## 14.2 Problems

### *14.2.1 The RLC Circuits*

**14.1** A 2.5 μF capacitor is connected in series with a non-inductive resistor of 300 Ω across a source of PD of rms value 50 V, alternating at 1000/2$\pi$ Hz. Calculate

**(a)** thermal values of the current in the circuit and the PD across the capacitor.
**(b)** the mean rate at which the energy is supplied by the source.

[Joint Matriculation Board of UK]

**14.2** A 3 Ω resistor is joined in series with a 10 mH inductor of negligible resistance, and a potential difference (rms) of 5.0 V alternating at 200/$\pi$ Hz is applied across the combination.

**(a)** Calculate the PD $V_R$ across the resistor and $V_L$ across the inductor.
**(b)** Determine the phase difference between the applied PD and the current.

[Joint Matriculation Board of UK]

**14.3** An inductance stores 10 J of energy when the current is 5 A. Find its value.

**14.4** A tuning circuit in a radio transmitter has a $4 \times 10^{-6}$ H inductance in series with a $5 \times 10^{-11}$ F capacitance. Find

**(a)** the frequency of the waves transmitted.
**(b)** their wavelength.

**14.5** A 6 Ω resistor, a 12 Ω inductive reactance and a 20 Ω capacitive reactance are connected in series to a 250 V rms AC generator. (a) Find the impedance. (b) Estimate the power dissipated in the resistor.

**14.6** At 600 Hz an inductor and a capacitor have equal reactances. Calculate the ratio of the capacitive reactance to the inductive reactance at 60 Hz.

**14.7** A capacitance has a reactance of 4 Ω at 250 Hz. (a) Find the capacitance. (b) Calculate the reactance at 100 Hz. (c) What is the rms current, if it is connected to a 220 V 50 Hz line?

**14.8** When an impedance, consisting of an inductance $L$ and a resistance $R$ in series, is connected across a 12 V 50 Hz supply, a current of 0.05 A flows which differs in phase from that of the applied potential difference by 60°. Find the value of $R$ and $L$. Find the capacitance of the capacitor which when connected in series in the above circuit has the effect of bringing the current into phase with the applied potential difference.

[University of London]

**14.9** When a 0.6 H inductor is connected to a 220 V 50 Hz AC line, what is (a) the rms current and (b) peak current?

**14.10** A simple alternator, when rotating at 50 revolutions/s, gives a 50 Hz alternating voltage of rms value 24 V. A 4.0 Ω resistance $R$ and a 0.01 H inductance $L$ are connected in series across its terminals. Assuming that the internal impedance of the generator can be neglected, find (a) the rms current flowing; (b) the power converted into heat; (c) the rms potential difference across each component.

**14.11** An AC circuit consists of only a resistor $R = 100\,\Omega$ and a source voltage $V = 0.5\,V_\mathrm{m}$ at time $t = 1/360\,\mathrm{s}$. Assuming that at $t = 0$, $V = 0$, find the frequency.

**14.12** Given that for a series LCR circuit the equation is

$$\frac{\mathrm{d}^2 V}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}V}{\mathrm{d}t} + \frac{1}{LC}V = 0$$

If a similar equation is to be used for a parallel LCR circuit as in Fig. 14.1, then show that

$$R_\mathrm{p} = \frac{L}{CR}$$

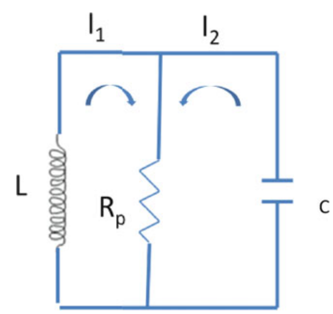



**Fig. 14.1**

**14.13** Verify the equation $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$.

**14.14** Show that in the usual notation the following combinations of physical quantities have the units of time: (a) $RC$, (b) $L/R$, (c) $\sqrt{LC}$.

**14.15** In an oscillating RLC circuit the amplitude of the charge oscillations drops to one-half its initial value in 4 cycles. Show that the fractional decrement of the resonance frequency is approximately given by $\frac{\Delta\omega}{\omega} = 0.00038$.

**14.16** Derive the equation for the current in a damped LC circuit for low damping.

**14.17** Set up the equation for the RC circuit in series and show that the input power is the sum of the powers delivered to the inductor and capacitor.

