1.62 A cube rests on a rough horizontal plane. A tension parallel to the plane is applied by a thread attached to the upper surface. Show that the cube will slide or topple according to the coefficient of friction is less or greater than 0.5.

1.63 A ladder leaning against a smooth wall makes an angle α with the horizontal when in a position of limiting equilibrium. Show that the coefficient of friction between the ladder and the ground is $\frac{1}{2}\cot\alpha$.

1.3 Solutions

1.3.1 Motion in One Dimension

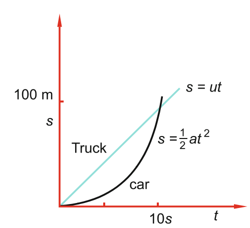
1.1 **(a)** Equation of motion for the truck: $s = ut$ (1)

Equation of motion for the car: $s = \dfrac{1}{2}at^2$ (2)

The graphs for (1) and (2) are shown in Fig. 1.13. Eliminating t between the two equations

$$s\left(1 - \frac{1}{2}\frac{as}{u^2}\right) = 0 \tag{3}$$

Fig. 1.13



Either $s = 0$ or $1 - \dfrac{1}{2}\dfrac{as}{u^2} = 0$. The first solution corresponds to the result that the truck overtakes the car at $s = 0$ and therefore at $t = 0$.

The second solution gives $s = \dfrac{2u^2}{a} = \dfrac{2 \times 10^2}{2} = 100\,\text{m}$

(b) $t = \dfrac{s}{u} = \dfrac{100}{10} = 10\,\text{s}$

(c) $v = at = 2 \times 10 = 20\,\text{m/s}$

1.2 When the stone reaches a height h above A

$$v_1^2 = u^2 - 2gh \tag{1}$$

and when it reaches a distance h below A

$$v_2^2 = u^2 + 2gh \tag{2}$$

since the velocity of the stone while crossing A on its return journey is again u vertically down.

Also, $v_2 = 2v_1$ (by problem) (3)

Combining (1), (2) and (3) $u^2 = \frac{10}{3}gh$ (4)

Maximum height

$$H = \frac{u^2}{2g} = \frac{10}{3}\frac{gh}{2g} = \frac{5h}{3}$$

1.3 Let the stones meet at a height s m from the earth after t s. Distance covered by the first stone

$$h - s = \frac{1}{2}gt^2 \tag{1}$$

where $h = 19.6$ m. For the second stone

$$s = ut = \frac{1}{2}gt^2 \tag{2}$$

$$v^2 = 0 = u^2 - 2gh$$

$$u = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 19.6} = 19.6\,\text{m/s} \tag{3}$$

Adding (1) and (2)

$$h = ut,\ t = \frac{h}{u} = \frac{19.6}{19.6} = 1\,\text{s}$$

From (2),

$$s = 19.6 \times 1 - \frac{1}{2} \times 9.8 \times 1^2 = 14.7\,\mathrm{m}$$

1.4 $x = A \sin \pi t = A \sin \omega t$

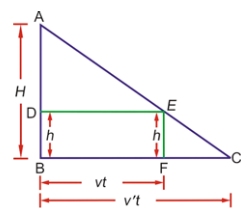
where ω is the angular velocity, $\omega = \pi$

Time period $T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2\,\mathrm{s}$

In $\frac{1}{2}$s (a quarter of the cycle) the distance covered is A. Therefore in 3 s the distance covered will be $6A$.

1.5 Let the lamp be at A at height H from the ground, that is AB $= H$, Fig. 1.14. Let the man be initially at B, below the lamp, his height being equal to BD $= h$, so that the tip of his shadow is at B. Let the man walk from B to F in time t with speed v, the shadow will go up to C in the same time t with speed v':

Fig. 1.14



$$\mathrm{BF} = vt;\ \mathrm{BC} = v't$$

From similar triangles EFC and ABC

$$\frac{\mathrm{FC}}{\mathrm{BC}} = \frac{\mathrm{EF}}{\mathrm{AB}} = \frac{h}{H}$$

$$\frac{\mathrm{FC}}{\mathrm{BC}} = \frac{\mathrm{EF}}{\mathrm{AB}} = \frac{h}{H} \rightarrow \frac{v't - vt}{v't} = \frac{h}{H}$$

or

$$v' = \frac{Hv}{H - h} = \frac{6 \times 7}{(6 - 1.8)} = 10\,\mathrm{m/s}$$

1.6 $\sqrt{3x} = 3t - 6$ (1)

Squaring and simplifying $x = 3t^2 - 12t + 12$ (2)

$$v = \frac{dx}{dt} = 6t - 12$$

$$v = 0 \text{ gives } t = 2\,\text{s} \quad (3)$$

Using (3) in (2) gives displacement $x = 0$

1.7 $$s = ut + \frac{1}{2}at^2 \quad (1)$$

$$\therefore \quad h = u \times 2 - \frac{1}{2}g \times 2^2 \quad (2)$$

$$h = u \times 10 - \frac{1}{2}g \times 10^2 \quad (3)$$

Solving (2) and (3) $h = 10g = 10 \times 9.8 = 98\,\text{m}$.

1.8 Take the origin at the position of A at $t = 0$. Let the car A overtake B in time t after travelling a distance s. In the same time t, B travels a distance $(s - 30)$ m:

$$s = ut + \frac{1}{2}at^2 \quad (1)$$

$$s = 13t + \frac{1}{2} \times 0.6\,t^2 \quad \text{(Car A)} \quad (2)$$

$$s - 30 = 20t - \frac{1}{2} \times 0.46\,t^2 \quad \text{(Car B)} \quad (3)$$

Eliminating s between (2) and (3), we find $t = 0.9\,\text{s}$.

