(iii) Obtain an expression for the emf induced in the coil if the wire is placed at $x = b/4$ and the current varies with time according to $I = 2t$.

13.83 Show that in the betatron the magnetic flux Ø linking an electron orbit of radius R is given instantaneously by $Ø = 2\pi R^2 B$ where B is the instantaneous magnetic field.

[University of Newcastle upon Tyne 1965]

13.2.6 Hall Effect

13.84 What is the Hall effect and what is the significance of a positive Hall coefficient?
A potential difference is applied between the ends of a strip of copper and a current of 100 A flows along its length. The strip is 20 cm long in the x-direction of a rectangular system of coordinates, 2 cm wide in the y-direction and 1 mm thick in the z-direction. A uniform magnetic field of 10 Wb/m^2 is applied across the strip in the positive y- direction and the hall EMF is found to be 5 μV
Derive (a) the magnitude and direction of the Hall field when the current flows in the positive x-direction and (b) the concentration of free electrons.
[University of Manchester 1972]

13.85 The Hall coefficient and electrical conductivity of an n-type silicon are -7.3×10^{-5} m^3/C and 2×10^7 mho/m, respectively. Calculate the magnitude of the mobility of the electrons.

[University of Durham 1962]

13.3 Solutions

13.3.1 Motion of Charged Particles in Electric and Magnetic Fields

13.1 $10^4\,\text{G} = 1\,\text{T}$

$$f = \frac{Bq}{2\pi m} = \frac{1 \times 1.6 \times 10^{-19}}{2\pi \times 4.0026 \times 1.66 \times 10^{-27}} = 3.83 \times 10^6\,\text{Hz} = 3.83\,\text{MHz}$$

13.2 **(a)** $K_p = \dfrac{1}{2}\dfrac{q^2 r^2 B^2}{m_p} = \dfrac{1}{2} \times \dfrac{(1.6 \times 10^{-19})^2 (0.25)^2 (1.5)^2}{1.66 \times 10^{-27}}$

$$= 2.17 \times 10^{-13}\,\text{J} = \frac{2.17 \times 10^{-12}}{1.6 \times 10^{-13}}\,\text{MeV} = 13.56\,\text{MeV}$$

$$f_\text{p} = \frac{Bq}{2\pi m_\text{p}} = \frac{1.5 \times 1.6 \times 10^{-19}}{2\pi \times 1.66 \times 10^{-27}} = 2.3 \times 10^7\,\text{Hz} = 23\,\text{MHz}$$

(b) $K_\alpha \simeq \frac{1}{2}\frac{(2e)^2 r^2 B^2}{4m_\text{p}} = K_\text{p} = 13.56\,\text{MeV}$

$$f_\alpha = \frac{(B)(2e)}{2\pi \times 4m_\text{p}} = \frac{f_\text{p}}{2} = 11.5\,\text{MHz}$$

13.3 $q\,v\,B = E\,q \rightarrow v = \frac{E}{B}$ (1)

$$\frac{mv^2}{r} = q\,v\,B' \rightarrow r = \frac{mv}{qB'} \quad (2)$$

Separation $= 2r_2 - 2r_1 = 2(r_2 - r_1)$

$$= \frac{2E}{qBB'}(m_2 - m_1) \quad (3)$$

where we have used (1) and (2).

13.4 $\frac{mv^2}{r} = Bev$

$$m = \frac{Ber}{v} = \frac{0.2 \times 1.6 \times 10^{-19} \times 0.2}{2.5 \times 10^7} = 2.56 \times 10^{-28}\,\text{kg}$$

In terms of electron mass

$$m = \frac{2.56 \times 10^{-28}}{9.1 \times 10^{-31}} = 281\,m_\text{e}$$

Hence the particle is a pion (π – meson)

13.5 Acceleration $a = \frac{qE}{m}$ (1)

$$y = \frac{1}{2}at^2 \quad (2)$$

$$x = vt \quad (3)$$

Combining (1), (2) and (3)

$$y = \frac{qEx^2}{2mv^2} \quad (4)$$

which is the equation to a parabola.

13.6 $r = \frac{\sqrt{2mK}}{qB} = \frac{(2 \times 9.1 \times 10^{-31} \times 5 \times 10^3 \times 1.6 \times 10^{-19})^{1/2}}{1.6 \times 10^{-19} \times 10^{-2}} = 0.0238\,\text{m}$

$$= 2.38\,\text{cm}$$

13.7 $qE = q\,v\,B$

$$\therefore \quad B = \frac{E}{v} = \frac{1500}{3000} = 0.5\,\text{T}$$

13.8 **(a)** $v = \dfrac{rqB}{m} = \dfrac{1.9 \times 1.6 \times 10^{-19} \times 3 \times 10^{-5}}{9.1 \times 10^{-31}} = 10^7\,\text{m/s}$

(b) $t = \dfrac{2\pi r}{v} = \dfrac{2\pi \times 1.9}{10^7} = 1.19 \times 10^{-6}\text{s} = 1.19\,\mu\text{s}$

13.9 $B = \dfrac{2\pi m f}{q} = \dfrac{2\pi \times 2.0141 \times 1.66 \times 10^{-27} \times 5 \times 10^6}{1.6 \times 10^{-19}} = 0.656\,\text{Wb/m}^2$

$$K = \frac{1}{2}\frac{q^2 r^2 B^2}{m} = \frac{1}{2} \times \frac{(1.6 \times 10^{-19})^2 (0.762)^2 (0.325)^2}{2.0141 \times 1.66 \times 10^{-27}} = 4.696 \times 10^{-13}\,\text{J}$$
$$= 2.9\,\text{MeV}$$

13.10 For deuterons

$$B = \frac{2\pi\ fm}{q} = \frac{2\pi \times 11.5 \times 10^6 \times 2.014 \times 1.66 \times 10^{-27}}{1.6 \times 10^{-19}} = 1.509\,\text{Wb/m}^2$$

For protons

$$f = \frac{qB}{2\pi m} = \frac{1.6 \times 10^{-19} \times 1.509}{2\pi \times 1.66 \times 10^{-27}} = 2.316 \times 10^7\,\text{c/s} = 23.16\,\text{Mc/s}$$

13.11 **(a)** $f = \dfrac{Bq}{2\pi m} = \dfrac{1.5 \times 1.6 \times 10^{-19}}{2\pi \times 3.32 \times 10^{-27}} = 11.5 \times 10^6\,\text{c/s} = 11.5\,\text{Mc/s}$

(b) $K = \dfrac{1}{2}\dfrac{q^2 r^2 B^2}{m} = \dfrac{1}{2}\dfrac{(1.6 \times 10^{-19} \times 0.5 \times 1.5)^2}{3.32 \times 10^{-27}} = 21.69 \times 10^{-13}\,\text{J} =$ $13.56\,\text{MeV}$

13.12 $i = \dfrac{q}{t} = \dfrac{1.6 \times 10^{-19}}{1.5 \times 10^{-16}} = 1.06 \times 10^{-3}\,\text{A}$

13.13 **(i)** Magnetic force, $F_M = qVB$
Electric force, $F_E = qE$
For no deflection, $F_M = F_E$

$$\therefore \quad qvB = qE$$

$$\therefore \quad v = \frac{E}{B} = \frac{80 \times 10^3}{0.4} = 2 \times 10^5\,\text{m/s}$$

(ii) $\dfrac{mv^2}{r} = q\,v\,B$

$$\therefore \quad \frac{q}{m} = \frac{v}{Br} = \frac{E}{B^2 r} = \frac{80 \times 10^3}{(0.4)^2 (1.14 \times 10^{-2})} = 4.38 \times 10^7\,\text{C/kg}$$

13.14 **(a)** $\boldsymbol{F} = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B}$

(b) In the electric field energy acquired, $K = qV$

$$\therefore \qquad p = \sqrt{2mK} = \sqrt{2mqV}. \tag{1}$$

In the magnetic field

$$p = qBr \tag{2}$$

Combining (1) and (2)

$$r = \sqrt{\frac{2mV}{qB^2}}$$

13.15 Let the isotopes ^{235}U and ^{238}U be called 1 and 2, respectively. In the magnetic field the momenta are given by

$$p_1 = qBr_1,\ \mathrm{p}_2 = \mathrm{q}Br_2$$

$$p_2 - p_1 = \mathrm{q}B\,(r_2 - r_1) = 1.6 \times 10^{-19} \times 0.2 \times 2 \times 10^{-3} \tag{13.26}$$

$$= 6.4 \times 10^{-21}\mathrm{kg\,m/s}. \tag{1}$$

In the electric field

$$qV = \frac{p_1^2}{2m_1} = \frac{p_2^2}{2m_2} \tag{2}$$

$$\therefore \quad p_2 = p_1\sqrt{\frac{m_2}{m_1}} = p_1\sqrt{\frac{3.95 \times 10^{-25}}{3.90 \times 10^{-25}}} = 1.00639\,p_1. \tag{3}$$

