CSE 544.01 Probability and Statistics for Data Scientists

Assignment - 4

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Assignment - 4

1 a) Given Gamma (x, y) distribution has

$\text{mean} = xy$ $\quad$ $\text{Variance} = xy^2$

$\hat{x}_{MME} = ?$ $\quad$ $\hat{y}_{MME} = ?$

Let $D = \{X_1, X_2 \cdots X_n\}$ be data sample

$K = 2$

Sample moments

$$\hat{\alpha}_1 = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$$

$$\hat{\alpha}_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

Theoretical based moments

$$\alpha_1 = E[X] = xy$$

$$\alpha_2 = E[X^2] = \text{Variance} + (E[X])^2$$
$$= xy^2 + x^2y^2$$
$$= xy^2(1+x)$$

Now Equating Theoretical ~~based~~ & Sample moments we get

$$\Rightarrow xy = \bar{X}$$

$$x = \frac{\bar{X}}{y} \quad — (1)$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i^2 = xy^2(1+x) \quad — (2)$$

Subotitute eq (1) in (2)

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \frac{\bar{X}}{y} y^2 \left(1 + \frac{\bar{X}}{y}\right)$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 = \bar{X}y\left(\frac{y+\bar{X}}{y}\right)$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 = \bar{X}y + \bar{X}^2$$

Now solving, the above equation, we get that the MME for y is

$$\hat{y} = \frac{1}{\bar{X}}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i)^2 - (\bar{X})^2\right)$$

From equation ① we could have MME for α

$$\hat{\alpha} = \frac{\overline{X}}{\hat{y}} = \frac{\overline{X}^2}{\frac{1}{n}\sum_{i=1}^{n}(X_i)^2 - (\overline{X})^2}$$

$$\therefore \left[y_{MME} = \frac{1}{\overline{X}}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i)^2 - (\overline{X})^2\right)\right], \quad \left[X_{MME} = \frac{(\overline{X})^2}{\frac{1}{n}\sum_{i=1}^{n}(X_i)^2 - (\overline{X})^2}\right]$$

1(b) Given Uniform distribution (a,b)

We know for Uniform distribution

$E[X] = \frac{a+b}{2}$ Variance $[X] = \left(\frac{b^2-a^2}{12}\right)$

Given Sample Variance $\bar{s}^2 = \frac{1}{n} \sum x_i^2 - \bar{x}^2$

$\underline{K=2}$

Sample based

$\hat{\alpha}_1 = \frac{\sum x_i}{n} = \bar{x}$

$\hat{\alpha}_2 = \frac{\sum x_i^2}{n}$

Theoretical based moments

$\alpha_1 = \frac{a+b}{2}$

$\alpha_2 = E[x^2] = \text{Variance} + (E[X])^2$

$= \bar{s}^2 + \bar{x}^2$

$\bar{s}^2 = \frac{(b-a)^2}{12}$

$12 s^2 = (b-a)^2$

$\sqrt{12}\, \bar{s} = b-a$

$b = \sqrt{12}\, s - a \quad -①$

Equating Sample & Theoretical moments

$\bar{x} = \frac{a+b}{2}$

$a = 2\bar{x} - b \quad -②$

Now, substitution eq ② ~~V~~value in ①, we get MME for b ④

$$\hat{b} = \sqrt{12}\,\bar{s} + (2\bar{x} - \hat{b})$$

$$2\hat{b} = 2\sqrt{3}\,\bar{s} + 2\bar{x}$$

$$\hat{b} = \sqrt{3}\,\bar{s} + \bar{x}$$

Similarly, MME for a.

$$\hat{a} = 2\bar{x} - \hat{b} = \bar{x} - \sqrt{3}\,\bar{s}$$

$$\therefore \hat{a}_{MME} = \bar{x} - \sqrt{3}\,\bar{s}, \quad \hat{b}_{MME} = \bar{x} + \sqrt{3}\,\bar{s}$$

