Assignment 7

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Question:
Show that if the random variables X_i are i.i.d. and normal, then their sample mean $\bar{X}$ and sample variances S^2 are two independent random variables.
Solution:

Given that the random variables X_i are i.i.d. and normal. We wish to show that RVs

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{1}$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2 \tag{2}$$

are independent.Since S^2 is a function of the n RVs $X_i - \bar{X}$, it suffices to show that each of these RVs is independent of $\bar{X}$.

We assume for simplicity that $E(X_i) = 0$.
Clearly,

$$E(X_i\bar{X}) = \frac{1}{n}E\left\{X_i^2\right\} = \frac{\sigma^2}{n} \tag{3}$$

$$E(\bar{X}\bar{X}) = \frac{1}{n^2}\sum_{i=1}^{n} X_i^2 = \frac{\sigma^2}{n} \tag{4}$$

because $E(X_iX_j) = 0$ for $i \neq j$.
Hence,

$$E((X_i - \bar{X})\bar{X}) = 0 \tag{5}$$

Thus, the RVs $X_i - \bar{X}$ and $\bar{X}$ are orthogonal and since they are jointly normal, they are independent.