Lecture 10

Sorting

Bringing Order to the World

Lecture Outline

- Iterative sorting algorithms (comparison based)
  - Selection Sort
  - Bubble Sort
  - Insertion Sort
- Recursive sorting algorithms (comparison based)
  - Merge Sort
  - Quick Sort
- Radix sort (non-comparison based)
- Properties of Sorting
  - In-place sort, stable sort
  - Comparison of sorting algorithms
- Note: we only consider sorting data in **ascending order**

Why Study Sorting?

- When an input is sorted, many problems become easy (e.g. searching, min, max, *k*-th smallest)

- Sorting has a variety of interesting algorithmic solutions that embody many ideas
  - Comparison vs non-comparison based
  - Iterative
  - Recursive
  - Divide-and-conquer
  - Best/worst/average-case bounds
  - Randomized algorithms

Applications of Sorting

- Uniqueness testing
- Deleting duplicates
- Prioritizing events
- Frequency counting
- Reconstructing the original order
- Set intersection/union
- Finding a target pair x, y such that $x+y = z$
- Efficient searching

Selection Sort

Selection Sort: Idea

- Given an array of *n* items
  1. Find the largest item *x*, in the range of [0…*n*−1]
  2. Swap *x* with the (*n*−1)th item
  3. Reduce *n* by 1 and go to Step 1

Selection Sort: Illustration

29	10	14	37	13

37 is the largest, swap it with the last element, i.e. 13.

Q: How to find the largest?

29	10	14	13	37

13	10	14	29	37

13	10	14	29	37

10	13	14	29	37

Sorted!

x **Unsorted items**

x **Largest item for current iteration**

x **Sorted items**

We can also find the smallest and put it the front instead

http://visualgo.net/sorting?create=29,10,14,37,13&mode=Selection

Selection Sort: Implementation

```
void selectionSort(int a[], int n) {
  for (int i = n-1; i >= 1; i--) {
    int maxIdx = i;
    for (int j = 0; j < i; j++)
      if (a[j] >= a[maxIdx])
        maxIdx = j;
    // swap routine is in STL <algorithm>
    swap(a[i], a[maxIdx]);
  }
}
```

Step 1: Search for maximum element

Step 2: Swap maximum element with the last item i

Selection Sort: Analysis

```
void selectionSort(int a[], int n) {
   for (int i = n-1; i >= 1; i--) {
     int maxIdx = i;
     for (int j = 0; j < i; j++)
       if (a[j] >= a[maxIdx])
         maxIdx = j;
     // swap routine is in STL <algorithm>
     swap(a[i], a[maxIdx]);
   }
}
```

Number of times executed

- $n-1$
- $n-1$
- $(n-1)+(n-2)+\ldots+1$
  $= n(n-1)/2$
- $n-1$

Total

$= c_1(n-1) +$
$c_2 * n * (n-1)/2$
$= \mathbf{O}(n^2)$

- c_1 and c_2 are cost of statements in outer and inner blocks

Bubble Sort

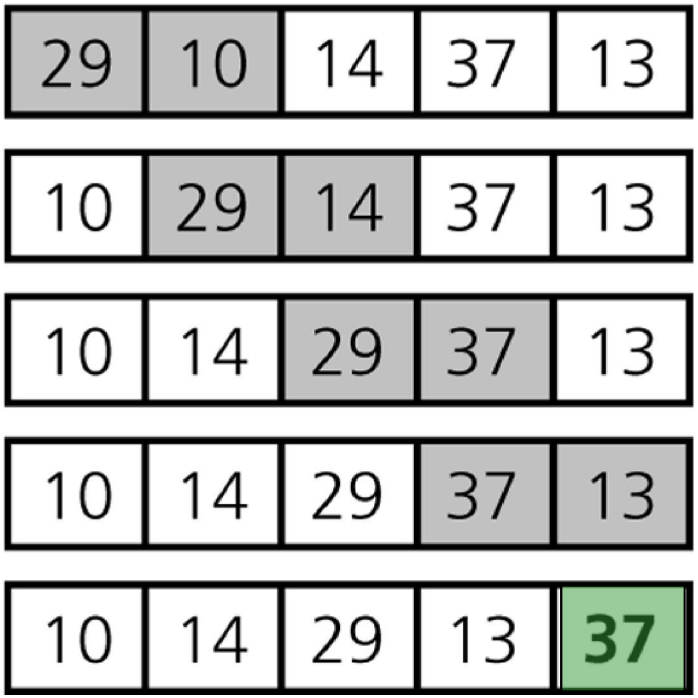
Bubble Sort: Idea

- Given an array of *n* items
  1. Compare pair of adjacent items
  2. Swap if the items are out of order
  3. Repeat until the end of array
     - The largest item will be at the last position
  4. Reduce *n* by 1 and go to Step 1

- Analogy
  - Large item is like “bubble” that floats to the end of the array

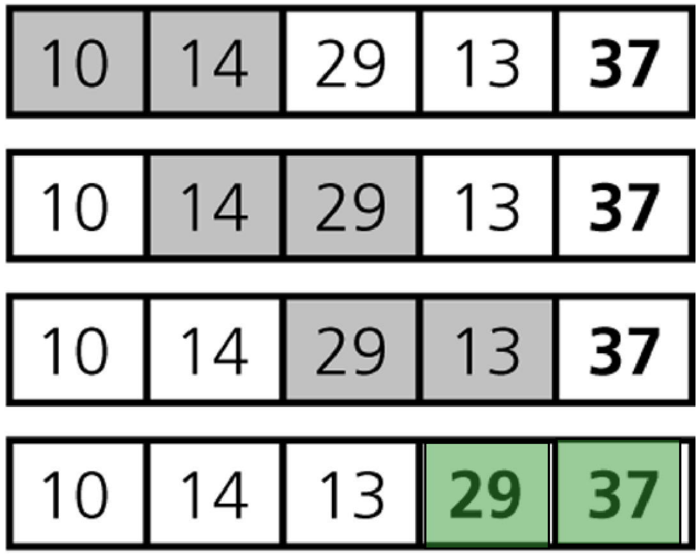
Bubble Sort: Illustration

(a) Pass 1



At the end of **Pass 1**, the largest item **37** is at the last position.

(b) Pass 2



At the end of **Pass 2**, the second largest item **29** is at the second last position.

Bubble Sort: Implementation

```
void bubbleSort(int a[], int n) {
  for (int i = n-1; i >= 1; i--) {
    for (int j = 1; j <= i; j++) {
      if (a[j-1] > a[j])
        swap(a[j], a[j-1]);
    }
  }
}
```

Step 1: Compare adjacent pairs of numbers

Step 2: Swap if the items are out of order

29	10	14	37	13

http://visualgo.net/sorting?create=29,10,14,37,13&mode=Bubble

Bubble Sort: Analysis

- 1 iteration of the inner loop (test and swap) requires time bounded by a constant c
- Two nested loops
  - Outer loop: exactly n iterations
  - Inner loop:
    - when i=0, (n−1) iterations
    - when i=1, (n−2) iterations
    - …
    - when i=(n−1), 0 iterations
- Total number of iterations = $0+1+\ldots+(n-1) = n(n-1)/2$
- Total time = $c\, n(n-1)/2 = \mathbf{O(n^2)}$

Bubble Sort: Early Termination

- Bubble Sort is inefficient with a $O(n^2)$ time complexity
- However, it has an interesting property
  - Given the following array, how many times will the inner loop swap a pair of item?

3	6	11	25	39

- Idea
  - If we go through the inner loop with no swapping
    ➔ the array is sorted
    ➔ can stop early!

Bubble Sort v2.0: Implementation

```
void bubbleSort2(int a[], int n) {
  for (int i = n-1; i >= 1; i--) {
    bool is_sorted = true;
    for (int j = 1; j <= i; j++) {
      if (a[j-1] > a[j]) {
        swap(a[j], a[j-1]);
        is_sorted = false;
      }
    } // end of inner loop
    if (is_sorted) return;
  }
}
```

Assume the array is sorted before the inner loop

Any swapping will invalidate the assumption

If the flag remains **true** after the inner loop ➔ sorted!