② Consistency of MLE

a) $D = \{x_1, x_2, \dots x_n\} \overset{iid}{\sim} Exp(1/\beta)$

MLE $(\hat{\beta}) = ?$

[MLE tries to find parameter estimates $(\vec{\theta})$ that ~~maxim~~ match Observation. MLE is an estimate that maximizes $Pr(D)$]

$$Pr(D) = Pr(\{x_1 \dots x_n\}) \overset{\perp}{=} Pr(x_1) \cdot Pr(x_2) \cdots = \prod_{i=0}^{n} Pr(x_i)$$

We know that $Pr(x)$ for exponential distribution is $\lambda e^{-\lambda x}$

$$Pr(D) = \prod_{i=1}^{n} \lambda e^{-\lambda x}$$

Given Value of λ in question $= 1/\beta$

$$Pr(D) = \prod_{i=1}^{n} \frac{1}{\beta} e^{-x_i/\beta}$$

Likelihood

$$\mathcal{L}(\beta) = \prod_{i=1}^{n} \frac{1}{\beta} e^{-\frac{1}{\beta}x_i}$$

$$= \frac{1}{\beta^n} e^{-\frac{1}{\beta}\sum_i x_i}$$

Take log on both sides

$$\log(\mathcal{L}(\beta)) = \log\left(\frac{1}{\beta^n} e^{-\frac{1}{\beta}\sum_{i=1}^{n} x_i}\right)$$

$$\log(\mathcal{L}(\beta)) = -n\log(\beta) - \frac{1}{\beta}\sum_{i=1}^{n} x_i$$

Differentiate it w.r.t β to maximize the function

For maximizing $\frac{d(\log(\mathcal{L}(\beta)))}{d\beta} = 0$

$$\Rightarrow 0 = \frac{d}{d\beta}\left(-n\log(\beta) - \frac{1}{\beta}\sum_{i=1}^{n} x_i\right)$$

$$0 = -\frac{n}{\beta} + \beta^{-2}\sum_{i=1}^{n} x_i$$

$$\beta = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\Rightarrow \hat{\beta} = \frac{\sum_{i=1}^{n} x_i}{n}$$

To show Consistency, prove $bias(\hat{\beta})$ & $se(\hat{\beta})$ tends to 0 as n tends to ∞.

$\left[\begin{array}{l} \text{Given mean} = 1/\lambda \quad \lambda = 1/\beta \\ \qquad\qquad\qquad\quad = \beta \end{array}\right]$

$$Bias(\hat{\beta}) = E[\hat{\beta}] - \beta$$

$$E[\hat{\beta}] = E\left[\frac{\sum_{i=1}^{n} X_i}{n}\right]$$

$$\overset{LOE}{=} \frac{1}{n} n E[X_1] = E[X_1] = \beta$$

$$\therefore Bias(\hat{\beta}) = \beta - \beta = 0$$

Thus $\boxed{Bias(\hat{\beta}) = 0}$

$\left[\text{Given Var} = 1/\lambda^2 = \frac{1}{(1/\beta)^2} = \beta^2\right]$

$$SE(\hat{\beta}) = \sqrt{Var(\hat{\beta})} = \sqrt{Var\left(\frac{\sum_{i=1}^{n} X_i}{n}\right)}$$

$$= \sqrt{\frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right)} \overset{iid}{=} \sqrt{\frac{1}{n^2} n Var(X_1)}$$

$$\left[\therefore Se(\hat{\beta}) = \frac{\beta}{\sqrt{n}}\right]$$

Thus as n tend to ∞ Se tends to 0

$\therefore$ As bias & Se tends to zero as $n \to \infty$

So MLE $(\hat{\beta})$ converges to β as $n \to \infty$

2(b) $D = \{X_1, X_2, \dots X_n\} \overset{iid}{\sim} Normal(\mu, \sigma^2)$

$\hat{\mu}_{MLE} = ?$ $\quad \sigma^2_{MLE} = ?$

$$L(\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(X_i - \mu)^2}{2\sigma^2}}$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\sum_{i=1}^{n} \frac{(X_i - \mu)^2}{2\sigma^2}}$$

