ASSIGNMENT - 1

① In a transmission system a 0 is encoded as 00000 and 1 as 11111 and these bits are sent through the binary symmetric channel where the bit error probability is p. At the receiving end the decoding is done by majority voting. What is the probability of error P_E assuming $p = 0.1$? When 0 is encoded as 0000000 and 1 as 1111111 and the decoding is done again by majority voting, what is the value of P_E for $p = 0.1$?

Sol :- Given

$0 \Rightarrow 0\ 0\ 0\ 0\ 0$

$1 \Rightarrow 1\ 1\ 1\ 1\ 1$

$P_E = ?$ $p = 0.1$

let us assume the cases of all inputs and the outputs.

Input	Output
0 0 0 0 0 $\Rightarrow$	0
0 0 0 0 1 $\Rightarrow$	0

0 0 0 1 0 ⇒ 0

0 0 1 0 0 ⇒ 0

0 1 0 0 0 ⇒ 0

1 0 0 0 0 ⇒ 0

0 0 0 1 1 ⇒ 0

0 0 1 1 1 ⇒ 1

0 1 1 1 1 ⇒ 1 } errors

1 1 1 1 1 ⇒ 1

Input: 0 0 0 0 0

output: 0 0 1 1 1

Probability:- 1-P 1-P P P P

$$\Rightarrow \binom{5}{3} \times P^3 \times (1-P)^2$$

Input: 0 0 0 0 0

output: 0 1 1 1 1

Probability: 1-P P P P P

$$\Rightarrow \binom{5}{4} \times P^4 \times (1-P)$$

Input: 0 0 0 0 0

output: 1 1 1 1 1

Probability P P P P P

$$\Rightarrow \binom{5}{5} \times P^5 \Rightarrow P^5$$

combined result from all the cases.

$$\Rightarrow \binom{5}{3} \times P^3 \times (1-P)^2 + \binom{5}{4} \times P^4 \times (1-P) + P^5$$

$$\Rightarrow 10P^3(1-P)^2 + 5P^4(1-P) + P^5$$

$$\Rightarrow 5P^3(1-P)[2(1-P)+P] + P^5$$

(Substituting $P = 0.1$)

$$\Rightarrow 5[0.001][0.9][1.9] + [0.00001]$$

$$\Rightarrow 0.00856 \Rightarrow 0.856\%$$

Similarly going through the other case

0 0 0 0 0 0 0 ⇒ 0 $\boxed{P = 0.1}$

1 1 1 1 1 1 1 ⇒ 1

errors all ⇒

Input 0 0 0 0 0 0 0 =

output
$\underset{1-p}{0}\ \underset{1-p}{0}\ \underset{1-p}{0}\ \underset{p}{1}\ \underset{p}{1}\ \underset{p}{1}\ \underset{p}{1} = {}^7C_4 (1-p)^3 p^4$

$\underset{1-p}{0}\ \underset{1-p}{0}\ \underset{p}{1}\ \underset{p}{1}\ \underset{p}{1}\ \underset{p}{1}\ \underset{p}{1} = {}^7C_5 (1-p)^2 p^5$

$\underset{1-p}{0}\ \underset{p}{1}\ \underset{p}{1}\ \underset{p}{1}\ \underset{p}{1}\ \underset{p}{1}\ \underset{p}{1} = {}^7C_6 (1-p)\, p^6$

$\underset{p}{1}\ \underset{p}{1}\ \underset{p}{1}\ \underset{p}{1}\ \underset{p}{1}\ \underset{p}{1}\ \underset{p}{1} \Rightarrow {}^7C_7 (1-p)^0 p^7$

result

$${}^7C_4 (1-p)^3 p^4 + {}^7C_5 (1-p)^2 p^5 + {}^7C_6 (1-p) p^6 + {}^7C_7 (1-p)^0 p^7$$

substituting $p = 0.1$

$$\Rightarrow 35(1-0.1)^3 \cdot (0.1) + 21(1-0.1)^2 \cdot (0.1)^5 + 7(1-0.1)(0.1)^6 + 1(0.1)^7$$

$\Rightarrow 0.00272801$

rounding up

$\Rightarrow 0.271$

$$\boxed{P_E = 0.271}.$$

② A father tells that he has two children and one of them is a girl. What is the probability that the other child is a boy?

