$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 \tag{5.26}$$

$$f = -\frac{h^2}{p^3}\frac{\mathrm{d}p}{\mathrm{d}r} \tag{5.27}$$

where p is the impact parameter and h is the angular momentum per unit mass.

5.2 Problems

5.2.1 Field and Potential

5.1 Calculate the gravitational force between two lead spheres of radius 10 cm in contact with one another, $G = 6.67 \times 10^{-11}$ MKS units. Density of lead $= 11{,}300\,\mathrm{kg/m^3}$.

[University of Dublin]

5.2 Considering Fig. 5.1, what is the magnitude of the net gravitational force exerted on the uniform sphere, of mass 0.010 kg, at point P by the other two uniform spheres, each of mass 0.260 kg, that are fixed at points A and B as shown.

[The University of Wales, Aberystwyth 2005]

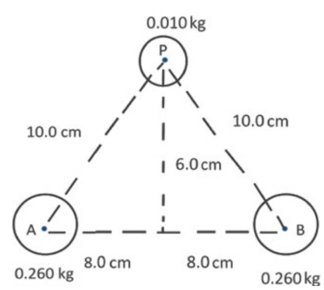


Fig. 5.1

5.3 Two bodies of mass m and M are initially at rest in an inertial reference frame at a great distance apart. They start moving towards each other under gravitational attraction. Show that as they approach a distance d apart ($d << r$), their relative velocity of approach will be $\sqrt{\dfrac{2G(M+m)}{d}}$, where G is the gravitational constant.

5.4 If the earth suddenly stopped in its orbit assumed to be circular, find the time that would elapse before it falls into the sun.

5.5 Because of the rotation of the earth a plumb bob when hung may not point exactly in the direction of the earth's gravitational force on the plumb bob. It may slightly deviate through a small angle.

(a) Show that at latitude λ, the deflection angle θ in radians is given by

$$\theta = \left(\frac{2\pi^2 R}{gT^2}\right)\sin 2\lambda$$

where R is the radius of earth and T is the period of the earth's rotation.
(b) At what latitude is the deflection maximum?
(c) What is the deflection at the equator?

5.6 Show that the gravitational energy of earth assumed to be the uniform sphere of radius R and mass M is $3GM^2/5R$. What is the potential energy of earth assuming it to be a uniform sphere of radius $R = 6.4 \times 10^6$ m and of mass $M = 6.0 \times 10^{24}$ kg.

5.7 Assuming that the earth has constant density, at what distance d from the earth's surface the gravity above the earth is equal to that below the surface.

5.8 Assuming the radius of the earth to be 6.38×10^8 cm, the gravitational constant to be $6.67 \times 10^{-8}\,\mathrm{cm^3\,g/m/s^2}$, acceleration due to gravity on the surface to be $980\,\mathrm{cm/s^2}$, find the mean density of the earth.

[University of Cambridge]

5.9 How far from the earth must a body be along a line towards the sun so that the sun's gravitational pull balances the earth? The sun is about 9.3×10^7 km away and its mass is $3.24 \times 10^5\, M_\mathrm{e}$, where M_e is the mass of the earth.

5.10 Assuming the earth to be a perfect sphere of radius 6.4×10^8 cm, find the difference due to the rotation of the earth in the value of g at the poles and at the equator.

[Northern Universities of UK]

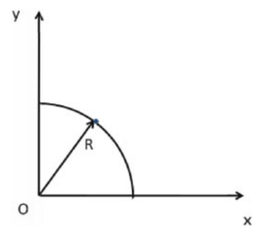
5.11 Derive an expression for the gravitational potential $V(r)$ due to a uniform solid sphere of mass M and radius R when $r < R$.

5.12 Derive an expression for the potential due to a thin uniform rod of mass M and length L at a point distant d from the centre of the rod on the axial line of the rod.

5.13 Show that for a satellite moving close to the earth's surface along the equator, moving in the western direction will require launching speed 11% higher than that moving in the eastern direction.

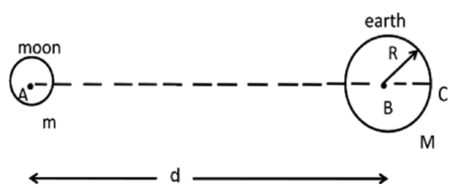
5.14 A thin wire of linear mass density λ is bent in the form of a quarter circle of radius R (Fig. 5.2). Calculate the gravitational intensity at the centre O.