**14.18** Set up the equation for the RLC circuit in parallel and show that at any time the Joule heat in the resistor comes from the energy stored in the inductor and capacitor.

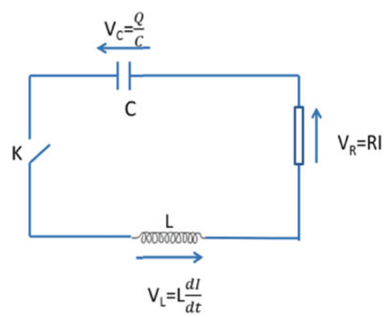
**14.19** For the electrical circuit shown below when the switch K is closed the charge and current are observed to oscillate (Fig. 14.2). Show that the differential equation governing the charge of the system can be written as

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}t^2} + 2\gamma \frac{\mathrm{d}Q}{\mathrm{d}t} + \omega_0^2 Q = 0$$

where $\gamma$ and $\omega_0^2$ are to be determined in terms of the capacitance, $C$, inductance, $L$, and resistance, $R$.
For a resistance $R = 100\,\Omega$, capacitance $C = 700\,\text{pF}$ and inductance $L = 80\,\text{mH}$ calculate

**(a)** The natural frequency, $f_0$, of the oscillation.
**(b)** The time constant, $\tau$, for the decay.

**Fig. 14.2**

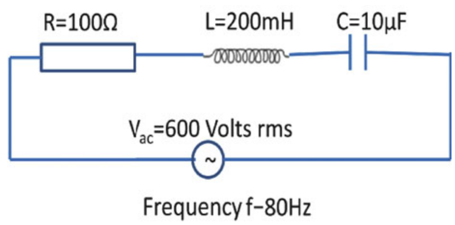



**14.20** Solve the differential equation given in prob. (14.19) and obtain the time period for damped harmonic motion.

**14.21** A resistor, capacitor and inductor are connected in series across an AC voltage source shown as in Fig. 14.3.

**(i)** Find the magnitude of the inductive reactance $X_\mathrm{L}$ of the inductor and capacitive reactance $X_\mathrm{C}$ of the capacitor.
**(ii)** Find the magnitude of the total impedance $Z$ of the circuit and sketch the impedance phasor diagram for this circuit.
**(iii)** Find the total current $I_\mathrm{T}$ through the circuit.

Fig. 14.3




**(iv)** Find the phase angle between supply voltage and current through the circuit.

**(v)** Find the voltages across $R$, $C$ and $L$ and show these on a phasor diagram.

**(vi)** What is the condition for resonance to occur in this type of circuit and at what frequency would this occur?

[University of Aberystwyth, Wales 2001]

**14.22** A 40 Ω resistor and a 50 μF capacitor are connected in series, and an AC source of 5 V at 300 Hz is applied. What is the magnitude of the current flowing through the circuit?

[University of Manchester 2007]

**14.23** For the circuits shown in Fig. 14.4a, b consisting of resistors, capacitors and inductors

**(i)** Derive the expression to represent the complex impedances for each of the networks.

**(ii)** Work out the magnitude of the impedance for each of the networks given that the frequency of the supply voltage is 150 Hz.

[University of Aberystwyth, Wales 2006]

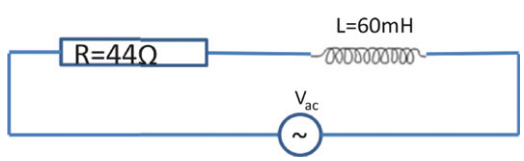



Fig. 14.4a

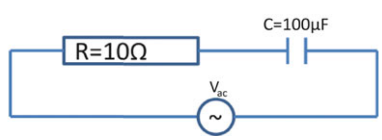



Fig. 14.4b

**14.24**

**(a)** Define what is meant by electric current and current density.
**(b)** When we refer to a quantity of charge we say that the value is quantized. Explain what is meant by quantized.
**(c)** A thin copper bar of rectangular cross-section of width 5.6 mm and height 50 μm has an electron density of $n = 8.5 \times 10^{28}/\text{m}^3$.

