① Assignment-3 CS-527

consider the discrete memoryless channel

$Y = X + Z \pmod 5$, where

$$Z = \begin{pmatrix} 0 & 1 \\ 1-P & P \end{pmatrix}. \quad \text{and } X \in \{0, 1, \ldots 4\}.$$

Assume that Z is independent of X.

Find the capacity (C).

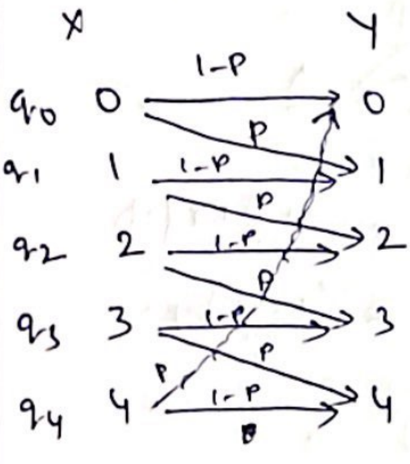
Sol:-

Given,

$$Y = X + Z \pmod 5).$$

where,

$$Z = \begin{cases} 0 & \text{with probability } 1-P, \\ 1 & \text{with probability } P. \end{cases}$$



Capacity (C) can be defined as,

$$C = \max(I(X,Y)).$$

∴ Where mutual information of X, Y can be defined as,

$$I(X;Y) = H(Y) - H(Y/X).$$

defining $H(Y)$,

$$H(Y) = P(Y=0) \cdot \log \frac{1}{P(Y=0)} + P(Y=1) \cdot \log \frac{1}{P(Y=1)} +$$

$P(y=2)\cdot\log\frac{1}{P(y=2)} + P(y=3)\cdot\log\frac{1}{P(y=3)} + P(y=4)\cdot\log\frac{1}{P(y=4)}.$

defining $P(y=i)$ where $i=\{0,1,2,3,4\}$

$\Rightarrow$

$P(y=0) \Rightarrow q_0(1-P) + q_4\cdot P$

$P(y=1) \Rightarrow q_1(1-P) + q_0\cdot P$

$P(y=2) \Rightarrow q_2(1-P) + q_1\cdot P$

$P(y=3) \Rightarrow q_3(1-P) + q_2\cdot P$

$P(y=4) \Rightarrow q_4(1-P) + q_3\cdot P$

The conditional entropy $H(Y/X)$ as,

$$H(Y/X) = \sum_{i=0}^{4} P(X=i)\, H(Y/X=i).$$

$$H(Y/x=i) = P\left(\frac{y=i}{x=i}\right)\cdot\log\frac{1}{P\left(\frac{y=i}{x=i}\right)} + P\left(\frac{y=i+1}{x=i+1}\right)\cdot\log\left(\frac{1}{P\left(\frac{y=i+1}{x=i+1}\right)}\right)$$

$$= (1-P)\cdot\log\left(\frac{1}{1-P}\right) + P\cdot\log\cdot\left(\frac{1}{P}\right).$$

$$\boxed{H(Y/x=i) = (1-P)\cdot\log\left(\frac{1}{1-P}\right) + P\cdot\log\left(\frac{1}{P}\right).}$$

defining,

$$H(Y/x=0) = P\left(\frac{y=0}{x=0}\right)\log\left(\frac{1}{P\left(\frac{y=0}{x=0}\right)}\right) + P\left(\frac{y=1}{x=0}\right)\cdot\log\left(\frac{1}{P\left(\frac{y=1}{x=0}\right)}\right).$$

$$= (1-P)\cdot\log\frac{1}{1-P} + P\cdot\log\frac{1}{P}.$$

$$H(Y/x=0) = (1-P)\cdot\log\frac{1}{1-P} + P\cdot\log\frac{1}{P} \,//.$$

$$H(Y|X) = \sum_{i=0}^{4} P(X=i)\, H(Y/X=i)$$

$$\boxed{\therefore \text{ sum of all probabilities } \sum_{i=0}^{4} P(X=i) = 1}$$

$$\boxed{H(Y/X) = (1-P).\log\frac{1}{1-P} + P.\log\frac{1}{P}}$$

$$H(Y) = \sum_{i=0}^{4} P(y).\log\frac{1}{P(y)}$$

$$\Rightarrow \sum_{i=0}^{4} \frac{1}{5}.\log\frac{1}{\left(\frac{1}{5}\right)} \Rightarrow \sum_{i=0}^{4} \frac{1}{5}.\log 5$$

