CSE 544.01 Probability and Statistics for Data Scientists

Assignment - 6

Team Members

Akhil Arradi - 114353508

Sai Vikas Balabadhrapatruni - 114777850

Varshith Adavala - 114778433

Akhila Juturu - 114777498

1)

a) Posterior $(\theta) \propto f(D/\theta) \cdot$ prior

$$\propto \Pi(f(x_i/\theta))\, f(\theta)$$

$$\propto f(x_1/\theta) . f(x_2/\theta) \ldots \ldots f(x_n/\theta) . f(\theta)$$

$$\propto \prod_{i=1}^{n} \left(\frac{e^{\frac{-(x_i-\theta)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} \right) \cdot \frac{e^{\frac{-(\theta-a)^2}{2b^2}}}{b\sqrt{2\pi}}$$

$$\propto \frac{e^{\frac{-(x_1-\theta)^2}{2\sigma^2}} \cdot e^{\frac{-(x_2-\theta)^2}{2\sigma^2}} \ldots e^{\frac{-(x_n-\theta)^2}{2\sigma^2}}}{(\sigma\sqrt{2\pi})^n} \cdot \frac{e^{\frac{-(\theta-a)^2}{2b^2}}}{b\sqrt{2\pi}}$$

As $(\sigma\sqrt{2\pi})^n . b\sqrt{2\pi}$ is a constant, removing it will not affect the proportionality.

$$\Rightarrow \propto e^{\frac{-\sum_{i=1}^{n}(x_i-\theta)^2}{2\sigma^2}} \cdot e^{\frac{-(\theta-a)^2}{2b^2}}.$$

$$\propto e^{\frac{-\sum x_i^2 - n\theta^2 + 2\sum x_i\theta}{2\sigma^2}} \cdot e^{\frac{-\theta^2 - a^2 + 2a\theta}{2b^2}}$$

$$\propto e^{\frac{-\sum x_i^2 - n\theta^2 + 2n\theta\bar{x}}{2\sigma^2}} \cdot e^{\frac{-\theta^2 - a^2 + 2a\theta}{2b^2}}$$

$$\propto e^{\frac{b^2(-\sum x_i^2 - n\theta^2 + 2nb\bar{x}\theta) + \sigma^2(-\theta^2 - a^2 + 2a\theta)}{2\sigma^2 b^2}}$$

$$\propto e^{\frac{-b^2\sum x_i^2 - b^2 n\theta^2 + 2nb^2\bar{x}\theta - \sigma^2\theta^2 - \sigma^2 a^2 + 2\sigma^2 a\theta}{2\sigma^2 b^2}}$$

$$\propto e^{\frac{-b^2\sum x_i^2}{2\sigma^2 b^2}} \cdot e^{\frac{-b^2 n\theta^2 + 2nb^2\bar{x}\theta - \theta^2\sigma^2 - a^2\sigma^2 + 2a\theta\sigma^2}{2\sigma^2 b^2}}$$

As $e^{\frac{-b^2\sum x_i^2}{2\sigma^2 b^2}}$ is constant, ignoring it will not change the proportionality.

$$\Rightarrow \propto e^{\frac{-b^2 n\theta^2 + 2nb^2\bar{x}\theta - \theta^2\sigma^2 - a^2\sigma^2 + 2a\theta\sigma^2}{2\sigma^2 b^2}}$$

$$\propto e^{\frac{-\theta^2(\sigma^2 + nb^2) + 2\theta(b^2\sum x + a\sigma^2)}{2\sigma^2 b^2}} \cdot e^{\frac{-\cancel{\sigma^2}a^2}{2\cancel{\sigma^2}b^2}}$$

As $e^{\frac{-a^2}{2b^2}}$ is a constant, ignoring it will not change the proportionality.

