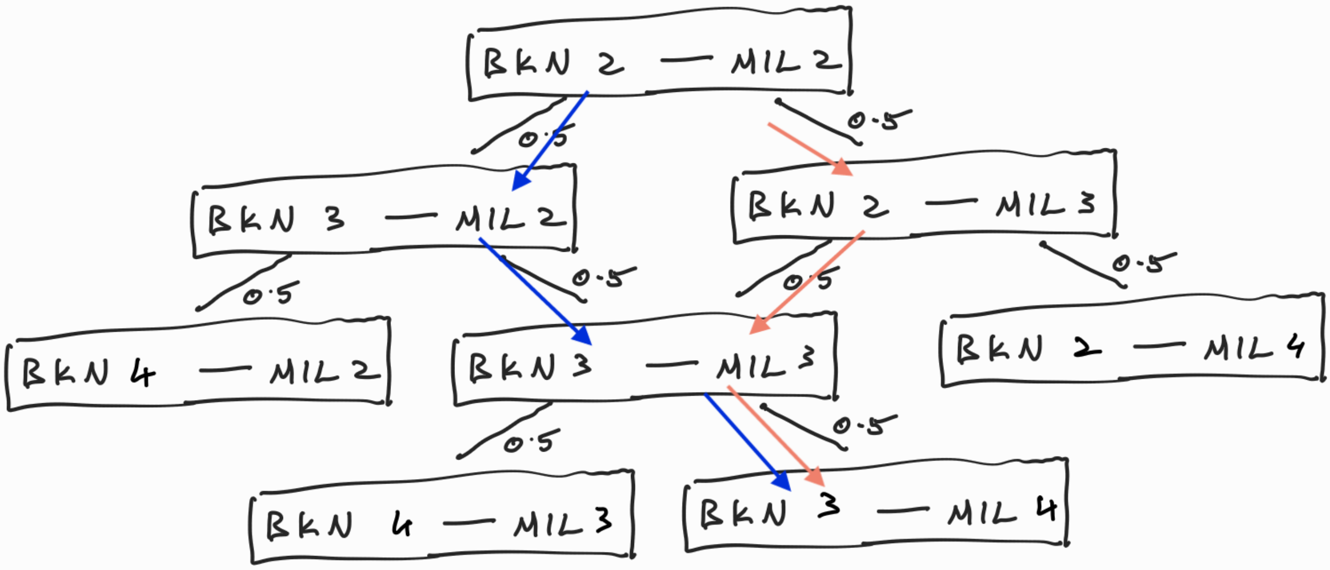
Problem 1

a) $$P = \left({}^4C_2\right) \times (0.5)^2 \times (0.5)^2$$

$$= 6 \times \frac{1}{16} = \frac{3}{8} = 0.375$$

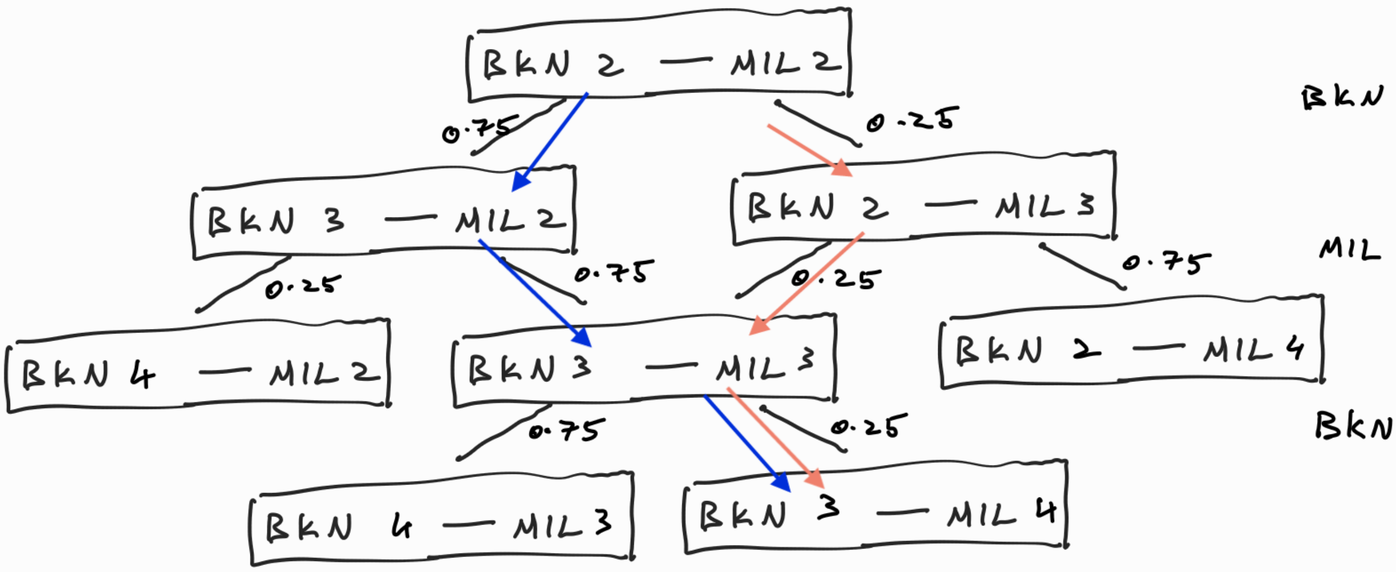
b)



