where l is the natural length of the spring, x is the coordinate of the wedge and s is the length of the spring.

(b) By using the Lagrangian derived in (a), show that the equations of motion are as follows:

$$(m + M)\ddot{x} + m\ddot{s}\cos\alpha = 0,$$
$$m\ddot{x}\cos\alpha + m\ddot{s} + k(s - s_0) = 0,$$

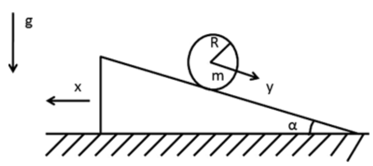
where $s_0 = l + (mg\sin\alpha)/k$.

(c) By using the equations of motion in (b), derive the frequency for a small amplitude oscillation of this system.

[University of Manchester 2008]

7.35 A uniform spherical ball of mass m rolls without slipping down a wedge of mass M and angle α, which itself can slide without friction on a horizontal table. The system moves in the plane shown in Fig. 7.12. Here g denotes the gravitational acceleration.

Fig. 7.12



(a) Find the Lagrangian and the equations of motion for this system.

(b) For the special case of $M = m$ and $\alpha = \pi/4$ find

(i) the acceleration of the wedge and

(ii) the acceleration of the ball relative to the wedge.

[Useful information: Moment of inertia of a uniform sphere of mass m and radius R is $I = \frac{2}{5}mR^2$.]

[University of Manchester 2007]

7.3 Solutions

7.1 This is obviously a two degree of freedom dynamical system. The square of the particle velocity can be written as

$$v^2 = \dot{r}^2 + (r\dot{\theta})^2 \tag{1}$$

Formula (1) can be derived from Cartesian coordinates

$$x = r\cos\theta, \quad y = r\sin\theta$$
$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \quad \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

We thus obtain

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

The kinetic energy, the potential energy and the Lagrangian are as follows:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \tag{2}$$

$$V = -\frac{\mu m}{r} \tag{3}$$

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\mu m}{r} \tag{4}$$

We take r, θ as the generalized coordinates q_1, q_2. Since the potential energy V is independent of $\dot{q}_i$, Lagrangian equations take the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad (i = 1, 2) \tag{5}$$

$$\text{Now} \quad \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \; \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\mu m}{r^2} \tag{6}$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \; \frac{\partial L}{\partial \theta} = 0 \tag{7}$$

Equation (5) can be explicitly written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \tag{8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{9}$$

Using (6) in (8) and (7) in (9), we get

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{\mu m}{r^2} = 0 \tag{10}$$

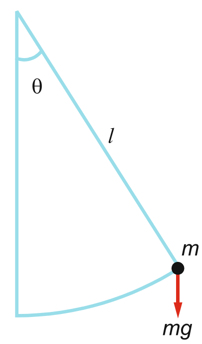
$$m\frac{\mathrm{d}(r^2\dot{\theta})}{\mathrm{d}t} = 0 \tag{11}$$

Equations (10) and (11) are identical with those obtained for Kepler's problem by Newtonian mechanics. In particular (11) signifies the constancy of areal velocity or equivalently angular momentum (Kepler's second law of planetary motion). The solution of (10) leads to the first law which asserts that the path of a planet describes an ellipse.

This example shows the simplicity and power of Lagrangian method which involves energy, a scalar quantity, rather than force, a vector quantity in Newton's mechanics.

7.2 The position of the pendulum is determined by a single coordinate θ and so we take $q = \theta$. Then (Fig. 7.13)

Fig. 7.13



$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 l^2 = \frac{1}{2}ml^2\dot{\theta}^2 \tag{1}$$

$$V = mgl(1 - \cos\theta) \tag{2}$$

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta) \tag{3}$$

$$\frac{\partial T}{\partial \dot{\theta}} = ml^2\dot{\theta}, \quad \frac{\partial T}{\partial \theta} = 0 \tag{4}$$

$$\frac{\partial V}{\partial \dot{\theta}} = 0, \qquad \frac{\partial}{\partial \theta} = mgl\sin\theta \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{\theta}}(T - V)\right) - \frac{\partial}{\partial \theta}(T - V) = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(ml^2\dot{\theta}) + mgl\sin\theta = 0$$

or $l\ddot{\theta} + g\sin\theta = 0$ (equation of motion)

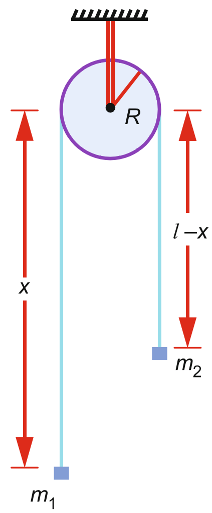
For small oscillation angles $\sin\theta \to \theta$

$$\ddot{\theta} = -\frac{g\theta}{l} \qquad \text{(equation for angular SHM)}$$

$$\therefore \quad \omega^2 = \frac{g}{l} \text{ or time period } T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

7.3 In this system there is only one degree of freedom. The instantaneous configuration is specified by $q = x$. Assuming that the cord does not slip, the angular velocity of the pulley is $\dot{x}/R$, Fig. 7.14.

Fig. 7.14



The kinetic energy of the system is given by

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}I\frac{\dot{x}^2}{R^2} \tag{1}$$

The potential energy of the system is

$$V = -m_1gx - m_2g(l - x) \tag{2}$$

And the Lagrangian is

$$L = \frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{x}^2 + (m_1 - m_2)gx + m_2gl \tag{3}$$

The equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{4}$$

yields

$$\left(m_1 + m_2 + \frac{I}{R^2}\right)\ddot{x} - g(m_1 - m_2) = 0$$

$$\text{or} \quad \ddot{x} = \frac{(m_1 - m_2)g}{m_1 + m_2 + \frac{I}{R^2}} \tag{5}$$

which is identical with the one obtained by Newton's mechanics.

7.4 By problem the masses of pulleys are negligible. The double machine is an Atwood machine in which one of the weights is replaced by a second Atwood machine, Fig. 7.15. The system now has two degrees of freedom, and its instantaneous configuration is specified by two coordinates x and x'. l and l' denote the length of the vertical parts of the two strings. Mass m_1 is at depth x below the centre of pulley A, m_2 at depth $l - x + x'$ and m_3 at depth $l + l' - x - x'$. The kinetic energy of the system is given by

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(-\dot{x} + \dot{x}')^2 + \frac{1}{2}m_3(-\dot{x} - \dot{x}')^2 \tag{1}$$

while the potential energy is given by

$$V = -m_1 g x - m_2 g(l - x + x') - m_3 g(l - x + l' - x') \tag{2}$$

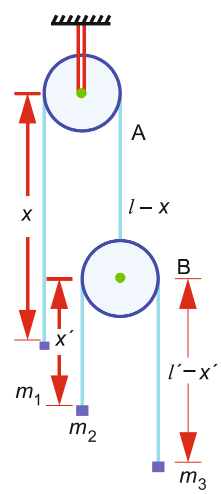


Fig. 7.15

The Lagrangian of the system takes the form

$$L = T - V = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(-\dot{x} + \dot{x}')^2 + \frac{1}{2}m_3(-\dot{x} - \dot{x}')^2 \\ + (m_1 - m_2 - m_3)gx + (m_2 - m_3)gx + m_2gl + m_3g(l + l') \tag{3}$$

The equations of motion are then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}'}\right) - \frac{\partial L}{\partial x'} = 0 \tag{5}$$

which yield

$$(m_1 + m_2 + m_3)\ddot{x} + (m_3 - m_2)\ddot{x}' = (m_1 - m_2 - m_3)g \tag{6}$$

$$(m_3 - m_2)\ddot{x} + (m_2 + m_3)\ddot{x}' = (m_2 - m_3)g \tag{7}$$

Solving (6) and (7) we obtain the equations of motion.

7.5 $T = v^2\dot{u}^2 + 2\dot{v}^2$ (1)

$$V = u^2 - v^2 \tag{2}$$

$$L = T - V = v^2\dot{u}^2 + 2\dot{v}^2 - u^2 + v^2 \tag{3}$$

$$\frac{\partial L}{\partial \dot{u}} = 2v^2\dot{u},\ \frac{\partial L}{\partial u} = -2u \tag{4}$$

$$\frac{\partial L}{\partial \dot{v}} = 4\dot{v},\ \frac{\partial L}{\partial v} = 2v(\dot{u}^2 + 1) \tag{5}$$

The equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{u}}\right) - \frac{\partial L}{\partial u} = 0 \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{v}}\right) - \frac{\partial L}{\partial v} = 0 \tag{7}$$

yield

$$2\frac{\mathrm{d}}{\mathrm{d}t}(v^2\dot{u}) + 2u = 0$$

$$\text{or} \quad v^2\ddot{u} + 2\dot{u}\dot{v} + 2u = 0 \tag{8}$$

$$2\ddot{v} + v(\dot{u}^2 + 1) = 0 \tag{9}$$

7.6 Here we need a single coordinate $q = x$:

$$T = \frac{1}{2}m\dot{x}^2, \quad V = \frac{1}{2}kx^2 \tag{1}$$

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{2}$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial x} = -kx \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{4}$$

$$m\ddot{x} + kx = 0 \tag{5}$$

$$\text{Let } x = A\sin\omega\mathrm{t} \tag{6}$$

$$\ddot{x} = -A\omega^2\sin\omega\mathrm{t} \tag{7}$$

Inserting (6) and (7) and simplifying

$$-m\omega^2 + k = 0$$

$$\omega = \sqrt{\frac{k}{m}} \quad \text{or} \quad T_0 = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

where T_0 is the time period.

