15.3 Solutions

15.3.1 Geometrical Optics

General

15.1 Let the point source be located at the centre O of a sphere of radius R, located within the medium. Light proceeding within a cone of semi-angle equal to the critical angle C can alone escape from a plane surface on the top, Fig. 15.14.

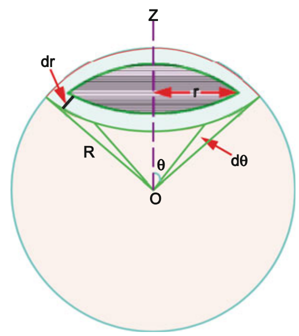


Fig. 15.14 Light from a point source escaping through a plane surface on the top

Consider a circular strip of radius r and width $\mathrm{d}r$, symmetrical over the sphere's surface. The angle θ is measured with respect to the z-axis.

Area of the circular strip $= 2\pi r \mathrm{d}r$

Surface area of the sphere $= 4\pi R^2$

$$\text{Fraction } \mathrm{d}f = \frac{\text{area of the strip}}{\text{area of the sphere}} = \frac{2\pi r \mathrm{d}r}{4\pi R^2}$$
$$= \frac{1}{2}\frac{(R\sin\theta)(R\mathrm{d}\theta)}{R^2} = \frac{1}{2}\sin\theta\,\mathrm{d}\theta$$

Fraction of light escaping within the semi-angle θ is then given by

$$f = \int \mathrm{d}f = \int_0^\theta \frac{1}{2}\sin\theta\mathrm{d}\theta = \frac{1}{2}(1-\cos\theta)$$

Substituting $\theta = C$, the critical angle

$$\cos\theta = \cos C = \sqrt{1-\sin^2 C} = \sqrt{1-\frac{1}{\mu^2}} = \frac{1}{\mu}\sqrt{\mu^2-1}$$

$$\therefore \quad f = \frac{1}{2}\left[1-\frac{1}{\mu}\sqrt{\mu^2-1}\right]$$

15.2 The momentum carried by each photon is $h\nu/c$. If it is incident normally on a black surface, it exerts an impulse of $h\nu/c$. The total pressure exerted would be $\frac{1}{c}\sum h\nu$, where the summation extends over photons of all frequencies incident on the surface per unit area per second. If W is the power of the source, then at distance r, the intensity $I = W/4\pi r^2$. Then pressure

$$P = \frac{I}{c} = \frac{W}{4\pi r^2 c} = \frac{1000}{(4\pi)(2^2)(3\times 10^8)} = 6.63\times 10^{-8}\,\text{Pa}$$

15.3 Suppose a ray in going from A to B traverses distance, $s_1, s_2, s_3, \ldots, s_p$ in media of indices $n_1, n_2, n_3, \ldots, n_p$, respectively. The total time of flight is then (Fig. 15.15)

$$t = \sum_{i=1}^{p}\frac{s_i}{v_i} = \frac{1}{c}\sum_{i=1}^{p} n_i s_i \tag{1}$$

The last summation is known as the optical path length (O.P.L).

Fermat's principle states that a ray of light traverses from one point to another by a route which takes least time.

A more, stringent formulation of Fermat's principle is as follows. A ray of light in traversing from one point to another, regardless of the media, adopts such a route which corresponds to stationary value of the optical path length:

$$\text{O.P.L} = \int_A^B n(s)\,\mathrm{d}s \tag{2}$$

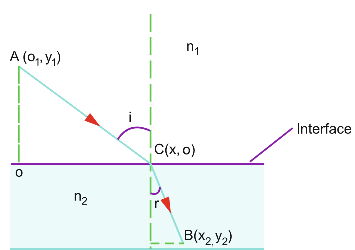


Fig. 15.15

A function $f(x)$ is said to have a stationary value at $x = x_0$, if its derivation $\mathrm{d}f/\mathrm{d}x$ vanishes at $x = x_0$. A stationary value could correspond to a maximum or minimum.

Let a ray of light proceed from $A(0, y_1)$ in medium of index n_1 and be incident at $C(x, 0)$ on the interface, get refracted and reach $B(x_2, y_2)$ in the medium of index n_2:

$$\text{O.P.L.} = n_1(\text{AC}) + n_2(\text{CB}) = n_1\sqrt{x^2 + y_1^2} + n_2\sqrt{(x_2 - x)^2 + y_2^2}$$

$$\frac{\mathrm{d}(\text{O.P.L})}{\mathrm{d}x} = 0$$

$$\therefore \quad \frac{n_1 x}{\sqrt{x^2 + y_1^2}} - \frac{n_2(x_2 - x_1)}{\sqrt{(x_2 - x)^2 + y_2^2}} = 0$$

$$\therefore \quad n_1 \sin i - n_2 \sin r = 0$$

$$\therefore \quad \frac{\sin i}{\sin r} = \frac{n_2}{n_1} \qquad \text{(Snell's law)} \qquad (3)$$

15.4 A mirage is a type of illusion formed by light rays coming from the low region of the sky in front of the observer. On a sunny day, a road gets heated and a temperature gradient is established in the vertical direction with the upper air layer being slightly less warmer and the corresponding indices of refraction being slightly larger. As the rays penetrate the atmospheric depth they start bending due to refraction, becoming horizontal to the road and then bending upwards. The blue of the sky in the background produces a virtual image of a water pool, and the turbulence of air close to the road enhances the effect of a water pool with waves on the surface. This phenomenon is known as mirage.

(a) Referring to Fig. 15.5, optical path length (O.P.L) $= \sum_{i=1}^{2} n_i s_i = n_1\text{AB} + n_2\text{BC} + n_1\text{CD}$

$$= n_1\sqrt{(d - x)^2/4 + h^2} + n_2 x + n_1\sqrt{(d - x)^2/4 + h^2}$$

$$\text{O.P.L} = n_2 x + n_1\sqrt{(d - x)^2 + 4h^2} \qquad (1)$$

(b) $\dfrac{\mathrm{d}(\text{O.P.L})}{\mathrm{d}x} = 0 \qquad$ (Fermat's principle)

Using (1), $\quad n_2 - \dfrac{n_1(d - x)}{\sqrt{(d - x)^2 + 4h^2}} = 0$

Substituting $n_1 = 1.00030$, $n_2 = 1.00020$ and $h = 2$ m and solving

$$d - x = 282.8 \qquad (2)$$

or $\quad x = 500 - 282.8 = 217.2$ m

(c) From (2) it is observed that as d decreases, x also decreases. The smallest value for x is zero, in which case the distance of the observer from the tree would be $d = 282.8$ m when the mirage disappears.

15.5 **(a)** The purpose of the cladding is to improve the transmission efficiency of the optical fibre. If cladding is not used then the signal is attenuated dramatically.

(b) Let a ray be incident at an angle θ, Fig. 15.6, the angle of refraction at P being θ_p. Let C be the critical angle at Q, interface of core and cladding:

$$\sin C = \frac{n_1}{n_2}$$

where n_1 and n_2 are the indices of the cladding and core, respectively:

$$\theta_p = 90 - \theta_{in}$$

where θ_{in} is the angle of incidence at Q:

$$n_0 \sin\theta = n_2 \sin\theta_p = n_2 \sin(90 - \theta_{in}) = n_2 \cos\theta_{in}$$

For internal reflection $\theta_{in} > C$ or $\cos\theta_r < \cos C$

$$\therefore \quad n_0 \sin\theta \le n_2 \cos C$$

$$\text{But} \quad n_2 \cos C = n_2\sqrt{1 - \sin^2 C} = \sqrt{n_2^2 - n_1^2}$$

$$\therefore \quad n_0 \sin\theta \le \sqrt{n_2^2 - n_1^2}$$

This shows that there is a maximum angle of acceptance cone outside of which entering rays will not be totally reflected within the fibre. For the largest acceptance cone, it is desirable to choose the index of refraction of the cladding to be as small as possible. This is achieved if there is no cladding at all. However, this leads to other problems associated with the loss of intensity.

(c) The transmission is reduced due to multiple reflections and the absorption of the fibre core material due to impurities.

15.3.2 Prisms and Lenses

15.6 With reference to Fig. 15.6, using Snell's law

$$n_2 \sin r_1 = n_1 \sin 20^\circ$$

$$\therefore \quad \sin r_1 = \frac{n_1}{n_2} \sin 20^\circ = \frac{1.1}{1.6} \times 0.342 = 0.2351$$

$$\therefore \quad r_1 = 13.6^\circ$$

From the geometry of Fig. 15.16

$$r_2 = 40 - 13.6^\circ = 26.4^\circ \quad (\because\ r_1 + r_2 = A,\ \text{the apex angle})$$

$$n_1 \sin i_2 = n_2 \sin r_2$$

$$\sin i_2 = \frac{n_2}{n_1} \sin r_2 = \frac{1.6}{1.1} \sin 26.4^\circ = 0.6467$$

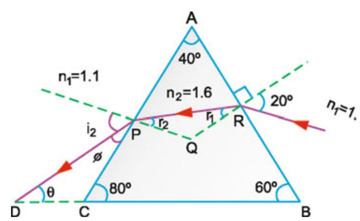
$$\therefore \quad i_2 = 40.3^\circ$$

$$\therefore \quad \phi = 90^\circ - 40.3^\circ = 49.7^\circ$$

$$\theta = 80 - 49.7^\circ = 30.3^\circ$$

($\because$ The exterior angle is equal to the sum of the interior angles).