c) Look for the blue and orange arrows in the above tree.

$$P(MIL\ 4\text{-}3) = 0.5^3 + 0.5^3 = 0.25$$

d)



e) Look for the blue and orange arrows in the above tree.

$$P(MIL\ 4\text{-}3) = 0.75 \times 0.75 \times 0.25 + 0.25 \times 0.25 \times 0.25$$

$$= \frac{10}{4^3} = 0.15625$$

Problem 2

Let E_i be the event that you pick jPhone i at the ith step. We need to find $P(\cup_{i=1}^n E_i)$. By the principle of inclusion-exclusion we have

$$P(\cup_{i=1}^n E_i) = \sum_{i=1}^{n}\sum_{i} P(E_i) - \sum_{i<j} P(E_i \cap E_j) + \sum_{i<j<k} P(E_i \cap E_j \cap E_k) - \cdots + (-1)^{n+1} P(E_i \cap E_j \cdots \cap E_n)$$

Note that $P(E_i) = 1/n$ for all i. One way to see this is by using the full sample space: there are $n!$ possible orderings of the jPhones, all equally likely, and $(n-1)!$ of these are favorable to E_i.

Similarly

$$P(E_i \cap E_j) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$

,

$$P(E_i \cap E_j \cap E_k) = \frac{(n-3)!}{n!} = \frac{1}{n(n-1)(n-2)}$$

and so on.

In the inclusion-exclusion formula, there are n terms involving one event, $\binom{n}{2}$ terms involving two events, $\binom{n}{3}$ terms involving three events, and so forth. By the symmetry of the problem, all n terms of the form $P(E_i)$ are equal, all $\binom{n}{2}$ terms of the form $P(E_i \cap E_j$ are equal. Therefore one has

$$\begin{aligned} P(\cup_{i=1}^n E_i) &= \frac{n}{n} - \frac{\binom{n}{2}}{n(n-1)} + \frac{\binom{n}{3}}{n(n-1)(n-2)} - \cdots + (-1)^{n+1}\frac{1}{n!} \\ &= 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n+1}\frac{1}{n!} \end{aligned}$$

Problem 3

a) Let 'A' represent 'the owner have an above-avg lifespan'
'R' represent 'the ring is the one ring'

Given that: $P(A|R) = 0.92$ $\quad$ $P(\bar{A}|\bar{R}) = 0.70$

$P(\bar{A}|R) = 0.08$ $\quad$ $P(A|\bar{R}) = 0.30$

$P(R) = 10^{-4}$

From Bayes theorem

$$P(R|A) = \frac{P(A|R)\,P(R)}{P(A)}$$

$$= \frac{P(A|R)\,P(R)}{P(A|R)P(R) + P(A|\bar{R})\,P(\bar{R})} \quad \left[\because P(A) = P(A|R)\,P(R) + P(A|\bar{R})P(\bar{R})\right]$$

$$= \frac{0.92 \times 10^{-4}}{0.92 \times 10^{-4} + 0.30 \times (1-10^{-4})} = \frac{0.92 \times 10^{-4}}{0.300062} \simeq 0.0003066$$

b) Let 'W' represent 'writing will appear on it'

Given that $P(W|R) = 0.95$ $\quad$ $P(W|\bar{R}) = 0.1$

$P(\bar{W}|R) = 0.05$ $\quad$ $P(\bar{W}|\bar{R}) = 0.9$

A and W are <u>conditionally</u> independent

$$P(R|WA) = \frac{P(WA|R)\,P(R)}{P(WA)} = \frac{P(WA|R)\,P(R)}{P(WA|R)\,P(R) + P(WA|\bar{R})\,P(\bar{R})}$$

