4.3 Solutions

4.3.1 Moment of Inertia

4.1 Imagine the sphere of mass M and radius R to be made of a series of circular discs, a typical one being of thickness dx at distance x from the centre, Fig. 4.16. The area of the disc is $\pi(R^2 - x^2)$, and if the density of the sphere is ρ, the mass of the disc is $\rho\,\pi(R^2 - x^2)\,\mathrm{d}x$. The elementary moment of inertia of the disc about the axis OX is $\frac{1}{2}$ (mass)(radius)^2

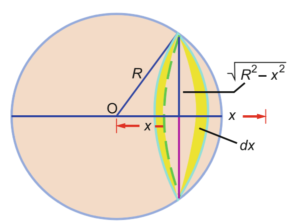
$$\therefore \quad \mathrm{d}I = \frac{1}{2}\pi\rho\,(R^2 - x^2)\mathrm{d}x(R^2 - x^2)$$

Hence the moment of inertia of the sphere is

$$I = \int \mathrm{d}I = \int_0^R \frac{\pi\rho}{2}\,(R^2 - x^2)^2\mathrm{d}x = \frac{8\pi\rho}{15}R^5 = \frac{2}{5}MR^2$$

as $\quad \rho = \dfrac{3M}{4\pi R^3}$

Fig. 4.16



4.2 Let the mass m_1 and m_2 be at distance r_1 and r_2, respectively, from the centre of mass. Then

$$r_1 = \frac{m_2 r}{m_1 + m_2}, \quad r_2 = \frac{m_1 r}{m_1 + m_2}$$

Moment of inertia of the masses about the centre of mass is given by

$$I = m_1 r_1^2 + m_2 r_2^2$$
$$= m_1\left(\frac{m_2 r}{m_1 + m_2}\right)^2 + m_2\left(\frac{m_1 r}{m_1 + m_2}\right)^2 = \frac{m_1 m_2}{m_1 + m_2}\,r^2 = \mu\,r^2$$

where $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$

4.3 Consider the cone to be made up of a series of discs, a typical one of radius r and of thickness dz at distance z from the apex. Volume of the disc is $\mathrm{d}V = \pi r^2 \mathrm{d}z$. Its mass will be $\mathrm{d}m = \rho \mathrm{d}V = \pi \rho r^2 \mathrm{d}z$, where ρ is the mass density of the cone. The moment of inertia dI of the disc about the z-axis is given by (Fig. 4.17)

$$\mathrm{d}I = \frac{1}{2} r^2 \mathrm{d}m = \frac{\pi}{2} \rho r^4 \mathrm{d}z$$

But $\frac{r}{a} = \frac{z}{h}$ (from the geometry of the figure)

or $r = \frac{az}{h}$

where h is the height of the cone and a is the radius of the base

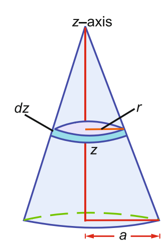
$$\therefore \quad \mathrm{d}I = \frac{\pi}{2} \rho \frac{a^4}{h^4} z^4 \mathrm{d}z$$

$$I = \int \mathrm{d}I = \frac{\pi}{2} \rho \frac{a^4}{h^4} \int_0^h z^4 \mathrm{d}z = \frac{\pi}{10} \rho a^4 h$$

But $\rho = \frac{3M}{\pi a^2 h}$

$$\therefore \quad I = \frac{3Ma^2}{10}$$

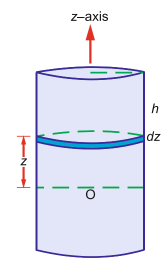
Fig. 4.17



4.4 Consider a slice of the cylinder of thickness dz at distance z from the centre of mass of cylinder O. The moment of inertia about an axis passing through the centre of the slice and perpendicular to z-axis will be

$$\mathrm{d}I = \frac{1}{4} \mathrm{d}m R^2$$

Fig. 4.18



Then the moment of inertia about an axis parallel to the slice and passing through the centre of mass is given by the parallel axis theorem, Fig. 4.18.

$$\mathrm{d}I_\mathrm{C} = \frac{1}{4}\,\mathrm{d}m\,R^2 + \mathrm{d}m\,z^2$$
$$= \pi R^2 \rho \left(\frac{R^2}{4} + z^2\right)\mathrm{d}z$$

where $\mathrm{d}m = \pi R^2 \rho \mathrm{d}z$ is the mass of the slice and ρ is the density.

$$\therefore \quad I_\mathrm{C} = \int \mathrm{d}I_\mathrm{C} = \pi\, R^2 \rho \int\limits_{-h/2}^{+h/2} \left(\frac{R^2}{4} + z^2\right)\mathrm{d}z$$
$$= \pi\, R^2 \rho \left[\frac{R^2}{4}h + \frac{h^3}{12}\right]$$

But $\quad \rho = \dfrac{M}{\pi\, R^2 h}$

$$\therefore \quad I_\mathrm{C} = \frac{M}{12}\left(3R^2 + h^2\right)$$

4.5 The moment of inertia of the larger solid sphere of mass M

$$I_1 = \frac{2}{5}\,Ma^2 \tag{1}$$

The moment of inertia of the smaller solid sphere of mass m, which is removed to hollow the sphere, is

$$I_2 = \frac{2}{5}\,mb^2 \tag{2}$$

As the axis about which the moment of inertia is calculated is common to both the spheres, the moment of inertia of the hollow sphere will be

$$I = I_1 - I_2 = \frac{2}{5}(Ma^2 - mb^2) = (M - m)k^2 \tag{3}$$

where $(M - m)$ is the mass of the hollow sphere and k is the radius of gyration.

$$\text{Now} \quad M = \frac{4}{3}\pi a^3 \rho \text{ and } m = \frac{4}{3}\pi b^3 \rho \tag{4}$$

Using (4) in (3) and simplifying we get

$$k = \sqrt{\frac{2}{5}\frac{(a^5 - b^5)}{(a^3 - b^3)}}$$

4.6 **(a)** Let AB represent a thin rod of length L and mass M, Fig. 4.19. Choose the x-axis along length of the rod and y-axis perpendicular to it and passing through its centre of mass O. Consider a differential element of length dx at a distance x from O. The mass associated with it is $M\,(\mathrm{d}x/L)$. The contribution to moment of inertia about the y-axis by this element of length will be $M\,(\mathrm{d}x/L)\,x^2$. The moment of inertia of the rod about y-axis passing through the centre of mass is

$$I_C = \int \mathrm{d}I_C = \int_{-L/2}^{+L/2} M\frac{\mathrm{d}x}{L}x^2 = \frac{ML^2}{12}$$

(b) Moment of inertia about y-axis passing through the end of the rod (A or B) is given by the parallel axis theorem:

$$I_A = I_B = I_C + M\left(\frac{L}{2}\right)^2 = \frac{ML^2}{3}$$

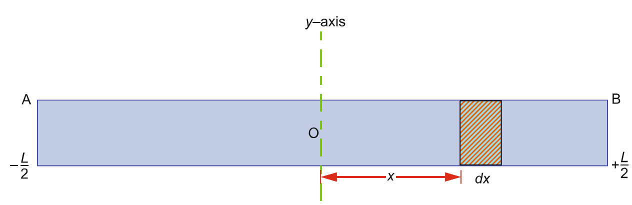


Fig. 4.19

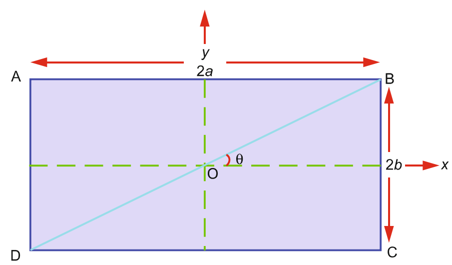
4.7 The moment of inertia of the plate about x-axis is $I_x = (1/3)\ Mb^2$ and about y-axis is $I_y = (1/3)\ Ma^2$. It can be shown that the moment of inertia about the line BD is

$$I_{\mathrm{BD}} = \frac{1}{3} Mb^2 \cos^2\theta + \frac{1}{3} Ma^2 \sin^2\theta \tag{1}$$

where θ is the angle made by BD with the x-axis. From Fig. 4.20, $\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$.

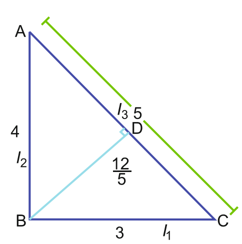
$$\therefore \quad I_{\mathrm{BD}} = \frac{1}{3}\frac{M\,b^2a^2}{\left(a^2+b^2\right)} + \frac{1}{3}\frac{M\,a^2b^2}{\left(a^2+b^2\right)} = \frac{2}{3}M\frac{a^2\,b^2}{\left(a^2+b^2\right)}$$

Fig. 4.20



4.8 The moment of inertia about any side of a triangle is given by the product of the one-sixth mass m of the triangle and the square of the distance (p) from the opposite vertex, i.e. $I = mp^2/6$. The perpendicular BD on AC is found to be equal to 12/5 from the geometry of Fig. 4.21.