Fig. 15.16 Triangular glass prism



15.7 **(a)** $\sin C = \frac{1}{n_g} = \frac{1}{1.45} = 0.6896$

$\therefore\ C = 43.6^\circ$

(b) In applying Snell's law instead of measuring angles with respect to the normal, we will measure them with respect to the interface, for convenience. Then

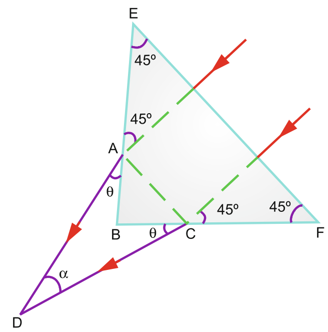
$$1.2 \cos 45^\circ = 1.0 \cos \theta$$

whence $\theta = 31.9^\circ$.

From Fig. 15.17 AC is parallel to EF and so $B\hat{A}C = B\hat{C}A = 45^\circ$. Thus in the triangle, ADC,

$$\alpha = 180^\circ - 2(\theta + 45^\circ) = 26.2^\circ$$

Fig. 15.17 Refraction in a prism

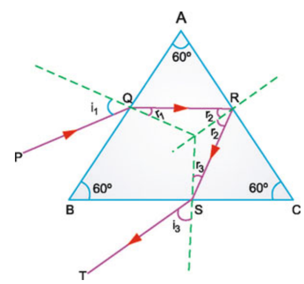


15.8 The net deviation of the ray

$$\delta = (i_1 - r_1) + (360^\circ - 2r_2) + (i_3 - r_3) \tag{1}$$

In the minimum angle position, $r_1 = r_2 = A/2 = 60/2 = 30^\circ$
From the geometry of Fig. 15.18, $r_3 = 30^\circ$.

Fig. 15.18 Deviation of a ray incident in the minimum deviation position of the prism after suffering one internal reflection



By Snell's law

$\text{Sin}\, i_1 = n \sin r_1 = 1.5 \times \sin 30^\circ = 0.75$
$\therefore \quad i_1 = 48.59^\circ$

Similarly, $i_3 = 48.59^\circ$

$\therefore \qquad \delta = (48.59 - 30^\circ) + (360 - 2 \times 30^\circ) + (48.59 - 30^\circ) = 337.18^\circ$

15.9 $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$

where u, the object distance, and v, the image distance, are measured from the centre of the lens. For real object and image u and v are positive, for virtual object or image u and v are negative. f is positive for convex lens and negative for concave lens.

Let the object o be placed at distance u_1 from lens L_1 of focal length f_1. The second tens L_2 is placed at distance d behind L_1, Fig. 15.19. The image I_1 is formed at distance v_1 from the lens L_1, alone. Then from the lens equation

$$\frac{1}{u_1} + \frac{1}{v_1} = \frac{1}{f_1} \text{ or } v_1 = \frac{f_1 u_1}{u_1 - f_1} \tag{1}$$

The image acts as an object for the second lens L_2 (real or virtual)

$$-\frac{1}{v_1 - d} + \frac{1}{v_2} = \frac{1}{f_2}$$

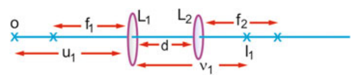


Fig. 15.19 Image due to combination of two lenses, a distance d apart

whence $v_2 = \frac{f_2(v_1 - d)}{f_2 + v_1 - d}$

Substituting v_1 from (1) and simplifying

$$v_2 = f_2 \frac{[f_1 u_1 - d(u_1 - f_1)]}{f_1 u_1 + (u_1 - f_1)(f_2 - d)} \tag{2}$$

In the limit $d \to 0$, (2) reduces to

$$v_2 = \frac{f_1 f_2 u_1}{f_1 u_1 + f_2 u_1 - f_1 f_2}$$

or $\frac{1}{u_1} + \frac{1}{v_2} = \frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$

15.10 The location of the object, the screen and the positions of the lens are indicated in Fig. 15.20

(a) $u_1 + v_1 = D$ (1)

$u_1 - u_2 = d$ (2)

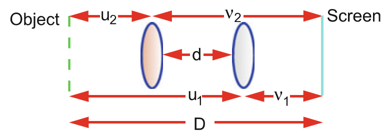


Fig. 15.20 Real images formed by a convex lens in two positions

By symmetry $\quad v_2 = u_1 \quad (3)$

$v_1 = u_2 \quad (4)$

Then (2) becomes

$$u_1 - v_1 = d \quad (5)$$

$$\frac{\text{size of image}}{\text{size of object}} = \frac{I_1}{O} = \frac{v_1}{u_1} \quad (6)$$

Similarly, $\quad \dfrac{I_2}{O} = \dfrac{v_2}{u_2} \quad (7)$

$$\therefore \quad \frac{I_1}{I_2} = \frac{v_1}{v_2}\frac{u_2}{u_1} = \frac{v_1^2}{v_2^2} \quad (8)$$

From (1) and (5), $v_2 = \dfrac{D+d}{2}$ and $v_1 = \dfrac{D-d}{2}$

Therefore, $\dfrac{I_1}{I_2} = \dfrac{(D-d)^2}{(D+d)^2}$

(b) Multiplying (6) and (7)

$$\frac{I_1}{O}\frac{I_2}{O} = \frac{v_1}{u_1}\frac{v_2}{u_2} = \frac{u_2}{u_1}\frac{u_1}{u_2} = 1$$

$$\therefore \quad O = \sqrt{I_1 I_2}$$

(c) $\dfrac{1}{u_1} + \dfrac{1}{v_1} = \dfrac{1}{f}$

$$\therefore \quad \frac{2}{D+d} + \frac{2}{D-d} = \frac{1}{f}$$

where we have used (1), (2) and (4).

$$\therefore \quad f = \frac{D^2 - d^2}{4D}$$

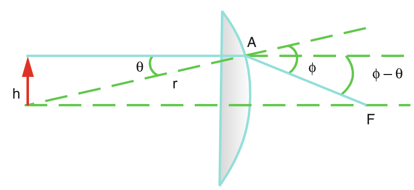
(d) From the result of **(c)**, we have

$$d^2 = \frac{D}{f}(D - 4f)$$

Since d^2 must be positive, it follows that $D > 4f$.

15.11 **(a)** Consider a plano-convex lens of focal length f. Let a paraxial ray be incident on the lens from a small object of height h. After striking the plano-convex lens normally, it gets refracted at the convex surface and passes through the principle focus F, behind the lens, Fig. 15.21. Let θ be the angle of incidence at A, the angle being measured with the radius of curvature of the curved surface r. Let ϕ be the angle of refraction:

Fig. 15.21



$$\sin\phi = n\sin\theta (\text{Snell's law})$$

$$\phi = n\theta (\because \text{ angles are small}) \tag{1}$$

$$\text{Also} \quad \theta = h/r, \phi - \theta = h/f \tag{2}$$

Combining (1) and (2)

$$\frac{1}{f} = \frac{n-1}{r} \tag{3}$$

We use the convention that r is positive if the refracting surface facing the object is convex and r is negative if the refracting index facing the object is concave.

A thin lens may be considered as two plano-convex or two plano-concave lenses in contact. Thus a thin biconvex lens whose faces have radii of curvature r_1 and r_2 are considered as two plano-convex lenses with their plane surfaces cemented together:

$$\frac{1}{f_1} = \frac{n-1}{r_1}, \ \frac{1}{f_2} = \frac{n-1}{-r_2} \tag{4}$$

$$\therefore \quad \frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{F} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \quad \text{(lens maker's formula)} \tag{5}$$

For a biconvex lens r_1 is positive and r_2 negative. For double-concave lens r_1 is negative and r_2 positive.

(b) If the lens of refractive index n_1 is immersed in a medium of index n_2, then $n = n_1/n_2$, $n = 1.2/1.33 = 0.9$, and the first bracket in (5) becomes $n - 1 = -0.1$ resulting in a negative value of F. Thus the plastic lens immersed in water acts as a diverging lens.

15.12 **(a)** The first image due to L_1 alone is formed at I_1 at a distance v_1 given by $\frac{1}{u_1} + \frac{1}{v_1} = \frac{1}{f_1}$

$$\therefore \quad v_1 = \frac{u_1 f_1}{u_1 - f_1} = \frac{5 \times 10}{5 - 10} = -10\,\text{cm}$$

The image is formed at 10 cm in front of the lens (Fig. 15.22).

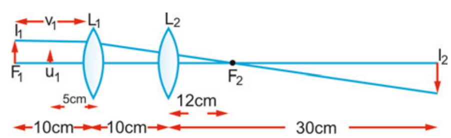


Fig. 15.22 Image due to combination of two convex lenses

(b) The image is virtual ($\because$ v is negative) and erect.

(c) The final image I_2 is located at distance v_2 from lens L_2 given by the result of prob. (15.9),

$$v_2 = \frac{f_2[f_1 u_1 - d(u_1 - f_1)]}{f_1 u_1 + (u_1 - f_1)(f_2 - d)} \tag{1}$$

Here $u_1 = 5\,\text{cm}$, $d = 10\,\text{cm}$, $f_1 = 10$ and $f_2 = 12\,\text{cm}$.
Substituting these values in (1) we find $v_2 = 30\,\text{cm}$ behind the lens L_2.

(d) The final image is real ($\because$ v_2 is positive) and inverted.