Combining (1) and (3)

$$p_1 = 1.00159 \times 10^{-19}\,\mathrm{kg\,m/s}. \tag{4}$$

Substituting (4) into (2)

$$V = \frac{(10^{-19})^2}{2 \times 1.6 \times 10^{-19} \times 3.90 \times 10^{-25}} = 8 \times 10^4\,\mathrm{V}$$

13.16 **(a)** $F = q\,v \times B$
If B is perpendicular to v, then the particle would move in a circle.
Centripetal force = magnetic force

$$\frac{mv^2}{r} = qvB \rightarrow r = \frac{mv}{qB}$$

(b) $\xi = -\dfrac{\mathrm{d}}{\mathrm{d}t}(N\phi) = -\dfrac{\mathrm{d}}{\mathrm{d}t}(NBA) = -NA\dfrac{\mathrm{d}B}{\mathrm{d}t}$

$= -NA\dfrac{\mathrm{d}}{\mathrm{d}t}(B_0\cos(15t)) = 15\,NAB_0\sin(15t)$

13.17 **(a)** $E = \dfrac{q}{4\pi\varepsilon_0 r}$

$\therefore \quad q = 4\pi\varepsilon_0 r E = 4\pi \times 8.85\times 10^{-12}\times 10^{-6}\times 5.8\times 10^{-3}$
$= 6.447\times 10^{-19}\,\mathrm{C}$

Number of electrons

$$n = \frac{q}{e} = \frac{6.447\times 10^{-19}}{1.6\times 10^{-19}} = 4.029 \text{ or } 4$$

(b) Minimum electric field E required to prevent the droplet from falling is conditioned by equating the electric force to the gravitational force:

$qE = mg$

$$\therefore \quad E = \frac{mg}{q} = \frac{4}{3}\pi r^3\frac{\rho g}{q} = \frac{4}{3}\pi \times \frac{(10^{-6})^3\times 1000\times 9.8}{6.447\times 10^{-19}}$$
$= 6.37\times 10^4\,\mathrm{V/m}$

13.18 When a charged particle moves at an angle θ to the field direction, the particle will move in a helical path. The vector velocity v of the particle can be resolved into two components, one parallel to B and one perpendicular to it:

$$v_{\parallel} = v\cos\theta \text{ and } v_{\perp} = v\sin\theta \tag{1}$$

The parallel component determines the pitch of the helix, that is, the distance between the adjacent turns. The perpendicular component determines the radius r of the helix:

$$v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{2\times 1\times 1.6\times 10^{-19}}{9.1\times 10^{-31}}} = 5.93\times 10^5\,\mathrm{m/s}$$

$$r = \frac{mv\sin\theta}{|q|\;B} = \frac{9.1\times 10^{-31}\times 5.93\times 10^5\sin 60^\circ}{1.6\times 10^{-19}\times 10^{-3}} = 2.92\times 10^{-3}\mathrm{m}$$
$= 2.92\,\mathrm{mm}$

Time period $T = \dfrac{2\pi\, m}{|q|\;B} = \dfrac{2\pi\times 9.1\times 10^{-31}}{1.6\times 10^{-19}\times 10^{-3}} = 3.57\times 10^{-8}\,\mathrm{s}$

Pitch $= (v\cos\theta)T = 5.93\times 10^5\times \cos 60^\circ\times 3.57\times 10^{-8} = 0.1\,\mathrm{m}$

13.19 **(i)** Choose the origin at o, Fig. 13.21. The electric field E acts along the y-direction perpendicular to the plates which are located in the x-direction. The electric force on the electron is directed along the y-axis and the magnetic force along the z-axis. If the component of initial velocity in the direction of B is zero then the path of electron will be contained entirely in the xy-plane.

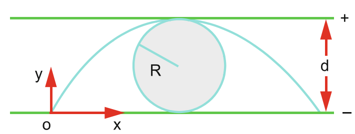


Fig. 13.21 Generating a cycloid

Writing Lorentz force $\boldsymbol{F} = q\boldsymbol{E} + qv\boldsymbol{B}$, in the component form

$$F_y = m\frac{\mathrm{d}v_y}{\mathrm{d}t} = qE - qBv_x \tag{1}$$

$$F_x = m\frac{\mathrm{d}v_x}{\mathrm{d}t} = qBv_y \tag{2}$$

Writing for convenience

$$\omega = \frac{qB}{m} \text{ and } \gamma = \frac{E}{B} \tag{3}$$

Equations (1) and (2) can be rewritten as

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = \omega\gamma - \omega v_x \tag{4}$$

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = \omega v_y \tag{5}$$

Differentiating (4) and using (5)

$$\frac{\mathrm{d}^2 v_y}{\mathrm{d}t^2} = -\omega\frac{\mathrm{d}v_x}{\mathrm{d}t} = -\omega^2 v_y$$

$$\text{or} \qquad \frac{\mathrm{d}^2 v_y}{\mathrm{d}t^2} + \omega^2 v_y = 0 \tag{6}$$

With the initial conditions $v_x = v_y = 0$, at $t = 0$, (6) has the solution

$$v_y = A \sin\omega t \tag{7}$$

where $A =$ constant:

$$\therefore \qquad \frac{\mathrm{d}v_y}{\mathrm{d}t} = A\omega \cos \omega t = \omega\gamma - \omega v_x$$

At $t = 0$, $v_x = 0$

$$\therefore \quad A\omega = \omega\gamma \to A = \gamma$$

$$\therefore \quad v_y = \gamma \sin \omega t \tag{8}$$

Substituting (8) into (5), integrating and using the initial condition $v_x = 0$ at $t = 0$

$$v_x = \gamma(1 - \cos\, \omega t) \tag{9}$$

(ii) The coordinates x and y at any time t can be found out by integrating separately (8) and (9) with the initial condition $x = y = 0$ at $t = 0$

$$y = \frac{\gamma}{\omega}(1 - \cos \omega t) \tag{10}$$

$$x = \gamma \left(t - \frac{\sin \omega t}{\omega} \right) \tag{11}$$

Using (9) and (10) we get

$$v_x = \omega y \tag{12}$$

(iii) The energy of the particle is unaffected in the static magnetic field. Under the electric field in the y-direction the energy picked up will be

$$\frac{qVy}{d} = \frac{1}{2}mv^2 = \frac{1}{2}m\left(v_x^2 + v_y^2\right)$$

$$\text{or} \qquad v_y^2 = \frac{2qVy}{md} - \omega^2 y^2 \tag{13}$$

where we have used (12).

13.20 Referring to Fig. 13.21 and setting $\theta = \omega t$ and $R = \gamma/\omega$ and (10) and (11) of prob. (13.19) we get the parametric equations of cycloid

$$y = R(1 - \cos \theta) \tag{1}$$

$$x = R(\theta - \sin \theta) \tag{2}$$

These equations define the path generated by a point on the circumference of a circle which rolls along the x-axis. The maximum displacement of electron along the y-axis is equal to the diameter of the rolling circle, that is, $2R$. Identifying $2R = d$

$$\frac{d}{2} = R = \frac{\gamma}{\omega} = \frac{E/B}{eB/m} = \frac{Em}{eB^2} \tag{3}$$

where we have set $q = e$, for the electron charge.

$$\text{Also} \qquad E = \frac{V}{d} \tag{4}$$

Using (4) in (3)

$$d^2 = \frac{2mV}{eB^2} \tag{5}$$

Thus the condition that the electrons are able to arrive at the positive plate is

$$d^2 < \frac{2mV}{eB^2}$$

13.3.2 Magnetic Induction

13.21 $B = \dfrac{\mu_0 i}{2\pi r}$

$$\therefore \quad i = \frac{2\pi Br}{\mu_0} = \frac{2\pi \times 10^{-6} \times 0.4}{4\pi \times 10^{-7}} = 2\,\text{A}$$

13.22 **(a)** Consider a typical current element dx. The magnitude of the contribution dB of this element to the magnetic field at P is found from Biot–Savart law and is given by (Fig. 13.22)

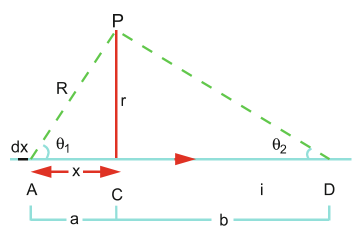


Fig. 13.22 Magnetic induction due to a current-carrying wire of finite length

$$\text{d}B = \frac{\mu_0 i \ \text{d}x \ \sin\theta}{4\pi \ R^2} \tag{1}$$

Since the direction of the contribution dB at point for all such elements is identical, i.e. at right angles to the plane of paper, the resultant field is obtained by integrating dB from A to D in (1). Writing $\sin\theta = \dfrac{r}{R}$

$$B = \int \text{d}B = \frac{\mu_0 i r}{4\pi}\left[\int_{-a}^{b} \frac{\text{d}x}{(x^2+r^2)^{3/2}}\right] = \frac{\mu_0 i r}{4\pi r^2}\, \frac{x}{(x^2+r^2)^{1/2}}\Bigg|_{-a}^{b}$$

$$\therefore \qquad B = \frac{\mu_0 i}{4\pi r}(\cos\theta_1 + \cos\theta_2) \tag{2}$$

(b) For infinite wire $\theta_1 \to 0$ and $\theta_2 \to 0$, in this limit (2) becomes

$$B = \frac{\mu_0 i}{2\pi r}$$

which is the expression for a long wire.