If a uniform current of $i = 2.4 \times 10^{-4}$ A flows through the strip

**(i)** Find the magnitude of the current density in the strip.
**(ii)** Find the magnitude of the drift speed of the charge carriers.
**(iii)** Briefly explain why the current is relatively high for such a small drift speed.

[University of Aberystwyth, Wales 2008]

**14.25** Two series resonant circuits with component values $L_1C_1$ and $L_2C_2$, respectively have the same resonant frequency. They are then connected in series; show that the combination has the same resonant frequency.

[University of Manchester 1972]

**14.26** An inductance and condenser in series have a capacitative impedance of 500 Ω at 1 kHz and an inductive impedance of 100 Ω at 5 kHz. Find the values of inductance and capacitance.

[University of Manchester 1972]

**14.27** A condenser of 0.01 μF is charged to 100 V. Calculate the peak current that flows when the charged condenser is connected across an inductance of 10 mH

[University of Manchester 1972]

**14.28** An inductance of 1 mH has a resistance of 5 Ω. What resistance and condenser must be put in series with the inductance to form a resonant circuit with a resonant frequency of 500 kHz and a $Q$ of 150?

[University of Manchester 1972]

**14.29** A parallel resonant circuit consists of a coil of inductance 1 mH and resistance 10 Ω in parallel with a capacitance of 0.0005 μF. Calculate the resonant frequency and the $Q$ of the circuit.

[University of Manchester 1972]

**14.30** The voltage on a capacitor in a certain circuit is given by $V(t) = V_0 \mathrm{e}^{-t/RC}$. Find the fractional error in the voltage at $t = 50\,\mu\text{s}$ if $R = 50\,\text{k}\Omega \pm 5\%$ and $C = 0.01\,\mu\text{F} \pm 10\%$.

[University of Manchester 1972]

**14.31** A condenser of 10 μF capacitance is charged to 3000 V and then discharged through a resistor of 10, 000 Ω. If the resistor has a temperature coefficient of 0.004/°C and a thermal capacity of 0.9 cal/°C, find **(a)** the time taken for the

voltage on the condenser to fall to $1/e$ of its initial value; **(b)** the percentage error which would have been introduced if thermal effects had been ignored.
[University of Manchester 1958]

**14.32** Show that the fractional half-width of the resonance curve of an RLC circuit is given by

$$\frac{\Delta\omega}{\omega} = \frac{\sqrt{3}}{Q}$$

where $Q$ is the quality factor given by $Q = \omega\,\mathrm{L/R}$.

### *14.2.2 Maxwell's Equations, Electromagnetic Waves, Poynting Vector*

**14.33** A plane em wave $E = 100\cos(6\times10^8\,t + 4x)$ V/m propagates in a medium. What is the dielectric constant of the medium?
[Indian Administrative Services]

**14.34** An infinite wire with charge density λ and current $I$ is at rest in the Lorentz frame $S$. Show that the speed of reference frame $S'$ where the electric field is zero, i.e. that frame in which one observes pure magnetic field, is given by $v = \frac{\lambda c^2}{I}$.

**14.35** Show that for a magnetic field $\boldsymbol{B}$ the wave equation has the form $\nabla^2\boldsymbol{B} = \mu_0\varepsilon_0\frac{\partial^2\boldsymbol{B}}{\partial t^2}$

**14.36** Use Maxwell's equation to show that $\nabla.\left(\boldsymbol{j} + \frac{1}{\varepsilon_0}\frac{\partial\boldsymbol{E}}{\partial t}\right) = 0$.

**14.37** The free-space wave equation for a medium without absorption is

$$\nabla^2\boldsymbol{E} - \mu_0\varepsilon_0\frac{\partial^2\boldsymbol{E}}{\partial t^2} = 0$$

Show that this equation predicts that electromagnetic waves are propagated with velocity of light given by $c = 1/\sqrt{\mu_0\varepsilon_0}$.

**14.38** An electromagnetic wave of wavelength 530 nm is incident onto a sheet of aluminium with resistivity $\rho = 26.5\times10^{-9}\,\Omega$m. Estimate the depth that the wave penetrates into the aluminium. The expression for the skin depth, $\delta$, is $\delta = \sqrt{2/\mu_0\sigma\omega}$.
[University of Manchester 2008]

**14.39** Consider an electromagnetic wave with its $E$-field in the $y$-direction. Apply the relation $\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$ to the harmonic wave

$\boldsymbol{E} = \boldsymbol{E}_0\cos(kx - \omega t), \boldsymbol{B} = \boldsymbol{B}_0\cos(kx - \omega t)$

to show that $E_0 = cB_0$.

