Hamiltonian's Canonical Equations

$$\frac{\partial H}{\partial p_r} = \dot{q}_r, \quad \frac{\partial H}{\partial q_r} = -\dot{p}_r \tag{7.4}$$

7.2 Problems

7.1 Consider a particle of mass m moving in a plane under the attractive force $\mu m/r^2$ directed to the origin of polar coordinates r, θ. Determine the equations of motion.

7.2 **(a)** Write down the Lagrangian for a simple pendulum constrained to move in a single vertical plane. Find from it the equation of motion and show that for small displacements from equilibrium the pendulum performs simple harmonic motion.

(b) Consider a particle of mass m moving in one dimension under a force with the potential $U(x) = k(2x^3 - 5x^2 + 4x)$, where the constant $k > 0$. Show that the point $x = 1$ corresponds to a stable equilibrium position of the particle. Find the frequency of a small amplitude oscillation of the particle about this equilibrium position.

[University of Manchester 2007]

7.3 Determine the equations of motion of the masses of Atwood machine by the Lagrangian method.

7.4 Determine the equations of motion of Double Atwood machine which consists of one of the pulleys replaced by an Atwood machine. Neglect the masses of pulleys.

7.5 A particular mechanical system depending on two coordinates u and v has kinetic energy $T = v^2\dot{u}^2 + 2\dot{v}^2$, and potential energy $V = u^2 - v^2$. Write down the Lagrangian for the system and deduce its equations of motion (do not attempt to solve them).

[University of Manchester 2008]

7.6 Write down the Lagrangian for a simple harmonic oscillator and obtain the expression for the time period.

7.7 A particle of mass m slides on a smooth incline at an angle α. The incline is not permitted to move. Determine the acceleration of the block.

7.8 A block of mass m and negligible size slides on a frictionless inclined plane of mass M at an angle θ with the horizontal. The plane itself rests on a smooth horizontal table. Determine the acceleration of the block and the inclined plane.

7.9 A bead of mass m is free to slide on a smooth straight wire of negligible mass which is constrained to rotate in a vertical plane with constant angular speed ω about a fixed point. Determine the equation of motion and find the distance x from the fixed point at time t. Assume that at $t = 0$ the wire is horizontal.

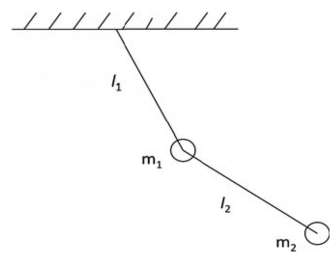
7.10 Consider a pendulum consisting of a small mass m attached to one end of an inextensible cord of length l rotating about the other end which is fixed. The pendulum moves on a spherical surface. Hence the name spherical pendulum. The inclination angle φ in the xy-plane can change independently.

(a) Obtain the equations of motion for the spherical pendulum.
(b) Discuss the conditions for which the motion of a spherical pendulum is converted into that of **(i)** simple pendulum and **(ii)** conical pendulum.

7.11 Two blocks of mass m and M connected by a massless spring of spring constant k are placed on a smooth horizontal table. Determine the equations of motion using Lagrangian mechanics.

7.12 A double pendulum consists of two simple pendulums of lengths l_1 and l_2 and masses m_1 and m_2, with the cord of one pendulum attached to the bob of another pendulum whose cord is fixed to a pivot, Fig. 7.1. Determine the equations of motion for small angle oscillations using Lagrange's equations.

Fig. 7.1



7.13 Use Hamilton's equations to obtain the equations of motion of a uniform heavy rod of mass M and length $2a$ turning about one end which is fixed.

7.14 A one-dimensional harmonic oscillator has Hamiltonian $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2$. Write down Hamiltonian's equation and find the general solution.

7.15 Determine the equations for planetary motion using Hamilton's equations.

7.16 Two blocks of mass m_1 and m_2 coupled by a spring of force constant k are placed on a smooth horizontal surface, Fig. 7.2. Determine the natural frequencies of the system.

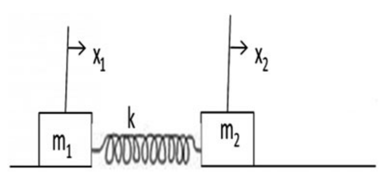
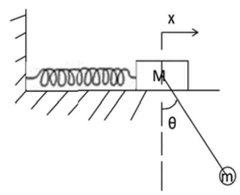


Fig. 7.2

7.17 A simple pendulum of length l and mass m is pivoted to the block of mass M which slides on a smooth horizontal plane, Fig. 7.3. Obtain the equations of motion of the system using Lagrange's equations.