13.23 $B = \dfrac{\mu_0 i}{2r}$

$$\therefore \qquad i = \frac{2Br}{\mu_0} = \frac{2 \times 10^{-5} \times 0.5}{4\pi \times 10^{-7}} = 7.96\,\text{A}$$

13.24 Magnetic field B_1 due to current i in one segment is (Fig. 13.23)

$$B_1 = \frac{\mu_0 i}{4\pi R}(\cos\theta_1 + \cos\theta_2)$$

Putting $\theta_1 = \theta_2 = 45°$ and $R = a/2$

$$B_1 = \frac{\mu_0 i}{\sqrt{2}\pi a}$$

Fields due to four sides are equal and additive. Therefore net field

$$B = 4B_1 = \frac{2\sqrt{2}\mu_0 i}{\pi a}$$

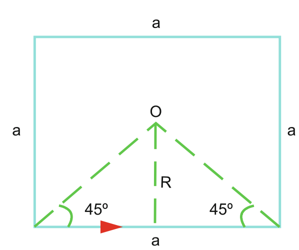


Fig. 13.23 Magnetic induction at the centre of a square conducting loop

13.25 The magnetic fields in the upper branch and lower branch act in the opposite direction. The current in the upper branch is $\dfrac{4I}{5}$ and in the lower branch is $\dfrac{I}{5}$. As the current is flowing through semicircles,

$$B_1 = \frac{\mu_0}{4a}\left(\frac{4I}{5}\right) = \frac{\mu_0 I}{5a}$$
$$B_2 = \frac{\mu_0}{4a}\left(\frac{I}{5}\right) = \frac{\mu_0 I}{20\,a}$$

Net field $B = B_1 - B_2 = \dfrac{3\mu_0 I}{20\,a}$

13.26 Field B at the centre is due to three-fourths of the circle (B_1) added to that due to the straight segment (B_2)

$$B_1 = \frac{3}{4} \times \frac{\mu_0 i}{2R}$$
$$B_2 = \frac{\mu_0 i}{4\pi d}(\cos\theta_1 + \cos\theta_2)$$

From the geometry of Fig. 13.24, $\theta_1 = \theta_2 = 45°$ and $d = \dfrac{R}{\sqrt{2}}$.

Then $\quad B = B_1 + B_2 = \dfrac{\mu_0\, i}{2\,\pi\, R}\left(1 + \dfrac{3\pi}{4}\right)$

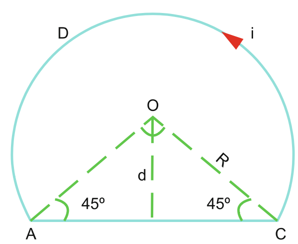


Fig. 13.24 Magnetic induction at the centre of a current-carrying wire made of three-fourths of a circle and a chord

13.27 **(a)** The magnetic induction due to straight wires is

$$B_1 = \frac{\mu_0\, i}{4\pi\, R} + \frac{\mu_0 i}{4\pi\, R} = \frac{\mu_0 i}{2\pi\, R} \tag{1}$$

because straight wires are of infinite length only on left side.
Induction at 0 due to semicircular portion is

$$B_2 = \frac{\mu_0 i}{4R} \tag{2}$$

Total magnetic induction

$$B = B_1 + B_2 = \frac{\mu_0 i}{2\pi\, R} + \frac{\mu_0 i}{4R} = \frac{\mu_0 i}{4\pi\, R}(2 + \pi)$$

(b) The straight portions of the wire do not contribute to the field at O as the current is directed towards C and makes an angle $\theta = 0°$, for which the Biot–Savart formula gives $B = 0$. Thus the entire induction comes from the semicircular portion of the wire for which $B = \dfrac{\mu_0 i}{4R}$.

13.28 **(a)** Let the angle θ be subtended at the centre by one side AC of a regular n-sided polygon, Fig. 13.25. Then

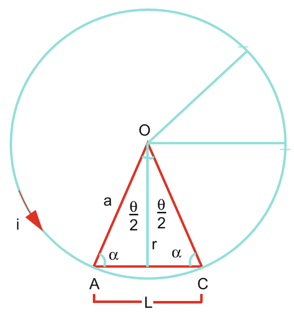


Fig. 13.25 Magnetic induction at the centre of a current-carrying regular n-sided polygon

$$\theta = \frac{2\pi}{n} \text{ or } \frac{\theta}{2} = \frac{\pi}{n} \tag{1}$$

The magnetic induction due to one side AC at the centre O is

$$B_1 = \frac{\mu_0 i}{4\pi r}(\cos\alpha + \cos\alpha) = \frac{\mu_0 i \cos\alpha}{2\pi r} \tag{2}$$

where r is the distance of O from AC.
The field B due to n sides will be additive and is given by

$$B = nB_1 = \frac{\mu_0 n i \cos\alpha}{2\pi r} \tag{3}$$

Now $r = a\sin\alpha$, so that in (3)

$$\frac{\cos\alpha}{r} = \frac{\cos\alpha}{a\sin\alpha} = \frac{1}{a}\cot\alpha = \frac{1}{a}\tan\left(\frac{\theta}{2}\right) = \frac{1}{a}\tan\left(\frac{\pi}{n}\right) \tag{4}$$

where we have used (1). Using (4) in (3)

$$B = \frac{\mu_0 n i}{2\pi a}\tan\left(\frac{\pi}{n}\right) \tag{5}$$

(b) In the limit $n \to \infty$, $\tan\left(\frac{\pi}{n}\right) \to \frac{\pi}{n}$ and (5) becomes

$$B = \frac{\mu_0 i}{2a}$$

an expression which is identical for B for a circular loop. This is reasonable since as $n \to \infty$, polygon $\to$ circle.

13.29 For square $l = 4a$ and for circle $l = 2\pi r$

$$\therefore \qquad \frac{a}{r} = \frac{\pi}{2} \tag{1}$$

At the centre of the circle, $B_c = \frac{\mu_0 i}{2r}$.

At the centre of the square, $B_s = \frac{2\sqrt{2}\,\mu_0 i}{\pi a}$ via prob. (13.24).

$$\therefore \qquad \frac{B_c}{B_s} = \frac{\pi}{4\sqrt{2}}\frac{a}{r} = \frac{\pi^2}{8\sqrt{2}} = 0.87$$

13.30 **(a)** The magnetic induction B at the centre of a circular wire is $B = \frac{\mu_0 i}{2r}$. Hence for the arc which subtends an angle θ at the centre

$$B = \frac{\mu_0 i}{2r}\frac{\theta}{2\pi} = \frac{\mu_0\, i\, \theta}{4\pi\, r}$$

Induction at C due to the inner arc is

$$B_1 = \frac{\mu_0 i\theta}{4\pi\, R_1}$$

and due to the outer arc

$$B_2 = -\frac{\mu_0 i\theta}{4\pi\, R_2}$$

The negative sign arises due to the fact that the current has reversed. As the radial part of the path points towards C, it does not contribute to B. Therefore, the resultant induction is

$$B = B_1 + B_2 = \frac{\mu_0 i\theta}{4\pi}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Note that for clockwise current we take B as positive and for counterclockwise we take B as negative.