**14.40** Using Maxwell's equations, show that in a conducting medium, the wave equation can be written as

$$\nabla^2 E = \mu\sigma \frac{\partial E}{\partial t} + \mu\varepsilon \frac{\partial^2 E}{\partial t^2}$$

and a similar expression for the $B$-field.

[University of Aberystwyth, Wales 2005]

**14.41** Let $l$ be the length of the coaxial cylindrical capacitor, $a$ the radius of the central wire and $b$ the radius of the tube. The conductors are connected to a battery of $V$ volts and a current $I$ is passed. Calculate (a) the capacitance per unit length of the cable, (b) the inductance per unit length.

**14.42** Consider a coaxial cable with radius $a$ for the central wire and radius $b$ for the tube connected to a resistance $R$ and battery of emf $\xi$. Calculate (a) $\boldsymbol{E}$; (b) $\boldsymbol{B}$; and (c) $\boldsymbol{S}$ for the region $a < r < b$.

**14.43** The general expression for magnetic energy density has the form $u_{\mathrm{B}} = {}^1\!/_2\,\boldsymbol{B.H.}$ Show that in the vacuum the above expression is reduced to $u_B = \frac{B^2}{2\mu_0}$.

**14.44** Show that at any point in the electromagnetic field the energy density stored in the electric field is equal to that stored in the magnetic field.

**14.45** A current $I$ is passed through a coaxial cable with inner radius $a$ and outer radius $b$. The cable can function both as a capacitor and as an inductor. If the stored electric and magnetic energy is equal then show that the resistance $R$ is approximately given by

$$R = \frac{377}{2\pi} \ln\left(\frac{b}{a}\right) \Omega$$

**14.46** A proton of kinetic energy 20 MeV circulates in a cyclotron with 0.5 m radius. Calculate its energy loss to radiation per orbit and show that it is negligible.

**14.47** A 40-W point source radiates equally in all directions. Find the amplitude of the $E$-field at a distance of 1 m.

**14.48** A laser beam has a cross-sectional area of 4.0 mm$^2$ and a power of 1.2 mW. Find (a) intensity $I$, (b) $E_0$, (c)$B_0$.

**14.49** A laser emits a 1-mm diameter highly collimated beam at a power level 314 mW. Calculate the irradiance.

**14.50** Beginning with the expression for the Poynting vector show that the time-averaged power per unit area carried by a plane electromagnetic wave in free space is given by

$$S_{\text{ave}} = \frac{E_0^2}{2\mu_0 c}$$

[University of Durham 2003]

**14.51** A plane electromagnetic wave has $E_x = E_y = 0$ and $E_z = 50 \sin\left[4\pi \times 10^{14}\left(t - \frac{x}{3\times 10^8}\right)\right]$. Calculate the irradiance (flux density).

**14.52** Show that $\boldsymbol{E} \times \boldsymbol{H}$ is in the same direction as the wave propagates and has magnitude equal to $|\boldsymbol{E} \times \boldsymbol{H}| = \frac{|\boldsymbol{E}|^2}{\mu_0 c}$.

[University of Aberystwyth, Wales]

**14.53** An electromagnetic plane wave in vacuum has $E$-field given by

$$E_z = 10 \sin \pi(2 \times 10^6 x - 6 \times 10^{14} t),\ E_x = E_y = 0.$$

Find (a) frequency; (b) wavelength; (c) speed; (d) $\boldsymbol{E}$-field amplitude; (e) polarization.

**14.54** Write down the equation for the associated magnetic field for the wave given in prob. (14.53).

**14.55** A radar monitors the speed $v$ of approaching cars by sending out waves of frequency $\nu$. If the frequency received is $\nu''$, find the speed of the car. How would the beat frequency change if the car is receding?

**14.56** Microwaves of frequency 800 MHz are beamed by a stationary police man towards a receding car speeding at 90 km/h. What beat frequency was registered by the radar?