Bubble Sort v2.0: Analysis

- Worst-case
  - Input is in descending order
  - Running time remains the same: $O(n^2)$

- Best-case
  - Input is already in ascending order
  - The algorithm returns after a single outer iteration
  - Running time: $O(n)$

Insertion Sort

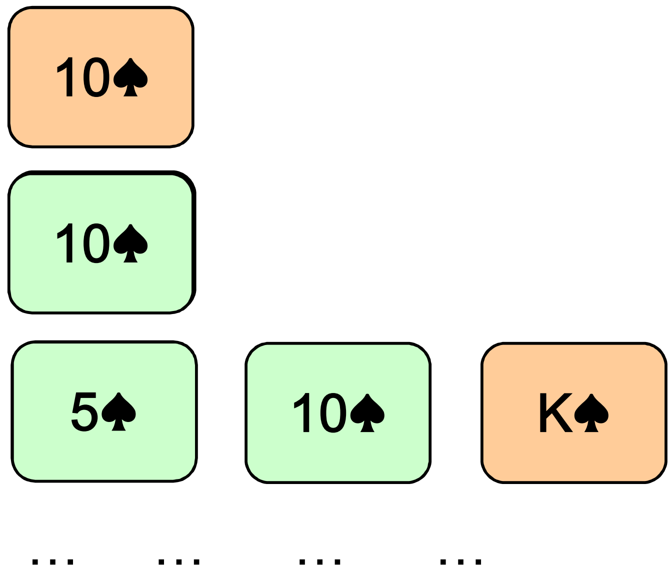
Insertion Sort: Idea

- Similar to how most people arrange a hand of poker cards
  - Start with one card in your hand
  - Pick the next card and insert it into its proper sorted order
  - Repeat previous step for all cards

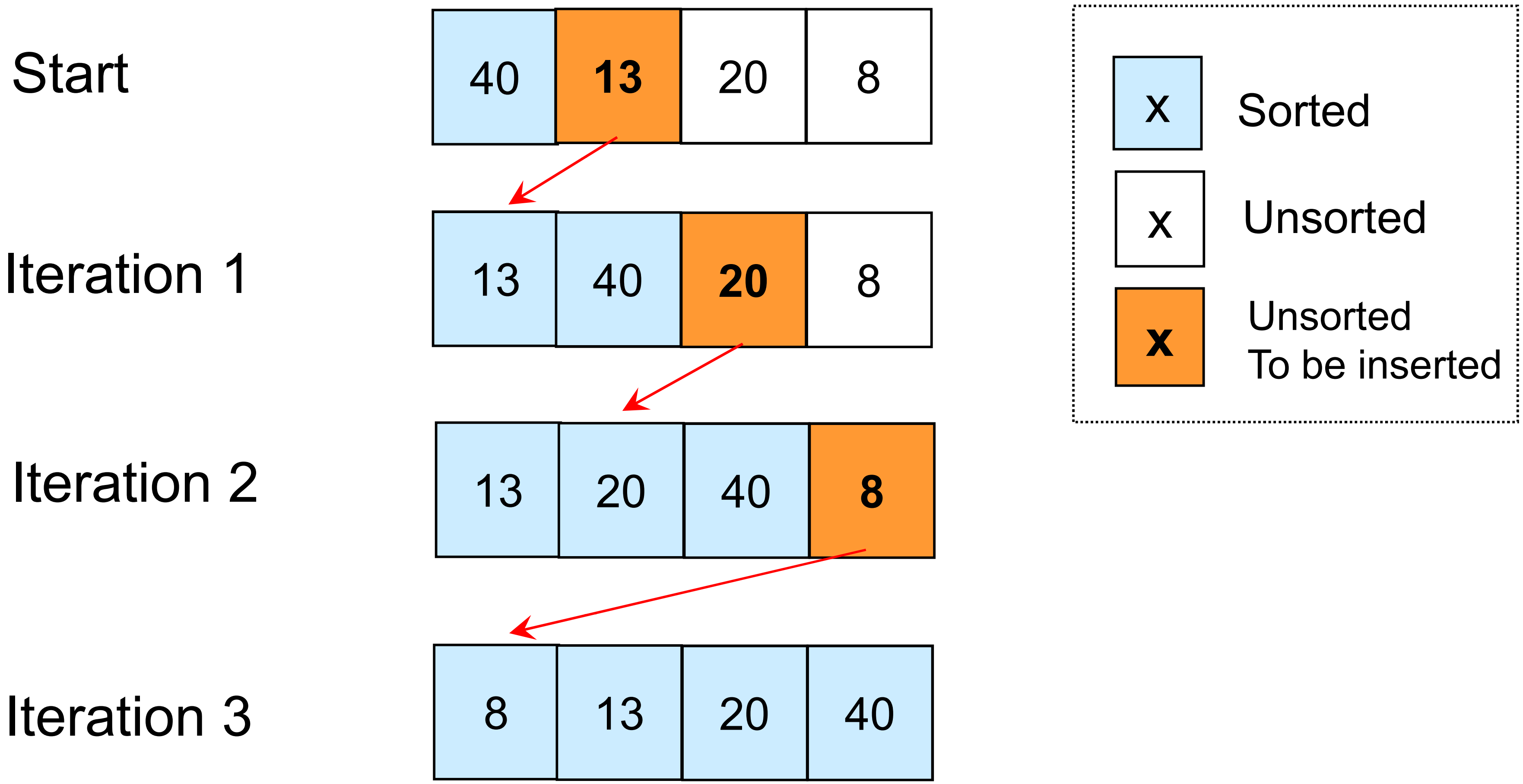
1st card: **10♠**

2nd card: **5♠**

3rd card: **K♠**



Insertion Sort: Illustration



http://visualgo.net/sorting?create=40,13,20,8&mode=Insertion

Insertion Sort: Implementation

```
void insertionSort(int a[], int n) {
  for (int i = 1; i < n; i++) {
    int next = a[i];
    int j;

    for (j = i-1; j >= 0 && a[j] > next; j--)
      a[j+1] = a[j];

    a[j+1] = next;
  }
}
```

next is the item to be inserted

Shift sorted items to make place for **next**

Insert **next** to the correct location

29	10	14	37	13

http://visualgo.net/sorting?create=29,10,14,37,13&mode=Insertion

Insertion Sort: Analysis

- Outer-loop executes ($n-1$) times
- Number of times inner-loop is executed depends on the input
  - **Best-case**: the array is already sorted and (a[j] > next) is always false
    - No shifting of data is necessary
  - **Worst-case**: the array is reversely sorted and (a[j] > next) is always true
    - Insertion always occur at the front
- Therefore, the **best-case** time is **O(n)**
- And the **worst-case** time is **O(n^2)**

Merge Sort

Merge Sort: Idea

- Suppose we only know how to merge two sorted sets of elements into one
  - Merge {1, 5, 9} with {2, 11} ➔ {1, 2, 5, 9, 11}
- Question
  - Where do we get the two sorted sets in the first place?
- Idea (use **merge** to sort *n* items)
  - Merge each pair of elements into sets of 2
  - Merge each pair of sets of 2 into sets of 4
  - Repeat previous step for sets of 4 …
  - Final step: merge 2 sets of $n/2$ elements to obtain a fully sorted set

Divide-and-Conquer Method

- A powerful problem solving technique
- Divide-and-conquer method solves problem in the following steps
  - **Divide** step
    - Divide the large problem into smaller problems
    - Recursively solve the smaller problems
  - **Conquer** step
    - Combine the results of the smaller problems to produce the result of the larger problem

Divide and Conquer: Merge Sort

- Merge Sort is a divide-and-conquer sorting algorithm
- Divide step
  - Divide the array into two (equal) halves
  - Recursively sort the two halves
- Conquer step
  - Merge the two halves to form a sorted array

Merge Sort: Illustration

7	2	6	3	8	4	5

Divide into two halves

7	2	6	3

8	4	5

Recursively sort the halves

2	3	6	7

4	5	8

Merge them

2	3	4	5	6	7	8

- Question
  - How should we sort the halves in the 2nd step?