1.9 Let BD $= x$. Time t_1 for crossing the field along AD is

$$t_1 = \frac{\text{AD}}{v_1} = \frac{\sqrt{x^2 + (600)^2}}{1.0} \quad (1)$$

Time t_2 for walking on the road, a distance DC, is

$$t_2 = \frac{\text{DC}}{v_2} = \frac{800 - x}{2.0} \quad (2)$$

$$\text{Total time } t = t_1 + t_2 = \sqrt{x^2 + (600)^2} + \frac{800 - x}{2} \quad (3)$$

Minimum time is obtained by setting $dt/dx = 0$. This gives us $x = 346.4\,\text{m}$. Thus the boy must head toward D on the round, which is 800–346.4 or 453.6 m away from the destination on the road.
The total time t is obtained by using $x = 346.4$ in (3). We find $t = 920\,\text{s}$.

1.10 Time taken for the first drop to reach the floor is

$$t_1 = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 2.45}{9.8}} = \frac{1}{\sqrt{2}}\,\text{s}$$

As the time interval between the first and second drop is equal to that of the second and the third drop (drops dripping at regular intervals), time taken by the second drop is $t_2 = \frac{1}{2\sqrt{2}}$ s; therefore, distance travelled by the second drop is

$$S = \frac{1}{2}gt_2^2 = \frac{1}{2} \times 9.8 \times \left(\frac{1}{2\sqrt{2}}\right)^2 = 0.6125\,\text{m}$$

1.11 Height h = area under the $\upsilon - t$ graph. Area above the t-axis is taken positive and below the t-axis is taken negative. h = area of bigger triangle minus area of smaller triangle.
Now the area of a triangle = base $\times$ altitude

$$h = \frac{1}{2} \times 3 \times 30 - \frac{1}{2} \times 1 \times 10 = 40\,\text{m}$$

1.12 **(a)** Time for the ball to reach water $t_1 = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 4.9}{9.8}} = 1.0\,\text{s}$
Velocity of the ball acquired at that instant $v = gt_1 = 9.8 \times 1.0 = 9.8\,\text{m/s}$.
Time taken to reach the bottom of the lake from the water surface

$$t_2 = 5.0 - 1.0 = 4.0\,\text{s}.$$

As the velocity of the ball in water is constant, depth of the lake,

$$d = vt_2 = 9.8 \times 4 = 39.2\,\text{m}.$$

(b) $< v >= \frac{\text{total displacement}}{\text{total time}} = \frac{4.9 + 39.2}{5.0} = 8.82\,\text{m/s}$

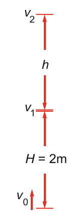
1.13 For the first stone time $t_1 = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 44.1}{9.8}} = 3.0\,\text{s}$.
Second stone takes $t_2 = 3.0 - 1.0 = 2.0\,\text{s}$ to strike the water

$$h = ut_2 + \frac{1}{2}gt_2^2$$

Using $h = 44.1$ m, $t_2 = 2.0$ s and $g = 9.8\,\text{m/s}^2$, we find $u = 12.25\,\text{m/s}$

1.14 Transit time for the single journey = 0.5 s.
When the ball moves up, let υ_0 be its velocity at the bottom of the window, v_1 at the top of the window and $v_2 = 0$ at height h above the top of the window (Fig. 1.15)

Fig. 1.15



$$v_1 = v_0 - gt = v_0 - 9.8 \times 0.5 = v_0 - 4.9 \quad (1)$$
$$v_1^2 = v_0^2 - 2gh = v_0^2 - 2 \times 9.8 \times 2 = v_0^2 - 39.2 \quad (2)$$

Eliminating v_1 between (1) and (2)

$$v_0 = 6.45\,\text{m/s} \quad (3)$$
$$v_2^2 = 0 = v_0^2 - 2g\,(H + h)$$
$$H + h = \frac{v_0^2}{2g} = \frac{(6.45)^2}{2 \times 9.8} = 2.1225\,\text{m}$$
$$h = 2.1225 - 2.0 = 0.1225\,\text{m}$$

Thus the ball rises 12.25 cm above the top of the window.

1.15 **(a)** $S_n = g\left(n - \frac{1}{2}\right)$ $\qquad S = \frac{1}{2}gn^2$

By problem $S_n = \frac{3s}{4}$

$g\left(n - \frac{1}{2}\right) = \left(\frac{3}{4}\right)\left(\frac{1}{2}\right)gn^2$

Simplifying $3n^2 - 8n + 4 = 0$, $n = 2$ or $\frac{2}{3}$

The second solution, $n = \frac{2}{3}$, is ruled out as $n < 1$.

(b) $s = \frac{1}{2}gn^2 = \frac{1}{2} \times 9.8 \times 2^2 = 19.6\,\text{m}$

1.16 In the triangle ACD, CA represents magnitude and apparent direction of wind's velocity w_1, when the man walks with velocity DC $= v = 4\,\text{km/h}$ toward west, Fig. 1.16. The side DA must represent actual wind's velocity because

$$\boldsymbol{W}_1 = \boldsymbol{W} - \boldsymbol{v}$$

When the speed is doubled, DB represents the velocity $2\boldsymbol{v}$ and BA represents the apparent wind's velocity $\boldsymbol{W}_2$. From the triangle ABD,

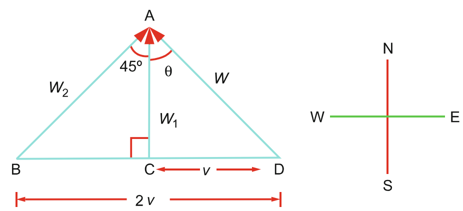


Fig. 1.16

$$\boldsymbol{W}_2 = \boldsymbol{W} - 2\boldsymbol{v}$$

By problem angle CAD $= \theta = 45°$. The triangle ACD is therefore an isosceles right angle triangle:

$$\mathrm{AD} = \sqrt{2}\mathrm{CD} = 4\sqrt{2}\,\mathrm{km/h}$$

Therefore the actual speed of the wind is $4\sqrt{2}$ km/h from southeast direction.

