14.100 Calculate the number of states of electromagnetic radiation between 5000 and 6000 Å in wavelength using periodic boundary conditions in a cubical region 0.5 cm on a side.

[University of Manchester 1972]

14.101 Consider a car entering a tunnel of dimensions 15 m wide and 4 m high. Assuming the walls are good conductors, can AM radio waves (530–1600 kHz) propagate in the tunnel?

[The University of Aberystwyth, Wales 2006]

14.102 Calculate the least cut-off frequency for TE_{mn} waves for a rectangular waveguide of dimensions 5 cm × 4 cm.

14.103 Calculate how the wave and group velocities of the TE_{01} wave in a rectangular waveguide with $a = 1$ cm and $b = 2$ cm vary with frequency.

[The University of Wales, Aberystwyth 2004]

14.104 Consider a rectangular waveguide of dimensions $x = a$ and $y = b$, the TM_{mn} wave travelling in the z-direction which is the axis of the guide. Given that the z-component E_z satisfies the equation

$$\left(\frac{\partial^z}{\partial x^z} + \frac{\partial^z}{\partial y^z}\right) E_z = (k^2 - \omega^2 \mu \varepsilon) E_z$$

obtain (a) the solution for E_z and (b) the cut-off frequency.

14.105 Consider a rectangular waveguide of dimensions $x = a$ and $y = b$, the wave travelling along the z-direction, the axis of the guide. Given that the z-component H_z satisfies the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) H_z = (k^2 - \omega^2 \mu \varepsilon) H_z$$

(a) obtain the solution for H_z. (b) Obtain the cut-off frequency. (c) What are the similarities and differences between TM_{mn} mode and TE_{mn} mode?

14.3 Solutions

14.3.1 The RLC Circuits

14.1 **(a)** Reactance of capacitor

$$X_c = \frac{1}{2\pi f c} = \frac{1}{2\pi \times (1000/2\pi) \times 2.5 \times 10^{-6}} = 400\ \Omega$$

Impedance of the circuit

$$Z = \sqrt{R^2 + X_c^2} = \sqrt{(300)^2 + (400)^2} = 500\,\Omega$$

The rms current, $I_e = \dfrac{V_e}{Z} = \dfrac{50}{500} = 0.1\,\mathrm{A}$
PD across the capacitor, $V_c = I_e X_c = 0.1 \times 400 = 40\,\mathrm{V}$

(b) Power, $P = V_e I_e \cos\alpha = V_e I_e \dfrac{R}{Z} = 50 \times 0.1 \times \dfrac{300}{500} = 3.0\,\mathrm{W}$

14.2 **(a)** $X_L = 2\pi f L = 2\pi \times \dfrac{200}{\pi} \times 10 \times 10^{-3} = 4\,\Omega$

$$Z = \sqrt{R^2 + X^2} = \sqrt{3^2 + 4^2} = 5\,\Omega$$

$$I_e = V_e/Z = 5/5 = 1.0\,\mathrm{A}$$

$$V_R = I_e R = 1.0 \times 3 = 3.0\,\mathrm{V}$$

$$V_L = I_e X_L = 1.0 \times 4 = 4.0\,\mathrm{V}$$

(b) Phase angle between V_e and I_e is given by

$$\tan\alpha = \frac{X_L}{R} = \frac{4}{3} \Rightarrow \alpha = 53^\circ$$

V_e leads I_e by 53°

14.3 $U = \dfrac{1}{2} L I^2$

$$L = \frac{2U}{I^2} = \frac{2 \times 10}{5^2} = 0.8\,\mathrm{H}$$

14.4 **(a)** $f = \dfrac{1}{2\pi\sqrt{LC}} = \dfrac{1}{2\pi\sqrt{4 \times 10^{-6} \times 5 \times 10^{-11}}} = 1.125 \times 10^{-7}\,\mathrm{Hz}$

(b) $\lambda = \dfrac{c}{f} = \dfrac{3 \times 10^8}{1.125 \times 10^7} = 26.67\,\mathrm{m}$

14.5 **(a)** $Z = \sqrt{(X_L - X_C)^2 + R^2} = \sqrt{(12 - 20)^2 + 6^2} = 10\,\Omega$

(b) $P = I_e V_e \dfrac{R}{Z} = \dfrac{V_e^2 R}{Z^2} = \dfrac{(250)^2 \times 6}{10^2} = 3750\,\mathrm{W}$

14.6 At $\omega_0 = 600\,\mathrm{rad/s}$, $X_L = X_C$

$$\therefore \quad \omega_0 L = \frac{1}{\omega_0 C} \Rightarrow \frac{1}{LC} = \omega_0^2$$

At $\omega = 60\,\mathrm{rad/s}$, $\dfrac{X_C}{X_L} = \dfrac{1/\omega c}{\omega L} = \dfrac{1}{\omega^2 LC} = \dfrac{\omega_0^2}{\omega^2} = \dfrac{(600)^2}{60^2} = 100$

14.7 **(a)** $X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$

$$C = \frac{1}{2\pi f\, X_C} = \frac{1}{2\pi \times 250 \times 4} = 1.57 \times 10^{-4}\,\text{F}$$

(b) $X'_C = \frac{1}{2\pi f'\, C} = \frac{1}{2\pi \times 100 \times 1.57 \times 10^{-4}} = 10\,\Omega$

(c) $I_e = \frac{V_e}{X'_C} = \frac{220}{10} = 22\,\text{A}$

14.8 $\frac{V_e}{I_e} = \sqrt{\omega^2 L^2 + R^2} = \sqrt{4\pi^2 f^2 L^2 + R^2}$

$$\therefore \quad \frac{12}{0.05} = \sqrt{4\pi^2 \times (50)^2 L^2 + R^2} \tag{1}$$

Also $\quad \tan\alpha = \frac{\omega L}{R} = \frac{2\pi f L}{R}$

$$\therefore \quad \tan 60^\circ = \sqrt{3} = 2\pi \times 50 \frac{L}{R} \tag{2}$$

Solving (1) and (2) we find $R = 120\,\Omega$ and $L = 0.66\,\text{H}$.
When the capacitor of capacitance C is connected in series with the above circuit $\alpha = 0$.

$$\tan\alpha = \tan 0^\circ = \frac{1}{R}\left(\omega L - \frac{1}{\omega c}\right)$$

$$\therefore \quad C = \frac{1}{\omega^2 L} = \frac{1}{(2\pi \times 50)^2 \times 0.66} = 15.37 \times 10^{-6}\,\text{F} = 15.37\,\mu\text{F}$$

14.9 **(a)** The rms current, $I_e = \frac{V_e}{\omega L} = \frac{V_e}{2\pi f L} = \frac{220}{2\pi \times 50 \times 0.6} = 1.17\,\text{A}$

(b) Peak current, $I_0 = \sqrt{2} I_e = 1.414 \times 1.17 = 1.65\,\text{A}$.

14.10 (**a**) $I_e = \frac{V_e}{\sqrt{4\pi^2 f^2 L^2 + R^2}} = \frac{24}{\sqrt{4\pi^2 \times (50)^2 \times (0.01)^2 + 4^2}} = 4.72\,\text{A}$

(**b**) Power, $P = I_e^2 R = (4.72)^2 \times 4 = 89\,\text{W}$

(**c**) $V_R = I_e R = 4.72 \times 4 = 18.88\,\text{V}$

$V_L = 2\pi f L I_e = 2\pi \times 50 \times 0.01 \times 4.72 = 13.82\,\text{V}$

14.11 $V = V_m \sin\omega t$

$$0.5\, V_m = V_m \sin\left(2\pi f \frac{1}{360}\right)$$

$$\therefore \quad \frac{2\pi f}{360} = \frac{\pi}{6} \Rightarrow f = 30\,\text{Hz}$$

14.12 For the parallel RLC circuit, Fig. 14.1

$$Q = CV,\ I_2 = -\frac{\mathrm{d}Q}{\mathrm{d}t} = -C\frac{\mathrm{d}V}{\mathrm{d}t}$$
$$V = R_\mathrm{p}(I_2 + I_1) = -L\frac{\mathrm{d}I_1}{\mathrm{d}t}$$
$$\frac{\mathrm{d}V}{\mathrm{d}t} = R_\mathrm{p}\left(\frac{\mathrm{d}I_2}{\mathrm{d}t} + \frac{\mathrm{d}I_1}{\mathrm{d}t}\right) = -CR_\mathrm{p}\frac{\mathrm{d}^2V}{\mathrm{d}t^2} - R_\mathrm{p}\frac{V}{L}$$

$$\therefore \quad \frac{\mathrm{d}^2V}{\mathrm{d}t^2} + \frac{1}{R_\mathrm{p}C}\frac{\mathrm{d}V}{\mathrm{d}t} + \frac{V}{LC} = 0 \qquad \text{(parallel arrangement)}$$

Compare the above equation with the given equation

$$\frac{\mathrm{d}^2V}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}V}{\mathrm{d}t} + \frac{V}{LC} = 0$$
$$\therefore \quad R_\mathrm{p} = \frac{L}{CR}.$$

14.13 We can find the dimensions of μ_0 and ε_0 from the following set of formulae:

$$F = ilB \quad \text{(force on the current-carrying wire)} \tag{1}$$
$$B = \frac{\mu_0 i}{2\pi r} \quad \text{(magnetic field due to a current-carrying wire)} \tag{2}$$
$$F = \frac{Q^2}{4\pi\varepsilon_0 r^2} \quad \text{(electrostatic force between charges)} \tag{3}$$
$$i = Q/t \tag{4}$$

Combining (1)–(4), $[\mu_0\varepsilon_0] = [T^2/L^2]$

$$\text{or} \quad \left[\frac{1}{\sqrt{\mu_0\varepsilon_0}}\right] = \left[\frac{L}{T}\right] = [v] = [c]$$

14.14 We first derive the dimensional formulae for R, C and L from the defining equation Power $= i^2R$

$$[\text{Power}] = [Ml^2T^{-3}] = [A^2R] = [A^2][R]$$
$$\therefore \quad [R] = [Ml^2T^{-3}A^{-2}] \tag{1}$$

Energy of a capacitor $E = \dfrac{1}{2}\dfrac{Q^2}{C}$ and $Q = it$

$$\therefore [C] = [M^{-1}l^{-2}T^4A^2] \tag{2}$$

Energy of an inductance, $E = \frac{1}{2} Li^2$

$$\therefore [L] = [Ml^2 T^{-2} A^{-2}] \tag{3}$$

Using (1), (2) and (3), it is observed that the given combinations have the dimension of time and therefore are expressed in seconds

$$[RC] = \left[L/R\right] = \left[\sqrt{LC}\right] = [T]$$

14.15 $$\omega = \frac{1}{\sqrt{LC}} \tag{1}$$

$$\omega' = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \tag{2}$$

$$\frac{\omega - \omega'}{\omega} = 1 - \frac{\omega'}{\omega} = 1 - \sqrt{1 - \frac{R^2 C}{4L}} \simeq 1 - \left(1 - \frac{R^2 C}{8L}\right) = \frac{R^2 C}{8L} \tag{3}$$

where we have used (1) and (2) and expanded the radical binomially.

$$q = q_{\mathrm{m}} \mathrm{e}^{-Rt/2L}, \; q/q_{\mathrm{m}} = \frac{1}{2}$$

whence $t = \frac{2L}{R} \ln\left(\frac{q_{\mathrm{m}}}{q}\right) = \frac{2L}{R} \ln 2$ (4)

If n is the number of cycles and ν the oscillating frequency

$$t = \frac{n}{\nu} = 2\pi n\sqrt{LC} = \frac{2L}{R} \ln 2$$

or $\frac{CR^2}{L} = \frac{(\ln 2)^2}{(\pi n)^2}$ (5)

Combining (3) and (5)

$$\frac{\omega - \omega'}{\omega} = \frac{\Delta\omega}{\omega} = \frac{(\ln 2)^2}{8\pi^2 n^2} = \frac{0.006085}{n^2} = 0.00038$$

where we have put $n = 4$.

14.16 $q = q_{\mathrm{m}} \mathrm{e}^{-Rt/2L} \cos\omega' t$ (charge oscillation of damped oscillator)
Differentiating with respect to time

$$i = \frac{\mathrm{d}q}{\mathrm{d}t} = -q_{\mathrm{m}}\omega' \mathrm{e}^{-Rt/2L}\left(\frac{R}{2L\omega'}\cos\omega' t + \sin\omega' t\right)$$

$$= -q_m \omega' e^{-Rt/2L} \quad (\tan\phi \cos\omega' t + \sin\omega' t)$$

where we have set $R/2L\omega' = \tan\phi$. This gives

$$i = \frac{-q_{\mathrm{m}}\omega' \mathrm{e}^{-Rt/2L}}{\cos\phi} \quad (\sin\phi\cos\omega' t + \cos\phi\sin\omega' t)$$
$$= -q_{\mathrm{m}}\omega' \mathrm{e}^{-Rt/2L}\frac{\sin(\omega' t + \phi)}{\cos\phi}$$

But for low damping $\phi \to 0$ as $R/2L\omega' \to 0$.

$$\therefore \quad \cos\phi \to 1, \text{ so that}$$
$$i = -q_{\mathrm{m}}\omega' e^{-Rt/2L}\sin(\omega' t + \phi)$$

14.17 Equation of the circuit is

$$L\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + \frac{1}{C}q = \xi \tag{1}$$

Multiply (1) by $i = \mathrm{d}q/\mathrm{d}t$

$$Li\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{C}q\frac{\mathrm{d}q}{\mathrm{d}t} = \xi t$$
$$\text{or} \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}Li^2\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{q^2}{2C}\right) = P_{\text{input}} \tag{2}$$

Thus, the input power is the sum of the powers delivered to the inductor and the capacitor.

