4.2 Problems

4.2.1 Moment of Inertia

4.1 Calculate the moment of inertia of a solid sphere about an axis through its centre.

4.2 Two particles of masses m_1 and m_2 are connected by a rigid massless rod of length r to constitute a dumbbell which is free to move in a plane. Show that the moment of inertia of the dumbbell about an axis perpendicular to the plane passing through the centre of mass is μr^2 where μ is the reduced mass.

4.3 Show that the moment of inertia of a right circular cone of mass M, height h and radius 'a' about its axis is $3Ma^2/10$.

4.4 Calculate the moment of inertia of a right circular cylinder of radius R and length h about a line at right angles to its axis and passing through the middle point.

4.5 Show that the radius of gyration about an axis through the centre of a hollow cylinder of external radius 'a' and internal radius 'b' is $\sqrt{\frac{2}{5}\left(\frac{a^5 - b^5}{a^3 - b^3}\right)}$.

4.6 Calculate the moment of inertia of a thin rod **(a)** about an axis passing through its centre and perpendicular to its length **(b)** about an end perpendicular to the rod.

4.7 Show that the moment of inertia of a rectangular plate of mass m and sides $2a$ and $2b$ about the diagonal is $\frac{2}{3}\frac{ma^2b^2}{(a^2 + b^2)}$

4.8 Lengths of sides of a right angle triangular lamina are 3, 4 and 5 cm, and the moment of inertia of the lamina about the sides I_1, I_2 and I_3, respectively (Fig. 4.1). Show that $I_1 > I_2 > I_3$.

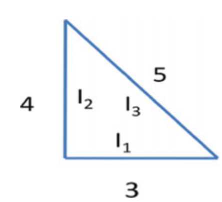


Fig. 4.1

4.9 A circular disc of radius R and thickness $R/6$ has moment of inertia I about the axis perpendicular to the plane and passing through its centre. The disc is melted and recasted into a solid sphere. Show that the moment of inertia of the sphere about its diameter is $I/5$.

4.10 Calculate the moment of inertia of a hollow sphere of mass M and radius R about its diameter.

4.11 Use the formula for moment of inertia of a uniform sphere about its diameter $\left(I = \frac{2}{5}MR^2\right)$ to deduce the moment of inertia of a thin hollow sphere about the axis passing through the centre.

4.2.2 Rotational Motion

4.12 A solid cylinder of mass m and radius R rolls down an inclined plane of height h without slipping. Find the speed of its centre of mass when the cylinder reaches the bottom.

4.13 A star has initially a radius of 6×10^8 m and a period of rotation about its axis of 30 days. Eventually it evolves into a neutron star with a radius of only 10^4 m and a period of 0.1 s. Assuming that the mass has not changed, find the ratio of initial and final **(a)** angular momentum and **(b)** kinetic energy.

4.14 A uniform solid ball rolls down a slope. If the ball has a diameter of 0.5 m and a mass of 0.1 kg, find the following:

(a) The equation which describes the velocity of the ball at any time, given that it starts from rest. Clearly state any assumptions you make.
(b) If the slope has an incline of 30° to the horizontal, what is the speed of the ball after it travels 3 m?
(c) At this point, what is the angular momentum of the ball?
(d) If the coefficient of friction between the ball and the slope is 0.26, what is the maximum angle of inclination the slope could have which still allows the ball to roll?

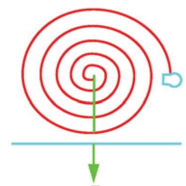
[University of Durham 2000]

4.15 **(a)** Show that the least coefficient of friction for an inclined plane of angle θ in order that a solid cylinder will roll down without slipping is $\frac{1}{3}\tan\theta$. **(b)** Show that for a hoop the least coefficient of friction is $\frac{2}{3}\tan\theta$.