15.13 Let a ray PA enter the sphere at A, refract at A and B and intersect the axis at F (Fig. 15.23). Let ϕ be the angle of incidence and θ the angle of refraction at A:

$$\text{Sin}\,\phi = \mu \sin\theta \qquad \text{(Snell's law)} \tag{1}$$

$$\phi = \mu\theta \qquad (\because \text{angles are small}) \tag{2}$$

In ΔBFC,

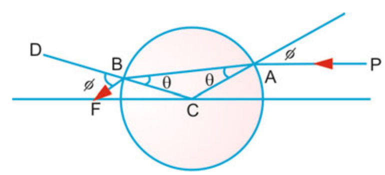
$$\frac{\mathrm{FC}}{\mathrm{BC}} = \frac{\sin \mathrm{F\hat{B}C}}{\sin \mathrm{B\hat{F}C}} = \frac{\sin(\pi - \phi)}{\sin 2(\phi - \theta)} = \frac{\sin \phi}{\sin 2(\phi - \theta)} = \frac{\phi}{2(\phi - \theta)} \tag{3}$$

since the angles are small. Hence the equivalent focal length

$$F_{\mathrm{eq}} = \mathrm{FC} = \frac{\mathrm{BC}\phi}{2(\phi - \theta)} = \frac{\mu r}{2(\mu - 1)} \tag{4}$$

where we have used (2), (3) and (4).

Fig. 15.23



15.14 The intensity at a distance r is given by

$$I = \frac{W}{4\pi r^2} = \frac{100}{4\pi \times 5^2} = 0.318\ \mathrm{W/m^2}$$

$$I = \frac{1}{2}\varepsilon_0 c E_0^2$$

$$\therefore \quad E_0 = \sqrt{\frac{2I}{\varepsilon_0 c}} = \sqrt{\frac{2 \times 0.318}{8.85 \times 10^{-12} \times 3 \times 10^8}} = 15.48\ \mathrm{V/m}$$

15.15 **(a)** If the object distance is u then the distance v_1 at which the image is formed by objective is given by

$$\frac{1}{u} + \frac{1}{v_1} = \frac{1}{f_o}$$

$$\text{or} \quad v_1 = -\frac{u f_o}{u - f_o} \tag{1}$$

The distance of this image from the eyepiece, u_1 is given by

$$u_1 = f_o + f_e - \frac{u f_o}{u - f_o} = \frac{u f_e - f_o^2 - f_o f_e}{u - f_o} \tag{2}$$

where $f_o + f_e$ is the distance between the objective and eyepiece.

If the final image is formed at a distance v from the eyepiece then

$$\frac{1}{v} + \frac{1}{u_1} = \frac{1}{f_\mathrm{e}}$$

$$\text{or} \quad \frac{1}{v} = \frac{1}{f_\mathrm{e}} - \frac{u - f_\mathrm{o}}{u f_\mathrm{e} - f_\mathrm{e}^2 - f_\mathrm{o} f_\mathrm{e}}$$

$$\text{or} \quad v = -\frac{f_\mathrm{e}(u f_\mathrm{e} - f_\mathrm{o}^2 - f_\mathrm{o} f_\mathrm{e})}{f_\mathrm{o}^2} \tag{3}$$

Now $u >> f_\mathrm{e}, f_\mathrm{o}$

$$\therefore \qquad v = -u\left(\frac{f_\mathrm{e}}{f_\mathrm{o}}\right)^2 = 10^4\left(\frac{1}{8}\right)^2 = 156.25\,\mathrm{m}$$

(b) $$M = \frac{\text{Height of the final image}}{\text{Height of the object}} = \frac{v_1}{u} \times \frac{v}{u_1}$$

$$\simeq -\frac{f_\mathrm{e}}{f_\mathrm{o}} = -\frac{1}{8} = -0.125$$

where we have used (1), (2) and (3).

$\therefore$ Height of final image = (height of the object).M

$= -100 \times 0.125 = -12.5\,\mathrm{m}$

The negative sign indicates that the final image is virtual.

Note that the height of the image is only one-eighth of that of the object, but the image is closer than the object by a factor of 64, so it subtends an angle eight times large, that is, the image appears eight times larger.

15.16 **(a)** The Intensity is given by

$$I = \frac{\text{power}}{\text{cross-section}} = \frac{W}{A}$$

$$= \frac{1000}{10^{-5} \times 10^{-4}} = 10^{12}\,\mathrm{W/m^2}$$

(b) $$I = \frac{1}{2}\varepsilon_0 c E_0^2$$

$$E_0 = \sqrt{\frac{2I}{\varepsilon_0 c}} = \sqrt{\frac{2 \times 10^{12}}{8.85 \times 10^{-12} \times 3 \times 10^8}}$$

$$= 2.74 \times 10^7\,\mathrm{V/m}$$

15.3.3 Matrix Methods

15.17 Assume that light is always incident from the left, then the refracting surface is convex if r is positive (from V to C) and concave if r is negative (from V to C'), Fig. 15.24.

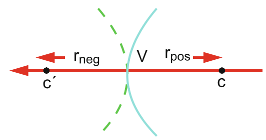


Fig. 15.24 Light is incident from left. Sign convention for refraction at a curved optical surface

The symbols used are object distance x_0, object size y_0, image distance x_i, image size y_i and radius of curvature, r. The object–image equation can be written as

$$\frac{n_1}{-x_0} + \frac{n_2}{x_i} = \frac{n_2 - n_1}{r} \tag{1}$$

The convention results in negative value for the object distance. Paraxial rays are considered so that α_1 and α_2 are small. Each ray is given a height and an angle, Fig. 15.25. The distance ε on the axis, called sagitta, is nearly zero. Considering counterclockwise angles as positive and clockwise angles as negative (1) becomes

$$\frac{n_1\alpha_1}{l_1} + n_2\left(-\frac{\alpha_2}{l_2}\right) = \frac{n_2 - n_1}{r} \tag{2}$$

As $l_2 = l_1$, two simultaneous equations can be written in the variables α_j and $l_j (j = 1, 2)$:

$$l_2 = l_1 \tag{3}$$

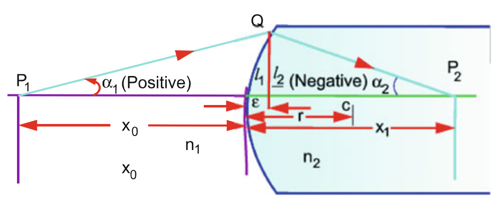


Fig. 15.25 Passage of a light ray P_1 in medium of refractive index n_1 into medium of refractive index n_2 onto P_2

$$\alpha_2 = \frac{n_1}{n_2}\alpha_1 - \left(\frac{n_2 - n_1}{n_2 r}\right) l_1 \tag{4}$$

In the matrix form (3) and (4) can be written as

$$\begin{pmatrix} l_2 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{r}\left(\frac{n_1}{n_2} - 1\right) & \frac{n_1}{n_2} \end{pmatrix} \begin{pmatrix} l_1 \\ \alpha_1 \end{pmatrix} \tag{5}$$

The initial image in medium 1 described by the column vector $I_1 = \begin{pmatrix} l_1 \\ \alpha_1 \end{pmatrix}$ is transformed into the final image in medium 2 described by the column vector $I_2 = \begin{pmatrix} l_2 \\ \alpha_2 \end{pmatrix}$. The transformation is accomplished by the refraction matrix

$$R_{12} = \begin{pmatrix} 1 & 0 \\ \frac{1}{r}\left(\frac{n_1}{n_2} - 1\right) & \frac{n_1}{n_2} \end{pmatrix} \tag{6}$$

In the matrix notation (5) is written as

$$I_2 = R_{12}\, I_1 \tag{7}$$

Next consider a parallel translation of a ray through a distance d in some homogeneous medium, Fig. 15.26.

Since $\alpha_1 = \alpha_2$, for small angles

$$l_2 = l_1 + \alpha_1 d \tag{8}$$

In matrix form (8) can be written as

$$\begin{pmatrix} l_2 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} l_1 \\ \alpha_1 \end{pmatrix} \tag{9}$$

$$\text{or} \qquad I_2 = T_2 I_1 \tag{10}$$

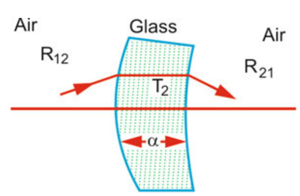


Fig. 15.26 Sequences of refraction, translation and refraction for a thick lens placed in air

where the translation matrix is

$$T_2 = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \tag{11}$$

In Fig. 15.26, the overall transformation can be written as

$$I_2 = R_{21}\, T_2 R_{12} I_1 \tag{12}$$

where R_{12} is the refraction matrix at the first surface (air to glass), T_2 is the translation matrix in the second medium (glass) and R_{21} is the refraction matrix at the second surface (glass to air). The matrix $R_{21} T_2 R_{12}$ is known as the system matrix.