7.7 Only one coordinate $q = x$ (distance on the surface of the incline) is adequate to describe the motion:

$$T = \frac{1}{2}m\dot{x}^2, \quad V = -mgx\sin\alpha, \quad L = \frac{1}{2}m\dot{x}^2 + mgx\sin\alpha$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial x} = mg\sin\alpha$$

Equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) - \frac{\partial}{\partial x}(mgx\sin\alpha) = 0$$

or $\quad \ddot{x} = g\sin\alpha$

7.8 This is a two degree of freedom system because both mass m and M are moving. The coordinate on the horizontal axis is described by x' for the inclined plane and x for the block of mass m on the incline. The origin of the coordinate system is fixed on the smooth table, Fig. 7.16. The x- and y-components of the velocity of the block are given by

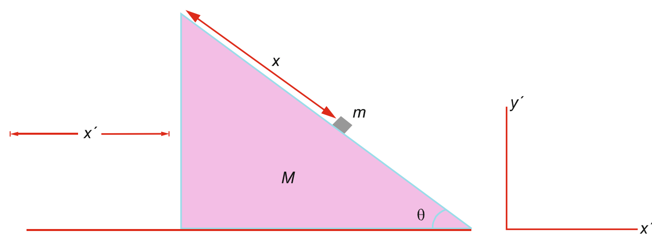


Fig. 7.16

$$v_x = \dot{x}' + \dot{x}\cos\theta \tag{1}$$

$$v_y = -\dot{x}\sin\theta \tag{2}$$

$$\therefore \quad v^2 = v_x^2 + v_y^2 = \dot{x}'^2 + 2\dot{x}\dot{x}'\cos\theta + \dot{x}^2 \tag{3}$$

Hence, the kinetic energy of the system will be

$$T = \frac{1}{2}M\dot{x}'^2 + \frac{1}{2}m(\dot{x}'^2 + 2\dot{x}\dot{x}'\cos\theta + \dot{x}^2) \tag{4}$$

while the potential energy takes the form

$$V = -mgx\sin\theta \tag{5}$$

and the Lagrangian is given by

$$L = \frac{1}{2}M\dot{x}'^2 + \frac{1}{2}m(\dot{x}'^2 + 2\dot{x}\dot{x}'\cos\theta + \dot{x}^2) + mgx\sin\theta \tag{6}$$

The equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}'}\right) - \frac{\partial L}{\partial x'} = 0 \tag{8}$$

yield

$$m(\ddot{x}'\cos\theta + \ddot{x}) - mg\sin\theta = 0 \tag{9}$$

$$M\ddot{x}' + m(\ddot{x}' + \ddot{x}\cos\theta) = 0 \tag{10}$$

Solving (9) and (10)

$$\ddot{x} = \frac{g\sin\theta}{1 - m\cos^2\theta/(M+m)} = \frac{(M+m)g\sin\theta}{M + m\sin^2\theta} \tag{11}$$

$$\ddot{x}' = -\frac{g\sin\theta\cos\theta}{(M+m)/m - \cos^2\theta} = -\frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta} \tag{12}$$

which are in agreement with the equations of prob. (2.14) derived from Newtonian mechanics.

7.9 $$T = \frac{1}{2}m(\dot{x}^2 + \omega^2 x^2) \tag{1}$$

because the velocity of the bead on the wire is at right angle to the linear velocity of the wire:

$$V = mgx\sin\omega\mathrm{t} \tag{2}$$

because $\omega t = \theta$, where θ is the angle made by the wire with the horizontal at time t, and $x\sin\theta$ is the height above the horizontal position:

$$L = \frac{1}{2}m(\dot{x}^2 + \omega^2 x^2) - mgx\sin\omega t \tag{3}$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial x} = m\omega^2 x - mg\sin\omega t \tag{4}$$

Lagrange's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} \tag{5}$$

then becomes

$$\ddot{x} - \omega^2 x + g\sin\omega\mathrm{t} = 0 \quad \text{(equation of motion)} \tag{6}$$

which has the solution

$$x = A\mathrm{e}^{\omega t} + B\mathrm{e}^{-\omega t} + \frac{g}{2\omega^2}\sin\omega\mathrm{t} \tag{7}$$

where A and B are constants of integration which are determined from initial conditions.

$$\text{At } t = 0, x = 0 \text{ and } \dot{x} = 0 \tag{8}$$

$$\text{Further } \dot{x} = \omega(A\mathrm{e}^{\omega t} - B\mathrm{e}^{-\omega t}) + \frac{g}{2\omega}\cos\omega\mathrm{t} \tag{9}$$

Using (8) in (7) and (9) we obtain

$$0 = A + B \tag{10}$$

$$0 = \omega(A - B) + \frac{g}{2\omega} \tag{11}$$

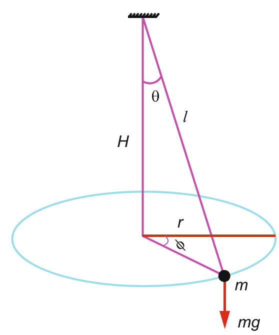
Solving (10) and (11) we get $A = -\frac{g}{4\omega^2}, \quad B = \frac{g}{4\omega^2}$ (12)

The complete solution for x is

$$x = \frac{g}{4\omega^2}(\mathrm{e}^{-\omega t} - \mathrm{e}^{\omega t} + 2\sin\omega \mathrm{t})$$

7.10 Let the length of the cord be l. The Cartesian coordinates can be expressed in terms of spherical polar coordinates (Fig. 7.17)

Fig. 7.17



$$x = l\sin\theta\cos\phi$$
$$y = l\sin\theta\sin\phi$$
$$z = -l\cos\theta$$
$$V = mgz = -mgl\cos\theta \tag{1}$$
$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = l^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$$
$$T = \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \tag{2}$$
$$L = \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + mgl\cos\theta \tag{3}$$
$$\frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta}, \quad \frac{\partial L}{\partial\theta} = ml^2\sin\theta\cos\theta\dot{\phi}^2 - mgl\sin\theta \tag{4}$$
$$\frac{\partial L}{\partial\dot{\phi}} = ml^2\sin^2\theta\dot{\phi}, \quad \frac{\partial L}{\partial\phi} = 0 \tag{5}$$

The Lagrangian equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \text{ and } \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \tag{6}$$

$$\text{give} \quad \ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 + \frac{g}{l}\sin\theta = 0 \tag{7}$$

$$ml^2\frac{\mathrm{d}}{\mathrm{d}t}(\sin^2\theta\dot{\phi}) = 0 \tag{8}$$

$$\text{Hence} \quad \sin^2\theta\dot{\phi} = \text{constant} = C \tag{9}$$

and eliminating $\dot{\phi}$ in (7) with the use of (9) we get a differential equation in θ only.

$$\ddot{\theta} + \frac{g}{l}\sin\theta - C^2\frac{\cos\theta}{\sin^3\theta} = 0 \tag{10}$$

$$\text{The quantity } P_\phi = \frac{\partial L}{\partial \dot{\phi}} = ml^2\sin^2\theta\dot{\phi} \tag{5}$$

is a constant of motion and is recognized as the angular momentum of the system about the z-axis. It is conserved because torque is not produced either by gravity or the tension of the cord about the z-axis. Thus conservation of angular momentum is reflected in (5).

Two interesting cases arise. Suppose $\phi = \text{constant}$. Then $\dot{\phi} = 0$ and $C = 0$. In this case (10) reduces to

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{11}$$

which is appropriate for simple pendulum in which the bob oscillates in the vertical plane.

Suppose $\theta = \text{constant}$, then from (9) $\dot{\phi} = \text{constant}$. Putting $\ddot{\theta} = 0$ in (7) we get

$$\omega = \dot{\phi} = \sqrt{\frac{g}{l\cos\theta}} = \sqrt{\frac{g}{H}} \tag{12}$$

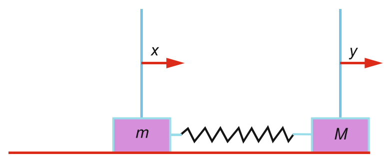
and time period

$$T = \frac{2\pi}{\omega} = \sqrt{\frac{H}{g}} \tag{13}$$

appropriate for the conical pendulum in which the bob rotates on horizontal plane with uniform angular velocity with the cord inclined at constant angle θ with the vertical axis.

7.11 The two generalized coordinates x and y are indicated in Fig. 7.18. The kinetic energy of the system comes from the motion of the blocks and potential energy from the coupling spring:

Fig. 7.18



$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\dot{y}^2 \tag{1}$$

$$V = \frac{1}{2}k(x-y)^2 \tag{2}$$

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\dot{y}^2 - \frac{1}{2}k(x-y)^2 \tag{3}$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial x} = -k(x-y) \tag{4}$$

$$\frac{\partial L}{\partial \dot{y}} = M\dot{y}, \quad \frac{\partial L}{\partial y} = k(x-y) \tag{5}$$

Lagrange's equations are written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \; \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \tag{6}$$

Using (4) and (5) in (6) we obtain the equations of motion

$$m\ddot{x} + k(x-y) = 0 \tag{7}$$

$$m\ddot{y} + k(y-x) = 0 \tag{8}$$

7.12 This problem involves two degrees of freedom. The coordinates are θ_1 and θ_2 (Fig. 7.19)

$$T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \tag{1}$$

$$v_1^2 = (l_1\dot{\theta}_1)^2 \tag{2}$$

$$v_2^2 = (l_1\dot{\theta}_1)^2 + (l_2\dot{\theta}_2)^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1) \text{ (by parallelogram law)} \tag{3}$$