4.16 A small mass m tied to a non-stretchable thread moves over a smooth horizontal plane. The other end of the thread is drawn through a hole with constant velocity, Fig. 4.2. Show that the tension in the thread is inversely proportional to the cube of the distance from the hole.

[Osmania University]

Fig. 4.2



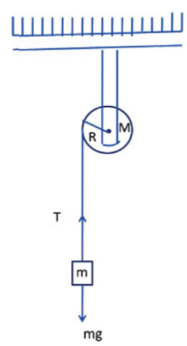
4.17 An ice skater spins at 4π rad/s with her arms extended.

(a) If her moment of inertia with arms folded is 80% of that with arms extended, what is her angular velocity when she folds her arms?
(b) Find the fractional change in kinetic energy.

4.18 A sphere of radius R and mass M rolls down a horizontal plane. When it reaches the bottom of an incline of angle θ it has velocity v_0. Assuming that it rolls without slipping, how far up the incline would it travel?

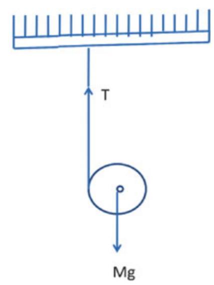
4.19 A body of mass m is attached to a light string wound around a pulley of mass M and radius R mounted on an axis supported by fixed frictionless bearings (Fig. 4.3). Find the linear acceleration 'a' of m and the tension T in the string.

Fig. 4.3



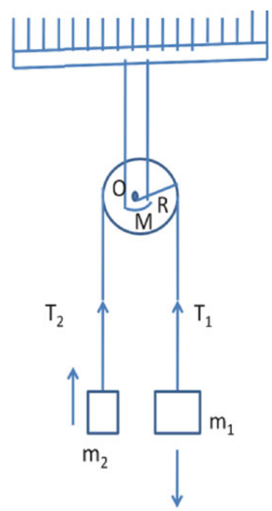
4.20 A light string is wound several times around a spool of mass M and radius R. The free end of the string is attached to a fixed point and the spool is held so that the part of the string not in contact with it is vertical (see Fig. 4.4). If the spool is let go, find the acceleration and the tension of the string.

Fig. 4.4



4.21 Two unequal masses m_1 and $m_2 (m_1 > m_2)$ are suspended by a light string over a pulley of mass M and radius R as in Fig. 4.5. Assuming that slipping does not occur and the friction of the axle is negligible, **(a)** find the acceleration with which the masses move; **(b)** angular acceleration of the pulley; **(c)** ratio of tensions T_1/T_2 in the process of motion.

Fig. 4.5



4.22 Two wheels of moment of inertia I_1 and I_2 are set in rotation with angular speed ω_1 and ω_2. When they are coupled face to face they rotate with a common angular speed ω due to frictional forces. Find **(a)** ω and **(b)** work done by the frictional forces.

4.23 Consider a uniform, thin rod of length l and mass M.

(a) The rod is held vertically with one end on the floor and is then allowed to fall. Use energy conservation to find the speed of the other end just before it hits the floor, assuming the end on the floor does not slip.
(b) You have an additional point mass m that you have to attach to the rod. Where do you have to attach it, in order to make sure that the speed of the falling end is not altered if the experiment in (a) is repeated?

[University of Durham 2005]

4.24 A thin circular disc of mass M and radius R is rotated with a constant angular velocity ω in the horizontal plane. Two particles each of mass m are gently attached at the opposite end of the diameter of the disc. What is the new angular velocity of the disc?

4.25 If the velocity is $\mathbf{v} = 2\hat{i} - 3\hat{j} + \hat{k}$ and the position vector is $\mathbf{r} = \hat{i} + 2\hat{j} - 3\hat{k}$, find the angular momentum for a particle of mass m.

4.26 A ball of mass 0.2 kg and radius 0.5 m starting from rest rolls down a 30° inclined plane. **(a)** Find the time it would take to cover 7 m. **(b)** Calculate the torque acting at the end of 7 m.

