal{M}(1,q)$; $|\mathcal{M}(1,q)_{s,t}| = 1/\sqrt{q}$ and is thus independent of s or t.

The s-th column of $\mathcal{M}(p,q)$ yields the vector of outputs $\overrightarrow{A(p)}$ when a unit amplitude, zero phase signal is sent into input s. By property 4), if the conjugate of the output vector is then used as the input for the MMI, there will be a single image at the s-th output. As an example, sending an input in with $\overrightarrow{A(0)} = \frac{1}{2}\left(-1\ e^{\frac{-i\pi}{4}}\ e^{\frac{-i3\pi}{4}}\ 1\right)$ will yield a single image at the $t = 2$ output with phase amplitude $e^{-ikn_r\frac{L}{q}}$. Thus the MMI matrix $\mathcal{M}(1,4)$ displays almost all relevant information for the 4×4 MMI very conveniently, as do all other MMI matrices.

4.2. Half-integer regular $N \times N$ MMIs

In appendix C it is shown that

$$\begin{aligned} E_y\left(x, \frac{1}{2q}\right) e^{-ikn_r\frac{L}{2q}} E_0(x') * \left(\sum_{s=1}^{q}\sum_{t=1}^{q} \mathcal{M}\left(\frac{1}{2},q\right) A_s(0)\,\delta\left(x' + \frac{1}{4q} - \frac{t}{2q}\right)\right) \\ = E_0(x') * \left(\sum_{t=1}^{q} A_t\left(\frac{1}{2}\right)\delta\left(x' + \frac{1}{4q} - \frac{t}{2q}\right)\right) \end{aligned} \tag{77}$$

where

$$\mathcal{M}\left(\frac{1}{2},q\right)_{s,t} = \sqrt{\frac{2}{q}}\sin\left(\frac{\pi}{4q}(2t-1)(2s-1)\right) e^{\frac{i\pi}{4}\left(3-\frac{2\left(t^2+s^2-t-s\right)-1}{q}\right)} \tag{78}$$

while

$$\widetilde{\mathcal{M}}\left(\frac{1}{2},q\right)_{s,t} = \sqrt{\frac{2}{q}}\cos\left(\frac{\pi}{4q}(2t-1)(2s-1)\right) e^{\frac{i\pi}{4}\left(1-\frac{2\left(t^2+s^2-t-s\right)-1}{q}\right)} \tag{79}$$

Furthermore it is also shown that $\mathcal{M}\left(\frac{1}{2},q\right)$ will never have a zero entry. For $s = 1$, $|\mathcal{M}\left(\frac{1}{2},q\right)_{s,t}| = |\sqrt{\frac{2}{q}}\sin\left(\frac{\pi}{4q}(2t-1)(2s-1)\right)| = \sqrt{\frac{2}{q}}|\sin\left(\frac{\pi}{4q}(2t-1)\right)|$. But $\frac{\pi}{4q}(2t-1)$ varies monotonically from $\frac{\pi}{4q}$ to $\frac{\pi}{4q}(2(q-1)-1) = \frac{\pi}{4q}(2q-3) < \frac{2q\pi}{4q} = \frac{\pi}{2}$. Thus, $|\mathcal{M}\left(\frac{1}{2},q\right)_{1,t}|$ increases monotonically from $t = 1$ to $t = q-1$. This holds true for all q, but can be seen graphically in Fig. 16 for $q = 4$.

Example 4.2 (Semi-regular 4×4 MMI)**.** We consider the case $(p,q) = (1/2,4)$. This is a continuation of example 4.1 where we stop the waveguide at $z' = 1/8$ instead of $z' = 1/4$, as displayed in Figs. 16 and 17. The MMI matrix in this case is

$$\mathcal{M}\left(\frac{1}{2},4\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{\frac{i\pi 13}{16}}\sin\left(\frac{\pi}{16}\right) & e^{\frac{i\pi 9}{16}}\sin\left(\frac{3\pi}{16}\right) & e^{\frac{i\pi}{16}}\sin\left(\frac{5\pi}{16}\right) & e^{\frac{-i\pi 11}{16}}\sin\left(\frac{7\pi}{16}\right) \\ e^{\frac{i\pi 9}{16}}\sin\left(\frac{3\pi}{16}\right) & e^{\frac{i5\pi}{16}}\sin\left(\frac{7\pi}{16}\right) & e^{\frac{-3i\pi}{16}}\sin\left(\frac{\pi}{16}\right) & e^{\frac{i\pi}{16}}\sin\left(\frac{5\pi}{16}\right) \\ e^{\frac{i\pi}{16}}\sin\left(\frac{5\pi}{16}\right) & e^{\frac{-3i\pi}{16}}\sin\left(\frac{\pi}{16}\right) & e^{\frac{i5\pi}{16}}\sin\left(\frac{7\pi}{16}\right) & e^{\frac{i9\pi}{16}}\sin\left(\frac{3\pi}{16}\right) \\ e^{\frac{-11i\pi}{16}}\sin\left(\frac{7\pi}{16}\right) & e^{\frac{i\pi}{16}}\sin\left(\frac{5\pi}{16}\right) & e^{\frac{i9\pi}{16}}\sin\left(\frac{3\pi}{16}\right) & e^{\frac{i13\pi}{16}}\sin\left(\frac{\pi}{16}\right) \end{pmatrix}$$
$$\approx \begin{pmatrix} 0.14e^{\frac{i\pi 13}{16}} & 0.4e^{\frac{i\pi 9}{16}} & 0.6e^{\frac{i\pi}{16}} & 0.7e^{\frac{-i\pi 11}{16}} \\ 0.4e^{\frac{i\pi 9}{16}} & 0.7e^{\frac{i5\pi}{16}} & 0.14e^{\frac{-3i\pi}{16}} & 0.6e^{\frac{i\pi}{16}} \\ 0.6e^{\frac{i\pi}{16}} & 0.14e^{\frac{-3i\pi}{16}} & 0.7e^{\frac{i5\pi}{16}} & 0.4e^{\frac{i9\pi}{16}} \\ 0.7e^{\frac{i3\pi}{16}} & 0.6e^{\frac{i\pi}{16}} & 0.4e^{\frac{i9\pi}{16}} & 0.14e^{\frac{i13\pi}{16}} \end{pmatrix} \tag{80}$$