14.18 The circuit equation is

$$L\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{C}q = 0 \tag{1}$$

Multiply (1) by $i = \dfrac{\mathrm{d}q}{\mathrm{d}t}$

$$Li\frac{\mathrm{d}i}{\mathrm{d}t} + Ri^2 + \frac{1}{C}q\frac{\mathrm{d}q}{\mathrm{d}t} = 0$$
$$\text{or} \quad \frac{d}{dt}\left(\frac{1}{2}Li^2 + \frac{q^2}{2C}\right) = -Ri^2$$
$$\text{or} \quad \frac{\mathrm{d}E}{\mathrm{d}t} = -i^2 R \tag{2}$$

where $E = \dfrac{1}{2}Li^2 + \dfrac{q^2}{2C}$ = total energy

14.19 If U is the total field energy then

$$U = U_B + U_E = \frac{1}{2}Li^2 + \frac{1}{2}\frac{Q^2}{C} \tag{1}$$

which shows that at any time the energy is stored partly in the magnetic field in the conductor and partly in the electric field in the capacitor. In the presence of the resistance R the energy is transferred to Joule heat, being given by

$$\frac{dU}{dt} = -i^2 R \tag{2}$$

the minus sign signifying that the stored energy U decreases with time. Differentiating (1) with respect to time and equating the result with (2) gives

$$Li\frac{di}{dt} + \frac{Q}{C}\frac{dQ}{dt} = -i^2 R \tag{3}$$

Substituting $i = dQ/dt$ and $di/dt = d^2Q/dt^2$, (3) becomes

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = 0 \tag{4}$$

Writing $R/L = 2\gamma$ and $1/LC = \omega_0^2$, (4) takes the required form

$$\frac{d^2Q}{dt^2} + 2\gamma\frac{dQ}{dt} + \omega_0^2 Q = 0 \tag{5}$$

(a) $f_0 = \dfrac{1}{2\pi\sqrt{LC}} = \dfrac{1}{2\pi\sqrt{80 \times 10^{-3} \times 700 \times 10^{-12}}} = 2.128 \times 10^4\,\text{Hz}$

(b) $\tau = \dfrac{L}{R} = \dfrac{80 \times 10^{-3}}{100} = 8 \times 10^{-4}\,\text{s}$

14.20
$$\frac{d^2Q}{dt^2} + 2\gamma\frac{dQ}{dt} + \omega_0^2 Q = 0 \tag{1}$$

Let $Q = e^{\lambda t}$ so that $dQ/dt = \lambda e^{\lambda t}$ and $d^2Q/dt^2 = \lambda^2 e^{\lambda t}$.
The characteristic equation is then

$$\lambda^2 + 2\gamma\lambda + \omega_0^2 = 0$$

whose roots are $\lambda = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$ (2)

Calling $\alpha = \sqrt{\gamma^2 - \omega_0^2}$

$\lambda_1 = -\gamma + \alpha, \lambda_2 = -\gamma - \alpha$

The general solution becomes

$$Q = C_1 e^{(-\gamma+\alpha)t} + C_2 e^{(-\gamma-\alpha)t} \tag{3}$$

The constants C_1 and C_2 are determined from the initial conditions. Suppose at $t = 0$, $Q = Q_0$ and

$$i = dQ/dt = 0$$

$$Q_0 = C_1 + C_2 \tag{4}$$

$$\frac{dQ}{dt} = C_1(-\gamma + \alpha) - C_2(\gamma + \alpha) = 0 \tag{5}$$

Solving (4) and (5), $C_1 = Q_0(\gamma + \alpha)/2\alpha$ and $C_2 = Q_0(\alpha - \gamma)/2\alpha$
Substituting C_1 and C_2 in (3)

$$Q = \frac{1}{2} Q_0 e^{-\gamma t}\left[(1 + \gamma/\alpha)e^{\alpha t} + (1 - \gamma/\alpha)e^{-\alpha t}\right] \tag{6}$$

For underdamping condition resistance R is small so that $\gamma < \omega_0$ and α is imaginary and may be written as $\alpha = j\omega'$, where j is imaginary. The roots of the characteristic equation are complex conjugate.

$$\omega'^2 = \omega_0^2 - \gamma^2 \tag{7}$$

Equation (6) reduces to

$$Q = Q_0 e^{-\gamma t}[\cos\omega' t + (\gamma/\omega')\sin\omega' t] \tag{8}$$

Calling $\sin\varepsilon = -\gamma/\omega_0$ and $\cos\varepsilon = \omega'/\omega_0$, (8) becomes

$$Q = \left(\frac{\omega_0 Q_0}{\omega'}\right) e^{-\gamma t}\cos(\omega' t + \varepsilon) \tag{9}$$

$$\text{or} \quad Q = A e^{-\gamma t}\cos(\omega' t + \varepsilon) \tag{10}$$

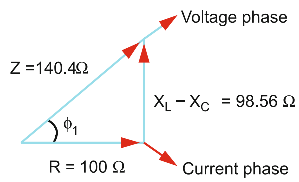
where the amplitude $A = \omega_0 Q_0/\omega'$ and the phase $\varepsilon = \tan^{-1}(-\gamma/\omega')$.
The constants A and ε which are real are determined by initial conditions. Equation (9) represents a damped harmonic motion of period

$$T' = \frac{2\pi}{\sqrt{\omega_0^2 - \gamma^2}} \tag{11}$$

As in the case of an undamped oscillation, the frequency is independent of the amplitude but is always lower than that of the undamped oscillator. The amplitude of oscillations $Ae^{-\gamma t}$ decreases exponentially and is no longer constant.

14.21 Phasor diagram, Fig. 14.5

Fig. 14.5



(i) $X_L = \omega L = 2\pi f L = (2\pi)(80)(0.2) = 100.48\,\Omega$

$$X_C = \frac{1}{\omega c} = \frac{1}{2\pi f c} = \frac{1}{(2\pi)(80)(10 \times 10^{-6})} = 199.04\,\Omega$$

(ii) $Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(100)^2 + (100.48 - 199.04)^2} = 140.4\Omega$

(iii) $I_T = \dfrac{V}{Z} = \dfrac{600}{140.4} = 4.27\,\mathrm{A(rms)}$

(iv) $\cos\varphi = \dfrac{R}{Z} = \dfrac{100}{140.4} = 0.71225 \to \varphi = 44.58°$

(v) $V_R = I_T R = 4.27 \times 100 = 427\,V_{rms}$

$V_C = I_T X_C = 4.27 \times 199.04 = 850\,V_{rms}$

$V_L = I_T X_L = 4.27 \times 100.48 = 429\,V_{rms}$

$V^2 = V_R^2 = (V_L - V_C)^2$

The voltages on R, C and L are shown in the phasor diagram, Fig. 14.6. Here the voltage lags the current as $X_C > X_L$.

(vi) $\omega^2 = \dfrac{1}{LC}$

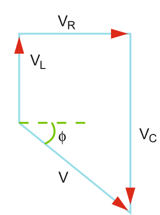


Fig. 14.6

(vii) The circuit will be in resonance when $X_L = X_C$, that is, $\omega L = \dfrac{1}{\omega c}$ or $\omega^2 = \dfrac{1}{LC}$.

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.2\times 10\times 10^{-6}}} = 112.6\,\text{Hz}$$

14.22 $X_C = \frac{1}{2\pi fC} = \frac{1}{(2\pi)(300)(50\times 10^{-6})} = 10.6\,\Omega$

$$Z = \sqrt{R^2 + X_C^2} = \sqrt{(300)^2 + (10.6)^2} = 300.19\,\Omega$$

$$I = \frac{V}{Z} = \frac{5}{300.19} = 0.0166\,\text{A}$$

14.23 Impedance of a capacitor

i. Let an AC emf be applied across a capacitor. The potential difference across the capacitor will be

$$V_C = V_0 \sin\omega t \tag{1}$$

where V_0 is the amplitude of the AC voltage of angular frequency $\omega t = 2\pi f$, across the capacitor.

$$q_C = CV_C = CV_0 \sin\omega t \tag{2}$$

The current $i_c = \frac{dq_C}{dt} = \omega CV_0 \cos\,\omega t = \omega CV_0 \sin(\omega t + 90^\circ)$ (3)

or $i_c = \frac{V_0}{X_c}\sin\,(\omega t + 90^\circ)$ (4)

where $X_C = 1/\omega C$ (5)

Comparison of (4) with (1) shows that i_c leads V_C by 90° or quarter of a cycle. Further, the current amplitude

$$I_0 = \frac{V_0}{X_C} \tag{6}$$

By Ohm's law X_C is to be regarded as impedance offered by the capacitor. In complex plane

$$Z_c = \frac{-j}{\omega C} \tag{7}$$

where j is imaginary.

Impedance of an inductance

On applying an AC across an inductance the potential difference will be

$$V_L = V_0 \sin\omega t \tag{8}$$

where V_0 is the amplitude of V_L. By Faraday's law of induction ($\xi = -L\ di/dt$) we can write

$$V_L = L\frac{di_L}{dt} \tag{9}$$

Combining (8) and (9)

$$\frac{di_L}{dt} = \frac{V_0}{L}\sin\omega t \tag{10}$$

$$\therefore \quad i_L = \int di_L = \frac{V_0}{L}\int \sin\omega t\, dt = -\left(\frac{V_0}{\omega L}\right)\cos\omega t$$

$$\therefore \quad i_L = \left(\frac{V_0}{X_L}\right)\sin(\omega t - 90°) \tag{11}$$

$$\text{where} \quad X_L = \omega L \tag{12}$$

is known as the inductive impedance. In complex plane $X_L = j\omega L$, where j is imaginary. Comparison of (11) with (8) shows that the current in the inductance lags behind the voltage by 90° or quarter of a cycle.

ii. $Z_L = \sqrt{R^2 + \omega^2 L^2} = \sqrt{(44)^2 + (2\pi \times 150 \times 0.06)^2} = 71.63\,\Omega$

$$Z_C = \sqrt{R^2 + \frac{1}{\omega^2 C^2}} = \sqrt{10^2 + \frac{1}{(2\pi \times 150 \times 10^{-4})^2}} = 14.58\,\Omega$$

14.24 **(a)** Electric current is the time rate of flow of charge; in symbols $I = dQ/dt$.
(b) The charge Q is an integral multiple of the unit of electron's charge e, that is, $Q = ne$, where n is a number. The charge Q is said to be quantized

(c) **(i)** Current density $j = \frac{i}{A} = \frac{2.4 \times 10^{-4}}{(5.6 \times 10^{-3})(50 \times 10^{-6})} = 857\,\text{A/m}^2$

(ii) Drift speed $V_d = \frac{j}{ne} = \frac{857}{(8.5 \times 10^{28})(1.6 \times 10^{-19})} = 6.3 \times 10^{-8}\,\text{m/s}$

(iii) Collisions with atoms and ions of the conductor makes possible large currents to pass.

14.25 By problem $\quad \omega = \frac{1}{\sqrt{L_1 C_1}} = \frac{1}{\sqrt{L_2 C_2}} \quad (1)$

When the combinations L_1C_1 and L_2C_2 are connected in series, the combination will have inductance L and capacitance C given by

$$L = L_1 + L_2 \tag{2}$$

$$C = \frac{C_1 C_2}{C_1 + C_2} \tag{3}$$

$$\text{Now} \quad LC = (L_1 + L_2)\frac{C_1 C_2}{C_1 + C_2} \tag{4}$$

From (1) we have $L_2 = \dfrac{L_1 C_1}{C_2}$ (5)

Substituting (5) in (4) we get on simplification $LC = L_1 C_1 = \dfrac{1}{\omega^2}$

It follows that $\omega = \dfrac{1}{\sqrt{LC}}$

14.26 $Z_c = \dfrac{1}{\omega c} = \dfrac{1}{2\pi f c}$

$$\therefore \quad C = \frac{1}{2\pi f z_c} = \frac{1}{2\pi \times 1000 \times 500} = 3.18 \times 10^{-7}\,\mathrm{F} = 0.318\,\mu\mathrm{F}$$

$$Z_L = \omega L = 2\pi f L$$

$$\therefore \quad L = \frac{Z_L}{2\pi f} = \frac{100}{2\pi \times 5000} = 3.18 \times 10^{-3}\,\mathrm{H} = 3.18\,\mathrm{mH}$$

14.27 The maximum stored energy in the capacitor must equal the maximum stored energy in the inductor, from the principle of energy conservation.

$$\therefore \quad \frac{1}{2}\frac{q_m^2}{C} = \frac{1}{2} L i_m^2 \tag{1}$$

where i_m is the maximum current and q_m is the maximum charge. Substituting CV_0 for q_m and solving for i_m in (1)

$$i_m = V_0 \sqrt{\frac{C}{L}} = 100 \sqrt{\frac{0.01 \times 10^{-6}}{10 \times 10^{-3}}} = 0.1\,\mathrm{A}$$

14.28 For resonance

$$\omega = \frac{1}{\sqrt{LC}}$$

$$\therefore \quad C = \frac{1}{4\pi^2 f^2 L} = \frac{1}{4\pi^2 (5 \times 10^5)^2 \times 10^{-3}} = 1.013 \times 10^{-9}\,\mathrm{F}$$

Quality factor

$$Q = \frac{\omega L}{R}$$

$$\therefore \quad R = \frac{2\pi f L}{Q} = \frac{2\pi (500 \times 10^3)(10^{-3})}{150} = 20.944\,\Omega$$

$\therefore$ Resistance to be included in series is $20.944 - 5.0 = 15.944\,\Omega$.

14.29 For parallel resonance circuit

$$\omega = \omega_0\sqrt{1 - \frac{CR^2}{L}}$$
$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-3} \times 5 \times 10^{-6}}} = 1.414214 \times 10^6 \text{ rad/s}$$
$$f = \frac{\omega}{2\pi} = \frac{\omega_0}{2\pi}\sqrt{1 - \frac{CR^2}{L}}$$
$$= \frac{1.414214 \times 10^6}{2\pi}\sqrt{1 - \frac{5 \times 10^{-10} \times 10^2}{10^{-3}}}$$
$$= 2.250736 \times 10^5/\text{s}$$

Quality factor

$$Q = \frac{\omega L}{R} = \frac{1.414 \times 10^6 \times 10^{-3}}{10} = 141.4$$

14.30 $V = V_0 \mathrm{e}^{-t/RC}$

$$\ln V = \ln V_0 - \frac{t}{RC}$$
$$\frac{\Delta V}{V} = 0 - \frac{t}{C}\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{1}{R}\right)\Delta R - \frac{t}{R}\frac{\mathrm{d}}{\mathrm{d}C}\left(\frac{1}{C}\right)\Delta C$$
$$= \frac{t}{CR}\left(\frac{\Delta R}{R} + \frac{\Delta C}{C}\right) = \frac{50 \times 10^{-6}}{10^{-8} \times 5 \times 10^4}\left(\frac{5}{100} + \frac{10}{100}\right) = 0.015$$

14.31 **(a)** $V = \xi\, \mathrm{e}^{-t/RC}$
The voltage on the condensor will fall to l/e of its initial value when the time

$$t = RC = 10^4 \times 10^{-5} = 0.1 \text{ s}$$

(b) Error on t will result from error on R.

$$\Delta t = C\Delta R$$
$$\therefore \quad \Delta t/t = \Delta R/R$$

$$\text{Power } P = i^2 R = \frac{\xi^2}{e^2 R} = \frac{(3000)^2}{(2.718)^2 \times 10^4} = 121.8 \text{ W}$$

