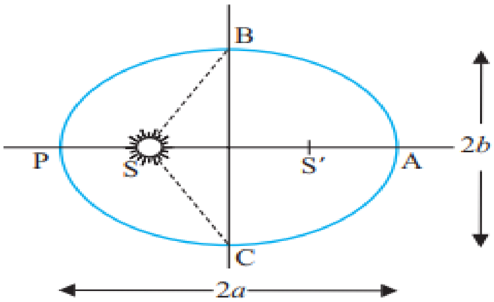
Chapter 7
Gravitation

Kepler's Laws

1.Law of orbits

All planets move in elliptical orbits with the Sun situated at one of the foci of the ellipse.



PA is the major axis
BC is the minor axis

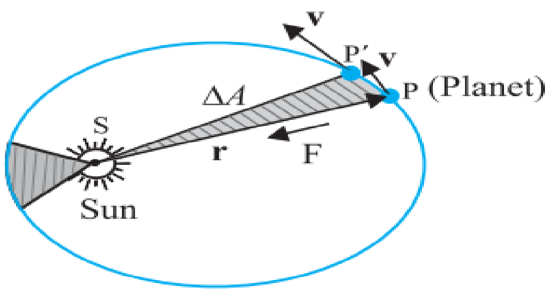
2.Law of areas

The line that joins any planet to the sun sweeps equal areas in equal intervals of time. i.e, areal velocity $\frac{\Delta \vec{A}}{\Delta t}$ is constant

The planets move slower when they are farther from the sun than when they are nearer.

The law of areas is a consequence of conservation of angular momentum.

Proof



The area swept out by the planet of mass m in time interval Δt is

$$\Delta \vec{A} = \frac{1}{2}(\vec{r} \times \vec{v}\Delta t)$$

$$\vec{p} = m\vec{v},$$

$$\vec{v} = \frac{\vec{p}}{m}$$

$$\frac{\Delta \vec{A}}{\Delta t} = \frac{1}{2}\left(\vec{r} \times \frac{\vec{p}}{m}\right)$$

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\frac{\Delta \vec{A}}{\Delta t} = \frac{\vec{L}}{2m}$$

For a central force, which is directed along r, angular momentum, $\vec{L}$ is a constant.

$$\frac{\Delta \vec{A}}{\Delta t} = \text{constant}$$

This is the law of areas.

3.Law of periods

The square of the time period of revolution of a planet is proportional to the cube of the semi- major axis of the ellipse traced out by the planet.

$$T^2 \propto a^3$$

Universal Law of Gravitation

Every body in the universe attracts every other body with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them .

$$F = G\frac{m_1 m_2}{r^2}$$

where G is the universal gravitational constant.

In vector Form

$$\vec{F} = G\frac{m_1 m_2}{r^2}(-\hat{r}) = -G\frac{m_1 m_2}{r^2}\hat{r}$$

$\hat{r}$ is the unit vector from m_1 to m_2.

The gravitational force is attractive, as the force $\vec{F}$ is along – r.

By Newton's third law the, gravitational force $\vec{F}_{12}$ on the body 1 due to 2 and $\vec{F}_{21}$ on the body 2 due to 1 are related as $\vec{F}_{12} = -\vec{F}_{21}$.

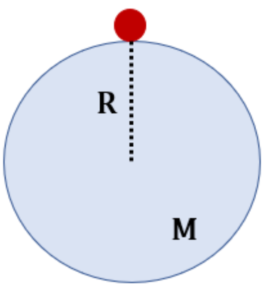
The Gravitational Constant

The value of the gravitational constant G was determined experimentally by English scientist Henry Cavendish in 1798.

$$G = 6.67\times10^{-11}\ N\ m^2/kg^2$$

Acceleration due to gravity of the Earth

Consider a body of mass m on the surface of earth of mass M and radius R. The gravitational force between body and earth is given by



$$F = \frac{GMm}{R^2} \text{ -----------(1)}$$

By Newton's second law

$$F=mg \text{ -----------(2)}$$

where g is acceleration due to gravity

From Eq (1) and (2)

$$mg = \frac{GMm}{R^2}$$

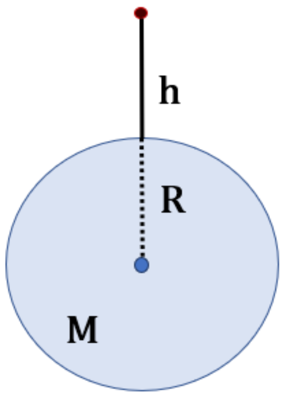
$$g = \frac{GM}{R^2}$$

Acceleration due to gravity is independent of mass of the body.