Fig. 7.3



7.18 Determine the equations of motion of an insect of mass m crawling at a uniform speed v on a uniform heavy rod of mass M and length $2a$ which is turning about a fixed end. Assume that at $t = 0$ the insect is at the middle point of the rod and it is crawling downwards.

7.19 A uniform rod of mass M and length $2a$ is attached at one end by a cord of length l to a fixed point. Calculate the inclination of the string and the rod when the string plus rod system revolves about the vertical through the pivot with constant angular velocity ω.

7.20 A particle moves in a horizontal plane in a central force potential $U(r)$. Derive the Lagrangian in terms of the polar coordinates (r, θ). Find the corresponding momenta p_r and p_θ and the Hamiltonian. Hence show that the energy and angular momentum of the particle are conserved.

[University of Manchester 2007]

7.21 Consider the system consisting of two identical masses that can move horizontally, joined with springs as shown in Fig. 7.4. Let x, y be the horizontal displacements of the two masses from their equilibrium positions.

(a) Find the kinetic and potential energies of the system and deduce the Lagrangian.

(b) Show that Lagrange's equation gives the coupled linear differential equations

$$\begin{cases} m\ddot{x} = -4kx + 3ky \\ m\ddot{y} = 3kx - 4ky \end{cases}$$

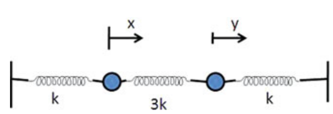
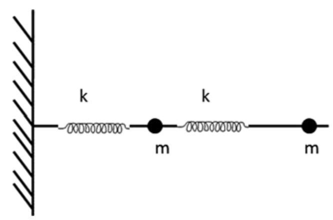


Fig. 7.4

(c) Find the normal modes of oscillation of this system and their period of oscillation.

7.22 Two identical beads of mass m each can move without friction along a horizontal wire and are connected to a fixed wall with two identical springs of spring constant k as shown in Fig. 7.5.

Fig. 7.5



(a) Find the Lagrangian for this system and derive from it the equations of motion.
(b) Find the eigenfrequencies of small amplitude oscillations.
(c) For each normal mode, sketch the system when it is at maximum displacement.

Note: Your sketch should indicate the relative sizes as well as the directions of the displacements.

[University of Manchester 2007]

7.23 Two beads of mass $2m$ and m can move without friction along a horizontal wire. They are connected to a fixed wall with two springs of spring constants $2k$ and k as shown in Fig. 7.6:

(a) Find the Lagrangian for this system and derive from it the equations of motion for the beads.
(b) Find the eigenfrequencies of small amplitude oscillations.
(c) For each normal mode, sketch the system when it is at the maximum displacement.

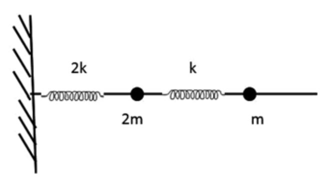


Fig. 7.6

7.24 Three identical particles of mass m, M and m with M in the middle are connected by two identical massless springs with a spring constant k. Find the normal modes of oscillation and the associated frequencies.

7.25 **(a)** A bead of mass m is constrained to move under gravity along a planar rigid wire that has a parabolic shape $y = x^2/l$, where x and y are, respectively, the horizontal and the vertical coordinates. Show that the Lagrangian for the system is

$$L = \frac{m(\dot{x})^2}{2}\left(1 + \frac{4x^2}{l^2}\right) - \frac{mgx^2}{l}$$

(b) Derive the Hamiltonian for a single particle of mass m moving in one dimension subject to a conservative force with a potential $U(x)$.

[University of Manchester 2006]

7.26 A pendulum of length l and mass m is mounted on a block of mass M. The block can move freely without friction on a horizontal surface as shown in Fig. 7.7.

(a) Show that the Lagrangian for the system is

$$L = \left(\frac{M+m}{2}\right)(\dot{x})^2 + ml\cos\theta\,\dot{x}\dot{\theta} + \frac{m}{2}l^2(\dot{\theta})^2 + mgl\cos\theta$$

(b) Show that the approximate form for this Lagrangian, which is applicable for a small amplitude swinging of the pendulum, is

$$L = \left(\frac{M+m}{2}\right)(\dot{x})^2 + ml\dot{x}\dot{\theta} + \frac{m}{2}l^2(\dot{\theta})^2 + mgl\left(1 - \frac{\theta^2}{2}\right)$$

(c) Find the equations of motion that follow from the simplified Lagrangian obtained in part (b),

(d) Find the frequency of a small amplitude oscillation of the system.