1.17 Choose the floor of the elevator as the reference frame. The observer is inside the elevator. Take the downward direction as positive.
Acceleration of the bolt relative to the elevator is

$$a' = g - (-a) = g + a$$

$$h = \frac{1}{2}a't^2 = \frac{1}{2}(g + a)t^2 \qquad t = \sqrt{\frac{2h}{g + a}}$$

1.18 In 2 s after the truck driver applies the brakes, the distance of separation between the truck and the car becomes

$$d_{\mathrm{rel}} = d - \frac{1}{2}\mathrm{at}^2 = 10 - \frac{1}{2} \times 2 \times 2^2 = 6\,\mathrm{m}$$

The velocity of the truck 2 becomes $20 - 2 \times 2 = 16$ m/s.
Thus, at this moment the relative velocity between the car and the truck will be

$$u_{\mathrm{rel}} = 20 - 16 = 4\,\mathrm{m/s}$$

Let the car decelerate at a constant rate of a_2. Then the relative deceleration will be

$$a_{\mathrm{rel}} = a_2 - a_1$$

If the rear-end collision is to be avoided the car and the truck must have the same final velocity that is

$v_{\mathrm{rel}} = 0$

Now $v_{\mathrm{rel}}^2 = u_{\mathrm{rel}}^2 - 2\,a_{\mathrm{rel}}\,d_{\mathrm{rel}}$

$$a_{\mathrm{rel}} = \frac{v_{\mathrm{rel}}^2}{2\,d_{\mathrm{rel}}} = \frac{4^2}{2 \times 6} = \frac{4}{3}\,\mathrm{m/s^2}$$

$$\therefore \quad a_2 = a_1 + a_{\mathrm{rel}} = 2 + \frac{4}{3} = 3.33\,\mathrm{m/s^2}$$

1.19 $\boldsymbol{v}_{\mathrm{BA}} = \boldsymbol{v}_{\mathrm{B}} - \boldsymbol{v}_{\mathrm{A}}$
From Fig. 1.17a

$$\boldsymbol{v}_{\mathrm{BA}} = \sqrt{v_{\mathrm{B}}^2 + v_{\mathrm{A}}^2 - 2v_{\mathrm{B}}v_{\mathrm{A}}\cos 60^\circ}$$

$$= \sqrt{20^2 + 30^2 - 2 \times 20 \times 30 \times 0.5} = 10\sqrt{7}\,\mathrm{km/h}$$

The direction of $\boldsymbol{v}_{\mathrm{BA}}$ can be found from the law of sines for ΔABC, Fig. 1.17a:

(i) $$\frac{\mathrm{AC}}{\sin\theta} = \frac{\mathrm{BC}}{\sin 60}$$

$$\text{or } \sin\theta = \frac{\mathrm{AC}}{\mathrm{BC}}\sin 60 = \frac{v_{\mathrm{B}}}{v_{\mathrm{BA}}}\sin 60^\circ = \frac{20 \times 0.866}{10\sqrt{7}} = 0.6546$$

$\theta = 40.9^\circ$

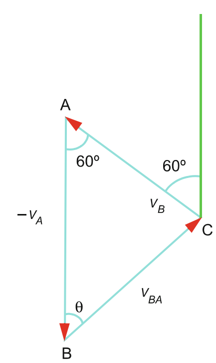


Fig. 1.17a

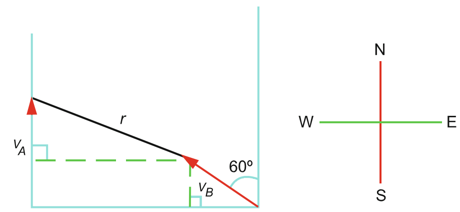


Fig. 1.17b

Thus $\boldsymbol{v}_{\mathrm{BA}}$ makes an angle 40.9° east of north.

(ii) Let the distance between the two ships be r at time t. Then from the construction of Fig. 1.17b

$$r = [(v_{\mathrm{A}}t - v_{\mathrm{B}}t\cos 60^\circ)^2 + (10 - v_{\mathrm{B}}t\sin 60^\circ)^2]^{1/2} \tag{1}$$

Distance of closest approach can be found by setting $\mathrm{d}r/\mathrm{d}t = 0$. This gives $t = \dfrac{\sqrt{3}}{7}h$. When $t = \dfrac{\sqrt{3}}{7}$ is inserted in (1) we get $r_{\min} = 20/\sqrt{7}$ or 7.56 km.

1.20 The initial velocity of the packet is the same as that of the balloon and is pointing upwards, which is taken as the positive direction. The acceleration due to gravity being in the opposite direction is taken negative. The displacement is also negative since it is vertically down:

$$u = 9.8\,\mathrm{m/s},\ a = -g = -9.8\,\mathrm{m/s^2};\ S = -98\,\mathrm{m}$$

$$s = ut + \frac{1}{2}at^2;\ -98 = 9.8t - \frac{1}{2} \times 9.8\,t^2 \quad \text{or} \quad t^2 - 2t - 20 = 0,$$

$$t = 1 \pm \sqrt{21}$$

The acceptable solution is $1 + \sqrt{21}$ or 5.58 s. The second solution being negative is ignored. Thus the packet takes 5.58 s to reach the ground.

1.3.2 Motion in Resisting Medium

1.21 Physically the difference between t_1 and t_2 on the one hand and v and u on other hand arises due to the fact that during ascent both gravity and air resistance act downward (friction acts opposite to motion) but during descent gravity and air resistance are oppositely directed. Air resistance F actually increases with the velocity of the object ($F \propto v$ or v^2 or v^3). Here for simplicity we assume it to be constant.