Merge Sort: Implementation

```
void mergeSort(int a[], int low, int high)  {
  if (low < high) {
    int mid = (low+high) / 2;

    mergeSort(a, low   , mid );
    mergeSort(a, mid+1, high);

    merge(a, low, mid, high);
  }
}
```

Merge sort on **a[low...high]**

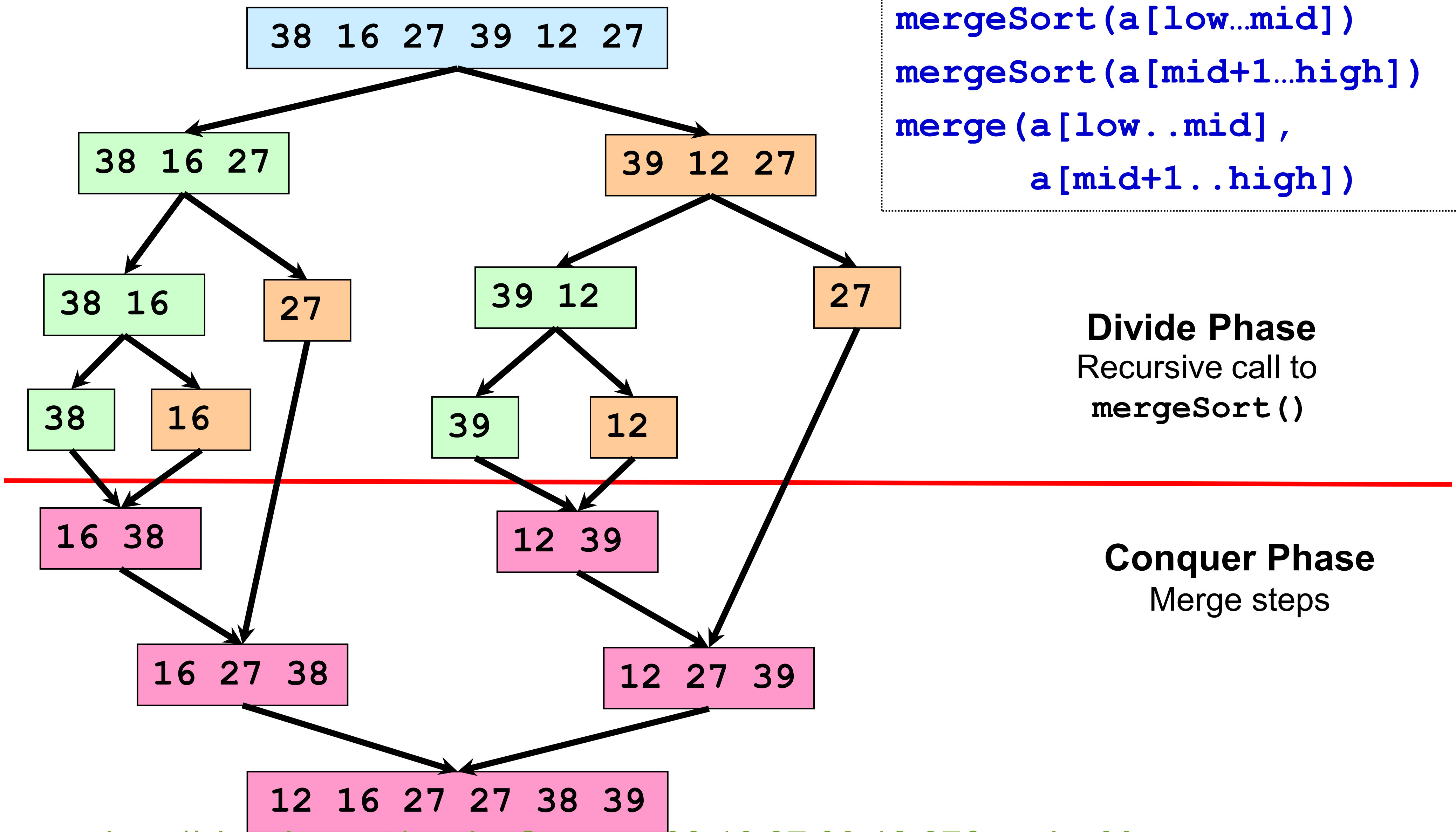
Divide a[ ] into two halves and **recursively** sort them

Conquer: merge the two sorted halves

Function to merge **a[low…mid]** and **a[mid+1…high]** into **a[low…high]**

- Note
  - **mergeSort()** is a recursive function
  - **low >= high** is the base case, i.e. there is 0 or 1 item

Merge Sort: Example



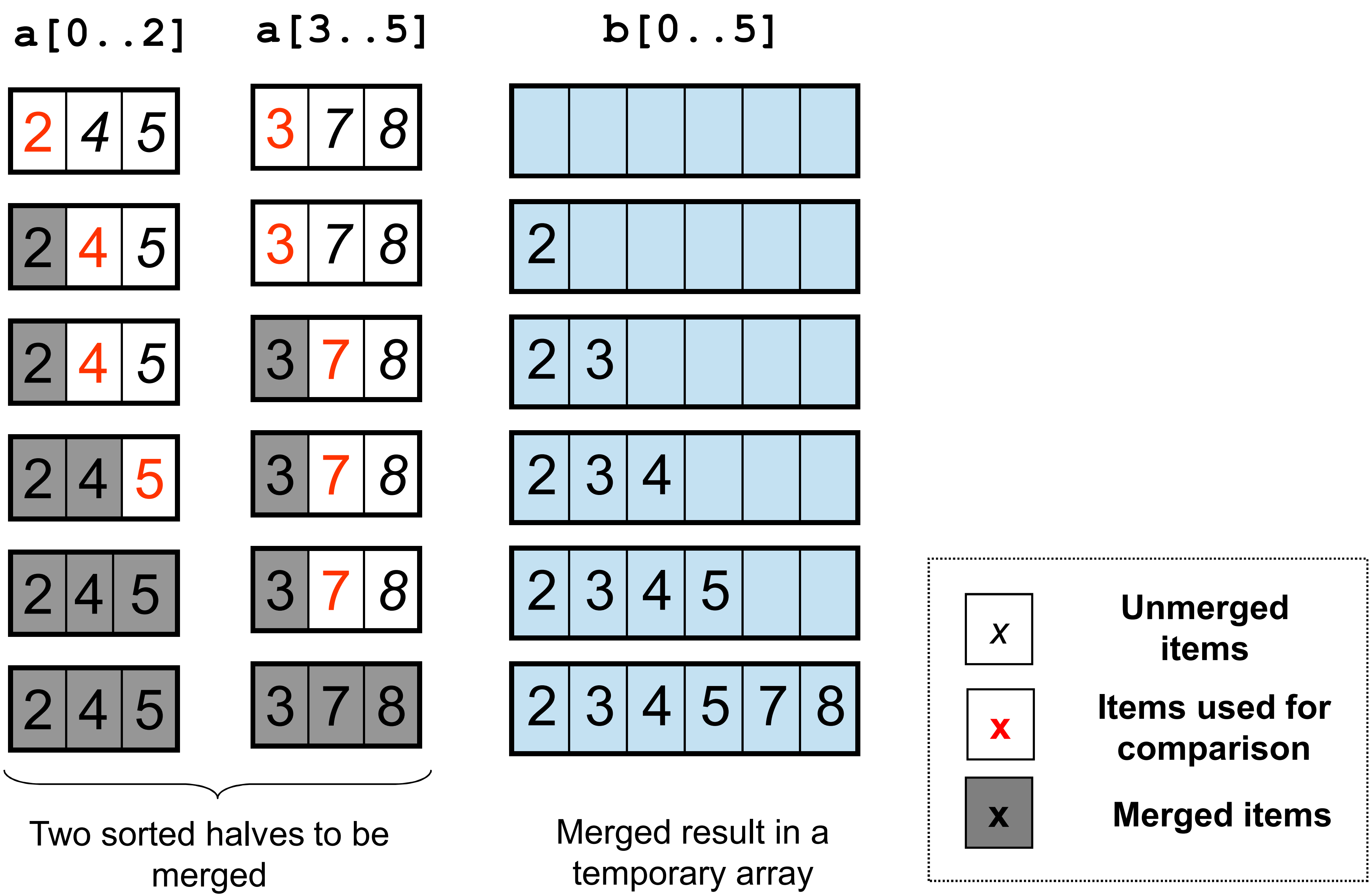

```
mergeSort(a[low…mid])
mergeSort(a[mid+1…high])
merge(a[low..mid],
      a[mid+1..high])
```