[University of Manchester 2006]

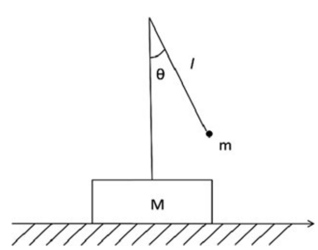


Fig. 7.7

7.27 **(a)** A particle of mass m slides down a smooth spherical bowl, as in Fig. 7.8. The particle remains in a vertical plane (the xz-plane). First, assume that the bowl does not move. Write down the Lagrangian, taking the angle ϑ with respect to the vertical direction as the generalized coordinate. Hence, derive the equation of motion for the particle.

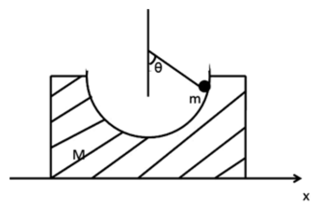


Fig. 7.8

(b) Assume now that the bowl rests on a smooth horizontal table and has a mass M, the bowl can slide freely along the x-direction.

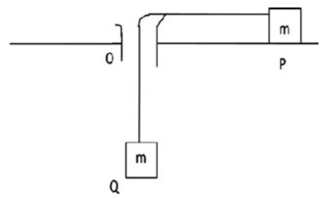
(i) Write down the Lagrangian in terms of the angle θ and the x-coordinate of the bowl, x.
(ii) Starting from the corresponding Lagrange's equations, obtain an equation giving $\ddot{x}$ in terms of θ, $\dot{\theta}$ and $\ddot{\theta}$ and an equation giving $\ddot{\theta}$ in terms of $\ddot{x}$ and θ.
(iii) Hence, and assuming that $M >> m$, show that for small displacements about equilibrium the period of oscillation of the particle is smaller by a factor $[M/(M+m)]^{1/2}$ as compared to the case where the bowl is fixed. [You may neglect the terms in $\theta^2\ddot{\theta}$ or $\theta\dot{\theta}^2$ compared to terms in $\ddot{\theta}$ or θ.]

[University of Durham 2004]

7.28 A system is described by the single (generalized) coordinate q and the Lagrangian $L(q, \dot{q})$. Define the generalized momentum associated with q and the corresponding Hamiltonian, $H(q, p)$. Derive Hamilton's equations from Lagrange's equations of the system. For the remainder of the question, consider the system whose Lagrangian, $L(q, \dot{q})$. Find the corresponding Hamiltonian and write down Hamilton's equations.

7.29 Briefly explain what is the "generalized (or canonical) momentum conjugate to a generalized coordinate". What characteristic feature should the Lagrangian function have for a generalized momentum to be a constant of motion? A particle P can slide on a frictionless horizontal table with a small opening at O. It is attached, by a string of length l passing through the opening, to a particle Q hanging vertically under the table (see Fig. 7.9). The two particles have equal mass, m. Let τ denote the distance of P to the opening, θ the angle between OP and some fixed line through O and g the acceleration of gravity. Initially, $r = a$, Q does not move, and P is given an initial velocity of magnitude $(ag)^{1/2}$ at right angles to OP.

Fig. 7.9



(a) Write the Lagrangian in terms of the coordinates r and θ and derive the corresponding equations of motion.

(b) Using these equations of motion and the initial conditions, show that $\ddot{r} = a^3 g/r^3 - g$.

(c) Hence, (i) show that the trajectory of P is the circle $r = a$, (ii) show that P describes small oscillations about this circle if it is slightly displaced from it and (iii) calculate the period of these oscillations:

$[v_p^2 = \dot{r}^2 + r^2\dot{\theta}^2,$ where v_p is the speed of P]

7.30 A particle of mass m is constrained to move on an ellipse E in a vertical plane, parametrized by $x = a\cos\theta$, $y = b\sin\theta$, where $a, b > 0$ and $a \neq b$ and the positive y-direction is the upward vertical. The particle is connected to the origin by a spring, as shown in the diagram, and is subject to gravity. The potential energy in the spring is $\frac{1}{2}kr^2$ where r is the distance of the point mass from the origin (Fig. 7.10).