Since A and W are conditionally independent

$$P(R|WA) = \frac{P(W|R)\,P(A|R)\,P(R)}{P(W|R)\,P(A|R)\,P(R) + P(W|\bar{R})\,P(A|\bar{R})\,P(\bar{R})}$$

$$= \frac{0.95 \times 0.92 \times 10^{-4}}{0.95 \times 0.92 \times 10^{-4} + 0.1 \times 0.3 \times (1-10^{-4})} \simeq 0.0029052$$

Problem 4

Way 1

First introduce a random variable I_x, where $I_x = \begin{cases} 1, & if\ X > x \\ 0, & otherwise \end{cases}$.

Now assume $X = \sum\limits_{x=0}^{\infty} I_x$. Then $E[X] = E[\sum\limits_{x=0}^{\infty} I_x] = \sum\limits_{x=0}^{\infty} E[I_x] = \sum\limits_{x=0}^{\infty} P(X > x)$

QED.

Way 2

$$\begin{aligned}
\sum_{x=0}^{\infty} Pr[X > x] &= \sum_{x=0}^{\infty} \sum_{y=x+1}^{\infty} Pr[X = y] \\
&= \sum_{x=0}^{\infty} \sum_{y=x+1}^{\infty} P_x(y) \\
&= \sum_{y=1}^{\infty} \sum_{x=0}^{y-1} Pr[X = y] \\
&= \sum_{y=1}^{\infty} P_x(y) \sum_{x=0}^{y-1} 1 \\
&= \sum_{y=1}^{\infty} P_x(y) y \\
&= E[X]
\end{aligned}$$

Problem 5

(a)

$$\begin{aligned} E[I_E] &= 1.Pr(I_E = 1) + 0.Pr(I_E = 0) \\ &= Pr(I_E = 1) \\ &= Pr(E) \end{aligned}$$

(b)

$$\begin{aligned} Var(I_E) &= E[I_E^2] - E[I_E]^2 \\ E[I_E^2] &= 1^2.Pr(I_E = 1) + 0^2.Pr(I_E = 0) \\ &= pr(I_E = 1) \\ &= Pr(E) \\ Var(I_E) &= pr(E) - pr(E)^2 \\ &= Pr(E)(1 - Pr(E) \end{aligned}$$

(c)

$$P_X = (1-p)^{x-1}.p$$

$$\begin{aligned} E[X] &= \sum_{x=1}^{\infty} x(1-p)^{x-1}.p \\ &= p.[1 + 2.(1-p) + 3.(1-p)^2 + ...] \ \textit{Say eq}(1) \\ (1-p).E[X] &= p.[0 + 1.(1-p) + 2.(1-p)^2 + 3.(1-p)^3 + ...] \ \textit{Say eq}(2) \end{aligned}$$

$let's\ take\ eq(1)\ -\ eq(2)$

$$\begin{aligned} E[X] - (1-p).E[X] &= p.[1 + (1-p) + (1-p)^2 + ...] \\ p.E[X] &= p.[\frac{1}{1-(1-p)}] \\ E[X] &= \frac{1}{p} \end{aligned}$$

(d)

To find the variance we need to find E[x^2], we will start With E[X] and will find E[X^2]

$$E[X] = p.\sum_{x=1}^{\infty} x(1-p)^{x-1}$$
$$(1-p).E[X] = p.\sum_{x=1}^{\infty} x(1-p)^{x}$$
$$\frac{1-p}{p^2} = \sum_{x=1}^{\infty} x(1-p)^{x}$$
$$\frac{d}{dp}(\frac{1-p}{p^2}) = \frac{d}{dp}(\sum_{x=1}^{\infty} x(1-p)^{x})$$
$$\frac{d}{dp}(\frac{1-p}{p^2}) = \sum_{x=1}^{\infty} \frac{d}{dp}(x(1-p)^{x})$$
$$\frac{d}{dp}(\frac{1-p}{p^2}) = -1.\sum_{x=1}^{\infty} x^2(1-p)^{x-1}$$
$$\frac{-2}{p^3} + \frac{1}{p^2} = -1.\sum_{x=1}^{\infty} x^2(1-p)^{x-1}$$
$$\frac{2-p}{p^3} = \sum_{x=1}^{\infty} x^2(1-p)^{x-1}$$
$$p.\frac{2-p}{p^3} = p.\sum_{x=1}^{\infty} x^2(1-p)^{x-1}$$
$$\frac{2-p}{p^2} = E[X^2]$$

$$\begin{aligned} Var(X) &= E[X^2] - E[X]^2 \\ &= \frac{2-p}{p^2} - \frac{1}{p^2} \\ &= \frac{1-p}{p^2} \end{aligned}$$