Divide Phase
Recursive call to `mergeSort()`

Conquer Phase
Merge steps

http://visualgo.net/sorting?create=38,16,27,39,12,27&mode=Merge

Merge Sort: Merge



Merge Sort: Merge Implementation

PS: C++ STL <algorithm> has merge subroutine too

```
void merge(int a[], int low, int mid, int high) {
  int n = high-low+1;
  int* b = new int[n];
  int left=low, right=mid+1, bIdx=0;

  while (left <= mid && right <= high) {
    if (a[left] <= a[right])
      b[bIdx++] = a[left++];
    else
      b[bIdx++] = a[right++];
  }

  // continue on next slide
```

b is a temporary array to store result

Normal Merging
Where both halves have unmerged items

Merge Sort: Merge Implementation

```
  // continued from previous slide

  while (left <= mid) b[bIdx++] = a[left++];
  while (right <= high) b[bIdx++] = a[right++];

  for (int k = 0; k < n; k++)
    a[low+k] = b[k];

  delete [] b;
}
```

Remaining items are copied into **b[]**

Merged result are copied back into **a[]**

Remember to free allocated memory

- Question
  - Why do we need a temporary array **b[]**?

Merge Sort: Analysis

- In `mergeSort()`, the bulk of work is done in the **merge** step
- For `merge(a, low, mid, high)`
  - Let total items = k = (high – low + 1)
  - Number of comparisons $\leq k - 1$
  - Number of moves from original array to temporary array = k
  - Number of moves from temporary array back to original array = k
- In total, number of operations $\leq 3k - 1 = O(k)$
- The important question is
  - How many times is `merge()` called?

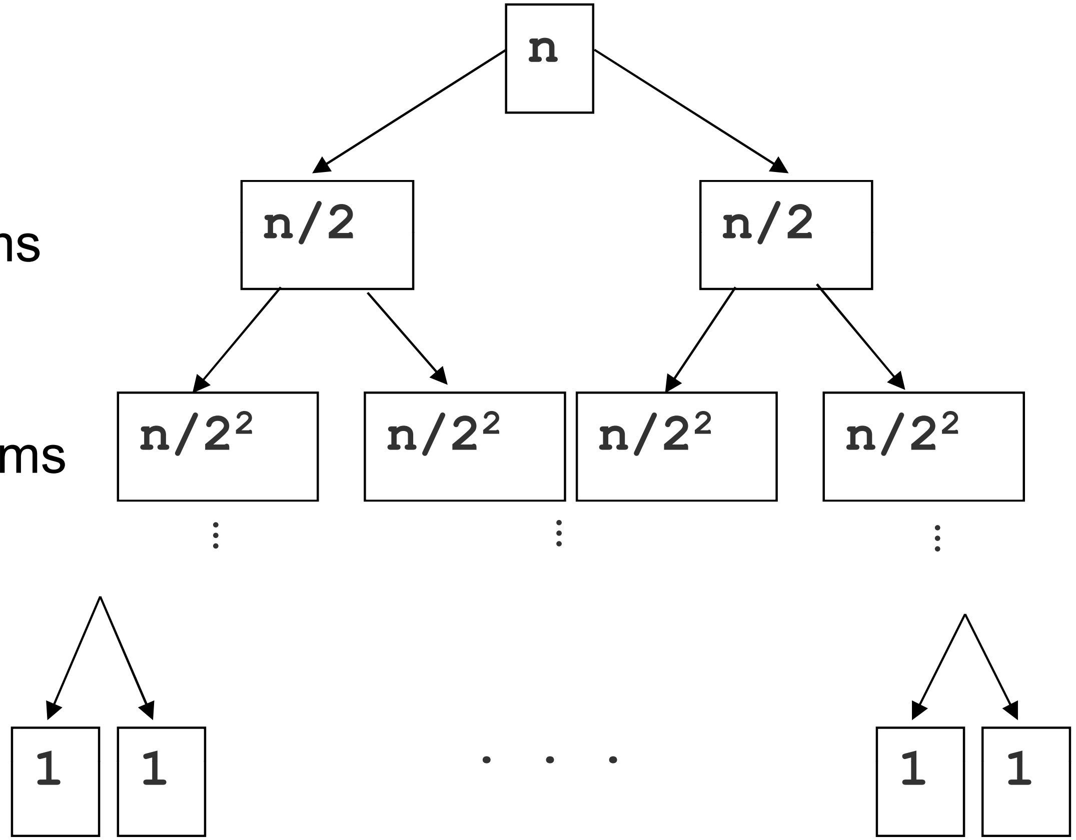
Merge Sort: Analysis

Level 0:
mergeSort **n** items

Level 1:
mergeSort **n/2** items

Level 2:
mergeSort **$n/2^2$** items

Level (**lg *n***):
mergeSort **1** item



Level 0:
1 call to mergeSort

Level 1:
2 calls to mergeSort

Level 2:
2^2 calls to mergeSort

Level (**lg *n***):
$2^{\lg n}$(= *n*) calls to mergeSort

$n/(2^k) = 1 \rightarrow n = 2^k \rightarrow k = \lg n$

Merge Sort: Analysis

- Level 0: **0** call to `merge()`
- Level 1: **1** calls to `merge()` with ***n*/2** items in each half,
  $O(1 \times 2 \times n/2) = O(n)$ time
- Level 2: **2** calls to `merge()` with $\boldsymbol{n/2^2}$ items in each half,
  $O(2 \times 2 \times n/2^2) = O(n)$ time
- Level 3: $\mathbf{2^2}$ calls to `merge()` with $\boldsymbol{n/2^3}$ items in each half,
  $O(2^2 \times 2 \times n/2^3) = O(n)$ time
- …
- Level ($\lg n$): $\mathbf{2^{\lg(n) - 1}(= n/2)}$ calls to `merge()` with $\boldsymbol{n/2^{\lg(n)}}$ **(= 1)** item in each half, $O(n)$ time
- Total time complexity = $\mathbf{O(n \lg(n))}$
- **Optimal** comparison-based sorting method

Merge Sort: Pros and Cons

- Pros
  - The performance is guaranteed, i.e. unaffected by original ordering of the input
  - Suitable for extremely large number of inputs
    - Can operate on the input portion by portion

- Cons
  - Not easy to implement
  - Requires additional storage during merging operation
    - $O(n)$ extra memory storage needed