Sol: If there are two children, then all possible cases are. {BB, BG, GB, GG}.

If a child is girl then we can exclude the case 'BB'.

the remaining {BG, GB, GG} are the cases

the probability of the other child is boy has 2 cases {BG, GB}.

The probability is $\frac{\text{no. of expected cases}}{\text{total no. of cases}}$ $\frac{(BG, GB)}{(BG, GB, GG)}$.

$$\boxed{P = \frac{2}{3}}$$

③ A random number is selected uniformly from 0; 1; 2; 3; 4; 5; 6; 7 without replacement until 3 is choosen. Let X denote the number of selection. Find the entropy $H(X)$ in bits.

Sol: Given numbers are,

$$\{0,1,2,3,4,5,6,7\}$$

selecting one number where

$$P(X=1) = \frac{1}{8}$$

$$P(X=2) = \frac{7}{8} \times \frac{1}{7} \Rightarrow \frac{\cancel{7}}{8 \times \cancel{7}} = \frac{1}{8}$$

$$P(X=3) = \frac{7}{8} \times \frac{6}{7} \times \frac{1}{6} = \frac{\cancel{7} \times \cancel{6}}{8 \times \cancel{7} \times \cancel{6}} = \frac{1}{8}$$

$$P(X=4) = \frac{7}{8} \times \frac{6}{7} \times \frac{5}{6} \times \frac{1}{5} \Rightarrow \frac{1}{8}$$

$$P(X=5) = \frac{7}{8} \times \frac{6}{7} \times \frac{5}{6} \times \frac{4}{5} \times \frac{1}{4} = \frac{1}{8}$$

$$P(X=6) = \frac{7}{8} \times \frac{6}{7} \times \frac{5}{6} \times \frac{4}{5} \times \frac{3}{4} \times \frac{1}{3} \Rightarrow \frac{1}{8}$$

$$P(X=7) = \frac{7}{8} \times \frac{6}{7} \times \frac{5}{6} \times \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} \Rightarrow \frac{1}{8}$$

$$P(x=8) = \frac{7}{8}\times\frac{6}{7}\times\frac{5}{6}\times\frac{4}{5}\times\frac{3}{4}\times\frac{2}{3}\times\frac{1}{2} = \frac{1}{8}$$

~~Entropy~~ of entropy of $H(x) = \sum_{i=1}^{8} P_i \log\left(\frac{1}{P_i}\right)$

$$\boxed{H(x) = \sum_{i=1}^{8} P_i \log\left(\frac{1}{P_i}\right)}$$

$$H(x) = \cancel{8}\times\frac{1}{\cancel{8}}\log(8)$$

$$= \log_2 2^3$$

$$= 3\log_2 2$$

$$= 3$$

$$\boxed{H(x) = 3 \text{ bits.}}$$

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(4) A box of 50 semiconductor chips includes 3 defective ones. 5 chips are randomly chosen from this box. Let X denote the number of defective chips. Find the entropy of H(x) in bits.

Sol

Sol^n

Total no. of chips (semi-conductor)

$= 50.$

Total no. of defective chips

$= 3$

Chips that need to be selected randomly

$= 5$.

If one chip is defective. out of 5.

$$P(X=1) \Rightarrow \frac{{}^{3}C_1 \times {}^{47}C_4}{{}^{50}C_5} = 0.25255$$

If two chips are defective out of 5

$$P(X=2) \Rightarrow \frac{{}^{3}C_2 \times {}^{47}C_3}{{}^{50}C_5} = 0.2296$$

If ~~two~~ three chips are defective out of 5.

$$P(X=3) \Rightarrow \frac{{}^{3}C_3 \times {}^{47}C_2}{{}^{50}C_5} = 0.00051.$$

The entropy is

$$\boxed{H(X) = \sum_{i=1}^{3} P_i \log\left(\frac{1}{P_i}\right)}$$

$$H(X) = 0.25255 \log\left(\frac{1}{0.25255}\right) + 0.2296 \times \log\left(\frac{1}{0.2296}\right) + 0.00051 \times \log\left(\frac{1}{0.00051}\right)$$

$$\Rightarrow 0.5014.. + 0.1250.. + 0.0055..$$

$$\Rightarrow \cancel{} \; 0.63196$$

$$\boxed{H(X) = 0.63196 \text{ bits.}}$$

(5) World series. The world series is a seven game-series that terminates as soon as either team wins four games. Let X be the random variable that represents the outcome of a world series between teams A and B; some possible values of X are AAAA, ABABBB and BABABAA. let Y be the number of games played, which ranges from 4 to 7. Assuming that A and B are equally matched and the games are independent, calculate H(X) and H(Y).