Fig. 4.21



$$I_1 = \frac{m}{6}\,(\mathrm{AB})^2 = \frac{m}{6}4^2 = \frac{8m}{3}$$
$$I_2 = \frac{m}{6}\,(\mathrm{BC})^2 = \frac{m}{6}3^2 = \frac{3m}{2}$$
$$I_3 = \frac{m}{6}\,(\mathrm{BD})^2 = \frac{m}{6}\left(\frac{12}{5}\right)^2 = \frac{24m}{25}$$
$$\therefore I_1 > I_2 > I_3$$

4.9 If the radius of the sphere is r then the volume of the sphere must be equal to that of the disc:

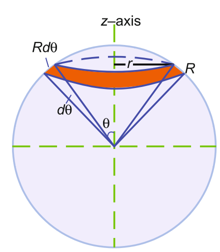
$$\frac{4}{3}\pi\, r^3 = \pi\, R^2 \frac{R}{6}$$
$$\therefore \quad r = \frac{R}{2}$$

The moment of inertia of the disc $I = I_\mathrm{D} = (1/2)\; m\, R^2$
The moment of inertia of the sphere

$$I_\mathrm{S} = \frac{2}{5}\, mr^2 = \frac{2}{5}\, m\, \frac{R^2}{4} = \frac{1}{5} \times \frac{1}{2}\, m\, R^2 = \frac{1}{5} I_\mathrm{D}$$

4.10 Consider a strip of radius r on the surface of the sphere symmetrical about the z-axis and width $R\mathrm{d}\theta$, where R is the radius of the hollow sphere, Fig. 4.22.

Fig. 4.22



Area of the strip is $2\pi r \cdot R\mathrm{d}\,\theta = 2\pi\, R^2 \sin\theta\, \mathrm{d}\,\theta$. If σ is the surface mass density (mass per unit area) then the mass of the strip is $\mathrm{d}m = 2\pi\, R^2 \sigma\, \sin\theta\, \mathrm{d}\,\theta$. Moment of inertia of the elementary strip about the z-axis

$$\mathrm{d}I = \mathrm{d}m\, r^2 = 2\pi\; R^4 \sigma\; \sin^3\theta\, \mathrm{d}\,\theta$$

Moment of inertia contributed by the entire surface will be

$$I = \int \mathrm{d}I = 2\pi\ R^4\ \sigma \int_0^{\pi} \sin^3\theta\ \mathrm{d}\theta$$

$$= 2\pi\ R^4 \sigma \frac{4}{3}$$

But $\sigma = \dfrac{M}{4\pi\ R^2}$

$$\therefore \quad I = \frac{2}{3} M R^2$$

4.11 By prob. (4.5), the radius of gyration of a hollow sphere of external radius a and internal radius b is

$$k = \sqrt{\frac{2}{5}\ \frac{(a^5 - b^5)}{(a^3 - b^3)}} \tag{1}$$

The derivation of (1) is based on the assumed value of moment of inertia for a solid sphere about its diameter $\left(I = \dfrac{2}{5}MR^2\right)$. Squaring (1) and multiplying by M, the mass of the hollow cylinder is

$$I = Mk^2 = \frac{2}{5}\ M\ \frac{(a^5 - b^5)}{(a^3 - b^3)} \tag{2}$$

Let $a = b + \Delta$ (3)

where Δ is a small quantity. Then (2) becomes

$$I = \frac{2}{5}\ M\ \frac{\left[(b + \Delta)^5 - b^5\right]}{\left[(b + \Delta)^3 - b^3\right]} = \frac{2}{5}\ M\ \frac{\left[b^5 + 5b^4\Delta + \cdots - b^5\right]}{\left[b^3 + 3b^3\Delta + \cdots - b^3\right]}$$

where we have neglected higher order terms in Δ. Thus

$$I = \frac{2}{5}\ M \frac{5b^4\Delta}{3b^2\Delta} = \frac{2}{3}Mb^2 = \frac{2}{3}MR^2$$

where $b = a = R$ is the radius of the hollow sphere.

4.3.2 Rotational Motion

4.12 Potential energy at height h is mgh and kinetic energy is zero. At the bottom the potential energy is assumed to be zero. The kinetic energy (K) consists of translational energy $\left(\dfrac{1}{2}mv^2\right)$ + rotational energy $\left(\dfrac{1}{2}I\omega^2\right)$.

$$K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2} \times \frac{1}{2}m\,R^2\frac{v^2}{R^2} = \frac{3}{4}mv^2$$

Gain in kinetic energy = loss of potential energy

$$\frac{3}{4}mv^2 = mgh$$

$$\text{or} \quad v = \sqrt{\frac{4g\,h}{3}}$$

4.13 **(a)** Initial angular momentum

$$L_1 = I_1\omega_1 = \frac{2}{5}MR_1^2\frac{2\pi}{T_1}$$

Final angular momentum $L_2 = I_2\omega_2 = \frac{2}{5}MR_2^2\frac{2\pi}{T_2}$

$$\frac{L_1}{L_2} = \frac{R_1^2}{R_2^2}\frac{T_2}{T_1} = \left(\frac{6 \times 10^8}{10^4}\right)^2\left(\frac{0.1}{30 \times 86{,}400}\right) = 138.9$$

(b) Initial kinetic energy (rotational)

$$K_1 = \frac{1}{2}I_1\omega_1^2 = \frac{1}{2} \times \frac{2}{5}MR_2^2\left(\frac{2\pi}{T_1}\right)^2$$

Final kinetic energy $K_2 = \frac{1}{2}I_2\omega_2^2 = \frac{1}{2} \times \frac{2}{5}MR_2^2\left(\frac{2\pi}{T_2}\right)^2$

$$\frac{K_1}{K_2} = \left(\frac{R_1}{R_2}\frac{T_2}{T_1}\right)^2 = \left(\frac{6 \times 10^8}{10^4} \times \frac{0.1}{30 \times 86{,}400}\right)^2 = 5.36 \times 10^{-6}$$

4.14 **(a)** Let M be the mass of the sphere, R its radius, θ the angle of incline. Let F and N be the friction and normal reaction at A, the point of contact, Fig. 4.23. Denoting the acceleration $\mathrm{d}x^2/\mathrm{d}t^2$ by $\ddot{x}$, the equations of motion are

$$M\ddot{x} = Mg\sin\theta - F \quad (1)$$

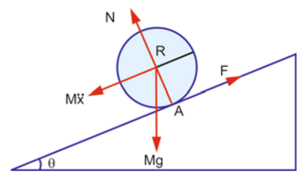
$$Mg\cos\theta - N = 0 \quad (2)$$

$$\text{Torque } I\alpha = FR \quad (3)$$

$$\text{or} \quad \frac{2}{5}MR^2\frac{a}{R} = FR$$

$$\text{or} \quad F = \frac{2}{5}M\ddot{x} \quad (4)$$

Fig. 4.23



Using (4) in (1)

$$a = \ddot{x} = \frac{5}{7} g \sin\theta \tag{5}$$

Thus the centre of the sphere moves with a constant acceleration. The assumption made in the derivation is that we have pure rolling without sliding

(b) $v = \sqrt{2as} = \sqrt{2 \times \frac{5}{7} \times 9.8 \times \sin 30^\circ \times 3} = 4.58\,\mathrm{m/s}$

(c) $L = I\omega = \frac{2}{5} MR^2 \frac{v}{R} = \frac{2}{5} MvR$

$$= \frac{2}{5} \times 0.1 \times 4.58 \times 0.25 = 0.0458\,\mathrm{kg\ m^2/T}$$

(d) Using (5) in (1)

$$F = \frac{2}{7} Mg \sin\theta \tag{4.9}$$

$$\therefore \quad \frac{F}{N} = \frac{2}{7} \tan\theta \tag{4.10}$$

For no slipping F/N must be less than μ, the coefficient of friction between the surfaces in contact. Therefore, the condition for pure rolling is that μ must exceed $(2/7)\tan\theta$.

$$\mu = \frac{2}{7} \tan\theta \tag{4.11}$$

$$\therefore \quad \tan\theta = \frac{7\mu}{2} = \frac{7}{2} \times 0.26 = 0.91 \tag{4.12}$$

$$\text{or} \quad \theta = 42.3^\circ \tag{4.13}$$

4.15 **(a)** Equation of motion of the cylinder for sliding down the incline is

$$ma_s = mg\sin\theta - \mu mg\cos\theta \tag{1}$$

$$\text{or} \quad a_s = g(\sin\theta - \mu\cos\theta) \tag{2}$$

When the cylinder rolls down without slipping, the linear acceleration is given by

$$a_R = R\alpha = R\,\frac{\tau}{I_{CM}} = R\frac{(\mu\, mg\ \cos\theta\ R)}{(1/2)mR^2} = 2\mu\, g\ \cos\theta \tag{3}$$