Energy $U = P.t = 121.8 \times 0.1 = 12.18$ J

Heat $H = 12.18/4.18 = 2.914$ cal

Rise in temperature

$$\Delta T = \text{Heat / thermal capacity} = 2.914/0.9 = 3.24^\circ\,\text{C}$$
$$R = R_0(1 + \alpha\Delta T) = 10^4(1 + 0.004 \times 3.24)$$
$$= 1.01296 \times 10^4$$
$$\Delta R = R - R_0 = 129.6$$
$$\therefore \quad \frac{\Delta t}{t} = \frac{\Delta R}{R_0} = \frac{129.6}{10^4} = 0.01296$$

$$\text{Percentage error} = \frac{\Delta t}{t} \times 100 = 1.3\%$$

14.32 The amplitude i_m of the current oscillations is given by

$$i_\text{m} = \frac{E_\text{m}}{\sqrt{(\omega' L - 1/\omega' c)^2 + R^2}}$$

At resonance, $\omega' = \omega$ and $i_\text{m} = \dfrac{E_\text{m}}{R}$

Set $\quad i_\text{m} = \dfrac{E_\text{m}}{\sqrt{(\omega L - 1/\omega c)^2 + R^2}} = \dfrac{1}{2}\dfrac{E_\text{m}}{R}$

Squaring and simplifying

$$\left(\omega L - \frac{1}{\omega c}\right)^2 = 3\,R^2$$
$$\text{or} \quad \omega^2\,LC \pm \sqrt{3}\omega RC - 1 = 0$$

The only acceptable solutions are

$$\omega_1 = \frac{\sqrt{3}}{2}\frac{R}{L} + \sqrt{\frac{3R^2}{4L^2} + \frac{1}{LC}}$$
$$\omega_2 = -\frac{\sqrt{3}}{2}\frac{R}{L} + \sqrt{\frac{3R^2}{4L^2} + \frac{1}{LC}}$$

Subtracting the last equation from the previous one

$$\Delta\omega = \omega_1 - \omega_2 = \sqrt{3}\frac{R}{L}$$
$$\therefore \quad \frac{\Delta\omega}{\omega} = \frac{\sqrt{3}}{\omega}\frac{R}{L}$$

14.3.2 Maxwell's Equations and Electromagnetic Waves, Poynting Vector

14.33 $E_z = 100\cos(6\times 10^8 t + 4x)$ (by problem) (1)

$E_z = A\cos(\omega t + kx)$ (standard equation) (2)

Comparison of (1) and (2) shows that

$\omega = 6\times 10^8$ and $k = 4$

$$v = \frac{\omega}{k} = \frac{6\times 10^8}{4} = 1.5\times 10^8\ \text{m/s}$$

Dielectric constant, $K = \dfrac{c}{v} = \dfrac{3\times 10^8}{1.5\times 10^8} = 2.0$

14.34 $\oint \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{s} = q/\varepsilon_0$ (Gauss' law)

$E(2\pi rl) = q/\varepsilon_0$

$$\therefore \quad E = \frac{\lambda \hat{e}_\mathrm{r}}{2\pi r\,\varepsilon_0} \quad (1)$$

$\oint \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{s} = \mu_0 I$ (Ampere's law)

$2\pi rB = \mu_0 I$

$$\therefore \quad B = \frac{\mu_0 I}{2\pi r}\hat{e}_\phi \quad (2)$$

$$E'_\mathrm{r} = \gamma(E_\mathrm{r} - vB_\phi) \quad \text{(Lorentz transformation)} \quad (3)$$

$$= \gamma\left(\frac{\lambda}{2\pi r\,\varepsilon_0} - \frac{v\,\mu_0 I}{2\pi r}\right) = \frac{\gamma}{2\pi\,r}\left(\frac{\lambda}{\varepsilon_0} - v\mu_0 I\right)$$

Thus $E'_\mathrm{r} = 0$ if $v\mu_0 I = \dfrac{\lambda}{\varepsilon_0}$

or $v = \dfrac{\lambda}{I\mu_0\,\varepsilon_0} = \dfrac{\lambda c^2}{I}$

14.35 Maxwell's equations in vacuum are

$$\nabla\cdot\boldsymbol{E} = 0 \quad (1)$$

$$\nabla\cdot\boldsymbol{B} = 0 \quad (2)$$

$$\nabla\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad (3)$$

$$\nabla\times\boldsymbol{B} = \mu_0\varepsilon_0\frac{\partial \boldsymbol{E}}{\partial t} \quad (4)$$

Use the vector identify

$$\nabla \times (\nabla \times \boldsymbol{B}) = \nabla(\nabla \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B}$$

$$\therefore \quad \nabla \times \left(\mu_0 \varepsilon_0 \frac{\partial E}{\partial t}\right) = -\nabla^2 B \quad (\because \nabla \cdot \boldsymbol{B} = 0 \quad \text{by}(2)$$

$$\therefore \quad \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\nabla \times \boldsymbol{E}) = -\mu_0 \varepsilon_0 \frac{\partial}{\partial t}\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla^2 \boldsymbol{B}$$

$$\therefore \quad \nabla^2 \boldsymbol{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2}$$

14.36 $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$ (Ampere's law) (1)

Use the vector identity $\boldsymbol{A} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = 0$. Put $\boldsymbol{A} = \nabla$.

$$\therefore \quad \nabla \cdot (\nabla \times \boldsymbol{B}) = 0$$

$$\therefore \quad \nabla \cdot \boldsymbol{j} = 0 \tag{2}$$

More generally, $\nabla \cdot \boldsymbol{j} + \frac{\partial \rho}{\partial t} = 0$ (continuity equation) (3)

and $\nabla \cdot \boldsymbol{E} = \varepsilon_0 \rho$ (Gauss' law) (4)

Combining (3) and (4)

$$\nabla \cdot \left(\boldsymbol{j} + \frac{1}{\varepsilon_0}\frac{\partial \boldsymbol{E}}{\partial t}\right) = 0$$

14.37 $\nabla^2 \boldsymbol{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2}$ (free-space wave equation)

Compare with the standard three-dimensional wave equation

$$\nabla^2 \Psi = \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2} \tag{5}$$

$$\therefore \quad v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{(4\pi \times 10^{-7})(8.854 \times 10^{-12})}}$$

$$= 2.998 \times 10^8 \text{ m/s} = \text{c}$$

14.38 $\omega = 2\pi\nu = \frac{2\pi c}{\lambda} = \frac{2\pi \times 3 \times 10^8}{530 \times 10^{-9}} = 3.55 \times 10^{15}$

$$\sigma = \frac{1}{\rho} = \frac{1}{26.5 \times 10^{-9}} = 3.77 \times 10^7$$

$$\delta = \sqrt{\frac{2}{\mu_0 \sigma \omega}} = \sqrt{\frac{2}{4\pi \times 10^{-7} \times 3.77 \times 10^7 \times 3.55 \times 10^{15}}}$$

$$= 3.45 \times 10^{-9}\,\text{m} = 3.45\,\text{nm}$$

14.39

$$\boldsymbol{E} = \boldsymbol{E}_0 \cos(kx - \omega t) \tag{1}$$

$$\frac{\partial E_y}{\partial x} = -kE_0 \sin(kx - \omega t) \tag{2}$$

$$\boldsymbol{B} = \boldsymbol{B}_0 \cos(kx - \omega t) \tag{3}$$

$$\frac{\partial B_Z}{\partial t} = \omega B_0 \sin(kx - \omega t) \tag{4}$$

$$\text{But} \quad \frac{\partial E_y}{\partial x} = -\frac{\partial E_z}{\partial t} \tag{5}$$

Combining (2), (4) and (5), we get

$$E_0 = \frac{\omega}{k} B_0 = cB_0$$

14.40 Maxwell's equations for a non-ferromagnetic homogeneous isotropic medium can be written as

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon} \tag{1}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{2}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{3}$$

$$\nabla \times \boldsymbol{B} = \mu\sigma \boldsymbol{E} + \mu\varepsilon \frac{\partial \boldsymbol{E}}{\partial t} \tag{4}$$

Taking the curl of (4)

$$\nabla \times (\nabla \times \boldsymbol{B}) = \mu\sigma(\nabla \times \boldsymbol{E}) + \mu\varepsilon \frac{\partial}{\partial t} (\nabla \times \boldsymbol{E}) \tag{5}$$

where the time and space derivatives are interchanged as $\boldsymbol{E}$ is assumed to be a well-behaved function. Expression (3) can be substituted in (5) to obtain

$$\nabla \times (\nabla \times \boldsymbol{B}) = -\mu\sigma \frac{\partial \boldsymbol{B}}{\partial t} - \mu\varepsilon \frac{\partial^2 \boldsymbol{B}}{\partial t^2} \tag{6}$$

Using the vector identity

$$\nabla \times (\nabla \times \boldsymbol{B}) = \nabla (\nabla \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B} \tag{7}$$

By virtue of (2), $\nabla \cdot B = 0$ and (6) becomes

$$\nabla^2 \boldsymbol{B} = \mu\varepsilon \frac{\partial^2 \boldsymbol{B}}{\partial t^2} + \mu\sigma \frac{\partial \boldsymbol{B}}{\partial t} \tag{8}$$

A similar procedure applied to (3) yields a similar equation for the E-field. Taking the curl of (3)

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\frac{\partial}{\partial t}(\nabla \times \boldsymbol{B}) \tag{9}$$

Using (4) in (9)

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\mu\sigma \frac{\partial \boldsymbol{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2} \tag{10}$$

But $\quad \nabla \times (\nabla \times \boldsymbol{E}) = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E}$ (11)

and $\quad \nabla \cdot \boldsymbol{B} = \dfrac{\rho}{\varepsilon}$ (1)

Combining (10), (11) and (1) we obtain for uncharged medium ($\rho = 0$)

$$\nabla^2 \boldsymbol{E} = \mu\sigma \frac{\partial \boldsymbol{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

14.41 **(a)** $\varepsilon_0 \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = q$ (Gauss' law) (1)

$\varepsilon_0 E(2\pi r l) = q$

or $\quad E = q/2\pi\varepsilon_0 r l$ (2)

the flux being entirely through the cylindrical surface and zero through the end caps. The potential difference between the conductors is

$$V = \int_a^b E\,\mathrm{d}r = \int_a^b \frac{q}{2\pi\varepsilon_0 l}\frac{\mathrm{d}r}{r} = \frac{q}{2\pi\varepsilon_0 l}\ln\left(\frac{b}{a}\right)$$

Capacitance $C = \dfrac{q}{V} = \dfrac{2\pi\varepsilon_0 l}{\ln(b/a)}$ (3)

$\therefore$ Capacitance per unit length of the cable is

$$\frac{C}{l} = \frac{2\pi\varepsilon_0}{\ln(b/a)} \tag{3a}$$

(b) The magnetic induction between the conductors is

$$B = \frac{\mu_0 i}{2\pi\, r} \tag{4}$$

Energy density $u = \dfrac{1}{2\mu_0}B^2 = \dfrac{\mu_0 i^2}{8\pi^2 r^2}$ (5)

where we have used (4).

Consider a volume element dV for the cylindrical shell of radii r and $r + dr$ and of length l. The energy contained in the volume element is

$$\mathrm{d}U = u\,\mathrm{d}V = \frac{\mu_0 i^2}{8\pi^2 r^2}(2\pi r l\,\mathrm{d}r) = \frac{\mu_0 i^2 l}{4\pi}\frac{\mathrm{d}r}{r} \tag{6}$$

$$U = \int \mathrm{d}U = \frac{\mu_0 i^2 l}{4\pi}\int_a^b \frac{\mathrm{d}r}{r} = \frac{\mu_0 i^2 l}{4\pi}\ln(b/a) \tag{7}$$

But $\quad U = \frac{1}{2}Li^2 \quad (8)$

Comparing (7) and (8)

$$L = \frac{\mu_0 l}{2\pi}\ln(b/a)$$

$\therefore$ Inductance per unit length of the cable is

$$\frac{L}{l} = \frac{\mu_0}{2\pi}\ln(b/a)$$

14.42 **(a)** By prob. (14.41) the potential difference between the conductors is

$$V = \frac{q}{2\pi\varepsilon_0 l}\ln(b/a) = \frac{\lambda}{2\pi\varepsilon_0}\ln(b/a) \tag{1}$$

where $\lambda = q/l$ is the charge density.

Now $\quad E = \frac{\lambda}{2\pi\varepsilon_0 r} = \frac{\xi}{r\ln(b/a)} \quad (2)$

where we have put $V = \xi$.

(b) $B = \frac{\mu_0 i}{2\pi r} = \frac{\mu_0 \xi}{2\pi\, r\, R} \quad (a < r < b) \quad (3)$

(c) The Poynting vector

$$\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B} = \frac{1}{\mu_0}EB = \frac{\xi^2}{2\pi r^2 R\ln(b/a)}$$

where we have used (2) and (3).

14.43 $u_\mathrm{B} = \frac{1}{2}B\cdot H \quad (1)$

$B = \mu\boldsymbol{H} \quad (2)$

Substituting (2) in (1)

$$u_B = \frac{1}{2}\mu\, H \cdot H = \frac{1}{2}\mu\, H^2 = \frac{B^2}{2\mu} \tag{3}$$

In free space $\mu = \mu_0$. Therefore in vacuum

$$u_{\mathrm{B}} = \frac{B^2}{2\mu_0}$$

14.44 $u_{\mathrm{E}} = \frac{1}{2}\varepsilon_0 E^2$ (energy density in E-field) (1)

$$u_{\mathrm{B}} = \frac{1}{2}\frac{B^2}{\mu_0} \quad \text{(energy density in } B\text{-field)} \tag{2}$$

The fields for the plane wave are

$$E = E_{\mathrm{m}} \sin(kx - \omega t) \tag{3}$$

$$B = B_{\mathrm{m}} \sin(kx - \omega t) \tag{4}$$

Substituting (3) in (1) and (4) in (2)

$$u_{\mathrm{E}} = \frac{1}{2}\varepsilon_0 E_{\mathrm{m}}^2 \sin^2(kx - \omega t) \tag{5}$$

$$u_{\mathrm{B}} = \frac{1}{2}\frac{B_{\mathrm{m}}^2}{\mu_0} \sin^2(kx - \omega t) \tag{6}$$

Dividing (5) by (6)

$$\frac{u_{\mathrm{E}}}{u_{\mathrm{B}}} = \frac{\varepsilon_0 \mu_0 E_{\mathrm{m}}^2}{B_{\mathrm{m}}^2} \tag{7}$$

But $\varepsilon_0 \mu_0 = \frac{1}{c^2}$ and $E_{\mathrm{m}} = cB_{\mathrm{m}}$

$\therefore \quad \frac{u_{\mathrm{E}}}{u_{\mathrm{B}}} = 1$ or $u_{\mathrm{E}} = u_{\mathrm{B}}$

14.45 By prob. (14.41) the magnetic energy stored in a coaxial cable

$$U_{\mathrm{B}} = \frac{\mu_0 i^2 l}{4\pi} \ln(b/a) \tag{1}$$

where i is the current and l is the length of the cable. Further, its capacitance is given by

$$C = \frac{2\pi\varepsilon_0 l}{\ln(b/a)} \tag{2}$$

The electric energy stored in the cable

$$U_{\mathrm{E}} = \frac{1}{2}\xi^2 C = \frac{1}{2}(i^2 R^2)\left(\frac{2\pi\varepsilon_0 l}{\ln(b/a)}\right) \tag{3}$$