Applying log on both sides for log likelihood

$$l(\mu, \sigma) = -n\log(\sigma) - n\log(\sqrt{2\pi}) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 \quad \text{—} ①$$

As the above function is a function of two Variables we do partial derivation of ① w.r.t μ & equating it to 0

$$\frac{d(l(\mu,\sigma))}{d\mu} = 0 = \frac{-1}{2\sigma^2} \times 2 \sum_{i=1}^{n}(x_i-\mu)(-1)$$

$$0 = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)$$

$$\Rightarrow \sum_{i=1}^{n}(x_i-\mu) = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i - n\mu = 0$$

$$\sum_{i=1}^{n} x_i = n\mu$$

$$\therefore \hat{\mu}_{MLE} = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$$

Taking partial derivate of ① w.r.t σ & equating it to 0

$$\frac{d(l(\mu,\sigma))}{d\sigma} = 0 = \frac{-n}{\sigma} + \sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\sigma^3}$$

$$\Rightarrow \sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\sigma^3} - \frac{n}{\sigma} = 0$$

$$\Rightarrow \sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\sigma^{\cancel{3}2}} = \frac{n}{\cancel{\sigma}}$$

$$\therefore \hat{\sigma}^2_{MLE} = \frac{\sum_{i=1}^{n} (x_i - \mu)^2}{n}$$

↳ It is unncorrected Sample Variance.

As we can see from the above obtained MLE of μ & σ^2 is the same as the sample mean & sample variance respectively.

2) (c) Let $X_1, X_2, \dots X_n \sim Nor(\theta, 1)$ $\quad \mu = 0 \quad \sigma^2 = 1$

MLE for θ is $\hat{\theta} = \bar{X}$

Let $\delta = E[I_{X_1 > 0}]$ Thus

$$\delta = E[I_{X_1 > 0}]$$
$$= P(X_1 > 0)$$
$$= 1 - P(X_1 \leq 0)$$
$$= 1 - F_{X_1}(0)$$
$$= 1 - \Phi\left(\frac{0-\mu}{\sigma}\right)$$

(Given Φ is CDF of normal distribution)

$$= \Phi\left(\frac{\mu}{\sigma}\right)$$
$$= \Phi\left(\frac{\theta}{1}\right)$$
$$= \Phi(\theta)$$

$$\therefore \text{MLE of } \delta = \Phi(\hat{\theta}_{MLE}) = \underline{\underline{\Phi\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)}}$$

③ Practise With MLE

9) $\hat{\alpha}_{MLE} = ?$

$$f(y|\alpha) = (2/\alpha) * y * (e^{-y^2/\alpha}), \quad y>0,\ \alpha>0$$

$$L(y|\alpha) = \prod_{i=1}^{n} (2/\alpha)\, y e^{-y^2/\alpha}$$

$$= \frac{2^n}{\alpha^n} \prod_{i=1}^{n} y_i e^{-y_i^2/\alpha}$$

$$= \frac{2^n}{\alpha^n} e^{-\frac{\sum_{i=1}^{n} y_i^2}{\alpha}} \prod_{i=1}^{n} y_i$$

We want to maximize this function. Take log likelihood

$$\log(L(y|\alpha)) = \log\left(\frac{2^n}{\alpha^n} e^{-\frac{\sum_{i=1}^{n} y_i^2}{\alpha}} \prod_{i=1}^{n} y_i\right)$$

$$= n\log 2 - n\log\alpha + \left(\frac{-1}{\alpha}\sum_{i=1}^{n} y_i^2\right) + \sum_{i=1}^{n} \log y_i$$

Take derivative w.r.t α

$$\frac{d}{d\alpha}\log(L(y|\alpha)) = \frac{d}{d\alpha}\left(n\log 2 - n\log\alpha + \left(\frac{-1}{\alpha}\sum_{i=1}^{n} y_i^2\right) + \sum_{i=1}^{n}\log y_i\right)$$