Fig. 5.2



5.15 A tidal force is exerted on the ocean by the moon. This is estimated by the differential (Δg) which is the difference of the acceleration at B and that at C due to the moon (Fig. 5.3). If R is the radius of the earth, d the distance of separation of the centre of earth and moon, M and m the mass of the earth and moon, respectively, show that $\Delta g \approx \dfrac{2GmR}{d^3}$.

Fig. 5.3



5.16 Assume that a star has uniform density. Show that the gravitational pressure $P \propto V^{-4/3}$, where V is the volume.

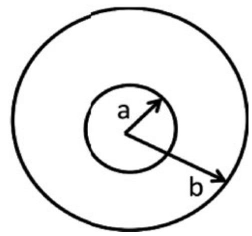
5.17 Find the gravitational field due to an infinite line mass of linear density λ, at distance R.

5.18 If the earth–moon distance is d and the mass of earth is 81 times that of the moon, locate the neutral point on the line joining the centres of the earth and moon.

5.19 A particle of mass m was taken from the centre of the base of a uniform hemisphere of mass M and radius R to infinity. Calculate the work done in overcoming gravitational force due to the hemisphere.

5.20 The cross-section of a spherical shell of uniform density and mass M and of radii a and b is shown in Fig. 5.4. How does the gravitational field vary in the region $a < r < b$?

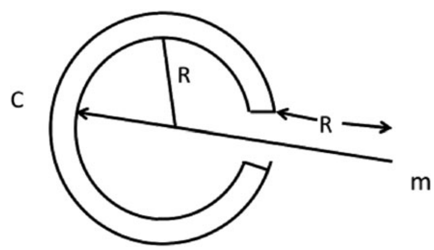
Fig. 5.4



5.21 Find the variation of the magnitude of gravitational field along the z-axis due to a disc of radius 'a' and surface density σ, lying in the xy-plane.

5.22 Figure 5.5 shows a spherical shell of mass M and radius R in a force-free region with an opening. A particle of mass m is released from a distance R in front of the opening. Calculate the speed with which the particle will hit the point C on the shell, opposite to the opening.

Fig. 5.5



5.2.2 Rockets and Satellites

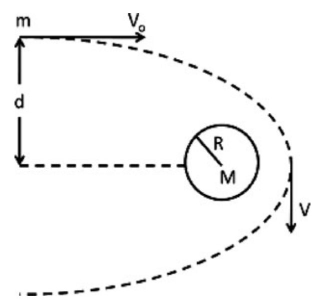
5.23 A particle of mass m is fired upwards from the surface of a planet of mass M and radius R with velocity $v = \sqrt{\dfrac{GM}{2R}}$. Show that the maximum height which the particle attains is $R/3$.

5.24 Consider a nebula in the form of a ring of radius R and mass M. A star of mass $m(m << M)$ is located at distance r from the centre of the ring on its axis, initially at rest. Show that the speed with which it crosses the centre of the ring will be $v = \sqrt{(2 - \sqrt{2})\frac{GM}{R}}$.

5.25 If W_1 is the work done in taking the satellite from the surface of the earth of radius R to a height h, and W_2 the extra work required to put the satellite in the orbit at altitude h, and if $h = R/2$ then show that the ratio $\frac{W_1}{W_2} = 1.0$.

5.26 An asteroid is moving towards a planet of mass M and radius R, from a long distance with initial speed v_0 and impact parameter d (Fig. 5.6). Calculate the minimum value of v_0 such that the asteroid does not hit the planet.

Fig. 5.6



5.27 The orbits of earth and Venus around the sun are very nearly circular with mean radius of the earth's orbit $r_E = 1.50 \times 10^{11}$ m and mean radius of Venus' orbit $r_v = 1.08 \times 10^{11}$ m. If the earth's period of orbit round the sun is 365.3 days and Venus is 224.7 days

(i) Show that these figures are approximately consistent with Kepler's third law.
(ii) Derive a formula to estimate the mass of the sun ($G = 6.67 \times 10^{-11}$ N m^2/kg^2).

[The University of Aberystwyth, Wales]

5.28 The greatest and least velocities of a certain planet in its orbit around the sun are 30.0 and 29.2 km/s. Find the eccentricity of the orbit.