**14.57** State Ampere's law.
A long solid conductor of radius $a$ lies on the axis of a long cylinder of inner radius $b$ and outer radius $c$. The central conductor carries a current $i$ while the outer conductor carries a current $-i$. The currents are uniformly distributed over the cross-sections of each conductor. By considering the current enclosed by a circular loop of radius $r$ centred on the axis of the inner conductor use Ampere's law to calculate the magnetic field in each of the four regions (a) $r < a$, (b) $a < r < b$, (c) $b < r < c$, (d) $r > c$.

**14.58** The CMS experiment at the Large Hadron Collider at CERN uses a large, cylindrical, superconducting solenoid. This magnet is 12.5 m in length with a diameter of 6 m. When powered, it generates a uniform magnetic field of 4 T. Estimate the energy stored in the magnetic field.

**14.59** Given that the total power radiated by the sun in the form of electromagnetic radiation is $4 \times 10^{26}$ W, estimate the electric and magnetic field amplitude at the surface of the sun. (The radius of the sun is $7 \times 10^8$ m).

[University of Durham 2003]

**14.60** At the orbit of the earth, the power of sunlight is 1,300 $\mathrm{Wm^{-2}}$. Estimate the amplitude of the electric field if we assume that all the power arrives on the earth in a monochromatic wave.

[University of Aberystwyth, Wales 2004]

**14.61** A typical value for the amplitude of the $E$-field for sunlight at the surface of Mars is 300 V/m Calculate the amplitude of the corresponding $B$-field and estimate the flux of radiation at the surface of Mars.

[University of Manchester 2006]

**14.62** Show that $\frac{|E|}{|H|} = 377\,\Omega$

[University of Aberystwyth, Wales]

**14.63** Calculate the skin depth in copper (conductivity $6 \times 10^7\ \Omega^{-1}/\mathrm{m}$) of radiation of frequency 20 kilocycles/s. Take $\mu$, the relative permeability of copper, as unity.

[University of Newcastle upon Tyne 1964]

**14.64** Copper has an electrical conductivity $\sigma = 5.6 \times 10^7\ \Omega^{-1}/\mathrm{m}$ and a magnetic permeability $\mu = 1$. On this basis estimate the order of magnitude of the depth to which radiation at a frequency of 3000 Mc/s can penetrate a large copper screen.

[University of Bristol 1959]

**14.65** Show that the skin depth in a good conductor is $\left[\frac{1}{2}\,\omega\sigma\mu\mu_0\right]^{-1/2}$ where the symbols have their usual meaning.

[University of Newcastle upon Tyne 1964]

**14.66** If the maximum electric field in a light wave is $10^{-3}$ V/m, find how much energy is transported by a beam of 1 $\mathrm{cm^2}$ cross-sectional area.

[University of Durham 1962]

**14.67** Prove Poynting's theorem, namely

$$\mathrm{div}\ (\boldsymbol{E} \times \boldsymbol{H}) + \boldsymbol{E}.\frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{H}.\frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{E}.\boldsymbol{j} = 0$$

What is the interpretation of this equation?

[University of Durham 1962][University of New Castle upon Tyne 1965]

**14.68** From Maxwell's equations, one can derive a wave equation for a dielectric of the form

$$\nabla^2 E - \mu_0\varepsilon_0\varepsilon_r\frac{\partial^2 E}{\partial t^2} - \mu_0\sigma_N\frac{\partial E}{\partial t} = 0$$

where $E$ is the electric field, $t$ is the time, $\varepsilon_r$ is the relative permittivity and $\sigma_N$ is the electrical conductivity. Hence by substituting a travelling wave solution into the wave equation derive a dispersion relation of the form

$$k^2 = \mu_0\varepsilon_0\varepsilon_r\omega^2 + i\mu_0\sigma_N\omega$$

where $k$ is the wave vector and $\omega$ is the angular frequency.

[University of Durham 2006]

**14.69** Show that the general identity

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\nabla^2\boldsymbol{E} + \nabla(\nabla.\boldsymbol{E})$$

is true for the specific vector field $F = x^2 z^3 \hat{i}$.

**14.70** Use the Poynting vector to determine the power flow in a coaxial cable by a DC current $I$ when voltage $V$ is applied. Neglect the resistance of the conductors. How are the results affected if this assumption is not made?