For small angles, $\cos(\theta_2 - \theta_1) \simeq 1$

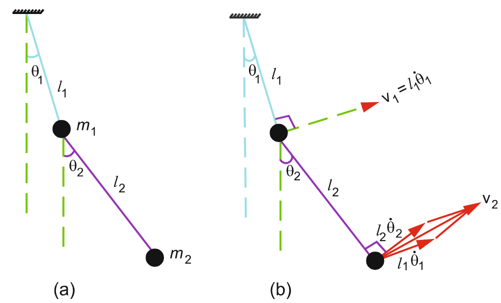


Fig. 7.19

$$T \simeq \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\right] \tag{4}$$

$$\begin{aligned} V &= m_1 g l_1(1-\cos\theta_1) + m_2 g l_1(1-\cos\theta_1) + m_2 g l_2(1-\cos\theta_2) \\ &\simeq m_1 g l_1\frac{\theta_1^2}{2} + \frac{m_2 g}{2}\left[l_1\theta_1^2 + l_2\theta_2^2\right] \end{aligned} \tag{5}$$

$$\begin{aligned} L =& \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\right] - m_1 g l_1\frac{\theta_1^2}{2} \\ &- \frac{m_2 g}{2}\left(l_1\theta_1^2 + l_2\theta_2^2\right) \end{aligned} \tag{6}$$

$$\frac{\partial L}{\partial\dot{\theta}_1} = m_1 l_1^2\dot{\theta}_1 + m_2 l_1^2\dot{\theta}_1 + m_2 l_1 l_2\dot{\theta}_2 \tag{7}$$

$$\frac{\partial L}{\partial\theta_1} = -m_1 g l_1\theta_1 - m_2 g l_1\theta_1 = -(m_1+m_2) g l_1\theta_1 \tag{8}$$

$$\frac{\partial L}{\partial\dot{\theta}_2} = m_2 l_2^2\dot{\theta}_2 + m_2 l_1 l_2\dot{\theta}_1 \tag{9}$$

$$\frac{\partial L}{\partial\theta_2} = -m_2 g l_2\theta_2 \tag{10}$$

Lagrange's equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) - \frac{\partial L}{\partial\theta_1} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right) - \frac{\partial L}{\partial\theta_2} = 0, \tag{11}$$

using (7, (8), (9) and 10) in (11) we obtain the equations of motion

$$(m_1+m_2) l_1\ddot{\theta}_1 + m_2 l_2\ddot{\theta}_2 + (m_1+m_2) g\theta_1 = 0 \tag{12}$$

$$l_2\ddot{\theta}_2 + g\theta_2 + l_1\ddot{\theta}_1 = 0 \tag{13}$$

7.13 Writing p_θ and p_ϕ for the generalized momenta, by (4) and (5) and V by (1) of prob. (7.10)

$$\frac{\partial L}{\partial \dot{\theta}} = P_\theta = ml^2\dot{\theta}, \quad \dot{\theta} = \frac{P_\theta}{ml^2} \tag{1}$$

$$\frac{\partial L}{\partial \dot{\phi}} = p_\phi = ml^2 \sin^2\theta\dot{\phi}, \quad \dot{\phi} = \frac{p_\phi}{ml^2 \sin^2\theta} \tag{2}$$

$$H = \Sigma \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \quad \text{or} \quad H + L = 2T = \Sigma \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}$$

$$2T = \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} + \dot{\phi}\frac{\partial L}{\partial \dot{\phi}} = \frac{1}{ml^2}\left(p_\theta^2 + \operatorname{cosec}^2\theta p_\phi^2\right) \tag{3}$$

$$\therefore \quad H = T + V = \frac{1}{2ml^2}\left(p_\theta^2 + \cos\mathrm{ec}^2\theta p_\phi^2\right) - mgl\cos\theta \tag{4}$$

The coordinate ϕ is ignorable, and therefore p_ϕ is a constant of motion determined by the initial conditions. We are then left with only two canonical equations to be solved. The canonical equations are

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2} \tag{5}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2}{ml^2}\frac{\cos\theta}{\sin^3\theta} - mgl\sin\theta \tag{6}$$

where p_ϕ is a constant of motion. By eliminating p_θ we can immediately obtain a second-order differential equation in θ as in prob. (7.10).

7.14 $$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 \tag{1}$$

$$\frac{\partial H}{\partial p} = \dot{q} = p \tag{2}$$

$$\frac{\partial H}{\partial q} = -\dot{p} = \omega^2 q \tag{3}$$

Differentiating (2)

$$\ddot{q} = \dot{p} = -\omega^2 q \tag{4}$$

Let $q = x$, then (4) can be written as

$$\ddot{x} + \omega^2 x = 0 \tag{5}$$

This is the equation for one-dimensional harmonic oscillator. The general solution is

$$x = A\sin(\omega t + \varepsilon) + B\cos(\omega t + \varepsilon) \tag{6}$$

which can be verified by substituting (6) in (5). Here A, B and ε are constants to be determined from initial conditions.

7.15 Let r, θ be the instantaneous polar coordinates of a planet of mass m revolving around a parent body of mass M:

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \tag{1}$$

$$V = GMm\left(\frac{1}{2a} - \frac{1}{r}\right) \tag{2}$$

where G is the gravitational constant and $2a$ is the major axis of the ellipse:

$$p_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r}, \quad \dot{r} = \frac{p_r}{m} \tag{3}$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \dot{\theta} = \frac{p_\theta}{mr^2} \tag{4}$$

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + GMm\left(\frac{1}{2a} - \frac{1}{r}\right) \tag{5}$$

and the Hamiltonian equations are

$$\frac{\partial H}{\partial p_r} = \frac{p_r}{m} = \dot{r}, \quad \frac{\partial H}{\partial r} = -\frac{p_\theta^2}{mr^3} + \frac{GMm}{r^2} = -\dot{p}_r \tag{6}$$

$$\frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}, \quad \frac{\partial H}{\partial \theta} = 0 = -\dot{p}_\theta \tag{7}$$

Two equations in (7) show that

$$p_\theta = \text{constant} = mr^2\dot{\theta} \tag{8}$$

meaning the constancy of angular momentum or equivalently the constancy of areal velocity of the planet (Kepler's second law of planetary motion).

Two equations in (6) yield

$$\ddot{r} = \frac{\dot{p}_r}{m} = \frac{p_\theta^2}{m^2r^3} - \frac{GMm}{r^2} = r\dot{\theta}^2 - \frac{GMm}{r^2} \tag{9}$$

Equation (9) describes the orbit of the planet (Kepler's first law of planetary motion)

7.16 $$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \tag{1}$$

$$V = \frac{1}{2}k(x_1 - x_2)^2 \tag{2}$$

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k(x_1 - x_2)^2 \tag{3}$$

Equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0 \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0 \tag{5}$$

Using (3) in (4) and (5)

$$m_1\ddot{x}_1 + k(x_1 - x_2) = 0 \tag{6}$$

$$m_2\ddot{x}_2 - k(x_1 - x_2) = 0 \tag{7}$$

It is assumed that the motion is periodic and can be considered as superposition of harmonic components of various amplitudes and frequencies. Let one of these harmonics be represented by

$$x_1 = A\sin\omega t, \quad \ddot{x}_1 = -\omega^2 A\sin\omega t \tag{8}$$

$$x_2 = B\sin\omega t, \quad \ddot{x}_2 = -\omega^2 B\sin\omega t \tag{9}$$

Substituting (8) and (9) in (6) and (7) we obtain

$$(k - m_1\omega^2)\,A - kB = 0$$
$$-\,kA + (k - m_2\omega^2)B = 0$$

The frequency equation is obtained by equating to zero the determinant formed by the coefficients of A and B:

$$\begin{vmatrix} (k - m_1\omega^2) & -k \\ -k & (k - m_2\omega^2) \end{vmatrix} = 0$$

Expansion of the determinant gives

$$m_1m_2\omega^4 - k(m_1 + m_2)\omega^2 = 0$$

or

$$\omega^2[m_1 m_2 \omega^2 - k(m_1 + m_2)] = 0$$

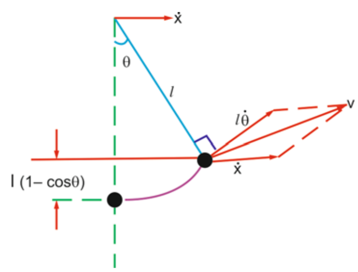
which yields the natural frequencies of the system:

$$\omega_1 = 0 \text{ and } \omega_2 = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}} = \sqrt{\frac{k}{\mu}}$$

where μ is the reduced mass. The frequency $\omega_1 = 0$ implies that there is no genuine oscillation of the block but mere translatory motion. The second frequency ω_2 is what one expects for a simple harmonic oscillator with a reduced mass μ.

7.17 Let $x(t)$ be the displacement of the block and $\theta(t)$ the angle through which the pendulum swings. The kinetic energy of the system comes from the motion of the block and the swing of the bob of the pendulum. The potential energy comes from the deformation of the spring and the position of the bob, Fig. 7.20.

