Mathematical foundations for data Science II

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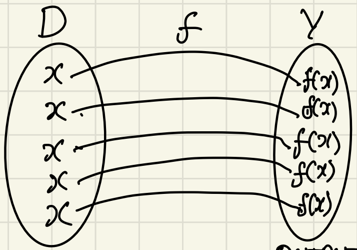
PROPERTIES OF FUNCTIONS 1D

A FUNCTION IS A RULE THAT ASSIGNS A UNIQUE ELEMENT $f(x) \in Y$ TO EACH ELEMENT $x \in D$



INPUT (DOMAIN) ↳ SET OF ELEMENTS AT D

OUTPUT (RANGE) ↳ SET OF ALL ELEMENTS

1. $f(x) = 1 + x^2$

$f(x): \mathbb{R} \to \mathbb{R}$

$\boxed{\text{Domain}: \mathbb{R}}$ (SINCE FOR NO $x \in \mathbb{R}$ f IS UNDEFINED)

$\boxed{\text{Range}: \{y \geq 1,\ y \in \mathbb{R}\}}$

2. $f(x) = 1 - \sqrt{x}$

$f(x): \mathbb{R} \to \mathbb{R}$

$\boxed{\begin{array}{l} \text{DOMAIN}: \{x \in \mathbb{R} \mid x \geq 0\} \\ \text{RANGE}: \{y \leq 1 \mid y \in \mathbb{R}\} \end{array}}$

3. $F(x) = \sqrt{5x+10}$

4. $g(x) = \sqrt{x^2 - 3x}$

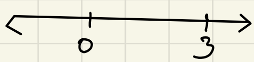
5. $f(t) = \dfrac{4}{3-t}$

6. $G(t) = \dfrac{2}{t^2 - 16}$.

4. DOMAIN: $\{x \in \mathbb{R} \mid x^2 - 3x \geq 0\}$

$x^2 - 3x = 0$

$\Rightarrow x = 0, 3$



DOM $\Rightarrow \{x \in \mathbb{R} \mid x \in (-\infty, 0] \cup [3, \infty)\}$

6. $f(x) = \dfrac{2}{x^2 - 16}$

$\boxed{\begin{array}{l} \text{DOM} = \{x \in \mathbb{R} \mid x \neq \pm 4\} \\ \mathbb{R} - \{-4, 4\} \end{array}}$

$y = \dfrac{2}{x^2 - 16}$

$y(x^2 - 16) = 2$

$yx^2 = 2 + 16y$

$x = \sqrt{\dfrac{2 + 16y}{y}}$ → PERMISSIBLE VALUES (INVERSE OF $f(x)$)

$\Rightarrow \quad 2+16y \geq 0 \quad \& \quad y>0$

$y \geq \dfrac{-2}{16}$

$y \geq \dfrac{1}{8} \quad \& \quad y>0$

$\Rightarrow \quad y>0$

$2+16y \leq 0 \quad \& \quad y<0$

$y \leq \dfrac{-1}{8} \quad \& \quad y<0$

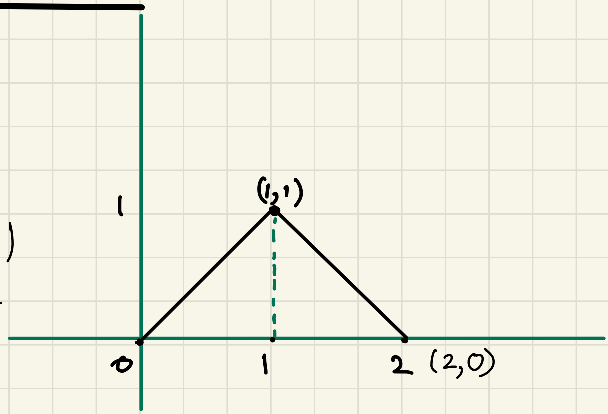
$\Rightarrow \quad y \leq \dfrac{-1}{8}$

$$\Rightarrow \boxed{\text{RAN} = \{ y \in \mathbb{R} \mid y \in (-\infty, -1/8] \cup [0, \infty) \}}$$

PIECEWISE FUNCTIONS

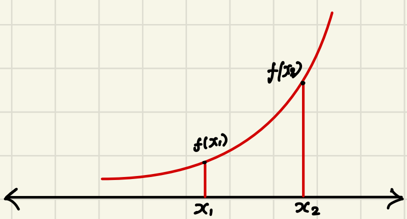
$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x<0 \end{cases}$$

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2-x \, ; & 1 < x \leq 2 \end{cases}$$



THIS IS A FUNCTION SINCE VERTICAL LINES SHOW ONLY ONE VALUE FOR x

INCREASING / DECREASING FUNCTION



f IS INCREASING IN INTERVAL (x_1, x_2)

$f(x) = x+5$
IS AN INCREASING FUNCTION

$$\boxed{\begin{array}{l} \text{EVEN} \Rightarrow f(-x) = f(x) \\ \text{ODD} \Rightarrow f(-x) = -f(x) \end{array}}$$

$y = x^3$

$y = x^2$

$\boxed{f(x) = x+7}$ NEITHER EVEN OR ODD

LIMITS

TO KNOW WHAT A FUNCTION APPROXIMATES TO TOWARDS A VALUE IS CALLED APPLYING LIMITS.

$$\boxed{\lim_{x \to 5} x^2+7 = 25+7 = 32}$$

$$\lim_{x \to -3} (x^2-13)$$
$$\Rightarrow 9-13$$
$$\Rightarrow -4$$

$$\lim_{x \to 1} \frac{x^2+x-2}{x^2-1}$$
$$\Rightarrow \frac{(x+2)\cancel{(x-1)}}{(x+1)\cancel{(x-1)}}$$
$$\Rightarrow \lim_{x \to 1} \frac{x+2}{x+1} = \frac{3}{2}$$

$$\lim_{x \to 0} \frac{\sin x}{x}$$
$$= \lim_{x \to 0} \frac{\cos x}{1} = 1$$

$$\lim_{x \to 1} f(x) \qquad f(x) = \begin{cases} 4-2x & x<1 \\ 6x-4 & x \geq 1 \end{cases}$$

LHL
$$f(x) = 4-2x$$
$$= 2$$

RHL
$$f(x) = 6x-4$$
$$= 2$$

CONTINUOUS FUNCTION

$$f(x) = x^2+7 \quad \text{at } x=0$$
$$f(0) = 7 \quad \text{(EXISTS)}$$
$$\lim_{x \to 0} (x^2+7) \quad \text{(EXISTS)}$$
$$\lim_{x \to 0} (x^2+7) = 7 == f(0) \quad \text{(EXISTS)}$$

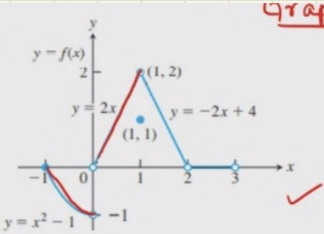
SINCE ALL THIS IS SATISFIED THE FUNCTION IS SAID TO BE CONT. AT 0

$$f(x) = \begin{cases} x^2 - 1, & -1 \le x < 0 \\ \cancel{2x}, & 0 < x < 1 \\ 1, & x = 1 \\ -2x + 4, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$$

NOT VALID AT $x=0$

NOT VALID AT $x=2$

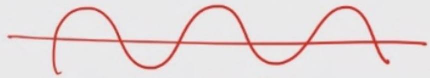
Graph



$x = 0, 1, 2$

not cont. at $x=0$

(No def. ~~for f(x)~~ at $x=0$)



cont.

HERE YOU SEE THAT THE GRAPH IS NOT DRAWN CONTINUOSLY

at $x = 1$

LHL = 2 RHL = 2

$$\lim_{x \to 1} f(x) = 2 \neq f(1)$$

DERIVATIVE

THE DERIVATIVE OF A FUNCTION f AT x_0, IS DENOTED $f'(x_0)$ IS:

$$\boxed{f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}}$$

GIVEN THAT LIMIT EXISTS

Ex: $f(x) = x^2$

$\Rightarrow$

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{(2+h)^2 - 4}{h}$$

$$= \lim_{h \to 0} \frac{4h + h^2}{h} = 4 + h$$

$$= 4$$

SO A GENERAL FORMULA

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$y = 2x^3 =$ SINCE $\boxed{f'(x^n) = n\ x^{n-1}}$

$$\frac{dy}{dx} = 2 * 3x^2 = 6x^2$$

EXAMPLE

$$\boxed{y = |x| = \begin{cases} -x, & x<0 \\ x, & x\geq 0 \end{cases}}$$

$$f'(0) = \lim_{\cancel{h} \to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)-f(0)}{h}$$

LHL

$$\lim_{\cancel{h}\to 0} \frac{-h}{h} = -1$$

RHL

$$\lim_{h\to 0} \frac{h}{h} = 1$$

SINCE LHL $\neq$ RHL WE SAY IT IS NOT POSSIBLE SINCE LIMIT DOES NOT EXISTS

THEOREM

IF A FUNCTION IS DERIVABLE THEN IT IS CONTINUOUS, BUT OTHER WAY IS NOT TRUE

NOTE: ALTHOUGH $|x|$ IS CONTINUOUS IT IS NOT DERIVABLE

DIFFERENTIATION RULES

General Formulas

Assume u and v are differentiable functions of x.

Constant: $\frac{d}{dx}(c) = 0$

Sum: $\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$

Difference: $\frac{d}{dx}(u - v) = \frac{du}{dx} - \frac{dv}{dx}$

Constant Multiple: $\frac{d}{dx}(cu) = c\frac{du}{dx}$

Product: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

Quotient: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Power: $\frac{d}{dx}x^n = nx^{n-1}$

Chain Rule: $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$

Trigonometric Functions

$\frac{d}{dx}(\sin x) = \cos x$ $\qquad$ $\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\qquad$ $\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\cot x) = -\csc^2 x$ $\qquad$ $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Exponential and Logarithmic Functions

$\frac{d}{dx}e^x = e^x$ $\qquad$ $\frac{d}{dx}\ln x = \frac{1}{x}$

$\frac{d}{dx}a^x = a^x \ln a$ $\qquad$ $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

Inverse Trigonometric Functions

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$ $\qquad$ $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$ $\qquad$ $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}}$

$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1 + x^2}$ $\qquad$ $\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2 - 1}}$

Hyperbolic Functions

$\frac{d}{dx}(\sinh x) = \cosh x$ $\qquad$ $\frac{d}{dx}(\cosh x) = \sinh x$

$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$ $\qquad$ $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$ $\qquad$ $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

Inverse Hyperbolic Functions

$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1 + x^2}}$ $\qquad$ $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}$

$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1 - x^2}$ $\qquad$ $\frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1 - x^2}}$

$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1 - x^2}$ $\qquad$ $\frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{1 + x^2}}$