$$\propto e^{\frac{-\theta^2(\sigma^2+nb^2)+2\theta(b^2\Sigma x+a\sigma^2)}{2\sigma^2 b^2}}$$

$$\propto e^{\frac{-\theta^2+\frac{2\theta}{\sigma^2+nb^2}(b^2\Sigma x+a\sigma^2)}{\frac{2\sigma^2 b^2}{\sigma^2+nb^2}}}$$

$$\propto e^{\frac{-\theta^2+2(\theta)\left(\frac{b^2\Sigma x+a\sigma^2}{\sigma^2+nb^2}\right)-\left(\frac{b^2\Sigma x+a\sigma^2}{\sigma^2+nb^2}\right)^2}{\frac{2\sigma^2 b^2}{\sigma^2+nb^2}}}\cdot\left(e^{\left(\frac{b^2\Sigma x+a\sigma^2}{\sigma^2+nb^2}\right)^2}\right)$$

As $\left(e^{\left(\frac{b^2\Sigma x+a\sigma^2}{\sigma^2+nb^2}\right)^2}\right)$ is a constant, ignoring it will not change the proportionality.

$$\Rightarrow \propto e^{\frac{-\left(\theta-\left(\frac{b^2\Sigma x+a\sigma^2}{\sigma^2+nb^2}\right)\right)^2}{\frac{2\sigma^2 b^2}{\sigma^2+nb^2}}}$$

$$\propto \frac{e^{\frac{-\left(\theta-\left(\frac{b^2\Sigma x+a\sigma^2}{\sigma^2+nb^2}\right)\right)^2}{\frac{2\sigma^2 b^2}{\sigma^2+nb^2}}}}{\sqrt{\frac{\sigma^2 b^2}{\sigma^2+nb^2}\cdot 2\pi}} \quad \rightarrow \text{constant.}$$

It is in the form of

$$\frac{e^{-\frac{(\theta - x)^2}{2y^2}}}{\sqrt{2\pi y^2}}$$

This is equal to Normal (x, y^2)

where

$$x = \frac{b^2 \Sigma x + a\sigma^2}{\sigma^2 + nb^2}$$

$$= \frac{b^2 \frac{\Sigma x}{n} + \frac{a\sigma^2}{n}}{\frac{\sigma^2}{n} + \frac{b^2 \cancel{n}}{\cancel{n}}}$$

$$x = \frac{b^2 \bar{x} + a\, se^2}{b^2 + se^2}$$

$$y^2 = \frac{\sigma^2 b^2}{\sigma^2 + nb^2}$$

$$= \frac{\frac{\sigma^2}{n} \cdot b}{\frac{\sigma^2}{n} + b^2} \Rightarrow y^2 = \frac{b^2 se^2}{b^2 + se^2}$$

b)

$$\theta_{posterior} = C \cdot e^{\frac{-(\theta - x)^2}{y^2}}$$

$1-\alpha$ interval

$\Rightarrow \theta \in [a, b]$ such that

$Pr(\theta < a) = \alpha/2$ and $Pr(\theta > b) = \alpha/2$.

since it is symmetric.

$Pr(\theta < a) = \alpha/2$

$$\Rightarrow Pr\left(\frac{\theta - x}{y} < \frac{a-x}{y}\right) = \frac{\alpha}{2}$$

$$\Rightarrow Pr\left(Z < \frac{a-x}{y}\right) = \frac{\alpha}{2}$$

$$\Rightarrow Z_{\alpha/2} = \frac{a-x}{y}$$

$$\Rightarrow a = x + y \cdot Z_{\alpha/2}.$$

$Pr(\theta > b) = \frac{\alpha}{2}$.

$$Pr\left(\frac{\theta - x}{y} > \frac{b-x}{y}\right) = \frac{\alpha}{2}$$

$$\Rightarrow -Z_{\alpha/2} = \frac{b-x}{y}$$

$$\Rightarrow b = x - y \cdot Z_{\alpha/2}.$$

$$\therefore [a, b] = [x + y \cdot z_{\alpha/2},\ x - y \cdot z_{\alpha/2}]$$

3a) Given Simple Linear Regression of n Sample points $(Y_1, X_1), (Y_2, X_2) \cdots (Y_3, X_3) \cdots\cdots (Y_n, X_n)$ is

$$Y = \beta_0 + \beta_1 X + \varepsilon_i, \quad \text{where } E[\varepsilon_i] = 0$$