The average value of g on the surface of earth is $9.8ms^{-2}$.

Acceleration due to gravity below and above the surface of earth

1.Acceleration due to gravity at a height h above the surface of the earth.



Acceleration due to gravity on the surface of earth

$$g = \frac{GM}{R^2} \text{------------}(1)$$

Acceleration due to gravity at a height above the surface of earth

$$g_h = \frac{GM}{(R+h)^2} \text{----------}(2)$$

for , h << R, $g_h = \frac{GM}{R^2(1+\frac{h}{R})^2}$

$$g_h = \frac{GM}{R^2}(1+\frac{h}{R})^{-2}$$

substituting from eq(1) , $g_h = g(1+\frac{h}{R})^{-2}$

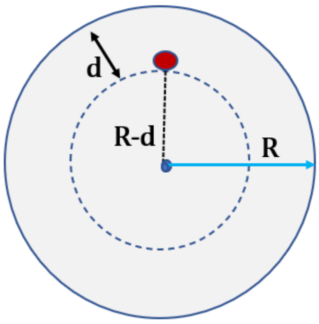
Using binomial expression and neglecting higher order terms.

$$\mathbf{g_h \cong g\left(1-\frac{2h}{R}\right)}$$

Thus, as we go above earth's surface, the acceleration due gravity decreases by a factor $(\mathbf{1}-\frac{\mathbf{2h}}{\mathbf{R}})$

2.Acceleration due to gravity at a depth d below the surface of the earth

We assume that the entire earth is of uniform density. Then mass of earth



Mass =volume x density

$$M = \frac{4}{3}\pi R^3\rho \text{------------}(1)$$

Acceleration due to gravity on the surface of earth

$$g = \frac{GM}{R^2} \text{-------------}(2)$$

Substituting the value of M in eq(2)

$$g = \frac{G}{R^2}\left(\frac{4}{3}\pi R^3\rho\right)$$

$$g = \frac{4}{3}\pi R\rho G \text{---------------}(3)$$

Acceleration due to gravity at a depth d below the surface of earth

$$g_d = \frac{4}{3}\pi(R-d)\rho G \text{---------}(4)$$

$$\frac{eq(4)}{eq(3)} \text{------} \quad \frac{g_d}{g} = \frac{\frac{4}{3}\pi(R-d)\rho G}{\frac{4}{3}\pi R\rho G}$$

$$\frac{g_d}{g} = \frac{(R-d)}{R}$$

$$\mathbf{g_d = g\left(1-\frac{d}{R}\right)}$$

Thus, as we go down below earth's surface, the acceleration due gravity decreases by a factor $\left(\mathbf{1}-\frac{\mathbf{d}}{\mathbf{R}}\right)$

- The value of acceleration due to earth's gravity is maximum on its surface and decreases whether you go up or down.
- At the centre of earth acceleration due to earth's gravity is **zero.**

Example

At what height the value of acceleration due to gravity will be half of that on surface of earth. (Given the radius of earth R =6400km)

$$g_h = g(1+\frac{h}{R})^{-2}$$

$$g_h = \frac{g}{2}$$

$$\frac{g}{2} = g(1+\frac{h}{R})^{-2}$$

$$\frac{1}{2} = (1+\frac{h}{R})^{-2}$$

$$2 = (1+\frac{h}{R})^{2}$$

$$\sqrt{2} = 1+\frac{h}{R}$$

$$\frac{h}{R} = \sqrt{2}-1$$

$$h = (\sqrt{2}-1)\, R$$

$$h = (1.414-1)\, 6400 = 2650 \text{ km}$$

Example

Calculate the value of acceleration due to gravity at a height equal to half of the radius of earth.

$$g_h = \frac{GM}{(R+h)^2}$$

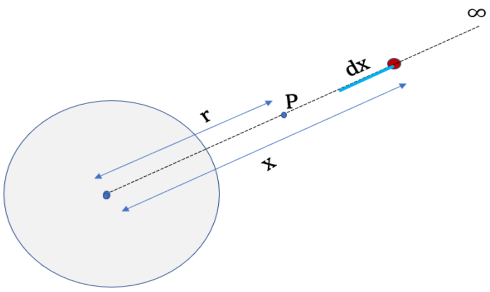
$$h = \frac{R}{2}$$

$$g_h = \frac{GM}{\left(R+\frac{R}{2}\right)^2} = \frac{GM}{\left(\frac{3}{2}R\right)^2}$$

$$= \frac{GM}{\frac{9}{4}R^2} = \frac{4}{9}\frac{GM}{R^2} = \frac{4}{9}\, g$$

Gravitational Potential Energy

Gravitational potential energy at point is defined as the work done in displacing the particle from infinity to that point without acceleration.