The least coefficient of friction when the cylinder would roll down without slipping is obtained by setting

$$a_R = a_s$$

$$\therefore \quad 2\mu g \cos\theta = g(\sin\theta - \mu\cos\theta)$$

$$\text{or} \quad \mu = \frac{1}{3}\tan\theta$$

(b) For the loop (2) is the same for sliding. But for rolling

$$a_R = \frac{R\tau}{I_{CM}} = R\frac{(\mu\, mg\ \cos\theta\ R)}{mR^2} = \mu\, g\ \cos\theta$$

Setting $a_R = a_s$

$$\mu g\cos\theta = g(\sin\theta - \mu\cos\theta) \tag{4.14}$$

$$\mu = \frac{1}{2}\tan\theta \tag{4.15}$$

4.16 Since the thread is being drawn at constant velocity v_0, angular momentum of the mass may be assumed to be constant. Further the particle velocities $\boldsymbol{v}$ and $\boldsymbol{r}$ are perpendicular. The angular momentum

$$J = mvr = \text{constant}$$

$$\therefore \quad v\,\alpha\,\frac{1}{r}$$

Now the tension T arises from the centripetal force

$$T = \frac{mv^2}{r}$$

$$\therefore \quad T\,\alpha\,\frac{1}{r^2}\frac{1}{r} \quad \text{or} \quad \alpha\,\frac{1}{r^3}$$

4.17 **(a)** Conservation of angular momentum gives

$$I_1\omega_1 = I_2\omega_2 \tag{4.16}$$

$$(I_1)(4\pi) = \frac{80}{100}\,I_1\omega_2 \tag{4.17}$$

$$\therefore \quad \omega_2 = 5\pi \tag{4.18}$$

(b) $$\frac{\Delta K}{K_1} = \frac{K_2 - K_1}{K_1} = \frac{K_2}{K_1} - 1$$
$$= \frac{(1/2)\, I_2\omega_2^2}{(1/2)\, I_1\omega_1^2} - 1 = (0.8)\left(\frac{5\pi}{4\pi}\right)^2 - 1 = \frac{1}{4}$$

4.18 At the bottom of the incline translational energy is $(1/2)\, M v_0^2$ while the rotational energy is

$$\frac{1}{2} I\omega^2 = \frac{1}{2} \times \frac{2}{5} M R^2 \frac{v_0^2}{R^2} = \frac{1}{5} M v_0^2$$

Total initial kinetic energy $= \frac{1}{2} M v_0^2 + \frac{1}{5} M v_0^2 = \frac{7}{10} M v_0^2$

Let the sphere reach a distance s up the incline or a height h above the bottom of the incline. Taking potential energy at the bottom of the incline as zero, the potential energy at the highest point reached is *Mgh*. Since the entire kinetic energy is converted into potential energy, conservation of energy gives

$$\frac{7}{10} M v_0^2 = Mgh$$

But $h = s \sin\theta$, so that

$$s = \frac{7}{10} \frac{v_0^2}{g \sin\theta}$$

4.19 Equation of motion is

$$Ma = mg - T \tag{1}$$

The resultant torque τ on the wheel is *TR* and the moment of inertia is $(1/2)\, M R^2$.

Now $\tau = I\alpha$

$$\therefore \quad TR = \frac{1}{2} M R^2 \frac{a}{R}$$

or $$T = \frac{1}{2} Ma \tag{2}$$

Solving (1) and (2)

$$a = \frac{2mg}{M + 2m} \quad T = \frac{Mmg}{M + 2m}$$

4.20 Equation of motion is

$$Ma = Mg - T \tag{1}$$

$$\text{Torque } \tau = TR = I\alpha = \frac{1}{2}MR^2\frac{a}{R} \tag{2}$$

$$\therefore \quad T = \frac{1}{2}Ma \tag{3}$$

Solving (1) and (3) $a = \frac{2}{3}g \quad T = \frac{Mg}{3}$

4.21 **(a)** Obviously m_1 moves down and m_2 up with the same acceleration 'a' if the string is taut. Let the tension in the string be T_1 and T_2 (Fig. 4.5). The equations of motion are

$$m_1 a = m_1 g - T_1 \tag{1}$$

$$m_2 a = T_2 - m_2 g \tag{2}$$

Taking moments about the axis of rotation O

$$T_1 R - T_2 R = I\alpha = \frac{MR^2}{2}\alpha \tag{3}$$

where α is the angular acceleration of the pulley and I is the moment of inertia of the pulley about the axis through O.

$$\text{But} \quad \alpha = \frac{a}{R}$$

$$\therefore \quad T_1 - T_2 = \frac{Ma}{2} \tag{4}$$

Adding (1) and (2)

$$(m_1 + m_2)a = T_2 - T_1 + (m_1 - m_2)g \tag{5}$$

Using (4) in (5) and solving for 'a', we find

$$a = \frac{(m_1 - m_2)g}{m_1 + m_2 + (1/2)\,M} \tag{6}$$

(b) $\alpha = \frac{a}{R} = \frac{(m_1 - m_2)g}{(m_1 + m_2 + (1/2)\,M)\,R}$

(c) Using (5) in (1) and (2), the values of T_1 and T_2 can be obtained from which the ratio T_1/T_2 can be found.

$$\frac{T_1}{T_2} = \frac{m_1(4m_2 + M)}{m_2(4m_1 + M)}$$

4.22 **(a)** Conservation of angular momentum gives

$$I_1\omega_1 + I_2\omega_2 = I\omega = (I_1 + I_2)\omega$$

The two moments of inertia I_1 and I_2 are additive because of common axis of rotation.

$$\therefore \quad \omega = \frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}$$

(b) Work done = loss of energy

$$\begin{aligned} W &= \frac{1}{2}(I_1 + I_2)\omega^2 - \left(\frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2\right) \\ &= \frac{1}{2}(I_1 + I_2)\frac{(I_1\omega_1 + I_2\omega_2)^2}{(I_1 + I_2)^2} - \frac{1}{2}I_1\omega_1^2 - \frac{1}{2}I_2\omega_2^2 \\ &= -\frac{1}{2}\frac{I_1 I_2(\omega_1 - \omega_2)^2}{(I_1 + I_2)} \end{aligned}$$

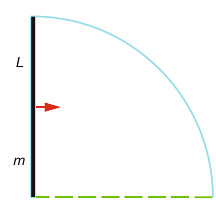
4.23 **(a)** Measure the potential energy from the bottom of the rod in the upright position, the height through which it falls is the distance of the centre of mass from the ground, i.e. $(1/2)\,L$ (Fig. 4.24). When it falls on the ground the potential energy is converted into kinetic energy (rotational).

$$mg\frac{1}{2}L = \frac{1}{2}I\omega^2 = \frac{1}{2} \times \frac{1}{3}mL^2\omega^2 = \frac{1}{6}mv^2$$

where I is the moment of inertia of the rod about one end and $v = \omega L$ is the linear velocity of the top end of the pole, $v = \sqrt{3gL}$.

(b) The additional mass has to be attached at the bottom of the rod.

Fig. 4.24



4.24 If I_1 and I_2 are the initial and final moments of inertia, ω_1 and ω_2 the initial and final angular velocity, respectively, the conservation of angular momentum gives

$$L = I_1\omega_1 = I_2\omega_2$$
$$MR^2\omega_1 = (MR^2 + 2mR^2)\omega_2$$
$$\therefore \quad \omega_2 = \frac{\omega_1 M}{M + 2m}$$

4.25 $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = m(\boldsymbol{r} \times \boldsymbol{v})$

$$= m\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & -3 & 1 \end{vmatrix} = (-7\hat{i} - 7\hat{j} - 7\hat{k})m = -7m(\hat{i} + \hat{j} + \hat{k})$$

4.26 **(a)** $a = \dfrac{g \sin\theta}{1 + \left(k^2/r^2\right)} = \dfrac{9.8 \sin 30°}{1 + (2/5)} = 3.5\,\mathrm{m/s^2}$

$$t = \sqrt{\frac{2s}{a}} = \sqrt{\frac{2 \times 7}{3.5}} = 2\,\mathrm{s}$$

(b) $\tau = I\alpha = \dfrac{2}{5}mR^2\dfrac{a}{R} = \dfrac{2}{5}mRa = \dfrac{2}{5} \times 0.2 \times 0.5 \times 3.5 = 0.14\,\mathrm{kg\ m^2/s^2}$.

