1 Question 1

1.1 MME for $\hat{x}$ and $\hat{y}$

First we know that, for Gamma distribution $Gamma(x, y)$

$$E[X] = xy \text{ and } Var(X) = xy^2$$

Equating the expectation and variance with the corresponding sample moments, we get:

$$E[X] = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X} = xy$$

$$E[X^2] = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

$$Var[X] = E[X^2] - (E[X])^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\frac{1}{n}\sum_{i=1}^{n} X_i)^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 = xy^2$$

For the first equation, we have

$$x = \frac{\bar{X}}{y}$$

Now, substituting it into the second equation, we get:

$$xy^2 = \bar{X}y = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

Now, solving for y in that last equation and we get that the MME for y,

$$\hat{y} = \frac{1}{n\bar{X}}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

Similarly, we could have the MME for x,

$$\hat{x} = \frac{\bar{X}}{\hat{y}} = \frac{n\bar{X}^2}{\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

1.2 MME for $\hat{a}$ and $\hat{b}$

First we know that, for Uniform distribution $Uniform(a, b)$

$$E[X] = \frac{a+b}{2} \text{ and } Var(X) = \frac{(b^2 - a^2)}{12}$$

From the first moment, we get:

$$E[X] = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X} = \frac{a+b}{2}$$

For the above equation, we have

$$a = 2\bar{X} - b$$

From the second moment we get,

$$E[X^2] = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

Using first and second moment, variance can be obtained as follows

$$Var[X] = E[X^2] - (E[X])^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\frac{1}{n}\sum_{i=1}^{n} X_i)^2 = \bar{S}^2 = \frac{(b-a)^2}{12}$$

In above statement, we can figure out the +/- based on the fact that b ¿ a in uniform distribution. Using above equation and taking square root:

$$b = \sqrt{12}\bar{S} + a$$

Now, substituting a into the above equation, we get MME for b:

$$\hat{b} = \sqrt{12}\bar{S} + 2\bar{X} - \hat{b}$$

$$2\hat{b} = 2\sqrt{3}\bar{S} + 2\bar{X}$$

$$\hat{b} = \sqrt{3}\bar{S} + \bar{X}$$

Similarly, we could have the MME for a,

$$\hat{a} = 2\bar{X} - \hat{b} = \bar{X} - \sqrt{3}\bar{S}$$

Q2. a) Given: $x_1, x_2, \ldots, x_n \sim Exp\left(\frac{1}{\beta}\right)$.

Step 1: Likelihood

$$L(\beta) = \prod_{i=1}^{n} \frac{1}{\beta} e^{-x_i/\beta}$$

$$L(\beta) = \frac{1}{\beta^n} e^{-\frac{1}{\beta}\sum_i x_i}$$

Step 2: Take log on both sides.

$$\log(L(\beta)) = \log\left(\frac{1}{\beta^n} e^{-\sum x_i/\beta}\right)$$

$$\log(L(\beta)) = -n\log(\beta) - \frac{1}{\beta}\sum_i x_i$$

Step 3: Differentiate wrt β and equate it to 0 to get MLE.

$$\Rightarrow \frac{-n}{\beta} + \beta^{-2}\sum_i x_i = 0.$$

$$\Rightarrow \hat{\beta} = \frac{\sum x_i}{n}$$

To show consistency:

Step 1: Bias$(\hat{\beta})$ tends to 0 as n tends to Infinity

$$Bias(\hat{\beta}) = E(\hat{\beta}) - \beta$$

$$E(\hat{\beta}) = E\left(\frac{\sum x_i}{n}\right)$$

Taking constant out and applying LOE

$$E(\hat{\beta}) = \frac{1}{n} n E(x_1) = E(x_1)$$

$$E(X_1) = B.$$

Thus, Bias $(\hat{B}) = 0$.

Step 2: $SE(\hat{B})$ tends to 0 as n tends to infinity.

$$se(\hat{B}) = \sqrt{var(\hat{B})}$$

$$= \sqrt{var\left(\frac{\sum_i x_i}{n}\right)}$$

$$= \sqrt{\frac{1}{n^2} var\left(\sum_i x_i\right)}.$$

As they are iid

$$\Rightarrow \sqrt{\frac{1}{n^2} n \, var(x_1)}$$

$$= \sqrt{\frac{var(x_1)}{n}} = \frac{B}{\sqrt{n}}.$$

Thus, $se(\hat{B}) = B/\sqrt{n}$ tends to 0 as n tends to ∞.