Quick Sort

Quick Sort: Idea

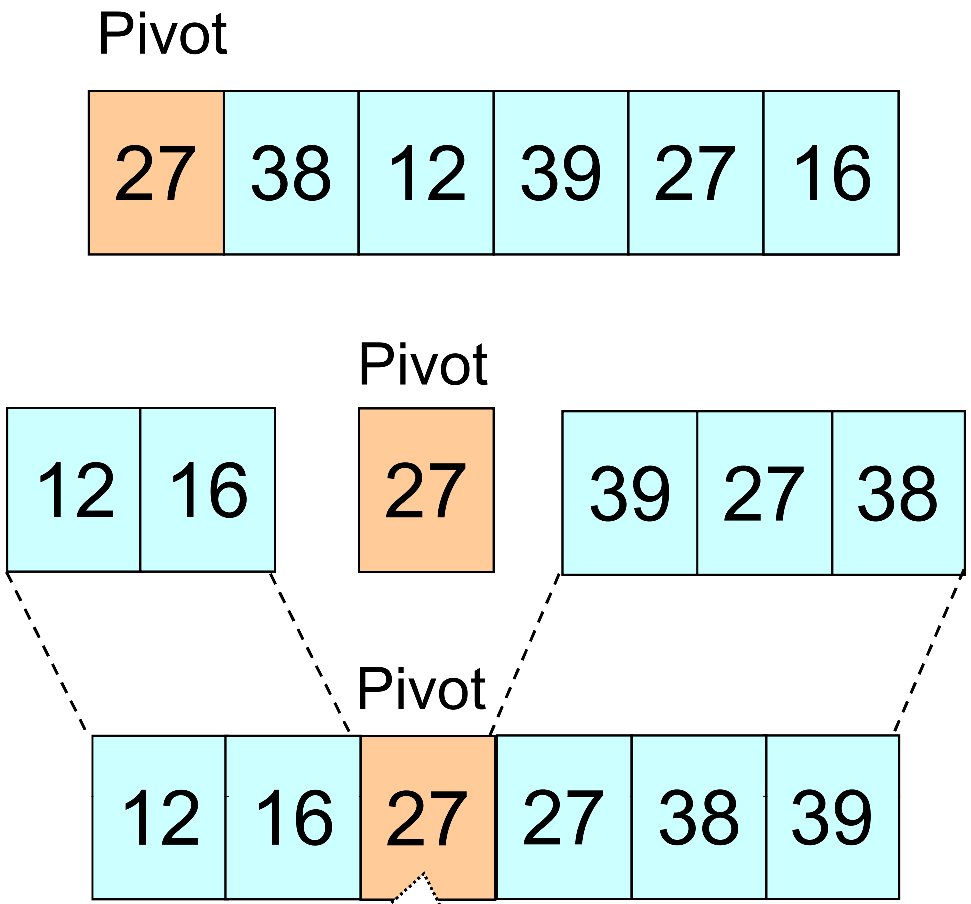
- **Quick Sort** is a divide-and-conquer algorithm
  - **Divide** step
    - Choose an item ***p*** (known as **pivot**) and partition the items of a[i...j] into two parts
      - Items that are smaller than ***p***
      - Items that are greater than or equal to ***p***
    - Recursively sort the two parts
  - **Conquer** step
    - Do nothing!
- In comparison, **Merge Sort** spends most of the time in conquer step but very little time in divide step

Quick Sort: Divide Step Example

Choose first element as pivot

Partition **a[]** about the pivot **27**

Recursively sort the two parts



Notice anything special about the position of pivot in the final sorted items?

Quick Sort: Implementation

```
void quickSort(int a[], int low, int high) {
  if (low < high) {
    int pivotIdx = partition(a, low, high);

    quickSort(a, low, pivotIdx-1);
    quickSort(a, pivotIdx+1, high);
  }
}
```

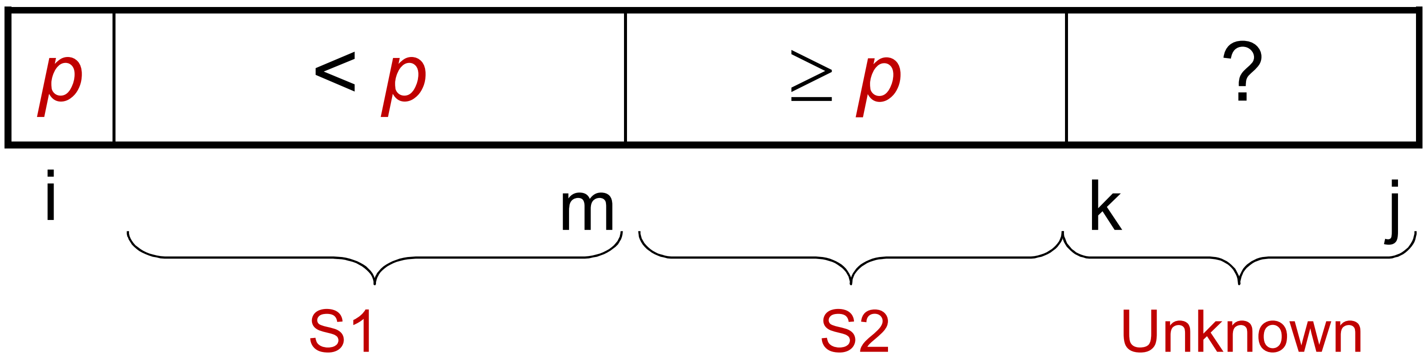
Partition **a[low...high]** and return the index of the pivot item

Recursively sort the two portions

- **partition()** splits **a[low...high]** into two portions
  - **a[low ... pivot–1]** and **a[pivot+1 ... high]**
- Pivot item does not participate in any further sorting

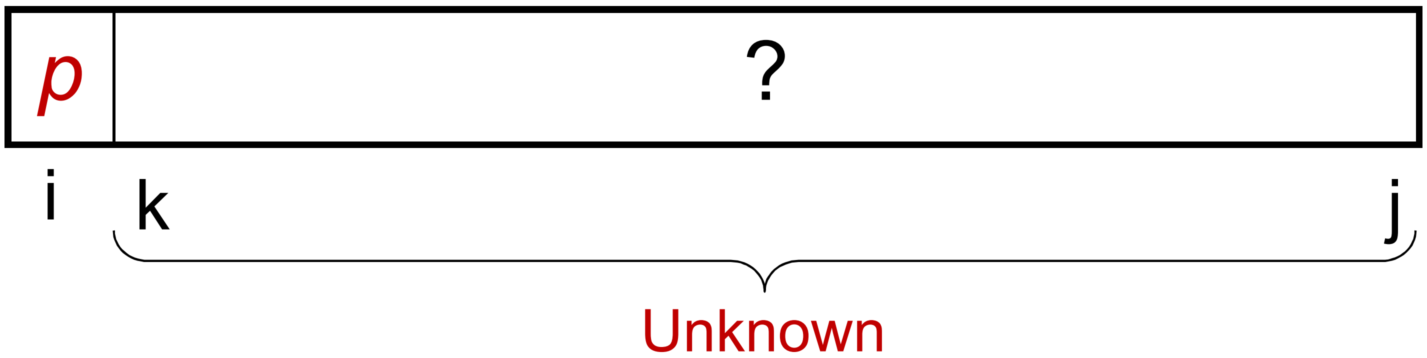
Quick Sort: Partition Algorithm

- To partition a[i...j], we choose a[i] as the pivot ***p***
  - Why choose a[i]? Are there other choices?
- The remaining items (i.e. a[i+1...j]) are divided into 3 regions
  - **S1** = a[i+1...m] where items < *p*
  - **S2** = a[m+1...k-1] where item ≥ *p*
  - **Unknown** (unprocessed) = a[k...j], where items are yet to be assigned to S1 or S2



Quick Sort: Partition Algorithm

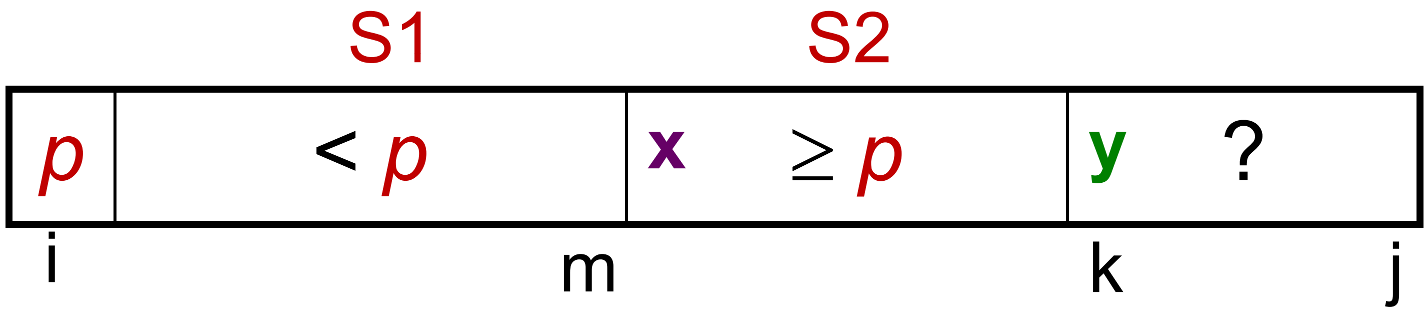
- Initially, regions S1 and S2 are empty
  - All items excluding *p* are in the unknown region
- For each item a[k] in the unknown region
  - Compare a[k] with *p*
    - If a[k] >= *p*, put it into S2
    - Otherwise, put a[k] into S1