(b) Put $\theta = \pi$ to obtain

$$B = \frac{\mu_0 i}{4}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

13.31 The total induction is given by adding B_s due to the straight conductor which contributes on both sides of P and B_c due to the circular path, both being directed out of page (Fig. 13.9)

$$B_s = \frac{\mu_0 i}{2\pi R}$$
$$B_c = \frac{\mu_0 i}{2R}$$
$$B = B_s + B_c = \frac{\mu_0 i}{2\pi R}(\pi + 1)$$

13.32 $$\mathrm{d}\boldsymbol{B} = \frac{\mu_0 i}{4\pi}\frac{\mathrm{d}\boldsymbol{s} \times \boldsymbol{r}}{r^3} \qquad \text{(Biot–Savart law)} \tag{1}$$

The magnetic induction d$\boldsymbol{B}$ at P on the z-axis due to an element of length dl of the ring is shown in Fig. 13.26. Resolving d$\boldsymbol{B}$ along z-axis and perpendicular to it, and summing over all such elements it is seen that the perpendicular components vanish for reasons of symmetry and the parallel components get added up.
Writing dl for ds and noting that the angle between $\boldsymbol{R}$ and d$\boldsymbol{l}$ is 90°, (1) can be written as

$$\mathrm{d}\boldsymbol{B}_z = (\mathrm{d}\boldsymbol{B})\cos\alpha = \frac{(\mu_0\, i r \mathrm{d}l)\cos\alpha}{4\pi r^3} \tag{2}$$

Writing dl = Rdϕ, where ϕ is the azimuth angle and r cos α = R, (2) becomes

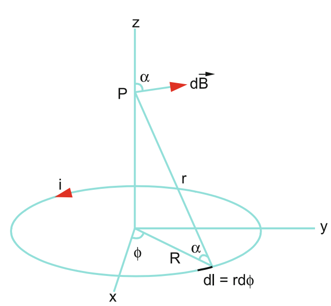


Fig. 13.26 Magnetic induction on the axis of a current-carrying ring

$$dB_z = \frac{\mu_0 i R^2 d\phi}{4\pi r^3} \tag{3}$$

Integrating

$$B_z = B = \int dB_z = \frac{\mu_0 i R^2}{4\pi r^3} \int_0^{2\pi} d\phi = \frac{\mu_0 i R^2}{2\,(R^2 + z^2)^{3/2}} \tag{4}$$

13.33 $B = \frac{\mu_0 N i}{l} = \frac{4\pi \times 10^{-7} \times 500 \times 5}{1.0} = 3.14 \times 10^{-3}\,\mathrm{T}$

13.34 $B = \frac{\mu_0 I}{2\pi}\left[\frac{1}{x} + \frac{1}{d - x}\right]$

13.35 Apply Ampere's law inside the hollow cylindrical conductor

$$(B)\,(2\pi\,r) = \frac{\mu_0\,i\,\pi\,(r^2 - a^2)}{\pi\,(b^2 - a^2)} \tag{1}$$

where the right-hand side includes only the fraction of the current that passes through the surface enclosed by the path of integration. Solving for B,

$$B = \frac{\mu_0 i}{2\,\pi\,r}\,\frac{(r^2 - a^2)}{(b^2 - a^2)}.$$

13.36 Magnetic field at P, due to loop A is

$$B_{\mathrm{A}} = \frac{\mu_0 INR^2}{2(R^2 + x^2)^{3/2}} \tag{1}$$

where $x = \mathrm{AP} = \frac{R}{2}$. Similarly the magnetic field B_{c} due to the second loop is given by an identical expression with $x = \mathrm{CP} = \frac{R}{2}$. As the currents are in the opposite direction these two fields are added:

$$\therefore \qquad B = B_{\mathrm{A}} + B_{\mathrm{C}} = \frac{\mu_0\,INR^2}{(R^2 + x^2)^{3/2}} \tag{2}$$

Put $x = R/2$ to find

$$B = \frac{8\,N\mu_0 I}{5^{3/2} R} \tag{3}$$

13.37 Consider a ring of radius r, width dr, concentric with the disc. The charge on the ring, Fig. 13.27

$$dq = q\,\frac{2\pi r\,dr}{\pi R^2} = \frac{2qr\,dr}{R^2} \tag{1}$$

The elementary current due to rotation of charge with frequency f is

$$di = f\,dq = \frac{\omega}{2\pi}\,\frac{2qr\,dr}{R^2} = \frac{\omega\,qr\,dr}{\pi\,R^2} \tag{2}$$

The induction at the centre due to the current in the ring is

$$dB = \frac{\mu_0 di}{2r} = \frac{\mu_0}{2r}\,\frac{\omega\,qr\,dr}{\pi\,R^2} = \frac{\mu_0\omega\,q\,dr}{2\pi\,R^2} \tag{3}$$

Total induction from the rotating disc

$$B = \int dB = \int_0^R \frac{\mu_0\omega\,q\,dr}{2\,\pi\,R^2} = \frac{\mu_0\omega\,q}{2\,\pi\,R} \tag{4}$$

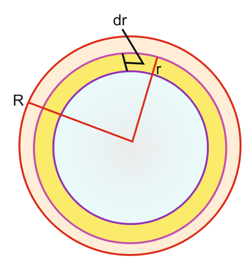


Fig. 13.27 Magnetic induction at the centre of a charged rotating disc

13.38 The field at any point P_1 at distance x from P, the middle point will be

$$B = \frac{\mu_0\,iNR^2}{2}\left\{\frac{1}{\left[R^2+\left(\frac{1}{2}R+x\right)^2\right]^{3/2}}+\frac{1}{\left[R^2+\left(\frac{1}{2}R-x\right)^2\right]^{3/2}}\right\} \tag{1}$$

by prob. (13.36).

Differentiate B with respect to x and evaluate $\left(\frac{\partial B}{\partial x}\right)_{x=0}$ to find the first derivative zero. Differentiate B once again to find $\left(\frac{\partial^2 B}{\partial x^2}\right)_{x=0} = 0$. Thus the field around P is seen to be fairly uniform.

13.39 **(i)** $B = \frac{\mu_0 i}{2\pi r} \qquad (r > R)$ (1)

$$= \frac{4\pi \times 10^{-7} \times 100}{2\pi \times 1.0} = 2 \times 10^{-5}\,\mathrm{T}$$

(ii) $B = \frac{\mu_0 i r}{2\pi R^2} \qquad (r < R)$ (2)

$$= \frac{4\pi \times 10^{-7} \times 100 \times 6 \times 10^{-3}}{2\pi \times (3 \times 10^{-2})^2} = 1.33 \times 10^{-6}\,\mathrm{T}$$

13.40 **(a)** $\Sigma B_t\, \Delta l = \mu_0\, i$ (Ampere's law)

$$\Sigma B \Delta l = B \Sigma \Delta l = B \cdot 2\pi r = \mu_0 i$$

where we enclose the current i by going round once the circle of radius r. The magnetic induction will be tangential to the circle and the summation is simply the circumference of the circle, Fig. 13.28a:

$$(B)(2\pi r) = \mu_0 i$$

$$\therefore \qquad B = \frac{\mu_0 i}{2\pi r} \qquad (r > R)$$

(b) Consider a circular path C at distance $r < R$, Fig. 13.28b. The current i_0 inside a cross-section of radius r is proportional to the cross-sectional area

$$i_0 = i\,\frac{\pi r^2}{\pi R^2} = \frac{i r^2}{R^2}$$

By Ampere's law

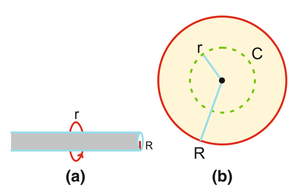


Fig. 13.28 Magnetic induction due to a current-carrying cylinder

$$\Sigma B_t \, \Delta l = (B)\,(2\pi\, r) = \mu_0 i_0 = \mu_0 i \frac{r^2}{R^2}$$

$$\therefore \qquad B = \frac{\mu_0 i r}{2\pi\, R^2} \qquad (r < R)$$

13.41 Let the radius of the inner arc of the loop be r and that of the outer arc $2r$. Both the arcs are quarter of a circle. The straight portions do not contribute to B as their directions pass through P. As the currents in the two arcs flow in the opposite sense, B will be down due to current in the outer arc and up due to the current in the inner wire:

$$\therefore \qquad B_{\text{net}} = \frac{1}{4}\frac{\mu_0 I}{2r} - \frac{1}{4}\frac{\mu_0 I}{2\times 2r} = \frac{\mu_0 I}{16\, r} \text{ down}$$

13.42 By prob. (13.22) at P, Fig. 13.29

$$B = \frac{\mu_0 I}{4\pi x}(\cos\,\theta_1 + \cos\theta_2), \tag{1}$$

put $\theta_1 = \theta_2$ then

$$\cos\theta_1 = \cos\theta_2 = \frac{L/2}{\sqrt{(L/2)^2 + x^2}}$$

So that (1) becomes

$$B = \frac{\mu_0 I}{4\pi x}\frac{L}{\sqrt{(L/2)^2 + x^2}} \tag{2}$$

(a) Let the square be of side $a = l/4$. The distance of the centre of square from the side is $a/2$. Put $x = a/2$ and $L = a = l/4$ in (2) to find for one side

$$B_1 = \frac{2\sqrt{2}\mu_0 I}{\pi l}$$

As there are four equal sides, $B = 4B_1 = \dfrac{8\sqrt{2}\mu_0 I}{\pi l}$.