15.18 Consider two curved surfaces of positive radii of curvature, r_1 and r_2. The final image in Fig. 15.27 is obtained from the equation:

$$\begin{pmatrix} l_3 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dfrac{n-1}{r_2} & n \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \dfrac{1}{r_1}\left(\dfrac{1-n}{n}\right) & \dfrac{1}{n} \end{pmatrix} \begin{pmatrix} l_1 \\ \alpha_1 \end{pmatrix}$$

Symbolically,

$$I_3 = R_{23}\, T_2\, R_{12}\, I_1$$

Let $P_{12} = \dfrac{1}{r_1} \dfrac{(1-n)}{n}$ and $P_{23} = \dfrac{n-1}{r_2}$

Then $$\begin{pmatrix} l_3 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ P_{23} & n \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ P_{12} & \dfrac{1}{n} \end{pmatrix} \begin{pmatrix} l_1 \\ \alpha_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + d P_{12} & \dfrac{d}{n} \\ P_{23} + d P_{23} P_{12} + n P_{12} & \dfrac{d P_{23}}{n} + 1 \end{pmatrix} \begin{pmatrix} l_1 \\ \alpha_1 \end{pmatrix}$$

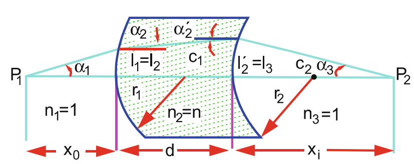


Fig. 15.27 Refraction in a double convex lens of thickness d placed in air

15.19 For a thin lens $d = 0$. The transformation matrix then becomes (see prob. 15.18)

$$\begin{pmatrix} 1 & 0 \\ P_{23} + nP_{12} & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ -(n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 0 \end{pmatrix}$$

Thus $-\dfrac{1}{f_1} - \dfrac{1}{f_2} = -\dfrac{1}{f}$ or $\dfrac{1}{f_1} + \dfrac{1}{f_2} = \dfrac{1}{f}$

15.3.4 Interference

15.20 A constant phase difference implies a constant difference in length between r_1 and r_2, Fig. 15.28:

$$r_1 - r_2 = 2a \tag{1}$$

$$\sqrt{(x+d)^2 + y^2} - \sqrt{(d-x)^2 + y^2} = 2a \tag{2}$$

Transposing the second radical

$$\sqrt{(x+d)^2 + y^2} = 2a + \sqrt{(d-x)^2 + y^2} \tag{3}$$

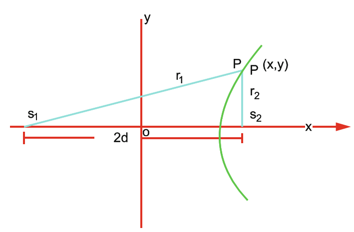


Fig. 15.28 Locus of a point with constant phase difference from two coherent point sources s_1 and s_2

Squaring and simplifying

$$x^2(d^2 - a^2) - y^2a^2 = a^2(d^2 - a^2) \tag{4}$$

Dividing by $a^2(d^2 - a^2)$

$$\frac{x^2}{a^2} - \frac{y^2}{d^2 - a^2} = 1 \tag{5}$$

Since $2d > 2a$ or $d > a$, $d^2 - a^2$ will be positive. Writing $d^2 - a^2 = b^2$, (5) becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{6}$$

This is the equation of a hyperbola with the centre at the origin and the foci on the x-axis. In three dimensions, the locus of P would be a hyperboloid, the figure of revolution of the hyperbola.

In an actual Young's experiment on the observation of interference fringes one looks at a limited field of view and consequently the central portions of hyperbolas appear as straight lines as in Fig. 15.29 (within the dotted lines).

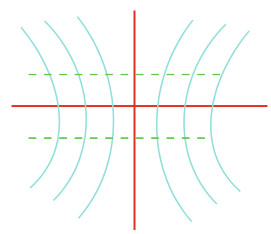


Fig. 15.29 Interference fringes in Young's experiment

15.21 At any point P on the screen at a distance y from the axis, the phase difference due to the waves coming from S_1 and S_2 will be due to the optical path difference $(S_2\mathrm{P} - S_1\mathrm{P})$, Fig. 15.11:

$$\delta = \frac{2\pi}{\lambda}(S_2\mathrm{P} - S_1\mathrm{P}) \tag{1}$$

If $(S_2\mathrm{P} - S_1\mathrm{P}) = n\lambda(n = 0, 1, 2, \ldots)$, the phase difference $\delta = 2n\pi$, and the intensity is maximum at P:

If $(S_2\mathrm{P} - S_1\mathrm{P}) = \left(m + \frac{1}{2}\right)\lambda$, where $m = 0, 1, 2, \ldots$, the phase difference, $\delta = (2m + 1)\pi$ and the intensity will be zero at P. With reference to Fig. 15.11,

$$(S_2\mathrm{P})^2 = L^2 + (x + d/2)^2$$
$$(S_1\mathrm{P})^2 = L^2 + (x - d/2)^2$$
$$\therefore \quad (S_2\mathrm{P})^2 - (S_1\mathrm{P})^2 = (S_2\mathrm{P} + S_1\mathrm{P})(S_2\mathrm{P} - S_1\mathrm{P}) = 2xd$$

In practice d is quite small, typically 0.5 mm, compared to L, typically 1.0 m, and P is close to the axis, $y << L$, so that $S_1\mathrm{P}$ as well as $S_2\mathrm{P}$ are only slightly greater than L. We can then set $S_2\mathrm{P} + S_1\mathrm{P} = 2L$. Therefore

$$S_2\mathrm{P} - S_1\mathrm{P} = \frac{2xd}{2L} = \frac{xd}{L} \tag{2}$$

Therefore, the condition for maximum intensity at P is

$$\frac{xd}{L} = m\lambda \qquad \text{(bright fringes)} \tag{3}$$

and the condition for zero intensity at P is

$$\frac{xd}{L} = \left(m + \frac{1}{2}\right)\lambda \qquad \text{(dark fringes)} \tag{4}$$

m is called the order of fringe system. On the axis, at $y = 0$, $m = 0$ and we have the intensity maximum. The central bright band on the screen is flanked on either side by a series of b bright and dark bands corresponding to $m = 1, 2, 3, \ldots$, the mth bright fringe being at a distance y_m from the axis:

$$y_m = \frac{m\lambda L}{d} \tag{5}$$

and the $(m + 1)$th bright fringe being at a distance

$$y_{(m+1)} = \frac{(m+1)\ \lambda L}{d} \tag{6}$$

The separation of the fringes β called the bandwidth is given by

$$\Delta y = y_{(m+1)} - y_m = \beta = \frac{\lambda L}{d} \tag{7}$$

$$\beta = \frac{\lambda L}{d} = \frac{612 \times 10^{-9} \times 4}{2 \times 10^{-3}} = 1.124 \times 10^{-3}\ \text{m} = 1.124\ \text{mm}$$

15.22 Let a transparent plate of thickness t and refractive index μ be introduced in the path of one of the two interfering beams of monochromatic light, Fig. 15.30. A ray travelling from S_1 to O covers a distance t in the plate while the rest of the distance $(S_1\text{O} - t)$ is covered in air. The effective optical path length would be

$$\mu t + (S_1\text{O} - t) \text{ or } S_1\text{O} + (\mu - 1)t$$

The optical path length for the ray emanating from S_2 would be S_2O. Clearly

$$S_1\text{O} + (\mu - 1)t > S_2\text{O}$$

Consequently, the central fringe corresponding to zero path difference is not formed at O, the normal position of the central fringe in the absence of the plate. The new position of the central fringe would be at O′ such that

$$S_1\text{O}' + (\mu - 1)t = S_2\text{O}'$$

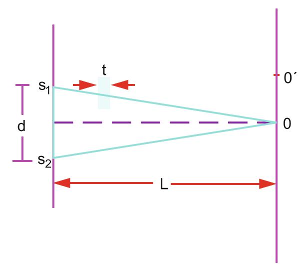


Fig. 15.30 Shift of fringes when a transparent plate is introduced in the path of one of the rays in Young's double-slit experiment

But $\quad S_2\mathrm{O}' - S_1\mathrm{O}' = \frac{d}{L} \cdot \mathrm{OO}'$

Calling $\mathrm{OO}' = \Delta$, the distance through which the central fringe shifts,

$$\Delta = \frac{L}{d}(\mu - 1)\,t$$

Furthermore, $(\mu - 1)t = n\lambda$.
This shift is towards the side on which the plate is placed.
Note that the bandwidth of the fringe system is unaffected and the entire fringe system undergoes a lateral shift.
With the use of monochromatic light it is not possible to detect shift of fringes. However, if white light is used then the central fringe being white is easily distinguished from the coloured fringes and its shift can be easily measured. Thus, by the use of the above procedure the thickness of the plate can be accurately measured.

15.23 (**a**) Let two light waves of the same wavelength λ and amplitude A pass through a given point and be represented by

$$y_1 = A\sin\omega t \tag{1}$$
$$y_2 = A\sin(\omega t - \delta) \tag{2}$$

where $\omega = 2\pi\nu$ and $\delta =$ constant phase difference between the two waves. The resulting displacement is then given by

$$\begin{aligned} y &= y_1 + y_2 \\ &= A\sin\omega t + A\sin(\omega t - \delta) \\ &= 2A\cos\frac{\delta}{2}\sin\left[\omega t - \frac{\delta}{2}\right] \end{aligned} \tag{3}$$

where we have used the identity

$$\sin B + \sin C = 2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right) \tag{4}$$

Equation (3) represents simple harmonic vibration of frequency $\omega/2\pi$ and amplitude $A' = 2\text{A}\cos(\delta/2)$. The amplitude of the resulting wave varies from $2A$ through 0 to $-2A$ according to the value of δ. The resulting intensity I at the given point is proportional to the square of the amplitude or A'^2:

$$I \propto 4A^2\cos^2\frac{\delta}{2} \tag{5}$$

If $\delta = (2n+1)\pi (n = 0, 1, 2, \ldots)$ then $A' = 0$, i.e. the crests of one wave coincide with the troughs of the other, the two waves interfere destructively to give zero intensity, i.e. darkness.