$$\Rightarrow \cancel{5} \times \frac{1}{\cancel{5}}.\log 5$$

$$\Rightarrow \boxed{H(Y) = \log 5}$$

mutual information, $I(X,Y)$ is

$$I(X,Y) = H(Y) - H(Y|X)$$

$$= \log 5 - \left[(1-P).\log\frac{1}{1-P} + P.\log\frac{1}{P}\right]$$

$$= \log 5 - \left[-(1-P).\log(1-P) - P.\log P\right]$$

$$= \log 5 + \underbrace{(1-P)\log(1-P) + P.\log P}_{H(P).}$$

$$= \log 5 - H(P).$$

$\Rightarrow$ case 1 $\Rightarrow$ $P=0$

then $H(P)=0$

$C = \max(\log 5 + 0)$

$= \max(\log 5)$

$C = \log 5$

$\therefore H(P) = (1-P)\log\frac{1}{1-P} + P.\log\frac{1}{P}.$

case 2 $\Rightarrow$ $P = 1$

then $H(P)=0$

$\max(\log 5 + 0)$

$C = \log 5$

case 3 $\Rightarrow$ $P = 0.5$

then $H(P)=1$

$\Rightarrow \max(\log 5 - 1)$

$C = \log 5$

$\therefore$ Hence the capacity of the discrete memoryless channel,

$C_{max} = \log 5.$

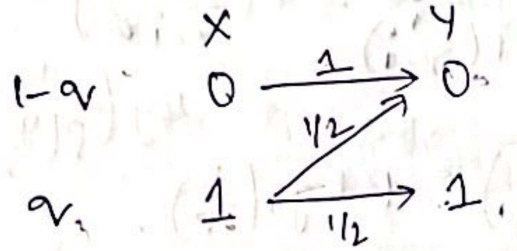
(2) The z channel has binary input and output alphabets and transition probabilities $P(Y/x)$ given by the following matrix:

$$Q = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix} \quad x, y \in \{0, 1\}.$$

Find the capacity of the z channel and the maximizing input probability distribution.

sol:- Given

$$Q = \begin{cases} 1 & \text{where probability } 1/2 \\ 0 & \text{where probability } 1/2. \end{cases}$$



defining,

$$P(y=0) = P(x=0) \cdot P\left(\frac{y=0}{x=0}\right) + P(x=1) \cdot P\left(\frac{y=0}{x=1}\right)$$

$$= (1-q) \cdot 1 + q \cdot \frac{1}{2}$$

$$\Rightarrow 1 - q + \frac{q}{2}$$

$$\Rightarrow 1 - \frac{q}{2} \;//.$$

defining,

$$P(y=1) = P(x=0) \cdot P\left(\frac{y=1}{x=0}\right) + P(x=1) \cdot \frac{P(y=1)}{P(x=1)}$$

$$= (1-q)(0) + q/2$$

$$= q/2$$

defining entropy $H(Y/X)$

$$\Rightarrow H(Y/X) = P(X=0) \cdot H(Y/X=0) + P(X=1) \cdot H(Y/X=1).$$

$$\Rightarrow H(Y/X=0) = P(y=0/x=0) \cdot \log \frac{1}{P\left(\frac{y=0}{x=0}\right)} + P\left(\frac{y=1}{x=0}\right) \cdot \log \frac{1}{P(y=1/x=0)}.$$

$$= 1 \cdot \log 1 + 0.$$

$$= 0$$

$$\Rightarrow H(Y/X=1) = P(y=0/x=1) \cdot \log \frac{1}{P(y=0/x=1)} + P(y=1/x=1) \cdot \log \frac{1}{P(y=1/x=1)}.$$

$$= \frac{1}{2} \cdot \log \frac{1}{\left(\frac{1}{2}\right)} + \frac{1}{2} \cdot \log \frac{1}{\left(\frac{1}{2}\right)}$$

$$= \frac{1}{2} \log 2 + \frac{1}{2} \log 2$$

$$= \log 2 \Rightarrow 1$$

$$H(y/x) = P(x=0).H(y/x=0) + P(x=1).H(y/x=1).$$

$$\Rightarrow (1-q).(0) + q.1$$

$$\Rightarrow q.$$

mutual information $I(X,Y)$,

$$= H(Y) - H(Y/x)$$

$$= P(y=0).\log\frac{1}{P(y=0)} + P(y=1).\log\frac{1}{P(y=1)}$$

$$- H(y/x)$$

$$\Rightarrow (1-q/2).\log\frac{1}{(1-q/2)} + \frac{q}{2}.\log\frac{1}{\left(\frac{q}{2}\right)} - q$$