Parametric Equations

If $x = f(t)$ and $y = g(t)$ are differentiable, then

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}.$$

EXAMPLE

$$\frac{d}{dx}(x^2 + \cos x) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\cos x)$$
$$= 2x - \sin x$$

$$\frac{d}{dx}(\sin x^2) = \cos x^2 \, \frac{d}{dx}(x^2$$
$$= (\cos x^2)(2x)$$

$$\frac{d}{dy}(1 + x^2)(x^{3/4} - x^{-3}) = (2x)(x^{3/4} - x^{-3}) + (1 + x^2)\left(\frac{3}{4}x^{-1/4} + 3x^{-4}\right)$$
$$= 2x^{7/4} - 2x^{-2}$$

$$f(t) = \frac{t^2-1}{t^2+t-2} = \frac{u}{v} \Rightarrow \frac{d(u)}{d(t)} = 2t \qquad \frac{dv}{dt} = 2t+1$$

$$\frac{df}{dt} = \left(v\frac{du}{dt} - u\frac{dv}{dt}\right) \Big/ v^2$$

$$= \frac{(t^2+t-2)\,2t - (t^2-1)(2t+1)}{(t^2+t-2)^2}$$

$$= \frac{2t^3+2t^2-4t-2t^3-t^2+2t+1}{(t^2+t-2)^2}$$

Leibniz's rule for higher-order derivatives of products Leibniz's rule for higher-order derivatives of products of differentiable functions says that

a. $$\frac{d^2(uv)}{dx^2} = \frac{d^2u}{dx^2}v + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}.$$

$$\rightarrow {}^2C_0\, u^{(2)}v + {}^2C_1\, u^{(1)}v^{(1)} + {}^2C_2\, u v^{(2)}$$

b. $$\frac{d^3(uv)}{dx^3} = \frac{d^3u}{dx^3}v + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + 3\frac{du}{dx}\frac{d^2v}{dx^2} + u\frac{d^3v}{dx^3}.$$

$$\rightarrow {}^3C_0\, u^{(3)}v + {}^3C_1\, u^{(2)}v^{(2)} + {}^3C_2\, u^{(1)}v^{(3)}$$

c. $$\frac{d^n(uv)}{dx^n} = \frac{d^nu}{dx^n}v + n\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx} + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!}\frac{d^{n-k}u}{dx^{n-k}}\frac{d^kv}{dx^k} + \cdots + u\frac{d^nv}{dx^n}.$$

$$\frac{d^3}{dx^3}(\sin x\ e^x)$$

$$= {}^3C_0 (\sin x)''' e^x + {}^3C_1 (\sin x)''(e^x)' + {}^3C_2 (\sin x)'(e^x)'' + {}^3C_3 (\sin x)(e^x)'''$$

$=$

$$\boxed{\begin{array}{ccc} & e^x & \sin x \\ & \downarrow & \downarrow \\ ' & e^x & \cos x \\ & \downarrow & \downarrow \\ '' & e^x & -\sin x \\ & \downarrow & \downarrow \\ ''' & e^x & -\cos x \end{array}}$$

H.W.

$$\frac{d^3}{dx}\left(\frac{\sin x}{x}\right)$$

Taylor Series Expansion

DEFINITIONS Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin series of f** is the Taylor series generated by f at $x = 0$, or

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots.$$

$\sin x \approx x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} \ldots$ (TAYLOR SERIES FOR $\sin x$)

EXAMPLE

$f(x) = \sin x$, $a=0$ (MACLURIN)

$f(x) = \sin x$, $f(0) = 0$

$f'(x) = \cos x$, $f'(0) = 1$

$f''(x) = -\sin x$, $f''(0) = 0$

$f'''(x) = -\cos x$, $f'''(0) = -1$

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0)$

$f(x) = 0 + x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots$

DEFINITION Let f be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the **Taylor polynomial of order n** generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

EXAMPLE $f(x) = 1/x$, $a=2$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots$$

$$= \frac{1}{2} - \frac{(x-2)}{4} + \frac{(x-2)^2}{2!} \times \frac{1}{4} + \dots$$

$f(x) = \frac{1}{x} = x^{-1}$ $f(a) = \frac{1}{2}$

$f'(x) = \frac{-1}{x^2}$ $f'(a) = \frac{-1}{4}$

$f''(x) = \frac{2}{x^3}$ $f''(a) = \frac{2}{8} = \frac{1}{4}$

CONVERGENCE

Taylor's Formula

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x), \quad (1)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x. \quad (2)$$

If $R_n(x) \to 0$ as $n \to \infty$ for all $x \in I$, we say that the Taylor series generated by f at $x = a$ **converges** to f on I, and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

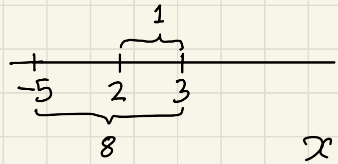
EXAMPLE $(1-x)^{-1}$ ROC : $|x|<1$

$f(x) = \frac{x}{(x-2)(x+5)}$ about $x=3$ (H.W. FOR T.S.)

ROC: $|x-3| < 1$ (SINCE SINGULARITY IS REACHED IN 1 DISTANCE)

$\Rightarrow -1 < x-3 < 1$

$\Rightarrow 2 < x < 4$



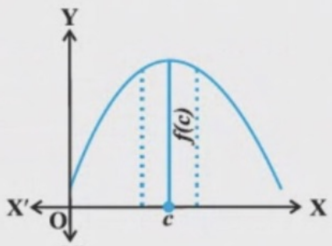
$x = 6$

MAXIMA & MINIMA

Definition 3 Let f be a function defined on an interval I. Then

(a) f is said to have a *maximum value* in I, if there exists a point c in I such that $f(c) > f(x)$, for all $x \in$ I.

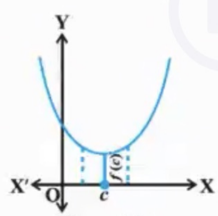
The number $f(c)$ is called the maximum value of f in I and the point c is called a *point of maximum value* of f in I.



(a)

(b) f is said to have a minimum value in I, if there exists a point c in I such that $f(c) < f(x)$, for all $x \in$ I.

The number $f(c)$, in this case, is called the minimum value of f in I and the point c, in this case, is called a *point of minimum value* of f in I.



(c)

$f(x) > f(c)$

Definition 4 Let f be a real valued function and let c be an interior point in the domain of f. Then

(a) c is called a point of *local maxima* if there is an $h > 0$ such that

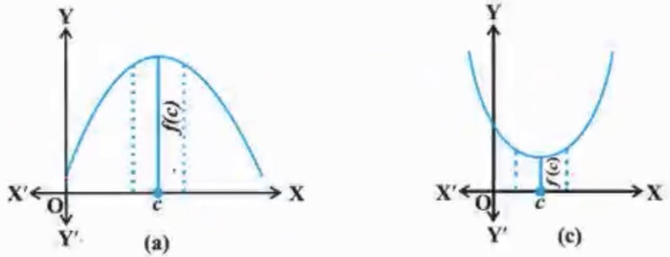
$$f(c) \geq f(x), \text{ for all } x \text{ in } (c-h, c+h), x \neq c$$

The value $f(c)$ is called the *local maximum value* of f.

(b) c is called a point of *local minima* if there is an $h > 0$ such that

$$f(c) \leq f(x), \text{ for all } x \text{ in } (c-h, c+h)$$

The value $f(c)$ is called the *local minimum value* of f.



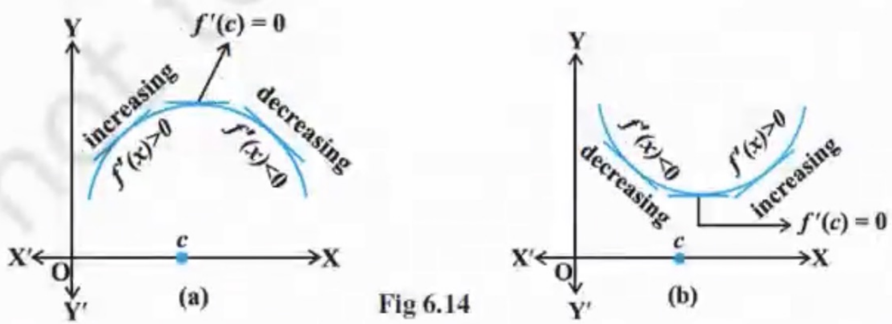


Fig 6.14

Theorem 2 Let f be a function defined on an open interval I. Suppose $c \in$ I be any point. If f has a local maxima or a local minima at $x = c$, then either $f'(c) = 0$ or f is not differentiable at c. when $f(x) = |x|$

$f(x) = x^2 - 2x + 5 \Rightarrow f'(x) = 2x - 2$

at $x = 1$ $x = 1$ IS A CRITICAL POINT

$f'(x) = 0$

WAYS TO CHECK TYPE OF EXTREME

Theorem 3 (First Derivative Test) Let f be a function defined on an open interval I. Let f be continuous at a critical point c in I. Then

(i) If $f'(x)$ changes sign from positive to negative as x increases through c, i.e., if $f'(x) > 0$ at every point sufficiently close to and to the left of c, and $f'(x) < 0$ at every point sufficiently close to and to the right of c, then c is a point of *local maxima*.

(ii) If $f'(x)$ changes sign from negative to positive as x increases through c, i.e., if $f'(x) < 0$ at every point sufficiently close to and to the left of c, and $f'(x) > 0$ at every point sufficiently close to and to the right of c, then c is a point of *local minima*.

(iii) If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Infact, such a point is called *point of inflection* (Fig 6.15).

TAKING PREVIOUS EXAMPLE

LHS OF $x=1$ $\Rightarrow f'(0) = -2$ (DECREASING)

RHS OF $x=1$ $\Rightarrow f'(2) = 2$ (INCREASING)

SINCE IT DECREASES FIRST AND THEN INCREASES SO IT IS A LOCAL MINIMA AT THE POINT $x=1$

Find the maximum and minimum values, if any, of the following functions given by

(i) $f(x) = |x + 2| - 1$

(ii) $g(x) = -|x + 1| + 3$

(iii) $h(x) = \sin(2x) + 5$

(iv) $f(x) = |\sin 4x + 3|$

(v) $h(x) = x + 1, x \in (-1, 1)$

(i) $f(x) = x+1$

$f'(x) = 1$

(NO EXTREME)

(ii) $g(x) = -|x+1|+3$

$g'(x)$ NOT POSSIBLE AT $\boxed{x=-1}$

$\therefore$ $x=-1$ IS A CRITICAL POINT

(iii) $h(x) = \sin(2x) + 5$

$h'(x) = 2\cos(2x)$

$h'(x) = 0 = 2\cos(2x)$

$\Rightarrow$

Theorem 4 (Second Derivative Test) Let f be a function defined on an interval I and $c \in$ I. Let f be twice differentiable at c. Then

(i) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$

The value $f(c)$ is local maximum value of f.

(ii) $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$

In this case, $f(c)$ is local minimum value of f.

(iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$.

In this case, we go back to the first derivative test and find whether c is a point of local maxima, local minima or a point of inflexion.

