ASSIGNMENT 2

AKHILA, CS21BTECH11031

Question: If $x = tan\,(logy)$, prove that

$$\left(1 + x^2\right)\frac{d^2y}{dx^2} + (2x - 1)\frac{dy}{dx} = 0$$

Solution: Given, $x = tan\,(logy)$

$$\implies \tan^{-1}x = logy \tag{1}$$
$$\implies y = e^{\tan^{-1}x} \tag{2}$$

Differentiating on both sides with respect to x..

$$\implies \frac{dy}{dx} = \frac{d}{dx}\left(e^{\tan^{-1}x}\right) \tag{3}$$
$$\implies \frac{dy}{dx} = \left(e^{\tan^{-1}x}\right)\frac{d}{dx}\left(\tan^{-1}x\right) \tag{4}$$
$$\implies \frac{dy}{dx} = \left(e^{\tan^{-1}x}\right)\left(\frac{1}{1 + x^2}\right) \tag{5}$$
$$\implies \frac{dy}{dx} = \frac{e^{\tan^{-1}x}}{1 + x^2} \tag{6}$$

From (2)...

$$\therefore \frac{dy}{dx} = \frac{y}{1 + x^2} \tag{7}$$

Again differentiating on both sides with respect to x..

$$\implies \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y}{1 + x^2}\right) \tag{8}$$
$$= \frac{d}{dx}\left(\frac{1}{1 + x^2}\right)y + \frac{1}{1 + x^2}\frac{dy}{dx} \tag{9}$$
$$= \frac{-1(2x)}{(1 + x^2)^2}\,(y) + \frac{1}{1 + x^2}\frac{dy}{dx} \tag{10}$$
$$= \frac{-2x}{(1 + x^2)}\frac{y}{(1 + x^2)} + \frac{1}{1 + x^2}\frac{dy}{dx} \tag{11}$$

From (7)...

$$\implies \frac{d^2y}{dx^2} = \frac{-2x}{(1 + x^2)}\frac{dy}{dx} + \frac{1}{(1 + x^2)}\frac{dy}{dx} \tag{12}$$
$$\implies \left(1 + x^2\right)\frac{d^2y}{dx^2} = \frac{dy}{dx}\,(1 - 2x) \tag{13}$$
$$\implies \left(1 + x^2\right)\frac{d^2y}{dx^2} = -\frac{dy}{dx}\,(2x - 1) \tag{14}$$
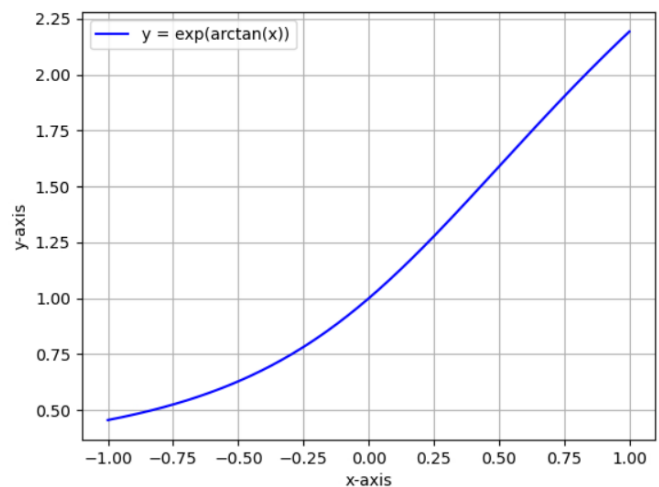$$\therefore \left(1 + x^2\right)\frac{d^2y}{dx^2} + \frac{dy}{dx}\,(2x - 1) = 0 \tag{15}$$

Hence proved.

Fig. 1: Graph of $y = e^{\tan^{-1}x}$