The B-field will be perpendicular to the plane containing the wire and the field point.

(b) Let each side of the equilateral triangle be $a = l/3$. Distance of the centre of the triangle from any side is $x = a/2\sqrt{3}$. Put $L = a = l/3$ and $x = a/2\sqrt{3}$ in (1) to find for one side $B_1 = \dfrac{9\mu_0 I}{2\pi l}$. Hence for three sides

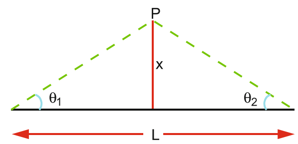


Fig. 13.29 Magnetic field due to a current-carrying straight wire of finite length

$$B = 3B_1 = \frac{27\mu_0 I}{2\pi l}$$

$$\therefore \quad \frac{B\text{ (square)}}{B\text{ (triangle)}} = \frac{27/2}{8\sqrt{2}} = 1.19$$

13.43 By prob. (13.36) $B = \frac{8N\mu_0 I}{5^{3/2}R} = \frac{(8)(100)(4\pi \times 10^{-7})(2)}{5^{3/2}(0.2)} \simeq 9 \times 10^{-4}\,\text{T}$

13.44 $\oint \boldsymbol{B}\,\mathrm{d}l = \mu_0 i \qquad$ (Ampere's law)

By prob. (13.43)

$$B = \frac{\mu_0 I r}{2\pi R^2}\,(r < R) \tag{1}$$

$$\therefore \quad \text{For } r = \frac{R}{2}, \quad B = \frac{\mu_0 I}{4\pi R} \tag{2}$$

$$\text{Further } B = \frac{\mu_0 I}{2\pi r}\,(r > R). \tag{3}$$

Equating (2) and (3) we find $r = 2R$. Thus at $r = 2R$, the magnetic field is the same as at $r = R/2$.

13.45 **(a)** In the absence of magnetic material the number of flux linkages $N\phi_B$(N being the number of turns) is proportional to the current

$$N\phi_\text{B} = Li \tag{1}$$

If n is the number of turns per unit length, A the cross-sectional area and l the length of the solenoid and B the magnetic induction

$$N\phi_\text{B} = (nl)(BA) \tag{2}$$

By Ampere's theorem

$$B = \mu_0 ni \tag{3}$$

Combining (1), (2) and (3)

$$L = \mu_0 n^2 Al \tag{4}$$

(b) $B = \mu_0 ni = \dfrac{\mu_0 Ni}{l}$

$$U_B = u_B Al = \frac{B^2}{2\mu_0} \cdot Al = \frac{(\mu_0 Ni)^2 Al}{2\mu_0 l^2} = \frac{\mu_0 N^2 i^2 \pi r^2}{2l}$$
$$= \frac{(4\pi \times 10^{-7})\,(100)^2 (5)^2 \pi\,(0.01)^2}{2 \times 0.1} = 4.93 \times 10^{-4}\,\text{J}$$

(c) $\phi = BA = \dfrac{\mu_0 Ni}{l} \cdot \pi r^2 = \dfrac{4\pi \times 10^{-7} \times 100 \times 5 \times \pi \times (0.1)^2}{0.1}$

$= 1.972 \times 10^{-4}\,\text{Wb}$

$$\xi = \frac{-\Delta\phi}{\Delta t} = \frac{-1.972 \times 10^{-4}}{5} = -3.94 \times 10^{-5}\,\text{V}$$

13.46 **(a)** $B = \dfrac{\mu_0 i}{2\pi r} = \dfrac{4\pi \times 10^{-7} \times 3}{2\pi \times 50 \times 10^{-3}} = 1.2 \times 10^{-7}\,\text{T}$

(b) In the vector form the Biot and Savart law can be written as

$$\mathrm{d}\boldsymbol{B} = \frac{\mu_0 i}{4\pi} \frac{\mathrm{d}\boldsymbol{l} \times \boldsymbol{r}}{r^3}$$

(i) $\mathrm{d}\boldsymbol{B} = \dfrac{(4\pi \times 10^{-7})(2)(2\hat{k} \times 3\hat{i})}{4\pi \qquad 3^3} = 4.44 \times 10^{-8}\,\hat{j} \qquad (\because\ \hat{k} \times \hat{i} = \hat{j})$

Thus $\mathrm{d}B = 4.44 \times 10^{-8}$ T along positive y-axis

(ii) $\mathrm{d}\boldsymbol{B} = \dfrac{(4\pi \times 10^{-7})(2)(2\hat{k} \times (-6\hat{i}))}{4\pi \qquad 6^3} = -1.11 \times 10^{-8}\,\hat{j}$

Thus $\mathrm{d}\boldsymbol{B} = 1.11 \times 10^{-8}$ T along negative y-axis

(iii) $\mathrm{d}\boldsymbol{B} = 0\ (\because\ \hat{k} \times \hat{k} = 0)$

(iv) $\mathrm{d}\boldsymbol{B} = \dfrac{(4\pi \times 10^{-7})(2)(2\hat{k} \times 3\hat{j})}{4\pi \qquad 3^3} = -4.44 \times 10^{-8}\hat{i}$

$\mathrm{d}\boldsymbol{B} = 4.44 \times 10^{-8}$ T along negative x-axis.

13.47 $H = n_0 i = \dfrac{Ni}{2\pi r} = \dfrac{100 \times 2}{2\pi \times 0.1} = 318.47$ A/m for both vacuum and material.

$B = K\mu_0 H = 1 \times 4\pi \times 10^{-7} \times 318.47 = 4 \times 10^{-4}$ T (vacuum)

$B = 500 \times 4\pi \times 10^{-7} \times 318.47 = 0.2$ T (material)

$M = \dfrac{B - \mu_0 H}{\mu_0} = 0 \qquad$ (vacuum)

$M = \dfrac{B}{\mu_0} - H = \dfrac{0.2}{4\pi \times 10^{-7}} - 318 = 1.59 \times 10^5 \qquad$ (material)

13.48 Use the formula for $B(z)$ on the axis of a circular coil of radius r carrying current i:

$$B(z) = \frac{\mu_0 i r^2}{2(r^2 + z^2)^{3/2}} \tag{1}$$

Use the following values:
$B = 6 \times 10^{-5}$ T, $\mu_0 = 4\pi \times 10^{-7}$ A/m, $r = 10^6$ m and z $= 6.4 \times 10^6$ m (distance of the pole from earth's centre) and solve for the current i. We find $i = 2.6 \times 10^{10}$ A. Thus the order of magnitude of current is 10^{10} A.

13.49 The induction midway between Helmholtz coils is (prob. 13.36)

$$B = \frac{8N\mu_0 I}{5^{3/2} R} \tag{1}$$

Given $N = 50$, $I = 10$ A, $R = 0.5$ m and $\mu_0 = 4\pi \times 10^{-7}$ H/m

$$\therefore \quad B = 8.94 \times 10^{-4}\ \text{T} \tag{2}$$

Emf generated between the centre and the rim of the disc is

$$\xi = \pi r^2 B f = \pi \times (0.1)^2 \times 8.94 \times 10^{-4} \times 16.66 = 468 \times 10^{-6}\ \text{V} = 468\ \mu\ \text{V}.$$

13.50 Given

$$l = 1.0\ \text{m},\ v = 3\hat{i} + 2\hat{j} + 3\hat{k},\ B = \hat{i} + 2\hat{j} + 3\hat{k}$$

Voltage developed

$$\xi = |\boldsymbol{v} \times \boldsymbol{B}|\, l \sin\phi = |\boldsymbol{v}|\,|B|\,(\sin\theta) l \sin\phi$$

where θ is the angle between v and B, ϕ is the angle which l makes with B:

$$|v| = \left(3^2 + 2^2 + 1^2\right)^{1/2} = \sqrt{14}$$
$$|\mathrm{B}| = \left(1^2 + 2^2 + 3^2\right)^{1/2} = \sqrt{14}$$
$$\cos\theta = \frac{v \cdot B}{|v|\,|B|} = \frac{3 + 4 + 3}{\left(\sqrt{14}\right)\left(\sqrt{14}\right)} = \frac{5}{7}$$
$$\therefore \quad \sin\theta = 0.4898$$
$$\therefore \quad \xi = \left(\sqrt{14}\right)\left(\sqrt{14}\right)(0.4898)\ (1)\ \sin\phi = 6.857\ \sin\phi$$

ξ will be maximum for $\phi = 90°$ and zero for $\phi = 0$ or $180°$.

