Assignment 9

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Question:
Suppose the Conditional distribution of X given Y = n is binomial with parameters n and p_1. Further, Y is a binomial random variable with parameters M and p_2. Show that the distribution of X is also binomial. Find its parameters.

Solution:
If X has the binomial distribution B(m, p) given by

$$p_n = \Pr(X = n) = \binom{m}{n} p^n q^{m-n}, 0 \leq n \leq m \tag{1}$$

then
Moment generating function

$$\Gamma(z) = E\left\{z^X\right\} = \sum_{n=0}^{m} \binom{m}{n} p^n q^{m-n} z^n = (pz+q)^m \tag{2}$$

Given that the Conditional distribution of X given Y = n is binomial with parameters n and p_1.

$$\Pr(X = k|Y = n) = \binom{n}{k} p_1^k q_1^{n-k}, k = 0, 1, 2, \ldots n$$

$$E[z^X|Y = n] = \sum_{k=0}^{n} z^k \Pr(X = k|Y = n)$$
$$= (p_1 z + q_1)^n \tag{3}$$

Also,We have

$$\Gamma(z) = E\left\{z^X\right\} = E\left\{E[z^X|Y = n]\right\} \tag{4}$$
$$= \sum_{n=0}^{M} E[z^X|Y = n] \Pr(Y = n) \tag{5}$$
$$= \sum_{n=0}^{M} (p_1 z + q_1)^n \binom{M}{n} p_2^n q_2^{M-n} \tag{6}$$
$$= \sum_{n=0}^{M} \binom{M}{n} \left[p_2 (p_1 z + q_1)\right]^n q_2^{M-n} \tag{7}$$
$$= (p_1 p_2 z + p_2 q_1 + q_2)^M \tag{8}$$

But

$$1 - p_1 p_2 = 1 - (1 - q_1)(1 - q_2) = q_1 p_2 + q_2 \tag{9}$$

Hence

$$\Gamma(z) = (pz + q)^M \tag{10}$$

Where $p = p_1 p_2$.
Therefore, X $\sim$ Binomial$(M, p_1 p_2)$
i.e, The distribution of X is also binomial with parameters M and $p_1 p_2$.