4.3. Restricted $N \times N$ MMIs

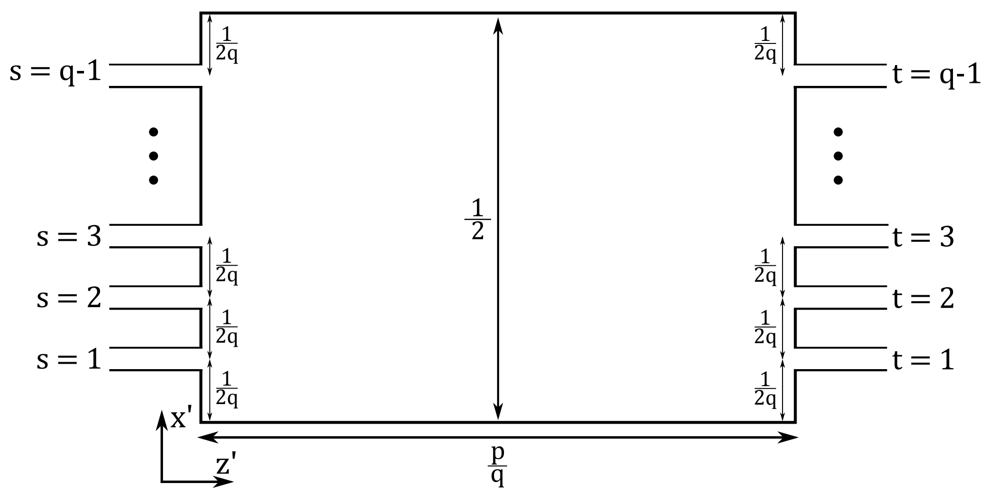

Fig. 18. Diagram for a restricted MMI.

Next we will consider the *restricted* $N \times N$ MMIs as in Fig. 18. These are so called because they contain symmetric ($p = 1/2, q = 2$) and paired ($p = 1/2, q = 3$) interference as special cases. In the same notation as the previous section, suppose that for a given integer q:

$$E_y\left(x', 0\right) = E_0\left(x'\right) * \left(\sum_{s=1}^{q-1} A_s\left(0\right)\delta\left(x' - \frac{s}{2q}\right)\right) \tag{81}$$

with $E_0\left(x'\right)$ and $E_0\left(x'\right)$ having a width less than or equal to $1/2q$. It is shown in appendix C that

$$E_y\left(x', z' = \frac{p}{q}\right) = E_0\left(x'\right) * \left(\sum_{t=1}^{q-1} A_t\left(p\right)\delta\left(x' - \frac{t}{2q}\right)\right) \tag{82}$$

where p is some half-integer, and

$$\overrightarrow{A(p)} = e^{-ikn_r\frac{pL}{q}}\mathcal{N}\left(p, q\right)\overrightarrow{A(0)} \tag{83}$$

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where $\mathcal{N}(p,q)$ is a $q-1\times q-1$ complex matrix. $\mathcal{N}(p,q)$ are denoted as the *restricted MMI matrices*. These matrices obey all the same properties as $\mathcal{M}(p,q)$ in table 1.

The following expressions for the coefficients of $\mathcal{N}(p,q)$ have been derived in appendix C.

$$\mathcal{N}(1,q)_{s,t} = (1-\sigma(s,t))\frac{2}{\sqrt{q}}\sin\left(\frac{\pi t s}{2q}\right)e^{\frac{i\pi}{4}\left(3-\frac{t^2+s^2}{q}\right)} \tag{84}$$

$$\widetilde{\mathcal{N}}(1,q)_{s,t} = (1-\sigma(s,t))\frac{2}{\sqrt{q}}\cos\left(\frac{\pi t s}{2q}\right)e^{\frac{i\pi}{4}\left(1-\frac{t^2+s^2}{q}\right)} \tag{85}$$

4.4. Half-integer restricted $N\times N$ MMIs

As for the regular MMIs, there also exists half integer restricted MMIs with matrices $\mathcal{N}(1/2,q)$. The following expressions for the coefficients of $\mathcal{N}(p,q)$ have been derived in appendix C.

$$\mathcal{N}\left(\frac{1}{2},q\right)_{s,t} = \sqrt{\frac{2}{q}}\sin\left(\frac{\pi t s}{q}\right)e^{\frac{i\pi}{4}\left(3-\frac{2\left(t^2+s^2\right)}{q}\right)} \tag{86}$$

$$\widetilde{\mathcal{N}}\left(\frac{1}{2},q\right)_{s,t} = \sqrt{\frac{2}{q}}\cos\left(\frac{\pi t s}{q}\right)e^{\frac{i\pi}{4}\left(1-\frac{2\left(t^2+s^2\right)}{q}\right)} \tag{87}$$

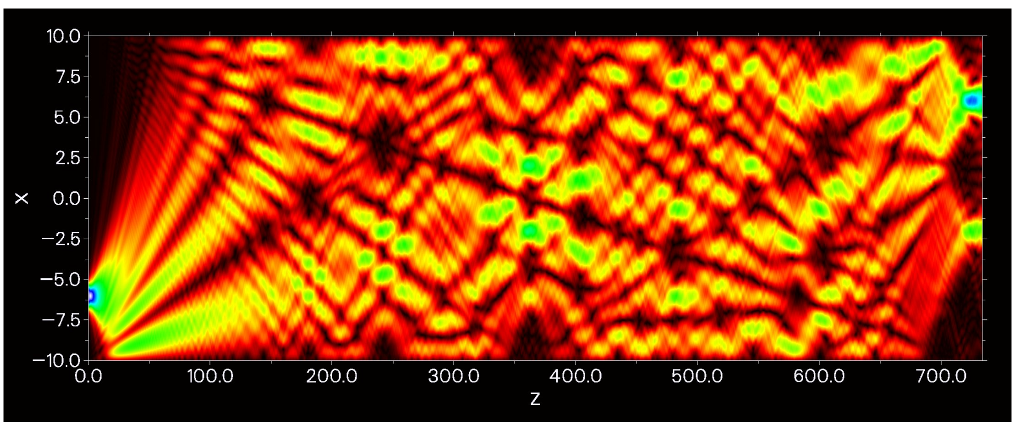

Fig. 19. $s=1$ output for a restricted 4×4 MMI.