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b) $X_1 \dots X_n \overset{iid}{\sim} \text{Normal}(\mu, \sigma^2)$.

$$P_\mu(D) = \prod_{i=1}^{n} P_\mu(x_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2}$$

$$P_\mu(D) = \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

$$\ln(P_\mu(D)) = -\ln(\sqrt{2\pi}\sigma)^n = \frac{-1}{2\sigma^2}\Sigma(x_i-\mu)^2$$

$$\frac{\partial(.)}{\partial\mu} = 0 - \frac{1}{2\sigma^2} 2\Sigma(x_i-\mu)(-1) = 0.$$

$$\Sigma x_i - n\mu = 0.$$

$$\hat{\mu}_{MLE} = \frac{\Sigma x_i}{n} \; // \rightarrow \text{sample mean.}$$

$$\frac{\partial(.)}{\partial\sigma} = \frac{-n}{\sigma} - \frac{(-2)}{2\sigma^3}\Sigma(x_i-\mu)^2 = 0.$$

$$\frac{n}{\sigma} = \frac{1}{\sigma^3}\Sigma(x_i-\mu)^2.$$

$$\sigma^2 = \frac{1}{n}\Sigma(x_i-\mu)^2 \; // \rightarrow \text{uncorrected sample variance.}$$

c) Let $X_1 \dots X_n \sim N(\theta, 1)$. The MLE for θ is $\hat{\theta} = \bar{X}$. Let $\delta = E[I_{X_1} > 0]$.

Thus, $\delta = E[I_{X_1} > 0]$

$$= P(X_1 > 0)$$

$$= 1 - P(X_1 \leq 0)$$

$$= 1 - F_{X_1}(0)$$

$$= 1 - \Phi\left(\frac{0 - \mu}{\sigma}\right)$$

$$= \Phi\left(\frac{\mu}{\sigma}\right)$$

$$= \Phi(\theta)$$

So, the MLE for δ is $\Phi(\bar{X}) = \Phi\left(\frac{\sum X_i}{n}\right)$.

Q3. a)

$$f(y|\alpha) = \frac{2}{\alpha} \cdot y \cdot \exp\left(\frac{-y^2}{\alpha}\right), y>0, \alpha$$

The likelihood function of the data is the joint distribution viewed as a function of the parameter, so we have:

$$L(\alpha) = \frac{2^n}{\alpha^n} \left\{ \prod_{i=1}^{n} y_i \right\} \exp\left\{ -\frac{1}{\alpha} \sum_{i=1}^{n} y_i^2 \right\}.$$

We want to maximize this function. First, we take the log:

$$\log L(\alpha) = n\log 2 - n\log\alpha + \sum_{i=1}^{n} \log y_i - \frac{1}{\alpha}\sum_{i=1}^{n} y_i^2$$

And then the derivative:

$$\frac{d}{d\alpha} \log L(\alpha) = \frac{-n}{\alpha} + \frac{1}{\alpha^2}\sum_{i=1}^{n} y_i^2$$

$$\frac{n}{\alpha} = \frac{1}{\alpha^2}\sum_{i=1}^{n} y_i^2$$

$$\boxed{\hat{\alpha}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} y_i^2}$$

Verify that the second derivate is < 0.

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$$\frac{d^2 \log L(\alpha)}{d\alpha^2} = \frac{n}{\alpha^2} - \frac{2}{\alpha^3}\sum_{i=1}^{n} y_i^2$$

$$\frac{d^2 \log L(\alpha)}{d\alpha^2} = \frac{n}{\alpha^2} - \frac{2n}{\alpha^3}\hat{\alpha} = \frac{-n^2}{\hat{\alpha}^2} < 0.$$

Hence, $\boxed{\hat{\alpha}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} y_i^2}$

b) If $Z_i = Y_i^2$, then $y_i = \sqrt{z_i}$.

$$\frac{dy_i}{dz_i} = \frac{1}{2\sqrt{z_i}}.$$

$$f_Z(z) = \frac{2}{\alpha}\sqrt{z}\exp\left(\frac{-z}{\alpha}\right) \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{z}}$$

$f_Z(z) = \frac{1}{\alpha}\exp\left(\frac{-z}{\alpha}\right)$; which is the pdf of an exponential distribution with parameter α.

Hence,

$$E\left[\frac{1}{n}\sum_{i=1}^{n} Y_i^2\right] = E(\bar{Z}) = E(Z_1) = \alpha.$$

so that $\hat{\alpha}_{MLE}$ is unbiased for α.

