1) Given,

$a \in \mathbb{Z}_p$

$(a+p)^n \pmod p = a^n \pmod p$

$\left({}^nC_0\, a^0 p^n + {}^nC_1\, a^1 p^{n-1} + {}^nC_2\, a^2 p^{n-2} \cdots + {}^nC_n\, a^n p^0\right) \bmod p$

$= (0 + 0 + \cdots + 0 + a^n) \bmod p$

$= a^n \bmod p$

2) $\mathbb{Z}_5$

$a = \{1, 2, 3, 4\}$

$a^{-1} = \{1, 3, 2, 4\}$

$\mathbb{Z}_{11}$:

$a = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$a^{-1} = \{1, 6, 4, 3, 9, 2, 8, 7, 5, 10\}$

3) Euclidian algorithm to find gcd:-

$gcd(56245, 43159) = ?$

$56245 = 1 \times 43159 + 13086$

$43159 = 3 \times 13086 + 3901$

$13086 = 3 \times 3901 + 1383$

$1383 = 1 \times 1135 + 248$

$248 = 1 \times 143 + 105$

$105 = 1 \times 65 + 38$

$105 = 2 \times 38 + 29$

$29 = 3 \times 9 + 2$

$9 = 4 \times 2 + 1$

$2 = 2 \times 1 + 0$

$\therefore gcd = 1$

4) $\phi(3^4)$

$\because$ 3 is a Prime, W.K.T $\phi(p^e) = p^e - p^{e-1}$

$$\Rightarrow \phi(3^4) = 3^4 - 3^{4-1}$$
$$= 3^4 - 3^3$$
$$= 3^3(3-1)$$
$$= 27 \times 2 = 54$$

$$\phi(2^{10}) = 2^{10} - 2^9$$
$$= 1024 - 512$$
$$= 512$$

5) $3^{100} \bmod (31319)$

$$100 = 1100100$$
$$= 2^6 + 2^5 + 2^2$$
$$(3^{100}) = (3)^{2^6 + 2^5 + 2^2}$$
$$= (3)^{2^6} \times (3)^{2^5} + (3)^{2^2}$$

$$\Rightarrow 3^{100}(\bmod(31319)) = \left((3)^{2^6} \times (3)^{2^5} \times (3)^{2^2}\right)(\bmod\ 313$$

$(3)^{2^0} \pmod{31319} = 3$

$(3)^{2^1} \equiv (3^2)^1$
$\equiv 9$
$\equiv 9 \pmod{31319}$

$(3)^{2^2} = (3^2)^2$
$= 9^2 \pmod{31319}$
$= 81 \bmod (31319)$

$(3^2)^3 = (81)^2 \pmod{31319}$
$= 6561 \pmod{31319}$

$(3^2)^4 = (3^{2^3})^2$
$= (6561)^2 \pmod{31319}$
$= 14415$

$(3^2)^5 = (3^{2^4})^2 = (14415)^2 \pmod{31319}$
$= 207792225 \pmod{31319}$
$= 21979$