Setting this equal to 0

$$0 = 0 - \frac{n}{\alpha} + \frac{1}{\alpha^2}\cdot\sum_{i=1}^{n} y_i^2 + 0$$

$$\Rightarrow \frac{n}{\cancel{\alpha}} = \frac{1}{\alpha^2}\sum_{i=1}^{n} y_i^2$$

$$\hat{\alpha} = \frac{1}{n}\sum_{i=1}^{n} y_i^2$$

MLE for α is
$$\hat{\alpha}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} y_i^2$$

(b) Let $Z_1 = Y_1^2$

Distribution for $Z_1 = ?$

For positive z, CDF of Z
$$F_Z(z) = P(Z \le z) = P(Y^2 \le z)$$
$$\Rightarrow P(Y \le \sqrt{z}) = F_Y(\sqrt{z})$$

Take derivative to find the pdf of Z
$$\frac{d}{dz} F_Z(z) = \frac{d}{dz}(P(Z \le z)) = \frac{d}{dz} F_Y(\sqrt{z})$$
$$f_Z(z) = f_Y(\sqrt{z}) \frac{d}{dz}\sqrt{z}$$
$$= f_Y(\sqrt{z}) \frac{1}{2\sqrt{z}}$$

$$\therefore f_Z(z) = f_Y(\sqrt{z}) \frac{1}{2\sqrt{z}}$$
$$= \frac{\cancel{2}}{\alpha}\cancel{\sqrt{z}}\, e^{-\frac{(\sqrt{z})^2}{\alpha}} \cdot \frac{1}{\cancel{2}\cancel{\sqrt{z}}}$$
$$f_Z(z) = \frac{1}{\alpha} e^{-\frac{z}{\alpha}} \quad -①$$

eq ① is the pdf of exponential distribution with parameter $\frac{1}{\alpha}$

[exp. distribution $\lambda e^{-\lambda x}$
w.k.T $E[X] = \frac{1}{\lambda}$]

$$f_Z(z) = \frac{1}{\alpha} e^{-\frac{z}{\alpha}}$$

W.K.T

$$\text{bias}(\hat{\alpha}) = E[\hat{\alpha}] - \alpha$$

From 3(a)

$$E[\hat{\alpha}] = E\Big[\underbrace{\frac{1}{n}\sum_{i=1}^{n} Y_i^2}_{\text{mean of } Z}\Big]$$

$$= E[\bar{Z}] = \bar{Z}$$

W.K.T $\quad f_Z(z) = \frac{1}{\alpha} e^{-z/\alpha}$

i.e Exponential $(1/\alpha)$ $\quad$ [$\because$ $E[X] = 1/\lambda$ for exp distribution]

$$\Rightarrow \bar{Z} = \frac{1}{1/\alpha} = \alpha$$

$$\therefore E[\hat{\alpha}] = \alpha$$

$$\Rightarrow \cancel{\text{bias}}\ E[\hat{\alpha}] - \alpha = \alpha - \alpha = 0$$

$$\therefore \text{bias}(\hat{\alpha}) = 0$$

$\therefore$ MLE for α an unbiased estimator of α.

Hence proved.

4. Parametric Inference With Data Samples

(a) Given $X = \begin{cases} 2 & \text{with prob } \theta \\ 3 & \text{otherwise} \end{cases}$ $\quad D = \{2, 3, 2\}$

$\hat{\theta}_{MME} = ?$

Step ① No. of unknown's $K = 1$

② $\hat{\alpha}_1 = \frac{1}{n} \sum X_i$ → estimated sample mean — ①

③ $\alpha(\theta) = E[X(\theta)] = \sum_{x \in \Omega} x \cdot P(x)$ — ②

$= 2 \cdot \theta + 3(1-\theta) = 3 - \theta$

④ Equating sample moment with Theoretical based moments

① = ②

$\frac{1}{n} \sum X_i = 3 - \hat{\theta}$

$\hat{\theta} = 3 - \frac{1}{n} \sum X_i = 3 - \frac{(2+3+2)}{3} = \frac{2}{3}$

$\therefore \hat{\theta}_{MME} = 2/3$

4(b) $\hat{se}(\hat{\theta}_{MME}) = ?$

$Se(\hat{\theta}) = \sqrt{Var(\hat{\theta})}$

$= \sqrt{Var(3 - \frac{1}{n} \sum X_i)} \underset{COV}{=} \sqrt{Var(3) + Var(\frac{\sum X_i}{n})}$