If $\delta = 2n\pi$, then $A' = 2A$, i.e. the two waves interfere constructively to produce maximum intensity of $4A^2$. Here the crests of one wave are an integral number of wavelengths ahead of crest of the other so that the waves are reinforced:

By eqn (1), prob. (15.21), $\delta = \left(\frac{2\pi}{\lambda}\right)(S_2P - S_1P) = \left(\frac{2\pi}{\lambda}\right)\left(\frac{\mathrm{d}x}{L}\right)$

$$\therefore \quad I = 4A^2\cos^2\left(\frac{\pi\,\mathrm{d}x}{\lambda L}\right) \tag{6}$$

Figure 15.31 shows the intensity distribution of Young's fringes. Here x is measured on the screen. The bright central fringe occurs at $x = 0$, in the centre of the fringe system. The other bright fringes are separated by distance, $x = \lambda L/d$, $2\,\lambda L/d$, $3\,\lambda L/d \ldots$ on either side. Halfway between two neighbouring bright fringes, the centres of dark fringes occur. The light intensity does not drop off suddenly but varies as $\cos^2(\delta/2)$. At maxima the intensity reaches a value of $4A^2$, and at minima, it is equal to zero. At other points it is given by (6). Thus the intensity in the interference pattern varies between $4A^2$ and zero. In the absence of interference each beam would contribute A^2 so that from two incoherent sources, there would be a uniform intensity of $2A^2$, which is indicated by the horizontal dotted line.

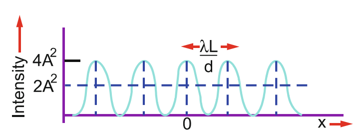


Fig. 15.31 Intensity distribution of fringes in young's double-slit experiment

(**b**) Now the average intensity of the interference pattern is obtained by averaging over the $\cos^2$ function in (6)

$$< I >= \frac{\int_0^{\lambda L/d} I\,\mathrm{d}x}{\int_0^{\lambda L/d} \mathrm{d}x} = \frac{4A^2 \int_0^{\lambda L/d} \cos^2(\pi\,\mathrm{d}x/\lambda L)\mathrm{d}x}{\lambda L/d} = 2A^2 \qquad (7)$$

The average intensity is equal to $2A^2$ as expected. Although at maximum the intensity is double the average value, at minima, it becomes zero and on the whole it averages out to $2A^2$. Hence there is no violation of energy conservation.

15.24 $u + v = D = 100\,\text{cm} \rightarrow v = 100 - u = 100 - 30 = 70\,\text{cm}$

$$\frac{I}{O} = \frac{0.7}{O} = \frac{v}{u} = \frac{70}{30}$$

$\therefore \quad 2d = O = 0.30\,\text{cm}$ (distance between two coherent sources)

$\beta = \frac{.\lambda D}{2d}$ (bandwidth)

$$\therefore \quad \lambda = \frac{2\mathrm{d}\beta}{D} = \frac{(0.3)(0.0195)}{100} = 5.85 \times 10^{-5}\text{cm} = 5850\,\text{Å}$$

15.25 $D = y_1 + y_2 = 10 + 100 = 110\,\text{cm}$

$$2d = 2(\mu - 1)y_1\alpha = 2 \times (1.5 - 1) \times \left(\frac{2}{57.3}\right) \times 10$$
$$= 0.349\,\text{cm}$$
$$\lambda = \frac{(2d)\beta}{D} = \frac{0.349 \times 0.018}{110} = 5.711 \times 10^{-5}\text{cm}$$
$$= 5711\,\text{Å}.$$

15.26 $\theta = 20\,\text{s} = \left(\frac{20}{3600}\right) \times \frac{1}{57.3}\text{rad}$

$$= 9.695 \times 10^{-5}\,\text{rad}$$
$$\beta = \frac{\lambda}{2\mu\theta} = \frac{5.82 \times 10^{-5}\text{cm}}{2 \times 1.5 \times 9.695 \times 10^{-5}} = 0.2\,\text{cm}$$

Number of fringes per centimetre$= \frac{1}{\beta} = \frac{1}{0.2} = 5.$

15.27 $\theta = \frac{\lambda}{2\mu\beta} = \frac{6 \times 10^{-5}\text{cm}}{2 \times 1.4 \times (0.2\,\text{cm})} = 1.07 \times 10^{-4}\,\text{rad}$

$$= 1.07 \times 10^{-4} \times 57.3^\circ = 22'' \text{ of arc}.$$

15.28 The radius of the nth dark ring is given by

$$r_n = \sqrt{n\lambda R} \tag{1}$$

The radius of the $(n+m)$th dark ring is given by

$$r_{n+m} = \sqrt{(n+m)\lambda R} \tag{2}$$

Squaring (1) and (2), subtracting and solving for R, the radius of curvature of lower lens

$$R = \frac{r_{n+m}^2 - r_n^2}{m\lambda} = \frac{r_{n+20}^2 - r_n^2}{20\lambda}$$

$$= \frac{(0.368)^2 - (0.162)^2}{20 \times 5.46 \times 10^{-5}} = 100\,\text{cm}$$

15.29 $r_n = \sqrt{n\lambda R}$ (dark ring in air)

$r'_n = \sqrt{n\lambda R/\mu}$ (dark ring in liquid)

$$\therefore \quad \mu = \frac{r_n^2}{r_n'^2} = \frac{60^2}{50^2} = 1.44$$

15.30 First, we calculate the air thickness t of the air gap between the horizontal surface and the lower surface of the lens where Newton's ring is formed as in Fig. 15.32.

DE $= t$ is the thickness of air gap; CB $= R$, the radius of curvature of the lens; and DA $=$ DB $= r$ is the radius of the ring. From a theorem in geometry on intersecting chords

$$\text{DE} \times \text{DG} = \text{DA} \times \text{DB}$$

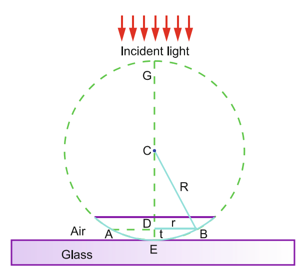


Fig. 15.32 Newton's rings with convex lens on a flat surface

or $\quad t \times (2R - t) = r^2$

$$\therefore \quad t = \frac{r^2}{2R} \qquad (\because\ t << 2R) \tag{1}$$

(a) Centres of curvature on the same side.
An air film BC is formed sandwiched between two curved surfaces of radii R_1 and R_2 in contact at O. The centres of curvature of lenses are on the same side, Fig. 15.33a. The thickness of air film

$$t = \mathrm{BC} = \mathrm{AC} - \mathrm{AB}$$

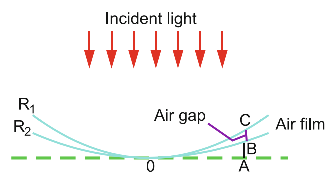


Fig. 15.33 a Newton's rings formed by two curved surfaces with the centres of curvature on the same side

Using (1)

$$\mathrm{AC} = \frac{r_m^2}{2R_1}; \qquad \mathrm{AB} = \frac{r_m^2}{2R_2}$$

$$t = \frac{r_m^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{2}$$

where r_m is the radius of the mth ring:

$$2t = m\lambda \qquad \text{(dark rings)} \tag{3}$$

Eliminating t between (2) and (3)

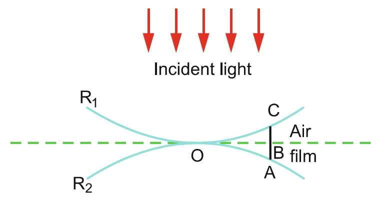
$$r_m^2\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = m\lambda \qquad (m = 0, 1, 2, \ldots \text{ dark rings}) \tag{4}$$

$$2t = \left(m + \frac{1}{2}\right)\lambda \qquad \text{(bright rings)} \tag{5}$$

$$r_m^2\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \left(m + \frac{1}{2}\right)\lambda \qquad (m = 0, 1, 2, \ldots \text{ bright rings}) \tag{6}$$

(b) Centres of curvature on the opposite side
The surfaces in contact at O are as in Fig. 15.33b; the thickness of air film t is

Fig. 15.33 b Newton's rings formed by two curved surfaces with centres of curvature on the opposite side



$$AC = BC + AB$$

$$t = \frac{r_m^2}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{7}$$

$$2t = m\lambda \qquad \text{(dark fringes)} \tag{8}$$

Eliminating t between (7) and (8)

$$r_m^2\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = m\lambda \qquad (m = 0, 1, 2 \ldots,\ \text{dark fringes}) \tag{9}$$

$$2t = \left(m + \frac{1}{2}\right)\lambda \qquad \text{(bright fringes)} \tag{10}$$

Eliminating t between (7) and (10)

$$r_m^2\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \left(m + \frac{1}{2}\right)\lambda \qquad (m = 0, 1, 2 \ldots,\ \text{bright fringes}) \tag{11}$$