Example 4.3 (Restricted 4×4 MMI)**.** We will briefly examine $\mathcal{N}(1,5)$ and $\mathcal{N}(1/2,5)$ as in Figs. 19 and 20.

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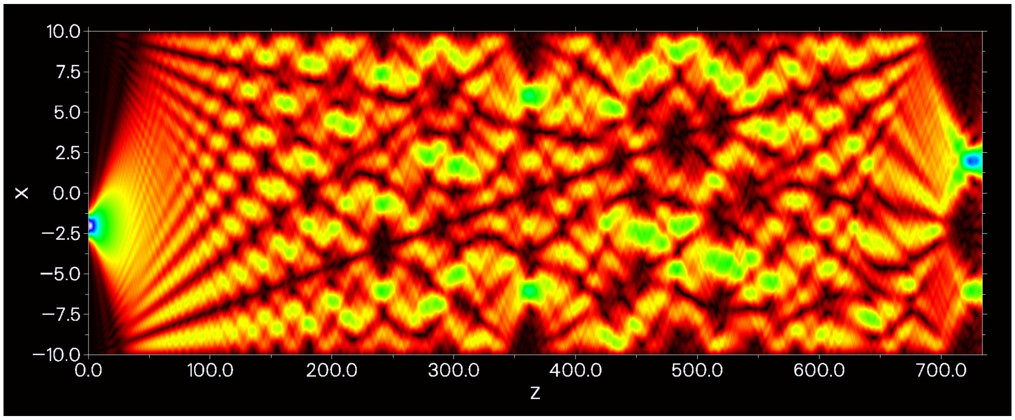

Fig. 20. $s = 2$ output for a restricted 4×4 MMI.

$$\mathcal{N}(1,5) = \frac{2}{\sqrt{5}} \begin{pmatrix} 0 & e^{\frac{i\pi}{2}} \sin\left(\frac{\pi}{5}\right) & 0 & e^{\frac{-i\pi}{10}} \sin\left(\frac{2\pi}{5}\right) \\ e^{\frac{i\pi}{2}} \sin\left(\frac{\pi}{5}\right) & 0 & e^{\frac{i\pi}{10}} \sin\left(\frac{2\pi}{5}\right) & 0 \\ 0 & e^{\frac{i\pi}{10}} \sin\left(\frac{2\pi}{5}\right) & 0 & e^{\frac{i\pi}{2}} \sin\left(\frac{\pi}{5}\right) \\ e^{\frac{-i\pi}{10}} \sin\left(\frac{2\pi}{5}\right) & 0 & e^{\frac{i\pi}{2}} \sin\left(\frac{\pi}{5}\right) & 0 \end{pmatrix} \approx \begin{pmatrix} 0 & 0.52e^{\frac{i\pi}{2}} & 0 & 0.85e^{\frac{-i\pi}{10}} \\ 0.52e^{\frac{i\pi}{2}} & 0 & 0.85e^{\frac{i\pi}{10}} & 0 \\ 0 & 0.85e^{\frac{i\pi}{10}} & 0 & 0.52e^{\frac{i\pi}{2}} \\ 0.85e^{\frac{-i\pi}{10}} & 0 & 0.52e^{\frac{i\pi}{2}} & 0 \end{pmatrix} \tag{88}$$

$$\mathcal{N}\left(\frac{1}{2},5\right) = \frac{2}{\sqrt{10}} \begin{pmatrix} \sin\left(\frac{\pi}{5}\right) e^{\frac{i11\pi}{20}} & \sin\left(\frac{2\pi}{5}\right) e^{\frac{i\pi}{4}} & \sin\left(\frac{2\pi}{5}\right) e^{-\frac{i\pi}{4}} & \sin\left(\frac{\pi}{5}\right) e^{-\frac{i19\pi}{20}} \\ \sin\left(\frac{2\pi}{5}\right) e^{\frac{i\pi}{4}} & \sin\left(\frac{\pi}{5}\right) e^{-\frac{i\pi}{20}} & \sin\left(\frac{\pi}{5}\right) e^{\frac{i9\pi}{20}} & \sin\left(\frac{2\pi}{5}\right) e^{-\frac{i\pi}{4}} \\ \sin\left(\frac{2\pi}{5}\right) e^{-\frac{i\pi}{4}} & \sin\left(\frac{\pi}{5}\right) e^{\frac{i9\pi}{20}} & \sin\left(\frac{\pi}{5}\right) e^{-\frac{i\pi}{20}} & \sin\left(\frac{2\pi}{5}\right) e^{\frac{i\pi}{4}} \\ \sin\left(\frac{\pi}{5}\right) e^{\frac{-i19\pi}{20}} & \sin\left(\frac{2\pi}{5}\right) e^{-\frac{i\pi}{4}} & \sin\left(\frac{2\pi}{5}\right) e^{\frac{i\pi}{4}} & \sin\left(\frac{\pi}{5}\right) e^{\frac{i11\pi}{20}} \end{pmatrix} \approx \begin{pmatrix} 0.37e^{\frac{i11\pi}{20}} & 0.60e^{\frac{i\pi}{4}} & 0.60e^{-\frac{i\pi}{4}} & 0.37e^{-\frac{i19\pi}{20}} \\ 0.60e^{\frac{i\pi}{4}} & 0.37e^{-\frac{i\pi}{20}} & 0.37e^{\frac{i9\pi}{20}} & 0.60e^{-\frac{i\pi}{4}} \\ 0.60e^{-\frac{i\pi}{4}} & 0.37e^{\frac{i9\pi}{20}} & 0.37e^{-\frac{i\pi}{20}} & 0.60e^{\frac{i\pi}{4}} \\ 0.37e^{\frac{-i19\pi}{20}} & 0.60e^{-\frac{i\pi}{4}} & 0.60e^{\frac{i\pi}{4}} & 0.37e^{\frac{i11\pi}{20}} \end{pmatrix} \tag{89}$$