Sol:- X is a random variable.

All possible scenarios of random variable X are AAAA, ABABBB & BABABAA, BBBB, ABAABA ....,

⇒ 4. no. of games played and the condition will be

$$4 < y < 7.$$

① if only 4 matches are played,
then the possibility are

AAAA and BBBB

$P(AAAA) = \frac{1}{2^4}$, $P(BBBB) = \frac{1}{2^4}$

$= \frac{1}{16}$. $= \frac{1}{16}$

then $2 \times \frac{1}{16} = \frac{1}{8}$ //.

⇒ ② If 5 matches are played then

$\underbrace{\_\ \_\ \_\ \_}\ \underset{w}{\_}$ if team A wins the last match

→ then out of first 4 matches 3 would have been won by team A.

so. it will be $^4C_3$.

$\Rightarrow {}^4C_3 \times 2 \Rightarrow 4 \times 8 = \underline{\underline{8 \text{ ways}}}$.

⇒ ~~If 6 matches are played~~

Probability $= \frac{1}{2^5}$

Probability for 5 game series $= 8 \times \frac{1}{2^5}$

$= \frac{8}{32} = \frac{1}{4}$ //.

(3) If 6 matches are played.

$$\underbrace{\_\ \_\ \_\ \_\ \_}_{^5C_3}\ \_$$

↓ if A wins the last match.

3 out of 5 matches A have won.

$\Rightarrow {}^5C_3 \times 2 \Rightarrow 5 \times 4 \times 2 = 20$ ways

Probability $= \frac{1}{2^6}$

Probability for 6 game series $= 20 \times \frac{1}{2^6}$

$= \frac{\cancel{20}^5}{\cancel{64}_{16}} = \frac{5}{16}$ //.

(4) If 7 matches are played.

$$\underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{^6C_3}\ \_$$

$\Rightarrow {}^6C_3 \times 2 = 5 \times 4 \times 2 \Rightarrow 40$ ways

Probability for 7 game series $= 40 \times \frac{1}{2^7}$

$= \frac{\cancel{40}}{\cancel{128}} \Rightarrow \frac{5}{16}$ //.

$$\boxed{\text{Entropy of } (X), \quad H(X) = \sum p(x) \cdot \log \frac{1}{p(x)}}$$

$$\Rightarrow H(X) = \cancel{2} \times \frac{1}{\cancel{16}_{8}} \times \log\left(\frac{1}{\left(\frac{1}{16}\right)}\right) + \cancel{8} \times \frac{1}{\cancel{32}_{4}} \times \log\left(\frac{1}{\left(\frac{1}{32}\right)}\right)$$

$$+ \overset{5}{\cancel{20}} \times \frac{1}{\underset{16}{\cancel{64}}} \times \log\left(\frac{1}{\left(\frac{1}{64}\right)}\right) + \overset{5}{\cancel{20}} \times \frac{1}{\underset{16}{\cancel{128}}} \times$$

$$\log\left(\frac{1}{\left(\frac{1}{128}\right)}\right)$$

$$\Rightarrow \frac{1}{8} \times \log 16 + \frac{1}{4} \log 32 + \frac{5}{16} \times \log 64$$

$$+ \frac{5}{16} \times \log 128$$

$$\Rightarrow \frac{1}{8} \times 4 + \frac{1 \times 5}{4} + \frac{5}{16} \times 6 + \frac{5}{16} \times 7$$

$$= 0.5 + \overset{1.25}{\cancel{0.025}} + \overset{1.875}{\cancel{0.0625}} + 2.1875$$

$$\Rightarrow 5.8125 \text{ bits.}$$

$$\boxed{H(X) = 5.8125 \text{ bits.}}$$

$$H(Y) = \frac{1}{2^3} \log 2^3 + \frac{1}{2^2} \log 2^2 + \frac{5}{16} \log \frac{16}{5}$$