5.29 A binary star is formed when two stars bound by gravity move around a common centre of mass. Each component of a binary star has period of revolution about their centre of mass, equal to 14.4 days and the velocity of each component of 220 km/s. Further, the orbit is nearly circular. Calculate **(a)** the separation of the two components and **(b)** the mass of each component.

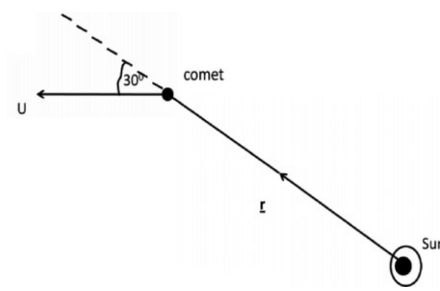
5.30 A satellite is fired from the surface of the moon of mass M and radius R with speed v_0 at 30° with the vertical. The satellite reaches a maximum distance of $5R/2$ from the centre of the planet. Show that $v_0 = (5GM/4R)^{1/2}$.

5.31 If a satellite has its largest and smallest speeds given by $v_{\max}$ and $v_{\min}$, respectively, and has time period equal to T, then show that it moves on an elliptic path of semi-major axis $\frac{T}{2\pi}\sqrt{v_{\max}v_{\min}}$.

5.32 A satellite of radius 'a' revolves in a circular orbit about a planet of radius b with period T. If the shortest distance between their surfaces is c, prove that the mass of the planet is $4\pi^2(a+b+c)/GT^2$.

5.33 When a comet is at a distance 1.75 AU from the sun, it is moving with velocity $u = 30\,\text{km/s}$ and its velocity vector is at an angle of 30° relative to its radius vector **r** centred on the sun (see Fig. 5.7).
What is the angular momentum per unit mass of the comet about the sun?
The closest distance from the sun that the comet reaches is 0.39 AU. What is the speed of the comet at this point?
Is the comet's orbit bound or unbound?
(1 AU $= 1.5 \times 10^{11}$ m, mass of the sun $= 2 \times 10^{30}$ kg)
[University of Durham 2002]

Fig. 5.7



5.34 **(a)** Assuming that the earth (mass M_E) orbits the sun (mass M_S) in a circle of radius R and with a speed v, write down the equation of motion for the earth. Hence show that $GM_S = v^2R$

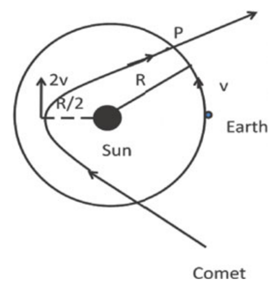
(b) A comet is in orbit around the sun in the same plane as the earth's orbit, as shown in Fig. 5.8. Its distance of closest approach to the sun's centre is $R/2$, at which point it has speed $2v$.
Using the condition for the Earth's orbit given in **(a)**, show that the comet's total energy is zero. (Neglect the effect of the earth on the comet.)

(c) Use conservation of angular momentum to determine the component of the comet's velocity which is tangential to the earth's orbit at the point P, where the comet's orbit crosses that of the earth.

(d) Use conservation of energy to find its speed at the point P. Hence show that the comet crosses the earth's orbit at an angle of 45°.

[University of Manchester 2008]

Fig. 5.8



5.35 The geocentric satellite 'Apple' was first launched into an elliptic orbit with the perigee (nearest point) of $r_{\rm p} = 6570$ km and apogee (farthest point) at $r_{\rm A} = 42{,}250$ km. The respective velocities were $v_{\rm p} = 10.25$ km/s and $v_{\rm A} = 1.594$ km/s. Show that the above data are consistent with the conservation of angular momentum of the satellite about the centre of the earth.

5.36 **(a)** Assuming that the earth is a sphere of radius 6400 km, with what velocity must a projectile be fired from the earth's surface in order that its subsequent path be an ellipse with major axis equal to 80,000 km?

(b) If the projectile is fired upwards at an angle 45° to the vertical, what would be the eccentricity of this ellipse?

5.37 A satellite of mass m is orbiting in a circular orbit of radius r and velocity v around the earth of mass M. Due to an internal explosion, the satellite breaks into two fragments each of mass $m/2$. In the frame of reference of the satellite, the two fragments appears to move radially along the line joining the original satellite and the centre of the earth, each with the velocity $v_0/2$. Show that immediately after the explosion each fragment has total energy $-3GM/16r$ and angular momentum $\dfrac{m}{2}\sqrt{GMr}$, with reference to the centre of the earth.