Q4

(a)

$$X = \begin{cases} 2 & \text{with prob } \theta \\ 3 & \text{otherwise} \end{cases} \qquad D = \{2, 3, 2\}$$

① $K = 1$

② $\hat{\alpha}_1 = \frac{1}{n} \sum X_i$

③ $\alpha_1(\theta) = E[X(\theta)] = \sum_{x \in \Omega} x \cdot p(x)$

$$= 2 \cdot \theta + 3(1-\theta) = 3 - \theta$$

④ $\frac{1}{n} \sum X_i = 3 - \hat{\theta}$

$$\hat{\theta} = 3 - \frac{\sum X_i}{n} = 3 - \frac{(2+3+2)}{3} = \frac{2}{3}$$

b) $se(\hat{\theta}_{MME}) = ?$, $\hat{\theta}_{MME} = 3 - \bar{X}$

$\hookrightarrow = \sqrt{Var(3-\bar{X})} = \sqrt{Var(\bar{X})} = \sqrt{Var\left(\frac{1}{n}\Sigma x_i\right)}$

$\stackrel{LOV}{=} \sqrt{\frac{1}{n^2}(n)Var(X)} = \sqrt{\frac{Var(X)}{n}}$

$E(X^2) = 4(\theta) + 9(1-\theta) = 9 - 5\theta$

$E(X) = 3 - \theta$

$Var(X) = 9 - 5\theta - (3-\theta)^2 = 9 - 5\theta - (9 + \theta^2 - 6\theta) = \theta - \theta^2 = \theta(1-\theta)$

$se(\hat{\theta}_{MME}) = \sqrt{\frac{\theta(1-\theta)}{n}}$; $\hat{se}(\hat{\theta}_{MME}) = \sqrt{\frac{\hat{\theta}_{MME}(1-\hat{\theta}_{MME})}{n}}$

$\hat{se}(\hat{\theta}_{MME}) = \sqrt{\frac{2/3 \times 1/3}{3}} = \frac{1}{3}\sqrt{\frac{2}{3}} = 0.2721$

$\boxed{\hat{\theta}_{MME} \text{ is AN}}$

This applies Normal CI

Estimate of 95%ile CI-Normal $= 2/3 \pm 1.96(0.2721) = [0.1332, 1.2034]$

(4) (c).

Let $X_1, \ldots, X_n$ be i.i.d from the given distribution.

$$P(X_1, \ldots, X_n | \theta) = \prod_{i=1}^{n} P(x_i | \theta).$$

$$\text{i.e., } L(\theta) = \prod_{i=1}^{n} P(x_i | \theta)$$

We have $P(x_i | \theta) = \theta^{(3-x_i)} (1-\theta)^{(x_i-2)}$

$$\therefore \ell(\theta) = \log L(\theta)$$

$$= \sum_{i=1}^{n} \log P(x_i | \theta)$$

$$= \sum_{i=1}^{n} \log \theta^{(3-x_i)} (1-\theta)^{(x_i-2)}$$

$$= \sum_{i=1}^{n} (3-x_i) \log \theta + (x_i-2) \log (1-\theta)$$

$$= \log \theta \sum_{i} (3-x_i) + \log (1-\theta) \sum_{i} (x_i-2)$$

$$\therefore \frac{\partial \ell}{\partial \theta} = 0$$

$$\Rightarrow \frac{1}{\theta} \sum_i (3 - x_i) - \frac{1}{1-\theta} \sum_i (x_i - 2) = 0$$

$$(1-\theta) \sum_i (3 - x_i) - \theta \sum_i (x_i - 2) = 0$$

$$\sum_i (3 - x_i) - \theta \left[\sum_i (3 - x_i) + \sum_i (x_i - 2) \right] = 0$$

$$\Rightarrow \hat{\theta} = \frac{\sum_i 3 - x_i}{\sum_i 3 - x_i + \sum_i (x_i - 2)} = \frac{\sum_i 3 - x_i}{n}$$

One can verify $\left. \frac{\partial \ell^2}{\partial \theta^2} \right|_{\theta = \hat{\theta}} < 0$

For $D = \{2, 3, 2\}$

$$\hat{\theta} = \frac{(3-2) + (3-3) + (3-2)}{3}$$

$$= 2/3$$

Q5

a)

MME of $Exp(\lambda)$, $\hat{\lambda}_{MME}$

Step 0: $k = 1$

Step 1: $\hat{\alpha}_1 = \frac{1}{n} \sum_{i=1}^{n} x_i$

Step 2: $\alpha_1(\lambda) = E[exp(\lambda)] = \frac{1}{\lambda}$

Step 3: $\alpha_1(\hat{\lambda}) = \hat{\alpha}_1$

$$\frac{1}{\hat{\lambda}} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$\Rightarrow \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i}$$

$$\therefore \hat{\lambda}_{MME} = \frac{n}{\sum_{i=1}^{n} x_i}$$

b)