4.27 A string is wrapped around a cylinder of mass m and radius R. The string is pulled vertically upwards to prevent the centre of mass from falling as the cylinder unwinds the string. Find

(a) the tension in the string.
(b) the work done on the cylinder when it acquires angular velocity ω.
(c) the length of the string unwound in the time the angular speed reaches ω.

4.28 Two cords are wrapped around the cylinder, one near each end and the cord ends which are vertical are attached to hooks on the ceiling (Fig. 4.6). The cylinder which is held horizontally has length L, radius R and weight W. If the cylinder is released find

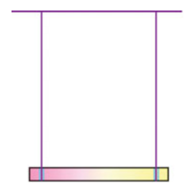
(a) the tension in the cords.
(b) acceleration of the cylinder.

[Osmania University]

4.29 A body of radius R and mass M is initially rolling on a level surface with speed u. It then rolls up an incline to a maximum height h. If $h = 3u^2/4g$, figure out the geometrical shape of the body.

4.30 A solid cylinder, a hollow cylinder, a solid sphere and a hollow sphere of the same mass and radius are placed on an incline and are released simultaneously from the same height. In which order would these bodies reach the bottom of the incline?

Fig. 4.6

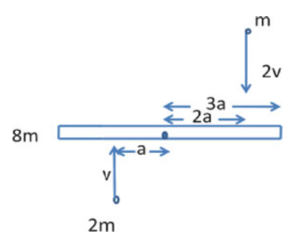


4.31 A tube of length L is filled with an incompressible liquid of mass M and closed at both the ends. The tube is then rotated in a horizontal plane about one of its ends with a uniform angular velocity ω. Show that the force exerted by the liquid at the other end is $F = \frac{1}{2}m\omega^2 L$.

4.32 A uniform bar of length $6a$ and mass $8m$ lies on a smooth horizontal table. Two point masses m and $2m$ moving in the same horizontal plane with speed $2v$ and v, respectively, strike the bar and stick to the bar (Fig. 4.7). The bar is set in rotation. Show that

(a) the centre of mass velocity $v_c = 0$
(b) the angular momentum $J = 6mva$
(c) the angular velocity $\omega = v/5a$
(d) the rotational energy $E = 3mv^2/5$

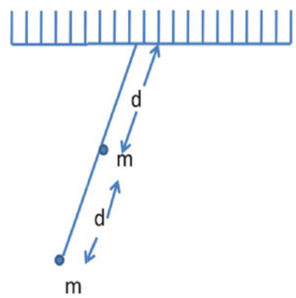
Fig. 4.7



4.33 A thin rod of negligible weight and of length $2d$ carries two point masses of m each separated by distance d, Fig. 4.8. If the rod is released from a horizontal position show that the speed of the lower mass when the rod is in the vertical position will be $v = \sqrt{\dfrac{24\,gd}{5}}$.

4.34 If the radius of the earth suddenly decreases to half its present value, the mass remaining constant, what would be the duration of day?

Fig. 4.8



4.35 A tall pole cracks and falls over. If θ is the angle made by the pole with the vertical, show that the radial acceleration of the top of the pole is $a_{\mathrm{R}} = \frac{3}{2}g(1-\cos\theta)$ and its tangential acceleration is $a_{\mathrm{T}} = \frac{3}{2}g\sin\theta$.

4.36 The angular momentum of a particle of a point varies with time as $\mathbf{J} = at^2\hat{i} + b\hat{j}$, where a and b are constants. When the angle between the torque about the point and the angular momentum is 45°, show that the magnitude of the torque and angular momentum will be $2\sqrt{ab}$ and $\sqrt{2}b$, respectively.

4.37 A uniform disc of radius R is spun about the vertical axis and placed on a horizontal surface. If the initial angular speed is ω and the coefficient of friction μ show that the time before which the disc comes to rest is given by $t = 3\omega R/4\mu g$.