**14.71** A wire with radius $a$ and of conductivity $\sigma_{\mathrm{E}}$ carries a constant, uniformly distributed current $I$ in the $z$-direction. Apply Poynting's theorem to show that power dissipated in the wire is given by the familiar expression $I^2R$ for Joule's heat.

**14.72** A super-conductor is a material which offers no DC resistance and satisfies the equation

$$\boldsymbol{B} = -\frac{m_e}{ne^2}\nabla \times \boldsymbol{J}$$

where $n$ is the number of conduction electrons per unit volume, $\boldsymbol{B}$ is the magnetic field and $\boldsymbol{J}$ is the current density. Using Maxwell's equations, show that the equation for a superconductor leads to the relation

$$\nabla^2\boldsymbol{B} = \frac{\mu_0 ne^2}{m_e}\boldsymbol{B}$$

[University of Durham 2004]

**14.73** An oscillating voltage of high frequency is applied to a load by means of copper wire of radius 1 mm. Given that the skin depth is $6.6 \times 10^{-5}$ m for this frequency, what is the high-frequency resistance per unit length of the wire in terms of its direct current resistance per unit length?

[University of Durham 1966]

**14.74** Use Stokes' theorem to derive the expression

$$\text{Curl } \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

[University of New Castle upon Tyne 1964]

**14.75** Explain the difference between the vectors $\boldsymbol{B}$ and $\boldsymbol{H}$ in the theory of magnetism. Derive the expression $B = \mu_0(H + M)$. Indicate briefly how $B$ depends on it in the case of (a) paramagnetics and (b) ferromagnetics.

[University of Durham 1962]

**14.76** Show that Gauss' and Ampere's laws in free space, subject to the Lorentz condition, can be expressed in the usual notation as

$$-\nabla^2\phi + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\rho}{\varepsilon_0} \text{ and } -\nabla^2\boldsymbol{A} + \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = \mu_0\boldsymbol{J}$$

respectively.

[The University of Aberystwyth, Wales 2005]

**14.77** Using Faraday's law, $\nabla \times \boldsymbol{E} = -\partial B/\partial t$, for the propagation of electromagnetic waves travelling along the $z$-axis, show that

$$E_y = Z_0 H_x$$

where $Z_0 = \sqrt{\mu_0/\varepsilon_0} = 376.6\,\Omega$, is the wave impedance of free space.

**14.78** **(a)** The electric field of an electromagnetic wave propagating in free space is described by the equation

$$E(z, t) = E_0[\hat{x} sin(kz - \omega t) + \hat{y} cos(kz - \omega t)]$$

where $\hat{x}$ and $\hat{y}$ are unit vectors in the $x$- and $y$-direction, respectively. What is this wave's direction of propagation? What is the polarization of the wave?

**(b)** State and prove the boundary conditions satisfied by the magnetic intensity $H$ and the magnetic field $B$ at the boundary between two media with different magnetic properties.

**(c)** Show that

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}$$

where $\theta_1$ and $\theta_2$ are incident and refraction angles.

**14.79** A plane wave is normally incident on a dielectric discontinuity. Use appropriate boundary conditions to calculate $R$, the reflectance, and $T$, the transmittance, and show that $T + R = 1$.

**14.80** An electromagnetic wave, propagating and linearly polarized in the $xz$-plane, is incident onto an interface between two non-conducting media as shown in Fig. 14.10. The electric fields and propagation vectors of the incident,

reflected and transmitted waves are denoted by $\boldsymbol{E}_\mathrm{I}$, $\boldsymbol{E}_\mathrm{R}$, $\boldsymbol{E}_\mathrm{T}$, $\boldsymbol{k}_\mathrm{I}$, $\boldsymbol{k}_\mathrm{R}$ and $\boldsymbol{k}_\mathrm{T}$ respectively. The wave is incident onto the interface at the origin and makes an angle $\theta$ to the normal. Both incident and reflected waves propagate in the medium with refractive index $n_1$. The transmitted wave propagates in the medium with refractive index $n_2$ at refraction angle $\phi$.
Use the boundary conditions satisfied by the electric field at the interface to show that the reflectance, $R$, is given by

$$R = \left(\frac{n_2 \cos\theta - n_1 \cos\phi}{n_2 \cos\theta + n_1 \cos\phi}\right)^2$$

**14.81** **(a)** Using the expressions for $R$ in prob. (14.80) and Snell's law, show that the reflectance is zero when $\tan\theta = \frac{n_2}{n_1}$
**(b)** Calculate this angle for electromagnetic radiation in air incident onto glass, which has a refractive index $n = 1.5$.