GLOBAL MAXIMA

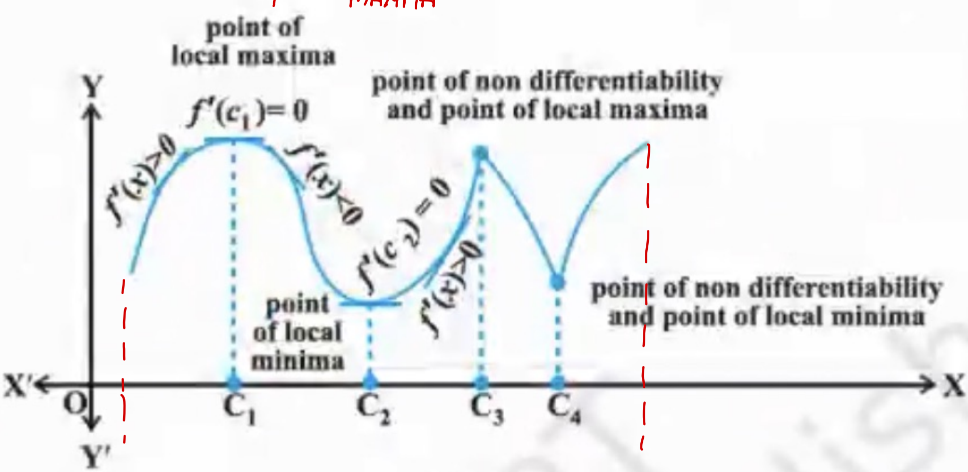


Fig 6.16

EXAMPLE

$f(x) = |x+2| - 1$

$f'(x)$ DOES NOT EXIST AT $x = -2$

$$\boxed{f(x) = \begin{cases} -(x+2)-1 = -x-3 & , x < -2 \\ (x+2) - 1 = x+1 & , x \geq -2 \end{cases}}$$

$$\Rightarrow \boxed{f'(x) = \begin{cases} -1 & , x < -2 \\ 1 & , x \geq -2 \end{cases}}$$

$f(x)$ HAS A MINIMA AT $x = -2$

---

4. Prove that the following functions do not have maxima or minima:

(i) $f(x) = e^x$ (ii) $g(x) = \log x$

(iii) $h(x) = x^3 + x^2 + x + 1$

(iii) $h(x) = x^3 + x^2 + x + 1$

$$h'(x) = 3x^2 + 2x + 1 = 0$$

$$x = \frac{-2 \pm \sqrt{4-12}}{2}$$

$\Rightarrow$ x ARE COMPLEX VALUES SO NO CRITICAL POINTS

How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

$f(x) = 10x(2 - \ln x)$ ON INTERVAL $[1, e^2]$

$$f'(x) = 10\left(2 - \ln x + x\left(\frac{-1}{x}\right)\right)$$

$$= 10(1 - \ln x)$$

$\Rightarrow f'(x) = 0 = 1 - \ln x$

$$\ln x = 1$$

$\Rightarrow$ $x = e$

SINCE e IS IN GIVEN INTERVAL WE SAY THAT IT IS A C.P.

$$f''(x) = 10\left(\frac{-1}{x}\right) = \frac{-10}{x}$$

$$f''(e) < 0$$

SO $f(x)$ HAS A MAXIMA AT $x = e$

$f(e) = 10e(2 - \ln e)$
$= 10e$

ABSOLUTE MAXIMA AT $x = e$

$f(1) = 20$ & $f(e^2) = 0$ ABSOLUTE MINIMA AT $x = e^2$

INTEGRALS

Derivatives	Integrals (Anti derivatives)
(i) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$;	$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
Particularly, we note that	
$\frac{d}{dx}(x) = 1$;	$\int dx = x + C$
(ii) $\frac{d}{dx}(\sin x) = \cos x$;	$\int \cos x\, dx = \sin x + C$
(iii) $\frac{d}{dx}(-\cos x) = \sin x$;	$\int \sin x\, dx = -\cos x + C$
(iv) $\frac{d}{dx}(\tan x) = \sec^2 x$;	$\int \sec^2 x\, dx = \tan x + C$
(v) $\frac{d}{dx}(-\cot x) = \mathrm{cosec}^2 x$;	$\int \mathrm{cosec}^2 x\, dx = -\cot x + C$
(vi) $\frac{d}{dx}(\sec x) = \sec x \tan x$;	$\int \sec x \tan x\, dx = \sec x + C$

(vii) $\frac{d}{dx}(-\operatorname{cosec} x) = \operatorname{cosec} x \cot x$; $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$

(xiv) $\frac{d}{dx}(e^x) = e^x$; $\int e^x \, dx = e^x + C$

(xv) $\frac{d}{dx}(\log|x|) = \frac{1}{x}$; $\int \frac{1}{x} \, dx = \log|x| + C$

(xvi) $\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x$; $\int a^x \, dx = \frac{a^x}{\log a} + C$

PROPERTIES

LINEARITY PROPERTY:

$$\int [f(x) + g(x)] \, dx = \int f(x) \, dx + \int g(x) \, dx$$

For any real number k, $\int k \, f(x) \, dx = k \int f(x) \, dx$

EXAMPLE:

$f'(x) = \sin 2x - 4e^{3x}$

$$f(x) = \int f'(x) dx = \int (\sin 2x - 4e^{3x}) = \int \sin 2x \, dx - 4 \int e^{3x} dx$$

$$= \frac{-\cos 2x}{2} - 4 \frac{e^{3x}}{3} + C$$

Find the following integrals in Exercises

12. $\int \frac{x^3+3x+4}{\sqrt{x}}\,dx$ **13.** $\int \frac{x^3-x^2+x-1}{x-1}\,dx$ **14.** $\int (1-x)\sqrt{x}\,dx$

15. $\int \sqrt{x}(3x^2+2x+3)\,dx$ **16.** $\int (2x-3\cos x+e^x)\,dx$

17. $\int (2x^2-3\sin x+5\sqrt{x})\,dx$ **18.** $\int \sec x(\sec x+\tan x)\,dx$

12. $\int \frac{x^3+3x+4}{\sqrt{x}}\,dx$

$= \int x^{5/2} + 3x^{1/2} + 4x^{-1/2}\,dx$

$= \frac{2}{7}x^{7/2} + \frac{6}{3}x^{3/2} + 8x^{1/2} + C$

$$\int \tan x \, dx = \log|\sec x| + C$$

$$\int \cot x \, dx = \log|\sin x| + C$$

$$\int \sec x \, dx = \log|\sec x + \tan x| + C$$

$$\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + C$$

Q $\int \frac{1}{x + x\log x}$

$$= \int \frac{1}{x(1+\log x)} dx$$

$$t = 1 + \log(x)$$

$$dt = \frac{1}{x} dx$$

$$\Rightarrow \int \frac{dt}{t} = \log|t| + c$$

$$= \log|(1+\log x)| + C$$

METHOD OF PARTIAL FRACTION

(1) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C$

(2) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + C$

(3) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\frac{x}{a} + C$

(4) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$

(5) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$

(6) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + C$

$$\int \frac{1}{x^2 - a^2} = \int \frac{1}{(x-a)(x+a)}$$

$$= \frac{A}{x-a} + \frac{B}{x+a}$$

$$\Rightarrow 1 = A(x+a) + B(x-a)$$

put $x = -a$

$$-B(2a) = 1$$

$$\boxed{B = \frac{-1}{2a}}$$

put $x = a$

$$\boxed{A = \frac{1}{2a}}$$

$$\Rightarrow \int \frac{1}{2a(x-a)} - \frac{1}{2a(x+a)} \, dx$$

INTEGRATION BY PARTS

INV ALG EXPO
ILATE
LOG TRIG

SHORTCUT

WHEN FIRST u IS A POLYNOMIAL

$$\int uv\,dx = u\int v\,dx - \int u'\left(\int v\,dx\right)dx$$

EXAMPLE:

Q $\int x \sin 3x \, dx$

$u = x \qquad v = \sin x$

$$x\int \sin(3x) - \int\int \sin 3x \, dx \, dx$$

$$= x\left(-\frac{\cos 3x}{3}\right) - \int -\frac{\cos 3x}{3}\,dx$$

$$= \frac{-x\cos 3x}{3} + \frac{\sin(3x)}{9} + C$$

Q $\int x^2 e^x \, dx$

$(x^2)\,(e^x) \; -$
$(2x)\,(e^x) \; +$
$(2)\,(e^x)$

DEFINITE INTEGRALS

$$\int_0^1 x^2 e^x \, dx$$

$$= x^2 e^x - \int 2xe^x \Big]_0^1$$

$$= x^2 e^x - 2xe^x + 2e^x \Big]_0^1$$

$$= e - 2e + 2e - (2)$$

$$= e - 2$$

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A definition

2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — A definition when $f(a)$ exists

3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any constant k

4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. *Max-Min Inequality:* If f has maximum value max f and minimum value min f on $[a, b]$, then

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a).$$

7. *Domination:* $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ (Special case)

Applications

DEFINITION If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$

DEFINITION If f is integrable on $[a, b]$, then its **average value on $[a, b]$**, also called its **mean**, is

$$\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

The **cumulative distribution function** $F(x)$ for a continuous rv X is defined for every number x by

Random Variable

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(y)\,dy$$

random Variable

Discrete

X	H	T

cont. , $a \le x \le b$

X	cont
Temp	

X	0
$f(x)$	$f(0)$

Probability density fn.

(i) $f(x) \geq 0$

(ii) $\int_a^b f(x)\,dx = 1$

Q FIND $F(x)$ IF $f(x) = kx$, $0 \le x \le 1$

$$\int_0^1 f(x)\,dx = 1$$

$$\frac{k}{2} = 1$$

$$k = 2 \Rightarrow f(x) = 2x, \quad 0 \le x \le 1$$

$$F(x) = \int_0^x f(x)\,dx$$

$$= \int_0^x 2x\,dx$$

$$= x^2 \Big]_0^x$$

$$\boxed{F(x) = x^2}$$

Q $f(x) \begin{cases} \frac{1}{8} + \frac{3x}{8} & 0 \le x \le 2 \\ 0 & \text{OTHERWISE} \end{cases} \Rightarrow F(x) = \int_0^x \frac{1}{8} + \frac{3}{8}x\,dx \Rightarrow \boxed{F(x) = \frac{x}{8} + \frac{3x^2}{16}}$

CALCULUS OF SEVERAL VARIABLES

DEFINITIONS Suppose D is a set of n-tuples ($\mathbb{R}^n$) of real numbers $(x_1, x_2, \ldots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number ($\mathbb{R}$)

$$w = f(x_1, x_2, \ldots, x_n)$$

to each element in D. The set D is the function's **domain**. The set of w-values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** x_1 to x_n. We also call the x_j's the function's **input variables** and call w the function's **output variable**.

These are functions of two variables. Note the restrictions that may apply to their domains in order to obtain a real value for the dependent variable z.

Function	Domain	Range
$z = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
$z = \frac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$z = \sin xy$	Entire plane	$[-1, 1]$

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

These are functions of three variables with restrictions on some of their domains.

Function	Domain	Range
$w = \sqrt{x^2 + y^2 + z^2}$	Entire space	$[0, \infty)$
$w = \frac{1}{x^2 + y^2 + z^2}$	$(x, y, z) \neq (0, 0, 0)$	$(0, \infty)$
$w = xy \ln z$	Half-space $z > 0$	$(-\infty, \infty)$

$f: \mathbb{R}^3 \rightarrow \mathbb{R}$

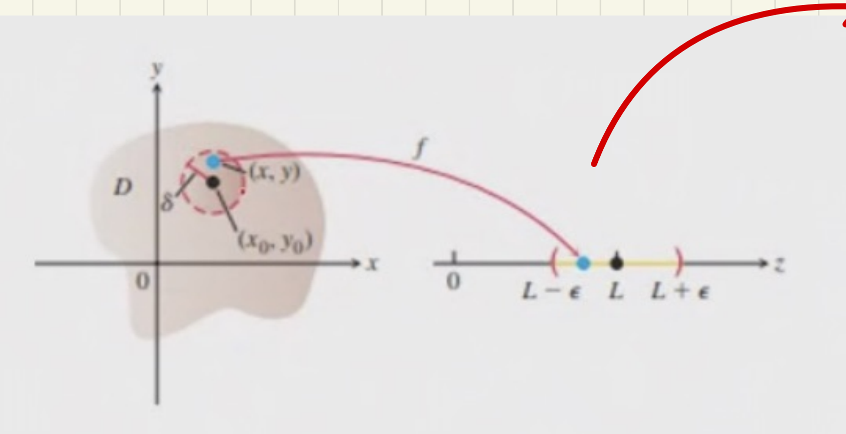
LIMITS

DEFINITION We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$



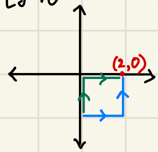
HERE THE NEIGHBORHOOD IS A CIRCLE IN CASE OF 2 VARIABLES

Q $\underset{(x,y)\to(2,0)}{\mathrm{Lt}} \; x^2+y^2$

$= \underset{x\to 2\; y\to 0}{\mathrm{Lt}} (x^2+y^2)$

$= \underset{x\to 2}{\mathrm{Lt}} \left\{ \underset{y\to 0}{\mathrm{Lt}} (x^2+y^2) \right\}$

$= 4$



THEOREM 1—Properties of Limits of Functions of Two Variables The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$
2. *Difference Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$
3. *Constant Multiple Rule:* $\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number k)
4. *Product Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) \cdot g(x,y)) = L \cdot M$
5. *Quotient Rule:* $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$,

   n a positive integer, and if n is even, we assume that $L > 0$.