4.38 A small homogeneous solid sphere of mass m and radius r rolls without slipping along the loop-the-loop track, Fig. 4.9. If the radius of the circular part of the track is R and the sphere starts from rest at a height $h = 6R$ above the bottom, find the horizontal component of the force acting on the track at Q at a height R from the bottom.

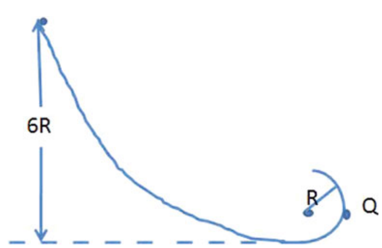
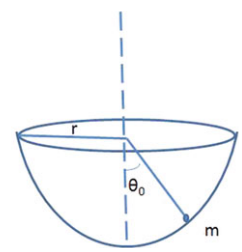


Fig. 4.9

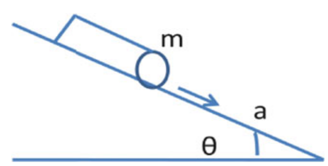
4.39 A particle is projected horizontally along the interior of a smooth hemispherical bowl of radius r. If the initial angular position of the particle is θ_0, find the initial velocity required by the particle just to reach the top of the bowl (Fig. 4.10).

Fig. 4.10



4.40 A spool of mass m, with a thread wound on it, is placed on an incline of 30° to the horizontal. The free end of the thread is attached to a nail, Fig. 4.11. Find the acceleration of the spool.

Fig. 4.11



4.41 A flywheel with initial angular velocity ω_0 undergoes deceleration due to frictional forces, with the torque on the axle being proportional to the square root of its angular velocity. Calculate the mean angular velocity of the wheel averaged over the total deceleration time.

4.42 A conical pendulum consisting of a thin uniform rod of length L and mass m with the upper end of the rod freely hanging rotates about a vertical axis with angular velocity ω. Find the angle which the rod makes with the vertical.

4.43 A billiard ball is initially struck such that it slides across the snooker table with a linear velocity V_0. The coefficient of friction between the ball and table is μ. At the instant the ball begins to roll without sliding calculate

(a) its linear velocity
(b) the time elapsed after being struck
(c) the distance travelled by the ball

State clearly what assumptions you have made about the forces acting on the ball throughout.

4.44 Consider a point mass m with momentum p rotating at a distance r about an axis. Starting from the definition of the angular momentum $L \equiv r \times p$ of this point mass, show that

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \tau,$$

where τ is the torque.
A uniform rod of length l and mass M rests on a frictionless horizontal surface. The rod pivots about a fixed frictionless axis at one end. The rod is initially at rest. A bullet of mass m travelling parallel to the horizontal surface and perpendicular to the rod with speed v strikes the rod at its centre and becomes embedded in it. Using the result above, show that the angular momentum of the rod after the collision is given by

$$|L| = \frac{1}{2} l v$$

Is $L = (l/2)\mathrm{m}v$ also correct?
What is the final angular speed of the rod?
Assuming $M = 5m$, what is the ratio of the kinetic energy of the system after the collision to the kinetic energy of the bullet before the collision?
[University of Durham 2008]

4.45 A uniform sphere of radius r initially at rest rolls without slipping down from the top of a sphere of radius R. Find the angular velocity of the ball at the instant it breaks off the sphere and show that the angle $\theta = \cos^{-1}(10/17)$ with the vertical.

4.46 A uniform rod of mass m and length $2a$ is placed vertically with one end in contact with a smooth horizontal floor. When it is given a small displacement, it falls. Show that when the rod is about to strike, the reaction is equal to $mg/4$.
[courtesy from R.W. Norris and W. Seymour, Mechanics via Calculus, Longmans & Co.]