$= \sqrt{0 + Var(\frac{\sum X_i}{n})} \quad \{ \because Var(3) = 0 \}$

$\underset{COV}{\overset{iid}{=}} \sqrt{\frac{1}{n^2} Var(\sum X_i)} = \sqrt{\frac{1}{n^2} \cdot n \; var(X_i)}$

$$= \sqrt{\frac{Var(X_1)}{n}}$$

$E[X] = 3-\theta,$ $\quad E[X^2] = 2^2 \cdot \theta + 3^2(1-\theta)$

$$= 4\theta + 9 - 9\theta$$

$$= 9 - 5\theta$$

$$Var[X] = (9-5\theta) - (3-\theta)^2 = 9 - 5\theta - (9 + \theta^2 - 6\theta)$$

$$= \theta(1-\theta)$$

$$\therefore se(\hat{\theta}_{mme}) = \sqrt{\frac{\theta(1-\theta)}{n}}$$

$$\hat{se}(\hat{\theta}_{mme}) = \sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}} = \sqrt{\frac{\frac{2}{3} \cdot \frac{1}{3}}{3}} = \sqrt{\frac{2}{27}}$$

$$\hat{se}(\hat{\theta}_{mme}) = \frac{1}{3}\sqrt{\frac{2}{3}}$$

Now Normal based CI can be applied

$$\hat{\theta}_n \simeq Normal(\theta, se^2)$$

CI is $(\hat{\theta} - z_{\alpha/2} \cdot se, \; \hat{\theta} + z_{\alpha/2}\, se)$

$1-\alpha = 0.95 \Rightarrow \alpha = 0.05$

$$\Rightarrow z_{\alpha/2} = z_{0.025} = 1.96$$

$$95\% \; CI = \left(\frac{2}{3} - 1.96 \times \frac{1}{3}\sqrt{\frac{2}{3}}, \; \frac{2}{3} + 1.96 * \frac{1}{3}\sqrt{\frac{2}{3}}\right)$$

$$= (0.13322, \; 1.20011)$$

$$= (0.1332, \; 1.200)$$

4) $\hat{\theta}_{MLE}$
c)

Let $x_1, \dots x_n$ be i.i.d from the given distribution

$$P(x_1, \dots, x_n \mid \theta) = \prod_{i=1}^{n} P(x_i \mid \theta)$$

$$L(\theta) = \prod_{i=1}^{n} P(x_i \mid \theta)$$

To derive continuous form of $P_x(x_i)$

When $x=2$ then $ax+b=1$ — ①

When $x=3$ then $ax+b=0$ — ②

$2a+b=1$ sub $x=2$

$3a+b=0$ sub $x=3$

$-a=-1$

$a=1$ $b=3$

4) c) $\hat{\theta}^{n}_{MLE} = ?$ $D = \{2,3,2\}$

$$L(\theta) = \prod_{i=1}^{n} P_x(x_i)$$

We need to find $P_x(x_i)$ function that satisfies the probability distribution.

$$P_x(x) = \theta^{3-x}(1-\theta)^{x-2}$$

$$L(\theta) = \prod_{i=1}^{n} (1-\theta)^{x_i-2}\,\theta^{3-x_i}$$

$$= (\theta^{3-x_1})(1-\theta)^{x_1-2} \cdot \theta^{3-x_2} \cdot (1-\theta)^{x_2-3} \dots$$