$$+ \frac{5}{16} \log \frac{16}{5}$$

$$\Rightarrow \frac{3}{8} + \frac{2}{4} + \frac{5}{16}(1.68) + \frac{5}{16}(1.68)$$

$$= 1.924 \text{ bits}$$

$$\boxed{H(Y) = 1.924 \text{ bits}}$$ //

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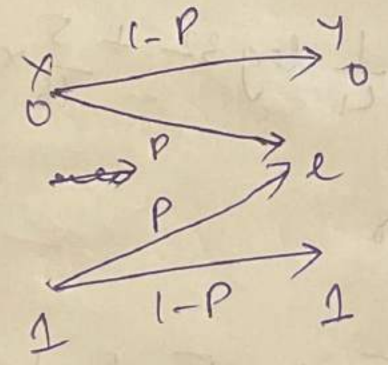
⑥ Erasure channel. Consider the discrete memoryless channel as shown.



Assuming $P(X=0) = \frac{2}{3}$ and $P(X=1) = \frac{1}{3}$

and $P = 1/4$.

find, a) $H(X)$, $H(Y)$

b) $H(Y/X)$, $H(X/Y)$

c) $H(X,Y)$

d) $I(X,Y)$

Sol: Given, $P(X=0) = \frac{2}{3}$ $\quad P = \frac{1}{4}$.

$P(X=1) = \frac{1}{3}$

a) $H(X) = \sum P(x) \log \frac{1}{P(x)}$

$\Rightarrow P(X=0) \cdot \log\left(\frac{1}{P}\right) + P(X=1) \cdot \log(1/P)$

$= \frac{2}{3} \cdot \log\left(\frac{1}{\frac{2}{3}}\right) + \frac{1}{3} \log\left(\frac{1}{\frac{1}{3}}\right)$

$= \frac{2}{3} \cdot \log \frac{3}{2} + \frac{1}{3} \log 3$

$= 0.6666 + 0.2517$

$= 0.9183$ bits.

$\boxed{H(X) = 0.9183 \text{ bits}}$

$H(Y) = P(Y=0) \cdot \log \frac{1}{P} + P(Y=1) \cdot \log \frac{1}{P}$

$P(Y=2) \log(1/P)$.

$\Rightarrow \frac{2}{3} \times \frac{3}{4} \times \log\left(\frac{1}{\frac{2}{3} \times \frac{3}{4}}\right) + \frac{1}{3} \times \frac{3}{4} \times \log\left(\frac{1}{\frac{1}{3} \times \frac{3}{4}}\right)$

$+ \frac{1}{4} \cdot \log\left(\frac{1}{\left(\frac{1}{4}\right)}\right)$

$\Rightarrow \frac{1}{2} \times \log 2 + \frac{1}{4} \times \log 4 + \frac{1}{4} \log 4$

$\Rightarrow \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \Rightarrow \frac{3}{2} \Rightarrow 1.5$ //.

$$\boxed{H(Y) = 1.5 \text{ bits}}$$

b) $\boxed{H(Y/X) = \sum P(x_i) H(Y/x_i)}$

$H(Y/x_1=0) \Rightarrow P(Y=0/x=0)\log(1/P) +$

$P(Y=1/x=0)\log(1/P) +$

$P(Y=e/x=0)\log(1/P)$.

$= 3/4 \log \frac{4}{3} + 0 + \frac{1}{4}\log 4.$

$\Rightarrow 0.8113.$

$H(Y/x) = P(x=0)H(Y/x=0) + P(x=1)H(Y/x=1)$

$\Rightarrow \frac{2}{3} \times 0.8113 + \frac{1}{3} \times 0.8113$

$\Rightarrow 0.8113 \times \left(\frac{2}{3} + \frac{1}{3}\right)$

$= 0.8113$ bits.