4.47 The double pulley shown in Fig. 4.12 consists of two wheels which are fixed together and turn at the same rate on a frictionless axle. A rope connected to mass m_1 is wound round the circumference of the larger wheel and a second rope connected to mass m_2 is wound round the circumference of the smaller wheel. Both ropes are of negligible mass. The moment of inertia, I, of the double pulley is $38\,\mathrm{kg\,m^2}$. The radii of the wheels are $R_1 = 1.2\,\mathrm{m}$ and $R_2 = 0.5\,\mathrm{m}$.

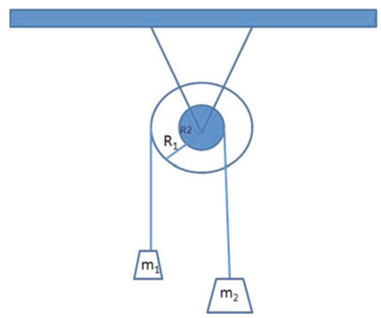
(a) If $m_1 = 25\,\mathrm{kg}$, what should the value of m_2 be so that there is no angular acceleration of the double pulley?
(b) The mass m_1 is now increased to 35 kg and the system released from rest.

(i) For each mass, write down the relationship between its linear acceleration and the angular acceleration of the pulley. Which mass has the greater linear acceleration?

(ii) Determine the angular acceleration of the double pulley and the tensions in both ropes.

[University of Manchester 2008]

Fig. 4.12



4.48 Two particles, each of mass m and speed v, travel in opposite directions along parallel lines separated by a distance d. Show that the vector angular momentum of this system of particles is independent of origin.

4.49 A small sphere of mass m and radius r rolls without slipping on the inside of a large hemisphere of radius R, the axis of symmetry being vertical. It starts from rest. When it arrives at the bottom show that

(a) the fraction $K(\text{rot})/K(\text{total}) = 2/7$

(b) the normal force exerted by the small sphere is given by $N = 17mg/7$

4.50 A solid sphere, a hollow sphere, a solid disc and a hoop with the same mass and radius are spinning freely about a diameter with the same angular speed on a table. For which object maximum work will have to be done to stop it?

4.51 In prob. (4.50) the four objects have the same angular momentum. For which object maximum work will have to be done to stop it?

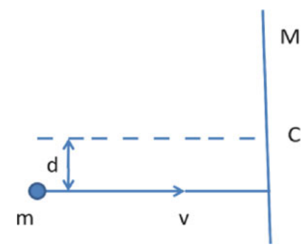
4.52 In prob. (4.50) the four objects have the same angular speed and same angular momentum. Compare the work to be done to stop them.

4.53 A solid sphere, a hollow sphere, a solid cylinder and a hollow cylinder roll down an incline. For which object the torque will be least?

4.54 A particle moves with the position vector given by $\mathbf{r} = 3t\hat{i} + 2\hat{j}$. Show that the angular momentum about the origin is constant.

4.55 A metre stick of length l and mass M is placed on a frictionless horizontal table. A hockey ball of mass m sliding along the table perpendicular to the stick with speed v strikes the stick elastically at distance d from the centre of the metre stick. Find d if the ball is to be brought to rest immediately after the collision (Fig. 4.13).

Fig. 4.13

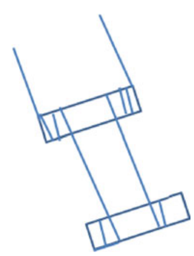


4.56 A uniform solid cylinder of mass m and radius R is set in rotation about its axis and lowered with the lateral surface on to the horizontal plane with initial centre of mass velocity v_0. If the coefficient of friction between the cylinder and the plane is μ, find

(a) how long the cylinder will move with sliding friction.
(b) the total work done by the sliding friction force on the cylinder.

4.57 Two identical cylinders, each of mass m, on which light threads are wound symmetrically are arranged as in Fig. 4.14. Find the tension of each thread in the process of motion. Neglect the friction in the axle of the upper cylinder.

Fig. 4.14



4.58 A uniform circular disc of radius r and mass m is spinning with uniform angular velocity ω in its own plane about its centre. Suddenly a point on its circumference is fixed. Find the new angular speed ω' and the impulse of the blow at the fixed point.