For upward motion, the equation of motion is

$$ma_1 = -(F + mg)$$

or

$$a_1 = -\left(\frac{F}{m} + g\right) \tag{1}$$

For downward motion, the equation of motion is

$$ma_2 = mg - F$$

or

$$a_2 = g - \frac{F}{m} \tag{2}$$

For ascent

$$v_1 = 0 = u + a_1 t = u - \left(\frac{F}{m} + g\right) t_1$$

$$t_1 = \frac{u}{g + \frac{F}{m}} \tag{3}$$

$$v_1{}^2 = 0 = u^2 + 2a_1 h$$

$$u = \sqrt{\frac{2h}{\left(g + \frac{F}{m}\right)}} \tag{4}$$

where we have used (1). Using (4) in (3)

$$t_1 = \sqrt{\frac{2h}{g + \frac{F}{m}}} \tag{5}$$

For descent $v^2 = 2a_2 h$

$$v = \sqrt{2h\left(g - \frac{F}{m}\right)} \tag{6}$$

where we have used (2)

$$t_2 = \frac{v}{a_2} = \sqrt{\frac{2h}{g - \frac{F}{m}}}, \tag{7}$$

where we have used (2) and (6)
From (5) and (7)

$$\frac{t_2}{t_1} = \sqrt{\frac{g + \frac{F}{m}}{g - \frac{F}{m}}} \tag{8}$$

It follows that $t_2 > t_1$, that is, time of descent is greater than the time of ascent. Further, from (4) and (6)

$$\frac{v}{u} = \sqrt{\frac{g - \frac{F}{m}}{g + \frac{F}{m}}} \tag{9}$$

It follows that $v < u$, that is, the final speed is smaller than the initial speed.

1.22 Taking the downward direction as positive, the equation of motion will be

$$\frac{dv}{dt} = g - kv \tag{1}$$

where k is a constant. Integrating

$$\int \frac{dv}{g - kv} = \int dt$$
$$\therefore \quad -\frac{1}{k} \ln\left(\frac{g - kv}{c}\right) = t$$

where c is a constant:

$$g - kv = ce^{-kt} \tag{2}$$

This gives the velocity at any instant.
As t increases e^{-kt} decreases and if t increases indefinitely $g - kv = 0$, i.e.

$$v = \frac{g}{k} \tag{3}$$

This limiting velocity is called the terminal velocity. We can obtain an expression for the distance x traversed in time t. First, we identify the constant c in (2). Since it is assumed that $v = 0$ at $t = 0$, it follows that $c = g$.
Writing $v = \frac{dx}{dt}$ in (2) and putting $c = g$, and integrating

$$g - k\frac{dx}{dt} = ge^{-kt}$$
$$\int g dt - k \int dx = g \int e^{-kt} dt + D$$
$$gt - kx = -\frac{g}{k}e^{-kt} + D$$

At $x = 0$, $t = 0$; therefore, $D = \frac{g}{k}$

$$x = \frac{gt}{k} - \frac{g}{k^2}\left(1 - e^{-kt}\right) \tag{4}$$

1.23 The equation of motion is

$$\frac{d^2x}{dt^2} = g - k\left(\frac{dx}{dt}\right)^2 \tag{1}$$

$$\frac{dv}{dt} = g - kv^2 \tag{2}$$

$$\therefore \quad \frac{1}{k}\int \frac{dv}{\frac{g}{k} - v^2} = t + c \tag{3}$$

writing $V^2 = \frac{g}{k}$ and integrating

$$\ln\frac{V+v}{V-v} = 2kV(t+c) \tag{4}$$

If the body starts from rest, then $c = 0$ and

$$\ln\frac{V+v}{V-v} = 2kVt = \frac{2gt}{V}$$

$$\therefore \quad t = \frac{V}{2g}\ln\frac{V+v}{V-v} \tag{5}$$

which gives the time required for the particle to attain a velocity $\upsilon = 0$. Now

$$\frac{V+v}{V-v} = e^{2kVt}$$

$$\therefore \quad \frac{v}{V} = \frac{e^{2kVt} - 1}{e^{2kVt} + 1} = \tanh kVt \tag{6}$$

i.e.

$$v = V\tanh\frac{gt}{V} \tag{7}$$

The last equation gives the velocity υ after time t. From (7)

$$\frac{dx}{dt} = V\tanh\frac{gt}{V}$$

$$x = \frac{V^2}{g}\ln\cosh\frac{gt}{V} \tag{8}$$

$$x = \frac{V^2}{g}\ln\frac{e^{gt/v} + e^{-gt/v}}{2} \tag{9}$$

no additive constant being necessary since $x = 0$ when $t = 0$. From (6) it is obvious that as t increases indefinitely υ approaches the value V. Hence V is the terminal velocity, and is equal to $\sqrt{g/k}$.
The velocity υ in terms of x can be obtained by eliminating t between (5) and (9).

From (9),

$$e^{kx} = \frac{e^{kVt} + e^{-kVt}}{2}$$

Squaring $4e^{2kx} = e^{2kVt} + e^{-2kVt} + 2$

$$= \frac{V+\upsilon}{V-\upsilon} + \frac{V-\upsilon}{V+\upsilon} + 2 \quad \text{from (5)}$$

$$= \frac{4V^2}{V^2 - \upsilon^2}$$

$$\therefore \quad \upsilon^2 = V^2(1 - e^{-2kx})$$

$$= V^2\left(1 - e^{-\frac{2gx}{V^2}}\right) \tag{10}$$

1.24 Measuring x upward, the equation of motion will be

$$\frac{d^2x}{dt^2} = -g - k\left(\frac{dx}{dt}\right)^2 \tag{1}$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d\upsilon}{dt} = \frac{d\upsilon}{dx}\cdot\frac{dx}{dt} = \upsilon\frac{d\upsilon}{dx}$$

$$\therefore \quad \upsilon\frac{d\upsilon}{dx} = -g - k\upsilon^2 \tag{2}$$

$$\therefore \quad \frac{1}{2k}\int \frac{d\left(\upsilon^2\right)}{(g/k) + \upsilon^2} = -\int dx$$