$$= (1-\theta)^{\sum x_i - 2n}\,\theta^{3n-\sum x_i}$$

$$= (1-\theta)^{(2+3+2)-2(3)}\,\theta^{3(3)-(2+3+2)}$$

$$= (1-\theta)\,\theta^{2}$$

$\theta^{ax+b}(1-\theta)^{1-(ax+b)}$

$x=2$ then $ax+b=1$

$x=3$ then $ax+b=0$

$2a+b=1$

$3a+b=0$

$-a=+1$

$[a=-1]$ $[b=3]$

Differentiate it w.r.t θ to maximize the function

$$\frac{d(L(\theta))}{d\theta} = 0 = \frac{d}{d\theta}\left(\theta^{2}(1-\theta)\right)$$

$$0 = \frac{d}{d\theta}(\theta^2 - \theta^3)$$

$$0 = 2\theta - 3\theta^2$$

$$2\theta = 3\theta^2$$

$$\therefore \hat{\theta} = \frac{2}{3}$$

$$\boxed{\therefore \hat{\theta}_{MLE} = \frac{2}{3}}$$

5. MME Versus MLE using real data

(a) $\text{EXP}(\lambda)$, $\hat{\lambda}_{MME} = ?$ $(k=1)$

Sample moments, Theoretical

$\hat{\alpha}_1 = \frac{\Sigma x_i}{n}$, $\alpha_1 = \frac{1}{\lambda}$

[For exponential distribution $E[X] = \frac{1}{\lambda}$]

$\cancel{\hat{\alpha}_1(\hat{\lambda})}$ $\alpha_1(\hat{\lambda}) = \hat{\alpha}_1$

$\frac{1}{\lambda} = \frac{\Sigma x_i}{n}$

$\Rightarrow \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i}$

$\therefore \hat{\lambda}_{MME} = \frac{n}{\sum_{i=1}^{n} x_i}$

(b) MLE of $\text{Exp}(\lambda)$, $\hat{\lambda}_{MLE} = ?$

$L(\lambda) = \prod P_x(x_i)$

$P_x(x_i) = \lambda e^{-\lambda x_i}$

$\Rightarrow L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$

$= \lambda^n e^{-\lambda \Sigma x_i}$

Take log on both sides for log likelihood

$\mathcal{L}(\lambda) = \log(L(\lambda)) = \log(\lambda^n e^{-\lambda \Sigma x_i})$

$= n \log \lambda - \lambda \Sigma x_i$

Apply derivative

$\frac{d(\mathcal{L}(\lambda))}{d\lambda} = \frac{d}{d\lambda}(n \log \lambda - \lambda \Sigma x_i)$

$0 = \frac{n}{\lambda} - \Sigma x_i$

$$\Rightarrow \quad \frac{n}{\lambda} = \Sigma x_i$$

$$\hat{\lambda} = \frac{n}{\Sigma x_i}$$

$$\therefore \quad \hat{\lambda}_{MLE} = \frac{n}{\Sigma x_i}$$

6.) a.) H_0 (Null hypothesis): The new drug has no effect.

H_1 (Alternative hypothesis): The new drug increases the probability of eliminating virus from an affected individual.

b.) Type-I Error:- is the Pr (test rejects H_0 | H_0 is true) ($\because$ from definition)

from the problem above Type-I Error:- Pr (New drug removes/eliminates virus from an affected individual | New drug has no effect on virus)

6)

c) Power = Pr(Reject Ho when Ho is false)

Given success rate is 0.8

=> Ho is false (Ground Truth)

=> Power = Pr (Ho is rejected)

= Pr (# patients cured ≥ 375)

= Pr(X = 375) + Pr (X = 376) + - - - Pr(X = 500)

$$= \sum_{i=375}^{500} Pr(X = i)$$

where X = # patients cured

X ~ Bernoulli (0.8)

$$\Rightarrow \text{power} = \sum_{i=375}^{500} {}^{500}C_i \, (0.8)^i \, (0.2)^{500-i}$$

power = 0.997.