4.27 **(a)** The equation of motion is

$$ma = mg - T \tag{1}$$
$$\tau = TR = I\alpha = \frac{1}{2}mR^2\frac{a}{R}$$
$$\therefore \quad T = \frac{1}{2}ma \tag{2}$$

Solving (1) and (2), $a = \dfrac{2g}{3}\,\cdot$ (3)

(b) Work done = increase in the kinetic energy

$$W = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}mR^2\right)\omega^2 = \frac{1}{4}mR^2\omega^2 \tag{4}$$

(c) $W = \displaystyle\int \tau\,\mathrm{d}\theta = \tau\theta = mgR\theta$ (5)

(where θ is the angular displacement) is an alternative expression for the work done. Equating (4) and (5) and simplifying

$$\theta = \frac{1}{4}\omega^2\frac{R}{g} \tag{6}$$

Length of the string unwound $= \theta R = \dfrac{1}{4}\dfrac{\omega^2 R^2}{g}$ (7)

4.28 As there are two strings, the equation of motion is

$$ma = mg - 2T \tag{1}$$

The net torque

$$\begin{aligned}\tau &= \tau_1 + \tau_2 = 2TR = I\alpha \\ &= \frac{1}{2}mR^2\frac{a}{R} = \frac{1}{2}maR \\ \therefore \quad T &= \frac{ma}{4}\end{aligned} \tag{2}$$

Solving (1) and (2)
(a) $T = \dfrac{mg}{6}$ **(b)** $a = \dfrac{2}{3}g$

4.29 The total kinetic energy (translational + rotational) at the bottom of the incline is

$$K = \frac{1}{2}mu^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mu^2 + \frac{1}{2}mk^2\frac{u^2}{R^2} = \frac{1}{2}mu^2\left(1 + \frac{k^2}{R^2}\right) \tag{1}$$

where k is the radius of gyration.
At the maximum height the kinetic energy is transformed into potential energy.

$$\frac{1}{2}mu^2\left(1 + \frac{k^2}{R^2}\right) = mgh = mg\frac{3u^2}{4g}$$

Solving we get $k = R/\sqrt{2}$. Therefore the body can be either a disc or a solid cylinder.

4.30 Time taken for a body to roll down an incline of angle θ over a distance s is given by

$$t = \sqrt{\frac{2s}{a}}$$

where $a = \dfrac{g\sin\theta}{1 + \left(k^2/R^2\right)}$. The quantity k^2/R^2 for various bodies is as follows:

Solid cylinder $\frac{1}{2}$ hollow cylinder 1
Solid sphere $\frac{2}{5}$ hollow sphere $\frac{2}{3}$

These bodies reach the bottom of the incline in the ascending order of acceleration 'a' or equivalently ascending order of k^2/R^2. Therefore the order in which the bodies reach is solid sphere, solid cylinder, hollow sphere and hollow cylinder. The physical reason is that the larger the value of k the greater will be I, and larger fraction of kinetic energy will go into rotational motion. Consequently less energy will be available for the translational motion and greater will be the travelling time.

4.31 Consider an element of length dx at distance x from the axis of rotation (Fig. 4.25). The corresponding mass will be

$$\mathrm{d}m = \rho A \mathrm{d}x$$

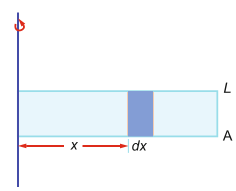
where ρ is the liquid density and A is the area of cross-section of the tube. The centrifugal force arising from the rotation of dm will be

$$\mathrm{d}F = (\mathrm{d}m)\omega^2 x = \omega^2 \rho A x \mathrm{d}x$$

The total force exerted at A, the other end of the tube, will be

$$F = \int \mathrm{d}F = \omega^2 \rho A \int_0^L x\ \mathrm{d}x = \frac{1}{2}\omega^2 \rho A L^2;\ \rho = \frac{M}{LA}$$
$$\therefore \quad F = \frac{1}{2} M \omega^2 L$$

Fig. 4.25



4.32 **(a)** Total initial momentum

$$= (2m)v - m(2v) = 0$$

Therefore the centre of mass system is the laboratory system and $v_c = 0$

(b) $J = (2m)(v)(a) + (m)(2v)(2a) = 6mva$

(c) $J = I\omega$ (conservation of angular momentum)

$$6mva = \left[\frac{1}{12} 8m(6a)^2 + 2ma^2 + m(2a)^2\right]\omega = 30ma^2\omega$$

The first term in square brackets is the M.I. of the bar, the second and the third terms are for the M.I. of the particles which stick to the bar.

Thus $\omega = \frac{v}{5a}$

(d) $E = \frac{1}{2} I\omega^2 = \frac{1}{2} 30\, ma^2 \left(\frac{v}{5a}\right)^2 = \frac{3}{5} mv^2$

4.33 Let the potential energy be zero when the rod is in the horizontal position. In the vertical position the loss in potential energy of the system will be $mg(d + 2d) = 3mgd$. The gain in rotational kinetic energy will be

$$\frac{1}{2} I\omega^2 = \frac{1}{2}(I_1 + I_2)\omega^2 = \frac{1}{2}\left[md^2 + m(2d)^2\right]\omega^2 = \frac{5}{2} md^2\omega^2$$

Gain in kinetic energy = loss of potential energy

$$\frac{5}{2} md^2\omega^2 = 3mgd$$

$$\therefore \quad \omega = \sqrt{\frac{6g}{5\alpha}}$$

The linear velocity of the lower mass in the vertical position will be

$$v = (\omega)(2d) = \sqrt{\frac{24}{5} gd}$$

4.34 Conservation of J gives

$$I_1\omega_1 = I_2\omega_2$$

$$\frac{2}{5} MR^2 \frac{2\pi}{T_1} = \frac{2}{5} M \left(\frac{R}{2}\right)^2 \frac{2\pi}{T_2}$$

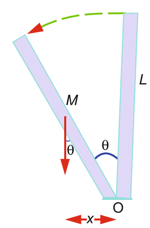
$$\therefore \quad T_2 = \frac{T_1}{4} = \frac{24}{4} = 6\text{ h}$$

4.35 The moment of inertia of the pole of length L and mass M about O is (Fig. 4.26)

$$I = \frac{ML^2}{3} \tag{1}$$

The torque $\tau = I\alpha = Mgx$ (2)

Fig. 4.26



where x is the projection of the centre of mass on the ground from the point O and α is the angular acceleration.

$$\text{Now } x = \frac{L}{2}\sin\theta \tag{3}$$

Using (1) and (3) in (2)

$$\alpha = \frac{3}{2}\frac{g}{L}\sin\theta \tag{4}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta} = \frac{3g}{2L}\sin\theta$$

Integrating

$$\int \omega\, d\omega = \frac{3}{2}\frac{g}{L}\int \sin\theta\, d\theta + C$$

where $C =$ constant.

$$\frac{\omega^2}{2} = -\frac{3}{2}\frac{g}{L}\cos\theta + C \tag{5}$$

When $\theta = 0,\ \omega = 0$

$$\therefore \quad C = \frac{3}{2}\frac{g}{L} \tag{6}$$

Using (6) in (5) $\quad \omega^2 = \dfrac{3g}{2L}(1-\cos\theta)$

Radial acceleration $a_R = \omega^2 L = \dfrac{3}{2}g(1-\cos\theta)$

Tangential acceleration of the top of the pole $a_T = \alpha L = \dfrac{3}{2}g\sin\theta$

4.36 $\boldsymbol{J} = at^2\hat{i} + b\hat{j}$ (1)

$$\therefore \quad \boldsymbol{\tau} = \frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} = 2at\hat{i} \qquad (2)$$

Take the scalar product of $\boldsymbol{J}$ and $\boldsymbol{\tau}$.

$$\boldsymbol{J} \cdot \boldsymbol{\tau} = 2a^2t^3 = \left(\sqrt{a^2t^4 + b^2}\right)(2at)\cos 45^\circ$$

Simplify and solve for t. We get

$$t = \sqrt{\frac{b}{a}} \qquad (3)$$

Using (3) in (2), $\quad |\boldsymbol{\tau}| = 2\sqrt{ab}$

Using (3) in (1), $\quad |\boldsymbol{J}| = \sqrt{2}b$

4.37 Consider a ring of radii r and $r + \mathrm{d}r$, concentric with the disc $(r < R)$. If the surface density is σ, the mass of the ring is $\mathrm{d}m = 2\pi r\mathrm{d}r\sigma$. The moment of inertia of the ring about the central axis will be

$$\mathrm{d}I = (2\pi r\ \mathrm{d}r\sigma)r^2 = 2\pi\sigma r^3\mathrm{d}r \qquad (1)$$

and the corresponding torque will be

$$\mathrm{d}\tau = \alpha\mathrm{d}I = 2\pi\sigma\alpha r^3\ \mathrm{d}r \qquad (2)$$

The frictional force on the ring is $\mu\mathrm{d}m\ g = \mu(2\pi r\ \mathrm{d}r\sigma)g$ and the corresponding torque will be

$$\mathrm{d}\tau = \mu(2\pi r\ \mathrm{d}r\sigma)gr = 2\pi\sigma\mu gr^2\mathrm{d}r \qquad (3)$$