The restricted MMI matrices can be used to quickly find the images of a given input. If an input is located at $x' = 1/2(p/q) = 1/2(pn/qn)$, then the np-th column of $\mathcal{N}(1, nq)$ will instantly yield the images located at $z' = 1/qn$. To obtain the images at $z' = m/qn$, powers of $\mathcal{N}(1, nq)$ may be taken. Indeed, the results of section 3 may be derived by calculating the vectors $\mathcal{N}(1, 4q)_{2q,t}$ for symmetric interference and $\mathcal{N}(1, 3q)_{q,t}$ or $\mathcal{N}(1, 3q)_{2q,t}$ for paired interference. This motivates our decision to refer to $\mathcal{N}(p, q)$ as the restricted MMI matrices. Indeed, the inputs of the MMI corresponding to $\mathcal{N}(p, q)$ will not excite any mode with a mode number divisible by q. This manifests itself in the following way. As $|\mathcal{N}(1/2, q)_{s,t}| \sim \sin$ âĄą $\left(\frac{\text{ÏĂ}ts}{q}\right)$, $\mathcal{N}(1/2, q)_{s,t}$ will vanish if and only if $q|ts$. Thus depending on how many divisors q has, both $\mathcal{N}(1/2, q)$ and $N(1, q)$ may have many zero entries. What's more, for a given pair of integers (p, q) the inputs and outputs of the restricted and regular (p, q) MMIs are completely disjoint. Thus, the two may be used simultaneously in a single MMI to create a $2q - 1 \times 2q - 1$ MMI of length $z' = 1/q$.

However the restricted MMIs generalise the regular MMIs somewhat, as the restricted MMIs allow us to compute the output of any input with a rational location.

4.5. $\mathcal{M}$ as an MMI transfer function

We will now examine how an $N \times N$ MMI may approximate an MMI with arbitrary input field $E_y(x, 0)$ if N is large enough.

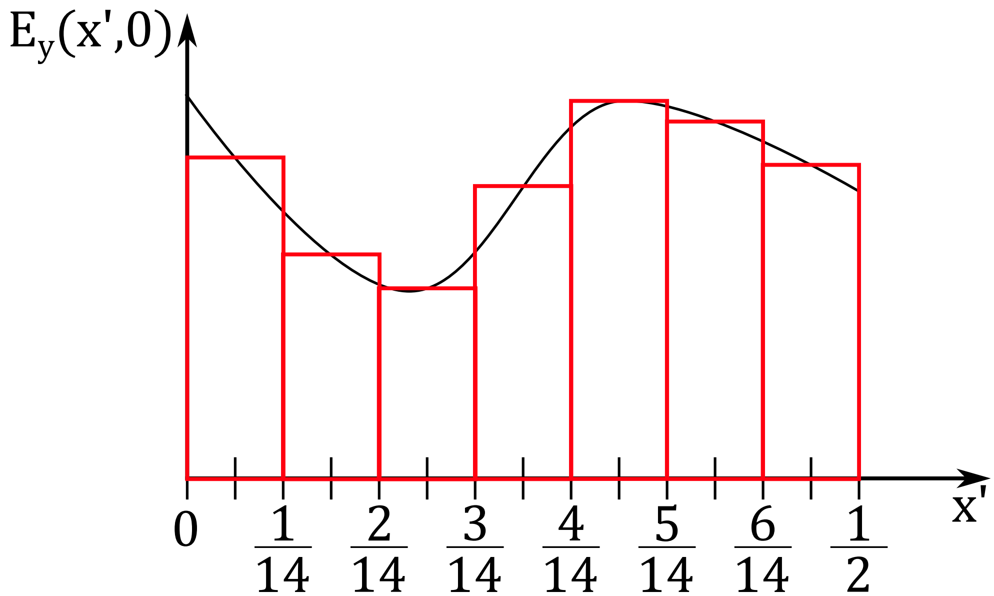

Fig. 21. The rectangular approximation for $q = 7$.

Define the *rectangular function* $\Pi(x')$ by

$$\Pi(x') = \begin{cases} 1 & |x'| < \frac{1}{2} \\ \frac{1}{2} & |x'| = \frac{1}{2} \\ 0 & |x'| > \frac{1}{2} \end{cases} \tag{90}$$

Let $q \gg 1$ be an integer. Then for large q, we may make the *rectangular approximation*

$$\begin{aligned} E_y(x', z') &\approx \sum_{s=1}^{q} E_y\left(\frac{s}{2q} - \frac{1}{4q}, z'\right) \Pi\left(2q\left(x' + \frac{1}{4q} - \frac{s}{2q}\right)\right) \\ &= \sum_{s=1}^{q} \Pi(2qx') * \left(E_y\left(\frac{s}{2q} - \frac{1}{4q}, z'\right) \delta\left(x' + \frac{1}{4q} - \frac{s}{2q}\right)\right) \end{aligned} \tag{91}$$

The rectangular approximation is illustrated for $q = 7$ in Fig. 21 where the red rectangles approximate the field $E_y(x', 0)$. For $z' = 0$, Eq. (91) is identical to Eq. (18) with $E_0(x')$ and $A_s(0)$ being replaced by $\Pi(2q'x')$ and $E_y(s/2q - 1/4q, 0)$ respectively. Thus, let $\overrightarrow{E_y(z')}$ be the discrete representation of $E_y(x', z')$ with $\overrightarrow{E_y(z')}_s = E_y(s/2q - 1/4q, z')$. As $\Pi(2qx')$ is an even function, by section 4 it follows that

$$\overrightarrow{E_y\left(z' + \frac{p}{q}\right)} = e^{-ikn_r\frac{pL}{q}} \mathcal{M}(p, q)\overrightarrow{E_y(0)} = \left(e^{-ikn_r\frac{L}{q}} \mathcal{M}(1, q)\right)^p \overrightarrow{E_y(0)} \tag{92}$$

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Equation (92) provides a novel numerical technique for evaluating the propagation of an electric field through an MMI, of which an example is provided in Fig. 22.