MLE of $Exp(\lambda)$, $\hat{\lambda}_{MLE}$

$$L(\theta) = L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$$

$$= \lambda^n e^{-\lambda \sum x_i}$$

$$l(\theta) = \log [\lambda^n e^{-\lambda \sum x_i}]$$

$$= n \log \lambda - \lambda \sum x_i$$

Now, $\frac{dl(\theta)}{d\theta} = 0$

$$\therefore n \cdot \frac{1}{\lambda} - \sum x_i = 0$$

$$\Rightarrow \lambda = \frac{n}{\sum x_i}$$

$$\therefore \hat{\lambda}_{MLE} = \frac{n}{\sum x_i}$$

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c) mme(normal):

$\hat{\mu} = 15.57$

$\hat{\sigma}^2 = 7.59$

mme(uniform):

$a = 69.61$

$b = 82.41$

mme(exp):

$\hat{\lambda} = 0.0425$

d) mle (normal):

$\hat{\mu} = 15.57$

$\hat{\sigma}^2 = 7.59$

mle (uniform):

$a = 70$

$b = 82$

mle (exp):

$\hat{\lambda} = 0.0425$

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Q6. a) $X \sim Bi(500, p)$

$H_0 : \mu = 0.7.$
$H_A : \mu > 0.7.$

b) Pr(Type I error) = $P(X \geq 375 | H_0)$.

$$\approx P\left(\frac{X - 500 \times 0.7}{\sqrt{500 \times 0.7 \times 0.3}}\right) \geq \left(\frac{375 - 500 \times 0.7}{\sqrt{500 \times 0.7 \times 0.3}} \middle| H_0\right)$$

$$\approx P(z > 2.4398)$$
$$= 0.073$$

c) Power = $P(X > 375 | c = 0.8)$.

$$\approx P\left(z \geq \frac{375 - 500 \times 0.8}{\sqrt{500 \times 0.8 \times 0.2}}\right)$$

$$= P(z \geq -2.80)$$
$$= 0.9974.$$

Q7.

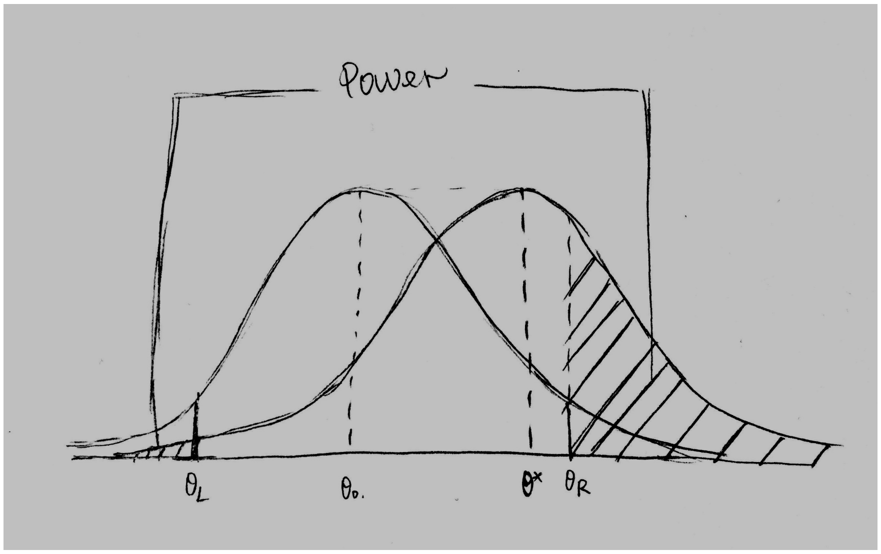
(a)

Based on the hypothesis, we know that this is a 2-sided test and the type II Error is failing to reject H_0 even when H_1 is true. Given

- $H_0 : \theta = \theta_0$
- $H_1 : \theta \neq \theta_0$

we know that we reject H_0 when $|W| > z_{\alpha/2}$, which means that

- Case 1: $W > z_{\alpha/2} \Rightarrow \frac{\theta-\theta_0}{\hat{se}} > z_{\alpha/2} \Rightarrow \theta > \theta_0 + \hat{se} \cdot z_{\alpha/2} = \theta_R$
- Case 2: $W < -z_{\alpha/2} \Rightarrow \frac{\theta-\theta_0}{\hat{se}} < -z_{\alpha/2} \Rightarrow \theta < \theta_0 - \hat{se} \cdot z_{\alpha/2} = \theta_L$