**14.82** **(a)** For normal incidence the reflection coefficient, $R$, at the planar surface between two dielectric media is given by

$$R = \frac{(n_1 - n_2)^2}{(n_1 + n_2)^2}$$

where $n_1$ and $n_2$ are the refractive indices of the media. Sketch the form of $R$ against $n_1/n_2$. On the same figure sketch the form of $T$ against $n_1/n_2$ where $T$ is the transmission coefficient. Indicate numerical values of $T$ and $R$ where appropriate.
**(b)** At what value of $n_1/n_2$ does $R = T$?

[University of Durham 2000]

**14.83** Consider the solutions of linearly polarized harmonic plane waves

$$\boldsymbol{E} = \boldsymbol{E}_0 \mathrm{e}^{i(\omega t - k.r + \varphi)}, \ \boldsymbol{B} = \boldsymbol{B}_0 \mathrm{e}^{i(\omega t - k.r + \varphi)}$$

where $\boldsymbol{E}_0$ and $\boldsymbol{B}_0$ are constant vectors associated with maximum amplitude of oscillations. Show that (a) $\boldsymbol{B}$ is perpendicular to $\boldsymbol{E}$; (b) $\boldsymbol{B}$ is in phase with $\boldsymbol{E}$ and (c) the magnitudes of $\boldsymbol{B}$ and $\boldsymbol{E}$ are related by $B = E/c$ for free space in the SI system.

**14.84** An uncharged dielectric cube of material of relative permittivity 6 contains a uniform electric field $E$ of 2 kV/m, which is perpendicular to one of the faces. What is the surface charge density induced on this face?

[University of Manchester 2006]

### *14.2.3 Phase Velocity and Group Velocity*

**14.85** Using the results of prob. (14.93) and the following table, estimate the fractional difference between the phase and group velocity in air at a wavelength of 5000 Å

| Free space wavelength (Å) | $(n-1)$ for air |
|---|---|
| 4800 | $2.786 \times 10^{-4}$ |
| 5000 | $2.781 \times 10^{-4}$ |
| 5200 | $2.777 \times 10^{-4}$ |

[University of Manchester 1972]

**14.86** Given the dispersion relation $\omega = ak^2$, calculate (a) phase velocity and (b) group velocity.

**14.87** **(a)** Write down an expression for the phase velocity $v_\mathrm{p}$ of an electromagnetic wave in a medium with permittivity $\varepsilon$ and permeability $\mu$.

**(b)** The relative permittivity, $\varepsilon_\mathrm{r}$, in an ionized gas is given by

$$\varepsilon_\mathrm{r} = 1 - \frac{D^2}{\omega^2}$$

where $D$ is a constant and $\omega$ is the angular frequency.
Find an expression for the refractive index $n$ and thus show that

$$\omega^2 = D^2 + c^2k^2$$

where $k$ is the wavenumber and $c$ is the speed of light in vacuum.

**(c)** Hence show that $v_\mathrm{p}v_\mathrm{g} = c^2$.

**(d)** In a particular gas, $D$ has the value $1.2 \times 10^{11}$/s. Determine the phase and group velocities at 20 GHz.
Comment on the result.

**14.88** Show that the group velocity $v_\mathrm{g}$ can be expressed as

$$v_\mathrm{g} = v_\mathrm{p} + k\frac{\mathrm{d}v_p}{\mathrm{d}k}$$

where $v_\mathrm{p}$ is the phase velocity and $k = 2\pi/\lambda$.

**14.89** Show that the group velocity can be expressed in the form

$$v_\mathrm{g} = \frac{c}{n} + \frac{\lambda c}{n^2}\frac{\mathrm{d}n}{\mathrm{d}\lambda}$$

where $n$ is the refractive index.

**14.90** Show that if the phase velocity varies inversely with the wavelength then the group velocity is twice the phase velocity.