Calculating the torques from (2) and (3) for the whole disc and equating them

$$\int_0^R 2\pi\sigma\alpha r^3\mathrm{d}r = \int_0^R 2\pi\sigma\mu gr^2\mathrm{d}r$$

$$\therefore \quad \alpha = \frac{4\mu g}{3R} \qquad (4)$$

but $\quad 0 = \omega - \alpha t$

$$\therefore \quad t = \frac{\omega}{\alpha} = \frac{3\omega R}{4\mu g}$$

4.38 The horizontal component of force at Q is mv^2/R. The drop in height in coming down to Q is

$$(6R - R) = 5R$$

Gain in kinetic energy = loss in potential energy

$$\frac{7}{10}mv^2 = (mg)(5R)$$
$$\therefore \quad \frac{mv^2}{R} = \frac{50}{7}mg$$

4.39 Let the velocity on the top be v. Energy conservation gives

$$\frac{1}{2}mv_0^2 = mgr\cos\theta_0 + \frac{1}{2}mv^2 \qquad (1)$$

where $r\cos\theta_0$ is the height to which the particle is raised. Angular momentum conservation gives

$$mvr = mv_0 r\sin\theta_0 \qquad (2)$$

Eliminating v between (1) and (2) and simplifying

$$v_0 = \sqrt{\frac{2gr}{\cos\theta_0}}$$

4.40 Equation of motion is

$$ma = mg\sin\theta - T \qquad (1)$$

Torque $\tau = TR = I\alpha = \frac{1}{2}mR^2\frac{a}{R}$

$$\therefore \quad T = \frac{1}{2}ma \qquad (2)$$

Using (2) in (1)

$$a = \frac{2}{3}g\sin\theta = \frac{2}{3}g\sin 30^\circ = \frac{g}{3}$$

4.41 $$< \omega > = \frac{\int \omega \mathrm{d}t}{\int \mathrm{d}t} \qquad (1)$$

$$\tau = I\alpha = C\sqrt{\omega}$$

where C is a constant.

$$\therefore \quad \alpha = C_1\sqrt{\omega}$$

where $C_1 =$ constant

$$\alpha = \frac{d\omega}{dt} = C_1\sqrt{\omega}$$

$$\therefore \quad dt = \frac{d\omega}{C_1\sqrt{\omega}} \tag{2}$$

Using (2) in (1)

$$< \omega > = \frac{\int_0^{\omega_0} \sqrt{\omega} d\omega}{\int_0^{\omega_0} \frac{d\omega}{\sqrt{\omega}}} = \frac{\omega_0}{3}$$

4.42 OC $= L$ is the length of the rod with the centre of mass G at the midpoint, Fig. 4.27. As the rod rotates with angular velocity ω it makes an angle θ with the vertical OA through O. Drop a perpendicular GD $= r$ on the vertical OA and a perpendicular GB on OC.

$$r = \frac{L}{2}\sin\theta$$

The acceleration of the rod at G at any instant is $\omega^2 r = \omega^2 (L/2)\sin\theta$, horizontally and in the plane containing the rod and OA. The component at right angles to OG is $\omega^2 (L/2)\sin\theta\cos\theta$ and the angular acceleration α about O in the vertical plane containing the rod and OA will be $\omega^2 \sin\theta\cos\theta$

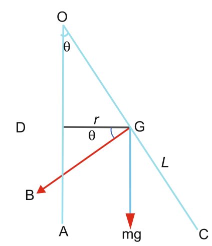
$$\text{Torque } m\,g\,r = mg\frac{L}{2}\sin\theta = I\alpha = m\frac{L^2}{3}\omega^2\sin\theta\cos\theta$$

$$\frac{mL}{2}\sin\theta\left[g - \frac{2L}{3}\omega^2\cos\theta\right] = 0$$

$$\theta = 0 \text{ or } \cos^{-1}\left(\frac{3g}{2\omega^2 L}\right)$$

If $3g > 2\omega^2 L$, i.e. $\omega^2 < \dfrac{3g}{2L}$, the only possible solution is $\theta = 0$, i.e. the rod hangs vertically. If $\omega^2 > \dfrac{3g}{2L}$, then $\theta = \cos^{-1}\dfrac{3g}{2\omega^2 L}$.

Fig. 4.27



4.43 **(a)** For pure sliding equation of motion is

$$\text{ma} = -\mu mg$$

$$\text{or} \quad a = -\mu g \tag{1}$$

$$v = v_0 - \mu gt \tag{2}$$

At the instant pure rolling sets in

$$\text{Torque } I\alpha = FR \tag{3}$$

$$\frac{2}{5}mR^2\alpha = \mu mgR$$

$$\therefore \quad \alpha = \frac{5}{2}\frac{\mu g}{R} \tag{4}$$

$$\omega = \alpha t = \frac{5}{2}\frac{\mu gt}{R} \tag{5}$$

Using (5) in (2)

$$v = v_0 - \frac{2}{5}\omega R = v_0 - \frac{2}{5}v$$

$$\therefore \quad v = \frac{5}{7}v_0$$

(b) $v = v_0 - \mu gt$

$$t = \frac{v_0 - v}{\mu g} = \frac{v_0 - (5/7)\, v_0}{\mu g} = \frac{2v_0}{7\mu g}$$

(c) $v^2 = v_0^2 + 2as$

$$= v_0^2 - 2\mu g s$$
$$\left(\frac{5}{7}v_0\right)^2 = v_0^2 - 2\mu g s$$
$$s = \frac{12}{49}\frac{v_0^2}{\mu g}$$

The assumption made is that we have either pure sliding or pure rolling. Actually in the transition both may be present.

4.44 $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$

Differentiating

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \boldsymbol{r} \times \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} + \boldsymbol{p} \times \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{r} \times \boldsymbol{F} + \boldsymbol{p} \times \boldsymbol{v} = \boldsymbol{\tau} + 0 = \boldsymbol{\tau}$$

because the momentum and velocity vectors are in the same direction.
Angular momentum conservation requires that

$$|\boldsymbol{L}_\mathrm{i}| = |\boldsymbol{L}_\mathrm{f}| = mv\left(\frac{l}{2}\right)$$

$\boldsymbol{L} = (l/2)m\boldsymbol{v}$ is not correct because $\boldsymbol{L}$ is perpendicular to $\boldsymbol{v}$.

L conservation gives

$$mv\frac{l}{2} = \frac{1}{3}Ml^2\omega + m\omega\frac{l^2}{4}$$
$$\therefore \quad \omega = \frac{6mv}{(4M + 3m)l} \tag{1}$$
$$K_\mathrm{rot} = \frac{1}{2}I\omega^2 + \frac{1}{2}m\left(\frac{\omega l}{2}\right)^2 = \frac{3m^2v^2}{2(4M + 3m)}$$

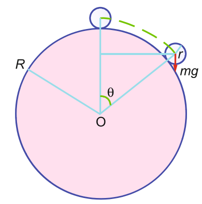
where we have used (1).

$$\therefore \quad \frac{K_\mathrm{rot}}{\frac{1}{2}mv^2} = \frac{3m}{4M + 3m} = \frac{3}{23}$$

where we have used $M = 5m$ (by problem).

4.45 Let the small sphere break off from the large sphere at angle θ with the vertical, Fig. 4.28. At that point the component of (gravitational force) – (centrifugal force) = reaction = 0

Fig. 4.28



$$mg\cos\theta = \frac{mv^2}{R+r} \tag{1}$$

Loss in potential energy = gain in kinetic energy

$$mg(R+r)(1-\cos\theta) = \frac{7}{10}mv^2 \tag{2}$$

Solving (1) and (2)

$$g(R+r) = \frac{17}{10}v^2 = \frac{17}{10}\omega^2 r^2$$

$$\therefore \quad \omega = \sqrt{\frac{10}{17}g\frac{(R+r)}{r^2}} \text{ and } \theta = \cos^{-1}\left(\frac{10}{17}\right)$$

4.46 Let N be the reaction of the floor and θ the angle which the rod makes with the vertical after time t, Fig. 4.29. The only forces acting on the rod are the weight and the reaction which act vertically and consequently the centre of mass moves in a straight line vertically downwards.
Equation of motion for the centre of mass is

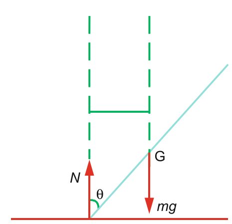


Fig. 4.29

$$mg - N = m\frac{d^2}{dt^2}(a - a\cos\theta)$$

$$\text{or} \quad mg - N = ma\left[\cos\theta\left(\frac{d\theta}{dt}\right)^2 + \sin\theta\frac{d^2\theta}{dt^2}\right] \tag{1}$$

The work–energy theorem gives

$$mga(1 - \cos\theta) = \frac{1}{2}m\left(\frac{d\theta}{dt}\right)^2\left[\frac{a^2}{3} + a^2\sin^2\theta\right]$$

where the square bracket has been written using the parallel axis theorem.