Q $\lim_{(x,y)\to(0,4)} \frac{x}{\sqrt{y}}$

$$= \lim_{x\to 0}\left\{\lim_{y\to 4} \frac{x}{\sqrt{y}}\right\} = 0$$

$$= \lim_{y\to 4}\left\{\lim_{x\to 0} \frac{x}{\sqrt{y}}\right\} = 0$$

Q $\lim_{(x,y)\to(3,4)} \sqrt{x^2+y^2-1}$

$$\lim_{x\to 3}\left\{\lim_{y\to 4} \sqrt{x^2+y^2-1}\right\} = \sqrt{24}$$

$$\lim_{y\to 4}\left\{\lim_{x\to 3} \sqrt{x^2+y^2-1}\right\} = \sqrt{24}$$

Q $\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2-2xy+y^2}{x-y}$

$$\lim \frac{(x-y)^2}{x-y} = \lim_{\substack{x\to 1\\ y\to 1}} x-y$$

$$= 0$$

Q $\mathrm{lt}_{P\to(1,3,4)} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}$

$$= 1+\frac{1}{3}+\frac{1}{4}$$

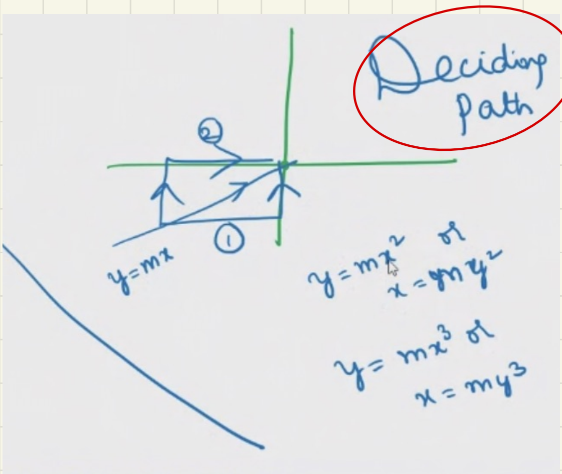
Two-Path Test for Nonexistence of a Limit

If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0), then $\lim_{(x, y)\to(x_0, y_0)} f(x, y)$ does not exist.

Q $f(x,y) = \frac{2x^2y}{x^4+y^2}$

$$\mathrm{lt}_{x\to 0}\left\{\mathrm{lt}_{y\to 0} \frac{2x^2y}{x^4+y^2}\right\} = 0$$

$$\mathrm{lt}_{y\to 0}\left\{\mathrm{lt}_{x\to 0} \frac{2x^2y}{x^4+y^2}\right\} = 0$$



$$\boxed{y = x^2}$$

$$\Rightarrow \mathrm{lt}_{x\to 0}\left\{\mathrm{lt}_{y\to 0} \frac{2x^4}{2x^4}\right\} = 1$$

SINCE DIFFERENT PARTS GIVE DIFFERENT VALUES LIMITS DOES NOT EXISTS

Q $\lim_{\substack{x\to 0\\ y\to 0}} \dfrac{x^2+y^2}{x^2-y^2}$

$\lim_{x\to 0}\left\{\lim_{y\to 0} \dfrac{x^2+y^2}{x^2-y^2}\right\} = 1$

$\lim_{y\to 0}\left\{\lim_{x\to 0} \dfrac{x^2+y^2}{x^2-y^2}\right\} = -1$

SINCE DIFF. NO LIMIT EXIST

Q $\lim_{\substack{x\to 0\\ y\to 0}} \dfrac{3x^2-y^2+5}{x^2+y^2+2}$

$$\lim_{x\to 0}\left\{\lim_{y\to 0} \frac{3x^2-y^2+5}{x^2+y^2+2}\right\} = \frac{5}{2}$$

Q $\lim_{\substack{x\to 0\\ y\to 0}} \dfrac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}} = \dfrac{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})+2(\sqrt{x}-\sqrt{y})}{\sqrt{x}-\sqrt{y}}$

$= \sqrt{x}+\sqrt{y}+2$

$\lim_{x\to 0}\left\{\lim_{y\to 0} \sqrt{x}+\sqrt{y}+2\right\}$

$= 2$

DEFINITION A function $f(x, y)$ is **continuous at the point** (x_0, y_0) if

1. f is defined at (x_0, y_0),
2. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists,
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

Show that

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at every point except the origin

CONTINOUS EVERYWHERE

At $(0,0)$ $\lim_{\substack{x \to 0 \\ y \to 0}} f(x,y) = \lim_{x \to 0} \left\{ \lim_{y \to 0} \dfrac{2xy}{x^2+y^2} \right\} = 0$

$= \lim_{y \to 0} \left\{ \lim_{x \to 0} \dfrac{2xy}{x^2+y^2} \right\} = 0$

LET $y = x \Rightarrow \lim_{x \to 0} \left\{ \lim_{y \to 0} \dfrac{2x^2}{2x^2} \right\}$

$= 1$

SINCE VALUES ARE DIFFERENT LIMIT DOES NOT EXISTS

PARTIAL DERIVATIVE

DEFINITION The **partial derivative of $f(x, y)$ with respect to x at the point** (x_0, y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists.

Q $f(x,y) = xy \Rightarrow$ FIND $\left.\dfrac{\partial f}{\partial x}\right|_{x=2}$

$= \dfrac{\partial xy}{\partial x} = y \times \dfrac{d(x)}{dx}$

$= y$

Q $f(x,y) = \dfrac{2y}{y+\cos x}$

$$f_x = 2y\left(\frac{-1}{(y+\cos x)^2}\right)(-\sin x)$$

$$f_y = \frac{(y+\cos x)2 - 2y}{(y+\cos x)^2}$$

Q $f(x,y,z) = xy + yz + xz$

$f_x = y+z \qquad f_y = x+z \qquad f_z = x+y$

THEOREM 5—Chain Rule For Functions of One Independent Variable and Two Intermediate Variables If $w = f(x, y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dw}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t),$$

or

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Q $w = \ln(x^2+y^2+z^2)$ $\quad x = \cos t$, $y = \sin t$ $\quad z = 4\sqrt{t}$ $\quad$ at $t=3$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\frac{\partial w}{\partial x} = \frac{1\,(2x)}{x^2+y^2+z^2} \qquad \frac{\partial w}{\partial y} = \frac{2y}{x^2+y^2+z^2} \qquad \frac{\partial w}{\partial z} = \frac{2z}{x^2+y^2+z^2}$$

$$\frac{dw}{dt} = \frac{2\cos t}{16t}(-\sin t) + \frac{2\sin t}{16t}(\cos t)$$

Q Find $\partial w/\partial v$ when $u = -1, v = 2$ if $w = xy + \ln z$, $x = v^2/u, y = u + v, z = \cos u$.

$$w = \frac{v^2}{u} \times (u + v) + \ln(\cos u)$$

$$= v^2 + \frac{v^3}{u} + \ln(\cos u)$$

$$\frac{\partial w}{\partial v} = 2v + \frac{3v^2}{u}$$

TAYLOR SERIES

$$f(x,y) = f(a,b) + (x-a)\frac{\partial f}{\partial x}\Big|_{(a,b)} + (y-b)\frac{\partial f}{\partial y}\Big|_{(a,b)}$$

$$+ \frac{1}{2!}\Big\{ (x-a)^2 \frac{\partial^2 f}{\partial x^2}\Big|_{(a,b)} + (y-b)^2 \frac{\partial^2 f}{\partial y^2}\Big|_{(a,b)}$$

$$+ 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y}\Big|_{(a,b)}\Big\}$$

Q $f(x,y) = \sin xy$ about $(1,1)$

$f(1,1) = \sin 1$

$f_x = (\cos xy)\, y$ $\qquad f_{xy} = \cos xy - y \sin xy$

$f_x(1,1) = \cos 1$

$f_y = (\cos xy)\, y$ $\qquad f_{yy} = -\sin xy \cdot x^2$

$f_y(1,1) = \cos 1$

$$\sin xy = \sin 1 + (x-1)\cos 1 + (y-1)\cos 1 + \frac{1}{2!}\Big\{ (x-1)^2(-\sin 1) + (y-1)^2(-\sin 1) + 2(x-1)(y-1)(\cos 1 - \sin$$

CONSTRAINED OPTIMIZATION (OR) MAXIMA AND MINIMA

$Z = f(x,y)$

$f_x = 0$, $f_y = 0$ > SOLVE THESE 2 TO GET C.P.

NOW COMPUTE

$r = f_{xx}$, $s = f_{xy}$, $t = f_{yy}$

$$\boxed{\begin{array}{ll} rt - s^2 > 0 \ \& \ r > 0 & \Rightarrow \text{MINIMA} \\ rt - s^2 > 0 \ \& \ r < 0 & \Rightarrow \text{MAXIMA} \\ rt - s^2 < 0 & \Rightarrow \text{SADDLE} \\ rt - s^2 = 0 & \Rightarrow \text{MORE WORK} \end{array}}$$

EXAMPLE

Q $f(x,y) = 2x^2 + 2xy + 2y^2 - 6x$

$f_x \Rightarrow 4x + 2y - 6 = 0$

$\Rightarrow 4x + 2y = 6$

$f_y \Rightarrow 2x + 4y = 0$

$6x = 12$

$x = 2$ $\quad y = -1$

$f_{xx} = 4 = r$

$f_{xy} = 2 = s$

$f_{yy} = 4 = t$

$rt - s^2 = 12 > 0$

$r = 4 > 0$

SO MINIMA AT $(2,-1)$

$f_{min} = 8 - 4 + 2 - 12$

$f_{min} = -6$

CONSTRAINED MAXIMA/MINIMA

The Method of Lagrange Multipliers

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq \mathbf{0}$ when $g(x, y, z) = 0$. To find the local maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$ (if these exist), find the values of x, y, z, and λ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y, z) = 0. \tag{1}$$

For functions of two independent variables, the condition is similar, but without the variable z.

EXTREMIZE $f(x,y,z)$ / $f(x,y)$

SUBJECT TO CONSTRAINTS $g_1(x,y,z)=0$ / $g_2(x,y)=0$ .... (MORE THAN 1)

CONSTRUCT:

$$\boxed{L = f - \lambda_1 g_1 - \lambda_2 g_2}$$

NOW COMPUTE

① $\frac{\partial L}{\partial x} = 0$ ② $\frac{\partial L}{\partial y} = 0$

③ $g_1 = 0$ ④ $g_2 = 0$

SOLVE TO FIND λ_1, λ_2, C.P. (P_1, P_2)

Q MAXIMIZE $f(x,y) = 8x^2 - 2y$ SUBJECT TO $x^2+y^2=1$

$$g = x^2+y^2-1$$

$$L = f - \lambda g$$

$$= (8x^2-2y) - \lambda(x^2+y^2-1)$$

$$\frac{\partial L}{\partial x} = 16x - \lambda(2x) = 0 \qquad \boxed{\lambda = 8}$$

$$\frac{\partial L}{\partial y} = -2 - 2\lambda y = 0 \qquad \boxed{y = -1/8}$$

$$g \Rightarrow x^2+y^2 = 1$$

$$x^2 = 1 - \frac{1}{64} \Rightarrow \boxed{x = \sqrt{\frac{63}{64}} = \pm\frac{\sqrt{63}}{8}}$$

$$P_1 = \left(\frac{\sqrt{63}}{8}, -\frac{1}{8}\right) \ \& \ P_2 = \left(-\frac{\sqrt{63}}{8}, -\frac{1}{8}\right)$$

$$f|_{P_1} = 8\left(\frac{63}{64}\right) - 2\left(\frac{-1}{8}\right) \qquad f|_{P_2} = 8\left(\frac{63}{64}\right) - 2\left(\frac{-1}{8}\right)$$

$$= \frac{65}{8} \qquad\qquad = \frac{65}{8}$$

Lagrange Multipliers with Two Constraints

Many problems require us to find the extreme values of a differentiable function $f(x, y, z)$ whose variables are subject to two constraints. If the constraints are

$$g_1(x, y, z) = 0 \qquad \text{and} \qquad g_2(x, y, z) = 0$$

$$\boxed{\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \qquad g_1(x, y, z) = 0, \qquad g_2(x, y, z) = 0 \qquad (2)}$$

Q

Extrema on a circle Find the extreme values of $f(x, y) = xy$ subject to the constraint $g(x, y) = x^2 + y^2 - 10 = 0$.