4.59 A uniform thin rod of mass m and length L is rotating on a smooth horizontal surface with one end fixed. Initially it has an angular velocity Ω and the motion slows down only because of air resistance which is $k\,\mathrm{d}x$ times the square of the velocity on each element of the rod of length $\mathrm{d}x$. Find the angular velocity ω after time t.

4.60 A sphere of radius a oscillates at the bottom of a hollow cylinder of radius b in a plane at right angles to the axis which is horizontal. If the cylinder is fixed and the sphere does not slide, find T, the time period of oscillations in terms of a, b and g, the acceleration due to gravity.

4.61 **(a)** Show that the moment of inertia of a disc of radius R and mass M about an axis through the centre perpendicular to its plane is

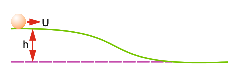
$$I = \frac{1}{2}MR^2$$

(b) A disc rolls without slipping along a horizontal surface with velocity u. The disc then encounters a smooth drop of height h, after which it continues to move with velocity v. At all times the disc remains in a vertical plane (Fig. 4.15).

Show that $v = \sqrt{u^2 + \dfrac{4gh}{3}}$

[University of Manchester 2008]

Fig. 4.15



4.62 A circular ring of mass m and radius r lies on a smooth horizontal surface. An insect of mass m sits on it and crawls round the ring with a uniform speed v relative to the ring. Obtain an expression for the angular velocity of the ring.

[With courtesy from R.W. Norris and W. Seymour, Longmans, Green and Co., 1923]

4.2.3 Coriolis Acceleration

4.63 **(a)** Given that earth rotates once every 23 h 56 min around the axis from the North to South Pole, calculate the angular velocity, ω, of the earth. When viewed from above the North Pole, the earth rotates counterclockwise (west to east). Which way does ω point?

(b) Foucault's pendulum is a simple pendulum suspended by a long string from a high ceiling. The effect of Coriolis force on the motion of the pendulum is to produce a precession or rotation of the plane of oscillation with time. Find the time for one rotation for the plane of oscillation of the Foucault pendulum at 30° latitude.

4.64 An object is dropped at the equator from a height of 400 m. How far does it hit the earth's surface from a point vertically below?

4.65 An object at the equator is projected upwards with a speed of 20 m/s. How far from its initial position will it land?

4.66 With what speed must an object be thrown vertically upwards from the surface of the earth on the equator so that it returns to the earth 1 m away from its original position?

4.67 A body is dropped from a height at latitude λ in the northern hemisphere. Show that it strikes the ground a distance $d = \frac{1}{3}\omega\cos\lambda\sqrt{\frac{8h^3}{g}}$ to the west, where ω is the earth's angular velocity.

4.68 An iceberg of mass 5×10^5 tons near the North Pole moves west at the rate of 8 km/day. Neglecting the curvature of the earth, find the magnitude and direction of the Coriolis force.

4.69 A tidal current is running due north in the northern latitude λ with velocity v in a channel of width b. Prove that the level of water on the east coast is raised above that on the western coast by $(2bv\omega\sin\lambda)g$ where ω is the earth's angular velocity.

4.70 If an object is dropped on the earth's surface, prove that its path is a semicubical parabola, $y^2 = z^3$.

4.71 A train of mass 1000 tons moves in the latitude 60° north. Find the magnitude and direction of the lateral force that the train exerts on the rails if it moves with a velocity of 15 m/s.

4.72 A train of mass m is travelling with a uniform velocity v along a parallel latitude. Show that the difference between the lateral force on the rails when it travels towards east and when it travels towards west is $4mv\omega\cos\lambda$, where λ is latitude and ω is the angular velocity of the earth.

4.73 A body is thrown vertically upwards with a velocity of 100 m/s at a 60° latitude. Calculate the displacement from the vertical in 10 s.