15.31 $x_m = \dfrac{m\,\lambda_1 D}{d} = (m+1)\lambda_2\dfrac{D}{d}$

$$m \times 780 = (m+1) \times 520$$

$$\therefore \quad m = 2$$

15.32 Let the amplitude of the incident beam be a and intensity I:

$$I = a^2$$

The intensity of the reflected beam from the first face, Fig. 15.13

$$I_1 = \frac{1}{4}I \qquad \text{(by problem)}$$

$$\therefore \quad a_1 = \frac{a}{2}$$

The intensity of the transmitted beam at the first face will be

$$I'_1 = \frac{3}{4} I$$

The corresponding amplitude will be

$$a'_1 = \frac{\sqrt{3}}{2} a$$

The reflected beam at the second face will have amplitude

$$a''_1 = \frac{1}{2} \times \frac{\sqrt{3}}{2} a = \frac{\sqrt{3}}{4} a$$

The emerging beam from the first face will have amplitude

$$a_2 = \frac{\sqrt{3}}{2} a''_1 = \frac{\sqrt{3}}{2} \frac{\sqrt{3}}{4} a = \frac{3}{8} a$$

The two beams reflected from the first face will interfere:

$$\frac{I_{\min}}{I_{\max}} = \frac{(a_1 - a_2)^2}{(a_1 + a_2)^2} = \frac{\left(\frac{a}{2} - \frac{3}{8} a\right)^2}{\left(\frac{a}{2} + \frac{3}{8} a\right)^2} = \frac{1}{49}$$

15.33 For constructive interference

$$2\,\mu t = \left(m + \frac{1}{2}\right) \lambda$$

$$\therefore \quad \lambda = \frac{2\mu t}{m + \frac{1}{2}} = \frac{2 \times 1.5 \times 4 \times 10^{-5}}{m + \frac{1}{2}} = \frac{12 \times 10^{-5}}{m + \frac{1}{2}} \text{ cm}$$

Only for $m = 2$, we get $\lambda = 4.8 \times 10^{-5}$ cm or 4800 Å, corresponding to blue colour in the visible region.

15.34 $2\mu t \cos r = m\lambda$ (minima)

For smallest thickness, $m = 1$

$$t = \frac{m\lambda}{2\mu \cos r} = \frac{1 \times 5890}{2 \times 1.5 \times \cos 60^\circ} = 3927 \text{ Å}$$

15.35 $2\mu t\cos r = \left(m + \frac{1}{2}\right)\lambda \qquad \text{(maxima)}, m = 0, 1, 2\ldots$

$$r = 0, \mu = 1$$

$$\lambda = \frac{2t}{m+\frac{1}{2}} = \frac{2 \times (2945\,\text{Å})}{m+\frac{1}{2}} = \frac{5890}{m+\frac{1}{2}}\text{Å}$$

For $m = 0, \quad \lambda = 11{,}980\,\text{Å}$

$m = 1, \quad \lambda = 3927\,\text{Å}$

$m = 2, \quad \lambda = 2356\,\text{Å}$

$\lambda = 3927\,\text{Å}$ falls within the range of visible spectrum. The colour is violet.

15.36 $2\mu t\cos r = \left(m + \frac{1}{2}\right)\lambda$

$r = 0, m = 0$

$t = \lambda/4\mu$

15.37 For constructive interference

$$2\mu t\cos r = \left(m + \frac{1}{2}\right)\lambda, \quad m = 0, 1, 2\ldots$$

$$\sin r = \frac{\sin i}{\mu} = \frac{\sin 30^\circ}{1.5} = 0.3333$$

$$\therefore \quad \cos r = 0.9428$$

$$\lambda = \frac{2\mu t\cos r}{m+\frac{1}{2}} = \frac{2 \times 1.5 \times 4 \times 10^{-5} \times 0.9428}{m+\frac{1}{2}}\,\text{cm}$$

For $m = 1$, we find $\lambda = 7.542 \times 10^{-5}$ cm $= 7542$ Å.

15.38 **(a)** The two mirrors M_1 and M_2 are adjusted such that their distance from the beam splitter are approximately equal, Fig. 15.4. The wavelengths λ_1 and λ_2 for the D_1 and D_2 lines of sodium differ only by a few angstroms. Two different sets of fringes arise due to two wavelengths. If M_1 is slowly moved away there is a gradual separation of the two sets, and finally the bright band of the one lies over the dark band of the other, resulting in uniform illumination. Thus the fringes disappear and reappear periodically. Between two successive disappearances the mirror has to be moved by, say d cm. This corresponds to a path difference of $2d$ cm. Assuming that $\lambda_2 < \lambda_1$, this path difference must contain exactly one more wavelength of λ_2 than of λ_1. Thus expressing λ_1 and λ_2 in cm,

$$\frac{2d}{\lambda_2} - \frac{2d}{\lambda_1} = 1$$

$$\therefore \quad \lambda_1 - \lambda_2 = \Delta\lambda = \frac{\lambda_1\lambda_2}{2d}$$

(b) $\Delta\lambda = \dfrac{(5.89 \times 10^{-5})^2}{2 \times 2.89 \times 10^{-5}} = 6\,\text{Å}$

15.39 $\lambda = \dfrac{2d}{N} = \dfrac{2 \times 2.948 \times 10^{-3}\,\text{cm}}{100} = 5.896 \times 10^{-5}\,\text{cm} = 5896\,\text{Å}$

15.40 Resolving power $\dfrac{\lambda}{\Delta\lambda} = \dfrac{g\pi D}{\lambda}$

where D = distance between the plates = 2 mm = 0.2 cm:

$$\lambda = 600\,\text{nm} = 6 \times 10^{-5}\,\text{cm}$$

$$g = \frac{2r}{1 - r^2} = \frac{2 \times 0.9}{1 - (0.9)^2} = 9.4737$$

$$R.P = \frac{\lambda}{\Delta\lambda} = \frac{9.4737 \times 3.14 \times 0.2}{6 \times 10^{-5}} = 9.9 \times 10^4$$

15.3.5 Diffraction

15.41 $a \sin\theta = m\lambda$ (1)

$$\therefore \quad \sin\theta = \frac{m\lambda}{a} = 1 \times \frac{\lambda}{a} = \frac{\lambda}{a} \tag{2}$$

If D is the slit screen distance and x the distance of the first minimum from the central maximum, then

$$\tan\theta \simeq \sin\theta = x/D \tag{3}$$

Combining (2) and (3)

$$\lambda = \frac{ax}{D} = \frac{0.02 \times 0.5}{200} = 5 \times 10^{-5}\,\text{cm} = 5000\,\text{Å}$$

15.42 $I_\theta = I_\text{m}\left(\dfrac{\sin\alpha}{\alpha}\right)^2$

Differentiating I_θ with respect to α and setting $\dfrac{dI_\theta}{d\alpha} = 0$,

$$\frac{dI_\theta}{d\alpha} = \frac{I_\text{m}(2\alpha^2 \sin\alpha\cos\alpha - 2\alpha\sin^2\alpha)}{\alpha^4} = 0$$

$\therefore \quad 2\alpha \sin\alpha(\alpha\cos\alpha - \sin\alpha) = 0$

$\therefore \quad \tan\alpha = \alpha$

15.43 **(a)** The half-width is the angle between the two points in the pattern where the intensity is one-half the centre of the pattern:

$$\frac{I_\theta}{I_\mathrm{m}} = \frac{1}{2} = \left(\frac{\sin\alpha_x}{\alpha_x}\right)^2 \tag{1}$$

The solution of (1) found by numerical method is

$$\alpha_x = 1.40\,\mathrm{rad} \tag{2}$$

$$\alpha_x = \frac{\pi a}{\lambda}\sin\theta_x = 1.40 \tag{3}$$

where $\quad \theta_x = \frac{1}{2}\Delta\theta.$

$\sin\theta_x = \frac{1.4\lambda}{\pi a}$ or $\quad \theta_x = \sin^{-1}\left(\frac{1.4\lambda}{\pi a}\right)$

or $\quad \Delta\theta = 2\theta_x = 2\sin^{-1}\left(\frac{1.4\lambda}{\pi a}\right)$

(b) $\Delta\theta = 2\sin^{-1}\left(\frac{1.4}{4\pi}\right) = 12.8°$

15.44 $a\ \sin\theta = m\lambda$

$a\ \sin\theta_1 = 1\cdot\lambda_1$

$a\ \sin\theta_2 = 2\lambda_2$

But $\quad \theta_1 = \theta_2$

$\therefore \qquad \lambda_1 = 2\lambda_2$

15.45 $\sin\theta = \frac{x}{D} = \frac{n\lambda}{a}$

$\therefore \quad a = \frac{n\lambda D}{x} = \frac{2\times 5.6\times 10^{-5}\times 200}{1.6} = 0.014\,\mathrm{cm}$

$= 0.14\,\mathrm{mm}.$

15.46 **(a)** Let m be the order of interference. If the slit width a is maintained constant and the separation of the slits d is varied, the scale of the interference pattern varies, but that of the diffraction pattern remains unchanged. If the diffraction angle corresponds to the minimum given by (1) then a particular order of interference maxima may be absent. This is called a missing order. Thus, a missing order is realized for an angle θ for which the following two equations are simultaneously satisfied:

$$d \sin\theta = m\lambda (m = 1, 2, 3 \ldots) \tag{1}$$

$$a \sin\theta = n\lambda (n = 1, 2, 3 \ldots) \tag{2}$$

Dividing (1) by (2)

$$\frac{d}{a} = \frac{m}{n} \tag{3}$$

Since m and n are both integers, missing orders will occur when d/a is in the ratio of two integers. Expressing d as the sum of the slit width a and the opaque space b between consecutive slits, that is

$$d = a + b \tag{4}$$

(3) an be written as

$$\frac{a+b}{a} = \frac{m}{n} \tag{5}$$

In particular, if $\frac{a+b}{a} = 1,\ b = 0$. In this case the first-order spectrum will be absent and the resultant diffraction pattern will be similar to that of a single slit.