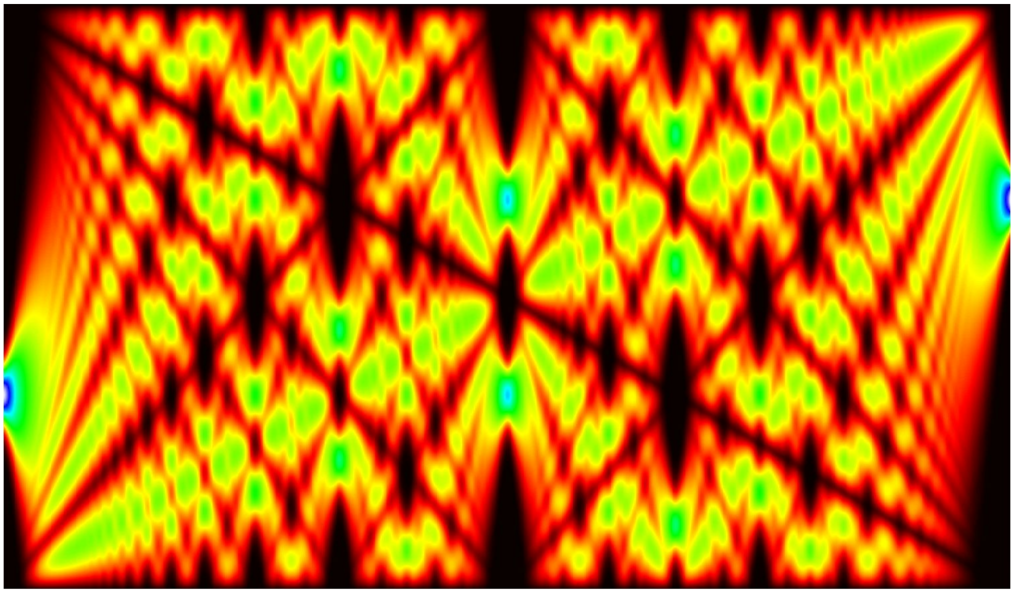

Fig. 22. A simulation of lightwave propagation in a waveguide using MMI matrices.

This application of the MMI matrices indicates a very strong connection to the discrete Fresnel transform [23–25]. This should not be surprising, as it was already observed that $F(x,z)$ propagates according to the Fresnel approximation in Eq. (7). Indeed, the MMI matrices may be thought of as a discrete Fresnel transform that accounts for the fictitious images induced by total internal reflection.

Appendices

A. An analytic expression for $\vartheta\left(x', \frac{1}{q}\right)$ using generalised quadratic Gauss sums

We wish to evaluate $\vartheta(x', z')$ for $z' = p/q$:

$$\vartheta\left(x', \frac{p}{q}\right) = \sum_{m=-\infty}^{\infty} e^{i\pi m^2 \frac{p}{q} + 2\pi i m x'} = \sum_{m=-\infty}^{\infty} e^{i2\pi m^2 \frac{p}{2q} + 2\pi i m x'} \tag{93}$$

The periodicity of the complex exponentials will be used to simplify this expression. Recall that $e^{ix} = e^{i2\pi m x} e^{ix} = e^{i(2\pi m + x)}$. As m varies, $e^{i2\pi m^2 \frac{p}{2q}}$ will vary periodically. In particular $e^{i2\pi(m+2q)^2 \frac{p}{2q}} = e^{i2\pi m^2 \frac{p}{2q}} e^{4\pi i p(m+q)} = e^{i2\pi m^2 \frac{p}{2q}}$.

Thus, $e^{i2\pi m^2 \frac{p}{2q}}$ may be completely determined by the values $m = 0, 1, 2 \ldots, 2q-1$. Let $m = n(2q) + r$, where n is some integer and $0 \leq r \leq 2q-1$. This is equivalent to dividing m by $2q$ with a remainder. Instead of summing over m in $\vartheta\left(x', \frac{p}{q}\right)$, it is equally valid to sum over n

and r. Then,

$$\vartheta\left(x', \frac{p}{q}\right) = \sum_{m=-\infty}^{\infty} e^{i\pi m^2 \frac{p}{q} + 2\pi i m x'} = \sum_{r=0}^{2q-1} \sum_{n=-\infty}^{\infty} e^{i2\pi(n2q+r)^2 \frac{p}{2q} + 2\pi i(n2q+r)x'}$$
$$= \sum_{r=0}^{2q-1} \sum_{n=-\infty}^{\infty} e^{i2\pi(n^2 4q^2 + 4nqr + r^2)\frac{p}{2q} + 4\pi i n q x' + 2\pi i r x'} = \sum_{r=0}^{2q-1} \sum_{n=-\infty}^{\infty} e^{2\pi i r^2 \frac{p}{2q}} e^{4\pi i n q x'} e^{2\pi i r x'}$$
$$= \sum_{r=0}^{2q-1} e^{2\pi i r^2 \frac{p}{2q}} e^{2\pi i r x'} \sum_{n=-\infty}^{\infty} e^{2\pi i n(2qx')} \quad (94)$$

The sum over n follows from standard Fourier theory:

$$\sum_{n=-\infty}^{\infty} e^{2\pi i n(2qx')} = \sum_{n=-\infty}^{\infty} \delta\left(2qx' - n\right) = \frac{1}{2q} \sum_{n=-\infty}^{\infty} \delta\left(x' - \frac{n}{2q}\right) \quad (95)$$

The remainder of n upon division by $2q$ dictates where the Dirac delta $\delta\left(x' - \frac{n}{2q}\right)$ is located in the unit interval $[0, 1]$. To find the coefficient of this Dirac delta for a given n, we evaluate:

$$\frac{1}{2q} \sum_{r=0}^{2q-1} e^{2\pi i r^2 \frac{p}{2q}} e^{2\pi i r x'} \quad (96)$$

at $x' = n/2q$. To simplify the sum, we perform division on n with respect to $2q$ to get $n = k(2q)+t$ with $0 \leq k < 2q$. Summing over n is equivalent to summing over k and t:

$$\vartheta\left(x', \frac{p}{q}\right) = \frac{1}{2q} \sum_{r=0}^{2q-1} e^{2\pi i r^2 \frac{p}{2q}} e^{2\pi i r x'} \sum_{t=0}^{2q-1} \sum_{k=-\infty}^{\infty} \delta\left(x' - \frac{(2q)k + t}{2q}\right)$$
$$= \frac{1}{2q} \sum_{r=0}^{2q-1} \sum_{t=0}^{2q-1} e^{2\pi i r^2 \frac{p}{2q}} e^{2\pi i r \frac{t}{2q}} \sum_{k=-\infty}^{\infty} \delta\left(x' - \frac{t}{2q} - k\right)$$
$$= \frac{1}{2q} \sum_{r=0}^{2q-1} \sum_{t=0}^{2q-1} e^{2\pi i r^2 \frac{p}{2q}} e^{2\pi i r \frac{t}{2q}} \delta\left(x' - \frac{t}{2q}\right) \quad (97)$$