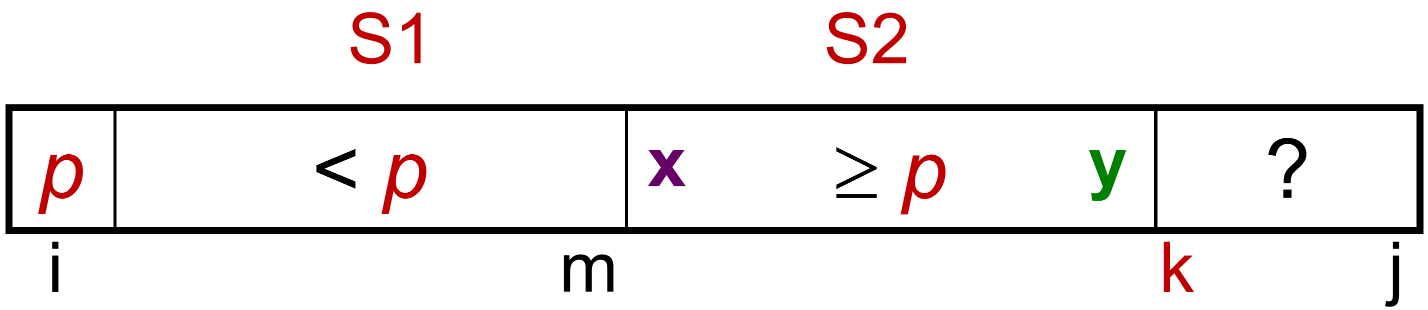
Quick Sort: Partition Algorithm

- Case 1: if a[k] >= *p*

If a[k]=y $\geq$ *p*,



Increment k



Quick Sort: Partition Algorithm

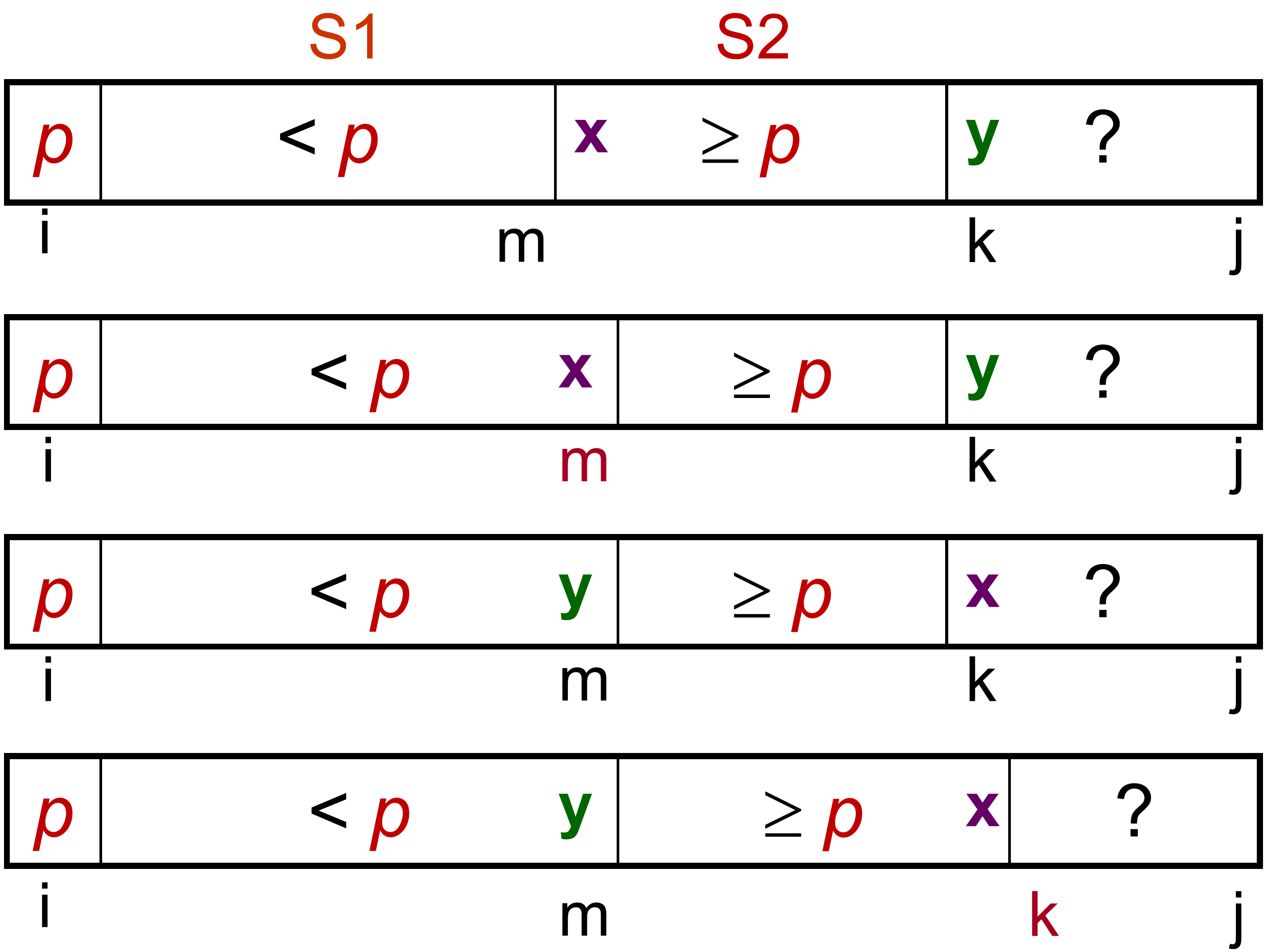
- Case 2: if a[k] < *p*

If a[k]=y < *p*

Increment m

Swap x and y

Increment k



Quick Sort: Partition Implementation

PS: C++ STL <algorithm> has partition subroutine too

```
int partition(int a[], int i, int j) {
  int p = a[i];
  int m = i;

  for (int k = i+1; k <= j; k++) {
    if (a[k] < p) {
      m++;
      swap(a[k], a[m]);
    }
    else {
    }
  }
  swap(a[i], a[m]);
  return m;
}
```