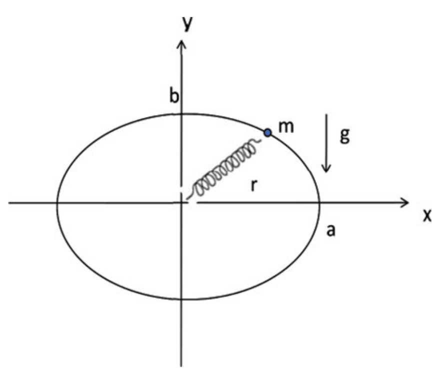


Fig. 7.10

(i) Using θ as a coordinate, find the kinetic and potential energies of the particle when moving on the ellipse. Write down the Lagrangian and show that Lagrange's equation becomes $m(a^2 \sin^2\theta + b^2 \cos^2\theta)\ddot{\theta} = (a^2 - b^2)(k - m\dot{\theta}^2)\sin\theta\cos\theta - mgb\cos\theta$.

(ii) Show that $\theta = \pm\pi/2$ are two equilibrium points and find any other equilibrium points, giving carefully the conditions under which they exist. You may either use Lagrange's equation or proceed directly from the potential energy.

(iii) Determine the stabilities of each of the two equilibrium points $\theta = \pm\pi/2$ (it may help to consider the cases $a > b$ and $a < b$ separately).

(iv) When the equilibrium point at $\theta = -\pi/2$ is stable, determine the period of small oscillations.

[University of Manchester 2008]

7.31 In prob. (7.12) on double pendulum if $m_1 = m_2 = m$ and $l_1 = l_2 = l$, obtain the frequencies of oscillation.

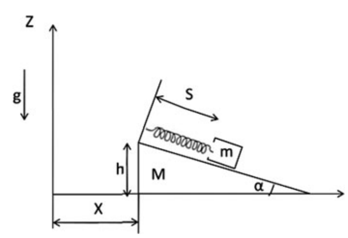
7.32 Use Lagrange's equations to obtain the natural frequencies of oscillation of a coupled pendulum described in prob. (6.46).

7.33 A bead of mass m slides freely on a smooth circular wire of radius r which rotates with constant angular velocity ω. On a horizontal plane about a point fixed on its circumference, show that the bead performs simple harmonic motion about the diameter passing through the fixed point as a pendulum of length $r = g/\omega^2$.

[with permission from Robert A. Becker, Introduction to theoretical mechanics, McGraw-Hill Book Co., Inc., 1954]

7.34 A block of mass m is attached to a wedge of mass M by a spring with spring constant k. The inclined frictionless surface of the wedge makes an angle α to the horizontal. The wedge is free to slide on a horizontal frictionless surface as shown in Fig. 7.11.

Fig. 7.11



(a) Show that the Lagrangian of the system is

$$L = \frac{(M+m)}{2}\dot{x}^2 + \frac{1}{2}m\dot{s}^2 + m\dot{x}\dot{s}\cos\alpha - \frac{k}{2}(s-l)^2 - mg(h - s\sin\alpha),$$

where l is the natural length of the spring, x is the coordinate of the wedge and s is the length of the spring.

(b) By using the Lagrangian derived in (a), show that the equations of motion are as follows:

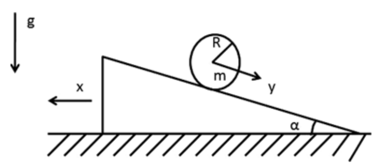
$$(m + M)\ddot{x} + m\ddot{s}\cos\alpha = 0,$$
$$m\ddot{x}\cos\alpha + m\ddot{s} + k(s - s_0) = 0,$$
where $s_0 = l + (mg\sin\alpha)/k$.

(c) By using the equations of motion in (b), derive the frequency for a small amplitude oscillation of this system.

[University of Manchester 2008]

7.35 A uniform spherical ball of mass m rolls without slipping down a wedge of mass M and angle α, which itself can slide without friction on a horizontal table. The system moves in the plane shown in Fig. 7.12. Here g denotes the gravitational acceleration.

Fig. 7.12



(a) Find the Lagrangian and the equations of motion for this system.

(b) For the special case of $M = m$ and $\alpha = \pi/4$ find

(i) the acceleration of the wedge and

(ii) the acceleration of the ball relative to the wedge.

[Useful information: Moment of inertia of a uniform sphere of mass m and radius R is $I = \frac{2}{5}mR^2$.]

[University of Manchester 2007]

7.3 Solutions

7.1 This is obviously a two degree of freedom dynamical system. The square of the particle velocity can be written as

$$v^2 = \dot{r}^2 + (r\dot{\theta})^2 \tag{1}$$

Formula (1) can be derived from Cartesian coordinates

$$x = r\cos\theta, \quad y = r\sin\theta$$
$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \quad \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$