7 a) Given the null hypothesis $H_0: \theta = \theta_0$

Given the true Value of θ is θ^*

Truth Table

Test Output →

	Accepts H_0	Reject H_0
H_0 True	True -ve	False +ve → Type-I
H_0 False	False -ve (Type II)	True +ve

$Pr(\text{Type II}) = pr(\text{Accepts } H_0 / H_0 \text{ false})$

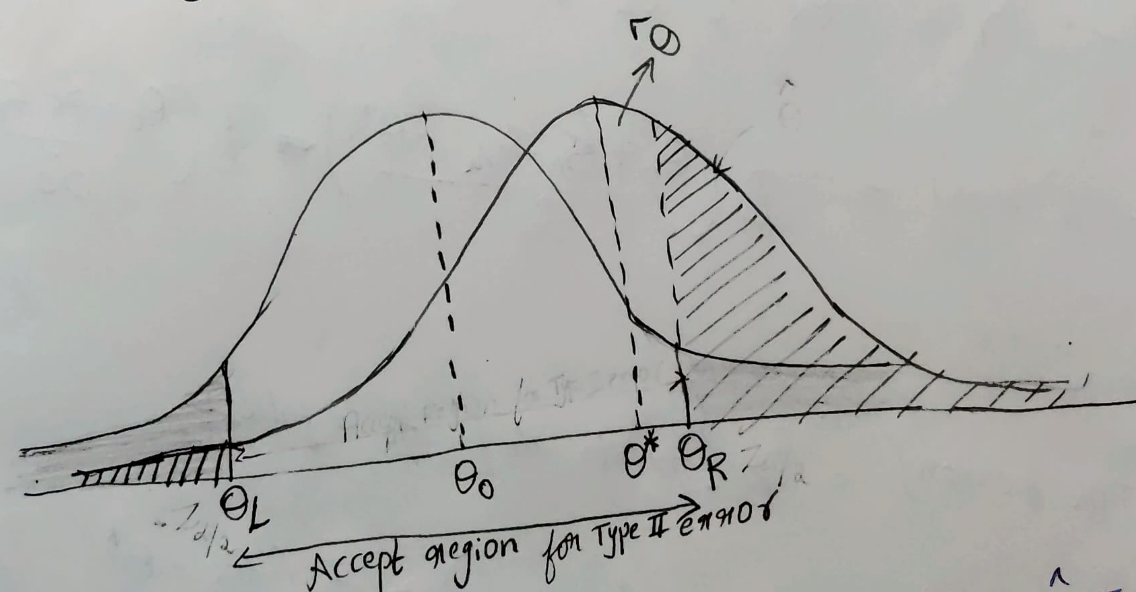
Now as the truth value is of H_0 is false, but we accepts the test.

$\therefore$ To accept the test

$$|W| = \left| \frac{\hat{\theta} - \theta_0}{\hat{se}(\hat{\theta})} \right| \leq Z_{\alpha/2}$$

We Know that we reject H_0 when $|W| > Z_{\alpha/2}$

$* \; W > Z_{\alpha/2} \Rightarrow \frac{\hat{\theta} - \theta_0}{\hat{se}} > Z_{\alpha/2}$ | $W < -Z_{\alpha/2} \Rightarrow \frac{\hat{\theta} - \theta_0}{\hat{se}}$



$\theta_R = \theta_0 + \hat{se}\, Z_{\alpha/2}$

$\theta_L = \theta_0 - \hat{se}\, Z_{\alpha/2}$

$\because H_0$ is false $\Rightarrow \theta \neq \theta_0$

Given true Value $\theta = \theta^*$

$$\Pr(\text{Type-II error}) = \Pr(\theta_L \le \hat{\theta} \le \theta_R)$$

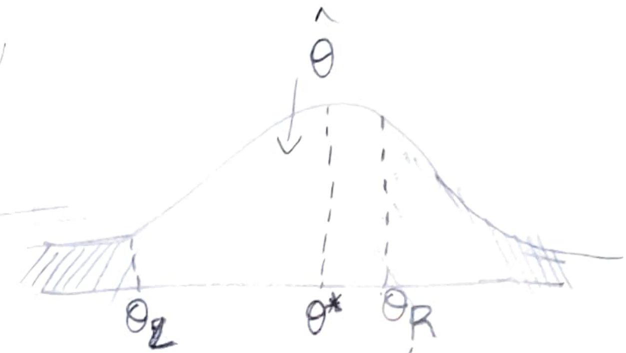
$$\Pr(\theta_L \le \hat{\theta} \le \theta_R)$$