Fig. 7.20



The velocity v of the bob is obtained by combining vectorially its linear velocity $(l\dot{\theta})$ with the velocity of the block $(\dot{x})$. The height through which the bob is raised from the equilibrium position is $l(1-\cos\theta)$, where l is the length of the pendulum:

$$v^2 = \dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta \tag{1}$$

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}) \tag{2}$$

$(\because \text{ for } \theta \to 0,\ \cos\theta \to 1)$

$$V = \frac{1}{2}kx^2 + mgl(1 - \cos\theta)$$

$$= \frac{1}{2}kx^2 + mgl\frac{\theta^2}{2} \tag{3}$$

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}) - \frac{1}{2}kx^2 - mgl\frac{\theta^2}{2} \tag{4}$$

Applying Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{5}$$

we obtain

$$(M + m)\ddot{x} + ml\ddot{\theta} + kx = 0 \tag{6}$$

$$l\ddot{\theta} + \ddot{x} + g\theta = 0 \tag{7}$$

7.18 Considering that at $t = 0$ the insect was in the middle of the rod, the coordinates of the insect x, y, z at time t are given by

$$x = (a + vt)\sin\theta\cos\phi$$
$$y = (a + vt)\sin\theta\sin\phi$$
$$z = (a + vt)\cos\theta$$

and the square of its velocity is

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v^2 + (a + vt)^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)$$

$$\therefore \quad T = \frac{2}{3}Ma^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{m}{2}\{v^2 + (a + vt)^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)\}$$
$$V = -Mga\cos\theta - mg(a + vt)\cos\theta + \text{constant}$$
$$L = T - V$$

The application of the Lagrangian equations to the coordinates θ and ϕ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{4}{3}Ma^2\dot{\theta} + m(a + vt)^2\dot{\theta}\right] - \left[\frac{4}{3}Ma^2 + m(a + vt)^2\right]\dot{\phi}^2\sin\theta\cos\theta$$
$$= -\{Ma + m(a + vt)\}g\sin\theta \tag{1}$$

$$\text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\left[\left\{\frac{4}{5}Ma^2 + m(a + vt)^2\right\}\dot{\phi}\sin^2\theta\right] = 0 \tag{2}$$

Equation (2) can be integrated at once as it is free from ϕ:

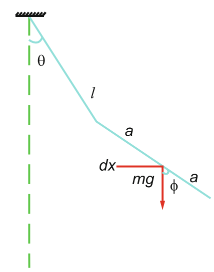
$$\left\{\frac{4}{5}Ma^2 + m(a + vt)^2\right\}\dot{\phi}\sin^2\theta = \text{constant} = C \tag{3}$$

Equation (3) is the equation for the constancy of angular momentum about the vertical axis.

When $\dot{\phi}$ in (2) is eliminated with the aid of (3) we obtain a second-order differential equation in θ.

7.19 Let ρ be the linear density of the rod, i.e. mass per unit length. Consider an infinitesimal element of length of the rod

Fig. 7.21



$$dT = \rho\omega^2 (l\sin\theta + x\sin\phi)^2 dx$$

$$T = \int dT = \rho\omega^2 \int_0^{2a} (l^2\sin^2\theta + 2lx\sin\theta\sin\phi + x^2\sin^2\phi)dx$$

$$= \omega^2\left(Ml^2\sin^2\theta + 2Mla\sin\theta\sin\phi + \frac{4}{3}Ma^2\sin^2\phi\right) \quad (1)$$

where we have substituted $\rho = M/2a$:

$$V = -Mg(l\cos\theta + a\cos\phi) \quad (2)$$

$$L = \omega^2\left(Ml^2\sin^2\theta + 2Mla\sin\theta\sin\phi + \frac{4}{3}Ma^2\sin^2\phi\right) + Mg(l\cos\theta + a\cos\phi) \quad (3)$$

$$\frac{\partial L}{\partial\dot{\theta}} = 0,\ \frac{\partial L}{\partial\theta} = \omega^2(2Ml^2\sin\theta\cos\theta + 2Mla\cos\theta\sin\phi) - Mgl\sin\theta \quad (4)$$

$$\frac{\partial L}{\partial\dot{\phi}} = 0,\ \frac{\partial L}{\partial\phi} = \omega^2\left(2Mla\sin\theta\cos\phi + \frac{8}{3}Ma^2\sin\phi\cos\phi\right) - Mga\sin\phi \quad (5)$$

The Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = 0 \quad (6)$$

yield

$$2\omega^2(l\sin\theta + a\sin\phi) = g\tan\theta \tag{7}$$

$$2\omega^2(l\sin\theta + \frac{4}{3}a\sin\phi) = g\tan\phi \tag{8}$$

Equation (7) and (8) can be solved to obtain θ and ϕ.

7.20 Express the Cartesian coordinates in terms of plane polar coordinates (r, θ)

$$x = r\cos\theta, \quad y = r\sin\theta \tag{1}$$

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \tag{2}$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \tag{3}$$

Square (2) and (3) and add

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2 \tag{4}$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \tag{5}$$

$$V = U(r) \tag{6}$$

$$\therefore \quad L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r) \quad \text{(Lagrangian)} \tag{7}$$

Generalized momenta:

$$p_k = \frac{\partial L}{\partial \dot{q}_k},\ p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r},\ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \tag{8}$$

Hamiltonian:

$$H = T + V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \tag{9}$$

Conservation of Energy: In general H may contain an explicit time dependence as in some forced systems. We shall therefore write $H = H(q, p, t)$. Then H varies with time for two reasons: first, because of its explicit dependence on t, second because the variable q and p are themselves functions of time. Then the total time derivative of H is

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} + \sum_{\beta=1}^{n}\frac{\partial H}{\partial q_\beta}\dot{q}_\beta + \sum_{\beta=1}^{n}\frac{\partial H}{\partial p_\beta}\dot{p}_\beta \tag{10}$$

Now Hamilton's equations are

$$\frac{\partial H}{\partial p_\beta} = \dot{q}_\beta, \;\; \frac{\partial H}{\partial q_\beta} = -\dot{p}_\beta \tag{11}$$

Using (11) in (10), we obtain

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} + \sum_{\beta=1}^{n}\left[\frac{\partial H}{\partial q_\beta}\frac{\partial H}{\partial p_\beta} - \frac{\partial H}{\partial p_\beta}\frac{\partial H}{\partial q_\beta}\right] \tag{12}$$

whence

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} \tag{13}$$

Equation (13) asserts that H changes with time only by virtue of its explicit time dependence. The net change is induced by the fact that the variation of q and p with time is zero.

Now in a conservative system, neither T nor V contains any explicit dependence on time.

Hence $\frac{\partial H}{\partial t} = 0$. It follows that

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0 \tag{14}$$

which leads to the law of conservation of energy

$$H = T + V = E = \text{constant} \tag{15}$$

The Hamiltonian formalism is amenable for finding various conservation laws.

Conservation of angular momentum: The Hamiltonian can be written as

$$H = \sum_{\beta=1}^{n} p_\beta \dot{q}_\beta - L \tag{16}$$

Using the polar coordinates (r, θ)

$$H = p_r\dot{r} + p_\theta\dot{\theta} - \left(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - U(r)\right) \tag{17}$$

Using (7) and (8) in (17)

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + U(r) \tag{18}$$

Now the second equation in (11) gives

$$-\dot{p}_\theta = \frac{\partial H}{\partial \theta} = 0 \qquad (\because \theta \text{ is absent in}(18)) \tag{19}$$

This leads to the conservation of angular momentum

$$p_\theta = J = \text{constant} \tag{20}$$

7.21 Let each mass be m.

(a)
$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \tag{1}$$

$$V = \frac{1}{2}kx^2 + \frac{1}{2}3k(x-y)^2 + \frac{1}{2}ky^2 = k(2x^2 - 3xy + 2y^2) \tag{2}$$

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - k(2x^2 - 3xy + 2y^2) \tag{3}$$

(b)
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{4}$$

and
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \tag{5}$$

yield

$$m\ddot{x} = -4kx + 3ky \tag{6}$$

$$m\ddot{y} = 3kx - 4ky \tag{7}$$

(c) Let $x = A\sin\omega\text{t}$ and $\quad y = B\sin\omega\text{t}$ (8)

$$\ddot{x} = -A\omega^2\sin\omega\text{t} \text{ and } \quad \ddot{y} = -B\omega^2\sin\omega\text{t} \tag{9}$$

Substituting (8) and (9) in (6) and (7) and simplifying we obtain

$$(4k - \omega^2 m)A - 3kB = 0 \tag{10}$$

$$-3kA + (4k - \omega^2 m)B = 0 \tag{11}$$

The frequency equation is obtained by equating to zero the determinant formed by the coefficients of A and B:

$$\begin{vmatrix} (4k - \omega^2 m) & -3k \\ -3k & (4k - \omega^2 m) \end{vmatrix} = 0 \tag{12}$$

Expanding the determinant

$$(4k - \omega^2 m)^2 - 9k^2 = 0 \tag{13}$$

This gives the frequencies

$$\omega_1 = \sqrt{\frac{k}{m}}, \quad \omega_2 = \sqrt{\frac{7k}{m}} \tag{14}$$

Periods of oscillations are

$$T_1 = \frac{2\pi}{\omega_1} = 2\pi\sqrt{\frac{m}{k}} \tag{15}$$

$$T_2 = \frac{2\pi}{\omega_2} = 2\pi\sqrt{\frac{m}{7k}} \tag{16}$$

If we put $\omega = \omega_1 = \sqrt{\frac{k}{m}}$ in (10) or (11) we get $A = B$ and if we put $\omega = \omega_2 = \sqrt{\frac{7k}{m}}$ in (10) or (11), we get $A = -B$. The first one corresponds to symmetric mode of oscillation and the second one to asymmetric one.

The normal coordinates q_1 and q_2 are formed by the linear combination of x and y:

$$q_1 = x - y, \quad q_2 = x + y \tag{17}$$

$$\therefore \quad x = \frac{q_1 + q_2}{2}, \quad y = \frac{q_2 - q_1}{2} \tag{18}$$

Substituting (18) in (6) and (7)

$$\frac{m}{2}(\ddot{q}_1 + \ddot{q}_2) = -2k(q_1 + q_2) + \frac{3k}{2}(q_2 - q_1) \tag{19}$$

$$\frac{m}{2}(\ddot{q}_2 - \ddot{q}_1) = \frac{3k}{2}(q_1 + q_2) - 2k(q_2 - q_1) \tag{20}$$

Adding (19) and (20), $m\ddot{q}_2 = -kq_2$ (21)

Subtracting (20) from (19), $m\ddot{q}_1 = -7kq_1$ (22)

Equation (21) is a linear equation in q_2 alone, with constant coefficients. Similarly (22) is a linear equation in q_1 with constant coefficients. Since the coefficients on the right sides are positive quantities, we note that both (21) and (22) are differential equations of simple harmonic motion having the frequencies given in (14). It is the characteristic of normal coordinates that when the equations of motion are expressed in terms of normal coordinates they are linear with constant coefficients, and each contains but one dependent variable.