$$\Rightarrow \left(1-\frac{q}{2}\right).\log\left(\frac{1}{1-q/2}\right) - \frac{q}{2}.\log\frac{q}{2} - q.$$

differentiating with q

$$\frac{dy}{dq} = \frac{\cancel{1-q/2}}{\cancel{1-q/2}}.\left(-1/2\right) - \frac{1}{2}\log(1-q/2) +$$

$$\frac{1}{2}\log\frac{q}{2} + q/2.\frac{1}{(q/2)}.\frac{1}{2} + 1)$$

$$= -\left[\frac{-1}{2} - \frac{1}{2}\log(1-q/2) + \frac{1}{2}\log\frac{q}{2} + 1/2 + 1\right]$$

$$\Rightarrow \frac{1}{2}\left[\log(1-q/2) - \log\frac{q}{2}\right) - 1 = 0$$

$$\log\left(1-\frac{q}{2}\right) - \log\frac{q}{2} = 2$$

$$\log\left(\frac{2}{q}-1\right) = 2 \Rightarrow \frac{2}{q}-1 = 4$$

$$\frac{2}{q} = 5 \Rightarrow \boxed{q = \frac{5}{2}} \text{ //.}$$

$$\text{capacity } (c) = \max\left(I(X,Y)\right)$$

$$\Rightarrow \left(1-\frac{1}{5}\right)\log\left(\frac{1}{1-1/5}\right) + \frac{1}{5}\log\frac{1}{5} - \frac{2}{5}$$

$$\Rightarrow (4/5)\cdot\log\left(\frac{5}{4}\right) + \frac{1}{5}\cdot\log\frac{1}{5} - \frac{2}{5}$$

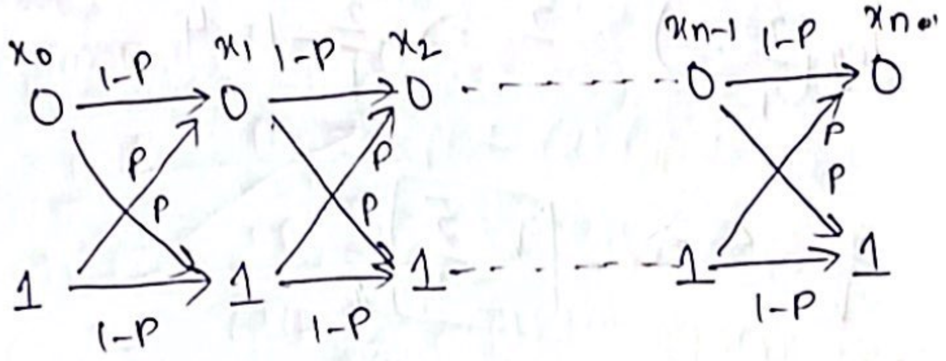
$$\Rightarrow 0.8 \times 0.32192 + 0.2(-2.32) - 0.4$$

$$\Rightarrow 0.322$$

$$\boxed{\text{maximum capacity} = 0.322 \text{ bits}}$$

3) a

sol: given,



using proof by induction,

$$P\left(\frac{x_n=1}{x_0=0}\right) = \frac{1}{2}\left(1-(1-2p)^n\right).$$

n=1

$$P\left(\frac{x_1=1}{x_0=0}\right) = \frac{1}{2}\left(\cancel{1}-\cancel{1}+2p\right) \Rightarrow P_{//}$$

n=2

$$P\left(\frac{x_2=1}{x_0=0}\right) = P\left(\frac{x_1=1}{x_0=0}\right)\cdot P\left(\frac{x_2=1}{x_1=1}\right) + P\left(\frac{x_1=0}{x_0=0}\right)\cdot P\left(\frac{x_2=1}{x_1=0}\right)$$

$$= P(1-P) + (1-P)P$$

$$= 2p - p^2$$

$$\Rightarrow \frac{1}{2}\left(1-(1-4p+4p^2)\right)$$

$$\Rightarrow \frac{1}{2}\left(1-(1-2p)^2\right)_{//}.$$

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$n=3$ Similarly,

$$P\left(\frac{x_3=1}{x_0=0}\right) = \frac{1}{2}\left(1-(1-2p)^3\right) \text{ //.}$$

then,

$$P\left(\frac{x_{r-1}=1}{x_0=0}\right) = \frac{1}{2}\left(1-(1-2p)^{r-1}\right).$$

$$P\left(\frac{x_r=1}{x_0=0}\right) = P\left(\frac{x_{r-1}=1}{x_0=0}\right) \cdot P\left(\frac{x_r=1}{x_{r-1}=1}\right) +$$