Note that the Dirac comb $\sum \delta\left(x - n\right)$ is being identified with the periodic Dirac delta $\delta(x)$ through an abuse of notation. $\vartheta\left(x', p/q\right)$ is thus a sum of $2q$ evenly spaced Dirac deltas, with the phase amplitude of the image at $x' = t/2q$ given by the sum:

$$\frac{1}{2q} \sum_{r=0}^{2q-1} e^{2\pi i r^2 \frac{p}{2q}} e^{2\pi i r \frac{t}{2q}} \quad (98)$$

A generalised quadratic Gauss sum $c_n(a, b)$ [26] is a sum of the form:

$$\frac{1}{b} \sum_{r=0}^{b-1} e^{2\pi i\left(\frac{ar^2 + nr}{b}\right)} = c_n(a, b) \quad (99)$$

Clearly (98) is such a sum:

$$\vartheta\left(x', \frac{p}{q}\right) = \sum_{t=0}^{2q-1} c_t(p, 2q)\delta\left(x' - \frac{t}{2q}\right) \quad (100)$$

Thus in the extended setting, an input has $2q$ evenly spaced images with a phase profile governed by $c_t(p, 2q)$. The coefficients possess a wide range of interesting properties which will not be explored here. In general, there is no closed form expression for $c_n(a, b)$, but there are algorithmic ways to compute the sum for given values of a, b and n. That said, there does exist a closed form expression for $c_t(p = 1, 2q)$ which we will now derive.

A generalisation of the quadratic Gauss sums obey the following reciprocity law [26]:

$$\sum_{r=0}^{|b|-1} e^{\pi i\left(\frac{ar^2+nr}{b}\right)} = \sqrt{\left|\frac{b}{a}\right|}e^{\pi i\left(\frac{|ab|-n^2}{4ab}\right)}\sum_{r=0}^{|a|-1} e^{-\pi i\left(\frac{br^2+nr}{a}\right)} \tag{101}$$

In the notation of (101), take $b = 2q$, $a = 2$ and $n = 2t$.Then by (101)

$$c_t(1, 2q) = \frac{1}{2q}\sqrt{q}e^{\pi i\left(\frac{q-t^2}{4q}\right)}\sum_{r=0}^{1} e^{-\pi i\left(\frac{2qr^2+2tr}{2}\right)} = \frac{1}{2\sqrt{q}}e^{\pi i\left(\frac{q-t^2}{4q}\right)}\left(1 + e^{-\pi i(q+t)}\right) \tag{102}$$

Unit amplitude complex numbers obey the following identity: $e^{i\alpha} + e^{i\beta} = 2\cos\left(\frac{\alpha-\beta}{2}\right)e^{i\frac{\alpha+\beta}{2}}$. Setting $\alpha = 0$ and $\beta = -\pi(q + t)$ yields $1 + e^{-\pi i(q+t)} = 2\cos\left(\frac{\pi(q+t)}{2}\right)e^{-i\frac{\pi}{2}(q+t)}$, which is zero unless t and q have the same parity, in which case $1 + e^{-\pi i(q+t)} = 2$. Thus, when t and q have the same parity:

$$c_t(1, 2q) = \frac{1}{\sqrt{q}}e^{\pi i\left(\frac{q-t^2}{4q}\right)} = \frac{1}{\sqrt{q}}e^{\frac{i\pi}{4}}e^{-i\frac{\pi t^2}{4q}} \tag{103}$$

Therefore an analytical expression is derived for $\vartheta\left(x', \frac{1}{q}\right)$

$$\vartheta\left(x', \frac{1}{q}\right) = \sum_{t=0}^{2q-1} c_t(1, 2q)\delta\left(x' - \frac{t}{2q}\right) = \frac{1}{\sqrt{q}}\sum_{\substack{t=0\\ t\equiv q \pmod 2}}^{2q-1} e^{\frac{i\pi}{4}\left(1-\frac{t^2}{q}\right)}\delta\left(x' - \frac{t}{2q}\right) \tag{104}$$

B. Impossibility of $N \times M$ MMIs with $M < N$.

We wish to prove that $N \times M$ MMIs with $M < N$ do not exist. We assume that there is an associated MMI matrix S and that there is no loss in the system. By the rank-nullity theorem from linear algebra, dim âĄą(Im(S)) + dim âĄą(ker âĄą(S)) $= N \Rightarrow$ dim âĄą(ker âĄą(S)) $=$ $N -$ dim âĄą(Im(S)) $> N - M \geq 1$. Thus, there must be some non-zero vector u contained in ker âĄą(S). But if the system is lossless, then $|u|^2 = |Su|^2 = |0|^2 = 0$, which is a contradiction. Thus, such an MMI cannot exist.

