CSE 544, Spring 2022, Probability and Statistics for Data Science

<u>Assignment 4: Parametric Inference & Hypothesis Testing</u> Due: 4/04, 8:15pm, via Blackboard
(8 questions, 70 points total)

I/We understand and agree to the following:
(a) Academic dishonesty will result in an 'F' grade and referral to the Academic Judiciary.
(b) Late submission, beyond the 'due' date/time, will result in a score of 0 on this assignment.

(write down the name of all collaborating students on the line below)

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1. Practice with MME **(Total 9 points)**

(a) The Gamma(x, y) distribution has mean $x \cdot y$ and variance $x \cdot y^2$. Find MME for $\hat{x}$ and $\hat{y}$. (4 points)

(b) Find MME $\hat{a}$ and $\hat{b}$ for the Uniform(a, b) distribution. Express your final answer in terms of the sample mean, $\bar{X} = (\sum X_i)/n$, and sample variance, $\overline{S^2} = ((\sum X_i^2)/n) - \bar{X}^2$. (5 points)

2. Consistency of MLE (Total 12 points)

(a) Let $X_1, X_2, \ldots, X_n$ be i.i.d. as Exponential(1/β). Show that the MLE($\hat{\beta}$) will converge to the unknown parameter β. Prove this by showing that bias($\hat{\beta}$) and se($\hat{\beta}$) tends to 0 as n tends to ∞. You can use the fact that the mean and variance of Exponential(λ) are 1/λ and $1/\lambda^2$, respectively. (4 points)

(b) Let $X_1, X_2, \ldots, X_n$ be distributed i.i.d. as Normal(μ, σ^2). Show that the MLE of μ and σ^2 is the same as the sample mean and (uncorrected) sample variance, respectively. (4 points)

(c) Let $X_1, X_2, \ldots, X_n \sim$ Normal(θ, 1). Let δ =$E[I_{X_1>0}]$. Use the Equivariance property to show that the MLE of δ is $\Phi(\frac{1}{n}\sum_{i=1}^{n} X_i)$, where $\Phi()$ is the CDF of the standard Normal. You can use the MLE of the Normal as provided in 2(b). (4 points)

3. Practice with MLE (Total 8 points)

Let $Y_1, Y_2, \ldots, Y_n$ be i.i.d. from a distribution with pdf:

$$f(y \,|\alpha\,) = (2/\alpha) \; * \; (y) \; * \; (e^{-y^2/\alpha}), y > 0, \alpha > 0$$

(a) Find the MLE of α. (4 points)

(b) Let $Z_1 = Y_1^2$. Find the distribution of Z_1. Is the MLE for α an unbiased estimator of α? (4 points)

4. Parametric Inference with Data Samples **(Total 9 points)**

Let $X = \begin{cases} 2 & with\ prob\ \theta \\ 3 & otherwise \end{cases}$, where θ is unknown. Let D = {2, 3, 2} be drawn i.i.d. from X.

(a) Derive $\hat{\theta}_{MME}$ using D as the sample data. Clearly show all your steps. (3 points)

(b) Provide a numerical estimate of the 95%ile confidence intervals for $\hat{\theta}_{MME}$. Start by deriving $\widehat{se}(\hat{\theta}_{MME})$: first derive $se(\hat{\theta}_{MME})$ in terms of θ, and then estimate $\widehat{se}(\hat{\theta}_{MME})$, as in class. Show all your steps. Your final answer should be a numerical range. (3 points)

(c) Derive $\hat{\theta}_{MLE}$ using D as the sample data. Clearly show all your steps. (3 points)

5. MME versus MLE using real data **(Total 8 points)**

For this question, we will use the acceleration, model, and mpg data from the Auto-mpg dataset (https://www.kaggle.com/uciml/autompg-dataset). Please use the data files on the class website. We will assume that acceleration is Normal(μ, σ^2) distributed, model year is Uniform(a, b) distributed, and mpg is Exponential(λ) distributed. You are to find the MME and MLE estimates of the parameters of the distributions for all 3 datasets. For the Normal MME and Uniform MLE, you can directly use the results from class. For the Normal MLE, use the result from Q2(b); for Uniform MME, use the result from Q1(b). For the Exponential, we will first derive the estimates.

(a) For the Exp(λ) distribution, find the $\hat{\lambda}_{MME}$. (2 points)

(b) For the Exp(λ) distribution, find the $\hat{\lambda}_{MLE}$. (2 points)

(c) For the 3 datasets, find the MME estimates. That is, find the MME for μ and σ^2 for the acceleration dataset, a and b for the model dataset, and λ for the mpg dataset. Provide your answer as a number with 3 significant digits. (2 points)

(d) Same as part (c), but this time find the MLE estimates. (2 points)

6. Clinical Testing (Total 4 points)

A standard drug is used to control a particular virus in human beings. The probability with which this drug eliminates the virus from an affected individual is 0.7. A new drug is developed which is known not to decrease the probability of virus elimination; it is desired to determine if the new drug performs better than the standard. An experiment is to be conducted in which 500 randomly selected patients (known to be affected with the virus) are treated with the new drug. After a fixed period of time, the patients for which the new drug eliminate the virus will be counted. The experiment will reject the null hypothesis that the drug has no effect if the drug eliminates the virus in 375 or more patients. The number of patients for which the drug eliminates the virus follows a binomial distribution.

(a) State symbolically the null and alternative hypotheses, defining all symbols in words. (1 point)

(b) What is the P_r(Type-I error) of the proposed test? (1 point)

(c) The power of a test is defined as the ability to correctly reject the null hypothesis when it is actually false (detecting that the null is wrong). If in fact the success rate of the drug is 0.8, what is the power of the test? (2 points)

7. Wald's test **(Total 10 points)**

(a) Suppose the null hypothesis is H_0: $\theta = \theta_0$, but the true value of θ is θ_*. Show that, under Wald's test, the probability of a Type II error is $\Phi(\frac{\theta_0-\theta_*}{\widehat{se}} + z_{\alpha/2}) - \Phi(\frac{\theta_0-\theta_*}{\widehat{se}} - z_{\alpha/2})$.

(Hints: (i) might help to draw a figure; (ii) think about the distribution of the estimate.) (5 points)

(b) You observe 46 successes in 100 trials of a coin. If the null hypothesis is that the coin is unbiased, use the Wald's test with the MLE or MME with $\alpha = 0.05$ to Reject/Accept the null. What if the null hypothesis is that the coin has p=0.7? (5 points)

8. More on Wald's test **(Total 10 points)**

(a) Use q8_a.csv dataset and assume it is distributed as Normal(θ, σ^2). Apply the Wald's test with α = 0.02 to check whether the true mean is $\theta_0 = 0.5$. Use sample mean to obtain $\hat{\theta}$ and corrected sample variance estimator for obtaining $\widehat{\sigma^2}$. (4 points)

(b) Use q8_b_X.csv and q8_b_Y.csv available at the class website for this question. Each contains 750 samples for X and Y drawn from two independent Normal distributions. Without worrying about the applicability of the test, use Wald's 2-population test with α = 0.05 to test whether the population means of X and Y are same (null) or not (alternative). Is this test applicable here? (6 points)