Gravitational force on a mass m at a distance x

$$F = \frac{GMm}{x^2}$$

The work done to give a displacement dx to the mass

$$dW = Fdx$$

$$dW = \frac{GMm}{x^2}dx$$

Total work done to move the mass from ∞ **to r**

$$W = \int_{\infty}^{r} \frac{GMm}{x^2} dx$$

$$W = GMm \int_{\infty}^{r} \frac{1}{x^2} dx$$

$$W = GMm \left[\frac{-1}{x}\right]_{\infty}^{r}$$

$$W = -GMm \left[\frac{1}{r} - \frac{1}{\infty}\right]$$

$$W = \frac{-GMm}{r}$$

This work is is stored as gravitational PE in the body.

$$\mathbf{U = \frac{-GMm}{r}}$$

Gravitational Potential

The gravitational potential due to the gravitational force of the earth is defined as the potential energy of a particle of unit mass at that point. The gravitational Potential energy of a bodyof mass m at a distance r

$$U = \frac{-GMm}{r}$$

For unit mass m=1

So gravitational potential, $\mathbf{V = \frac{-GM}{r}}$

Escape speed

The minimum speed required for an object to reach infinity i.e. to escape from the earth's gravitational pull is called escape speed.
Let the body thrown from the surface of earth to infinity.

Total initial energy of the body

$$TE = KE + PE$$

$$TE = \frac{1}{2}mv_i^2 - \frac{GMm}{R} \text{-------}(1)$$

Total final energy, $TE = \frac{1}{2}mv_f^2 + 0$ ------------(2)

By conservation of energy TE is constant.

$$\frac{1}{2}mv_i^2 - \frac{GMm}{R} = \frac{1}{2}mv_f^2$$

RHS is always always a positive quantity with minimum value zero
Taking the minimum value

$$\frac{1}{2}mv_i^2 - \frac{GMm}{r} = 0$$

$$\frac{1}{2}mv_i^2 = \frac{GMm}{R}$$

$$v_i^2 = \frac{2GM}{R}$$

$$\mathbf{v_e = \sqrt{\frac{2GM}{R}}}$$

But $g = \frac{GM}{R^2}$, $GM = gR^2$

$$v_e = \sqrt{\frac{2gR^2}{R}}$$

$$\mathbf{v_e = \sqrt{2gR}}$$

Escape velocity is independent of mass of the body.
Escape speed (or escape velocity) on the surface of earth is 11.2km/s

Moon has no atmosphere. Why?

The escape speed of moon is about 2.3 km/s. which is less than the average speed of gas molecules of moon. Thus gas molecules escape from surface of moon and it has no atmosphere.

Earth Satellites

Earth satellites are objects which revolve around the earth.
Their motion is very similar to the motion of planets around the Sun and hence Kepler's laws of planetary motion are equally applicable to them.

Satellites are of two types (1) Natural satellites and artificial satellites. Moon is the natural satellite of earth whose time period of revolution is 27.3 days.

Artificial satellites are used for telecommunication, geophysics and meteorology etc.

Orbital Speed

The speed with which a satellites revolves around earth is called orbital speed.