C. Derivation of the $N \times N$ matrix coefficients

C.1. Coefficients of $\mathcal{M}\left(\frac{1}{2}, q\right)$

We will proceed exactly as for the calculation of $\mathcal{M}(1, q)$. Analogously, Eq. (68) becomes

$$\begin{aligned} F\left(x', \frac{1}{2q}\right) &= \frac{e^{-ikn_r\frac{L}{2q}}}{\sqrt{2q}}E_0(x') \\ &* \left(\sum_{s=1}^{q}\sum_{\substack{t'=1\\ t'\equiv 2q \pmod 2}}^{4q} e^{\frac{i\pi}{4}\left(1-\frac{t'^2}{2q}\right)}A_s(0)\left(\delta\left(x' + \frac{1}{4q} - \frac{s}{2q} - \frac{t'}{4q}\right) - \delta\left(x' - \frac{1}{4q} + \frac{s}{2q} - \frac{t'}{4q}\right)\right)\right) \end{aligned} \tag{105}$$

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The condition $t' \equiv 2q \pmod 2$ is equivalent to enforcing t' to be even. Thus, let $t' = 2t$.

$$F\left(x', \frac{1}{2q}\right) = \frac{e^{-ikn_r \frac{L}{2q}}}{\sqrt{2q}} E_0(x') * \left(\sum_{s=1}^{q} \sum_{t=1}^{2q} e^{\frac{i\pi}{4}\left(1-\frac{2t^2}{q}\right)} A_s(0) \left(\delta\left(x' + \frac{1}{4q} - \frac{s}{2q} - \frac{t}{2q}\right) - \delta\left(x' - \frac{1}{4q} + \frac{s}{2q} - \frac{t}{2q}\right)\right)\right) \tag{106}$$

We perform the same shift in indices prior to Eq. (69) so as to align the Dirac deltas in the sum:

$$F\left(x', \frac{1}{2q}\right) = \frac{e^{-ikn_r \frac{L}{2q}}}{\sqrt{2q}} E_0(x') * \left(\sum_{s=1}^{q} \sum_{t=1}^{2q} \left(e^{\frac{i\pi}{4}\left(1-2\frac{(t-s)^2}{q}\right)} - e^{\frac{i\pi}{4}\left(1-2\frac{(t+s-1)^2}{q}\right)} \right) A_s(0) \delta\left(x' + \frac{1}{4q} - \frac{t}{2q}\right)\right) \tag{107}$$

Define

$$\beta(s,t) = \frac{1}{\sqrt{2q}} e^{\frac{i\pi}{4}\left(1-2\frac{(t-s)^2}{q}\right)} - e^{\frac{i\pi}{4}\left(1-2\frac{(t+s-1)^2}{q}\right)} \tag{108}$$

As $e^{i\alpha} - e^{i\beta} = 2i \sin\left(\frac{\alpha-\beta}{2}\right) e^{i\left(\frac{\alpha+\beta}{2}\right)}$:

$$\begin{aligned}
\beta(s,t) &= \frac{2i}{\sqrt{2q}} \sin\left(\frac{\pi}{8}\frac{2}{q}\left((t+s-1)^2 - (t-s)^2\right)\right) e^{\frac{i\pi}{8}\left(1-2\frac{(t-s)^2}{q}+1-2\frac{(t+s-1)^2}{q}\right)} \\
&= \sqrt{\frac{2}{q}} \sin\left(\frac{\pi}{4q}(2t-1)(2s-1)\right) e^{\frac{i\pi}{4}\left(3-\frac{(t-s)^2+(t+s-1)^2}{q}\right)} \\
&= \sqrt{\frac{2}{q}} \sin\left(\frac{\pi}{4q}(2t-1)(2s-1)\right) e^{\frac{i\pi}{4}\left(3-\frac{2(t^2+s^2-t-s)+1}{q}\right)}
\end{aligned} \tag{109}$$

Thus by Eq. (109), if the width of $E_0(x')$ is less than $1/2q$:

$$E_y\left(x', \frac{1}{2q}\right) = e^{-ikn_r \frac{L}{2q}} E_0(x') * \left(\sum_{s=1}^{q} \sum_{t=1}^{q} \mathcal{M}\left(\frac{1}{2}, q\right) A_s(0) \delta\left(x' + \frac{1}{4q} - \frac{t}{2q}\right)\right) \tag{110}$$

As $\mathcal{M}(1/2, q)_{s,t} = \beta(s,t)$, Eq. (78) is derived. Equations (75) and (79) may be derived in exactly the same way. The only difference comes from $E_0(-x) = -E_0(x')$, so that in Eqs. (75) and (79) the minus sign of the fictitious Dirac deltas becomes a plus. Furthermore, instead of using $e^{i\alpha} - e^{i\beta} = 2i \sin((\alpha-\beta)/2) e^{i(\alpha+\beta)/2}$, the identity $e^{i\alpha} + e^{i\beta} = 2\cos((\alpha-\beta)/2) e^{i(\alpha+\beta)/2}$ is needed.

C.2. Coefficients of $\mathcal{N}(1,q)$ and $\mathcal{N}\left(\frac{1}{2}, q\right)$

Proceeding from Eq. (81) we may evaluate $F(x', \frac{1}{q})$ using Eq. (15):

$$F\left(x', \frac{1}{q}\right) = \frac{e^{-ikn_r \frac{L}{q}}}{\sqrt{q}} E_0(x') * \left(\sum_{s=1}^{q-1} \sum_{\substack{t=1 \\ t \equiv q \pmod 2}}^{2q} e^{\frac{i\pi}{4}\left(1-\frac{t^2}{q}\right)} A_s(0) \left(\delta\left(x' - \frac{s}{2q} - \frac{t}{2q}\right) - \delta\left(x' + \frac{s}{2q} - \frac{t}{2q}\right)\right)\right) \tag{111}$$

For the real source, we shift t by $t \to t - s$ so that $\delta\left(x' - \frac{s}{2q} - \frac{t-s}{2q}\right) = \delta\left(x' - \frac{t}{2q}\right)$. For the fictitious source, we shift t by $t \to t + s$ so that $\delta\left(x' + \frac{s}{2q} - \frac{t+s}{2q}\right) = \delta\left(x' - \frac{t}{2q}\right)$. Recalling that $t + s \equiv t - s \pmod 2$, Eq. (111) becomes

$$F\left(x', \frac{1}{q}\right) = \frac{e^{-ikn_r\frac{L}{q}}}{\sqrt{q}} E_0(x') * \left(\sum_{s=1}^{q-1} \sum_{\substack{t=1 \\ t-s\equiv q \pmod 2}}^{2q} \left(e^{\frac{i\pi}{4}\left(1-\frac{(t-s)^2}{q}\right)} - e^{\frac{i\pi}{4}\left(1-\frac{(t+s)^2}{q}\right)} \right) A_s(0)\delta\left(x' - \frac{t}{2q}\right) \right)$$
$$= e^{-ikn_r\frac{L}{q}} E_0(x') * \left(\sum_{s=1}^{q-1} \sum_{t=1}^{2q} \gamma(s,t) A_s(0)\delta\left(x' - \frac{t}{2q}\right) \right) \tag{112}$$

with

$$\gamma(s,t) = \begin{cases} \frac{1}{\sqrt{q}} e^{\frac{i\pi}{4}\left(1-\frac{(t-s)^2}{q}\right)} - e^{\frac{i\pi}{4}\left(1-\frac{(t+s)^2}{q}\right)} & t - s \equiv q \pmod 2 \\ 0 & t - s \equiv q + 1 \pmod 2 \end{cases} \tag{113}$$