Integrating, $\ln\left(\dfrac{(g/k) + \upsilon^2}{c}\right) = -2kx$

$$\text{or} \qquad \frac{g}{k} + \upsilon^2 = ce^{-2kx} \tag{3}$$

When $x = 0$, $\upsilon = u$; $\therefore c = \dfrac{g}{k} + u^2$ and writing $\dfrac{g}{k} = V^2$, we have

$$\frac{V^2 + \upsilon^2}{V^2 + u^2} = e^{-\frac{2gx}{V^2}} \tag{4}$$

$$\therefore \quad \upsilon^2 = (V^2 + u^2)e^{-\frac{2gx}{V^2}} - V^2 \tag{5}$$

The height h to which the particle rises is found by putting $\upsilon = 0$ at $x = h$ in (5)

$$\frac{V^2+u^2}{V^2} = e^{\frac{2gh}{V^2}}$$

$$h = \frac{V^2}{2g} \ln\left(1 + \frac{u^2}{V^2}\right) \tag{6}$$

1.25 The particle reaches the height h given by

$$h = \frac{V^2}{2g} \ln\left(1 + \frac{u^2}{V^2}\right) \quad \text{(by prob. 1.24)}$$

The velocity at any point during the descent is given by

$$v^2 = V^2\left(1 - e^{-\frac{2gx}{V^2}}\right) \quad \text{(by prob. 1.23)}$$

The velocity of the body when it reaches the point of projection is found by substituting h for x:

$$\therefore \quad v^2 = V^2\left\{1 - \frac{V^2}{V^2+u^2}\right\} = \frac{u^2V^2}{V^2+u^2}$$

$$\text{Loss of kinetic energy} = \frac{1}{2}mu^2 - \frac{1}{2}mv^2$$

$$= \frac{1}{2}mu^2\left\{1 - \frac{V^2}{V^2+u^2}\right\} = \frac{1}{2}mu^2\left(\frac{u^2}{V^2+u^2}\right)$$

1.3.3 Motion in Two Dimensions

1.26 **(i)**

$$\frac{dx}{dt} = 6 + 2t$$

$$\int dx = 6\int dt + 2\int t\,dt$$

$$x = 6t + t^2 + C$$

$$x = 0, t = 0; C = 0$$

$$x = 6t + t^2$$

$$\frac{dy}{dt} = 4 + t$$

$$\int dy = 4\int dt + \int t\,dt$$

$$y = 4t + \frac{t^2}{2} + D$$
$$y = 0, t = u;\ D = u$$
$$y = u + 4t + \frac{t^2}{2}$$

(ii) $\vec{v} = (6 + 2t)\hat{i} + (4 + t)\hat{j}$

(iii) $\vec{a} = \dfrac{\mathrm{d}v}{\mathrm{d}t} = 2\hat{i} + \hat{j}$

(iv) $a = \sqrt{2^2 + 1^2} = \sqrt{5}$

$$\tan\theta = \frac{1}{2};\ \theta = 26.565^\circ$$

Acceleration is directed at an angle of $26^\circ 34'$ with the x-axis.

1.27 Take upward direction as positive, Fig. 1.18. At time t the velocities of the objects will be

$$\boldsymbol{v}_1 = u_1\hat{i} - gt\,\hat{j} \tag{1}$$
$$\boldsymbol{v}_2 = -u_2\hat{i} - gt\,\hat{j} \tag{2}$$

If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are to be perpendicular to each other, then $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 0$, that is

$$\left(u_1\hat{i} - gt\,\hat{j}\right) \cdot \left(-u_2\hat{i} - gt\,\hat{j}\right) = 0$$
$$\therefore \quad -u_1u_2 + g^2t^2 = 0$$
$$\text{or} \quad t = \frac{1}{g}\sqrt{u_1u_2} \tag{3}$$

The position vectors are $\boldsymbol{r}_1 = u_1t\,\hat{i} - \frac{1}{2}gt^2\,\hat{j}$, $\quad \boldsymbol{r}_2 = -u_2t\,\hat{i} - \frac{1}{2}gt^2\,\hat{j}$.

The distance of separation of the objects will be

$$r_{12} = |\vec{\boldsymbol{r}}_1 - \vec{\boldsymbol{r}}_2| = (u_1 + u_2)t$$

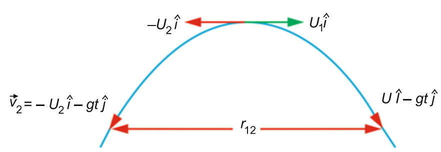


Fig. 1.18

or

$$r_{12} = \frac{(u_1 + u_2)}{g}\sqrt{u_1 u_2} \quad (4)$$

where we have used (2).

1.28 Consider the equation

$$s = ut + \frac{1}{2}at^2 \quad (1)$$

Taking upward direction as positive, $a = -g$ and let $s = h$, the height of the tower, (1) becomes

$$h = ut - \frac{1}{2}gt^2$$

or

$$\frac{1}{2}gt^2 - ut + h = 0 \quad (2)$$

Let the two roots be t_1 and t_2. Compare (2) with the quadratic equation

$$ax^2 + bx + c = 0 \quad (3)$$

The product of the two roots is equal to c/a. It follows that

$$t_1 t_2 = \frac{2h}{g} \text{ or } \sqrt{t_1 t_2} = \sqrt{\frac{2h}{g}} = t_3$$

which is the time taken for a free fall of an object from the height h.