Now for each observed response Y_i, with a corresponding Predictor Variable X_i, So we would like to minimize SSE of each observed response to its fitted value

We have $\quad Y_i/X_i = \beta_0 + \beta_1 X_i + \varepsilon_i$

$$E[Y_i/X_i] = E[\beta_0 + \beta_1 X_i + \varepsilon_i] \qquad [\text{Since } E[\varepsilon_i] = 0]$$

$$= \beta_0 + \beta_1 X_i$$

We have $\quad \hat{Y}_i = E[Y_i/X_i] = \hat{\beta}_0 + \hat{\beta}_1 X_i$

$$\hat{\varepsilon}_i = Y_i - \hat{Y}_i$$

$$S = \sum_{i=1}^{n} (\hat{\varepsilon}_i)^2 = \sum_{i=1}^{n} (Y_i - (\hat{\beta}_0 + \hat{\beta}_1 X_i))^2$$

We take derivatives & set them to 0 to find maximum value.

$$\frac{\partial S}{\partial \hat{\beta}_0} = 0 \Rightarrow \sum_{i=1}^{n} 2(Y_i - (\hat{\beta}_0 + \hat{\beta}_1 X_i))(-1)$$

$$\Rightarrow -2 \sum_{i=1}^{n} (Y_i - (\hat{\beta}_0 + \hat{\beta}_1 X_i)) = 0$$

$$\sum Y_i = \sum_{i=1}^{n} (\hat{\beta}_0 + \hat{\beta}_1 X_i)$$

$$\frac{\sum Y_i}{n} = \hat{\beta}_0 + \hat{\beta}_1 \frac{\sum X_i}{n}$$

$$[\bar{Y} = \hat{\beta}_0 + \hat{\beta}_1 \bar{X}]$$

$$\therefore \hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$$

3) a) Estimates of $\hat{\beta}_1$ derived in class

$$\hat{\beta}_1 = \frac{\Sigma(x_i y_i) - n\bar{x}\bar{y}}{\Sigma(x_i^2) - n(\bar{x})^2}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

S.T $\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$

Consider $\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \sum_{i=1}^{n} x_i y_i - \Sigma x_i \bar{y} - \Sigma \bar{x} y_i + \Sigma \bar{x}\bar{y}$

$$= \Sigma x_i y_i - \bar{y}\Sigma x_i - \bar{x}\Sigma y_i + \bar{x}\bar{y}(n)$$

$$= \Sigma x_i y_i - 2\bar{y}\bar{x}(n) + \bar{x}\bar{y}(n)$$

$$= \Sigma x_i y_i - n\bar{x}\bar{y}$$

$$= \Sigma x_i y_i - n\bar{x}\bar{y} \quad - (1)$$

$$\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \Sigma x_i y_i - n\bar{x}\bar{y}$$

Consider $\Sigma(x_i - \bar{x})^2 = \Sigma(x_i^2 + \bar{x}^2 - 2x_i\bar{x}) = \Sigma x_i^2 + \Sigma \bar{x}^2 - 2\Sigma x_i \bar{x}$

$$\bar{x} = \frac{\Sigma x_i}{n}$$

$$= \Sigma(x_i^2 + \bar{x}^2 - 2x_i\bar{x})$$

$$= \Sigma x_i^2 + n\bar{x}^2 - 2n x_i \bar{x}$$

$$= \Sigma x_i^2 + n\bar{x}^2 - 2\bar{x}\bar{x} = \Sigma x_i^2 + n\bar{x}^2 - 2\bar{x}^2 n$$

$$= \Sigma x_i^2 + n\bar{x}^2 \quad - (2)$$

$$= \Sigma x_i^2 - n\bar{x}^2$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{\Sigma x_i y_i - n\bar{x}\bar{y}}{\Sigma x_i^2 - n\bar{x}^2}$$

$$\therefore \hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$$\hat{\beta}_0 = \bar{y} - \left[\frac{\sum_{i=1}^{n}(x_i - \bar{x})(y - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right]\bar{x}$$