By problem $U_E = U_M$.

$$\therefore \quad \frac{1}{2}(i^2 R^2)\left(\frac{2\pi\varepsilon_0 l}{\ln(b/a)}\right) = \frac{\mu_0 i^2 l}{4\pi}\ln(b/a) \tag{4}$$

$$\therefore \quad R = \frac{1}{2\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}\ln(b/a) = \frac{1}{2\pi}\sqrt{\frac{4\pi\times 10^{-7}}{8.85\times 10^{-12}}}\ln(b/a)$$

$$= \frac{376.7}{2\pi}\ln(b/a)\,\Omega.$$

14.46 As the kinetic energy of proton (20 MeV) is much smaller than its rest mass energy (938 MeV), non-relativistic calculations will be valid. In the classical picture a charged particle undergoing acceleration 'a' emits electromagnetic radiation. The electromagnetic energy radiated per second is given by

$$P = \frac{q^2 a^2}{6\pi\varepsilon_0 c^3} \tag{1}$$

$$\text{Now } a = \frac{v^2}{R} = \frac{1}{2}mv^2\frac{2}{mR} = \frac{2K}{mR} \tag{2}$$

Substituting (2) in (1) and putting $q = e$ for the charge of proton

$$P = \frac{2e^2K^2}{3\pi\varepsilon_0 c^3 m^2 R^2} \tag{3}$$

The energy radiated per orbit is given by multiplying P by this time period $2\pi R/v = 2\pi R\sqrt{m/2K}$

$$\begin{aligned}\therefore \quad \Delta K &= \frac{4e^2K^{3/2}}{3\sqrt{2}\varepsilon_0 c^3 m^{3/2} R} \\ &= \frac{4(1.6\times 10^{-19})^2(20\times 1.6\times 10^{-13})^{3/2}}{3\sqrt{2}(8.85\times 10^{-12})(3\times 10^8)^3(1.67\times 10^{-27})^{3/2}(0.5)} \\ &= 1.89\times 10^{-31}\,\mathrm{J} = 1.18\times 10^{-12}\,\mathrm{eV}\end{aligned}$$

which is quite negligible

14.47 Consider a sphere of radius r with its centre at the point source of power P. Then the intensity of radiation at distance r will be $I = \dfrac{P}{4\pi r^2}$.

$$I = \frac{P}{4\pi r^2} = \frac{c\,\varepsilon_0 E_0^2}{2}$$

$$\therefore \quad E_0 = \left[\frac{P}{2\pi\, r^2 c\varepsilon_0}\right]^{1/2} = \left[\frac{40}{2\pi(1.0)^2(3\times 10^8 \times 8.85\times 10^{-12})}\right]^{1/2} = 49\,\mathrm{V/m}$$

14.48 **(a)** $I_0 = \dfrac{P}{A} = \dfrac{1.2\times 10^{-3}}{4\times 10^{-6}} = 300\,\mathrm{W/m^2}$

(b) $I_0 = \dfrac{c\varepsilon_0 E_0^2}{2} \to E_0 = \left[\dfrac{2I_0}{c\varepsilon_0}\right]^{1/2} = \left[\dfrac{2\times 300}{3\times 10^8 \times 8.85\times 10^{-12}}\right]^{1/2}$
$= 475\,\mathrm{V/m}$

(c) $B_0 = \dfrac{E_0}{c} = \dfrac{475}{3\times 10^8} = 1.58\times 10^{-6}\ \mathrm{T} = 1.58\,\mu\mathrm{T}$

14.49 $I = \dfrac{P}{A} = \dfrac{P}{\pi r^2} = \dfrac{314\times 10^{-3}}{3.14\times(0.5\times 10^{-3})^2} = 4\times 10^5\ \mathrm{W/m^2}$

14.50 The average value of the Poynting vector $< \boldsymbol{s} >$ over a period of oscillation of the electromagnetic wave is known as the radiant flux density, and if the energy is incident on a surface it is called irradiance.

$$\boldsymbol{E} = \boldsymbol{E}_0\cos(kx-\omega t),\ \boldsymbol{B} = \boldsymbol{B}_0\cos(kx-\omega t)$$

$$\therefore \quad \boldsymbol{S} = c^2\varepsilon_0 \boldsymbol{E}\times\boldsymbol{B} = c^2\varepsilon_0\boldsymbol{E}_0\times\boldsymbol{B}_0\cos^2(kx-\omega t)$$

$$\therefore \quad \langle \boldsymbol{S}\rangle = c^2\varepsilon_0\,|\boldsymbol{E}_0\times\boldsymbol{B}_0|\left\langle\cos^2(kx-\omega t)\right\rangle$$

$$\text{But} \quad \left\langle\cos^2(kx-\omega t)\right\rangle = \frac{1}{T}\int_0^T \cos^2(kx-\omega t)\mathrm{d}t = \frac{1}{2}$$

Further $\boldsymbol{E}\perp\boldsymbol{B}$ and $E_0 = cB_0$.

$$\therefore \quad < \boldsymbol{s} > = I = \frac{1}{2}c\varepsilon_0 E_0^2 = \frac{1}{2}\frac{E_0^2}{\mu_0 c}$$

14.51 $E_z = 50\sin\left[4\pi\times 10^{14}\left(t - \dfrac{x}{3\times 10^8}\right)\right]$

$$I = \frac{c\varepsilon_0 E_0^2}{2} = \frac{1}{2}(3\times 10^8)(8.85\times 10^{-12})(50^2) = 0.066\,\mathrm{W/m^2}$$

14.52 $|\boldsymbol{E}\times\boldsymbol{H}| = \dfrac{|\boldsymbol{E}\times\boldsymbol{B}|}{\mu_0} = \dfrac{\boldsymbol{E}^2}{\mu_0 c} \quad (\because \boldsymbol{B} = \boldsymbol{E}/c)$

$\boldsymbol{E}\times\boldsymbol{H}$ will be in the direction of $\boldsymbol{S} = (\boldsymbol{E}\times\boldsymbol{B})/\mu_0$, which is the direction of propagation.

14.53 $E_z = 10 \sin \pi \left(2 \times 10^6 x - 6 \times 10^{14} t\right)$
$= 10 \sin \left(2\pi \times 10^6 x - 6\pi \times 10^{14} t\right)$

Compare this with the standard equation

$E_z = E_{z_0} \sin(kx - \omega t)$

(a) $\omega = 2\pi f = 6\pi \times 10^{14} \rightarrow f = 3 \times 10^{14}$ Hz
(b) $k = 2\pi \times 10^6 \rightarrow \lambda = 2\pi/k = 10^{-6}$ m
(c) $v = \omega/k = 6\pi \times 10^{14}/2\pi \times 10^6 = 3 \times 10^8$ m/s
(d) $E_{z0} = 10$ V/m
(e) The wave is linearly polarized in the z-direction and propagates along the x-axis.

14.54 The wave propagates in the x-direction while the E-field oscillates along the z-direction, that is, the E-field is contained in the xz-plane. Since B is normal to both E and the direction of propagation it must be contained in the xy-plane.
Thus, $B_x = B_z = 0$, and $B = B_y(x, t)$. Now $B = E/c$.

$$\therefore \quad B_y(x, t) = 3.33 \times 10^{-8} \sin \pi (2 \times 10^6 x - 6 \times 10^{14} t) \text{ T}$$

14.55 Let ν be the frequency of the incident microwave beam. Let the car be approaching with speed v towards the observer. Then the frequency seen by the car is given by the formula for Doppler shift

$$\nu' = \nu(1 + v/c) \tag{1}$$

Upon reflection the microwave returns as if it was emitted by a moving source travelling with speed v towards the observer. Therefore the observed frequency is

$$\nu'' = \nu'(1 + v/c) = \nu(1 + v/c)^2 \tag{2}$$

where we have used (1).

$$\Delta\nu = \nu'' - \nu = \nu(1 + v/c)^2 - \nu = 2\nu \frac{v}{c}\left(1 + \frac{v}{2c}\right)$$

Assuming that $v/c << 1$

$$\Delta\nu = 2\nu \frac{v}{c} \qquad \text{(beat frequency)} \tag{3}$$

For a receding car, proceeding along similar lines,

$$\Delta\nu = \nu'' - \nu = -2\nu \frac{v}{c} \tag{4}$$

14.56 By prob. (14.55), $\Delta \nu = \nu'' - \nu = -2\nu \frac{v}{c}$

$$\Delta \nu = -\frac{2 \times 800 \times 10^6}{3 \times 10^8} \times \left(\frac{5}{18} \times 90\right) = -133\,\text{Hz}$$

14.57 The relation between the current i and the magnetic field B expressed as $\oint B \cdot \mathrm{d}l = \mu_0 i$ is known as Ampere's law. There are equal and opposite currents i in the conductors.

(a) $r < a$. The net current passing through the conductor bounded by the closed path corresponds to that flowing through the inner conductor, Fig. 14.7. Hence by Ampere's theorem

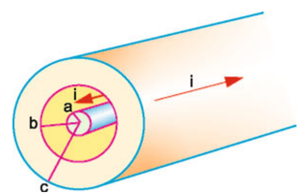


Fig. 14.7 Magnetic field due to current carrying coaxial cylinder

$$\oint B \cdot \mathrm{d}l = (B)(2\pi r) = \mu_0 i \frac{(\pi r^2)}{\pi a^2}$$

$$\text{or} \quad B = \frac{\mu_0 i r}{2\pi a^2}$$

(b) $a < r < b$. Here the current through the outer conductor does not contribute to B.

$$\oint B \cdot \mathrm{d}l = (B)(2\pi r) = \mu_0 i$$

$$\text{or} \quad B = \frac{\mu_0 i}{2\pi r}$$

(c) $b < r < c$. Here currents through both the conductors contribute to B.

$$\oint B \cdot \mathrm{d}l = (B)(2\pi r) = \mu_0 i - \mu_0 i \frac{\pi(r^2 - b^2)}{\pi(c^2 - b^2)}$$

$$\text{or} \quad B = \frac{\mu_0 i}{2\pi r} \frac{(c^2 - r^2)}{(c^2 - b^2)}$$

(d) $r > c$. As the net current flowing through the closed path is zero, $B = 0$.

14.58 Magnetic energy density

$$u_{\mathrm{B}} = \frac{1}{2}\frac{B^2}{\mu_0} = \frac{1}{2}\frac{4^2}{4\pi \times 10^{-7}} = \frac{2}{\pi} \times 10^7\,\mathrm{J/m^3}$$

Energy stored in the magnetic field $U_B = u_B \times \text{volume} = u_{\mathrm{B}}\pi r^2 l$

$$= 2 \times 10^7 \times 3^2 \times 12.5 = 2.25 \times 10^9\,\mathrm{J}$$

14.59 If P_0 is the power radiated by sun of radius r, then using the results of prob. (14.51)

$$P_0 = \langle S\rangle\, 4\pi r^2 = \frac{1}{2\mu_0 c} E_{\mathrm{m}}^2 4\pi r^2$$

$$\therefore \quad E_{\mathrm{m}} = \frac{1}{r}\sqrt{\frac{P_0\mu_0 c}{2\pi}}$$

$$\therefore \quad E_{\mathrm{m}} = \frac{1}{7 \times 10^8}\sqrt{\frac{4 \times 10^{26} \times 4\pi \times 10^{-7} \times 3 \times 10^8}{2\pi}} = 2.21 \times 10^5\,\mathrm{V/m}$$

$$B_{\mathrm{m}} = \frac{E_{\mathrm{m}}}{c} = \frac{2.21 \times 10^5}{3 \times 10^8} = 7.37 \times 10^{-4}\,\mathrm{T} = 7.37\,\mathrm{G}$$

14.60 By prob. (14.51)

$$\langle S\rangle = \frac{E_{\mathrm{m}}^2}{2\mu_0 c}$$

$$\therefore \quad E_{\mathrm{m}} = \sqrt{2\mu_0 c\,\langle S\rangle} = \sqrt{2 \times 4\pi \times 10^{-7} \times 3 \times 10^8 \times 1300} = 990\,\mathrm{V/m}$$

14.61 $B_{\mathrm{m}} = \dfrac{E_{\mathrm{m}}}{c} = \dfrac{300}{3 \times 10^8} = 1 \times 10^{-6}\,\mathrm{T}$

$$\langle S\rangle = \frac{E_{\mathrm{m}}^2}{2\mu_0 c} = \frac{(300)^2}{2 \times 4\pi \times 10^{-7} \times 3 \times 10^8} = 119.4\,\mathrm{W/m^2}$$

14.62 $\dfrac{|E|}{|H|} = \dfrac{E\,\mu_0}{B} = \dfrac{(Bc)\mu_0}{B} = \dfrac{\mu_0}{\sqrt{\mu_0\varepsilon_0}} = \sqrt{\dfrac{\mu_0}{\varepsilon_0}}$

where we have used the equations $B = \mu_0 H$ and $c = \dfrac{1}{\sqrt{\mu_0\varepsilon_0}}$.

$$\sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{4\pi \times 10^{-7}}{8.85 \times 10^{-12}}} = 377\,\Omega$$

Units of $\dfrac{|E|}{|H|}$ are $\dfrac{\text{Volt/metre}}{\text{Ampere/metre}} = \dfrac{\text{Volt}}{\text{Ampere}} = \text{Resistance}$

14.63 Skin depth $\delta = \sqrt{\dfrac{2}{2\pi f \sigma \mu_0 \mu}}$

$$= \sqrt{\frac{2}{2\pi \times 20 \times 10^3 \times 6 \times 10^7 \times 4\pi \times 10^{-7} \times 1}}$$
$$= 4.6 \times 10^{-4}\,\text{m} = 0.46\,\text{mm}$$

14.64 $\delta = \sqrt{\dfrac{2}{2\pi f \sigma \mu_0 \mu}}$

$$= \sqrt{\frac{1}{\pi \times 3 \times 10^8 \times 5.6 \times 10^7 \times 4\pi \times 10^{-7} \times 1}} = 1.23 \times 10^{-4}\,\text{m}$$
$$= 0.123\,\text{mm} = 3.88\,\mu\text{m}$$

14.65 Let us begin with the 'curl H' Maxwell's equation

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \tag{1}$$

Assume that all the fields and currents in this equation are sinusoidal at a single frequency ω. In that case (1) can be replaced by

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \mathrm{j}\omega \boldsymbol{D} = \boldsymbol{J} + \mathrm{j}\omega\varepsilon \boldsymbol{E} \tag{2}$$

where J is the current density, $\boldsymbol{D} = \varepsilon \boldsymbol{E}$ is the displacement vector and j is imaginary $\left(\sqrt{-1}\right)$.
The 'curl E' Maxwell's equation is obtained in a similar way as (1).