1.29 Let the shell hit the plane at $p(x, y)$, the range being AP $= R$, Fig. 1.19. The equation for the projectile's motion is

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta} \quad (1)$$

Now $\quad y = R\sin\alpha \quad (2)$

$$x = R\cos\alpha \quad (3)$$

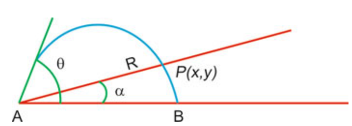


Fig. 1.19

Using (2) and (3) in (1) and simplifying

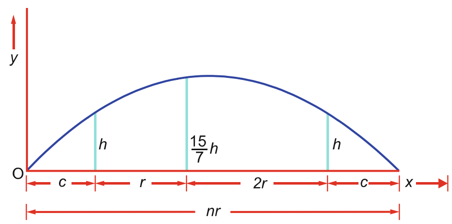
$$R = \frac{2u^2 \cos\theta \sin(\theta - \alpha)}{g \cos^2 \alpha}$$

The maximum range is obtained by setting $\frac{\mathrm{d}R}{\mathrm{d}\theta} = 0$, holding u, α and g constant. This gives $\cos(2\theta - \alpha) = 0$ or $2\theta - \alpha = \frac{\pi}{2}$

$$\therefore\ \alpha = \frac{\theta}{2} + \frac{\pi}{4}$$

1.30 As the outer walls are equal in height (h) they are equally distant (c) from the extremities of the parabolic trajectory whose general form may be written as (Fig. 1.20)

Fig. 1.20



$$y = ax - bx^2 \tag{1}$$

$y = 0$ at $x = R = nr$, when R is the range

This gives $a = bnr$ (2)

The range $R = c + r + 2r + c = nr$, by problem

$$\therefore \quad c = (n - 3)\frac{r}{2} \tag{3}$$

The trajectory passes through the top of the three walls whose coordinates are (c, h), $\left(c + r, \frac{15}{7}h\right)$, $(c + 3r, h)$, respectively. Using these coordinates in (1), we get three equations

$$h = ac - bc^2 \tag{4}$$

$$\frac{15h}{7} = a(c + r) - b(c + r)^2 \tag{5}$$

$$h = a(c + 3r) - b(c + 3r)^2 \tag{6}$$

Combining (2), (3), (4), (5) and (6) and solving we get $n = 4$.

1.31 The equation to the parabolic path can be written as

$$y = ax - bx^2 \tag{1}$$

$$\text{with } a = \tan\theta;\ b = \frac{g}{2u^2\cos^2\theta} \tag{2}$$

Taking the point of projection as the origin, the coordinates of the two openings in the windows are (5, 5) and (11, 7), respectively. Using these coordinates in (1) we get the equations

$$5 = 5a - 25b \tag{3}$$

$$7 = 11a - 121b \tag{4}$$

with the solutions, $a = 1.303$ and $b = 0.0606$. Using these values in (2), we find $\theta = 52.5°$ and $u = 14.8\,\text{m/s}$.

1.32 Let the rifle be fixed at A and point in the direction AB at an angle α with the horizontal, the monkey sitting on the tree top at B at height h, Fig. 1.21. The bullet follows the parabolic path and reaches point D, at height H, in time t.

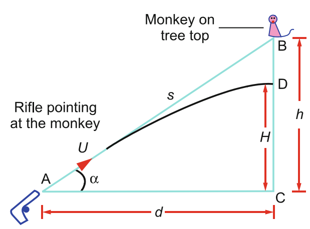


Fig. 1.21

The horizontal and initial vertical components of velocity of bullet are

$$u_x = u\cos\alpha;\ u_y = u\sin\alpha$$

Let the bullet reach the point D, vertically below B in time t, the coordinates of D being (d, H). As the horizontal component of velocity is constant

$$d = u_x t = (u\cos\alpha)t = \frac{udt}{s}$$

where $s = \text{AB}$:

$$t = \frac{s}{u}$$

The vertical component of velocity is reduced due to gravity.

In the same time, the y-coordinate at D is given by

$$y = H = u_y t - \frac{1}{2}gt^2 = u(\sin\alpha)t - \frac{1}{2}gt^2$$
$$H = u\left(\frac{h}{s}\right)\left(\frac{s}{u}\right) - \frac{1}{2}gt^2 = h - \frac{1}{2}gt^2$$
$$\text{or} \quad h - H = \frac{1}{2}gt^2$$
$$\therefore \quad t = \sqrt{\frac{2(h-H)}{g}}$$

But the quantity $(h–H)$ represents the height through which the monkey drops from the tree and the right-hand side of the last equation gives the time for a free fall. Therefore, the bullet would hit the monkey independent of the bullet's initial velocity.

1.33 $R = \dfrac{u^2 \sin 2\alpha}{g}, \quad h = \dfrac{u^2 \sin^2\alpha}{g}, \quad T = \dfrac{2u\sin\alpha}{g}$

(a) $\dfrac{h}{R} = \dfrac{1}{4}\tan\alpha \rightarrow \tan\alpha = \dfrac{4h}{R}$

(b) $\dfrac{h}{T^2} = \dfrac{g}{8} \rightarrow h = \dfrac{gT^2}{8}$

1.34 **(i)** $T = \dfrac{2u\sin\alpha}{g} = \dfrac{2\times 800\sin 60°}{9.8} = 141.4\,\text{s}$

(ii) $R = \dfrac{u^2\sin 2\alpha}{g} = \dfrac{(800)^2\sin(2\times 60)}{9.8} = 5.6568\times 10^4\text{m} = 56.57\,\text{km}$

(iii) Time to reach maximum height $= \frac{1}{2}T = \frac{1}{2}\times 141.4 = 707\,\text{s}$

(iv) $$x = (u\cos\alpha)t \tag{1}$$
$$y = (u\sin\alpha)t - \frac{1}{2}gt^2 \tag{2}$$

Eliminating t between (1) and (2) and simplifying

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{u^2\cos^2\alpha} \tag{3}$$

which is of the form $y = bx + cx^2$, with $b = \tan\alpha$ and $c = -\dfrac{1}{2}\dfrac{g}{u^2\cos^2\alpha}$.