5.38 A particle describes an ellipse of eccentricity e under a force to a focus. When it approaches the nearer apse (turning point) the centre of force is transferred to the other focus. Prove that the eccentricity of the new orbit is $\varepsilon(3+\varepsilon)/(1-\varepsilon)$.

5.39 A particle of mass m describes an elliptical orbit of semi-major axis 'a' under a force mk/r^2 directed to a focus. Prove that

(a) the time average of reciprocal distance

$$\frac{1}{T}\int \frac{\mathrm{d}t}{r} = \frac{1}{a}$$

(b) the time average of square of the speed $\frac{1}{T}\int v^2 \mathrm{d}t = \frac{GM}{a}$

5.40 A small meteor of mass m falls into the sun when the earth is at the end of the minor axis of its orbit. If M is the mass of the sun, find the changes in the major axis and in the time period of the earth.

5.41 A particle is describing an ellipse of eccentricity 0.5 under the action of a force to a focus and when it arrives to an apse (turning point) the velocity is doubled. Show that the new orbit will be a parabola or hyperbola accordingly as the apse is the farther or nearer one.

5.42 When a particle is at the end of the minor axis of an ellipse, the force is increased by half. Prove that the axes of the new orbit are $3a/2$ and $\sqrt{2b}$, where $2a$ and $2b$ are the old axes.

5.43 A satellite is placed in a circular orbit of radius R around the earth.

(a) What are the forces acting on the satellite? Write down the equilibrium condition.
(b) Derive an expression for the time period of the satellite.
(c) What conditions must be satisfied by a geocentric satellite?
(d) What is the period of a geosynchronous satellite?
(e) Calculate the radius of orbit of a geocentric satellite from the centre of the earth.

5.44 A satellite moves in an elliptic path with the earth at one focus. At the perigee (nearest point) its speed is v and its distance from the centre of the earth is r. What is its speed at the apogee (farthest point)?

5.45 A small body encounters a heavy body of mass M. If at a great distance the velocity of the small body is v and the impact parameter is p, and φ is the angle of encounter, prove that $\tan(\varphi/2) = GM/v^2 p$.

5.46 Obtain an expression for the time required to describe an arc of a parabola under the action of the force k/r^2 to the focus, starting from the end of the axis.

5.47 A comet describes a parabolic path in the plane of the earth's orbit, assumed to be circular. Show that the maximum time the comet is able to remain inside the earth's orbit is $2/3\pi$ of a year.

5.48 Find the law of force for the orbit $r = a \sin n\theta$.

5.49 Find the law of force to the pole when the orbit described by the cardioid $r = a(1 - \cos\theta)$.

5.50 In prob. (5.49) prove that if Q be the force at the apse and v the velocity, $3v^2 = 4aQ$.

5.51 A particle moves in a plane under an attractive force varying as the inverse cube of the distance. Find the equation of the orbit distinguishing three cases which may arise.

5.52 Show that the central force necessary to make a particle describe the lemniscate $r^2 = a^2 \cos 2\theta$ is inversely proportional to r^7.

5.53 Show that if a particle describes a circular orbit under the influence of an attractive central force directed towards a point on the circle, then the force varies as the inverse fifth power of distance.

5.54 If the sun's mass suddenly decreased to half its value, show that the earth's orbit assumed to be originally circular would become parabolic.

5.3 Solutions

5.3.1 Field and Potential

5.1 $F = \dfrac{GM_1M_2}{r^2}$

If R is the radius of either sphere, the distance between the centre of the spheres in contact is $r = 2R$:

$$M_1 = M_2 = M = \frac{4}{3}\pi R^3 \rho$$

$$F = \frac{GM^2}{4R^2} = \frac{4\pi^2 G R^4 \rho^2}{9}$$

$$= \frac{4\pi^2}{9} \times 6.67 \times 10^{-11} \times (0.2)^4 (11300)^2 = 5.98 \times 10^{-5}\,\mathrm{N}$$

5.2 As the mass of A and B are identical and the distance PA = PB, the magnitude of the force $F_{\mathrm{PA}} = F_{\mathrm{PB}}$. Resolve these forces in the horizontal and vertical direction. The horizontal components being in opposite direction get cancelled. The vertical components get added up.

$$F_{\mathrm{PA}} = F_{\mathrm{PB}} = G\frac{(0.01)(0.26)}{(0.1)^2} = 0.26G$$