$L = xy - \lambda(x^2 + y^2 - 10)$

1. $\frac{\partial L}{\partial x} = y - 2\lambda x = 0 \Rightarrow \lambda = \frac{y}{2x}$
2. $\frac{\partial L}{\partial y} = x - 2\lambda y = 0 \Rightarrow \lambda = \frac{x}{2y} \Rightarrow \lambda^2 = \frac{1}{4} \Rightarrow \boxed{\lambda = \pm\frac{1}{2}}$
3. $x^2 + y^2 = 10 \qquad \Rightarrow \boxed{x = \pm y}$

$\Rightarrow (\pm x)^2 + x^2 = 10$

$x = \pm\sqrt{5}$

$\Rightarrow$ C.P. $\Rightarrow$

$(\sqrt{5}, \sqrt{5})$	$f\|_{P_1} = 5$	MAXIMUM
$(\sqrt{5}, -\sqrt{5})$	$f\|_{P_2} = -5$	MINIMUM
$(-\sqrt{5}, \sqrt{5})$	$f\|_{P_3} = -5$	MINIMUM
$(-\sqrt{5}, -\sqrt{5})$	$f\|_{P_4} = 5$	MAXIMUM

Q

Maximize the function $f(x, y, z) = x^2 + 2y - z^2$ subject to the constraints $2x - y = 0$ and $y + z = 0$.

$$L = x^2 + 2y - z^2 - \lambda_1(2x - y) - \lambda_2(y + z)$$

$$\frac{\partial L}{\partial x} = 2x - 2\lambda_1 = 0 \Rightarrow \lambda_1 = x \quad ①$$

$$\frac{\partial L}{\partial y} = 2 + \lambda_1 - \lambda_2 = 0 \Rightarrow \lambda_2 = x + 2 \quad ②$$

$$\frac{\partial L}{\partial z} = -2z - \lambda_2 = 0 \Rightarrow x + 2 = -2z \quad ③$$

$$x + 2z = -2$$

$$2x = y \quad ④$$

$$y + z = 0 \quad ⑤$$

$$2x - y = 0$$

$$\Rightarrow 2x + z = 0$$

$$2x + 4z = -4$$

$$\Rightarrow 3z = -4$$

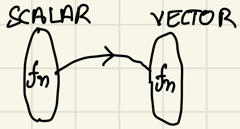
$$z = -4/3$$

$$x = -z/2 = 4/6 = 2/3$$

VECTOR CALCULUS

GRADIENT

$$\boxed{\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k} \longrightarrow \text{GRADIENT } (\nabla)$$



Q GRADIENT AT $(1,0,1)$

$$f(x,y,z) = x^2 + y^2 + z^2 \Rightarrow \frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = 2y \quad \frac{\partial f}{\partial z} = 2z$$

$$\Rightarrow \nabla f = 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$$

$$\boxed{\nabla f\,|_{(1,0,1)} = 2\hat{i} + 2\hat{k}}$$

NOTE

IF $f(x,y,z) = C$ IS A SURFACE THEN,

GEOMETRICALLY ∇f GIVES THE DIRECTION OF NORMAL TO SURFACE

DIVERGENCE

$$\bar{U} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$$

$$\boxed{\nabla \bar{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}}$$

Q $F = xy\hat{i} + yz\hat{j} + zx\hat{k} \Rightarrow \frac{\partial u_1}{\partial x} = y \quad \frac{\partial u_2}{\partial y} = z \quad \frac{\partial u_3}{\partial z} = x$

$$\underline{\operatorname{div} \bar{F} = y + z + x}$$

CURL

$$\nabla \times u = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u_1 & u_2 & u_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right) - j\left(\frac{\partial u_3}{\partial x} - \frac{\partial u_1}{\partial z}\right) + \hat{k}\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)$$

Q FIND CURL $\bar{u}$ WHERE

$\bar{u} = xy\hat{i} + yz\hat{j} + zx\hat{k}$

$$\nabla \times \bar{u} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix}$$

$$= \hat{i}(0 - y) - \hat{j}(z - 0) + \hat{k}(0 - x)$$

$$= -y\hat{i} - z\hat{j} - x\hat{k}$$

SOME PROPERTIES

$$\nabla \times (\nabla \times \boldsymbol{u}) = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla^2 \boldsymbol{u},$$

$$\nabla \cdot (f\boldsymbol{u}) = \boldsymbol{u} \cdot \nabla f + f \nabla \cdot \boldsymbol{u},$$

$$\nabla \times (f\boldsymbol{u}) = \nabla f \times \boldsymbol{u} + f \nabla \times \boldsymbol{u},$$

$$\nabla(\boldsymbol{u} \cdot \boldsymbol{v}) = (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{u} + \boldsymbol{u} \times (\nabla \times \boldsymbol{v}) + \boldsymbol{v} \times (\nabla \times \boldsymbol{u}),$$

$$\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot (\nabla \times \boldsymbol{u}) - \boldsymbol{u} \cdot (\nabla \times \boldsymbol{v}),$$

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{u}(\nabla \cdot \boldsymbol{v}) - \boldsymbol{v}(\nabla \cdot \boldsymbol{u}) + (\boldsymbol{v} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{v}.$$

Gradient Descent Algorithm

Gradient Descent is an iterative optimization algorithm for finding a local minimum of a differentiable function.

The function which is set to be minimised is called as an $\boxed{\textbf{objective function.}}$

For **machine learning**, the **objective function** is also termed as the **cost function** or **loss function**. It is the loss function which is optimized (minimised) and gradient descent is used to find the most optimal value of parameters / weights which minimises the loss function

Method Steps:

- Calculate the gradient by taking the derivative of the function with respect to the specific parameter.

- In case, there are multiple parameters, take the partial derivatives with respect to different parameters.

- Calculate the descent value for different parameters by multiplying the value of derivatives with learning or descent rate (step size) and -1.

- Update the value of parameter by adding up the existing value of parameter and the descent value.

MINIMIZE $f(x)$ $\quad$ x_0 IS WHERE WE ARE STARTING

$$\boxed{X_{k+1} = X_k + \gamma_k \nabla f(X_k)} \qquad f(x_k - \gamma_k \nabla f(x_k))$$

EXAMPLE:

Q MINIMIZE $f(x) = 4x_1^2 - 4x_1x_2 + 2x_2^2$

$$X_0 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$X_{k+1} = X_k - \gamma_k \nabla F|_{x_k} \qquad \boxed{\nabla F = (8x_1 - 4x_2)\hat{i} + (-4x_1 + 4x_2)\hat{j}}$$

$$\nabla F_{(2,3)} = 4\hat{i} + 4\hat{j}$$

$$\Rightarrow X_1 = X_0 - \gamma \nabla F|_{x_0}$$

$$X_1 = 2\hat{i} + 3\hat{j} - \gamma(4\hat{i} + 4\hat{j})$$

$$= (2 - 4\gamma)\hat{i} + (3 - 4\gamma)\hat{j}$$

γ GIVEN BY EXTREMIZING

$$f(x_1)$$

$$\Rightarrow 4(2-4\gamma)^2 - 4(2-4\gamma)(3-4\gamma) + 2(3-4\gamma)^2$$

$$f' = 10 - 32\gamma + 32\gamma^2 = 0$$

$$\Rightarrow \gamma = 1/2$$

$$X_1 = \begin{bmatrix} 2-4\gamma \\ 3-4\gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$X_2 = X_1 - \gamma \nabla f|_{x_1} \qquad \nabla F|_{x_1} = -4\hat{i} + 4\hat{j}$$

$$= (4\gamma)\hat{i} + (1 - 4\gamma)\hat{j}$$

$$f(\gamma) = 160\gamma^2 - 32\gamma + 2 = 0$$

$$\gamma = \frac{1}{10}$$

$$X_2 = \begin{bmatrix} 4\gamma \\ 1-4\gamma \end{bmatrix} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$$

LIMITATIONS

- Converging to a local minimum can be quite slow
- if there are multiple local minima, then there is no guarantee that the procedure will find the global minimum.
- Gradient descent can converge to a local minimum and slow down in a neighborhood of a saddle point.

Fast gradient methods

Momentum or *heavy ball* method

THESE ALGOS TRY TO OVERCOME THESE PITFALLS

INDUCTION PRINCIPLE

PRINCIPLE OF MATHEMATICAL INDUCTION To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: We verify that $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

Q

There are infinitely many stations on a train route. Suppose that the train stops at the first station and suppose that if the train stops at a station, then it stops at the next station. Show that the train stops at all stations.

RULE: $S1 \longrightarrow S2$

FROM THE ABOVE RULE WE KNOW THAT IF A TRAIN STOPS AT 1 IT WILL STOP AT 2 ASWELL

$P(1) \longrightarrow P(2)$

$\Rightarrow P(2) \longrightarrow P(3)$ THEN

$P(K) \longrightarrow P(K+1)$ FOR ALL $K > 0$

Q

4. Let $P(n)$ be the statement that $1^3 + 2^3 + \cdots + n^3 = (n(n+1)/2)^2$ for the positive integer n.

a) What is the statement $P(1)$?

b) Show that $P(1)$ is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step, identifying where you use the inductive hypothesis.

f) Explain why these steps show that this formula is true whenever n is a positive integer.