Now, if we know that the true value of θ is θ^* Therefore,

$$
\begin{aligned}
Power &= P[\text{case 1}|\text{true mean is } \theta^*] + P[\text{case 2}|\text{true mean is } \theta^*] \\
&= P(\theta > \theta_R|\theta = \theta^*) + P(\theta < \theta_L|\theta = \theta^*) \\
&= P(\theta > \theta_0 + \hat{se} \cdot z_{\alpha/2}|\theta = \theta^*) + P(\theta < \theta_0 - \hat{se} \cdot z_{\alpha/2}|\theta = \theta^*) \\
&= P(W > \frac{\theta_0 + \hat{se} \cdot z_{\alpha/2} - \theta^*}{\hat{se}}) + P(W < \frac{\theta_0 - \hat{se} \cdot z_{\alpha/2} - \theta^*}{\hat{se}}) \\
&= P(W > \frac{\theta_0 - \theta^*}{\hat{se}} + z_{\alpha/2}) + P(W < \frac{\theta_0 - \theta^*}{\hat{se}} - z_{\alpha/2}) \\
&= 1 - P(W < \frac{\theta_0 - \theta^*}{\hat{se}} + z_{\alpha/2}) + P(W < \frac{\theta_0 - \theta^*}{\hat{se}} - z_{\alpha/2}) \\
&= 1 - \Phi(\frac{\theta_0 - \theta^*}{se} + z_{\alpha/2}) + \Phi(\frac{\theta_0 - \theta^*}{se} - z_{\alpha/2})
\end{aligned}
$$

Thus,

$$
P[\text{Type II error}] = 1 - Power = \Phi(\frac{\theta_0 - \theta^*}{se} + z_{\alpha/2}) - \Phi(\frac{\theta_0 - \theta^*}{se} - z_{\alpha/2})
$$

Given experiment is Bernoulli Distribution, then $p_0 = 0.5$

(b). Given - $H_0 : p = p_0$, $H_1 : p \neq p_0$

we have 46 success

$$\Rightarrow \quad \hat{p} = \bar{X} = \frac{\Sigma X_i}{n} = \frac{46}{100} = 0.46$$

wald's test $\rightarrow$ $$W = \frac{\hat{p} - p_0}{\hat{se}} = \frac{\hat{p} - p_0}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}$$

$$W = \frac{0.46 - 0.5}{\sqrt{\frac{0.46(1-0.46)}{100}}} = -0.803$$

$$|W| = 0.803$$

$\alpha = 0.05$ $\quad Z_{\alpha/2} = Z_{0.025} = 1.96$

$|W| < Z_{\alpha/2}$ $\Rightarrow$ we accept the hypothesis.

p-value $= 2(1 - \Phi(|W|)) = 2\Phi(-|W|) = 2\Phi(-0.803)$
$= 0.42371$

Now, if $p = 0.7$, So $p_0 = 0.7$

$$W = \frac{0.46 - 0.7}{\sqrt{\frac{0.46(1-0.46)}{100}}} = -4.819$$

$\therefore |W| = 4.819$

Now, $|W| > Z_{\alpha/2}$ $\Rightarrow$ we reject the hypothesis

p-value $= 2\Phi(-|W|) = 2\Phi(-4.819)$
< 0.00001, i.e. we get a very small value from the p-value table.

8)

a) $\hat{\theta} = 0.541$

$\hat{\sigma} = 0.103$

$$W = \frac{0.041}{0.103} = 0.398 \leq Z_{\alpha/2} = 2.326$$

Since, the constraint satisfied, $\theta_0 = 0.5$ is the true mean

b) $W = \left| \frac{\hat{\theta} - 0}{\hat{se}(\hat{\theta})} \right|$

$\hat{\theta} = \bar{X} - \bar{Y}$, where X and Y data samples are normally distributed and are independent as well making the test applicable.

$$= \left| \frac{\bar{X} - \bar{Y}}{\hat{se}(\bar{X} - \bar{Y})} \right| = \frac{|\bar{X} - \bar{Y}|}{\sqrt{\frac{Var(X_i)}{750} + \frac{Var(Y_i)}{750}}}$$

$$= \frac{|5.0048 - 5.846|}{\sqrt{\frac{2.31}{750} + \frac{6.47}{750}}} = \frac{0.842}{\sqrt{0.0118}}$$

$$= \frac{0.842}{0.109} = 7.725$$

$|W| > Z_{\alpha/2}$ for $\alpha = 0.05$, hence we reject the null hypothesis, H_0.