Another feature of normal coordinates is that both kinetic energy and potential energy will have quadratic terms and the cross-products will be absent. Thus in this example, when (18) is used in (1) and (2) we get the expressions

$$T = \frac{m}{4}(\dot{q}_1^2 + \dot{q}_2^2), \ V = \frac{k}{4}(7q_1^2 + q_2^2)$$

The two normal modes are depicted in Fig. 7.22.

(a) Symmetrical with $\omega_1 = \sqrt{\frac{k}{m}}$ and (b) asymmetrical with $\omega_2 = \sqrt{\frac{7k}{m}}$

Fig. 7.22

7.22 **(a)** See Fig. 7.23 : $T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2)$ (1)

$$V = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_2 - x_1)^2$$

$$= k\left(x_1^2 - x_1x_2 + \frac{1}{2}x_2^2\right) \tag{2}$$

$$L = T - V = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - k\left(x_1^2 - x_1x_2 + \frac{1}{2}x_2^2\right) \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0 \tag{4}$$

$$m\ddot{x}_1 + k(2x_1 - x_2) = 0 \tag{5}$$

$$m\ddot{x}_2 + k(x_1 - x_1) = 0 \tag{6}$$

(5) and (6) are equations of motion

(b) Let the harmonic solutions be $x_1 = A\sin\omega t, \quad x_2 = B\sin\omega t$ (7)

Then $\ddot{x}_1 = -A\omega^2\sin\omega t, \quad \ddot{x}_2 = -B\omega^2\sin\omega t,$ (8)

Using (7) and (8) in (5) and (6) we get

$$(2k - m\omega^2)A - kB = 0 \tag{9}$$

$$-kA + (k - m\omega^2)B = 0 \tag{10}$$

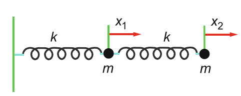


Fig. 7.23

The eigenfrequency equation is obtained by equating to zero the determinant formed by the coefficients of A and B:

$$\begin{vmatrix} (2k - m\omega^2) & -k \\ -k & (k - m\omega^2) \end{vmatrix} = 0 \tag{11}$$

Expanding the determinant, we obtain

$$m^2\omega^4 - 3mk\omega^2 + k^2 = 0$$

$$\omega^2 = \left(\frac{3 \pm \sqrt{5}}{2}\right)\frac{k}{m} \tag{12}$$

$$\therefore \quad \omega_1 = 1.618\sqrt{\frac{k}{m}}, \quad \omega_2 = 0.618\sqrt{\frac{k}{m}} \tag{13}$$

(c) Inserting $\omega = \omega_1$ in (10) we find $B = 1.618\,A$. This corresponds to a symmetric mode as both the amplitudes have the same sign.
Inserting $\omega = \omega_2$ in (10), we find $B = -0.618\,A$. This corresponds to asymmetric mode. These two modes of oscillation are depicted in Fig. 7.24 with relative sizes and directions of displacement.

(a) Symmetric mode $\omega_1 = 1.618\sqrt{\frac{k}{m}}$

(b) Asymmetric mode $\omega_2 = 0.618\sqrt{\frac{k}{m}}$

Fig. 7.24

7.23 **(a)** Let x_1 and x_2 be the displacements of the beads of mass $2m$ and m, respectively.

$$T = \frac{1}{2}(2m)\dot{x}_1^2 + \frac{1}{2}(m)\dot{x}_2^2 \tag{1}$$

$$V = \frac{1}{2} \cdot 2kx_1^2 + \frac{1}{2}k(x_2 - x_1)^2 \tag{2}$$

$$L = m\left(\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2\right) - k\left(\frac{3}{2}x_1^2 - x_1x_2 + \frac{1}{2}x_2^2\right) \tag{3}$$

Lagrange's equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0 \tag{4}$$

which yield equations of motion

$$2m\ddot{x}_1 + k(3x_1 - x_2) = 0 \tag{5}$$
$$m\ddot{x}_2 - k(x_1 - x_2) = 0 \tag{6}$$

(b) Let the harmonic solutions be

$$x_1 = A \sin \omega t, \quad x_2 = B \sin \omega t \tag{7}$$
$$\ddot{x}_1 = -A\omega^2 \sin \omega t, \quad \ddot{x}_2 = -B\omega^2 \sin \omega t \tag{8}$$

Substituting (7) and (8) in (5) and (6) we obtain

$$(3k - 2m\omega^2)A - kB = 0 \tag{9}$$
$$kA + (m\omega^2 - k)B = 0 \tag{10}$$

The frequency equation is obtained by equating to zero the determinant formed by the coefficients of A and B:

$$\begin{vmatrix} (3k - 2m\omega^2) & -k \\ k & m\omega^2 - k \end{vmatrix} = 0$$

Expanding the determinant

$$2m^2\omega^4 - 5km\,\omega^2 + 2k^2 = 0$$
$$\omega_1 = \sqrt{\frac{2k}{m}}, \quad \omega_2 = \sqrt{\frac{k}{2m}}$$

(c) Put $\omega = \omega_1 = \sqrt{\dfrac{2k}{m}}$ in (9) or (10). We find $B = -A$.

Put $\omega = \omega_2 = \sqrt{\dfrac{k}{2m}}$ in (9) or (10). We find $B = +2A$.

The two normal modes are sketched in Fig. 7.25.

Fig. 7.25

(a) Asymmetric mode

$$\omega_1 = \sqrt{\frac{2k}{m}} \quad B = -A$$

(b) Symmetric mode $\omega_2 = \sqrt{\dfrac{k}{2m}} \quad B = +2A$

7.24 There are three coordinates x_1, x_2 and x_3, Fig. 7.26:

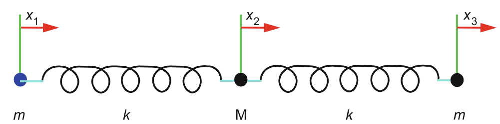


Fig. 7.26

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}Mx_2^2 + \frac{1}{2}m\dot{x}_3^2 \tag{1}$$

$$V = \frac{1}{2}k\left[(x_2 - x_1)^2 + (x_3 - x_2)^2\right]$$

$$= \frac{1}{2}k\left(x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + x_3^2\right) \tag{2}$$

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 - \frac{1}{2}k\left(x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + x_3^2\right) \tag{3}$$

Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0, \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0, \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_3}\right) - \frac{\partial L}{\partial x_3} = 0 \tag{4}$$

yield

$$m\ddot{x}_1 + k(x_1 - x_2) = 0 \tag{5}$$

$$M\ddot{x}_2 + k(-x_1 + 2x_2 - x_3) = 0 \tag{6}$$

$$m\ddot{x}_3 + k(-x_2 + x_3) = 0 \tag{7}$$

Let the harmonic solutions be

$$x_1 = A\sin\omega\mathrm{t}, \quad x_2 = B\sin\omega\mathrm{t}, \quad x_3 = C\sin\omega\mathrm{t} \tag{8}$$

$$\therefore \quad \ddot{x}_1 = -A\omega^2\sin\omega\mathrm{t},\ \ddot{x}_2 = -B\omega^2\sin\omega\mathrm{t},\ \ddot{x}_3 = -C\omega^2\sin\omega\mathrm{t} \tag{9}$$

Substituting (8) and (9) in (5), (6) and (7)

$$(k - m\omega^2)A - kB = 0 \tag{10}$$

$$-kA + (2k - M\omega^2)B - kC = 0 \tag{11}$$

$$-kB + (k - m\omega^2)C = 0 \tag{12}$$

The frequency equation is obtained by equating to zero the determinant formed by the coefficients of A, B and C

$$\begin{vmatrix} (k - m\omega^2) & -k & 0 \\ -k & (2k - M\omega^2) & -k \\ 0 & -k & (k - m\omega^2) \end{vmatrix} = 0$$

Expanding the determinant we obtain

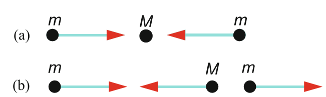
$$\omega^2 (k - m\omega^2)\,(\omega^2 Mm - 2km - Mk) = 0 \tag{13}$$

The frequencies are

$$\omega_1 = 0, \quad \omega_2 = \sqrt{\frac{k}{m}}, \quad \omega_3 = \sqrt{\frac{k(2m + M)}{Mm}} \tag{14}$$

The frequency $\omega_1 = 0$ simply means a translation of all the three particles without vibration. Ratios of amplitudes of the three particles can be found out by substituting ω_2 and ω_3 in (10), (11) and (12). Thus when $\omega = \omega_2\sqrt{\frac{k}{m}}$ is substituted in (10), we find the amplitude for the central atom $B = 0$. When $B = 0$ is used in (11) we obtain $C = -A$. This mode of oscillation is depicted in Fig. 7.27a.

Fig. 7.27



Substituting $\omega = \omega_3 = \sqrt{\frac{k(2m + M)}{Mm}}$ in (10) and (12) yields

$$B = -\left(\frac{2m}{M}\right) A = -\left(\frac{2m}{M}\right) C$$

Thus in this mode particles of mass m oscillate in phase with equal amplitude but out of phase with the central particle.