$$\nabla \times \boldsymbol{E} = -\mathrm{j}\omega \boldsymbol{B} \tag{3}$$

Applying the curl operation to each side of (3)

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\mathrm{j}\omega \nabla \times \boldsymbol{B} \tag{4}$$

Now $\nabla \times \boldsymbol{B}$ can be replaced using (2) and the relation $H = B/\mu$:

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\mathrm{j}\omega\mu(\boldsymbol{J} + \mathrm{j}\omega\varepsilon \boldsymbol{E}) \tag{5}$$

We now use the vector identity

$$\nabla \times \nabla \times \boldsymbol{E} = \nabla(\nabla \cdot \boldsymbol{E}) = -\nabla^2 \boldsymbol{E} \tag{6}$$

Conductive materials do not contain any real charge density because any real charge that many exist will repel itself and move outwards until it resides on the material's outer surface. Therefore $\nabla \cdot \boldsymbol{D} = 0$ and so also $\nabla \cdot \boldsymbol{E} = 0$. Thus the first term on the right by (6) vanishes. Therefore (5) becomes

$$\nabla^2 \boldsymbol{E} = \mathrm{j}\omega\mu(\boldsymbol{J} + \mathrm{j}\omega\varepsilon\boldsymbol{E}) \tag{7}$$

Assume that the material under consideration obeys Ohm's law, $\boldsymbol{J} = \sigma_{\mathrm{E}}\boldsymbol{E}$, where σ_{E} is the conductivity. Then (7) becomes

$$\nabla^2 \boldsymbol{E} = \mathrm{j}\omega\mu(\sigma_{\mathrm{E}} + \mathrm{j}\omega\varepsilon)\boldsymbol{E} \tag{8}$$

For simplicity assume that the given material is an excellent conductor, so that $\sigma_{\mathrm{E}} >> |\omega\varepsilon|$. In that case, the displacement current term, masked by the conduction current, can be neglected, yielding

$$\nabla^2 \boldsymbol{E} = \mathrm{j}\omega\mu\sigma_{\mathrm{E}}\boldsymbol{E} \tag{9}$$

Since $\boldsymbol{E} = \boldsymbol{J}/\sigma_{\mathrm{E}}$ we can write

$$\nabla^2 \boldsymbol{J} = j\omega\mu\sigma_{\mathrm{E}}\boldsymbol{J} \tag{10}$$

Suppose the current flows through this material in the z-direction. The current density is independent of x and y. In that case (10) simplifies to

$$\frac{\partial^2 J_z}{\partial z^2} = \mathrm{j}\omega\mu\sigma_{\mathrm{E}} J_z \tag{11}$$

which has the solution

$$\boldsymbol{J}_x = A\mathrm{e}^{-(1+\mathrm{j})z/\delta} + B\mathrm{e}^{(1+\mathrm{j})z/\delta} \tag{12}$$

where A and B are constants, and δ given by

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma_{\mathrm{E}}}} \tag{13}$$

is known as the skin depth. Thus the magnitude of current density decreases with depth. This effect is of practical importance as it affects resistive losses accompanying a high-frequency current flow in an electronic circuit.

14.66 The average value of the Poynting vector is

$$\begin{aligned}\langle S\rangle &= \frac{1}{2\mu_0}E_{\mathrm{m}}B_{\mathrm{m}} = \frac{1}{2\mu_0}E_{\mathrm{m}}\left(\frac{E_{\mathrm{m}}}{c}\right) = \frac{E_{\mathrm{m}}^2}{2\mu_0 c}\\ &= \frac{\left(10^{-3}\right)^2}{2\times 4\pi\times 10^{-7}\times 3\times 10^8} = 1.327\times 10^{-9}\,\mathrm{W/m^2}\\ &= 1.327\times 10^{-13.}\,\mathrm{W/cm^2}.\end{aligned}$$

14.67 Poynting's theorem is a mathematical statement based on Maxwell's equations. The theorem is interpreted as energy balance equation in situations where electromagnetic waves are present. We begin with the vector identity

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) = \boldsymbol{H} \cdot (\nabla \times \boldsymbol{E}) - \boldsymbol{E} \cdot (\nabla \times \boldsymbol{H}) \tag{1}$$

We now make use of Maxwell's 'curl' equations

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{2}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \tag{3}$$

Substituting (2) and (3) in the RHS of (1)

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) = -\boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} - \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} - \boldsymbol{E} \cdot \boldsymbol{J} \tag{4}$$

which is the mathematical statement of Poynting's theorem.

Further $\quad \boldsymbol{B} = \mu \boldsymbol{H}$ and $\boldsymbol{D} = \varepsilon \boldsymbol{E}$ (5)

$$\frac{\partial}{\partial t}(\boldsymbol{E} \cdot \boldsymbol{E}) = \frac{\partial}{\partial t} |\boldsymbol{E}|^2 = 2\boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t} \tag{6}$$

Using (5) and (6) in (4) we obtain

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) = -\frac{\partial}{\partial t}\left(\frac{\varepsilon\, |\boldsymbol{E}|^2}{2}\right) - \frac{\partial}{\partial t}\left(\frac{\mu\, |\boldsymbol{H}|^2}{2}\right) - \boldsymbol{E} \cdot \boldsymbol{J} \tag{7}$$

We can integrate each side over any arbitrary volume V. Applying the divergence theorem to the integral on the left

$$\int_S (\boldsymbol{E} \times \boldsymbol{H}) \cdot \mathrm{d}S = -\int_v \left[\frac{\partial}{\partial t}\left(\frac{\varepsilon}{2}\, |\boldsymbol{E}|^2\right) + \frac{\partial}{\partial t}\left(\frac{\mu}{2}\, |\boldsymbol{H}|^2\right) + \boldsymbol{E} \cdot \boldsymbol{J}\right] \mathrm{d}V \tag{8}$$

and changing the sign of the equation and the order of differentiation and integration we can rewrite (8) as

$$-\int_s (\boldsymbol{E} \times \boldsymbol{H}) \cdot \mathrm{d}\boldsymbol{S} = \frac{\partial}{\partial t} \int_V \frac{\varepsilon\, |\boldsymbol{E}|^2}{2} \mathrm{d}V + \frac{\partial}{\partial t} \int_V \frac{\mu\, |\boldsymbol{H}|^2}{2} \mathrm{d}V + \int_v \boldsymbol{E} \cdot \boldsymbol{J}\, \mathrm{d}V \tag{9}$$

The first term on the right represents the rate of increase of electric energy inside the volume V. The second term on the right represents the rate of increase of magnetic energy. The last term corresponds to the power converted into Joule's heat. Energy conservation demands that these three terms be balanced by flow of energy into the volume and this is accounted for by the term on the left.

The quantity $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$, representing the power density, is known as the Poynting vector. The minus sign in (9) means that $\mathrm{d}\boldsymbol{s}$ is the outward normal and that $(\boldsymbol{E} \times \boldsymbol{H}) \cdot \mathrm{d}\boldsymbol{s}$ represents power flowing outwards rather than inwards for energy balance.

14.68 In one dimension the given equation reduces to

$$\frac{\partial^2 E}{\partial x^2} - \mu_0 \varepsilon_0 \varepsilon_r \frac{\partial^2 E}{\partial t^2} - \mu_0 \sigma_N \frac{\partial E}{\partial t} = 0 \tag{1}$$

Let the travelling wave be given by

$$E = E_0 \mathrm{e}^{\mathrm{i}(kx - \omega t)} \tag{2}$$

$$\frac{\partial^2 E}{\partial x^2} = -k^2 E,\ \frac{\partial E}{\partial t} = -\mathrm{i}\omega E \text{ and } \frac{\partial^2 E}{\partial t^2} = -\omega^2 E \tag{3}$$

Inserting (3) in (1) and re-arranging and cancelling E

$$k^2 = \mu_0 \varepsilon_0 \varepsilon_r \omega^2 + i \mu_0 \sigma_N \omega \tag{4}$$

14.69 $\boldsymbol{F} = x^2 z^3 \hat{i}$

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 z^3 & 0 & 0 \end{vmatrix} = 3x^2 z^2 \hat{j}$$

$$\nabla \times 3x^2 z^2 \hat{j} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 3x^2 z^2 & 0 \end{vmatrix} = -6x^2 z \hat{i} + 6xz^2 \hat{k}$$

$$\therefore \quad \nabla \times (\nabla \times F) = -6x^2 z \hat{i} + 6xz^2 \hat{k} \tag{1}$$

$$-\nabla^2 F = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) x^2 z^3 \hat{i} = -(6x^2 z + 2z^3)\hat{i} \tag{2}$$

$$\nabla(\nabla \cdot F) = -\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (x^2 z^3 \hat{i})$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2xz^3) = 2z^3 \hat{i} + 6xz^2 \hat{k} \tag{3}$$

Thus $\nabla \times (\nabla \times F) = -6x^2 z\hat{i} + 6xz^2\hat{k}$ (1)

$$- \nabla^2 F + \nabla(\nabla \cdot F) = -(6x^2 z + 2z^3)\hat{i} + 2z^3\hat{i} + 6xz^2\hat{k}$$

$$= -6x^2 z\hat{i} + 6xz^2\hat{k} \quad (4)$$

Comparing (1) and (4), the identity

$\nabla \times (\nabla \times E) = -\nabla^2 E + \nabla(\nabla \cdot E)$ is verified.

14.70 The electric field of the cable is radial and is given by

$$E = E_{\mathrm{r}} = \frac{V}{r \ln(b/a)} \quad (1)$$

where a and b are the radii of the inner and outer cable. The corresponding magnetic intensity is tangential and is given by

$$H = H_\phi = \frac{I}{2\pi r} \quad (2)$$

As the angle between $\boldsymbol{E}$ and $\boldsymbol{H}$ is 90°, the Poynting vector

$$|\boldsymbol{S}| = |\boldsymbol{E} \times \boldsymbol{H}| = EH \quad (3)$$

So that $S = S_z = \dfrac{VI}{2\pi r^2 \ln(b/a)}$ (4)

and the direction of $\boldsymbol{S}$ is that of the current in the positive conductor.
The power flow is confined to the space between the conductors and for any plane perpendicular to the axis of the conductor

$$P = \int_a^b S_z 2\pi r \, \mathrm{d}r = \int_a^b \frac{VI}{\ln(b/a)} \frac{\mathrm{d}r}{r} \quad (5)$$

where we have substituted S_z from (4).

But $\displaystyle\int_a^b \frac{\mathrm{d}r}{r} = \ln(b/a)$ (6)

$\therefore \quad P = VI$ (7)

This is the entire power transmitted by the cable. It follows that the Poynting theorem indicates that the entire flow of energy resides in the space between the conductors.

If the resistance of the cable cannot be neglected then V is no longer constant. An axial component of $\boldsymbol{E}$ is necessary to maintain the flow of current to compensate for the Ohmic energy loss.

14.71 The current density in the wire is $(I/\pi a^2)\hat{e}_z$. Therefore the electric field in the wire, including on the surface of the wire, will be $(I/\pi a^2\sigma_{\mathrm{E}})\hat{e}_z$.
The magnetic field intensity by Ampere's theorem is $(I/2\pi a^2)\hat{e}_\phi$.
The Poynting vector at the surface is given by

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} = \left(\frac{I}{\pi a^2 \sigma_{\mathrm{E}}}\hat{e}_z\right) \times \left(\frac{I}{2\pi a}\hat{e}_\phi\right) = -\frac{I^2}{2\pi^2 a^3 \sigma_{\mathrm{E}}}\hat{e}_r \tag{1}$$

Now Poynting's theorem is

$$-\int_s (\boldsymbol{E} \times \boldsymbol{H}) \cdot \mathrm{d}\boldsymbol{s} = \frac{\partial}{\partial t}\int_V \frac{\varepsilon\,|\boldsymbol{E}|^2}{2}\mathrm{d}V + \frac{\partial}{\partial t}\int_V \frac{\mu\,|\boldsymbol{H}|^2}{2}\mathrm{d}V + \int_V \boldsymbol{E} \cdot \boldsymbol{J}\,\mathrm{d}V \tag{2}$$

Since the fields are constant in time, the first two terms on the right of (2) which contain time derivative $\partial/\partial t$ vanish. The power dissipated in the wire is then

$$P = \int \boldsymbol{E} \cdot \boldsymbol{J}\mathrm{d}V = -\int (\boldsymbol{E} \times \boldsymbol{H}) \cdot \mathrm{d}\boldsymbol{s} = -\frac{I^2}{2\pi^2 \sigma_{\mathrm{E}} a^3}(2\pi a L) = \frac{I^2 L}{\pi \sigma_{\mathrm{E}} a^2} = I^2 R$$

14.72 Given $\quad \boldsymbol{B} = -\dfrac{m_{\mathrm{e}}}{ne^2}\nabla \times \boldsymbol{J} \qquad (1)$

Use the vector identity

$$\nabla \times (\nabla \times \boldsymbol{B}) = -\nabla^2 \boldsymbol{B} + \nabla(\nabla \cdot \boldsymbol{B}) \tag{2}$$

Use Maxwell's equations

$$\nabla \cdot \boldsymbol{B} = 0 \tag{3}$$

$$\nabla \times \frac{\boldsymbol{B}}{\mu_0} = \boldsymbol{J} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \tag{4}$$

Here $\quad \dfrac{\partial \boldsymbol{D}}{\partial t} = \varepsilon_0 \dfrac{\partial \boldsymbol{E}}{\partial t} = 0 \qquad (5)$

so that (4) becomes

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \tag{6}$$

Using (3) and (6) in (2)

$$\mu_0 \nabla \times \boldsymbol{J} = -\nabla^2 \boldsymbol{B} \tag{7}$$

Using (1) in (7)

$$\nabla^2 \boldsymbol{B} = \frac{\mu_0 n e^2 \boldsymbol{B}}{m_\mathrm{e}}$$

14.73 High-frequency resistance

$$R_\mathrm{s} = \sqrt{\frac{\omega\mu}{2\sigma}} \tag{1}$$

Direct current resistance per metre

$$R = \frac{1}{\sigma A} = \frac{1}{\sigma \pi r^2} \tag{2}$$

Further the skin thickness

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}} \tag{3}$$

For metals assume $\mu = \mu_0$. Combining (1), (2) and (3)

$$\frac{R_\mathrm{s}}{R} = \frac{\pi r^2}{\delta}\sqrt{\frac{\mu_0}{2}} = \frac{\pi(10^{-3})^2}{6.6 \times 10^{-5}}\sqrt{\frac{4\pi \times 10^{-7}}{2}} = 3.77 \times 10^{-5}$$