Recalling the definition of $\sigma(s,t)$ from Eq. (71) γ may be simplified:

$$\gamma(s,t) = \frac{(1 - \sigma(s,t))}{\sqrt{q}} e^{\frac{i\pi}{4}\left(1-\frac{(t-s)^2}{q}\right)} - e^{\frac{i\pi}{4}\left(1-\frac{(t+s)^2}{q}\right)} \tag{114}$$

However

$$e^{\frac{i\pi}{4}\left(1-\frac{(t-s)^2}{q}\right)} - e^{\frac{i\pi}{4}\left(1-\frac{(t+s)^2}{q}\right)} = 2i \sin\left(\frac{\pi}{8q}\left((t+s)^2 - (t-s)^2\right)\right) e^{\frac{i\pi}{8}\left(2-\frac{(t-s)^2+(t+s)^2}{q}\right)}$$
$$= 2 \sin\left(\frac{\pi t s}{2q}\right) e^{\frac{i\pi}{4}\left(3-\frac{t^2+s^2}{q}\right)} \tag{115}$$

Note that if $t = 0$ or q, then $\gamma = 0$. This confirms that there are no images at the boundary of the waveguide. Thus, if the width of $E_0(x')$ is less than $1/2q$, then

$$E_y\left(x' \frac{1}{q}\right) = e^{-ikn_r\frac{L}{q}} E_0(x') * \left(\sum_{s=1}^{q-1} \sum_{t=1}^{q-1} \mathcal{N}(1,q) A_s(0)\delta\left(x' - \frac{t}{2q}\right) \right) \tag{116}$$

with $\mathcal{N}(1,q)_{s,t} = \gamma(s,t)$, as in Eq. (84).

To derive Eq. (86), we repeat the above steps for $p = 1/2$.

$$F\left(x', \frac{1}{2q}\right) = \frac{e^{-ikn_r\frac{L}{2q}}}{\sqrt{2q}} E_0(x') * \left(\sum_{s=1}^{q-1} \sum_{\substack{t'=1 \\ t'\equiv 2q \pmod 2}}^{4q} e^{\frac{i\pi}{4}\left(1-\frac{t'^2}{2q}\right)} A_s(0) \left(\delta\left(x' - \frac{s}{2q} - \frac{t'}{4q}\right) - \delta\left(x' + \frac{s}{2q} - \frac{t'}{4q}\right) \right) \right) \tag{117}$$

$t' \equiv 2q \pmod 2 \Rightarrow t' = 2t$. Thus

$$F\left(x', \frac{1}{2q}\right) = \frac{e^{-ikn_r\frac{L}{2q}}}{\sqrt{2q}} E_0(x') * \left(\sum_{s=1}^{q-1} \sum_{t=1}^{2q} e^{\frac{i\pi}{4}\left(1-\frac{2t^2}{q}\right)} A_s(0) \left(\delta\left(x' - \frac{s}{2q} - \frac{t}{2q}\right) - \delta\left(x' + \frac{s}{2q} - \frac{t}{2q}\right) \right) \right) \tag{118}$$

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The summation index t is shifted identically as in the $p = 1$ case:

$$F\left(x', \frac{1}{2q}\right) = \frac{e^{-ikn_r\frac{L}{2q}}}{\sqrt{2q}} E_0(x') * \left(\sum_{s=1}^{q-1}\sum_{t=1}^{2q}\left(e^{\frac{i\pi}{4}\left(1-\frac{2(t-s)^2}{q}\right)} - e^{\frac{i\pi}{4}\left(1-\frac{2(t+s)^2}{q}\right)}\right) A_s(0)\delta\left(x' - \frac{t}{2q}\right)\right)$$

$$= e^{-ikn_r\frac{L}{2q}} E_0(x') * \left(\sum_{s=1}^{q-1}\sum_{t=1}^{2q}\epsilon(s,t)A_s(0)\delta\left(x' - \frac{t}{2q}\right)\right) \quad (119)$$

where

$$\epsilon(s,t) = \frac{1}{\sqrt{2q}} e^{\frac{i\pi}{4}\left(1-\frac{2(t-s)^2}{q}\right)} - e^{\frac{i\pi}{4}\left(1-\frac{2(t+s)^2}{q}\right)} \quad (120)$$

But

$$e^{\frac{i\pi}{4}\left(1-\frac{2(t-s)^2}{q}\right)} - e^{\frac{i\pi}{4}\left(1-\frac{2(t+s)^2}{q}\right)} = 2i\sin\left(\frac{\pi}{8q}2\left((t+s)^2 - (t-s)^2\right)\right) e^{\frac{i\pi}{8}\left(2-\frac{2(t-s)^2+(t+s)^2}{q}\right)}$$

$$= 2\sin\left(\frac{\pi t s}{q}\right) e^{\frac{i\pi}{4}\left(3-\frac{2(t^2+s^2)}{q}\right)} \quad (121)$$

As with the previous case, there are no images at the edge of the waveguide, where $t = 0, q$. Thus, if 2.the width of $E_0(x')$ is less than $1/2q$,

$$E_y\left(x'\frac{1}{2q}\right) = e^{-ikn_r\frac{L}{2q}} E_0(x') * \left(\sum_{s=1}^{q-1}\sum_{t=1}^{q-1}\mathcal{N}(\frac{1}{2},q)A_s(0)\delta\left(x' - \frac{t}{2q}\right)\right) \quad (122)$$

with $\mathcal{N}(1/2, q)_{s,t}$ given by Eq. (86). Equations (85) and (87) are easily derived, as explained above.

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