This problem has a bearing on the vibrations of linear molecules such as CO_2. The middle particle represents the C atom and the particles on either side represent O atoms. Here too there will be three modes of oscillations. One will have a zero frequency, $\omega_1 = 0$, and will correspond to a simple translation of the centre of mass. In Fig. 7.27a the mode with $\omega_1 = \omega_2$ is such that the carbon atom is stationary, the oxygen atoms oscillating back and forth in opposite phase with equal amplitude. In the third mode which has frequency ω_3, the carbon atom undergoes motion with respect to the centre of mass and is in opposite phase from that of the two oxygen atoms. Of these two modes only ω_3 is observed optically. The frequency ω_2 is not observed because in this mode, the electrical centre of the system is always coincident with the centre of mass, and so there is no oscillating dipole moment Σer available. Hence dipole radiation is not emitted for this mode. On the other hand in the third mode characterized by ω_3 such a moment is present and radiation is emitted.

7.25 **(a)** $$y = \frac{x^2}{l} \tag{1}$$

$$\dot{y} = \frac{2x \cdot \dot{x}}{l} \tag{2}$$

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{x}^2\left(1 + \frac{4x^2}{l^2}\right)$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2\left(1 + \frac{4x^2}{l^2}\right)$$

$$V = mgy = \frac{mgx^2}{l}$$

$$L = T - V = \frac{1}{2}m\dot{x}^2\left(1 + \frac{4x^2}{l^2}\right) - \frac{mgx^2}{l}$$

(b) $$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad \dot{r} = \frac{p_r}{m}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \dot{\theta} = \frac{p_\theta}{mr^2}$$

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + U(r)$$

7.26 **(a)** At any instant the velocity of the block is $\dot{x}$ on the plane surface. The linear velocity of the pendulum with respect to the block is $l\dot{\theta}$, Fig. 7.28. The velocity $l\dot{\theta}$ must be combined vectorially with $\dot{x}$ to find the velocity v or the pendulum with reference to the plane:

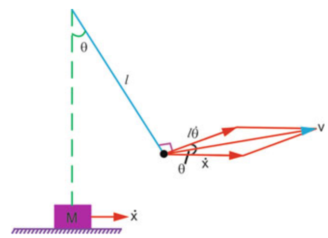
$$v^2 = \dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}\, l\dot{\theta}\, \cos\theta \tag{1}$$

The total kinetic energy of the system

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}\, l\dot{\theta}\, \cos\theta) \tag{2}$$

Taking the zero level of the potential energy at the pivot of the pendulum, the potential energy of the system which comes only from the pendulum is

Fig. 7.28



$$V = -mgl\cos\theta \tag{3}$$

$$\therefore \quad L = T - V = \frac{1}{2}(M+m)\dot{x}^2 + ml\cos\theta\,\dot{x}\dot{\theta} + \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta \tag{4}$$

where we have used (2) and (3).

(b) For small angles $\cos\theta \simeq 1 - \frac{\theta^2}{2}$, in the first approximation, and $\cos\theta \simeq 1$, in the second approximation. Thus in this approximation (4) becomes

$$L = \frac{1}{2}(M+m)\dot{x}^2 + ml\dot{x}\dot{\theta} + \frac{1}{2}ml^2\dot{\theta}^2 + mgl\left(1 - \frac{\theta^2}{2}\right) \tag{5}$$

(c) The Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0,\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{6}$$

lead to the equations of motion

$$\ddot{x} + l\ddot{\theta} + g\theta = 0 \tag{7}$$

$$(M+m)\ddot{x} + ml\ddot{\theta} = 0 \tag{8}$$

(d) Eliminating $\ddot{x}$ between (7) and (8) and simplifying

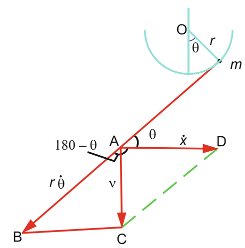
$$\ddot{\theta} + \frac{(M+m)}{M}\frac{g}{l}\theta = 0 \tag{9}$$

This is the equation for angular simple harmonic motion whose frequency is given by

$$\omega = \sqrt{\frac{(M+m)g}{Ml}} \tag{10}$$

7.27 **(a)** First, we assume that the bowl does not move. Both kinetic energy and potential energy arise from the particle alone. Taking the origin at O, the centre of the bowl, Fig. 7.29, the linear velocity of the particle is $v = r\dot{\theta}$. There is only one degree of freedom:

Fig. 7.29



$$T = \frac{1}{2}mv^2 = \frac{1}{2}mr^2\dot{\theta}^2 \tag{1}$$

$$V = -mgr\cos\theta \tag{2}$$

$$L = \frac{1}{2}mr^2\dot{\theta}^2 + mgr\cos\theta \tag{3}$$

Lagrange's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{4}$$

becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{5}$$

which yields the equation of motion

$$mr^2\ddot{\theta} + mgr\sin\theta = 0$$

$$\text{or}\quad \ddot{\theta} + \frac{g}{r}\sin\theta = 0 \quad \text{(equation of motion)} \tag{6}$$

For small angles, $\sin\theta \simeq \theta$. Then (6) becomes

$$\ddot{\theta} + \frac{g}{r}\theta = 0 \tag{7}$$

which is the equation for simple harmonic motion of frequency $\omega = \sqrt{\frac{g}{r}}$ or time period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r}{g}} \tag{8}$$

(b) **(i)** The bowl can now slide freely along the x-direction with velocity $\dot{x}$. The velocity of the particle with reference to the table is obtained by adding $l\dot{\theta}$ to $\dot{x}$ vectorially, Fig. 7.29. The total kinetic energy then comes from the motion of both the particle and the bowl. The potential energy, however, is the same as in (a):

$$v^2 = r^2\dot{\theta}^2 + x^2 - 2r\,\dot{\theta}\dot{x}\cos(180 - \theta) \tag{9}$$

from the diagonal AC of the parallelogram ABCD

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\,(r^2\dot{\theta}^2 + \dot{x}^2 - 2r\dot{x}\dot{\theta}\cos\theta) \tag{10}$$

$$V = -mgr\cos\theta \tag{11}$$

$$\begin{aligned} L &= T - V \qquad \text{(Lagrangian)} \\ &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(r^2\dot{\theta}^2 + \dot{x}^2 - 2r\dot{x}\dot{\theta}\cos\theta) + mgr\cos\theta \end{aligned} \tag{12}$$

(ii) and (iii).
In the small angle approximation the cos θ in the kinetic energy can be neglected as $\cos\theta \to 1$ but can be retained in the potential energy in order to avoid higher order terms.
Equation (12) then becomes

$$L = \frac{1}{2}(M + m)\dot{x}^2 - mr\dot{x}\dot{\theta} + \frac{1}{2}mr^2\dot{\theta}^2 + mgr\cos\theta \tag{13}$$

We have now two degrees of freedom, x and θ, and the corresponding Lagrange's equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{14}$$

which yield the equations of motion

$$(M+m)\ddot{x} - mr\,\ddot{\theta} = 0 \tag{15}$$

$$\ddot{x} - r\,\ddot{\theta} - g\,\theta = 0 \tag{16}$$

Equations (15) and (16) constitute the equations of motion. Eliminating $\ddot{x}$ we obtain

$$\ddot{\theta} + \left(\frac{M+m}{M}\right)\frac{g}{r}\theta = 0 \tag{17}$$

which is the equation for simple harmonic motion with frequency $\omega = \sqrt{\frac{(M+m)}{M}\frac{g}{r}}$ and time period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{M}{(M+m)}\frac{r}{g}} \tag{18}$$

On comparing (18) with (8) it is observed that the period of oscillation is smaller by a factor $[M/(M+m)^{1/2}]$ as compared to the case where the bowl is fixed.

7.28 Take the differential of the Lagrangian

$$L(q_1, \ldots, q_n,\ \dot{q}_1, \ldots, \dot{q}_n,\ t)$$

$$\mathrm{d}L = \sum_{r=1}^{n}\left(\frac{\partial L}{\partial q_r}\mathrm{d}q_r + \frac{\partial L}{\partial \dot{q}_r}\mathrm{d}\dot{q}_r\right) + \frac{\partial L}{\partial t}\mathrm{d}t \tag{1}$$

Now the Lagrangian equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0 \tag{2}$$

and the generalized momenta are defined by

$$\frac{\partial L}{\partial \dot{q}_r} = p_r \tag{3}$$

Using (3) in (2) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_r}\right) = \dot{p}_r \tag{4}$$

Using (2), (3) and (4) in (1)

$$\mathrm{d}L - \sum_{r=1}^{n} (\dot{p}_r \mathrm{d}q_r + p_r \mathrm{d}\dot{q}_r) + \frac{\partial L}{\partial t}\mathrm{d}t \tag{5}$$

Equation (5) can be rearranged in the form

$$\mathrm{d}\left(\sum_{r=1}^{n} p_r \dot{q}_r - L\right) = \sum_{r=1}^{n} (\dot{q}_r \mathrm{d}p_r - \dot{p}_r \mathrm{d}q_r) - \frac{\partial L}{\partial t}\mathrm{d}t \tag{6}$$

The Hamiltonian function H is defined by

$$H = \sum_{r=1}^{n} p_r \dot{q}_r - L(q_1, \ldots, q_n,\ \dot{q}_1, \ldots, \dot{q}_n,\ t) \tag{7}$$

Equation (6) therefore my be written as

$$\mathrm{d}H = \sum_{r=1}^{n} (\dot{q}_r dp_r - \dot{p}_r dq_r) - \frac{\partial L}{\partial t}\mathrm{d}t \tag{8}$$

While the Lagrangian function L is an explicit function of $q_1, \ldots, q_n$, $\dot{q}_1, \ldots, \dot{q}_n$ and t, it is usually possible to express H as an explicit function only of $q_1, \ldots, q_n,\ p_1, \ldots, p_n,\ t$, that is, to eliminate the n generalized velocities from (7). The n equation of type (3) are employed for this purpose. Each provides one of the p's in terms of the $\dot{q}'s$. Assuming that the elimination of the generalized velocities is possible, we may write

$$H = H(q_1, \ldots, q_n, p_1, \ldots, p_n, t) \tag{9}$$

H now depends explicitly on the generalized coordinates and generalized momenta together with the time. Therefore, taking the differential $\mathrm{d}H$, we obtain

$$\mathrm{d}H = \sum_{r=1}^{n} \left(\frac{\partial H}{\partial q_r}\mathrm{d}q_r + \frac{\partial H}{\partial p_r}\mathrm{d}p_r\right) + \frac{\partial H}{\partial t}\mathrm{d}t \tag{10}$$

Comparing (8) and (10), we have the relations

$$\frac{\partial H}{\partial p_r} = \dot{q}_r, \quad \frac{\partial H}{\partial q_r} = -\dot{p}_r \tag{11}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{12}$$

Equations (11) are called Hamilton's canonical equations. These are $2n$ is number. For a system with n degrees of freedom the n Lagrangian equations (2) of the second order are replaced by $2n$ Hamiltonian equations of the first order. We note from the second equation of (11) that if any coordinate q_l is not contained explicitly in the Hamiltonian function H, the conjugate momentum p_l is a constant of motion. Such coordinates are called ignorable coordinates.