Hence proved

③ Shown that the above estimators, given x_i's are unbiased.

b) We need to prove $E[\hat{\beta}_0] = \beta_0$ & $E[\hat{\beta}_1] = \beta_1$

W.K.T

$$E[\hat{\beta}_1 | x_i] = E\left[\frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\Sigma(x_i - \bar{x})^2} \middle| x_i\right]$$

$E[\hat{\beta}_1$ [x's are Constants] $\Rightarrow \dfrac{E(\Sigma x_i y_i) - n\bar{x}\bar{y}}{\Sigma x_i^2 - n\bar{x}^2}$

$\hookrightarrow \Sigma x_i^2 \to$ Const

$(\bar{x})^2 \to$ const

$\bar{y} \to$ const

$$E[\hat{\beta}_1] = \frac{\Sigma x_i E(y_i) - n\bar{x}\bar{y}}{\Sigma x_i^2 - n\bar{x}^2} \quad \text{--- ①}$$

W.K.T $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$ $\quad [\because \bar{x} = \frac{\Sigma x_i}{n}]$

$E[y_i] = \beta_0 + \beta_1 E[x_i] + \underbrace{E[\varepsilon_i]}_{0}$

$= \beta_0 + \beta_1 x_i$

$$\bar{y} = \frac{\Sigma y_i}{n} = \frac{\Sigma(\beta_0 + \beta_1 x_i + \varepsilon_i)}{n} = \frac{n\beta_0 + \beta_1 n\bar{x} + 0}{n}$$

$$\bar{y} = \beta_0 + \beta_1 \bar{x} \quad \text{--- ②}$$

Substitute eq(2) in eq(1)

$$= \frac{\Sigma x_i(\beta_0 + \beta_1 x_i) - n\bar{x}(\beta_0 + \beta_1 \bar{x})}{\Sigma x_i^2 - n\bar{x}^2}$$

$$= \frac{\beta_0 \Sigma x_i + \beta_1 \Sigma x_i^2 - n\bar{x}\beta_0 - n\beta_1\bar{x}}{\Sigma x_i^2 - n\bar{x}^2}$$

$$= \frac{\cancel{\beta_0 n\bar{x}} + \beta_1 \Sigma x_i^2 - \cancel{n\bar{x}\beta_0} - n\beta_1\bar{x}}{\Sigma x_i^2 - n\bar{x}^2}$$

3b)

$$= \frac{\beta_1 (\cancel{\Sigma x_i^2 - n\bar{x}})}{\cancel{\Sigma x_i^2 - n\bar{x}})} = \beta_1$$

$\therefore \left[E[\hat{\beta}_1] = \beta_1\right]$, $\hat{\beta}_1$ is unbiased.

[W.K.T from 3(a)) $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$

$$E[\hat{\beta}_0] = E[\bar{y} - \hat{\beta}_1 \bar{x}]$$

$$= \bar{y} - \beta_1 E[\bar{x}]$$

$$= \bar{y} - \beta_1 \bar{x} = \beta_0 + \beta_1 \bar{x} - \beta_1 \bar{x}$$

$$= \beta_0$$

$\left[\therefore E[\hat{\beta}_0] = \beta_0\right]$ $\hat{\beta}_0$ is unbiased

Hence proved.