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{6g(1 - \cos\theta)}{a(1 + 3\sin^2\theta)} \tag{2}$$

$$\therefore \quad \frac{d^2\theta}{dt^2} = \frac{3g}{a}\left[\frac{\sin\theta(7 - 6\cos\theta - 3\sin^2\theta)}{(1 + 3\sin^2\theta)^2}\right] \tag{3}$$

By substituting $\left(\frac{d\theta}{dt}\right)^2$ and $\left(\frac{d^2\theta}{dt^2}\right)$ from (2) and (3) in (1), the reaction N is obtained as a function of θ. When the rod is about to strike the floor,

$$\theta = \frac{\pi}{2};\ \left(\frac{d\theta}{dt}\right)^2 = \frac{3g}{2a} \text{ and } \frac{d^2\theta}{dt^2} = \frac{3g}{4a}$$

Thus the reaction from (1) will be

$$N = m\left[g - \frac{3g}{4}\right] \text{ or } \frac{1}{4}mg$$

4.47 **(a)** For $\alpha_{net} = 0$, the two torques which act in the opposite sense must be equal (Fig. 4.30), i.e.

$$\tau_1 = \tau_2$$

$$\text{or} \quad m_1 g R_1 = m_2 g R_2$$

$$m_2 = \frac{m_1 R_1}{R_2} = \frac{25 \times 1.2}{0.5} = 60\,\text{kg}$$

(b) **(i)** $a_1 = \alpha R_1, \quad a_2 = \alpha R_2$ (1)

as $R_1 > R_2, a_1 > a_2$

(ii) Equations of motion are

$$m_1 a_1 = m_1 g - T_1 \quad (2)$$
$$m_2 a_2 = T_2 - m_2 g \quad (3)$$
$$T_1 R_1 - T_2 R_2 = I\alpha \quad (4)$$

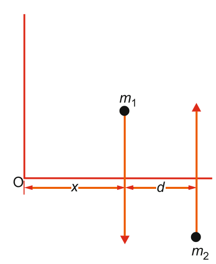
Combining (1), (2), (3) and (4) and substituting $m_1 = 35\,\text{kg}$, $m_2 = 60\,\text{kg}$, $R_1 = 1.2\,\text{m}$, $R_2 = 0.5\,\text{m}$, $I = 38\,\text{kg m}^2$ and $g = 9.8\,\text{m/s}^2$, we find

$$a_1 = \frac{(m_1 R_1 - m_2 R_2) R_1 g}{m_1 R_1^2 + m_2 R_2^2 + I}$$
$$= \frac{(35 \times 1.2 - 60 \times 0.5)1.2g}{35 \times 1.2^2 + 60 \times 0.5^2 + 38} = 0.139g$$
$$a_2 = a_1 \frac{R_2}{R_1} = 0.139 \times \frac{0.5}{1.2} = 0.058\,g$$
$$T_1 = m_1(g - a_1) = 35g(1 - 0.139) = 295.3\,\text{N}$$
$$T_2 = m_2(g + a_2) = 60g(1 + 0.058) = 622.1\,\text{N}$$
$$\alpha = \frac{T_1 R_1 - T_2 R_2}{I} = \frac{295.3 \times 1.2 - 622.1 \times 0.5}{38} = 1.14\,\text{rad/s}^2$$

4.48 $\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2$
$$= x\hat{i} \times (-mv\hat{j}) + (x + d)\hat{i} \times (mv\hat{j})$$
$$= mvd\hat{i} \times \hat{j} = mvd\hat{k}$$

which is independent of x and therefore independent of the origin.

Fig. 4.30



4.49 **(a)** $K_{\text{rot}} = \frac{1}{2}I\omega^2 = \frac{1}{2} \cdot \frac{2}{5}mr^2\frac{v^2}{r^2} = \frac{1}{5}mv^2$

$$K_{\text{total}} = \frac{1}{2}mv^2 + \frac{1}{5}mv^2 = \frac{7}{10}mv^2$$
$$\therefore \quad \frac{K_{\text{rot}}}{K_{\text{total}}} = \frac{(1/5)\,mv^2}{(7/10)\,mv^2} = \frac{2}{7}$$

(b) In coming down to the bottom of the hemisphere loss of potential energy $= mgh = mgR$. Gain in kinetic energy $= (7/10)\, mv^2$.

$$\therefore \quad \frac{7}{10} mv^2 = mgR$$
$$\text{or} \quad \frac{mv^2}{R} = \frac{10mg}{7}$$

The normal force exerted by the small sphere at the bottom of the large sphere will be

$$N = mg + \frac{mv^2}{R} = mg + \frac{10mg}{7} = \frac{17mg}{7}$$

4.50 Work done $W = \tau\theta = I\alpha\theta = \dfrac{I\omega^2}{2}$

Along the diameter for hoop, $I = mR^2/2$, while for the solid sphere, hollow sphere and the disc, $I = (2/5)\, mR^2$, $(2/3)\, mR^2$ and $(1/4)\, mR^2$, respectively, maximum work will have to be done to stop the hollow sphere, ω being identical as it has the maximum moment of inertia.

4.51 Work done $W = \tau\theta = I\alpha\theta = \dfrac{I\omega^2}{2} = \dfrac{J^2}{2I}$

where we have used the formula $J = I\omega$. Maximum work will have to be done for the disc since I is the least, τ being identical.

4.52 $W = \dfrac{1}{2} I\omega^2 = \dfrac{1}{2}(I\omega)\omega = \dfrac{1}{2} J\omega$

Since J and ω are the same for all the four objects, work done is the same.

4.53 $\tau = I\alpha = I\dfrac{a}{R} = \dfrac{MgR\sin\theta}{1 + \left(R^2/k^2\right)}$

For solid sphere, hollow sphere, solid cylinder and hollow cylinder the quantity $1 + (R^2/k^2)$ is $7/2$, $5/2$, 3, 2, respectively. Therefore τ will be least for solid sphere.

4.54 $\boldsymbol{r} = 3t\hat{i} + 2\hat{j}$

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = 3\hat{i}$$

$$L = \boldsymbol{r} \times \boldsymbol{p} = m(\boldsymbol{r} \times \boldsymbol{v}) = m(3t\hat{i} + 2\hat{j}) \times 3\hat{i}$$

$$= 6\,m(\hat{j} \times \hat{i}) = -6\,m\hat{k} \quad \text{(constant)}$$

4.55 Angular momentum conservation gives

$$J = mvd = I\omega \tag{1}$$

Linear momentum conservation gives

$$mv = Mv_{\mathrm{c}} \tag{2}$$

Energy conservation gives

$$\frac{1}{2}mv^2 = \frac{1}{2}I\omega^2 + \frac{1}{2}Mv_{\mathrm{c}}^2 \tag{3}$$

$$I = \frac{Ml^2}{12} \tag{4}$$

Eliminating ω and v_{c} from (1) and (2) and using (3)

$$\frac{1}{2}mv^2 = \frac{1}{2}\frac{m^2v^2d^2}{I} + \frac{1}{2}\frac{m^2v^2}{M} \tag{5}$$

Simplifying and using (4) in (5)

$$d = \frac{l}{2}\sqrt{\frac{M-m}{3m}}$$

4.56 **(a)** Let the initial velocity be v_0, then at instant t the velocity

$$v = v_0 - at = v_0 - \mu gt \tag{1}$$

Torque $\tau = I\alpha = FR$

$$\frac{1}{2}mR^2\alpha = \mu\, mgR$$

$$\alpha = \frac{2\mu g}{R}$$

$$\mu gt = \frac{\alpha Rt}{2} = \frac{\omega R}{2}$$

Therefore (1) becomes

$$v = v_0 - \frac{\omega R}{2} = v_0 - \frac{v}{2}$$
$$\therefore \quad v = \frac{2}{3} v_0 \tag{2}$$

Using (2) in (1)

$$\frac{2}{3} v_0 = v_0 - \mu\, gt$$
$$\text{or} \quad t = \frac{v_0}{3\,\mu g}$$

(b) Work done $W = \Delta K = \frac{1}{2} m v^2 - \frac{1}{2} m v_0^2$

$$= \frac{1}{2} m \left[\frac{4}{9} v_0^2 - v_0^2 \right] = -\frac{5}{18} m v_0^2$$

4.57 Equation of motion is

$$ma = mg - 2T \tag{1}$$

where 'a' is the linear acceleration and T the tension in each thread.

$$\text{Torque} \quad I\alpha = 2Tr \qquad (\because \text{ there are two threads})$$
$$\frac{1}{2} m r^2 \alpha = 2Tr$$
$$\text{or} \quad \alpha = \frac{4T}{mr} \tag{2}$$

As both the cylinders are rotating,

$$a = 2\alpha r = \frac{8T}{m} \tag{3}$$
$$\text{or} \quad ma = 8T \tag{4}$$

Using (4) in (1) we get

$$T = \frac{1}{10} mg$$

Note that if the lower cylinder is not wound then

$$a = \frac{4T}{m} \quad \text{and} \quad T = \frac{1}{6}mg$$

4.58 C is the centre of the disc and A the point which is fixed, Fig. 4.31. The forces acting at A have no torque at A, so that the angular momentum is conserved. Initially the moment of inertia of the disc about the axis passing through its centre and perpendicular to its plane is

$$I_c = I = \frac{1}{2}mr^2 \tag{1}$$

When the point A is fixed the moment of inertia about an axis parallel to the central axis and passing through A will be

$$I_A = I_c + mr^2 = m\left(\frac{1}{2}r^2 + r^2\right) = \frac{3}{2}mr^2 \tag{2}$$

by parallel axis theorem.
Angular momentum conservation requires

$$I_A\omega' = I_c\omega \tag{3}$$

Substituting (1) and (2) in (3) we obtain

$$\omega' = \frac{\omega}{3} \tag{4}$$

If X and Y are the impulses of the forces at A perpendicular and along CA, then

$$X = m\,r\omega' = mr\frac{\omega}{3} \quad \text{and} \quad Y = 0$$

Thus the impulse of the blow at A is $mr\dfrac{\omega}{3}$ at right angles to CA.