7.29 The generalized momentum p_r conjugate to the generalized coordinate q_r is defined as $\frac{\partial L}{\partial \dot{q}_r} = p_r$. If the Lagrangian of a dynamical system does not contain a certain coordinate, say q_s, explicitly then p_s is a constant of motion.

(a) The kinetic energy arises only from the motion of the particle P on the table as the particle Q is stationary. The potential energy arises from the particle Q alone.

When P is at distance r from the opening, Q will be at a depth l–x from the opening:

$$T = \frac{1}{2}mv_p^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \tag{1}$$

$$V = -mg(l - r) \tag{2}$$

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mg(l - r) \tag{3}$$

For the two coordinates r and θ, Lagrange's equations take the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{4}$$

Equations (4) yield

$$\ddot{r} = r\dot{\theta}^2 - g \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) = 0$$

$$\therefore \quad r^2\dot{\theta} = C = \text{constant} \tag{6}$$

Equations (5) and (6) constitute the equations of motion.

(b) Initial conditions: At $r = a$, $r\dot{\theta} = \sqrt{ag}$

$$\therefore \quad \dot{\theta} = \sqrt{\frac{g}{a}} \tag{7}$$

Using (7) in (6) with $r = a$, we obtain

$$C^2 = a^3 g \tag{8}$$

Using (6) and (8) in (5)

$$\ddot{r} = \frac{rc^2}{r^4} - g = \frac{a^3 g}{r^3} - g \tag{9}$$

(c) **(i)** Since Q does not move, P must be at constant distance $r = a$ from the opening. Therefore P describes a circle of constant radius a.

(ii) Let P be displaced by a small distance x from the stable circular orbit of radius a, that is

$$r = a + x \tag{10}$$

$$\therefore \quad \ddot{r} = \ddot{x} \tag{11}$$

Using (10) and (11) in (9)

$$\ddot{x} = g\left[\frac{a^3}{(a+x)^3} - 1\right] = g\left[\left(1 + \frac{x}{a}\right)^{-3} - 1\right]$$

$$\text{or} \quad \ddot{x} \simeq -\frac{3gx}{a}$$

$$\text{or} \quad \ddot{x} + \frac{3gx}{a} = 0 \tag{12}$$

which is the equation for simple harmonic motion. Thus the particle P when slightly displaced from the stable orbit of radius a executes oscillations around $r = a$.

This aspect of oscillations has a bearing on the so-called betatron oscillations of ions in circular machines which accelerate charged particles to high energies. If the amplitudes of the betatron oscillations are large then they may hit the wall of the doughnut and be lost, resulting in the loss of intensity of the accelerated particles.

7.30 **(i)**

$$x = a\cos\theta,\ \ y = b\sin\theta, \quad r^2 = a^2\cos^2\theta + b^2\sin^2\theta \tag{1}$$

$$\dot{x} = -a\dot{\theta}\sin\theta,\ \ \dot{y} = b\dot{\theta}\cos\theta \tag{2}$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(a^2\sin^2\theta + b^2\cos^2\theta)\dot{\theta}^2 \tag{3}$$

$$V = mgy + \frac{1}{2}kr^2 = mgb\sin\theta + \frac{1}{2}k(a^2\cos^2\theta + b^2\sin^2\theta) \tag{4}$$

$$L = \frac{1}{2}m(a^2\sin^2\theta + b^2\cos^2\theta)\dot{\theta}^2 - mgb\sin\theta - \frac{1}{2}k(a^2\cos^2\theta + b^2\sin^2\theta) \tag{5}$$

Lagrange's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

yields

$$m(a^2 \sin^2 \theta + b^2 \cos^2 \theta)\ddot{\theta} + m(a^2 \sin\theta \cos\theta - b^2 \sin\theta \cos\theta)\dot{\theta}^2$$
$$+ mgb \cos\theta + k(-a^2 \sin\theta \cos\theta + b^2 \sin\theta \cos\theta) = 0$$

or $\quad m(a^2 \sin^2 \theta + b^2 \cos^2 \theta)\ddot{\theta} - (a^2 - b^2)(k - m\dot{\theta}^2) \sin\theta \cos\theta$

$$+ mgb \cos\theta = 0 \tag{6}$$

(ii) Equilibrium point is located where the force is zero, or $\partial V/\partial \theta = 0$. Differentiating (4)

$$\frac{\partial V}{\partial \theta} = mgb \cos\theta + k(b^2 - a^2) \sin\theta \cos\theta \tag{7}$$

Clearly the right-hand side of (7) is zero for $\theta = \pm\frac{\pi}{2}$
Writing (7) as

$$[mgb + k(b^2 - a^2) \sin\theta] \cos\theta \tag{8}$$

Another equilibrium point is obtained when

$$\sin\theta = \frac{mgb}{k(a^2 - b^2)} \tag{9}$$

provided $a > b$.

(iii) An equilibrium point will be stable if $\frac{\partial^2 V}{\partial \theta^2} > 0$ and will be unstable if $\frac{\partial^2 V}{\partial \theta^2} < 0$. Differentiating (8) again we have

$$\frac{\partial^2 V}{\partial \theta^2} = k(b^2 - a^2)(\cos^2\theta - \sin^2\theta) - mgb \sin\theta \tag{10}$$

For $\theta = \frac{\pi}{2}$, (10) reduces to

$$k(a^2 - b^2) - mgb \tag{11}$$

Expression (11) will be positive if $a^2 > b^2 + \frac{mgb}{k}$, and $\theta = \frac{\pi}{2}$ will be a stable point.

For $\theta = -\dfrac{\pi}{2}$, (10) reduces to

$$k(a^2 - b^2) + mgb \tag{12}$$

Expression (12) will be positive if $a^2 > b^2 - \dfrac{mgb}{k}$, and $\theta = -\dfrac{\pi}{2}$ will be a stable point.

(iv) $T = 2\pi\sqrt{\dfrac{A(\theta)}{V''(\theta)}}$

$V''\left(-\dfrac{\pi}{2}\right) = k(a^2 - b^2) + mgb$, by (12)

$A(\theta)$ is the coefficient of $\dfrac{1}{2}\dot{\theta}^2$ in (3)

$\therefore \quad A(\theta) = m(a^2 \sin^2\theta + b^2\cos^2\theta)$

$\therefore \quad A\left(-\dfrac{\pi}{2}\right) = ma^2$

$\therefore \quad T = 2\pi\sqrt{\dfrac{ma^2}{k(a^2 - b^2) + mgb}}$

7.31 In prob. (7.12) the following equations were obtained:

$$(m_1 + m_2)\, l_1\ddot{\theta}_1 + m_2 l_2\ddot{\theta}_2 + (m_1 + m_2)g\theta_1 = 0 \tag{1}$$

$$l_2\ddot{\theta}_2 + g\theta_2 + l_1\ddot{\theta}_1 = 0 \tag{2}$$

For $l_1 = l_2 = l$ and $m_1 = m_2 = m$, (1) and (2) become

$$2l\ddot{\theta}_1 + l\ddot{\theta}_2 + 2g\theta_1 = 0 \tag{3}$$

$$l\ddot{\theta}_2 + l\ddot{\theta}_1 + g\theta_2 = 0 \tag{4}$$

The harmonic solutions of (3) and (4) are written as

$$\theta_1 = A\sin\omega\mathrm{t}, \quad \theta_2 = B\sin\omega\mathrm{t} \tag{5}$$

$$\ddot{\theta}_1 = -A\omega^2\sin\omega\mathrm{t}, \quad \ddot{\theta}_2 = -B\omega^2\sin\omega\mathrm{t} \tag{6}$$

Substituting (5) and (6) in (3) and (4) and simplifying

$$2(l\omega^2 - g)A + l\omega^2 B = 0 \tag{7}$$

$$l\omega^2 A + (l\omega^2 - g)B = 0 \tag{8}$$

The frequency equation is obtained by equating to zero the determinant formed by the coefficients of A and B:

$$\begin{vmatrix} 2(l\omega^2 - g) & l\omega^2 \\ l\omega^2 & (l\omega^2 - g) \end{vmatrix} = 0$$

Expanding the determinant

$$l^2\omega^4 - 4\,\mathrm{lg}\,\omega^2 + 2g^2 = 0$$

$$\omega^2 = \left(2 \pm \sqrt{2}\right)\frac{g}{l}$$

$$\therefore \quad \omega = \sqrt{\left(2 \pm \sqrt{2}\right)\frac{g}{l}}$$

$$\therefore \quad \omega_1 = 0.76\sqrt{\frac{g}{l}},\ \omega_2 = 1.85\sqrt{\frac{g}{l}}$$

7.32 While the method employed in prob. (6.46) was based on forces or torques, that is, Newton's method, the Lagrangian method is based on energy:

$$T = \frac{1}{2}m\,(\dot{x}_1^2 + \dot{x}_2^2) \tag{1}$$

For small angles $\dot{y}_1$ and $\dot{y}_2$ are negligibly small

$$V = \frac{1}{2}k(x_1 - x_2)^2 + mgb(1 - \cos\theta_1) + mgb(1 - \cos\theta_2)$$

For small angles $1 - \cos\theta_1 = \dfrac{\theta_1^2}{2} = \dfrac{x_1^2}{2b^2}$.