6)

a)

$$P(H=0/W) \geq P(H=1/W)$$

$$P(H=0/W) = \frac{P(W/H=0)\cdot P(H=0)}{P(W)}$$

$$P(W/H=0) = \prod_{i=1}^{n}\left(\frac{e^{\frac{-(x-(-\mu))^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}\right)$$

$$= \frac{e^{\frac{-(x_1+\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} \cdot \frac{e^{\frac{-(x_2+\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} \cdots \frac{e^{\frac{-(x_n+\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$$

$$= \frac{e^{\frac{-\sum(x_i+\mu)^2}{2\sigma^2}}}{(\sigma\sqrt{2\pi})^n}$$

$$= \frac{e^{\frac{-\sum(x_i^2+\mu^2+2x_i\mu)}{2\sigma^2}}}{(\sigma\sqrt{2\pi})^n}$$

Here the distribution is w.

$$\Rightarrow \frac{e^{\frac{-(\sum w_i^2 + \sum \mu^2 + 2\mu \sum w)}{2\sigma^2}}}{(\sigma\sqrt{2\pi})^n}$$

$$= \frac{e^{\frac{-\sum w^2 - n\mu^2 - 2\mu\sum w}{2\sigma^2}}}{(\sigma\sqrt{2\pi})^n}$$

$$P(w/H=1) = \prod_{i=1}^{n}\left(\frac{e^{\frac{-(w-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}\right)$$

$$= \frac{e^{\frac{-\sum(w-\mu)^2}{2\sigma^2}}}{(\sigma\sqrt{2\pi})^n}$$

$$= \frac{e^{\frac{-\sum(w^2+\mu^2-2\mu w)}{2\sigma^2}}}{(\sigma\sqrt{2\pi})^n}$$

$$= \frac{e^{\frac{-\sum w^2 - n\mu^2 + 2\mu\sum w}{2\sigma^2}}}{(\sigma\sqrt{2\pi})^n}$$

$P(w$

$$P(H=0|w) = \frac{e^{\frac{-\Sigma w^2 - n\mu^2 - 2\mu\Sigma w}{2\sigma^2}}}{(\sigma\sqrt{2\pi})^n \cdot P(w)} \cdot p$$

$$P(H=1|w) = \frac{e^{\frac{-\Sigma w^2 - n\mu^2 + 2\mu\Sigma w}{2\sigma^2}}}{(\sigma\sqrt{2\pi})^n \cdot P(w)} \cdot (1-p)$$

$$P(H=0|w) \geq P(H=1|w)$$

$$\Rightarrow \frac{e^{\frac{-\Sigma w^2 - n\mu^2 - 2\mu\Sigma w}{2\sigma^2}}}{\cancel{(\sigma\sqrt{2\pi})^n P(w)}} \cdot p \geq \frac{e^{\frac{-\Sigma w^2 - n\mu^2 + 2\mu\Sigma w}{2\sigma^2}}}{\cancel{(\sigma\sqrt{2\pi})^n P(w)}} \cdot (1-p)$$

$$\Rightarrow e^{\frac{-\Sigma w^2 - n\mu^2 - 2\mu\Sigma w}{2\sigma^2}} \cdot p \geq e^{\frac{-\Sigma w^2 - n\mu^2 + 2\mu\Sigma w}{2\sigma^2}} (1-p)$$

$$\frac{p}{1-p} \geq \frac{e^{\frac{-\Sigma w^2 - n\mu^2 + 2\mu\Sigma w}{2\sigma^2}}}{e^{\frac{-\Sigma w^2 - n\mu^2 - 2\mu\Sigma w}{2\sigma^2}}}$$

$$\frac{p}{1-p} \geq e^{\frac{\cancel{-\Sigma w^2} \cancel{- n\mu^2} + 2\mu\Sigma w \cancel{+ \Sigma w^2} \cancel{+ n\mu^2} + 2\mu\Sigma w}{2\sigma^2}}$$

$$\Rightarrow \frac{p}{1-p} \geq e^{\frac{4\mu\sum w}{2\sigma^2}}$$

$$\frac{p}{1-p} \geq e^{\frac{2\mu\sum w}{\sigma^2}}$$

$$\ln\left(\frac{p}{1-p}\right) \geq \frac{2\mu\sum w}{\sigma^2}$$

$$\Rightarrow \boxed{\sum w \leq \ln\left(\frac{p}{1-p}\right)\frac{\sigma^2}{2\mu}}$$