$$P\left(\frac{x_{r-1}=0}{x_0=0}\right) \cdot P\left(\frac{x_r=1}{x_{r-1}=0}\right)$$

$$= \left(\frac{1}{2}\left(1-(1-2p)^{r-1}\right) - p\left(1-(1-2p)^{r-1}\right)+p\right)$$

$$= \frac{1}{2}\left(1-(1-2p)^{r-1}\right) - p(1-2p)^{r-1}.$$

$$= \frac{1}{2}\left(1-(1-2p)^{r-1} - 2p(1-2p)^{r-1}\right)$$

$$= \frac{1}{2}\left(1-(1-2p)(1-2p)^{r-1}\right]$$

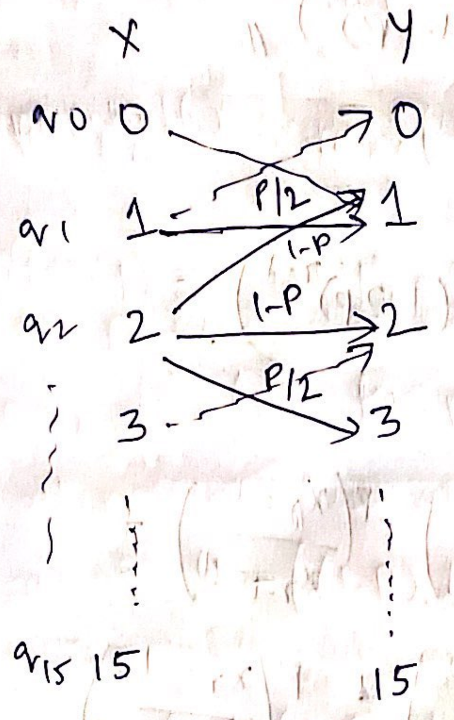
$$= \frac{1}{2}\left[1-(1-2p)^r\right] \text{ //}$$

$$\boxed{P\left(\frac{x_r=1}{x_0=0}\right) = \frac{1}{2}\left(1-(1-2p)^r\right]} \text{ //}$$

By induction proof.

(4)

Sol: Given,



defining,

$$H(Y) = P(y=0)\cdot\log\frac{1}{P(y=0)} + P(y=1)\cdot\log\frac{1}{P(y=1)} + \cdots\cdots P(y=15)\cdot\log\frac{1}{P(y=15)}.$$

$\Rightarrow$ ~~q0 (1-~~

$$\Rightarrow P(y=0) = P(x=0)\cdot P\left(\frac{y=0}{x=0}\right) + P(x=1)\cdot P\left(\frac{y=0}{x=1}\right) + \cdots\cdots P(x=15)\cdot P\left(\frac{y=0}{x=15}\right).$$

$$= q_0(1-P) + q_1(P/2) + \cdots\cdots q_{15}(P/2)$$

Similarly $P(y=1)$, $P(y=2)$, ---- $P(y=15)$.

$$\Rightarrow P(y=1) \Rightarrow q_1(1-P) + q_0(P/2) + q_2(P/2).$$

$P(y=15) \Rightarrow q_{15}(1-P) + q_{14}(P/2) + q_0(P/2)$.

To maximize the value,

$q_i = 1/16$, so, $P(y=i) = \frac{1}{16}(1-P+P/2+P/2)$.

$= 1/16$.

$H(Y) = \frac{1}{16}\log_2 16 \times 16 = 4$

$H(Y/X) = P(X=0) \cdot H(Y/x=0) + P(x=1) \cdot H(Y/x=1)$

$+ \ldots P(x=15) \cdot H(Y/x=15)$.

$H(y/x=0) = P(y=0/x=0) \cdot \log \frac{1}{P\left(\frac{y=0}{x=0}\right)} +$

$P\left(\frac{y=1}{x=0}\right) \cdot \log \frac{1}{P(y=1/x=0)} + \ldots\ldots$

$P\left(\frac{y=15}{x=0}\right) \cdot \log \frac{1}{P(y=15/x=0)}$.

$H(y/x=0) = (1-P) \cdot \log \frac{1}{1-P} + \frac{P}{2} \cdot \log \frac{1}{P/2}$

$+ \frac{P}{2} \cdot \log \frac{1}{P/2}$.

$H(y|x=0) = (1-P) \cdot \log \frac{1}{1-P} + P \log \frac{2}{P}$

$H(Y/X) = P(x=0) \cdot H(y|x=0) + \ldots$

$P(x=15) \cdot H(y|x=15)$.