14.74 When a charge q moves in a magnetic field, it experiences a magnetic force

$$F_\mathrm{m} = q\boldsymbol{v} \times \boldsymbol{B} \tag{1}$$

When an electric conductor is physically moved across a magnetic field, the free electrons in the conductor will experience a force on them in the direction of the force. The flow of electrons implies the existence of a potential difference between the ends of the conductor. The situation is the same as if an electric field had been set up in the conductor which is expressed by the relation

$$E_\mathrm{m} = F_\mathrm{m}/q = \boldsymbol{v} \times \boldsymbol{B}\ \mathrm{V/m} \tag{2}$$

Equation (2) implies that every moving magnetic field is accompanied by an electric field.
From (2) and the definition of the emf ξ of a source, the instant emf of the source is

$$\xi = \int \boldsymbol{E}_{\mathrm{m}} \cdot \mathrm{d}\boldsymbol{l} = \int \boldsymbol{v} \times \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l}\,\mathrm{V} \tag{3}$$

This is the general expression for motional emf. Now Faraday's law states that

$$\oint \boldsymbol{E}_{\mathrm{m}} \cdot \mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}\phi}{\mathrm{d}t} \tag{4}$$

This equation states that every time-changing magnetic field has an electric field associated with it. Now the total flux through the surface is

$$\phi = \int_s \boldsymbol{B} \cdot \hat{n}\mathrm{d}s \tag{5}$$

Therefore

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int \boldsymbol{B} \cdot \hat{n}\,\mathrm{d}s \tag{6}$$

If the source and only the induction are changing, $\mathrm{d}/\mathrm{d}t$ outside the integral may be replaced by $\partial/\partial t$ inside the integral. The expression becomes

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \int \frac{\partial \boldsymbol{B}}{\partial t} \cdot \hat{n}\,\mathrm{d}s \tag{7}$$

Combining (7) with (4)

$$\oint_c \boldsymbol{E}_{\mathrm{m}} \cdot \mathrm{d}\boldsymbol{l} = -\int \frac{\partial \boldsymbol{B}}{\partial t} \cdot \hat{n}\,\mathrm{d}s \tag{8}$$

Transforming the line integral in (8) into surface integral by the use of Stokes' theorem

$$\oint_c \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\int_s \hat{n} \cdot (\nabla \times \boldsymbol{E})\,\mathrm{d}s \tag{9}$$

Combining (8) and (9)

$$\int_s n \cdot (\nabla \times \boldsymbol{E})\,\mathrm{d}s = -\int \hat{n} \cdot \frac{\partial \boldsymbol{B}}{\partial t}\,\mathrm{d}s \tag{10}$$

Since this expression is true for any surface, the two integrals in (10) can be equated to yield

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{11}$$

This is known as Faraday's law in point form or differential form.

14.75 Ampere's law is expressed by

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 i \tag{1}$$

where B is known as magnetic induction or magnetic field. Its unit is weber per metre square or tesla. If magnetic materials are placed in the field of induction, the elementary magnetic dipoles, permanent or induced, will set up its own field that will modify the original field. A large value of B in an iron core is explained by a subsidiary vector, the magnetization M which is the magnetic moment per unit volume of the core material. A hypothetical current i_M is introduced and Ampere's law, (1), is modified accordingly:

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 (i + i_\mathrm{M}) \tag{2}$$

Writing

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 i + \mu_0 \oint M \cdot \mathrm{d}\boldsymbol{l} \tag{3}$$

we find

$$\oint \left(\frac{\boldsymbol{B} - \mu_0 \boldsymbol{M}}{\mu_0} \right) \cdot \mathrm{d}\boldsymbol{l} = i \tag{4}$$

or $$\oint \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = i \tag{5}$$

where $$H = \frac{B - \mu_0 M}{\mu_0} \tag{6}$$

is known as the magnetic field strength.

$$\therefore \quad B = \mu_0 (H + M) \tag{7}$$

The unit of H is henry/metre.

(**a**) For paramagnetic material $\boldsymbol{B}$ is directly proportional to $\boldsymbol{H}$, the relation being $B = k_m \ \mu_0 H$, where k_m is the permeability of the magnetic medium, which is a constant for a given temperature and density of the material.

(**b**) In ferromagnetic materials the relationship between B and H is far from linear. The B–H curve is known as the familiar hysteresis curve. k_m is a function not only of the value of H but also because of hysteresis and is a function of the magnetic and thermal history of the specimen.

14.76 Consider Maxwell's equations

$$\nabla \cdot \boldsymbol{E} = \rho / \varepsilon \tag{1}$$

$$\text{and} \quad \nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \tag{2}$$

Define a new electric potential function $\phi(r, t)$ such that

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} \tag{3}$$

The reason for redefining the scalar potential in this fashion is that (3) is consistent with Faraday's law, $\nabla \times \boldsymbol{E} = -(\partial \boldsymbol{B}/\partial t)$, as can be verified by substitution. On the other hand the electrostatic definition, $\boldsymbol{E} = -\boldsymbol{\nabla} V$ is inconsistent with Faraday's law and therefore cannot be used in electrodynamics. However, the relation $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, continues to be correct. Substituting (3) in (1), we obtain

$$\nabla^2\phi + \frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) = -\frac{\rho}{\varepsilon} \tag{4}$$

Substituting $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ in (2), we get

$$\nabla^2\boldsymbol{A} - \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) = -\mu\boldsymbol{J} - \mu\varepsilon\frac{\partial \boldsymbol{E}}{\partial t} \tag{5}$$

It is convenient to choose a Lorentz gauge given by

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = -\frac{1}{c^2}\frac{\partial \varphi}{\partial t} \quad \text{(Lorentz condition)} \tag{6}$$

With the use of (6), (4) and (5) are simplified to

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\varepsilon} \tag{7}$$

$$\nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = -\mu\boldsymbol{J} \tag{8}$$

14.77 $$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad \text{(Faraday's law)} \tag{1}$$

Writing the curl in rectangular form gives

$$\frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t}$$

Integrating in time and choosing the constant of integration as zero, we obtain

$$B_x = \frac{1}{v}f(z - vt) \tag{2}$$

Notice that the variation of B is exactly the same as the variation of E, except that E_y and B_x are at right angles to each other and perpendicular to the direction of propagation. From (2) and the relation $H_x = B_x/\mu_0$ we find

$$H_x = \frac{1}{\mu_0 v} f(x - vt) \quad (3)$$

so that $E_y = \mu_0 v H_x$ (4)

Using the relation $v = \sqrt{1/\mu_0 \varepsilon_0}$, we can write (4) in the form

$$E_y = Z_0 H_x$$

with $Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.6\,\Omega$.

14.78 **(a)** $\boldsymbol{E}(z, t) = E_0[\hat{x}\sin(kz - \omega t) + \hat{y}\cos(kz - \omega t)]$. This is a plane polarized wave polarized in the xy-plane and propagating in the positive z-direction.

(b) The magnetic lines will suffer refraction in passing from one magnetic medium to another.

(i) The continuity of B lines is first specified as a necessary condition. Figure 14.8 shows a bundle of B lines in passing through the interface between two magnetic media characterized by μ_1 and μ_2.

(ii) Since $di\text{v}\ B = 0$, it is required that the magnetic flux associated with the flux lines be constant in passing through the interface.

$$\phi = B_1\,\mathrm{d}s_1 = B_2\,\mathrm{d}s_2$$

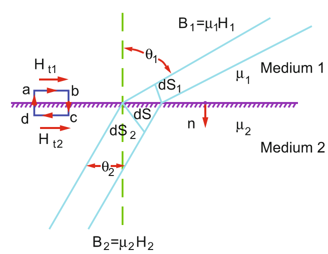


Fig. 14.8 The refraction of magnetic lines

where $\mathrm{d}s_1$ and $\mathrm{d}s_2$ are the cross-section of the flux lines in medium 1 and 2, respectively. Dividing by $\mathrm{d}s$, the corresponding area on the interface, we get

$$B_1 \frac{\mathrm{d}s_1}{\mathrm{d}s} = B_2 \frac{\mathrm{d}s_2}{\mathrm{d}s}$$

which from Fig. 14.8 may be written as

$$B_1 \cos\theta_1 = B_2 \cos\theta_2 \tag{1}$$

which may be written as

$$\boldsymbol{B}_1 \cdot \hat{n} = \boldsymbol{B}_2 \cdot \hat{n}$$

which shows that the normal component of the B vector is the same on both sides of the boundary.

(iii) Next we apply Ampere's circuital law to the path across the interface, Fig. 14.8. Assuming that no current exists in the interface, for the path considered

$$\oint \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = 0$$

Breaking the integral into individual parts of the path

$$\oint \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = \int_a^b \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} + \int_b^c \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} + \int_c^d \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} + \int_d^a \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = 0$$

In the limit the path shrinks approaching the interface

$$\int_b^c \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = \int_d^a \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = 0$$

$$\therefore \quad \int_a^b \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} + \int_c^d \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = 0$$

Thus $\quad H_{t1} = H_{t2}$

i.e. $\quad \boldsymbol{H}_1 \times \hat{n} = \boldsymbol{H}_2 \times \hat{n} \qquad (2)$

This implies that the tangential component of the $\boldsymbol{H}$ vector is the same on both sides of the boundary.

(c) Dividing (2) by (1)

$$\frac{H_1 \sin\theta_1}{B_1 \cos\theta_2} = \frac{H_2 \sin\theta_2}{B_2 \cos\theta_2}$$

$$\therefore \quad \frac{1}{\mu_1} \tan\theta_1 = \frac{1}{\mu_2} \tan\theta_2$$

$$\therefore \quad \frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2} \quad \text{(law of refraction)}$$

14.79 Consider a plane wave normally incident on a dielectric discontinuity, as in Fig. 14.9. In the region $z < 0$, $\varepsilon = \varepsilon_1$, and for $z > 0$, $\varepsilon = \varepsilon_2$. The boundary condition on E is that its tangential component is continuous, the boundary condition on H is that its tangential component is also continuous.

$$E_\mathrm{i}(z=0) + E_\mathrm{r}(z=0) = \boldsymbol{E}_\mathrm{t}(z=0) \quad (1)$$

$$H_\mathrm{i}(z=0) + H_\mathrm{r}(z=0) = \boldsymbol{H}_\mathrm{t}(z=0)$$

Letting $E_\mathrm{i} = E_0 \mathrm{e}^{-\mathrm{j}kz} e_x,\ E_\mathrm{r} = E_1 \mathrm{e}^{\mathrm{j}kz} e_x,\ E_\mathrm{t} = E_2 \mathrm{e}^{-\mathrm{j}kz} e_x,$

$$H_\mathrm{i} = \frac{E_0}{\eta} \mathrm{e}^{-\mathrm{j}kz} e_y \text{ and } H_\mathrm{r} = -\frac{E_1}{\eta} \mathrm{e}^{-\mathrm{j}kz} e_y \quad (2)$$

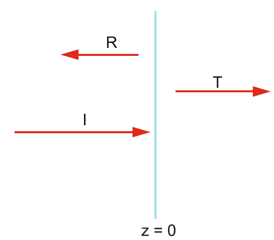


Fig. 14.9 Reflection of plane waves normally incident on the interface between two dielectrics

and substituting in (1), we obtain

$$E_0 + E_1 = E_2 \quad (3)$$

$$\frac{\mathrm{E}_0}{\eta_1} - \frac{\mathrm{E}_1}{\eta_1} = \frac{\mathrm{E}_2}{\eta_2} \quad (4)$$

where $\eta_1 = \sqrt{\mu_1/\varepsilon_1}$ and $\eta_2 = \sqrt{\mu_2/\varepsilon_2}$. Solving, we find

$$\frac{E_1}{E_0} = \rho = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad (5)$$

$$\frac{E_2}{E_0} = \tau = \frac{2\eta_2}{\eta_2 + \eta_1} \quad (6)$$

Note the minus sign in the last relation for H_r in (2) arises because the Poynting vector $S = E \times H$ must be in the direction of propagation (right-hand rule).

Substituting $\eta_1 = \sqrt{\mu_1/\varepsilon_1}$ and $\eta_2 = \sqrt{\mu_2/\varepsilon_2}$ in (5) and (6) and setting $\mu_1 = \mu_2 = \mu_0$ for non-magnetic substances, and putting $\sqrt{\varepsilon_1/\varepsilon_2} = n_1/n_2$ for the refractive index, we obtain

$$R = \rho^2 = \frac{(n_2 - n_1)^2}{(n_2 + n_1)^2} \tag{7}$$

$$T = \tau^2 \frac{n_1}{n_2} = \left(\frac{2n_2}{n_2 + n_1}\right)^2 \frac{n_1}{n_2} = \frac{4n_1 n_2}{(n_1 + n_2)^2} \tag{8}$$

Adding (7) and (8) it follows that $R + T = 1$. This is simply the consequence of conservation of energy.

14.80 Refer to Fig. 14.10. Consider a plane perpendicular to the propagation direction through the origin. Let the distance from this plane measured in the direction of propagation be called l. If the coordinates of a point are x, z, then

$$l_I = x \sin\theta + z \cos\theta \tag{1}$$

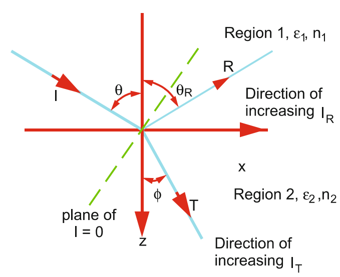


Fig. 14.10 Reflection and Refraction of electromagnetic wave

As the electric field is in the plane of incidence, for the incident wave

$$\begin{cases} E_\mathrm{I} = E_0 \mathrm{e}^{\mathrm{j}k_\mathrm{I} l_\mathrm{I}} (\cos\theta\, e_x - \sin e_z) \\ H_\mathrm{I} = \frac{E_0}{\eta_1} \mathrm{e}^{-\mathrm{j}k_\mathrm{I} l_\mathrm{I}} e_y \end{cases} \tag{2}$$

where $k_1 = \omega\sqrt{\nu\varepsilon_1}$ and $\eta_1 = \sqrt{\mu_1/\varepsilon_1}$ are the values of the propagation constant and characteristic impedance in region 1.