**14.91** Show that the group velocity can be expressed as

$$v_{\mathrm{g}} = \frac{c}{n + \omega\left(\frac{\mathrm{d}n}{\mathrm{d}\omega}\right)}$$

**14.92** For a rectangular guide of width 2.5 cm what free-space wavelength of radiation is required for energy to traverse 50 m of length of the guide in 1 μs. What would be the phase velocity under these conditions?

**14.93** Show that for light waves of angular frequency $\omega$ in a medium of refractive index $n$, the group velocity $v_{\mathrm{g}}$ and the phase velocity $v_{\mathrm{p}}$ are related by the expression

$$\frac{1}{v_{\mathrm{g}}} = \frac{1}{v_{\mathrm{p}}} + \frac{\omega}{c}\frac{dn}{d\omega}$$

where c is the velocity of light in free space.

[University of Manchester 1972]

**14.94** Prove that the usual expression for the group velocity of a light wave in a medium can be rearranged as $v_{\mathrm{g}} = c\frac{dv}{d(nv)}$, where $c$ is the phase velocity of the waves in free space, $v$ is the frequency and $n$ is the refractive index of the medium.

[University of Durham 1961]

**14.95** Show that the group velocity associated with a free non-relativistic particle is the classical velocity of the particle.

[University of Manchester 1972]

**14.96** Calculate the group velocity of light of wavelength 500 nm in glass for which the refractive index $\mu$ at wavelength λ (meters) is

$$\mu = 1.420 + \frac{3.60 \times 10^{-14}}{\lambda^2}$$

[University of Manchester 1972]

### *14.2.4 Waveguides*

**14.97** For a rectangular guide of width 2.5 cm, calculate (a) the phase velocity; (b) the group velocity; (c) guide wavelength for the free-space wavelength of 4 cm. Assume the dominant mode.

**14.98** A rectangular guide has a width $a = 3$ cm. What should be the free-space wavelength if the guide wavelength is to be thrice the free-space wavelength?

**14.99** **(a)** Calculate the guide wavelength for a rectangular waveguide of width $a = 5$ cm if the free-space wavelength is 8 cm. **(b)** What is the cut-off wavelength for the guide?

**14.100** Calculate the number of states of electromagnetic radiation between 5000 and 6000 Å in wavelength using periodic boundary conditions in a cubical region 0.5 cm on a side.

[University of Manchester 1972]

**14.101** Consider a car entering a tunnel of dimensions 15 m wide and 4 m high. Assuming the walls are good conductors, can AM radio waves (530–1600 kHz) propagate in the tunnel?

[The University of Aberystwyth, Wales 2006]

**14.102** Calculate the least cut-off frequency for $TE_{mn}$ waves for a rectangular waveguide of dimensions 5 cm × 4 cm.

**14.103** Calculate how the wave and group velocities of the $TE_{01}$ wave in a rectangular waveguide with $a = 1$ cm and $b = 2$ cm vary with frequency.

[The University of Wales, Aberystwyth 2004]

**14.104** Consider a rectangular waveguide of dimensions $x = a$ and $y = b$, the $TM_{mn}$ wave travelling in the $z$-direction which is the axis of the guide. Given that the $z$-component $E_z$ satisfies the equation

$$\left(\frac{\partial^z}{\partial x^z} + \frac{\partial^z}{\partial y^z}\right) E_z = (k^2 - \omega^2 \mu \varepsilon) E_z$$

obtain (a) the solution for $E_z$ and (b) the cut-off frequency.

**14.105** Consider a rectangular waveguide of dimensions $x = a$ and $y = b$, the wave travelling along the $z$-direction, the axis of the guide. Given that the $z$-component $H_z$ satisfies the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) H_z = (k^2 - \omega^2 \mu \varepsilon) H_z$$

(a) obtain the solution for $H_z$. (b) Obtain the cut-off frequency. (c) What are the similarities and differences between $TM_{mn}$ mode and $TE_{mn}$ mode?

## 14.3 Solutions

### *14.3.1 The RLC Circuits*

**14.1** **(a)** Reactance of capacitor

$$X_c = \frac{1}{2\pi f c} = \frac{1}{2\pi \times (1000/2\pi) \times 2.5 \times 10^{-6}} = 400\ \Omega$$

Impedance of the circuit

$$Z = \sqrt{R^2 + X_c^2} = \sqrt{(300)^2 + (400)^2} = 500\,\Omega$$