Assignment 8

AKHILA, CS21BTECH11031

Question:
We denote by X_m a random variable equal to the number of tosses of a coin until heads shows for the mth time. Show that if P$\{h\}$ = p. then $E\{x_m\} = \frac{m}{p}$.

Solution:

We know that,

$$1 + X + X^2 + ... + X^n ... = \frac{1}{1-X}, |X| < 1 \quad (1)$$

Differentiating on both sides we get..

$$\implies 1 + 2X + 3X^2 + ...nX^{n-1} + ... = \sum_{k=1}^{\infty} kX^{k-1} \quad (2)$$

$$= \frac{1}{(1-X)^2} \quad (3)$$

The RV X_1 equals to the number of tosses until head shows for the first time.
Hence, X_1 takes the values 1,2,.... with $\Pr\{X_1 = k\} = pq^{k-1}$.
Hence,

$$E\{X_1\} = \sum_{k=1}^{\infty} k \Pr\{X_1 = k\} = \sum_{k=1}^{\infty} kpq^{k-1} \quad (4)$$

$$= \frac{p}{(1-q)^2} = \frac{1}{p} \quad (5)$$

Starting the count after the first head shows,we conclude that the RV $X_2 - X_1$ has the same statistics as the RV X_1. Hence,

$$E\{X_2 - X_1\} = E\{X_1\} \quad (6)$$

$$E\{X_2\} = 2E\{X_1\} = \frac{2}{p} \quad (7)$$

Reasoning similarly,we conclude that

$$E\{X_n - X_{n-1}\} = E\{X_1\} \quad (8)$$

Hence,

$$E\{X_n\} = E\{X_{n-1}\} + E\{X_1\} = \frac{n-1}{p} + \frac{1}{p} = \frac{n}{p} \quad (9)$$