For the reflected wave $l_\mathrm{R} = x \sin\theta' - z\cos\theta'$ (3)

$$\begin{cases} E_\mathrm{R} = E_\mathrm{R} \mathrm{e}^{-\mathrm{j}k_\mathrm{I} l_\mathrm{R}} (\cos\theta_\mathrm{R}\, e_x + \sin\theta_\mathrm{R} e_z) \\ H_\mathrm{R} = -\dfrac{E_\mathrm{R}}{\eta_1} \mathrm{e}^{-\mathrm{j}k_\mathrm{I} l_\mathrm{R}} e_y \end{cases} \tag{4}$$

For the transmitted wave the relations are

$$l_T = x\sin\phi + z\cos\phi \tag{5}$$

$$\begin{cases} E_{\mathrm{T}} = E_{\mathrm{T}} e^{-jk_{\mathrm{T}} l_{\mathrm{T}}}(\cos\phi\, e_x - \sin\phi e_z) \\ H_{\mathrm{T}} = \dfrac{E_{\mathrm{T}}}{\eta_2} e^{-jk_{\mathrm{T}} l_{\mathrm{T}}} e_y \end{cases} \tag{6}$$

The boundary conditions at $z = 0$ require that the tangential E that is E_x and tangential H_y be continuous. Setting $E_x(z = 0)$ in region 1 equal to $E_x(z = 0)$ in region 2,

$$E_0\cos\theta\, e^{-jk_{\mathrm{I}} x\sin\theta} + E_{\mathrm{R}}\cos\theta_{\mathrm{R}}\, e^{-jk_{\mathrm{I}} x\sin\theta_{\mathrm{R}}} = E_{\mathrm{T}}\cos\phi e^{-jk_{\mathrm{T}} x\sin\phi} \tag{7}$$

If this equation is to hold for all values of x then

$$k_{\mathrm{I}}\sin\theta = k_{\mathrm{I}}\sin\theta_{\mathrm{R}} = k_{\mathrm{T}}\sin\phi \tag{8}$$

It follows that $\theta_{\mathrm{R}} = \theta$, that is, the angle of incidence is equal to angle of reflection as in a plane mirror.
Further

$$\frac{\sin\phi}{\sin\theta} = \frac{k_I}{k_T} = \frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}} = \frac{n_1}{n_2} \tag{9}$$

where n is the index of refraction of the material. Equation (9) then gives Snell's law ($n_1\sin\theta = n_2\sin\phi$) which holds irrespective of the nature of polarization. Using (9) in (7) and cancelling the exponential terms we have

$$E_0\cos\theta + E_{\mathrm{R}}\cos\theta = E_{\mathrm{T}}\cos\phi \tag{10}$$

Using (8), the boundary conditions on H_y yield

$$\frac{E_0}{\eta_1} - \frac{E_{\mathrm{R}}}{\eta_1} = \frac{E_{\mathrm{T}}}{\eta_2} \tag{11}$$

Solving (10) and (11) we get

$$\frac{E_{\mathrm{R}}}{E_0} = \frac{\eta_2\cos\phi - \eta_1\cos\theta}{\eta_2\cos\phi + \eta_1\cos\theta} \tag{12}$$

$$\frac{E_{\mathrm{T}}}{E_0} = \frac{2\eta_2\cos\theta}{\eta_2\cos\phi + \eta_1\cos\theta}. \tag{13}$$

Substituting $\eta_1 = \sqrt{\mu_1/\varepsilon_1}, \eta_2 = \sqrt{\mu_2/\varepsilon_2}$, $\mu_1 = \mu_2 = \mu_0$ for non-magnetic dielectric and using (9) in (12), we obtain

$$R = \left(\frac{E_1}{E_0}\right)^2 = \left(\frac{n_2\cos\theta - n_1\cos\phi}{n_2\cos\theta + n_1\cos\phi}\right)^2$$

14.81 **(a)** Reflectance $R = 0$ if $n_2\cos\theta - n_1\cos\phi = 0$ (1)

Given $\tan\theta = \dfrac{n_2}{n_1}$ (Brewster's law of polarization) (2)

$\dfrac{\sin\theta}{\sin\phi} = \dfrac{n_2}{n_1}$ (Snell's law of refraction) (3)

Combining (2) and (3), we have (Fig. 14.11)

$\sin\phi = \cos\theta$ (4)

or $\phi = 90° - \theta$ (5)

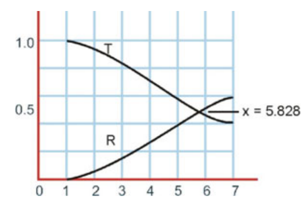


Fig. 14.11 R and T against n_1/n_2

Eliminating n_2 between (1) and (2)

$n_2\cos\theta - n_1\cos\phi = n_1(\sin\theta - \cos\phi) = n_1(\sin\theta - \sin\theta) = 0$ where we have used (5).

(b) $\tan\theta = \dfrac{n_2}{n_1} = \dfrac{1.5}{1} = 1.5$

$\therefore \quad \theta = 56.31°$

14.82 **(a)** $R = \dfrac{(n_1 - n_2)^2}{(n_1 + n_2)^2} = \dfrac{\left(\frac{n_1}{n_2} - 1\right)^2}{\left(\frac{n_1}{n_2} + 1\right)^2} = \dfrac{(x-1)^2}{(x+1)^2}$ (1)

where $x = \dfrac{n_1}{n_2}$

$T = 1 - R = \dfrac{4x}{(x+1)^2}$ (2)

(b) Setting $R = T$ yields the quadratic equation $x^2 - 6x + 1 = 0$, whose solution is $x = 3 + 2\sqrt{2}$ or 5.828. Thus for $n_1/n_2 = 5.828$, we get $R = T = 0.5$.

14.83 Using Maxwell's equations

$$\nabla \cdot \boldsymbol{E} = 0 \quad (1)$$

$$\nabla \cdot \boldsymbol{B} = 0 \quad (2)$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad (3)$$

(a) $\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\omega t - \boldsymbol{k}.\boldsymbol{r} + \phi)}$ (4)

$\boldsymbol{B} = \boldsymbol{B}_0 e^{i(\omega t - \boldsymbol{k}.\boldsymbol{r} + \phi)}$ (5)

Let the wave be propagated in the z-direction. Then

$\nabla \cdot \boldsymbol{E} = -i\,\boldsymbol{k} \cdot \boldsymbol{E} = 0$

and $\nabla \cdot \boldsymbol{B} = -i\,\boldsymbol{k} \cdot \boldsymbol{B} = 0$

This shows that both $\boldsymbol{E}$ and $\boldsymbol{B}$ are perpendicular to $\boldsymbol{k}$, which is the direction of propagation. Therefore both $\boldsymbol{E}$ and $\boldsymbol{B}$ are transverse oscillations.

(b) $\boldsymbol{B}$ and $\boldsymbol{E}$ are in phase

(c) Using (3)

$$\nabla \times \boldsymbol{E} = -i\,\boldsymbol{k} \times \boldsymbol{E} = -i\omega \boldsymbol{B}$$

$$\therefore \quad \boldsymbol{B} = \frac{\boldsymbol{k} \times \boldsymbol{E}}{\omega} = \frac{1}{c}\frac{\boldsymbol{k} \times \boldsymbol{E}}{k}$$

$$\therefore \quad \boldsymbol{B} = \frac{1}{c}\hat{\boldsymbol{s}} \times \boldsymbol{E} \quad (6)$$

where $\hat{s} = \boldsymbol{k}/k$ is a unit vector in the direction of propagation. The three vectors $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{k}$ form a right-handed rectangular coordinate system. From (6) we obtain $B = E/c$.

14.84 $E = \dfrac{\sigma}{2\varepsilon_r \varepsilon_0}$

$\therefore \quad \sigma = 2\varepsilon_0 \varepsilon_r E = 2 \times 8.85 \times 10^{-12} \times 6 \times 2 \times 10^3$

$= 2.124 \times 10^{-7}\,\mathrm{C/m^2}$

14.3.3 Phase Velocity and Group Velocity

14.85 Using the result of prob. 14.93

$$\frac{1}{V_g} - \frac{1}{V_p} = \frac{V_p - V_g}{V_g V_p} = \frac{V_p - V_g}{c^2} - = \frac{\omega}{c}\frac{dn}{d\omega}$$

$$\therefore \quad \frac{V_\mathrm{p} - V_\mathrm{g}}{V_\mathrm{p}} = \frac{\omega c}{V_\mathrm{P}} \frac{\mathrm{d}n}{\mathrm{d}\omega} \tag{1}$$

But $\omega = 2\pi\nu = 2\pi \mathrm{c}/\lambda$ (2)

$$V_\mathrm{p} = c/n \tag{3}$$

$$\frac{\mathrm{d}n}{\mathrm{d}\omega} = \frac{\mathrm{d}n}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}\omega} = -\frac{\lambda^2}{2\pi c}\frac{\mathrm{d}n}{\mathrm{d}\lambda} \tag{4}$$

Substituting (2), (3) and (4) in (1)

$$\frac{V_\mathrm{p} - V_\mathrm{g}}{V_\mathrm{p}} = -n\lambda \frac{\mathrm{d}n}{\mathrm{d}\lambda} \tag{5}$$

$n_1 - 1$	$n_2 - 1$	$n = \frac{n_1+n_2}{2}$	Δn	$\lambda_1(A)$	$\lambda_2(A)$	$\lambda = \frac{\lambda_1+\lambda_2}{2}$	$\Delta\lambda = \lambda_1 - \lambda_2$
2.786×10^{-4}	2.781×10^{-4}	$1 + 2.784 \times 10^{-4}$	5×10^{-7}	4800	5000	4900 A	-200 A
2.781×10^{-4}	2.777×10^{-4}	$1 + 2.779 \times 10^{-4}$	4×10^{-7}	5000	5200	5100 A	-200 A

Using formula (5), the first set of data gives $(V_\mathrm{p} - V_\mathrm{g})/V_\mathrm{p} = 1.22 \times 10^{-5}$ and the second set 1.02×10^{-5}.

14.86 $\omega = ak^2$

(a) $v_\mathrm{p} = \dfrac{\omega}{k} = ak$

(b) $v_\mathrm{g} = \dfrac{\mathrm{d}\omega}{\mathrm{d}k} = 2ak = 2v_\mathrm{p}$

14.87 **(a)** $v_\mathrm{p} = \dfrac{1}{\sqrt{\varepsilon\mu}}$ (1)

(b) $n = \sqrt{\varepsilon_\mathrm{r}}$

$$\varepsilon_\mathrm{r} = n^2 = 1 - \frac{D^2}{\omega^2}$$

$$\therefore \quad n = \sqrt{1 - \frac{D^2}{\omega^2}} \tag{2}$$

$$v_\mathrm{p} = \frac{c}{n} = \frac{c}{\sqrt{1 - \frac{D^2}{\omega^2}}} \tag{3}$$

Squaring (3) and re-arranging

$$\omega^2 = D^2 + \frac{\omega^2 c^2}{v_\mathrm{p}^2} = D^2 + k^2 c^2 \tag{4}$$

$$\therefore \quad v_\mathrm{p} = \omega/k$$

(c) Differentiating with D with c constant, $\omega \dfrac{\mathrm{d}\omega}{\mathrm{d}k} = c^2 k$

$$\therefore \quad v_\mathrm{g} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{c^2 k}{\omega} = \frac{c^2}{v_\mathrm{p}} = c\sqrt{1 - \frac{D^2}{\omega^2}}$$

$$\therefore \quad v_\mathrm{p} v_\mathrm{g} = c^2 \tag{5}$$

(d) Substituting $D = 1.2 \times 10^{11}/\mathrm{s}$, $\omega = 2\pi \times 20 \times 10^9$ Hz and $c = 3 \times 10^8$ m/s in (3), we find $v_\mathrm{p} = 1.016 \times 10^9$ m/s
Substituting $v_\mathrm{p} = 1.016 \times 10^9$ m/s in (5) we find $v_\mathrm{g} = 8.858 \times 10^7$ m/s.
It is observed that while $v_\mathrm{p} >$ c, $v_\mathrm{g} <$ c.

14.88 $$V_\mathrm{g} = \frac{\mathrm{d}\omega}{\mathrm{d}k} \tag{1}$$

$$\omega = vk \tag{2}$$

$$\therefore \quad V_\mathrm{g} = \frac{\mathrm{d}}{\mathrm{d}k}(vk) = v + k\frac{\mathrm{d}v}{\mathrm{d}k} \tag{3}$$

14.89 $$V_\mathrm{g} = v + k\frac{\mathrm{d}v}{\mathrm{d}k} \quad \text{(by prob. 14.88)} \tag{1}$$

$$V = c/n \tag{2}$$

Substituting (2) in (1)

$$V_\mathrm{g} = c/n + ck\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{1}{n}\right) = \frac{c}{n} - \frac{ck}{n^2}\frac{\mathrm{d}n}{\mathrm{d}k} \tag{3}$$

$$\text{Now} \quad \frac{\mathrm{d}n}{\mathrm{d}k} = \frac{\mathrm{d}n}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}k} = \frac{\mathrm{d}n}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{2\pi}{k}\right) = -\frac{2\pi}{k^2}\frac{\mathrm{d}n}{\mathrm{d}\lambda} \tag{4}$$

Using (4) in (3)

$$V_\mathrm{g} = \frac{c}{n} + \frac{\lambda c}{n^2}\frac{\mathrm{d}n}{\mathrm{d}\lambda}$$

14.90 $v \propto \dfrac{1}{\lambda}$ (by problem)

$$\therefore \quad v = Ak \quad \text{(where } A = \text{constant)}$$

$$v_\mathrm{g} = v + k\frac{\mathrm{d}v}{\mathrm{d}k} = Ak + k\frac{\mathrm{d}}{\mathrm{d}k}(Ak) = Ak + Ak = 2Ak = 2v$$

14.91 $$v_\mathrm{g} = v + k\frac{\mathrm{d}v}{\mathrm{d}k} = v + k\frac{\mathrm{d}v}{\mathrm{d}\omega}\frac{\mathrm{d}\omega}{\mathrm{d}k} = v + kv_\mathrm{g}\frac{\mathrm{d}v}{\mathrm{d}\omega} \tag{1}$$

Now $\quad v = c/n \qquad (2)$

$$\therefore \quad \frac{\mathrm{d}v}{\mathrm{d}\omega} = \frac{\mathrm{d}v}{\mathrm{d}n}\frac{\mathrm{d}n}{\mathrm{d}\omega} = -\frac{c}{n^2}\frac{\mathrm{d}n}{\mathrm{d}\omega} \tag{3}$$

Substituting (2) and (3) in (1) and using $\omega = kv$ and rearranging we get

$$v_\mathrm{g} = \frac{c}{n + \omega(\mathrm{d}n/\mathrm{d}\omega)}$$

14.92 $V_\mathrm{g} = \dfrac{\text{distance}}{\text{time}} = \dfrac{50}{1 \times 10^{-6}} = 5 \times 10^7\ \mathrm{m/s}$

$$V_\mathrm{g} = c\sqrt{1 - (\lambda/2a)^2}$$

$$\therefore \quad 5 \times 10^7 = 3 \times 10^8\sqrt{1 - (\lambda/5)^2}$$

Solving for λ, we find the free-space wavelength $\lambda = 4.93\,\mathrm{cm}$.

$$V_\mathrm{p} = \frac{c^2}{V_\mathrm{g}} = \frac{(3 \times 10^8)^2}{5 \times 10^7} = 1.8 \times 10^9\ \mathrm{m/s}$$