---

b)

c)

$$AEP = P(C=0|H=1)\cdot P(H=1) + P(C=1|H=0)\,P(H=0)$$

$$P(C=0|H=1) \Rightarrow \sum w \leq \frac{\sigma^2}{2\mu}\ln\left(\frac{p}{1-p}\right)$$

$$\text{Let } \frac{\sigma^2}{2\mu}\ln\left(\frac{p}{1-p}\right) = C \text{ (constant)}$$

$$\Rightarrow \sum w \leq C$$

As $w \sim \text{Normal}(\mu, \sigma^2)$

$$\Rightarrow \sum w \sim \text{Normal}(n\mu, n\sigma^2)$$

ϕ

Let $\sum w = W$

$$\Rightarrow P(W < c) = P\left(\frac{W - n\mu}{\sqrt{n}\sigma} \le \frac{c - n\mu}{\sigma\sqrt{n}}\right)$$

$$= \phi\left(\frac{c - n\mu}{\sigma\sqrt{n}}\right)$$

$$\Rightarrow P(c = 0 \mid H = 1) P(H = 1) = \phi\left(\frac{c - n\mu}{\sigma n}\right)(1 - p)$$

For

$$P(c = 1 \mid H = 0) \Rightarrow \sum w_i > \frac{\sigma^2}{2\mu} \ln\left(\frac{p}{1-p}\right)$$

For $H = 0$

$$w \sim \text{Normal}(-\mu, \sigma^2)$$

$$\Rightarrow W = \sum w_i \sim \text{Normal}(-n\mu, n\sigma^2)$$

$$\Rightarrow P(W > c) = 1 - P(W < c)$$

$$= 1 - P\left(\frac{W - (-n\mu)}{\sigma\sqrt{n}} < \frac{c - (-n\mu)}{\sigma\sqrt{n}}\right)$$

$$= 1 - \phi\left(\frac{c + n\mu}{\sigma\sqrt{n}}\right)$$

$$\Rightarrow AEP = \phi\left(\frac{c - n\mu}{\sigma\sqrt{n}}\right)(1-p) + \left(1 - \phi\left(\frac{c + n\mu}{\sigma\sqrt{n}}\right)\right)p.$$

$$\cancel{\therefore AEP = \phi\left(\frac{\frac{\sigma^2}{2\mu}\ln\left(\frac{p}{1-p}\right) - n\mu}{\sigma\sqrt{n}}\right)(1-p) + \left(1 - \phi\left(\frac{\frac{\sigma^2}{2\mu}\ln\left(\frac{p}{1-p}\right) + n\mu}{\sigma\sqrt{n}}\right)\right)(1-p)}$$

$$\therefore AEP = \phi\left(\frac{\frac{\sigma^2}{2\mu}\ln\left(\frac{p}{1-p}\right) - n\mu}{\sigma\sqrt{n}}\right)(1-p) + \left(1 - \phi\left(\frac{\frac{\sigma^2}{2\mu}\ln\left(\frac{p}{1-p}\right) + n\mu}{\sigma\sqrt{n}}\right)\right)(p)$$

6-B

```
For P(H0) = 0.1, the hypotheses selected are :: 0 1 0 1 1 1 0 1 1 1
For P(H0) = 0.3, the hypotheses selected are :: 0 1 0 1 1 1 0 0 1 1
For P(H0) = 0.5, the hypotheses selected are :: 0 0 0 1 0 1 0 0 1 0
For P(H0) = 0.8, the hypotheses selected are :: 0 0 0 1 0 0 0 0 0 0
```