13.51 The motion of the proton is equivalent to a current. The current density is given by

$$J = e\boldsymbol{v} \tag{1}$$

The magnetic field due to a current-carrying circuit is given by the Biot–Savart law

$$\mathrm{d}B = \frac{\mu_0 I}{4\pi}\left(\frac{\mathrm{d}l' \times \boldsymbol{R}}{R^3}\right) \tag{2}$$

When $\mathrm{d}l'$ is the circuit element, $\boldsymbol{R}$ is the vector which points from $\mathrm{d}l'$ to the field point. As there is only one proton there is no need to integrate to find B. Replacing the current by the current density J, the Biot–Savart law is modified as

$$B = \frac{\mu_0}{4\pi}\left(\frac{J \times \boldsymbol{R}}{R^3}\right) \tag{3}$$

Substituting (1) into (3)

$$B = \frac{\mu_0 e}{4\pi}\left(\frac{v \times \boldsymbol{R}}{R^3}\right)$$

$$\boldsymbol{R} = \left(\hat{i} + 2\hat{j}\right)\ \mathrm{m}$$

$$v = \left(\hat{i} + 3\hat{j}\right)\ 10^4\ \mathrm{m/s}$$

$$\boldsymbol{v} \times \boldsymbol{R} = 10^4 \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 0 \\ 1 & 2 & 0 \end{vmatrix} = -10^4\hat{k}$$

$$B = \frac{4\pi \times 10^{-7}}{4\pi} \times 1.6 \times 10^{-19} \times \frac{\left(-10^4\hat{k}\right)}{\left(\sqrt{5}\right)^3}$$

$$= 1.43 \times 10^{-23}\hat{k}\ \mathrm{T}$$

13.3.3 Magnetic Force

13.52 $\dfrac{F}{l} = \dfrac{\mu_0 i_1 i_2}{2\pi\, d} = \dfrac{4\pi \times 10^{-7} \times 2 \times 3}{2\pi \times 0.05} = 2.4 \times 10^{-5}\ \mathrm{N}$

13.53 $\dfrac{F}{l} = \dfrac{\mu_0 i_1 i_2}{2\,\pi\, d}$

$$\therefore \qquad i_2 = \frac{2\pi d}{\mu_0 i_1}\left(\frac{F}{l}\right) = \frac{2\pi \times 0.2 \times 10^{-5}}{4\pi \times 10^{-7} \times 10} = 1.0\ \mathrm{A}$$

The currents are parallel.

13.54 For equilibrium, magnetic force = gravitational force.

$$F = \frac{\mu_0 l i_1 i_2}{2\pi d} = \text{mg}$$

$$\therefore \quad d = \frac{\mu_0 l i_1 i_2}{2\pi mg} = \frac{4\pi \times 10^{-7} \times 2 \times 10 \times 15}{2\pi \times 4 \times 10^{-3} \times 9.8} = 1.53 \times 10^{-3}\,\text{m}$$

$$= 1.53\,\text{mm}$$

13.55 **(a)** Net force per metre on the outer wire

$$\frac{F}{l} = \frac{F_1}{l} + \frac{F_2}{l} = \frac{\mu_0 i_1 i_3}{2\pi d} + \frac{\mu_0 i_2 i_3}{2\pi \times 2d}$$

$$= \frac{\mu_0 i_3(2i_1 + i_2)}{4\pi d} = \frac{4\pi \times 10^{-7} \times 20(2 \times 20 + 20)}{4\pi \times 0.1} = 1.2 \times 10^{-3}\,\text{N}$$

(b) Zero

13.56 $F = ilB + (i)(\pi R)B + ilB = i(2l + \pi R)B.$

13.57 **(i)** The force on the horizontal segments is zero as they are perpendicular to the straight wire. For the vertical segments the force is repulsive for antiparallel currents and attractive for parallel currents, the magnitude being

$$F = \frac{\mu_0 l i_1 i_2}{2\pi d} \tag{1}$$

where d is the distance of separation.
Force on the nearer vertical segment

$$F_1 = -\frac{(4\pi \times 10^{-7})(0.1)(20)(5)}{(2\pi)(0.02)} = -1 \times 10^{-4}\,\text{N}$$

Force on the farther vertical segment

$$F_2 = +\frac{(4\pi \times 10^{-7})(0.1)(20)(5)}{(2\pi)(0.07)} = +2.86 \times 10^{-5}\,\text{N}$$

(ii) The net force on the coil, $F_{\text{net}} = F_1 + F_2 = -7.14 \times 10^{-5}\,\text{N}$

13.58 **(a)** Wire PQ will produce a field of induction B_1 at the segment AB of the coil. The magnitude of B_1 will be

$$B_1 = \frac{\mu_0 I_1}{2\pi d} \tag{1}$$

The right-hand rule shows that the direction of B_1 at the segment AB is down, wire AB, which carries a current I_2 finds itself immersed in an external field of magnetic induction B_1. A length l of the segment AB will experience a sideway magnetic force equal to i$\boldsymbol{l} \times \boldsymbol{B}$ whose magnitude is

$$F_2 = I_2 l B_1 = \frac{\mu_0 l I_1 I_2}{2\pi d} \tag{2}$$

The vector rule of signs shows that F_2 lies in the plane of the coil and points to the right. A similar reasoning shows that the force on the segment CD will be

$$F_3 = \frac{\mu_0 l I_1 I_2}{2\pi\,(d + l)} \tag{3}$$

to the left in the plane of the coil. There is no force on the segments BC and AD as they are perpendicular on PQ:

$$F_{\text{net}} = \frac{\mu_0 l I_1 I_2}{2\pi}\left[\frac{1}{d} - \frac{1}{d+l}\right] \tag{4}$$

to the right

$$F_{\text{net}} = \frac{(4\pi \times 10^{-7})(0.05)(1)(4)(10)}{2\pi}\left[\frac{1}{0.1} - \frac{1}{0.1 + 0.05}\right]$$
$$= 1.335 \times 10^{-5}\text{N}$$

where we have multiplied (4) by 10 for the number of coils.

13.59 **(a)** The flux ϕ_B enclosed by the loop, Fig. 13.30, is

$$\phi_B = Blx \tag{1}$$

where lx is the area of that part of the loop in which B is not zero. By Faraday's law

$$\xi = -\frac{\mathrm{d}\phi_\mathrm{B}}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}(Blx) = -Bl\frac{\mathrm{d}x}{\mathrm{d}t} = Bl\,v \tag{2}$$

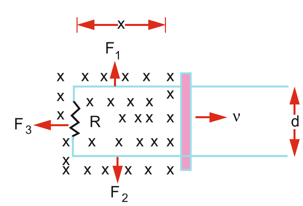


Fig. 13.30

where the speed $v = -\mathrm{d}x/\mathrm{d}t$ for the speed with which the connecting rod is pulled out of the magnetic field. The emf Blv sets up a current in the loop given by

$$i = \frac{\xi}{R} = \frac{Blv}{R} \tag{3}$$

(b) The total power delivered to the resistor is just the Joule heat given by

$$P_J = i^2 R = \frac{B^2 l^2 v^2}{R}$$

Note that energy conservation tells us that for steady motion of the rod the external agent must provide power equal to the Joule heat. That this is so is borne out from

$$P = F_3 v = \frac{B^2 l^2 v^2}{R}.$$

(c) The current the loop produces forces F_1, F_2 and F_3 on the three sides of the loop in accordance with

$$F = i\boldsymbol{l} \times \boldsymbol{B} \tag{4}$$

Because F_1 and F_2 are equal and opposite, they cancel each other, while F_3 which opposes the motion of the sliding rod is given by (4) and (3) in magnitude as

$$F_1 = ilB \sin 90^\circ = \frac{B^2 l^2 v}{R} \tag{5}$$

13.3.4 Magnetic Energy, Magnetic Dipole Moment

13.60 Elementary magnetic moment

$$\mathrm{d}\mu = (\mathrm{d}i)(\mathrm{d}A) = \left(\frac{\omega q r \mathrm{d}r}{\pi R^2}\right)(\pi r^2) = \frac{\omega q r^3 \mathrm{d}r}{R^2}$$

where $\mathrm{d}A$ is the area enclosed by r and the value of $\mathrm{d}i$ is used from (2) of prob. (13.37).
Therefore the magnetic moment of the disc is

$$\mu = \int \mathrm{d}\mu = \int_0^R \frac{\omega q r^3 \mathrm{d}r}{R^2} = \frac{\omega q R^2}{4}$$