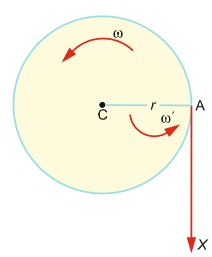


Fig. 4.31

4.59 The torque of the air resistance on an element dx at distance x from the fixed end, about this end, will be

$$\mathrm{d}\tau = k(\omega x)^2 x\,\mathrm{d}x = k\omega^2 x^3 \mathrm{d}x$$

$$\tau = \int \mathrm{d}\tau = -k\omega^2 \int_0^L x^3 \mathrm{d}x = I\alpha$$

i.e. $$-\frac{k\omega^2 L^4}{4} = \frac{1}{3} m L^2 \frac{\mathrm{d}\omega}{\mathrm{d}t}$$

$$\therefore \quad -3kL^2 \mathrm{d}t = 4m \frac{\mathrm{d}\omega}{\omega^2}$$

$$\therefore \quad -3kL^2 t = -\frac{4m}{\omega} + C$$

where C is the constant of integration. Initial condition: when $t = 0$, $\omega = \Omega$.

Therefore $C = \dfrac{4m}{\Omega}$

$$\therefore \quad -3kL^2 t = 4m\left(\frac{1}{\Omega} - \frac{1}{\omega}\right)$$

$$\therefore \quad \omega = \frac{4m\Omega}{4m + 3\Omega k L^2 t}$$

4.60 OA is the vertical radius b of the cylinder and a the radius of the sphere which is vertical in the lowest position and shown as CA, Fig 4.32.
In the time the centre of mass of the sphere C has moved to C′ through an angle θ, the sphere has rotated through ϕ so that the reference lime CA has gone into the place of C′D.

If there is no slipping

$$a(\phi + \theta) = b\theta \tag{1}$$

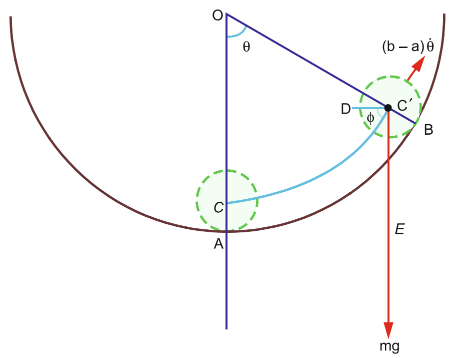
The velocity of the centre of mass is $(b - a)\dot{\theta}$ and the angular velocity of the sphere about its centre is

$$\dot{\phi} = \frac{(b - a)}{a}\dot{\theta} \tag{2}$$

Taking A as zero level, the potential energy

$$U = mg(b - a)(1 - \cos\theta) \tag{3}$$

Fig. 4.32



The kinetic energy $= T\,(\text{trans}) + T\,(\text{rot})$

$$
\begin{aligned}
T &= \frac{1}{2}m(b-a)^2\dot{\theta}^2 + \frac{1}{2}I\,\dot{\phi}^2 \\
&= \frac{1}{2}m(b-a)^2\dot{\theta}^2 + \frac{1}{2}\cdot\frac{2}{5}ma^2\frac{(b-a)^2}{a^2}\dot{\theta}^2 \\
&= \frac{7}{10}m(b-a)^2\dot{\theta}^2 \qquad (4)
\end{aligned}
$$

where we have used (2).
Total energy

$$E = T + U = \frac{7}{10}m(b-a)^2\dot{\theta}^2 + mg(b-a)(1-\cos\theta) = \text{constant} \qquad (5)$$

Differentiating with respect to time and cancelling common factors

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{7}{5}m(b-a)\ddot{\theta}\cdot\dot{\theta} + g\sin\theta\cdot\dot{\theta} = 0 \qquad (6)$$

$$\text{or} \quad \ddot{\theta} + \frac{5g}{7(b-a)}\sin\theta = 0 \qquad (7)$$

For small oscillation angles $\sin\theta \to \theta$.

$$\therefore \quad \ddot{\theta} + \frac{5g\,\theta}{7(b-a)} = 0 \qquad (8)$$

which is the equation for simple harmonic motion with frequency

$$\omega = \sqrt{\frac{5g}{7(b-a)}} \tag{9}$$

and time period

$$T = 2\pi\sqrt{\frac{7(b-a)}{5g}} \tag{10}$$

4.61 **(a)** Let the disc be composed of a number of concentric rings of infinitesimal width. Consider a ring of radius r, width dr and surface density σ (mass per unit area). Then its mass will be $(2\pi r \mathrm{d}r)\sigma$. The moment of inertia of the ring about an axis passing through the centre of the ring and perpendicular to its plane will be

$$\mathrm{d}I = (2\pi r\,\mathrm{d}r)\sigma r^2$$

Then the moment of inertial of the disc

$$I = \int \mathrm{d}I = 2\pi\sigma \int_0^R r^3 \mathrm{d}r = \frac{1}{2}\pi\,\sigma\,R^4 \tag{1}$$

If M is the mass of the disc, then

$$\sigma = \frac{M}{\pi R^2} \tag{2}$$

$$\therefore \quad I = \frac{1}{2}MR^2 \tag{3}$$

(b) The total kinetic energy T of the disc on the horizontal surface is

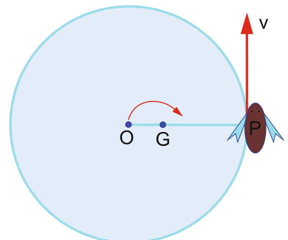
$$\begin{aligned} T\,(\text{initial}) &= \frac{1}{2}Mu^2 + \frac{1}{2}I\omega^2 \\ &= \frac{1}{2}Mu^2 + \frac{1}{2}\cdot\frac{1}{2}MR^2\frac{u^2}{R^2} = \frac{3}{4}Mu^2 \\ T\,(\text{final}) &= \frac{3}{4}Mv^2 = \frac{3}{4}Mu^2 + Mgh \end{aligned} \tag{4}$$

by energy conservation

$$\text{Solving,}\ \ v = \sqrt{u^2 + \frac{4}{3}gh}$$

4.62 In Fig. 4.33, O is the centre of the ring, P the instantaneous position of the insect and G the centre of mass of the system. Suppose the insect crawls

Fig. 4.33



around the ring in the counterclockwise sense. The only forces acting in a horizontal plane are the reactions at P which are equal and opposite. Consequently G will not move and the angular momentum about G which was zero initially will remain zero throughout the motion due to its conservation.

$$m \cdot \mathrm{PG}(v - \mathrm{PG}\omega) - I_G\omega = 0 \tag{1}$$

where ω is the angular velocity of the ring.

$$\text{Now} \quad \mathrm{PG} = \frac{Mr}{M+m}, \quad \mathrm{OG} = \frac{mr}{M+m} \tag{2}$$

$$I_\omega = I_{\mathrm{CM}} + M(\mathrm{OG})^2 = Mr^2 + M\frac{m^2r^2}{(M+m)^2} \tag{3}$$

Using (2) and (3) in (1) and simplifying we obtain

$$\omega = \frac{mv}{(M+2m)r} \tag{4}$$

4.3.3 Coriolis Acceleration

4.63 **(a)** $\boldsymbol{\omega}$ points in the south to north direction along the rotational axis of the earth.