14.93 $V_\mathrm{g} = \mathrm{d}\omega/\mathrm{d}k \qquad (1)$

Rewriting (1), $1/V_\mathrm{g} = \mathrm{d}k/\mathrm{d}\omega$

$$\therefore \quad \frac{1}{V_\mathrm{g}} = \frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{\omega}{V_\mathrm{p}}\right) = \frac{1}{V_\mathrm{p}} - \frac{\omega}{V_\mathrm{p}^2}\frac{\mathrm{d}V_\mathrm{p}}{\mathrm{d}\omega} \tag{2}$$

Substituting $V_\mathrm{p} = c/n$ in (2)

$$\frac{1}{V_\mathrm{g}} = \frac{1}{V_\mathrm{p}} - \frac{\omega n^2 c}{c^2}\left(-\frac{1}{n^2}\frac{\mathrm{d}n}{\mathrm{d}\omega}\right) = \frac{1}{V_\mathrm{p}} + \frac{\omega}{c}\frac{\mathrm{d}n}{\mathrm{d}\omega} \tag{3}$$

14.94 $V_\mathrm{g} = \dfrac{\partial\omega}{\partial k}$

$$\therefore \quad \frac{1}{V_\mathrm{g}} = \frac{\partial k}{\partial\omega} = \frac{\partial\,(2\pi/\lambda)}{\partial(2\pi\nu)} = \frac{\partial\,(1/\lambda)}{\partial\nu}$$

But $n = \dfrac{c}{v_\mathrm{p}} = \dfrac{c}{\nu\lambda} \to \dfrac{1}{\lambda} = \dfrac{n\nu}{c}$

$$\therefore \quad V_\mathrm{g} = \frac{\partial\nu}{\partial\left(\frac{1}{\lambda}\right)} = \frac{\partial\nu}{\partial\,(n\nu/c)} = \frac{c\partial\nu}{\partial(n\,\nu)}$$

14.95 $v_g = \frac{d\omega}{dk} = \frac{d(\omega\hbar)}{d(k\hbar)} = \frac{dE}{dp} = \frac{d\left(p^2/2m\right)}{dp} = \frac{1}{2m}\frac{dp^2}{dp} = \frac{2p}{2m} = \frac{mv}{m} = v$

14.96 By prob. (14.85)

$$\frac{V_p - V_g}{V_p} = -n\lambda\frac{dn}{d\lambda} \tag{1}$$

$$\text{where} \quad V_p = \frac{c}{n} \tag{2}$$

Re-arranging (1) with the aid of (2) and writing μ for n we find

$$V_g = c\left[\frac{1}{\mu} + \lambda\frac{d\,\mu}{d\,\lambda}\right] \tag{3}$$

$$\mu = 1.420 + \frac{3.60 \times 10^{-14}}{\lambda^2} \quad \text{(by problem)} \tag{4}$$

Substituting $\lambda = 500\,\text{nm} = 5 \times 10^{-7}$ m in (4)

$$\mu = 1.564 \tag{5}$$

Differentiating μ with respect to λ in (4)

$$\frac{d\mu}{d\lambda} = -\frac{7.2 \times 10^{-14}}{\lambda^3}$$

$$\text{or} \quad \lambda\frac{d\mu}{d\lambda} = -\frac{7.2 \times 10^{-14}}{\lambda^2} = -0.288 \tag{6}$$

Substituting (5) and (6) in (3), we find $V_g = 0.35\,$c.

14.3.4 Waveguides

14.97 **(a)** Assuming the dominant mode, for $a = 2.5$ cm, and $b = 2.5$ cm, for the rectangular waveguide, and $c = 3 \times 10^8$ m/s, the velocity of electromagnetic waves in free space, the phase velocity is given by

$$V_p = \frac{c}{\sqrt{1-(\lambda/2a)^2}} = \frac{3 \times 10^8}{\sqrt{1-(4/5)^2}} = 5 \times 10^8\,\text{m/s}$$

(b) $V_g = c\sqrt{1-(\lambda/2a)^2} = 3 \times 10^8\sqrt{1-(4/5)^2} = 1.8 \times 10^8$ m/s

(c) $\lambda_g = \frac{\lambda}{\sqrt{1-(\lambda/2a)^2}} = \frac{4}{\sqrt{1-(4/5)^2}} = 6.67\,\text{cm}$

14.98 $$\lambda_g = \frac{\lambda}{\sqrt{1-(\lambda/2a)^2}} \tag{1}$$

$$\lambda_g = 3\lambda \quad \text{(by problem)} \tag{2}$$

Combining (1) and (2) and solving for λ with $a = 3\,\text{cm}$ we find the free-space wavelength $\lambda = 4\sqrt{2}\,\text{cm}$.

14.99 **(a)** $$\lambda_g = \frac{\lambda}{\sqrt{1-(\lambda/2a)^2}} = \frac{8}{\sqrt{1-(8/10)^2}} = 13.33\,\text{cm}$$

(b) The cut-off wavelength $\lambda_c = 2a = 2 \times 5 = 10\,\text{cm}$.

14.100 $$N = \frac{8\pi \nu^2 \mathrm{d}\nu V}{c^3} = \frac{8\pi \mathrm{d}\lambda \cdot V}{\lambda^4}$$

Mean $\lambda = 5500\text{Å} = 5.5 \times 10^{-7}$ cm

$$\mathrm{d}\lambda = (6000 - 5000)\mathring{A} = 10^{-7}\,\text{cm}$$

$$V = (0.5)^3\,\text{cm}^3$$

$$N = \frac{8\pi \times 10^{-7} \times (0.5)^3}{(5.5 \times 10^{-7})^4} = 3.43 \times 10^{18}$$

14.101 The cut-off frequency of the TM_{mn} or TE_{mn} mode is

$$\omega_{mn} = c\left[\left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2\right]^{1/2}$$

The lowest value we can have for ν_{mn} is for the choice $m = 1$, $a = 15$ and $n = 0$, that is, for TM_{10} wave.

$$\therefore \quad \nu_{10} = \frac{c}{2} \times \frac{1}{15} = \frac{3 \times 10^8}{30} = 10^7\,\text{Hz} = 10^4\,\text{kHz}$$

This is much above the range of AM waves (530–1600 kHz). Hence AM waves cannot propagate in the tunnel.

14.102 The cut-off frequency will be least for TE_{10} waves. Of course TM_{10} waves do not exist.

$$\nu_{10} = \frac{c}{2\pi}\frac{\pi}{a} = \frac{c}{2a} = \frac{3 \times 10^8}{2 \times 0.05} = 3 \times 10^9\,\text{Hz} = 3\,\text{GHz}$$

Note that we take the higher dimension (5 cm) for the lower value of cut-off frequency.

14.103 The relation between ω and k in a rectangular waveguide is

$$k^2 = \frac{\omega^2}{c^2} - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 \tag{1}$$

For TE_{01} waves $m = 0$, $n = 1$, $a = 1\,\text{cm}$ and $b = 2\,\text{cm}$. Equation (1) is then reduced to

$$k^2 = \frac{\omega^2}{c^2} - \left(\frac{\pi}{0.02}\right)^2 \tag{2}$$

The phase velocity is given by

$$v_p = \frac{\omega}{k} \tag{3}$$

The group velocity v_g is given by

$$v_g = \frac{d\omega}{dk} \tag{4}$$

$$v_g = \frac{d\omega}{dk} = \frac{kc^2}{\omega} = \frac{c^2}{v_p}$$

$$\text{or } v_p v_g = c^2 \tag{5}$$

The cut-off frequency is given by

$$\omega_{01} = \frac{\pi c}{b} \tag{6}$$

The $\omega - k$ plot for $m = 0$, $n = 1$, $b = 0.02m$ is shown in Fig. 14.12. For convenience the variables are chosen as dimensionless.

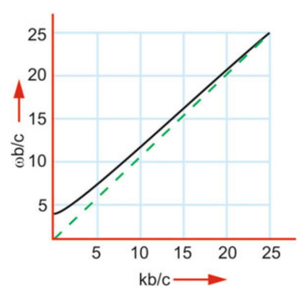


Fig. 14.12 Dispersion diagram for the TE_{01} mode of rectangular waveguide for $b = 2\,\text{cm}$

At high frequencies, the curve is asymptotic to the line $\omega - kc$. Thus at high frequencies, both the phase and group velocities approach c. However, the

$\omega - k$ plot is always above the $\omega = kc$ line. This implies that the phase velocity $v_\mathrm{p} = \omega/k$ is always larger than c.

The group velocity $v_\mathrm{g} = \mathrm{d}\omega/\mathrm{d}k$ is determined by the slope of the curve which is less than the slope of the line $\omega = kc$, and so v_g is always less than c.

There is a minimum frequency, known as the cut-off frequency, below which k becomes imaginary and the wave ceases to exist. As the frequency approaches the cut-off frequency the phase velocity becomes infinite and the group velocity becomes zero.

14.104 **(a)** $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E_z = (k^2 - \omega^2 \mu\varepsilon) E_z \quad (1)$$

For TM waves H_z vanishes, and E_z must be a solution of (1). Let

$$E_z = (A \cos k_x x + B \sin k_x x)(C \cos k_y y + D \sin k_y y)\, \mathrm{e}^{-\mathrm{j}kz} \quad (2)$$

where A, B, C, D are arbitrary constants and k_x and k_y are constants to be determined by boundary conditions. Substitution of (2) in (1) yields

$$\left(k_x^2 + k_y^2\right) = \omega^2 \mu\varepsilon - k^2 \quad (3)$$

The form of (2) is further constrained by the boundary conditions. Assuming that the walls of the waveguide are perfect conductors, E_z which is tangential to the walls must vanish at $x = 0$, $x = a$, $y = 0$ and $y = b$, Fig. 14.13. In (2) E_z will not vanish at $x = 0$ unless $A = 0$. Similarly the boundary condition at $y = 0$ is satisfied if $C = 0$. The boundary conditions at $x = a$ and $y = b$ are satisfied by putting.

$$k_x = \frac{m\pi}{a}$$

$$k_y = \frac{m\pi}{b} \quad (4)$$

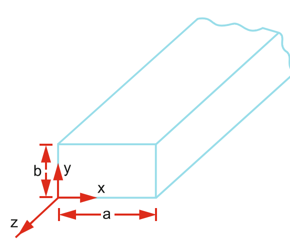


Fig. 14.13 Rectangular hollow metal waveguide

where m and n are positive integers. Finally, we get the result

$$E_z = K \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-jkz} \tag{5}$$

where $K = (B)(D)$ is another constant. Note that E_z vanishes if either $m = 0$ or $n = 0$. In that case TM_{m0} or TM_{0n} wave does not exist. The other field components are given from equations which are obtained by manipulating Maxwell's equations.

$$E_x = -\frac{j}{\omega^2\mu\varepsilon - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\mu\frac{\partial H_z}{\partial y}\right) \tag{6}$$

$$E_y = -\frac{j}{\omega^2\mu\varepsilon - k^2}\left(-k\frac{\partial E_z}{\partial y} + \omega\mu\frac{\partial H_z}{\partial x}\right) \tag{7}$$

Using (5) in (6) and (7) and putting $H_z = 0$

$$E_x = -\frac{jK\,km\pi}{(\omega^2\mu\varepsilon - k^2)a} \cos\frac{m\pi x}{a} \sin\frac{n\pi x}{b} e^{-jkz} \tag{8}$$

$$E_y = -\frac{jK\,kn\pi}{(\omega^2\mu\varepsilon - k^2)b} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-jkz} \tag{9}$$

The solutions (5), (8) and (9) represent an infinitely large family of waves, characterized by different values of the integers m and n. They differ from one another by the values of the integers m and n. They also differ in their velocity as well as field configuration.

(b) Combining (3) and (4) we get

$$k^2 = \frac{\omega^2}{c^2} - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 \tag{10}$$

where we have substituted $\mu\varepsilon = 1/c^2$. The cut-off frequency is obtained by setting $k = 0$.

$$\omega_{mn} = \pi c\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^{1/2} \tag{11}$$

14.105 **(a)** For TE waves there is no E_z. Here, we must find boundary conditions on H_z that cause the tangential component of E_z to vanish. The given equation is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) H_z = (k^2 - \omega^2\mu\varepsilon)\, H_z \tag{1}$$

The general solution is

$$H_z = (A\cos k_x x + B\sin k_x x)(C\cos k_y y + D\sin k_y y)e^{-jkz} \tag{2}$$

$$\therefore \quad \frac{\partial H_z}{\partial x} = (-k_2 A\sin k_x x + Bk_2\cos k_x x)$$
$$(C\cos k_y y + D\sin k_y y)e^{-jkz}$$

For $\dfrac{\partial H_z}{\partial x} = 0$ at $x = 0$, it is necessary that $B = 0$.
Similarly

$$\frac{\partial H_z}{\partial y} = (A\cos k_x x + B\sin k_x x)(-Ck_y\sin k_y y + D\cos k_y y)e^{-jkz}$$

For $\dfrac{\partial H_z}{\partial y} = 0$, $D = 0$. Therefore (2) becomes

$$H_z = K\cos(k_x x)\cos(k_y y)\, e^{-jkz} \tag{3}$$

where K is the product of A and B is another constant. Imposing boundary conditions at $x = a$ and $y = b$, we have

$$\frac{\partial H_z}{\partial x} = -K\, k_x \sin k_x x\cos k_y y = 0$$

yielding

$$k_x a = m\pi \tag{4}$$

and $\dfrac{\partial H}{\partial y} = -K\, k_y\cos k_x x\sin k_y y = 0$

yielding

$$k_y b = n\pi \tag{5}$$

$$\therefore \quad H_z = K\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \tag{6}$$

Substituting (2) in (1) we obtain

$$k_x^2 + k_y^2 = \omega^2\mu\varepsilon - k^2 \tag{7}$$

which is identical with (3) of prob. (14.104).

(b) Substituting the values of k_x and k_y from (4) and (5) in (7) and setting $k = 0$ gives the cut-off frequency.

$$\omega_{mn} = \pi c \left[\left(\frac{m}{a} \right)^2 + \left(\frac{n}{b} \right)^2 \right]^{1/2} \tag{8}$$

which is identical with (11) of prob. (14.104).

(c) Thus the features which are identical for the TM and TE modes are

(i) $\omega - k$ plot
(ii) Phase
(iii) Group velocity
(iv) Cut-off frequency

However the important difference is that when either $m = 0$ or $n = 0$, the TM mode fails to exist. On the other hand H_z does not vanish for $m = 0$ or $n = 0$ (see (6)). Because of this fact the TE_{10} is the mode which has the lowest cut-off frequency. Here we have assumed that $a > b$. The cut-off frequency for TE_{10} mode is given by

$$\omega_{10} = \frac{\pi c}{a} \tag{9}$$

the free space wavelength being $2a$.

The small dimension (b) has no bearing on the cut-off frequency for this mode. The other advantage is that single mode operation is feasible over a wide range of frequencies.