(a) $P(1)$: LHS RHS

$$1^3 = \left(\frac{1(1+1)}{2}\right)^2 \Rightarrow 1 = 1 \quad (b)$$

(c) $P(K) \rightarrow P(K+1)$

GIVEN:

$$1^3 + 2^3 + \cdots + K^3 = \frac{K^2(K+1)^2}{4}$$

TO PROVE

$$1^3 + 2^3 + \cdots + K^3 + (K+1)^3 = \frac{(K+1)^2 (K+2)^2}{4}$$

$$\Rightarrow \frac{K^2(K+1)^2}{4} + (K+1)^3 = \frac{(K+1)^2(K+2)^2}{4}$$

$$= (K+1)^2 \left\{ \frac{K^2}{4} + K + 1 \right\}$$

$$= \frac{(K+1)^2 (K+2)^2}{4} = RHS$$

HENCE PROVED

6. Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.

$$P(1) = \overset{LHS}{1 \cdot 1!} = (1+1)! - 1$$

$$= 1 = 1$$

$$P(K) = \underbrace{1 \cdot 1! + 2 \cdot 2! + \cdots + K \cdot K!}_{LHS} = \underbrace{(K+1)! - 1}_{RHS}$$

$$P(K+1) = P(K) + (K+1) \cdot (K+1)! = (K+2)! - 1$$

$$LHS \Rightarrow (K+1)! + (K+1)(K+1)! - 1$$

$$LHS \Rightarrow (K+1)! (K+1+1) - 1$$

$$\Rightarrow (K+2)(K+1)! - 1$$

$$LHS \Rightarrow (K+2)! - 1 = RHS$$

$$LHS = RHS$$

HENCE PROVED

20. Prove that $3^n < n!$ if n is an integer greater than 6.

$P(7) \Rightarrow 3^7 < 7!$

$\Rightarrow 2178 < 5040$

$P(k) \Rightarrow 3^k < k!$

$P(k+1) \Rightarrow 3^{k+1} < (k+1)!$

LHS RHS

$LHS \Rightarrow 3^{k+1}$

$\Rightarrow 3^k \cdot 3 < 3k!$

$\Rightarrow 3k! < (k+1)k!$

$\Rightarrow 3k! < (k+1)!$

$\Rightarrow 3^{(k+1)} < (k+1)!$

FOR ALL $k > 6$

32. Prove that 3 divides $n^3 + 2n$ whenever n is a positive integer.

$P(1)$: $1^3 + 2 \times 1 = 3 \times 1$ (DIV BY 3 TRUE)

$P(k)$: $k^3 + 2k = 3m$ (DIV BY 3)

$\Rightarrow P(k+1)$:

$(k+1)^3 + 2(k+1)$

$= k^3 + 3k^2 + 3k + 1 + 2k + 2$

$\Rightarrow = k^3 + 3k^2 + 5k + 3$

$= k^3 + 2k + 3k^2 + 3k + 3$

$= 3m + 3(k^2 + k + 3)$

$= 3(m + k^2 + k + 3)$

$\Rightarrow P(k+1) =$ TRUE

10. a) Find a formula for

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}$$

by examining the values of this expression for small values of n.

b) Prove the formula you conjectured in part (a).

(a) $n=1$: $\frac{1}{1\cdot 2} = \frac{1}{2}$

$n=2 = \frac{1}{2}+\frac{1}{6} = \frac{2}{3}$

$n=3 = \frac{2}{3}+\frac{1}{12} = \frac{9}{12} = \frac{3}{4}$

$n=k = \frac{k}{k+1}$

(b) $\Rightarrow$ $k = \frac{k}{k+1}$

LHS $\overset{T.P}{=}$ RHS

$P(k+1) \Rightarrow \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)} = \frac{k+1}{(k+2)}$

LHS $\Rightarrow$ $\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$

LHS $\Rightarrow$ $\frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)}$

$= \frac{k+1}{k+2} =$ RHS

Suppose that **A** and **B** are square matrices with the property **AB** = **BA**. Show that $\mathbf{AB}^n = \mathbf{B}^n\mathbf{A}$ for every positive integer n.

P(1) IS TRUE

SUPPOSE P(k) IS TRUE

$P(K) \Rightarrow AB^K = B^K A$

TPT $\Rightarrow P(K+1) \Rightarrow AB^{K+1} = B^{K+1}A$

LHS RHS

CONSIDER:

$$
\begin{aligned}
&AB^{K+1} \\
&= (AB^K)B \\
&\Rightarrow (B^K A)B = B^K(AB) \quad | \text{ ASSOCIATIVE} \\
&= B^K(BA) \\
&= B^{K+1}A = RHS
\end{aligned}
$$

HENCE PROVED

STRONG INDUCTION

STRONG INDUCTION To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: We verify that the proposition $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true for all positive integers k.

Claim: Let a_n be the sequence defined by $a_1 = 1$, $a_2 = 8$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. Prove that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all $n \in \mathbb{N}$.

$P(1): a_1 = 1 \qquad P(2): a_2 = 8$

P(K) IS TRUE SO: $a_k = 3 \cdot 2^{k-1} + 2(-1)^k$
P(K-1) IS TRUE SO: $a_{k-1} = 3 \cdot 2^{k-2} + 2(-1)^{k-1}$

TPT:

$a_{k+1} = 3 \cdot 2^k + 2(-1)^{k+1}$

WKT:

$$\begin{aligned}
a_{k+1} &= a_k + 2a_{k-1} \\
&= 3 \cdot 2^{k-1} + 2(-1)^k + 2(3 \cdot 2^{k-2} + 2(-1)^{k-1}) \\
&= 3 \cdot 2^{k-1} + 3 \cdot 2^{k-1} + 2(-1)^k + 4(-1)^{k-1} \\
&= 3 \cdot 2^k + (-2(-1)^{k-1}) + 4(-1)^{k-1} \\
&= 3 \cdot 2^k + 2(-1)^{k-1} \\
&= 3 \cdot 2^k + 2(-1)^{k-1} \times 1 \\
&= 3 \cdot 2^k + 2 \times (-1)(-1) \times (-1)^{k-1} \\
&= 3 \cdot 2^k + 2(-1)^{k+1}
\end{aligned}$$

---

5. a) Determine which amounts of postage can be formed using just 4-cent and 11-cent stamps.

b) Prove your answer to (a) using the principle of mathematical induction. Be sure to state explicitly your inductive hypothesis in the inductive step.

c) Prove your answer to (a) using strong induction. How does the inductive hypothesis in this proof differ from that in the inductive hypothesis for a proof using mathematical induction?

RECURSIVE RELATIONS

We use two steps to define a function with the set of nonnegative integers as its domain:

BASIS STEP: Specify the value of the function at zero.

RECURSIVE STEP: Give a rule for finding its value at an integer from its values at smaller integers.

SAMPLE QUESTIONS

2. Find $f(1)$, $f(2)$, $f(3)$, $f(4)$, and $f(5)$ if $f(n)$ is defined recursively by $f(0) = 3$ and for $n = 0, 1, 2, \ldots$

a) $f(n+1) = -2f(n)$.

b) $f(n+1) = 3f(n) + 7$.

c) $f(n+1) = f(n)^2 - 2f(n) - 2$.

d) $f(n+1) = 3^{f(n)/3}$.

(a) $f(0) = 3$, $f(n+1) = -2f(n)$

$f(1) = -2 \times 3 = -6$

$f(2) = -2 \times (-6) = 12$

$f(3) = -2 \times 12 = -24$

$f(4) = -2 \times -24 = 48$

$f(5) = -2 \times 48 = -96$

(d) $f(0) = 3$ $f(n+1) = 3^{f(n)/3}$

$f(1) = 3^{3/3)} = 3$

$f(2) = 3^{3/3} = 3$

$f(3) = 3^{3/3} = 1$

$f(4) = 3^{3/3} = 1$

$f(5) = 3^{3/3} = 1$

6. Determine whether each of these proposed definitions is a valid recursive definition of a function f from the set of nonnegative integers to the set of integers. If f is well defined, find a formula for $f(n)$ when n is a nonnegative integer and prove that your formula is valid.

a) $f(0) = 1$, $f(n) = -f(n-1)$ for $n \geq 1$ VALID

b) $f(0) = 1$, $f(1) = 0$, $f(2) = 2$, $f(n) = 2f(n-3)$ for $n \geq 3$ VALID

c) $f(0) = 0$, $f(1) = 1$, $f(n) = 2f(n+1)$ for $n \geq 2$

d) $f(0) = 0$, $f(1) = 1$, $f(n) = 2f(n-1)$ for $n \geq 1$

e) $f(0) = 2$, $f(n) = f(n-1)$ if n is odd and $n \geq 1$ and $f(n) = 2f(n-2)$ if $n \geq 2$

(a) $f(n) = -f(n-1)$

$f(1) = -1$

$f(2) = 1$

$f(3) = -1$

$\vdots$

$\boxed{f(n) = (-1)^n, \; n \geq 0}$

VALID

ASSUME $f(K) = (-1)^K$ AS TRUE

T.P. $f(K+1) = (-1)^{K+1}$

LHS $\Rightarrow$ $f(K+1) = -1 \times f(K)$ # GIVEN

$= -1 \times (-1)^K$

$= \underbrace{(-1)^{K+1} = \text{RHS}}_{\text{HENCE PROVED}}$

(b) $f(0)=1$ $f(1)=0$ $f(2)=2$ $f(n)=2f(n-3), n\geq 3$

$f(3)=2$ $f(7)=0$

$f(4)=0$ $f(8)=8$

$f(5)=4$ $f(9)=8$

$f(6)=4$ $f(10)=0$

$f(3n)=2^n, n\geq 1$

$f(3n+1)=0, n\geq 0$

$f(3n-1)=2^n, n\geq 1$

ASSUME $f(3k)=2^k$

TPT $f(3(k+1))=2^{k+1}$

LHS $\Rightarrow$ $2\times f(3k+3-1)$

$\Rightarrow$

(c) $f(0)=2$ $f(n)\begin{cases} f(n-1) & \text{IF ODD} \\ 2f(n-2) & \text{IF EVEN}\end{cases}$

$f(1)=2$

$f(2)=4$

$f(3)=4$

$f(4)=8$

$f(5)=8$

$f(n)=$

8. Give a recursive definition of the sequence $\{a_n\}$, $n = 1, 2, 3, \ldots$ if

a) $a_n = 4n - 2$. ✓

b) $a_n = 1 + (-1)^n$.

c) $a_n = n(n+1)$.

d) $a_n = n^2$.

(c) $a_n = n(n+1)$, $n = 1, 2$

$a_1 = 1 \cdot 2 = 2$

$a_2 = 2 \cdot 3 = 6$ $\quad 3/1$

$a_3 = 3 \cdot 4 = 12$ $\quad 2/1$

$a_4 = 4 \cdot 5 = 20$ $\quad 5/3$

d) $a_n = n^2$

$a_1 = 1$

$a_2 = 4$

$a_3 = 9$

$a_4 = 16$

$a_n = (\sqrt{a_n} + 1)^2$

STRUCTURAL INDUCTION

BASIS STEP: Show that the result holds for all elements specified in the basis step of the recursive definition to be in the set.

RECURSIVE STEP: Show that if the statement is true for each of the elements used to construct new elements in the recursive step of the definition, the result holds for these new elements.

In Exercises 12–19 f_n is the nth Fibonacci number.

12. Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ when n is a positive integer.

13. Prove that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$ when n is a positive integer.

***14.** Show that $f_{n+1} f_{n-1} - f_n^2 = (-1)^n$ when n is a positive integer.

12) $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$, $n \in N$

0 1 2 3 4 5 6 7

0, 1, 1, 2, 3, 5, 8, 13

$f(1) = f_1^2 = 0 \times 1$

$f(2) = 0^2 + 1^2 = 1 \times 1 =$

ASSUME $f(K) = f_K \; f_{K+1}$

T.P. $f(K+1) = f(K) + f_{K+1}^2 = f_{K+1} f_{K+2}$

$LHS = f_K f_{K+1} + f_{K+1}^2$

$= f_{K+1} (f_K + f_{K+1})$

$= f_{K+1} \times f_{K+2} = RHS$

HENCE PROVED

14) $f_{n+1} f_{n-1} - f_n^2 = (-1)^n$

$n=1$

$$f_2 f_0 - f_1^2 = -1$$

ASSUME

$$f_{k+1} f_{k-1} - f_k^2 = (-1)^k$$

T.P.T.

$$f_{k+2} f_k - f_{(k+1)}^2 = (-1)^{k+1}$$

$$\text{LHS} \Rightarrow (f_{k+1} + f_k) f_k - f_{k+1}^2$$
$$= f_{k+1} f_k + f_k^2 - f_{k+1}^2$$
$$= f_{k+1} f_k + f_{k+1} f_{k-1} - (-1)^k - f_{k+1}^2$$
$$= f_{k+1}(f_k + f_{k-1}) - (-1)^k - f_{k+1}^2$$
$$= f_{k+1}^2 - (-1)^k - f_{k+1}^2$$
$$= f_{k+1}^2 + (-1)^{k+1} - f_{k+1}^2$$
$$= (-1)^{k+1} = \text{RHS}$$

HENCE PROVED

18. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Show that

$$\mathbf{A}^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$$

when n is a positive integer.

$$A^1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

ASSUME $A^k = \begin{bmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{bmatrix}$

T.P.T. $A^{k+1} = \begin{bmatrix} f_{k+2} & f_{k+1} \\ f_{k+1} & f_k \end{bmatrix}$

$$\text{LHS} \Rightarrow A^k A$$
$$= \begin{bmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{bmatrix} * \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} f_{k+1} + f_k & f_{k+1} \\ f_k + f_{k-1} & f_k \end{bmatrix}$$
$$= \begin{bmatrix} f_{k+2} & f_{k+1} \\ f_{k+1} & f_k \end{bmatrix} = \text{RHS}$$

HENCE PROVED

***20.** Give a recursive definition of the functions max and min so that $\max(a_1, a_2, \ldots, a_n)$ and $\min(a_1, a_2, \ldots, a_n)$ are the maximum and minimum of the n numbers $a_1, a_2, \ldots, a_n$, respectively.