$$\Rightarrow \Pr(\hat{\theta} < \theta_R) - \Pr(\hat{\theta} < \theta_L)$$

$$\Rightarrow \Pr\left(\frac{\hat{\theta} - \theta^*}{\hat{se}} < \frac{\theta_R - \theta^*}{\hat{se}}\right) - \Pr\left(\frac{\hat{\theta} - \theta^*}{\hat{se}} < \frac{\theta_L - \theta^*}{\hat{se}}\right)$$

$\left(\frac{\hat{\theta} - \theta^*}{\hat{se}}\right.$ follows standard Normal distribution $\left.\sim N(0,1)\right]$

$$\Rightarrow \Phi\left(\frac{\theta_R - \theta^*}{\hat{se}}\right) - \Phi\left(\frac{\theta_L - \theta^*}{\hat{se}}\right)$$



$\hat{\theta}$ is AN. It follows Normal distribution

$$\hat{\theta} \sim Nor(\theta^*, se^2)$$

$$\frac{\hat{\theta} - \theta^*}{\hat{se}} \sim N(0,1)$$

Substitute θ_R & θ_L Values

$$\therefore \Pr[\text{Type II error}] = \Phi\left(\frac{\theta_0 + \hat{se}\, Z_{\alpha/2} - \theta^*}{\hat{se}}\right) - \Phi\left(\frac{\theta_0 - se\, Z_{\alpha/2} - \theta^*}{\hat{se}}\right)$$

$$\therefore \Pr[\text{Type II error}] = \Pr(\text{Accepts } H_0 \mid H_0 \text{ is false})$$

$$= \Phi\left(\frac{\theta_0 - \theta^*}{\hat{se}} + Z_{\alpha/2}\right) - \Phi\left(\frac{\theta_0 - \theta^*}{se} - Z_{\alpha/2}\right)$$

7) b) Check whether coin is unbaised or not

i) $H_0 : P = 0.5$ Vs $H_1 : P \neq 0.5$ [Considering MLE as an estimator]

Bernoulli trail

$$\hat{P}_{MLE} = \bar{X} = \frac{\Sigma X_i}{n} = \frac{46}{100} = 0.46$$

$\because$ MLE is AN, we can apply Wald's test

$$W = \frac{\hat{\theta} - \theta_0}{se(\hat{\theta})} = \frac{\frac{46}{100} - 0.5}{se(\hat{\theta})}$$

$|W| > Z_{\alpha/2} \Rightarrow |W| > 1.96$ reject H_0

Consider $se(\hat{\theta}) = \sqrt{Var(\hat{\theta})} = \sqrt{Var\left(\frac{\Sigma X_i}{n}\right)} \underset{\text{iid}}{\overset{\text{cov}}{=}} \sqrt{\frac{Var(X_i)}{n}} = \sqrt{\frac{P(1-P)}{n}}$

It is consistent by equivariance

$$\hat{se}(\hat{\theta}) = \sqrt{\frac{\hat{P}_{MLE}(1-\hat{P}_{MLE})}{100}} = \sqrt{\frac{(0.46)(1-0.46)}{100}} = 0.049839$$

$$\therefore |W| = \left|\frac{0.46 - 0.5}{0.049839}\right| = |-0.8025| = 0.8025$$

If $P = 0.5$

[$\therefore |W| < 1.96$ Accept H_0 (coin is unbiased)]

ii) If $H_0 : P = 0.7$ $H_1 : P \neq 0.7$

then $|W| = \left|\frac{\overset{\hat{P}_{MLE}}{0.46} - 0.7}{0.049839}\right| = \left|\frac{-0.24}{0.049839}\right| = |-4.815|$

$= -4.815$

[Here $|W| > 1.96$, reject H_0 (coin is biased)]