Similarly, $1 - \cos\theta_2 = \dfrac{x_2^2}{2b^2}$

$$\therefore \quad V = \frac{1}{2}\,k(x_1 - x_2)^2 + \frac{mg}{2b}(x_1^2 + x_2^2) \tag{2}$$

$$\therefore \quad L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}k(x_1^2 - 2x_1x_2 + x_2^2) - \frac{mg}{2b}(x_1^2 - x_2^2) \tag{3}$$

The Lagrange's equations for the coordinates x_1 and x_2 are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0 \tag{4}$$

Using (3) in (4) we obtain

$$m\ddot{x}_1 + \left(k + \frac{mg}{b}\right)x_1 - kx_2 = 0 \tag{5}$$

$$m\ddot{x}_2 - kx_1 + \left(k + \frac{mg}{b}\right)x_2 = 0 \tag{6}$$

Assuming that x_1 and $x_2$are periodic with the same frequency but different amplitudes, let

$$x_1 = A\sin\omega t, \quad \ddot{x}_1 = -A\omega^2 \sin\omega t \tag{7}$$

$$x_2 = B\sin\omega t, \quad \ddot{x}_2 = -B\omega^2 \sin\omega t \tag{8}$$

Substituting (7) and (8) in (5) and (6) and simplifying

$$\left(k + \frac{mg}{b} - m\omega^2\right) A - kB = 0 \tag{9}$$

$$-kA + \left(k + \frac{mg}{b} - m\omega^2\right) B = 0 \tag{10}$$

The frequency equation is obtained by equating to zero the determinant formed by the coefficients of A and B:

$$\begin{vmatrix} \left(k + \frac{mg}{b} - m\omega^2\right) & -k \\ -k & \left(k + \frac{mg}{b} - m\omega^2\right) \end{vmatrix} = 0$$

Expanding the determinant and solving gives

$$\omega_1 = \sqrt{\frac{g}{b}} \text{ and } \omega_2 = \sqrt{\frac{g}{b} + \frac{2k}{m}},$$

In agreement with the results of prob. (6.46).

7.33 Let the origin be at the fixed point O and OB be the diameter passing through C the centre of the circular wire, Fig. 7.30. The position of m is indicated by the angle θ subtended by the radius CP with the diameter OB. Only one general coordinate $q = \theta$ is sufficient for this problem. Let $\phi = \omega t$ be the angle which the diameter OB makes with the fixed x-axis. From the geometry of the diagram (Fig. 7.30) the coordinates of m are expressed as

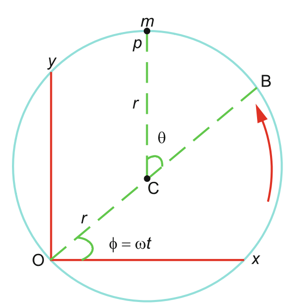


Fig. 7.30

$$x = r\cos\omega t + r\cos(\theta + \omega t) \tag{1}$$
$$y = r\sin\omega t + r\sin(\theta + \omega t) \tag{2}$$

The velocity components are found as

$$\dot{x} = -r\,\omega\sin\omega t - r(\dot{\theta} + \omega)\sin(\theta + \omega t) \tag{3}$$
$$\dot{y} = r\,\omega\cos\omega t - r(\dot{\theta} + \omega)\cos(\theta + \omega t) \tag{4}$$

Squaring and adding and simplifying we obtain

$$\dot{x}^2 + \dot{y}^2 = r^2\omega^2 + r^2(\dot{\theta} + \omega)^2 + 2r^2\omega(\dot{\theta} + \omega)\cos\theta \tag{5}$$
$$\therefore \quad T = \frac{1}{2}mr^2[\omega^2 + (\dot{\theta} + \omega)^2 + 2\omega(\dot{\theta} + \omega)\cos\theta] \tag{6}$$

Here $V = 0$, and so $L = T$. The Lagrange's equation then simply reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = 0 \tag{7}$$

Cancelling the common factor mr^2 (7) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\theta} + \omega + \omega\cos\theta) + \omega(\dot{\theta} + \omega)\sin\theta = 0 \tag{8}$$

which reduces to

$$\ddot{\theta} + \omega^2\sin\theta = 0 \tag{9}$$

which is the equation for simple pendulum. Thus the bead oscillates about the rotating line OB as a pendulum of length $r = a/\omega^2$.

7.34 **(a)** The velocity v of mass m relative to the horizontal surface is given by combining $\dot{s}$ with $\dot{x}$. The components of the velocity v are

$$v_x = \dot{x} + \dot{s}\cos\alpha \tag{1}$$
$$v_y = -\dot{s}\sin\alpha \tag{2}$$
$$\therefore \quad v^2 = v_x^2 + v_y^2 = \dot{x}^2 + \dot{s}^2 + 2\dot{x}\,\dot{s}\cos\alpha \tag{3}$$

Kinetic energy of the system

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\,(\dot{s}^2 + \dot{x}^2 + 2\dot{s}\dot{x}\cos\alpha) \tag{4}$$

Potential energy comes exclusively from the mass m (spring energy + gravitational energy)

$$V = \frac{k}{2}(s-l)^2 + mg(h - s\ \sin\alpha) \tag{5}$$

$$L = T - V = \frac{(M+m)}{2}\dot{x}^2 + \frac{1}{2}m\dot{s}^2 + m\dot{x}\dot{s}\cos\alpha - \frac{k}{2}(s-l)^2 - mg(h - s\sin\alpha) \tag{6}$$

(b) The generalized coordinates are $q_1 = x$ and $q_2 = s$. The Lagrange's equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = 0 \tag{7}$$

Using (6) in (7), equations of motion become

$$(M+m)\ddot{x} + m\ddot{s}\cos\alpha = 0 \tag{8}$$

$$m\ddot{x}\cos\alpha + m\ddot{s} + k(s - s_0) = 0 \tag{9}$$

where $s_0 = l + (mg\sin\alpha)/k$.

Let $x = A\sin\omega\mathrm{t}$ and $s - s_0 = B\sin\omega\mathrm{t}$ (10)

$$\ddot{x} = -\omega^2 A\sin\omega\mathrm{t}, \quad \ddot{s} = -B\omega^2\sin\omega\mathrm{t} \tag{11}$$

Substituting (10) and (11) in (8) and (9) we obtain

$$A(M+m) + B\cos\alpha = 0 \tag{12}$$

$$Am\omega^2\cos\alpha + B(m\omega^2 - k) = 0 \tag{13}$$

Eliminating A and B, we find

$$\omega = \sqrt{\frac{k(M+m)}{m(M + m\ \sin^2\alpha)}} \tag{14}$$

Components of the velocity of the ball as observed on the table are

7.35 $v_x = \dot{x} + \dot{y}\cos\alpha$ (1)

$$v_y = \dot{y}\sin\alpha \tag{2}$$

$$v^2 = v_x^2 + v_y^2 = \dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{y}\cos\alpha \tag{3}$$

$$T(\text{ball}) = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}\times\frac{2}{5}mr^2\omega^2 = \frac{1}{2}mv^2 + \frac{1}{5}mv^2 = \frac{7}{10}mv^2 \tag{4}$$

$$T(\text{wedge}) = \frac{1}{2}(M+m)\dot{x}^2 \tag{5}$$

$$\therefore \quad T\,(\text{system}) = \frac{7}{10}m(\dot{x}^2 + \dot{y}^2 + 2\dot{x}\,\dot{y}\cos\alpha) + \frac{1}{2}(M+m)\dot{x}^2 \tag{6}$$

$$V\,(\text{system}) = V\,(\text{ball}) = -mgy\sin\alpha \tag{7}$$

$$L = \frac{7}{10}m(\dot{x}^2 + \dot{y}^2 + 2\dot{x}\,\dot{y}\cos\alpha) + \frac{1}{2}(M+m)\dot{x}^2 + mgy\sin\alpha \tag{8}$$

Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \tag{9}$$

become

$$\frac{7m}{5}\ddot{x} + \frac{7}{5}m\ddot{y}\cos\alpha + (M+m)\ddot{x} = 0 \tag{10}$$

$$\frac{7m}{5}\ddot{y} + \frac{7}{5}m\ddot{x}\cos\alpha - mg\sin\alpha = 0 \tag{11}$$

Solving (10) and (11) and simplifying

$$\ddot{x} = -\frac{5mg\sin\alpha\cos\alpha}{5M + (5 + 7\sin^2\alpha)m} \quad \text{(for the wedge)}$$

$$\ddot{y} = \frac{5(5M + 12m)g\sin\alpha}{7(5M + (5 + 7\sin^2\alpha)m)} \quad \text{(for the ball)}$$

For $M = m$ and $\alpha = \pi/4$

$$\ddot{x} = \frac{5g}{27}$$

$$\ddot{y} = \frac{85\sqrt{2}}{189}$$