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{86,160} = 7.292 \times 10^{-5}\ \mathrm{rad/s}$$

(b) The period of rotation of the plane of oscillation is given by

$$T' = \frac{2\pi}{\omega'} = \frac{2\pi}{\omega \sin\lambda} = \frac{T_0}{\sin\lambda} = \frac{24}{\sin 30^\circ} = 48\ \mathrm{h}$$

4.64 The object undergoes an eastward deviation through a distance

$$d = \frac{1}{3}\omega\cos\lambda\sqrt{\frac{8h^3}{g}} = \frac{1}{3}\times 7.29\times 10^{-5}\times\cos 0^\circ\sqrt{\frac{8\times 400^3}{9.8}} = 0.1756\,\text{m}$$
$$= 17.56\,\text{cm}$$

4.65 $$y' = \frac{4}{3}\frac{u^3}{g^2}\omega\cos\lambda$$
$$= \frac{4}{3}\times\frac{(20)^3}{(9.8)^2}\times 7.29\times 10^{-5}\cos 0^\circ = 0.0081\,\text{m} = 8.1\,\text{mm}$$

4.66 $$y' = \frac{4}{3}\frac{u^3}{g^2}\omega\cos\lambda,\ \lambda = 0^\circ$$
$$u = \left[\frac{3y'g^2}{4\omega\cos\lambda}\right]^{1/3} = \left[\frac{3}{4}\times\frac{1\times(9.8)^2}{7.27\times 10^{-5}}\right]^{1/3} = 99.7\,\text{m/s}$$

4.67 Consider two coordinate systems, one inertial system S and the other rotating one S', which are rotating with constant angular velocity ω

$$\begin{array}{ccccccc} \text{Acceleration in} & & \text{Acceleration in} & & \text{Coriolis} & & \text{centrifugal} \\ \text{inertial frame} & = & \text{rotating frame} & + & \text{acceleration} & + & \text{acceleration} \\ \dfrac{d^2\boldsymbol{r}}{dt^2} & = & \dfrac{d^2\boldsymbol{r}'}{dt^2} & + & 2\vec{\omega}\times\dfrac{d\boldsymbol{r}'}{dt} & + & \boldsymbol{\omega}\times\boldsymbol{\omega}\times\boldsymbol{r} \end{array} \tag{1}$$

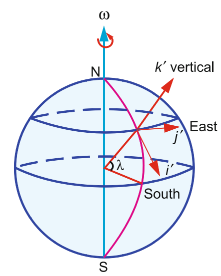
Let the k axis in the inertial frame S be directed along the earth's axis. Let the rotating frame S' be rigidly attached to the earth at a geographical latitude λ in the northern hemisphere. Let the k' axis be directed outwards at the latitude λ along the plumb line, whose direction is that of the resultant passing through the earth's centre. With the choice of a right-handed system, the i'-axis is in the southward direction and the j'-axis in the eastward direction, Fig. 4.34. Assume g the acceleration due to gravity to be constant. It includes the centrifugal term $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})$ since g is supposed to represent the resultant acceleration of a falling body at the given place.

$$\frac{d^2r'}{dt^2} = g - 2\boldsymbol{\omega}\times\boldsymbol{v}_R \tag{2}$$

Since we are considering the fall of a body in the northern hemisphere, the components of angular velocity are

$$\left.\begin{aligned} \omega_x &= -\omega\cos\lambda \\ \omega_y &= 0 \\ \omega_z &= \omega\sin\lambda \end{aligned}\right\} \tag{3}$$

Fig. 4.34



$$\boldsymbol{\omega} \times \boldsymbol{v}_\mathrm{R} = \begin{vmatrix} i' & j' & k' \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ \dot{x}' & \dot{y}' & \dot{z}' \end{vmatrix}$$

$$= -\omega\sin\lambda\,\dot{y}' i' + (\omega\sin\lambda\,\dot{x}' + \omega\cos\lambda\,\dot{z}')\,j' - (\omega\cos\lambda\,\dot{y}')k'$$

But $\dfrac{\mathrm{d}^2\boldsymbol{r}'}{\mathrm{d}t^2} = g - 2(\boldsymbol{\omega} \times \boldsymbol{v})$

$$\therefore \quad \ddot{x}'\,i' + \ddot{y}'\,j' + \ddot{z}'\,k' = -g\,k' + 2\omega\sin\lambda\,\dot{y}' i' - 2(\omega\sin\lambda\,\dot{x}' + \omega\cos\lambda\,\dot{z}')\,j' + 2\omega\cos\lambda\,\dot{y}' k' \tag{4}$$

Equating coefficients of i', j' and k'on both sides of (4), we obtain the equations of motion

$$\ddot{x}' = 2\omega\ \sin\lambda\,\dot{y}' \tag{5}$$

$$\ddot{y}' = -2(\omega\ \sin\lambda\,\dot{x}' + \omega\cos\lambda\,\dot{z}') \tag{6}$$

$$\ddot{z}' = -g + 2\omega\cos\lambda \tag{7}$$

Now the quantities $\dot{x}'$ and $\dot{y}'$ are small compared to $\dot{z}'$. To the first approximation we can write

$$(v_\mathrm{R})_x = 0; \quad (v_\mathrm{R})_y = 0; \quad (v_\mathrm{R})_z = \dot{z}' = -g \tag{8}$$

Setting $\dot{x}' = \dot{y}' = 0$ in (5), (6) and (7), we obtain the equations for the components of a_R:

$$(a_\mathrm{R})_x = \ddot{x}' = 0 \tag{9}$$

$$(a_\mathrm{R})_y = \ddot{y}' = -2\omega\,\dot{z}'\cos\lambda \tag{10}$$

$$(a_\mathrm{R})_z = \ddot{z}' = -g \tag{11}$$

Equation (9) shows that no deviation occurs in the north–south direction.

Integrating (11)

$$\dot{z}' = -gt \tag{12}$$

and $$z' = -\frac{1}{2}gt^2 \tag{13}$$

with the initial condition that at $t = 0$, $\dot{z}' = 0$, $z' = 0$.

Using (12) in (10) and integrating twice

$$\dot{y}' = \omega\, gt^2 \cos\lambda \tag{14}$$

because $(\dot{y}')_0 = 0$.

$$y' = \frac{1}{3}\omega\, gt^3 \cos\lambda \tag{15}$$

because $(y')_0 = 0$.

Setting $-z' = h = (1/2)\, gt^2$, or $t = \sqrt{2h/g}$, in (15) the body undergoes eastward deviation through a distance

$$d = y' = \frac{1}{3}\omega \cos\lambda \sqrt{\frac{8h^3}{g}} \tag{16}$$

4.68 $\boldsymbol{F}_{\text{coriolis}} = -2m\boldsymbol{\omega} \times \boldsymbol{v}_{\text{R}}$

$$F_{\text{cor}} = 2m\omega v_{\text{R}} \sin\theta = 2 \times 5 \times 10^8 \times 7.27 \times 10^{-5} \times \frac{8000}{86,400} \quad (\because \theta = 90°)$$

$= 6730\,\text{N}$ due north

4.69 Coriolis action on a mass m of water towards the eastern side (Fig. 4.35) is

$$m\ddot{y}' = 2mv\omega \sin\lambda \tag{1}$$

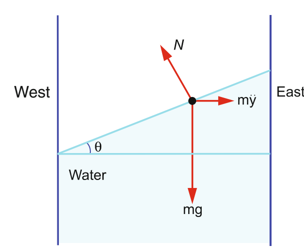


Fig. 4.35

Let N be the normal reaction and let the water level be tilted through an angle θ. Resolve N into horizontal and vertical components and balance them with the Coriolis force and the weight, respectively.

$$N \sin\theta = 2m\, v\omega\, \sin\lambda$$
$$N \cos\theta = mg$$

Dividing the equations, $\tan\theta = \dfrac{d}{b} = 2\, v\, \omega \sin\lambda$

or $\quad d = \dfrac{2bv\omega}{g} \sin\lambda$

4.70 By eqn. (15) prob. (4.67)

$$y' = \frac{1}{3}\omega g t^3 \cos\lambda \tag{1}$$
$$z' = \frac{1}{2} g t^2 \tag{2}$$

Eliminate t between (1) and (2) to find

$$\frac{y'^2}{z'^3} = \frac{8}{9}\, \frac{\omega^2 \cos^2\lambda}{g}$$

or $\quad y'^2 = C z'^3 \quad$ (semi-cubical parabola)

where $C =$ constant.

4.71 $F_{\text{cor}} = 2m\, v\, \omega\, \sin\lambda$
$= 2 \times 10^6 \times 15 \times 7.27 \times 10^{-5} \sin 60°$
$= 1889\,\text{N}$ on the right rail.

4.72 The difference between the lateral forces on the rails arises because when the train reverses its direction of motion Coriolis force also changes its sign, the magnitude remaining the same. Therefore, the difference between the lateral force on the rails will be equal to $2m\, v\, \omega\, \cos\lambda - (-2m\, v\, \omega \cos\lambda)$ or $4mv\omega\cos\lambda$.

4.73 The displacement from the vertical is given by

$$y' = \left(\frac{1}{3} g t^3 - u t^2\right) \omega \cos\lambda$$
$$= \left(\frac{1}{3} \times 9.8 \times 10^3 - 100 \times 10^2\right) \times 7.27 \times 10^{-5} \cos 60°$$
$$= -0.245\,\text{m} = -24.5\,\text{cm}$$

Thus the body has a displacement of 24.5 cm on the west.