***21.** Let $a_1, a_2, \ldots, a_n$, and $b_1, b_2, \ldots, b_n$ be real numbers. Use the recursive definitions that you gave in Exercise 20 to prove these.

a) $\max(-a_1, -a_2, \ldots, -a_n) = -\min(a_1, a_2, \ldots, a_n)$

20) $MAX(a_1, a_2) = \begin{cases} a_1 & , a_1 > a_2 \\ a_2 & , a_1 < a_2 \end{cases}$

$MIN(a_1, a_2) = \begin{cases} a_1 & , a_1 < a_2 \\ a_2 & , a_1 > a_2 \end{cases}$

$MAX(a_1, a_2, a_3) = MAX(MAX(a_1, a_2), a_3)$

$MAX(a_1, \ldots, a_k) = MAX(MAX(a_1, \ldots, a_{k-1}), a_k)$

21) ASSUME TO BE TRUE

$MAX(-a_1, -a_2, \ldots, -a_k) = -MIN(a_1, a_2, \ldots, a_k)$

T.P.T

$MAX(-a_1, \ldots, -a_{k+1}) = -MIN(a_1, \ldots, a_{k+1})$

LHS $= MAX(MAX(-a_1, \ldots, -a_k), -a_{k+1})$

$= MAX(-MIN(a_1, \ldots, a_k), -a_{k+1})$

$= -MIN(MIN(a_1, \ldots, a_k), a_{k+1})$

$= -MIN(a_1, a_2, \ldots, a_{k+1}) = RHS$

HENCE PROVED

26. Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis step: $(0, 0) \in S$.

Recursive step: If $(a, b) \in S$, then $(a + 2, b + 3) \in S$ and $(a + 3, b + 2) \in S$.

a) List the elements of S produced by the first five applications of the recursive definition.

b) Use strong induction on the number of applications of the recursive step of the definition to show that $5 \mid a + b$ when $(a, b) \in S$. *

c) Use structural induction to show that $5 \mid a + b$ when $(a, b) \in S$.

(a)

1 $(0,0) \in S$

2 $(2,3), (3,2) \in S$

3 $(4,6), (5,5), (6,4) \in S$

4 $(6,9), (7,8), (7,8), (8,7), (8,7), (9,6) \in S$

(b) $[P(1), P(2), \ldots, P(k)]$ ARE TRUE

T.P.T. $P(k+1)$ IS TRUE

$P(k+1)$ GIVES (a_1, b_1) THEN $\exists (a,b) \in P_k$

$a_1 = a+2,\ b_1 = b+3$

$a_1 = a+3,\ b_1 = b+2$

T.P.T. $(a_1, b_1) = 5 \mid a_1 + b_1$

$\Rightarrow (a+2, b+3)$

$\Rightarrow a_1 + b_1 = a + b + 5$

$\Rightarrow 5 \mid a_1 + b_1 = RHS$

HENCE PROVED

(c) $P_{k+1} = (a+2, b+3)$ OR $(a+3, b+2)$

WHERE

$(a,b) \in P(k)$

$\Rightarrow 5 \mid (a+b)$

$(a+2)+(b+3)$

$= (a+b+5)$ IS DIV BY 5

$(a+3)+(b+2)$

$= a+b+5$ IS DIV BY 5

COUNTING

THE PRODUCT RULE Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

THE SUM RULE If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different ports to a microcomputer in the center are there?

TOTAL NO. OF PORTS = 32×24

A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

TOTAL NO. OF PROJECTS = $23 + 15 + 19$

1. There are 18 mathematics majors and 325 computer science majors at a college.
   a) In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?
   b) In how many ways can one representative be picked who is either a mathematics major or a computer science major?

(a) 18×325

(b) $18 + 325$

4. A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of this shirt are made?

12×3

7. How many different three-letter initials can people have?

8. How many different three-letter initials with none of the letters repeated can people have?

9. How many different three-letter initials are there that begin with an A?

7. $26 \times 26 \times 26$

8. $26 \times 25 \times 24$

9. 26×26

20. How many positive integers between 5 and 31

a) are divisible by 3? Which integers are these?

b) are divisible by 4? Which integers are these?

c) are divisible by 3 and by 4? Which integers are these?

(a) $\lfloor 31 | 3 \rfloor - \lfloor 5 | 3 \rfloor$

$\Rightarrow 10 - 1 = 9$

(b) $\lfloor 31 | 4 \rfloor - \lfloor 5 - 4 \rfloor$

$7 - 1 = 6$

(c) $\lfloor 31 | 12 \rfloor - \lfloor 5 | 12 \rfloor$

$2 - 0 = 2$

28. How many license plates can be made using either three digits followed by three uppercase English letters or three uppercase English letters followed by three digits?

$$\underbrace{2 \times}_{\text{SINCE 2 TYPES EXISTS}} \underbrace{10}_{\text{DIGITS}}{}^{3} \times \underbrace{26}_{\text{LETTERS}}{}^{3}$$

26. How many strings of four decimal digits

a) do not contain the same digit twice?

b) end with an even digit?

c) have exactly three digits that are 9s?

(a) $10 \times 9 \times 8 \times 7$

(b) $10 \times 10 \times 10 \times 5 = 5 \times 10^3$

(c)

9 9 9 <>

9 9 <> 9

9 <> 9 9

<> 9 9 9

$= 4 \times 9$

$10000 - 810 \times 4$

$10000 - 3240$

35. How many one-to-one functions are there from a set with five elements to sets with the following number of elements?

a) 4 b) 5 c) 6 d) 7

(a) NONE (b) 5! (c) $6 \times 5 \times 4 \times 3 \times 2$ d) $7 \times 6 \times 5 \times 4 \times 3$

36. How many functions are there from the set $\{1, 2, \ldots, n\}$, where n is a positive integer, to the set $\{0, 1\}$?

2^n

---

53. How many positive integers not exceeding 100 are divisible either by 4 or by 6?

$\lfloor 100 / 6 \rfloor = 16$
$\lfloor 100 / 4 \rfloor = 25$
$\lfloor 100 / 12 \rfloor = 8$

RES $= 16 + 25 - 8 = 33$

SUBTRACTION & DIVISION RULE

THE SUBTRACTION RULE If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

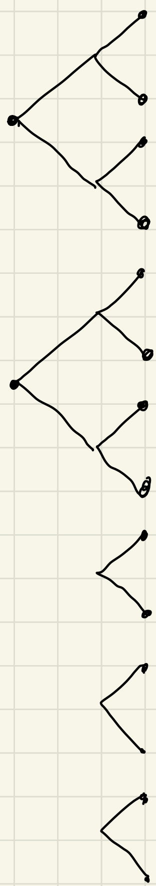
THE DIVISION RULE There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

TREE DIAGRAM

64. Use a tree diagram to find the number of bit strings of length four with no three consecutive 0s.

TOTAL POSSIBILITIES = 16

0 0 0 1
1 0 0 0 $\Rightarrow 16 - 3 = 13$
0 0 0 0



PIGEONHOLE PRINCIPLE

THEOREM 1 **THE PIGEONHOLE PRINCIPLE** If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

(a) (b) (c)

FIGURE 1 **There Are More Pigeons Than Pigeonholes.**

A function f from a set with $k + 1$ or more elements to a set with k elements is not one-to-one.

Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

THE GENERALIZED PIGEONHOLE PRINCIPLE If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

CEIL FUNC.

2. Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.

$\lceil 30/26 \rceil = 2$

4. A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.

a) How many balls must she select to be sure of having at least three balls of the same color?

b) How many balls must she select to be sure of having at least three blue balls?

(a) HOLES = 2

PIGEONS = BALLS

$$\left\lceil \frac{n}{2} \right\rceil \geq 3 \Rightarrow n = 5$$

(b) 13

6. Let d be a positive integer. Show that among any group of $d+1$ (not necessarily consecutive) integers there are two with exactly the same remainder when they are divided by d.

holes : d (reminders)

pigeons : d+1

$$\left\lceil \frac{d+1}{d} \right\rceil = 2$$

16. How many numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ to guarantee that at least one pair of these numbers add up to 16?

(15, 1) (13, 3) (11, 5) (9, 7)

HOLES: 4

PIGEONS: $^8C_2 = \frac{8!}{2! \times 6!} = \frac{8 \times 7}{2} = 49$

$7+6+5+4+3+2+1 = 28$

THEOREM 3 Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

22. Show that if there are 101 people of different heights standing in a line, it is possible to find 11 people in the order they are standing in the line with heights that are either increasing or decreasing.

PERMUTATIONS

A **permutation** of a set of distinct objects is an ordered arrangement of these objects. We also are interested in ordered arrangements of some of the elements of a set. An ordered arrangement of r elements of a set is called an ***r*-permutation**.

If n is a positive integer and r is an integer with $1 \le r \le n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) = {}^{n}P_{r} = \frac{n!}{(n-r)!}$$

r-permutations of a set with n distinct elements.

COMBINATIONS

An ***r*-combination** of elements of a set is an unordered selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.

The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \le r \le n$, equals

$$C(n, r) = \frac{n!}{r!\,(n-r)!}.$$

12. How many bit strings of length 12 contain

a) exactly three 1s?

b) at most three 1s?

c) at least three 1s?

d) an equal number of 0s and 1s?

(a) ${}^{12}C_3$

(b) ${}^{12}C_0 + {}^{12}C_1 + {}^{12}C_2 + {}^{12}C_3$

(c) ${}^{12}C_{12} - ({}^{12}C_0 + {}^{12}C_1 + {}^{12}C_2)$

(d) ${}^{12}C_6$

24. How many ways are there for 10 women and six men to stand in a line so that no two men stand next to each other? [*Hint:* First position the women and then consider possible positions for the men.]