Problem 6

(a)

$$\begin{aligned}
\sum_{i=0}^{\infty} P_X(i) &= \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} = e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!} \\
&= e^{-\lambda}\left(1 + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots\right) \\
&= e^{-\lambda}e^{\lambda} \text{ (From Taylor series expansion of } e^{\lambda}\text{)} \\
&= 1
\end{aligned}$$

(b)

$$\begin{aligned}
E[X] &= \sum_{i=0}^{\infty} iP_X(i) = \sum_{i=0}^{\infty} i\frac{e^{-\lambda}\lambda^i}{i!} \\
&= \sum_{i=1}^{\infty} i\frac{e^{-\lambda}\lambda^i}{i!} = e^{-\lambda}\sum_{i=1}^{\infty} i\frac{\lambda^i}{i!} \\
&= \lambda e^{-\lambda}\sum_{i=1}^{\infty}\frac{\lambda^{i-1}}{(i-1)!} \\
&= \lambda e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!} \\
&= \lambda e^{-\lambda}e^{\lambda} = \lambda
\end{aligned}$$

Problem 7

(a)

Let X be a geometric random variable with parameter p. We know that, $P(X \leq x) = F(x) = 1 - (1-p)^x$ where $x \in [1, 2, 3...]$ which implies $P(X \geq x) = 1 - P(X \leq x-1) = (1-p)^{(x-1)}$

$$\begin{aligned}
Pr[X \geq a+b | X \geq a] &= \frac{Pr[X \geq a+b \text{ and } X \geq a]}{Pr[X \geq a]} \\
&= \frac{Pr[X \geq a+b]}{Pr[X \geq a]} && (\because \{X \geq a+b\} \subset \{X \geq a\}) \\
&= \frac{(1-p)^{a+b-1}}{(1-p)^{a-1}} \\
&= (1-p)^b \\
&= P(X > b)
\end{aligned}$$

(b)

$$\begin{aligned}
&P(X \geq a+b | X \geq a) = P(X > b) && \Omega = \{1, 2, 3, ...\} \\
&\frac{P(X \geq a+b \cap X \geq a)}{P(X \geq a)} = P(X > b) \\
&\frac{P(X \geq a+b)}{P(X \geq a)} = P(X > b) && (\because \{x \geq a+b\} \subset \{x \geq a\}) \\
&P(X \geq a+b) = P(X \geq a)P(X > b)
\end{aligned}$$

By substituting a=1, b= [1,2,3...],

$P(X \geq 2) = P(X \geq 1)P(X > 1) = P(X > 1)$ $\quad (\because P(X \geq 1) = 1)$

$P(X \geq 3) = P(X \geq 2)P(X > 1) = P(X > 1)^2$

$P(X \geq 4) = P(X \geq 3)P(X > 1) = P(X > 1)^3$

By induction,

$P(X \geq x) = P(X > 1)^{x-1}$

$Let P(X = 1) = p \implies P(X > 1) = 1 - p$

$\therefore P(X \geq x) = P(X > 1)^{x-1} = (1-p)^{x-1}$

Problem 8

(a)

$$\int_1^{+\infty} f_X(x) = \int_1^{+\infty} \alpha x^{-\alpha-1} dx = -x^{-\alpha}|_1^{+\infty} = 0 + 1 = 1$$

(b)

$$E[x] = \int_1^{+\infty} x \cdot \alpha \cdot x^{-\alpha-1} dx = \int_1^{+\infty} \alpha x^{-\alpha} dx = \frac{\alpha}{-\alpha+1} x^{-\alpha+1}|_1^{+\infty} = \frac{\alpha}{\alpha-1}$$

(c)

Similarly, we have

$$E[x^2] = \int_1^{+\infty} \alpha x^{-\alpha+1} dx = \frac{\alpha}{-\alpha+2} x^{-\alpha+2}|_1^{+\infty}$$

Since $1 < \alpha < 2$, then $0 < -\alpha + 2 < 1$. Thus $x^{-\alpha+2} \to +\infty$, so $E[x^2] = +\infty$ and $Var[x] = +\infty$