1.35 **(i)** $T = \dfrac{u\sin\alpha}{g} = \dfrac{350\ \sin 55°}{9.8} = 29.25\,\text{s}$

(ii) At the highest point of the trajectory, the velocity of the particle is entirely horizontal, being equal to $u\cos\alpha$. The momentum of this particle at the highest point is $p = mu\cos\alpha$, when m is its mass. After the explosion, one fragment starts falling vertically and so does not carry any momentum initially. It would fall at half of the range, that is

$$\frac{R}{2} = \frac{1}{2}\frac{u^2\sin 2\alpha}{g} = \frac{(350)^2\sin(2\times 55°)}{2\times 9.8} = 5873\,\text{m, from the firing point.}$$

The second part of mass $\frac{1}{2}m$ proceeds horizontally from the highest point with initial momentum p in order to conserve momentum. If its velocity is v then

$$p = \frac{m}{2}v = mu\cos\alpha$$
$$v = 2u\cos\alpha = 2\times 350\cos 55° = 401.5\,\text{m/s}$$

Then its range will be

$$R' = v\sqrt{\frac{2h}{g}} \tag{1}$$

But the maximum height

$$h = \frac{u^2\sin^2\alpha}{2g} \tag{2}$$

Using (2) in (1)

$$R' = \frac{vu\sin\alpha}{g} = \frac{(401.5)(350)(\sin 55°)}{9.8} = 11746\,\text{m}$$

The distance form the firing point at which the second fragment hits the ground is

$$\frac{R}{2} + R' = 5873 + 11746 = 17619\,\text{m}$$

(iii) Energy released = (kinetic energy of the fragments) − (kinetic energy of the particle) at the time of explosion

$$= \frac{1}{2}\frac{m}{2}v^2 - \frac{1}{2}m(u\cos\alpha)^2$$
$$= \frac{20}{4}\times(401.5)^2 - \frac{20}{2}(350\cos\,55°)^2 = 4.03\times 10^5\,\text{J}$$

1.36 The radius of curvature

$$\rho = \frac{\left[1 + (\mathrm{d}y/\mathrm{d}x)^2\right]^{3/2}}{\mathrm{d}^2 y/\mathrm{d}x^2} \tag{1}$$

$$x = v_o t = 10 \times 3 = 30\,\mathrm{m}$$

$$y = \frac{1}{2} g t^2 = \frac{1}{2} \times 9.8 \times 3^2 = 44.1\,\mathrm{m}.$$

$$\therefore y = \frac{1}{2} g \frac{x^2}{v_0^2}$$

$$v_0^2 = \frac{9.8 \times 30}{10^2} = 2.94 \tag{2}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{g}{v_0^2} = \frac{9.8}{10^2} = 0.098 \tag{3}$$

Using (2) and (3) in (1) we find $\rho = 305\,\mathrm{m}$.

1.37 Let P be the position of the boat at any time, Let AP $= r$, angle $B\hat{A}P = \theta$, and let v be the magnitude of each velocity, Fig. 1.5:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -v + v \sin\theta$$

$$\text{and} \quad \frac{r\mathrm{d}\theta}{\mathrm{d}t} = v \cos\,\theta$$

$$\therefore \quad \frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{-1 + \sin\theta}{\cos\theta}$$

$$\therefore \quad \int \frac{\mathrm{d}r}{r} = \int [-\sec\theta + \tan\theta]\,\mathrm{d}\theta$$

$$\therefore \quad \ln r = -\ln\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) - \ln\cos\theta + \ln C \text{ (a constant)}$$

When $\theta = 0, r = a,$ so that $C = a$

$$\therefore \quad r = \frac{a}{\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)\cos\theta}$$

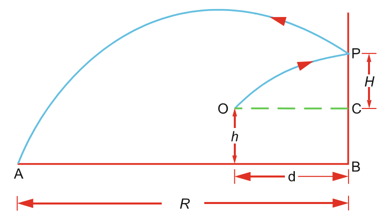
The denominator can be shown to be equal to $1 + \sin\theta$:

$$\therefore \quad r = \frac{a}{1 + \sin\theta}$$

This is the equation of a parabola with AB as semi-latus rectum.

1.38 Take the origin at O, Fig. 1.22. Draw the reference line OC parallel to AB, the ground level. Let the ball hit the wall at a height H above C. Initially at O,

Fig. 1.22



$$u_x = u\cos\alpha = u\cos 45^\circ = \frac{u}{\sqrt{2}}$$
$$u_y = u\sin\alpha = u\sin 45^\circ = \frac{u}{\sqrt{2}}$$

When the ball hits the wall, $y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{u^2\cos^2\alpha}$
Using $y = H$, $x = d$ and $\alpha = 45^\circ$

$$H = d\left(1 - \frac{gd}{u^2}\right) \tag{1}$$

If the collision of the ball with the wall is perfectly elastic then at P, the horizontal component of the velocity (u'_x) will be reversed, the magnitude remaining constant, while both the direction and magnitude of the vertical component v'_y are unaltered. If the time taken for the ball to bounce back from P to A is t and the range BA = R

$$y = v'_y t - \frac{1}{2}gt^2 \tag{2}$$

$$\text{Using} \quad t = \frac{R}{u\cos 45^\circ} = \sqrt{2}\frac{R}{u} \tag{3}$$

$$y = -(H + h) \tag{4}$$

$$v'_y t = u\sin 45^\circ - g\frac{d}{u\cos 45^\circ} = \frac{u}{\sqrt{2}} - \sqrt{2}\frac{gd}{u} \tag{5}$$

Using (3), (4) and (5) in (2), we get a quadratic equation in R which has the acceptable solution

$$R = \frac{u^2}{2g} + \sqrt{\frac{u^2}{4g^2} + H + h}$$

1.3.4 Force and Torque

1.39 Resolve the force into x- and y-components:

$$F_x = -80\cos 35^\circ + 60 + 40\cos 45^\circ = 22.75\,\text{N}$$
$$F_y = 80\sin 35^\circ + 0 - 40\sin 45^\circ = 17.6\,\text{N}$$

(i) $\boldsymbol{F}_{\text{net}} = \sqrt{F_x^2 + F_y^2} = \sqrt{(22.75)^2 + (17.6)^2} = 28.76\,\text{N}$

$$\tan\theta = \frac{F_y}{F_x} = \frac{17.6}{22.75} = 0.7736 \to \theta = 37.7^\circ$$

The vector $\boldsymbol{F}_{\text{net}}$ makes an angle of 37.7° with the x-axis.