BINOMIAL THEOREM

THE BINOMIAL THEOREM Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Q LET n BE

$$\sum \binom{n}{k} = 2^n$$

Q

$$\sum (-1)^k \binom{n}{k} = 0$$

$$(x+y)^n = \sum {}^nC_k \, x^{n-k} y^k$$

PASCALS IDENTITY

PASCAL'S IDENTITY Let n and k be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

VANDERMONDE IDENTITY

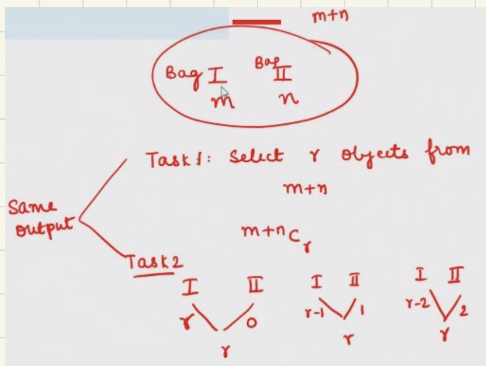
VANDERMONDE'S IDENTITY Let m, n, and r be nonnegative integers with r not exceeding either m or n. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.$$

$$^{m+n}C_r = \sum_{K=0}^{r} {}^{m}C_{r-K}\, {}^{n}C_K$$

$$= {}^{m}C_r\, {}^{n}C_0 + {}^{m}C_{r-1}\, {}^{n}C_1 + \cdots + {}^{m}C_0\, {}^{n}C_r$$

$$= \sum_{K=0}^{r} {}^{m}C_{r-K}\, {}^{n}C_K$$



Q. x^5y^8 FIND COEF.

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r$$

FOR x^5y^8

$$r = 8 \Rightarrow {}^{13}C_8\, x^5 y^8$$

$$\text{COEFF} \Rightarrow \frac{13!}{8! \times 5!}$$

$$\Rightarrow \frac{13 \times \cancel{12}^{3} \times 11 \times \cancel{10}^{2} \times \cancel{9}^{3}}{\cancel{5} \times \cancel{4} \times \cancel{3} \times 2}$$

Q. $(x + 1/x)^{100}$, x^k COEFF

$$T_{r+1} = {}^{100}C_r\, x^{100-r} \left(\frac{1}{x}\right)^r$$

$$= {}^{100}C_r\, x^{100-2r}$$

$$100 - 2r = k \Rightarrow r = \frac{100-k}{2}$$

COEFF $\quad {}^{100}C_{\frac{100-k}{2}}$

14. Show that if n is a positive integer, then $1 = \binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil} > \cdots > \binom{n}{n-1} > \binom{n}{n} = 1.$

$P(1): \quad 1 = {}^1C_0 = {}^1C_1 = 1$ TRUE

$$P(k): \quad 1 = {}^kC_0 < {}^kC_1 \cdots < \binom{k}{\lfloor k/2 \rfloor} = \binom{k}{\lceil k/2 \rceil} > \cdots > \binom{k}{k} = 1$$

${}^kC_0 < {}^kC_1$

$\Rightarrow {}^kC_0 < {}^kC_0 + {}^kC_1$

$\Rightarrow {}^{k+1}C_0 < {}^{k+1}C_1$

TPT $\quad 1 = {}^nC_0 < {}^nC_1 \cdots < \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil} > \cdots \binom{n}{n} = 1$

Step 1: For $n=1$ the statement is true

$P(1): \quad 1 = {}^1C_0 = {}^1C_1 = 1$ ✓ (True) $\qquad | \; {}^nC_r = {}^nC_{n-r}$ ✓

Induction step: $\quad P(k) \longrightarrow P(k+1)$

$1 = {}^kC_0 < {}^kC_1 \cdots < \binom{k}{\lfloor k/2 \rfloor} = \binom{k}{\lceil k/2 \rceil} > \cdots > \binom{k}{k} = 1$

TPT $\quad 1 = {}^{k+1}C_0 < {}^{k+1}C_1 \cdots < \binom{k+1}{\lfloor \frac{k+1}{2} \rfloor} < \binom{k+1}{\lceil \frac{k+1}{2} \rceil} > \cdots > \binom{k+1}{k+1} = 1$

${}^kC_0 < {}^kC_1$

$\Rightarrow {}^kC_0 < {}^kC_0 + {}^kC_1$ ✓

${}^{k+1}C_0 < {}^{k+1}C_1$

$\boxed{{}^kC_0 = {}^{k+1}C_0 = 1}$

${}^kC_1 < {}^kC_2$ Given

${}^kC_1 + {}^kC_0 < {}^kC_2 + {}^kC_1$ Given ${}^kC_0 < {}^kC_1$

${}^{k+1}C_1 < {}^{k+1}C_2$

Proceeding

$\boxed{{}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r}$

TPT $\quad {}^{k+1}C_0 < {}^{k+1}C_1$

Consider $\quad {}^kC_0 < {}^kC_1$

$${}^kC_0 \le {}^kC_1 + {}^kC_0$$
$${}^{k+1}C_0 < {}^{k+1}C_1 \checkmark$$

$1 < 2$
$1 < 2+1$

${}^kC_0 = 1$
Also ${}^{k+1}C_0 = 1$

$${}^kC_1 < {}^kC_2$$

also $\quad {}^kC_0 < {}^kC_1$

$${}^kC_0 + {}^kC_1 < {}^kC_1 + {}^kC_2$$
$${}^{k+1}C_1 < {}^{k+1}C_2$$

proceeding in this way &

A *linear homogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}, \checkmark$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

2. Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.

a) $a_n = 3a_{n-2}$

b) $a_n = 3$

c) $a_n = a_{n-1}^2$

d) $a_n = a_{n-1} + 2a_{n-3}$

e) $a_n = a_{n-1}/n$ L,H,VC,D=1

f) $a_n = a_{n-1} + a_{n-2} + n + 3$ L, NH, C

g) $a_n = 4a_{n-2} + 5a_{n-4} + 9a_{n-7}$ L, H, C

(a) $a_n = 3a_{n-2}$ LINEAR (COEFFICIENTS ARE CONST) HOMOGENOUS & CONST COEFF

(b) $a_n = 3$ LINEAR, NON-HOMOGENOUS, CONST. COEFF & DEGREE = 0

(c) $a_n = (a_{n-1})^2$ NON LINEAR

(d) $a_n = a_{n-1} + 2a_{n-3}$ LINEAR, HOMOGENOUS, CONST. COEFF, DEG = 3

SOLUTION TO LINEAR RR WITH WITH CONST. COEFF

Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

(1) $a_n = r^n$

$\Rightarrow a_{n-1} = r^{n-1}$

$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k} \Rightarrow r^k = c_1 r^{k-1} + \cdots + c_k$

$\boxed{r^k - c_1 r^{k-1} - \cdots - c_k = 0} \rightarrow$ CHARACTERISTIC EQN

Ex: $a_n = 3a_{n-1} - 2a_{n-2}$

Assume its sol as $a_n = r^n$

$r^n = 3r^{n-1} - 2r^{n-2}$

$\Rightarrow r^2 - 3r + 2 = 0$

$\Rightarrow r = 2, 1$

sol. $a_n = \underline{\alpha_1}(2)^n + \underline{\alpha_2}(1)^n$

4. Solve these recurrence relations together with the initial conditions given.

a) $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$
b) $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$, $a_0 = 2$, $a_1 = 1$
c) $a_n = 6a_{n-1} - 8a_{n-2}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 10$
d) $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 1$

(a) C.E. $\Rightarrow$ $\gamma^n = \gamma^{n-1} + 6\gamma^{n-2}$

$\Rightarrow \gamma^2 - \gamma - 6 = 0$

$\Rightarrow \gamma = 3, -2$

$a_n = \alpha_1 3^n + \alpha_2 (-2)^n$

$a_0 = 3 \Rightarrow n = 0 \quad a_n = 3$

$\Rightarrow 3 = \alpha_1 + \alpha_2$

$a_1 = 6 \Rightarrow n = 1 \quad a_n = 6$

$\Rightarrow 6 = 3\alpha_1 - 2\alpha_2$

$\alpha_1 = 9/4$

$\alpha_2 = 3/4$

$$a_n = \frac{9}{4} 3^n + \frac{3}{4}(-2)^n$$

$a_n = 6a_{n-1} - 8a_{n-2}$

$\gamma^n = \gamma^{n-1} + \gamma^{n-2}$

$\gamma^2 - 6\gamma + 8 = 0$

$\gamma = 2, 4$

$a_n = \alpha_1 2^n + \alpha_2 4^n$

$\alpha_1 + \alpha_2 = 4$

$2\alpha_1 + 4\alpha_2 = 10$

$2\alpha_2 = 2$

$\alpha_2 = 1$

$\alpha_1 = 3$

(d) $a_n = 2a_{n-1} - a_{n-2}$

C.Eq. $r^2 - 2r + 1 = 0$

$r = 1, 1$

$a_n = \alpha_1 (1)^n - \alpha_2 n (1)^n$

$a_n = \alpha_1 - n\alpha_2$

$\alpha_1 = 4$

$\alpha_1 + \alpha_2 = 1 \Rightarrow \alpha_2 = -3$

14. Find the solution to $a_n = 5a_{n-2} - 4a_{n-4}$ with $a_0 = 3$, $a_1 = 2$, $a_2 = 6$, and $a_3 = 8$.

C.Eq : $r^4 - 5r^2 + 4 = 0$

$\Rightarrow (r^2 - 4)(r^2 - 1) = 0$

$\Rightarrow r = \pm 2 \quad r = \pm 1$

$a_n = \alpha_1 (1)^n + \alpha_2 (-1)^n + \alpha_3 (2)^n + \alpha_4 (-2)^n$

SOLVING NON HOMOGENOUS

If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

$a_n = 3a_{n-1} - a_{n-2} + n^2$ non homog.

homo I, II

$a_n = 3a_{n-1} - a_{n-2}$ $\quad$ $a_n = 3a_{n-1} - a_{n-2} + n^2$

$a_n^{(h)} = (\ldots)$ $\quad$ $a_n^{(p)}$ is particular sol

$a_n = a_n^{h} + a_n^{p}$

innovate

Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0)s^n,$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers. When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0)s^n.$$

When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0)s^n.$$

$a_n = 3a_{n-1} - 2a_{n-2} + (-2)^n$

$a_n^{(h)} = \alpha_1 (2)^n + \alpha_2 (1)^n$

① $y_p = \alpha_3 (-2)^n$ $\quad [F(n) = (-2)^n]$

② $y_p = \alpha_3 n (2^n)$ $\quad [F(n) = 2^n]$

③ $y_p = \alpha_3 n^2 (2)^n$ $\quad [F(n) = n2^n]$

④ $F(n) = 5(2)^n$

$y_p = \alpha_3 n 2^n$

⑤ $F(n) = (n^2+n)(-2)^n$

$y_p = (P_1 n^2 + P_2 n)(-2)^n$

⑥ $F(n) = (n^2+n)(2)^n$

$y_p = n(P_1 n^2 + P_2 n)(2)^n$

26. What is the general form of the particular solution guaranteed to exist by Theorem 6 of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} + F(n)$ if

a) $F(n) = n^2$?
b) $F(n) = 2^n$?
c) $F(n) = n2^n$?
d) $F(n) = (-2)^n$?
e) $F(n) = n^2 2^n$?
f) $F(n) = n^3(-2)^n$?
g) $F(n) = 3$?

C.E. $\Rightarrow \gamma^3 - 6\gamma^2 + 12\gamma - 8 = 0$

$\gamma = 2, 2, 2$

$a_n^h = (\alpha_1 + \alpha_2 n + \alpha_3 n^2)(2)^n$

(a) $F(n) = n^2 (1)^n$
$a_n^P = \alpha_4 n^2 (1)^n$

(b) $F(n) = 2^n$
$a_n^P = \alpha_4 n^3 (2)^n$

(c) $F(n) = n2^n$
$a_n^P = \alpha_4 n^4 2^n$

(d) $F(n) = (-2)^n$
$a_n^P = \alpha_4 (-2)^n$

(e) $F(n) = n^2 2^n$
$a_n^P = \alpha_4 n^5 2^n$

(f) $F(n) = n^3(-2)^n$
$a_n^K = \alpha_4 n^3 (-2)^n$

(g) $F(n) = 3 (1)^n$
$a_n^P = \alpha_4 (1)^n$