13.61 Magnetic moment

$$\mu = NiA = (1)(i)(\pi r^2)$$

$$\therefore \quad i = \frac{\mu}{\pi r^2} = \frac{6.4 \times 10^{21}}{\pi\,(6.4 \times 10^6)^2} = 4.98 \times 10^7 \text{ A}$$

13.62 Magnetic energy density

$$u_\mathrm{B} = \frac{1}{2\mu_0} B^2$$

But $\quad B = \dfrac{\mu_0 i}{2r}$

$$\therefore \quad u_\mathrm{B} = \frac{1}{2\mu_0}\left(\frac{\mu_0 i}{2r}\right)^2 = \frac{1}{8}\frac{\mu_0 i^2}{r^2} = \frac{4\pi \times 10^{-7} \times (100)^2}{8 \times (0.1)^2}$$

$$= 0.157\,\mathrm{J/m^3}$$

13.63 $u_\mathrm{B} = \dfrac{1}{2\mu_0} B^2 = \dfrac{(13.5)^2}{2 \times 4\pi \times 10^{-7}} = 7.25 \times 10^7\ \mathrm{J/m^3}$

13.64 $\tau_{\max} = i\mathrm{AB} = (i)\,(\pi r^2)\,B$

But $\quad l = 2\pi r \rightarrow r = \dfrac{l}{2\pi}$

$$\therefore \qquad \tau_{\max} = \pi i \left(\frac{l}{2\pi}\right)^2 B = \frac{l^2 i B}{4\pi}$$

13.65 Let the sphere of radius R rotate about the z-axis, Fig. 13.31. Consider a spherical shell of radius $r(r < R)$ concentric with the sphere. Consider a volume element symmetrical about the z-axis.

$$\mathrm{d}V = 2\pi r^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}r \tag{1}$$

where θ is the polar angle and r is the distance of the volume element from the centre. The charge $\mathrm{d}q$ residing in the volume element is

$$\mathrm{d}q = \frac{3\,q}{4\,\pi\,R^3}\,\mathrm{d}V \tag{2}$$

The current due to rotation of charge is

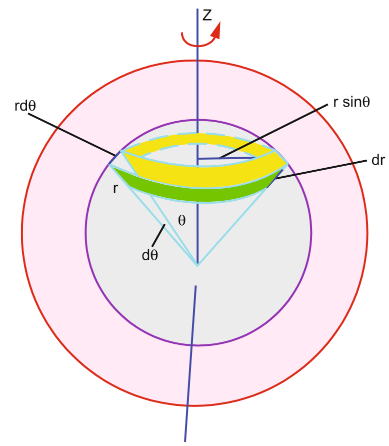


Fig. 13.31 Magnetic moment due to a rotating charged sphere

$$\mathrm{d}i = \frac{\omega}{2\pi}\,\mathrm{d}q = \frac{3q\omega}{4\pi}\,\frac{r^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}r}{R^3} \tag{3}$$

where we have used (2) and (1).
The magnetic moment due to di is

$$\mathrm{d}\mu = (\mathrm{d}i)(\mathrm{d}A) = \left(\frac{3q\,\omega r^2\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}r}{4\pi\,R^3}\right)(\pi\,r^2\sin^2\theta)$$
$$= \frac{3}{4}\,\frac{\omega\,q r^4 \mathrm{d}r\,\sin^3\theta\,\mathrm{d}\theta}{R^3} \tag{4}$$

where dA is the area enclosed by the circle of radius $r\,\sin\theta$.
Total magnetic moment

$$\mu = \int \mathrm{d}\mu = \frac{3}{4}\,\frac{\omega\,q}{R^3}\int_0^R r^4 \mathrm{d}r \int_0^\pi \sin^3\theta\,\mathrm{d}\theta = \frac{3}{4}\,\frac{\omega\,q}{R^3}\left(\frac{R^5}{5}\right)\left(\frac{4}{3}\right) = \frac{\omega\,q\,R^2}{5} \tag{5}$$

The angular momentum of a sphere about a diameter is

$$L = I\,\omega = \frac{2}{5}\,m\,R^2\omega \tag{6}$$

$$\therefore \qquad \frac{\mu}{L} = \frac{q}{2m} \tag{7}$$

13.66 $U = -p \cdot E = -p\,E\,\cos\theta = -(1.6 \times 10^{-29})(1000)\cos 30° = -1.38 \times 10^{-26}$ J.

13.3.5 Faraday's Law

13.67 $$\frac{\mathrm{d}r}{\mathrm{d}t}\,\alpha\, t \qquad \text{(by problem)} \tag{1}$$

$$\xi = -\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\,(\mathrm{BA}) \qquad \text{(by Faraday's law)} \tag{2}$$

$$\text{But} \qquad B \propto \frac{1}{r} \text{ and } A = \pi r^2 \tag{3}$$

$$\therefore \qquad \xi \propto \frac{\mathrm{d}r}{\mathrm{d}t} \tag{4}$$

$$\text{or} \qquad \xi \propto t \tag{5}$$

where we have used (1)

13.68 0.4 Oe $= 0.4 \times 80$ A/m $= 32$ A/m

$$B = \mu_0 H = 4\pi \times 10^{-7} \times 32 = 4.02 \times 10^{-5}\,\mathrm{T}$$
$$v = 720\,\mathrm{km/h} = 200\,\mathrm{m/s}$$
$$\xi = Blv = 4.02 \times 10^{-5} \times 30 \times 200 = 0.24\,\mathrm{V}$$

13.69 $V = \pi r^2 B f = \pi(0.1)^2 \times 2 \times 10^{-5} \times 5 = 3.14 \times 10^{-6}$ V

13.70 $R = \dfrac{BAN}{q} = \dfrac{0.1 \times 0.001 \times 30}{10^{-5}} = 300\,\Omega$

13.71 $\xi = -\dfrac{\Delta\phi}{\Delta t} = \dfrac{A\Delta B}{\Delta t} = \dfrac{0.2 \times 0.25}{10^{-4}} = 500$ V

$$i = \frac{\xi}{R} = \frac{500}{20} = 25\,\mathrm{A}$$

13.72 The amplitude of the induced voltage

$$\xi_0 = \omega BA = 2\pi fBA = 2\pi \times 40 \times 0.5 \times (0.25)^2 = 7.85\,\mathrm{V}$$

Amplitude of the induced current

$$I_0 = \frac{\xi_0}{R} = \frac{7.85}{4} = 1.96\,\mathrm{A}$$

13.73 $\xi = iR = 0.4 \times 5 = 2$ V

$$\xi = vlB$$

$$\therefore \quad v = \frac{\xi}{lB} = \frac{2}{0.2 \times 1.0} = 10\,\text{m/s}$$

13.74 $\xi = -\dfrac{\mathrm{d}\phi}{\mathrm{d}t} = -\dfrac{A\mathrm{d}B}{\mathrm{d}t}$

A = area of the triangle sandwiched between the ends of the rod and the radii connecting the centre with the ends, Fig. 13.18:

$$A = \frac{1}{2}\,(\text{base})(\text{altitude}) = \frac{1}{2} l \sqrt{R^2 - (l/2)^2}$$

Thus $\quad \xi = -\dfrac{\mathrm{d}B}{\mathrm{d}t}\dfrac{l}{2}\sqrt{R^2 - (l/2)^2}$

13.75 Gravitational force on the loop

$$F_\text{g} = mg\,\sin\theta$$

Magnetic force on the loop

$$F_\text{m} = (B\,\cos\theta)\,il = B\,\cos\theta\,\frac{\xi\,l}{R}$$

$$= \frac{B\,\cos\theta\,l}{R}\,v\,B\cos\theta \cdot l = \frac{vB^2l^2\cos^2\theta}{R}$$

For steady speed, $F_{net} = F_\text{m} - F_\text{g} = 0$

$$\therefore \quad \frac{v\,B^2l^2\cos^2\theta}{R} - mg\,\sin\theta = 0$$

or $\quad v = \dfrac{mg\,R\,\sin\theta}{B^2l^2\cos^2\theta}$

13.76 $\xi = \pi R^2 B f$

$$\therefore \quad B = \frac{\xi}{\pi R^2 f} = \frac{6.28 \times 10^{-3}}{3.14 \times (0.1)^2\,(1200/60)} = 0.01\,\text{T}$$