If $\frac{a+b}{a} = 2,\ a = b$, that is the width of the slit is equal to the width of the opaque space. Here the second-order spectrum will be absent.

(b) $\dfrac{a+b}{a} = \dfrac{m}{n}$

$$\frac{0.16 + 0.8}{0.16} = 6 = \frac{m}{n}$$

The above relation is satisfied for

$$n = 1, 2, 3, \ldots$$
$$m = 6, 12, 18, \ldots$$

Thus the order 6, 12, 18, etc. of the interference maxima will be missing, in the diffraction pattern.

15.47 The central diffraction peak is limited by the first minima. The angular locations of these minima are given by

$$a \sin\theta = \lambda \qquad (\because\ m = 1) \tag{1}$$

The angular locations of the bright interference fringes are given by

$$d \sin\theta = m\lambda \qquad (m = 0, 1, 2 \ldots) \tag{2}$$

We can locate the first diffraction minimum within the double-slit fringe pattern by dividing (2) by (1)

$$n = \frac{d}{a}$$
$$n = \frac{0.2}{0.012} = 16.66$$

Therefore 16 interference fringes will lie within the central maximum.

15.48 $m_1\lambda_1 = m_2\lambda_2$

$$\lambda_1 = \frac{m_2\lambda_2}{m_1} = \frac{3}{4} \times 5460 = 4095\,\text{Å}$$

15.49 $d \sin\theta = m\lambda$

For $\quad m = m_{\max}, \theta = \theta_{\max} = 90^\circ,$ Given $\dfrac{1}{d} = 5000$

$$m_{\max} = \frac{d}{\lambda} = \frac{1}{5000} \times \frac{1}{6 \times 10^{-5}} = 3.33$$
$$\therefore \quad m_{\max} = 3$$

15.50 $\dfrac{d}{a} = \dfrac{m}{n}$ (condition for missing orders)

$$\frac{0.3}{0.1} = \frac{3}{1} = \frac{m}{n}$$

where m and n are integers. The above relation is satisfied for

$$m = 3, 6, 9$$
$$n = 1, 2, 3$$

Thus the maxima will be missing in the third, sixth, ninth, etc. orders.

15.51 $\dfrac{d}{a} = \dfrac{a+b}{a} = \dfrac{a+2a}{a} = 3 = \dfrac{m}{n}$

The above relation is satisfied for

$$m = 3, 6, 9, \ldots$$
$$n = 1, 2, 3, \ldots$$

Thus the interference maxima will be missing in the third, sixth, ninth, etc. orders.

15.52 $n_1(\text{max}) = \dfrac{d}{\lambda_1} = \dfrac{1}{N\lambda_1} = \dfrac{1}{4000 \times 4 \times 10^{-5}} = 6.25$

$$n_2(\text{max}) = \frac{d}{\lambda_2} = \frac{1}{N\lambda_2} = \frac{1}{4000 \times 7 \times 10^{-5}} = 3.57$$

The maximum order of spectrum varies between 3 (towards red) and 6 (towards violet).

15.53 $\dfrac{d}{a} = \dfrac{a+b}{a} = \dfrac{a+a}{a} = 2 = \dfrac{m}{n}$ (condition for missing orders)

$\therefore \quad m = 2n$

The above condition is satisfied for

$n = 1, 2, 3 \ldots$

$m = 2, 4, 6 \ldots$

Thus all the even orders of interference fringes $(2, 4, 6, \ldots)$, except $m = 0$, are missing.

15.54 The secondary maxima lie approximately halfway between the minima. Now, the intensity at an angle θ is given by

$$I_\theta = I_\text{m} \left(\frac{\sin\alpha}{\alpha}\right)^2 \tag{1}$$

$$\text{where} \quad \alpha = \frac{\pi\, a}{\lambda} \sin\theta \tag{2}$$

Minima occur in (1) when

$$\alpha = m\pi \qquad (m = 1, 2, 3 \ldots) \tag{3}$$

Therefore the first secondary maximum would occur halfway between first minimum and second minimum. Therefore,

$$\alpha = \left(m + \frac{1}{2}\right)\pi = \frac{3\pi}{2} \tag{4}$$

Substituting (4) into (1)

$$\frac{I}{I_\text{m}} = \left(\frac{\sin(3\pi/2)}{3\pi l2}\right)^2 = 0.045 \text{ or } 4.5\%$$

15.55 In Fig. 15.34 the grating space AD $= d$. In the ΔABD, B$\hat{\text{A}}$D $= i$, the angle of incidence. Also D$\hat{\text{A}}$C $= \theta$, the angle of diffraction. In the ΔABD, BD the path difference between the incident rays, YD and XA is

$$\text{BD} = d \sin i$$

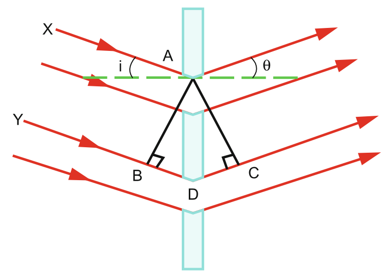


Fig. 15.34 Diffraction by a grating for a parallel beam of light which is obliquely incident

Similarly, the path difference between the diffracted rays is

$$DC = d \sin \theta$$

The total path difference

$$\text{BD} + \text{DC} = d(\sin i + \sin \theta) \tag{1}$$

For the mth primary maximum

$$d(\sin \theta_m + \sin i) = m\lambda \tag{2}$$

$$\text{or} \qquad \sin\left(\frac{\theta_m + i}{2}\right) = \frac{m\lambda}{2d \cos\left(\dfrac{\theta_m - i}{2}\right)} \tag{3}$$

The angle of deviation of the diffracted beam is

$$\delta_m = \theta_m + i \tag{4}$$

For δ_m to be minimum, $\cos \dfrac{\theta_m - i}{2}$ must be maximum, that is

$$(\theta_m - i)/2 = 0 \to \theta_m = i \tag{5}$$

Then δ_m will be minimum, say D_m, and is given by

$$D_m = \theta_m + i = 2i$$

$$\text{or} \qquad i = D_m/2 \tag{6}$$

Substituting (5) and (6) into (3)

$$2d \sin\left(\frac{D_m}{2}\right) = m\lambda \tag{7}$$

$$N = \frac{1}{d} = \frac{2\sin(20/2)}{1 \times 4.358 \times 10^{-5}} = 7969 \text{ lines/cm}$$

15.56 Condition for overlapping is

$$m\lambda_1 = (m+1)\lambda_2$$
$$5400\,m = 4050 \times (m+1)$$
$$\therefore \quad m = 3$$
$$d \sin\theta = m\lambda_1$$
$$N = \frac{1}{d} = \frac{\sin\theta}{m\lambda_1} = \frac{\sin 30°}{3 \times 5.4 \times 10^{-5}}$$
$$= 3086 \text{ lines/cm}$$

15.57 $\overline{\lambda} = 5893\,\text{Å} = 5.893 \times 10^{-5}\text{cm}$

$$\Delta\lambda = \lambda_1 - \lambda_2 = 5896 - 5890 = 6\,\text{Å} = 6 \times 10^{-8}\text{cm}$$

Resolving power

$$R = \frac{\overline{\lambda}}{\mathrm{d}\lambda} = \frac{5.893 \times 10^{-5}}{6 \times 10^{-8}} = 982$$
$$R = Nm$$
$$\therefore \quad N = \frac{R}{m} = \frac{982}{2} = 491 \qquad \text{(Total number of lines)}$$

If N' is the number of lines/cm, then the width of the grating

$$W = \frac{N}{N'} = \frac{491}{800} = 0.614\,\text{cm}$$

15.58 Total number of lines on the grating

$$N = N'W \tag{1}$$

where $N' =$ number of lines/in. and W is the grating width (in inch)

$$\therefore \quad N = 10,000 \times 3 = 30,000$$

$$R = \frac{\lambda}{d\lambda} = Nm$$

$$\therefore \quad d\lambda = \frac{\lambda}{Nm} = \frac{6 \times 10^{-5}}{3 \times 10^4 \times 1} = 2 \times 10^{-9}\,\text{cm} = 0.2\,\text{Å}$$

15.59 Total number of lines

$$N = N'W = 2 \times 425 = 850$$

$$\bar{\lambda} = 5893\,\text{Å}, \; d\lambda = 5896 - 5890 = 6\,\text{Å}$$

(**i**) First order:

$$R = \frac{\bar{\lambda}}{d\lambda} = Nm$$

$$N = \frac{1}{m}\frac{\bar{\lambda}}{d\lambda} = \frac{1}{1} \times \frac{5893}{6} = 982\,\text{lines}$$

As the required number of lines (982) exceeds the total number of lines (850) the lines are not resolved in the first order.

(**ii**) Second order

$$N = \frac{1}{m}\frac{\bar{\lambda}}{d\lambda} = \frac{1}{2} \times \frac{5893}{6} = 491$$

As the required number of lines (491) is less than the total number of lines, the lines are resolved in the second order.

15.60 The resolving power for a prism of base length B is given by

$$R = \frac{\lambda}{\Delta\lambda} = \frac{B\,d\mu}{d\lambda} \tag{1}$$

where $d\mu/d\lambda$ is the variation of refraction index of the prism with wavelength, λ is the mean wavelength and $\Delta\lambda$ is the difference in wavelengths to be resolved:

$$\lambda = 5893\,\text{Å}, \, \Delta\lambda = 5896 - 5890 = 6\,\text{Å}$$

$$\frac{d\mu}{d\lambda} = \frac{1.6635 - 1.6545}{(6563 - 5270) \times 10^{-8}\text{cm}} = 696/\,\text{cm}$$

Substituting the above values in (1) and solving for B, we find the length of the base of the prism, $B = 1.41$cm.