Consider a satellite of mass m in a circular orbit of a distance (R + h) from the centre of the earth. **The necessary centripetal force for revolution is provided by gravitational force between earth and satellite.**

$$F_{gravitational} = \frac{GMm}{(R+h)^2}$$

$$F_{centripetal} = \frac{mv^2}{R+h}$$

$$F_{centripetal} = F_{gravitational}$$

$$\frac{mv^2}{R+h} = \frac{GMm}{(R+h)^2}$$

$$v^2 = \frac{GM}{(R+h)}$$

$$v_o = \sqrt{\frac{GM}{(R+h)}}$$

Thus orbital velocity v_o decreases as height ,h increases.
If the satellite is very close to earth (R+h) ≈R

$$v_o = \sqrt{\frac{GM}{R}}$$

But $g = \frac{GM}{R^2}$, $GM = gR^2$

$$v_o = \sqrt{gR}$$

Relation Connecting Escape Velocity and Orbital Velocity

Orbital Velocity ,$v_0 = \sqrt{\frac{GM}{R}}$ or $v_o = \sqrt{gR}$

Escape Velocity ,$v_e = \sqrt{\frac{2GM}{R}}$ or $v_e = \sqrt{2gR}$

$$v_e = \sqrt{2}\, v_o$$

Escape Velocity = $\sqrt{2}$ x Orbital Velocity

Period of a Satellite

Period of a satellite is the time required for a satellite to complete one revolution around the earth in a fixed orbit.

$$\text{Time}=\frac{\text{Distance}}{speed}$$

For one revolution

$$\text{Period T}=\frac{\text{circumference of the orbit}}{orbital\ speed}$$

$$T=\frac{2\pi\,(R+h)}{\sqrt{\frac{GM}{(R+h)}}}$$

$$\mathbf{T=2\pi\sqrt{\frac{(R+h)^3}{GM}}}$$

If the satellite is very close to earth $(R+h) \approx R$

$$\mathbf{T=2\pi\sqrt{\frac{R^3}{GM}}}$$

$$\text{But } g=\frac{GM}{R^2}, \quad GM = gR^2$$

$$T=2\pi\sqrt{\frac{R^3}{gR^2}}$$

$$\mathbf{T=2\pi\sqrt{\frac{R}{g}}}$$

If we substitute the numerical values, $g = 9.8\ ms^{-2}$ and $R = 6400$ km.,

$$T=2\pi\sqrt{\frac{6400\ x\ 10^3}{9.8}}$$

$$T_o=85 \text{ minutes}$$

Proof of Kepler's third law

Period of a satellite, $T= 2\pi\sqrt{\frac{(R+h)^3}{GM}}$

$$T^2= 4\pi^2\,\frac{(R+h)^3}{GM}$$

$$T^2= \frac{4\pi^2}{GM}(R+h)^3$$

$$T^2=\text{constant x }(R+h)^3$$

$$T^2 \propto (R+h)^3$$

$$\mathbf{T^2 \propto a^3}$$

Which is Kepler's Law of Periods.

Weighing the Earth :

You are given following data: g = $9.81 s^{-2}$ m, $R_E = 6.37\times10^6$ m, the distance to the moon R = 3.84×10^8 m and the time period of the moon's revolution is 27.3 days. Obtain mass of the Earth M_E in two different ways.

First method

$$g=\frac{GM}{R^2}$$

$$M_E = \frac{g R_E^2}{G} = \frac{9.81\times(6.37\times10^6)^2}{6.67\times10^{-11}}$$

$$=5.97\times 10^{24}\text{ kg.}$$

Second method

$$T=2\pi\sqrt{\frac{R^3}{GM}}$$

$$T^2 = \frac{4\pi^2R^3}{G M_E}$$

$$M_E = \frac{4\pi^2R^3}{G T^2} = \frac{4\times3.14\times3.14\times(3.84)^3\times10^{24}}{6.67\times10^{-11}\times(27.3\times24\times60\times60)^2}$$

$$= 6.02\times10^{24}\text{ kg}$$

Both methods yield almost the same answer

Energy of an orbiting satellite

$$\text{KE}=\frac{1}{2}\text{m}v_o^2$$

$$v_o = \sqrt{\frac{GM}{R}}\ ,\quad v_o^2 = \frac{GM}{R}$$

$$\text{KE}=\frac{1}{2}\text{m}\ x\ \frac{GM}{R}$$

$$\text{KE}=\frac{GMm}{2R}$$

$$\text{PE} = \frac{-GMm}{R}$$

$$\text{Energy} = \text{KE} + \text{PE}$$

$$E = \frac{GMm}{2R} + \frac{-GMm}{R}$$

$$\mathbf{E = \frac{-GMm}{2R}}$$

The total energy of an circularly orbiting satellite is negative, which means that the satellite is bound to the planet .If the total energy is positive or zero, the object escapes to infinity. Satellites are always at finite distance from the earth and hence their energies cannot be positive or zero.

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