13.77

$$\begin{aligned}
[\text{emf}] &= [\text{electric field}][\text{distance}] \\
&= [\text{force/charge}][\text{distance}] \\
&= [MLT^{-2}\,Q^{-1}][L] = [ML^2T^{-2}Q^{-1}] \\
[\mathrm{d}\phi_B/\mathrm{d}t] &= [\phi_B][T^{-1}] = [B][\text{area}][T^{-1}] \\
&= [\text{force/(velocity) charge}][L^2][T^{-1}] \\
&= [MLT^{-2}/LT^{-1}Q][L^2][T^{-1}]
\end{aligned}$$

$$= [ML^2T^{-2}Q^{-1}]$$

$$\therefore \quad [\text{emf}] = [\mathrm{d}\phi_B/\mathrm{d}t]$$

13.78 The flux through the coil

$$\phi = B \cdot A = 15 \times 10^{-4} \times 3 \times 10^{-12}\ \sin(4 \times 10^6 t)$$

$$\xi = -N\frac{\mathrm{d}\phi}{\mathrm{d}t} = -200 \times 15 \times 10^{-4} \times 3 \times 4 \times 10^6 \times 10^{-12} \cos(4 \times 10^6 t)$$

$$= -3.6 \times 10^{-6} \cos(4 \times 10^6\, t)\ \mathrm{V}$$

13.79 $\xi \propto \dfrac{\mathrm{d}\phi}{\mathrm{d}t}$ or $\propto \dfrac{\mathrm{d}B}{\mathrm{d}t}$

$$B = B_0 \sin(\omega t + \phi)$$

$$\therefore \quad \xi \propto \omega B_0 \cos(\omega t + \phi)$$

$$\therefore \quad \frac{\xi_{\max}\ (\text{television})}{\xi_{\max}\ (\text{radio})} = \frac{\omega_1}{\omega_2} = \frac{f_1}{f_2} = \frac{100}{1} = 100$$

$\because$ B_0 is the same for both the waves.

13.80 The flux ϕ_B enclosed by a loop of area A is given by $\phi_B = BA$, where B is the magnetic field. Faraday's law of induction says that the induced emf ξ is a circuit equal to the negative rate at which the flux through the circuit is changing. In symbols $\xi = -\mathrm{d}\phi_B/\mathrm{d}t$, the current being $I = \xi/R$, where R is the resistance. The magnetic force on a straight wire is given by $F = i\boldsymbol{l} \times \boldsymbol{B}$:

$$\phi_B = BA = Blx\,(B \perp l)$$

By Faraday's law

$$\xi = -\frac{\mathrm{d}\varphi_B}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}(Blx) = -Bl\frac{\mathrm{d}x}{\mathrm{d}t} = Blv$$

$$i = \frac{\xi}{R} = \frac{Blv}{R}$$

$$\therefore \quad \text{Force } F = ilB = \frac{B^2l^2v}{R}$$

If the magnetic force acts as a resisting force then equation of motion will be

$$ma = -\frac{B^2l^2v}{R}$$

$$\text{or} \quad m\frac{\mathrm{d}v}{\mathrm{d}t} = m\frac{\mathrm{d}v}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = mv\frac{\mathrm{d}v}{\mathrm{d}s} = -\frac{B^2l^2v}{R}$$

$$\therefore \qquad \mathrm{d}v = -\left(\frac{B^2 l^2}{mR}\right)\mathrm{d}s$$

$$v = \int \mathrm{d}v = -\frac{B^2 l^2 s}{mR} + C$$

where C = constant of integration.
When $s = 0$, $v = u$

$$\therefore \qquad c = u$$

$$\therefore \qquad s = \frac{(u - v)mR}{B^2 l^2}$$

$$\therefore \qquad s_{\max} = \frac{u\,m\,R}{B^2 l^2}$$

where we have put $v = 0$.

13.81 $\xi = -N\dfrac{\mathrm{d}\phi}{\mathrm{d}t} = -N\dfrac{\mathrm{d}}{\mathrm{d}t}(BA) = -N\pi r^2\dfrac{\mathrm{d}B}{\mathrm{d}t}$ (Faraday's law)

$= -100 \times \pi(0.1)^2(0.1) = 0.314\,\mathrm{V}$

13.82 **(i)** The flux on one side is equal and opposite to that on the other for $0 < r < b - x$, where r is the distance of any point from the long wire. Only the flux through the portion $b - x < r < x$ is not cancelled:

$$\mathrm{d}\phi = (a\mathrm{d}r)\mathrm{d}b = \frac{\mu_0 I a \mathrm{d}r}{2\pi\, r}$$

$$\therefore \qquad \phi = \frac{\mu_0 I a}{2\pi}\int_{b-x}^{x}\frac{\mathrm{d}r}{r} = \frac{\mu_0 I a}{2\pi}\ln\left(\frac{x}{b - x}\right)$$

(ii) $\phi \to \infty$ for $x \to b$ and $\phi = 0$ for $x = b/2$

(iii) For $I = 2t$ and $x = b/4$

$$\phi = \frac{\mu_0 a}{2\pi}\ln\left(\frac{1}{3}\right)2t$$

$$\xi = -\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\mu_0 a \ln 3}{\pi}$$

13.83 Betatron is a machine to accelerate electrons to high energy. It consists of an evacuated 'doughnut' in which the electrons are made to circulate under the influence of changing magnetic field. If a magnetic flux ϕ changes in an electromagnet, it accelerates the electrons and at the same time holds them in an orbit of fixed radius. The average force acting on the particle during a single rotation is the work (induced voltage multiplied by charge) divided by the distance $2\pi R$. Equating this to the time rate of change of momentum, by Faraday's law of induction

$$e\frac{\mathrm{d}\phi}{\mathrm{d}t} = 2\pi R\frac{\mathrm{d}P}{\mathrm{d}t}$$
$$e\frac{\mathrm{d}\phi/\mathrm{d}t}{2\pi R} = \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(BeR)$$

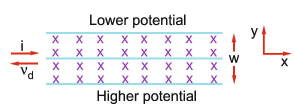
where B is the magnetic field and R is the radius.
Integrating and assuming that initially the flux is zero and R is constant

$$\phi = 2\pi R^2 B$$

13.3.6 Hall Effect

13.84 Consider a strip of a conductor of width W and thickness t carrying a d.c current i in the positive x-direction along its length, Fig. 13.32. A magnetic field B set up in the z-direction into the page produces a deflection force in the positive y-direction, as the drift velocity of electrons is in the negative x-direction. Consequently charge concentration builds up towards the upper edge of the strip. As the charges collect on one side of the strip they set up an electric field that opposes sideways motion of additional charge carriers inside the conductor. This build up of charges establishes a potential V_H across the width of the strip, called Hall potential and the phenomenon is known as Hall effect. Eventually equilibrium conditions are reached and a maximum voltage, known as Hall voltage, is quickly established. The sign of the voltage gives the sign of charge carriers and its magnitude the number density n of charge carriers:

Fig. 13.32 Hall effect



(a) $E_\mathrm{H} = v_\mathrm{d} B = \frac{iB}{ne} = \frac{iB}{newb}$

$$= \frac{100 \times 10}{1.25 \times 10^{30} \times 1.6 \times 10^{-19} \times 10^{-3} \times 0.02} = 2.5 \times 10^{-4}\ \mathrm{V/m}$$

The field is along positive y-direction.

(b) When equilibrium is established the Lorentz force is zero:

$$qE + q\boldsymbol{v}_\mathrm{d} \times \boldsymbol{B} = 0$$
$$\text{or} \qquad \boldsymbol{E} = -\boldsymbol{V}_\mathrm{d} \times \boldsymbol{B}$$
$$E = -v_\mathrm{d} B \qquad (\because\ \boldsymbol{v}_\mathrm{d} \perp \boldsymbol{B})$$
$$E = \frac{V_\mathrm{H}}{W} = v_\mathrm{d} B = \frac{jB}{ne} = \frac{iB}{W\,t\,ne}$$
$$\therefore \qquad n = \frac{iB}{etV_\mathrm{H}} = \frac{100 \times 10}{1.6 \times 10^{-19} \times 10^{-3} \times 5 \times 10^{-6}}$$
$$= 1.25 \times 10^{30}/\mathrm{m}^3$$

13.85 If R_H is the Hall coefficient, σ the electrical conductivity, then the mobility μ is given by

$$\mu = R_\mathrm{H}\sigma = (-7.3 \times 10^{-5})\,(2 \times 10^3) = -0.146\,\mathrm{m}^2/\mathrm{V/s}$$

The magnitude is $0.146\,\mathrm{m}^2/\mathrm{V/s}$.