$$\boxed{H(Y|x) = 0.8113 \text{ bits}}$$

$H(X/Y) = \sum P(y_i)\, H(X/y_i)$

$\Rightarrow \cancel{H(X/y_i) = H(XY) - H(Y)}$

$\Rightarrow \cancel{H(XY) =}$

$H(X/y_i) = H(X)$

$\Rightarrow H(X/Y) \Rightarrow \sum P(y_i) \cdot H(X)$

$\Rightarrow \frac{1}{4} \times 0.9183$

$\Rightarrow 0.2296$ bits

$\boxed{H(X/Y) = 0.2296 \text{ bits}}$

(c) $H(X,Y) = -\sum_x \sum_y P(x,y) \log P(x,y)$

$\Rightarrow -P(x_1,y_1) \log P - P(x,y_1) \log(P) -$

$- P(x_1,y_2) \log P$

$\Rightarrow -\frac{2}{3} \times \frac{3}{4} \log \frac{1}{2} - 0 - \frac{2}{3\times4} \log \frac{1}{6}$

$- \frac{1}{3} \times \frac{3}{4} \times \log \frac{1}{6}$

$\Rightarrow 0.6\cancel{887} \ldots$ //.

$(+0.5)(\cancel{-0.5}) - \frac{2}{3\times4} \times 4 \Rightarrow$

$\Rightarrow 0.5 + 0.4308 + 0.5 + 0.3617$

$\Rightarrow 1.7925$ bits

$\boxed{H(X,Y) = 1.7925 \text{ bits}}$

d) $\boxed{I(X,Y) = H(X) - H(X/Y)}$

$\Rightarrow 0.9183 - 0.2296$

$\Rightarrow 0.6887$ bits.

$I(X,Y)$ ∴ Mutual information between X, Y is 0.6887 bits.

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(7) A fair die is rolled until six occurs at the top. Let X denote the number of rolls required. Find $H(X)$.

Sol The probability of getting a 6 on top from the fair die is $P = \frac{1}{6}$ //.

no. of repetitions $r = n$.

$$P_n = q^{n-1} \cdot P$$

$$\Rightarrow P \cdot \log \frac{1}{p} + q \cdot p \log \frac{1}{qp} + q^2 p \cdot \log \frac{1}{q^2 p} + \cdots$$

$$\Rightarrow P \cdot \left\{ \log \frac{1}{p} + q \log \frac{1}{qp} + q^2 \cdot \log \frac{1}{q^2 p} + \cdots \right)$$

$$\Rightarrow P \left(\log \frac{1}{p} (1 + q + q^2 + \cdots) + q \cdot \log \frac{1}{q} + \cdots \right]$$

$$\Rightarrow P \left[\log \frac{1}{p} \left[\frac{1}{1-q} \right] + q \cdot \log \frac{1}{q} + q^2 \cdot \log \frac{1}{q^2} + \cdots \right)$$

$$\Rightarrow P \left[\left[\frac{1}{p} \times \log \frac{1}{p} \right] + q \cdot \log \frac{1}{q} \left[1 + 2q + 3q^2 + \cdots \right] \right]$$

$$\Rightarrow P \times \frac{1}{p} \times \log \frac{1}{p} + Pq \cdot \log \frac{1}{q} \cdots$$

$$\Rightarrow \underline{1} = 1 + 2q + 3q^2 + 4q^3 + \cdots$$

$$-q_1 = -q - 2q^3 - 3q^2 + \cdots$$

$$1 - q_1 = 1 + q + q^2 + q^3 + \cdots$$

$$1(1-q) = 1/1-q$$

$$P = \frac{1}{P} \Rightarrow \boxed{\cancel{[illegible]}} //$$

$$\underline{I} = \frac{1}{P^2} //$$

Entropy $H(x) =$

$$\log\frac{1}{p} + pq\log\frac{1}{q}\left(\frac{1}{p^2}\right)$$

$$= \log\frac{1}{p} + \frac{q}{p}\log\frac{1}{q}$$

$p = \frac{1}{6}$ $q = \frac{5}{6}$

$$= \log\frac{1}{\left(\frac{1}{6}\right)} + \frac{\left(\frac{5}{6}\right)}{\left(\frac{1}{6}\right)}\log\left(\frac{1}{\frac{5}{6}}\right)$$

$$\Rightarrow \log 6 + 5\log\frac{6}{5}$$

$$\Rightarrow 2.584 + 5(0.263)$$

$$\Rightarrow 3.91 \text{ bits}$$

$$\boxed{H(x) = 3.91 \text{ bits}}$$

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