$= (1-P) \cdot \log \frac{1}{1-P} + \ldots P \cdot \log \frac{2}{P}$.

$$\Rightarrow (1-P).\log \frac{1}{1-P} + P.\log \frac{2}{P} \,//.$$

$$\text{Capacity}(C) = \max(I(X,Y)).$$

$$I(X,Y) = H(Y) - H(Y|X).$$

$$\boxed{= \cancel{4 - (1-P)}\,(}$$

$$= 4 - \left((1-P).\log \frac{1}{1-P} + P.\log \frac{2}{P}\right)$$

$$= 4 + (1-P).\log(1-P) + P.\log \frac{P}{2})$$

<u>case 1</u>: $P=0$

$$\max(I(X,Y)) = 4 + (1-0)\log(1-0) + 0$$

$$= 4$$

<u>case 2</u>: $P=1$

$$= 4 + (1-1)\log(1-1) + 1\log\frac{1}{2}$$

$$= 4 + (-\log\frac{1}{2})$$

$$= 4+1$$

$$= 5$$

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case 3: $P = 0.5$

$= 4 - (1-0.5)\log(1-0.5) - 0.5\log 0.5 - 0.33$

$= 4 + \frac{1}{2} - \frac{1}{2}\log\frac{1}{2} + 0.33$

$= 4 + 0.5 + 0.5 + 0.33$

$= 5.33.$

$\therefore$ Hence maximum capacity $(4, 5, 5.33)$

$\boxed{C_{max} = 5.33 \text{ bits.}}$

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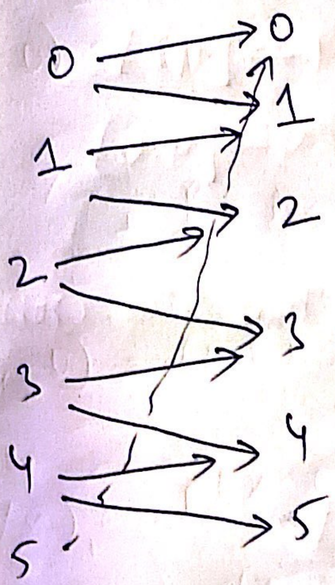
(5) Sol: (a) Assuming a flash memory of 6 levels

$\{0, 1, 2, 3, 4, 5\}$.

$\rightarrow$ with one digit we can choose $\{0, 2, 4\}$.



$\therefore$ Here maximum number of words

are $\frac{6 \times 6}{4} \Rightarrow \frac{36}{4} = 9$.

taking the first combination

00,

all possible cases are [00, 01, 10, 11].

02,

all possible cases are [02, 03, 12, 13].

04,

all possible cases are [04, 05, 14, 15].

20,

all possible cases are [20, 21, 31, 30].

22,

all possible cases are [22, 23, 32, 33].

24,

all possible cases are [24, 25, 34, 35].

~~2~~40,

all possible cases are [40, 41, 50, 51].

42,

all possible cases are [42, 43, 52, 53]

44, all possible cases are [44, 45, 54, 55].

Hence by taking the set {0, 2, 4}.

the 9 crosswords are formed.

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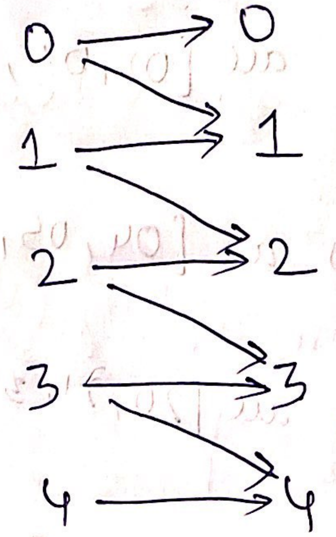
(b) Assuming, the number of levels

$\{0,1,2,3,4\}$.

hence,



the one letter crosswords are,

$0 \rightarrow [0,1]$ $\quad$ $1 \rightarrow [1,2]$

$2 \rightarrow [2,3]$ $\quad$ $3 \rightarrow [3,4]$

or $\quad$ or

$0 \rightarrow [0,1]$ $\quad$ $1 \rightarrow [1,2]$

$3 \rightarrow [3,4]$ $\quad$ $4 \rightarrow [4,0]$.

The combination of two letter crosswords are,

$11 \rightarrow [11, 21, 12, 22]$

$03 \rightarrow [03, 13, 04, 14]$

24 → [24, 34, 20, 30]

40 → [40, 41, 00, 01]

32 → [32, 42, 33, 43].

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