p is the pivot

S1 and **S2** empty initially

Go through each element in unknown region

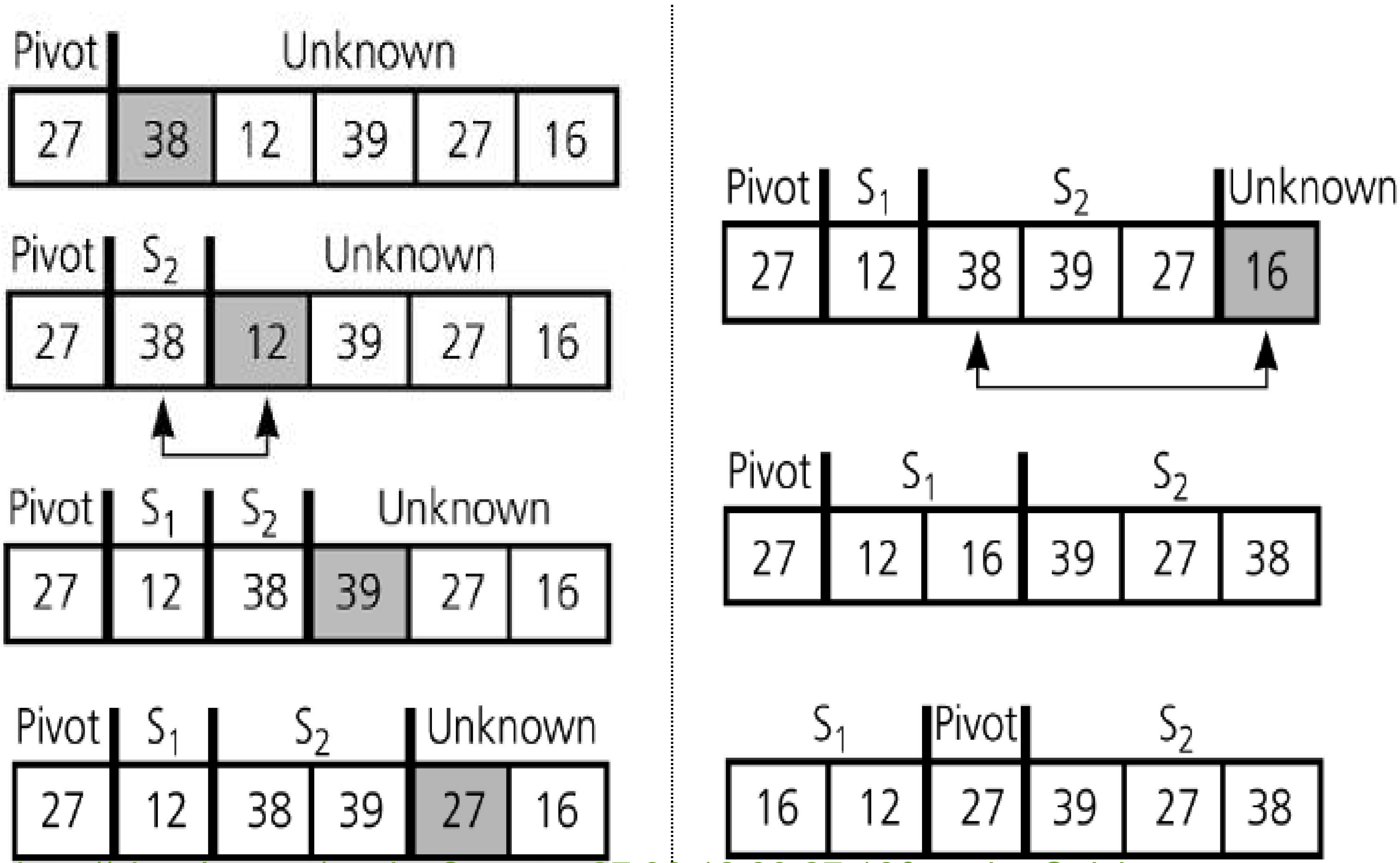
Case 2

Case 1: Do nothing!

Swap pivot with **a[m]**

m is the index of pivot

Quick Sort: Partition Example



http://visualgo.net/sorting?create=27,38,12,39,27,16&mode=Quick

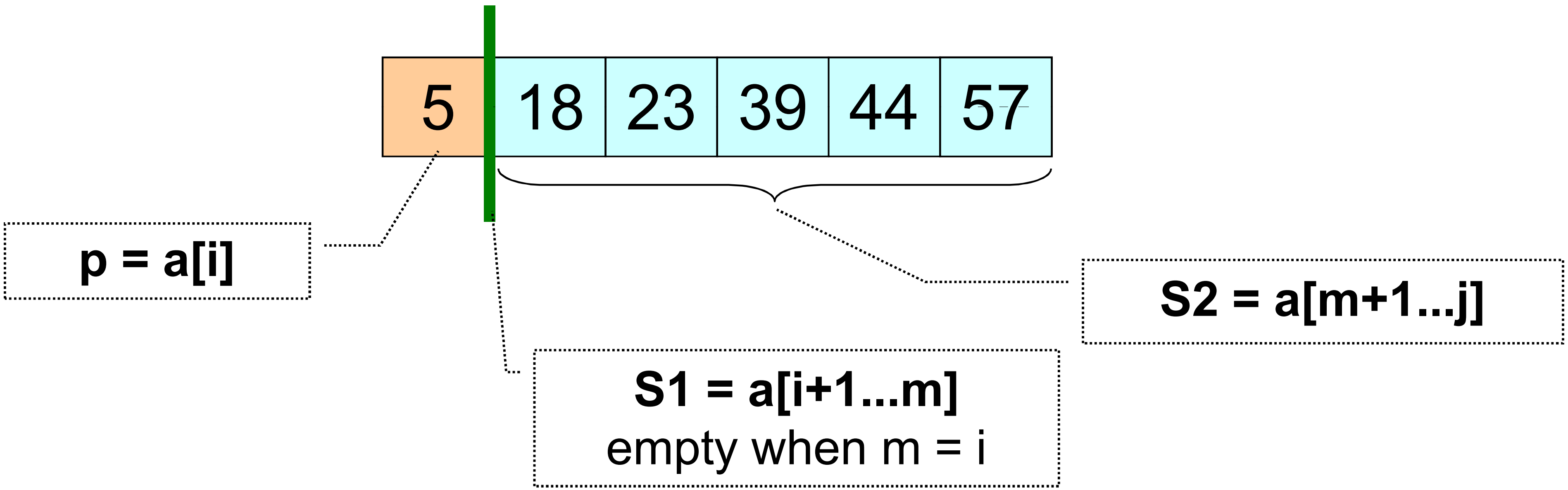
Quick Sort: Partition Analysis

- There is only a single for-loop
  - Number of iterations = number of items, n, in the unknown region
    - n = high – low
  - Complexity is O(n)

- Similar to **Merge Sort**, the complexity is then dependent on the number of times **partition()** is called