15.61 The limit of resolution of a telescope is

$$d\theta = 1.22\frac{\lambda}{D} = \frac{1.22 \times 55 \times 10^{-5}}{500} = 1.342 \times 10^{-7}\,\text{rad}$$

If the distance between two points is x and the moon–earth distance r, then

$$x = r\,d\theta = 3.8 \times 10^{8} \times 1.342 \times 10^{-7}\,\text{m} = 51\,\text{m}$$

15.62 $\theta = \dfrac{1.22\lambda}{d} = \dfrac{1.22 \times 5.89 \times 10^{-7}}{30 \times 10^{-6}} = 0.024\,\text{rad}$

15.63 $I = I_0\left[\dfrac{2J_1(\rho)}{\rho}\right]^2$

where $\rho = \dfrac{2\pi}{\lambda} a \sin\theta$ and $J_1(\rho)$ is the Bessel function of the first kind. According to the Rayleigh criterion, the separation of the peaks is equal to the distance between the first minimum and the centre of the diffraction pattern, that is, the first minimum of the Bessel function is at $\rho = 3.83$, and we have

$$\rho = 3.83 = \frac{2\pi a}{\lambda}\left(\frac{R_1}{X}\right)$$

$$\text{or}\qquad \gamma = \frac{R_1}{X} = 0.61\frac{\lambda}{a} = 1.22\frac{\lambda}{2a} = 1.22\frac{\lambda}{D}$$

where $2a = D$ is the diameter of the lens, γ is the angle between the two stars, the distance between the observation screen and the lens is X (equal to focal length of the lens) and the position of the details of the diffraction pattern is R_1, Fig. 15.35a, b.

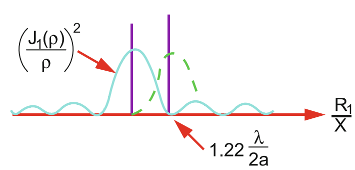


Fig. 15.35 a Intensity distribution for diffraction from a circular aperture described by Bessel function

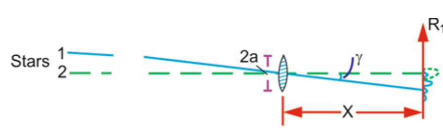


Fig. 15.35 b Resolution of diffraction patterns for two stars, the angle between them being γ. For the explanation of parameters X, R_1 and ρ, see the text

15.64 $r_n = \sqrt{f_n n \lambda}$

$$r_1 = \sqrt{50 \times 1 \times 5 \times 10^{-5}} = 0.05\,\text{cm}$$
$$r_{25} = \sqrt{50 \times 25 \times 5 \times 10^{-5}} = 0.25\,\text{cm}$$

15.3.6 Polarization

15.65 Path difference introduced by the plate

$$\Delta x = (n_0 - n_e)t = (1.642 - 1.478) \times 4 \times 10^{-5}$$
$$= 6.56 \times 10^{-6}\,\text{m}$$

Phase difference

$$\delta = \frac{2\pi\,\Delta x}{\lambda} = \frac{(2\pi)(6.56 \times 10^{-6})}{6 \times 10^{-7}} = 68.66 \ \text{rad}$$

15.66 $I_\theta = I_{\max} \cos^2\theta$ (Malus' law)

$$\cos\theta = \sqrt{\frac{I_\theta}{I_{\max}}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$\therefore \quad \theta = \pm 45, \pm 135°.$

15.67 $\tan\theta_p = \mu = 1.33$ (Brewster's law)

$\therefore \quad \theta_p = 53°$

Therefore the elevation of the sun is $90 - 53° = 37°$

15.68 $\tan Q_p = \mu_{wg} = \dfrac{\mu_g}{\mu_w} = \dfrac{1.5}{1.33} = 1.1278$

$\therefore Q_p = 48.4°$

15.69 $t = \dfrac{\lambda}{4(\mu_0 - \mu_e)} = \dfrac{6 \times 10^{-5}}{4(1.553 - 1.544)} = 1.67 \times 10^{-3}\,\text{cm}$

15.70 $\theta = \alpha L D$

where θ is the angle of rotation of plane of polarization, α is the specific rotation, L is the length of the tube in decimetres and D is the amount of solvent in grams per 100 c.c:

$L = 20\,\text{cm} = 2\,\text{dm}$

$$D = \frac{\theta}{\alpha L} = \frac{13.2^\circ}{66^\circ \times 2} = 0.1\,\text{g}/100\,\text{c.c.} = 1.0\ \text{g/L}$$

15.71 As the light passing through the first sheet is unpolarized, the intensity I_1 of the light transmitted by the first sheet is given by the one-half rule:

$$I_1 = \frac{1}{2} I_0 \tag{1}$$

Because the polarizing direction of the first sheet is parallel to the y-axis, the polarization of the light transmitted by it is also along y-axis.

Because the light reaching the second sheet is polarized, the intensity I_2 of the light transmitted by that sheet is given by the cosine-squared rule. The angle θ in the rule is the angle between the polarization direction of the incoming light, parallel to the y-axis, and the polarizing direction of the second sheet, θ being counterclockwise from the y-axis. Thus

$$I_2 = I_1 \cos^2\theta \qquad \text{(Malus' law)} \tag{2}$$

Because the light entering the third sheet is polarized and the polarizing between the second and the third sheets is $90 - \theta$, the transmitted intensity is again given by Malus' law. Thus

$$I_3 = I_2\,\text{Cos}^2(90 - \theta) = I_2 \sin^2\theta \tag{3}$$

From (3), (2) and (1)

$$I_3 = I_2 \sin^2\theta = I_1 \cos^2\theta \sin^2\theta = \frac{I_0}{2} \cos^2\theta\,(1 - \cos^2\theta) \tag{4}$$

or $\quad \dfrac{I_3}{I_0} = 0.1152 = \dfrac{1}{2}\cos^2\theta\ (1 - \cos^2\theta)$

or $\quad \cos^4\theta - \cos^2\theta + 0.2304 = 0$

$\therefore \quad \cos^2\theta = 0.64$ or 0.36

or $\quad \cos\theta = \pm 0.8$ or ± 0.6

Taking only the positive value

$\theta = 36.87^\circ$ or 53.13°

The polarization is along the x-axis.

15.72 Let an unpolarized light beam be incident from air on a dielectric such as glass. The E vector for the wave can be resolved into two components – one parallel to the plane of incidence, that is, the plane of paper and the second one perpendicular to the plane of incidence. The former one is represented by double arrow and is called π-component while the latter shown by dots is known as σ-component; Fig. 15.36.

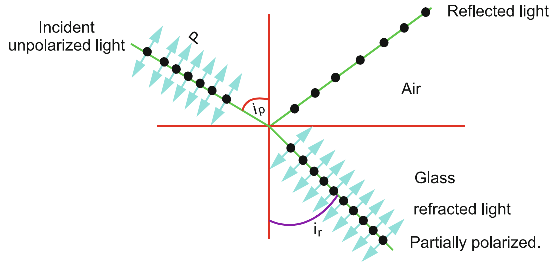


Fig. 15.36 Polarization by reflection • σ-component $\leftrightarrow$ π-component

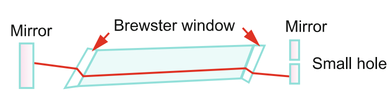
For a given dielectric, there is a particular angle of incidence i_{p}, called the polarizing angle or Brewster angle, at which the reflected beam is completely polarized with its plane of vibration perpendicular to the plane of incidence, that is, the reflected beam contains σ-component alone. Experiments show that at the polarizing angle, the reflected and refracted beams are at right angles. This leads to Brewster's law

$$\tan i_p = \mu$$

where μ is the refractive index. Note that at the polarizing angle the π-component is entirely refracted containing some amount of σ-component as well. Thus the transmitted light is only partially polarized. With the use of a stack of glass plates the proportion of σ-component can be increased in the reflected beam from multiple reflections and at the same time π-component can be made richer in the transmitted beam. Thus the transmitted beam is rendered plane polarized with the plane of vibration in the plane of incidence.

Dielectrics such as glass, water can partially or fully polarize light by reflection. If the surface is horizontal, the light is partially or fully polarized horizontally resulting in a bright spot (the glare) on the surface where reflection takes place. Such a glare from horizontal surfaces is eliminated by mounting the lenses in polarizing glasses with their polarizing direction vertical.

Fig. 15.37 Brewster windows mounted on a laser tube



Brewster windows are used in laser technology (Fig. 15.37). The tubes of gas lasers are sealed by mounting them in such a way that laser light is incident at Brewster's angle. Mirrors mounted outside the tube reflect the light back and forth through the tube. A small hole in one mirror permits the laser light to leave. Because the windows are tilted to the axis such that light is incident at Brewster's angle, the parallel component can traverse back and forth in the tube with minimum attenuation. The perpendicular component after several traversals gets substantially attenuated. Thus the light leaving the laser is polarized in the parallel direction

$$\tan\theta_B = \mu = 1.5$$

$$\therefore \quad \theta_B = 56.3^\circ, \text{ which is also the inclination of the Brewster window.}$$