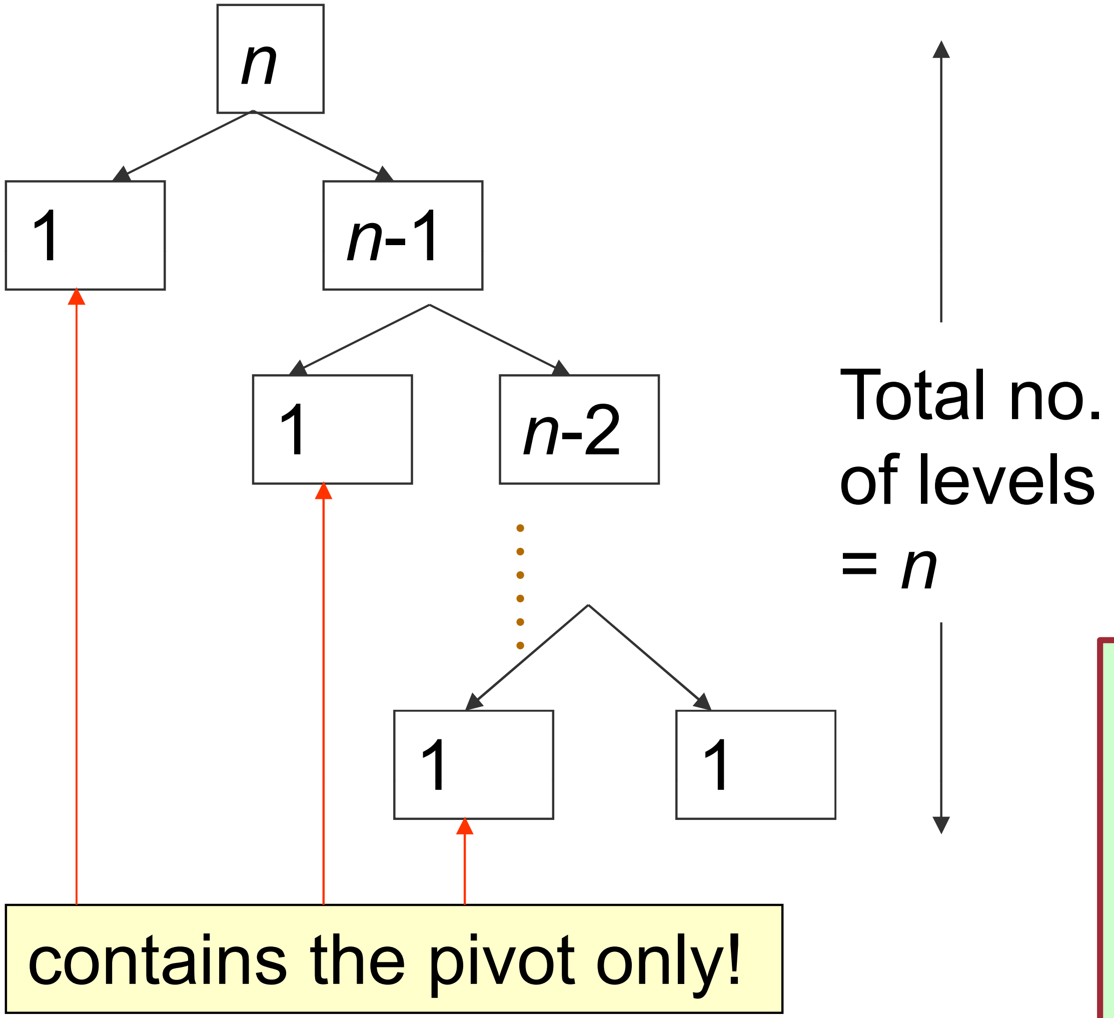
Quick Sort: Worst Case Analysis

- When the array is already in ascending order



- What is the pivot index returned by **partition()**?
  - What is the effect of swap(a, i, m)?
- S1 is empty, while S2 contains every item except the pivot

Quick Sort: Worst Case Analysis



As each partition takes linear time, the algorithm in its worst case has n levels and hence it takes time $n+(n-1)+...+1 = O(n^2)$

Quick Sort: Best/Average Case Analysis

- Best case occurs when partition always splits the array into **two equal halves**
  - Depth of recursion is $\log n$
  - Each level takes n or fewer comparisons, so the time complexity is $O(n \log n)$
- In practice, worst case is rare, and on the average we get some good splits and some bad ones (details in CS3230 :O)
  - Average time is also $O(n \log n)$

Lower Bound: Comparison-Based Sort

- It is known that
  - All **comparison-based** sorting algorithms have a complexity **lower bound** of $n \log n$

- Therefore, any comparison-based sorting algorithm with **worst-case complexity** O(n log n) is **optimal**

Radix Sort

Radix Sort: Idea

- Treats each data to be sorted as a **character string**
- It is not using comparison, i.e. **no comparison** between the data is needed
- In each iteration
  - Organize the data into groups according to the next character in each data
  - The groups are then "concatenated" for next iteration

Radix Sort: Example

0123, 2154, 0222, 0004, 0283, 1560, 1061, 2150	Original integers
(156**0**, 215**0**) (106**1**) (022**2**) (012**3**, 028**3**) (215**4**, 000**4**)	Grouped by fourth digit
1560, 2150, 1061, 0222, 0123, 0283, 2154, 0004	Combined
(00**0**4) (02**2**2, 01**2**3) (21**5**0, 21**5**4) (15**6**0, 10**6**1) (02**8**3)	Grouped by third digit
0004, 0222, 0123, 2150, 2154, 1560, 1061, 0283	Combined
(0**0**04, 1**0**61) (0**1**23, 2**1**50, 2**1**54) (0**2**22, 0**2**83) (1**5**60)	Grouped by second digit
0004, 1061, 0123, 2150, 2154, 0222, 0283, 1560	Combined
(**0**004, **0**123, **0**222, **0**283) (**1**061, **1**560) (**2**150, **2**154)	Grouped by first digit
0004, 0123, 0222, 0283, 1061, 1560, 2150, 2154	Combined (sorted)

Radix Sort: Implementation

```
void radixSort(vector<int>& v, int d) {
  int i;
  int power = 1;
  queue<int> digitQueue[10];

  for (i = 0; i < d; i++) {
    distribute(v, digitQueue, power);
    collect(digitQueue, v);
    power *= 10;
  }
}
```

10 groups. Each is a queue to retain the order of item

- **distribute()**: Organize all items in **v** into groups using digit indicated by the power
- **collect()**: Place items from the groups back into **v**, i.e. “concatenate” the groups

Radix Sort: Implementation

```
void distribute(vector<int>& v,
                queue<int> digitQ[], int power) {
  int digit;
  for (int i = 0; i < v.size(); i++){
    digit = (v[i]/power) % 10;
    digitQ[digit].push(v[i]);
  }
}
```

- Question
  - How do we extract the digit used for the current grouping?

Radix Sort: Implementation

```
void collect(queue<int> digitQ[], vector<int>& v) {
  int i = 0, digit;

  for (digit = 0; digit < 10; digit++)
    while (!digitQ[digit].empty()) {
      v[i] = digitQ[digit].front();
      digitQ[digit].pop();
      i++;
    }
}
```

- Basic Idea
  - Start with `digitQ[0]`
    - Place all items into vector **v**
  - Repeat with `digitQ[1]`, `digitQ[2]`, ...

Radix Sort: Analysis

- For each iteration
  - We go through each item once to place them into group
  - Then go through them again to concatenate the groups
  - Complexity is O(n)
- Number of iterations is d, the maximum number of digits (or maximum number of characters)
- Complexity is thus **O(dn)**

Properties of Sorting

In-Place Sorting

- A sort algorithm is said to be an **in-place** sort
  - If it requires only a **constant amount** (i.e. O(1)) of **extra space** during the sorting process

- Questions
  - **Merge Sort** is not in-place, why?
  - Is **Quick Sort** in-place?
  - Is **Radix Sort** in-place?

Stable Sorting

- A sorting algorithm is **stable** if the **relative order** of **elements with the same key value** is preserved by the algorithm

- Example application of stable sort
  - Assume that **names** have been sorted in alphabetical order
  - Now, if this list is sorted again by **tutorial group number**, a stable sort algorithm would ensure that all students in the same tutorial groups still appear in alphabetical order of their names

Non-Stable Sort

- Selection Sort

  $1285\ 5_a\ 4746\ 602\ 5_b\ (8356)$

  $1285\ 5_a\ 5_b\ 602\ (4746\ 8356)$

  $602\ 5_a\ 5_b\ (1285\ 4746\ 8356)$

  $5_b\ 5_a\ (602\ 1285\ 4746\ 8356)$

- Quick Sort
  - $\underline{1285}\ 5_a\ 150\ 4746\ 602\ 5_b\ 8356$ (pivot=1285)
  - $\underline{1285}\ (5_a\ 150\ 602\ 5_b)\ (4746\ 8356)$
  - $5_b\ 5_a\ 150\ 602\ 1285\ 4746\ 8356$

Sorting Algorithms: Summary

	Worst Case	Best Case	In-place?	Stable?
`Selection Sort`	$O(n^2)$	$O(n^2)$	Yes	No
`Insertion Sort`	$O(n^2)$	$O(n)$	Yes	Yes
`Bubble Sort`	$O(n^2)$	$O(n^2)$	Yes	Yes
`Bubble Sort 2`	$O(n^2)$	$O(n)$	Yes	Yes
`Merge Sort`	$O(n \lg n)$	$O(n \lg n)$	No	Yes
`Quick Sort`	$O(n^2)$	$O(n \lg n)$	Yes	No
`Radix sort`	$O(dn)$	$O(dn)$	No	yes

Summary

- Comparison-Based Sorting Algorithms
  - Iterative Sorting
    - Selection Sort
    - Bubble Sort
    - Insertion Sort
  - Recursive Sorting
    - Merge Sort
    - Quick Sort
- Non-Comparison-Based Sorting Algorithms
  - Radix Sort
- Properties of Sorting Algorithms
  - In-Place
  - Stable