(ii) $a = \dfrac{\boldsymbol{F}_{\text{net}}}{m} = \dfrac{28.76\,N}{3.8\,kg} = 7.568\,\text{m/s}^2$

(iii) $\boldsymbol{F}_4$ of magnitude 28.76 N must be applied in the opposite direction to $\boldsymbol{F}_{\text{net}}$

1.40 **(a)** **(i)** $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$

$$\tau = rF\sin\theta = (0.4\,\text{m})(50\,\text{N})\sin 90^\circ = 20\,\text{N} - \text{m}$$

(ii) $\tau = I\alpha$

$$\alpha = \frac{\tau}{I} = \frac{20}{20} = 1.0\,\text{rad/s}^2$$

(iii) $\omega = \omega_0 + \alpha t = 0 + 1 \times 3 = 3\,\text{rad/s}$

(iv) $\omega^2 = \omega_0^2 + 2\alpha\theta,\ \theta = \frac{3^2 - 0}{2\times 1} = 4.5\,\text{rad}$

(b) **(i)** $\tau = 0.4 \times 50 \times \sin(90 + 20) = 18.794\,\text{N m}$

(ii) $\alpha = \dfrac{\tau}{I} = \dfrac{18.794}{20} = 0.9397\,\text{rad/s}^2$

1.41 Force applied to the container $F = ma$
Frictional force = $F_\text{r} = \mu\, mg$

$$F_\text{r} = F$$
$$\mu\, mg = ma$$
$$\mu = \frac{a}{g} = \frac{1.5}{9.8} = 0.153$$

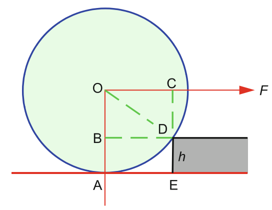


Fig. 1.23

1.42 Taking torque about D, the corner of the obstacle, $(F)\text{CD} = (W)\text{BD}$ (Fig. 1.23)

$$F = W\frac{\text{BD}}{\text{CD}} = \sqrt{\frac{\text{OD}^2 - \text{OB}^2}{\text{CE} - \text{DE}}}$$
$$= \sqrt{\frac{r^2 - (r-h)^2}{r-h}} = \frac{\sqrt{h(2r-h)}}{r-h}$$

1.3.5 Centre of Mass

1.43 Let λ be the linear mass density (mass per unit length) of the wire. Consider an infinitesimal line element $\mathrm{d}s = R\,\mathrm{d}\theta$ on the wire, Fig. 1.24. The corresponding mass element will be $\mathrm{d}m = \lambda \mathrm{d}s = \lambda R\,\mathrm{d}\theta$. Then

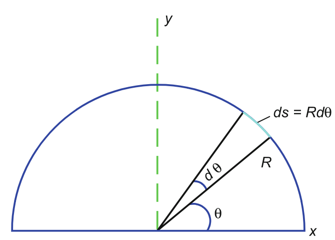
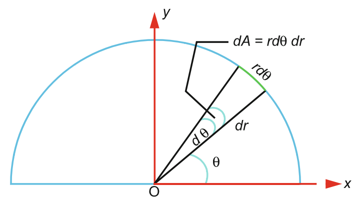


Fig. 1.24

$$y_{\text{CM}} = \frac{\int y\,\mathrm{d}m}{\int \mathrm{d}m} = \frac{\int_0^\pi (R\sin\theta)(\lambda R\,\mathrm{d}\theta)}{\int_0^\pi \lambda R\,\mathrm{d}\theta}$$
$$= \frac{\lambda R^2 \int_0^\pi \sin\theta\,\mathrm{d}\theta}{\lambda R \int_0^\pi \mathrm{d}\theta} = \frac{2R}{\pi}$$

1.44 Let the x-axis lie along the diameter of the semicircle. The centre of mass must lie on y-axis perpendicular to the flat base of the semicircle and through O, the centre of the base, Fig. 1.25.

Fig. 1.25



For continuous mass distribution

$$y_{CM} = \frac{1}{M}\int y\,\mathrm{d}m$$

Let σ be the surface density (mass per unit area), so that

$$M = \frac{1}{2}\pi R^2\sigma$$

In polar coordinates $\mathrm{d}m = \sigma\,\mathrm{d}A = \sigma r\,\mathrm{d}\theta\,\mathrm{d}r$
where $\mathrm{d}A$ is the element of area. Let the centre of mass be located at a distance y_{CM} from O along y-axis for reasons of symmetry:

$$y_{CM} = \frac{1}{\frac{1}{2}\pi R^2\sigma}\int_0^R\int_0^\pi (r\sin\theta)(\sigma r\,\mathrm{d}\theta\,\mathrm{d}r) = \frac{2}{\pi R^2}\int_0^R r^2\int_0^\pi \sin\theta\,\mathrm{d}\theta = \frac{4R}{3\pi}$$

1.45 Let O be the origin, the centre of the base of the hemisphere, the z-axis being perpendicular to the base. From symmetry the CM must lie on the z-axis, Fig. 1.26. If ρ is the density, the mass element, $\mathrm{d}m = \rho\,\mathrm{d}V$, where $\mathrm{d}V$ is the volume element:

$$Z_{CM} = \frac{1}{M}\int Z\,\mathrm{d}m = \frac{1}{M}\int Z\rho\,\mathrm{d}V \tag{1}$$

In polar coordinates, $Z = r\cos\theta$ (2)

$$\mathrm{d}V = r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}r \tag{3}$$

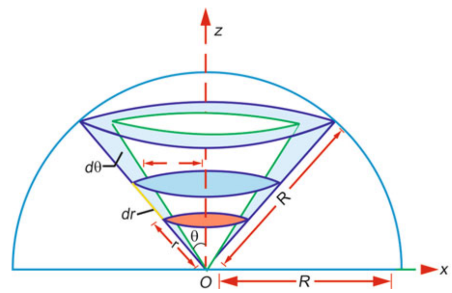
$$0 < r < R;\ 0 < \theta < \frac{\pi}{2};\qquad 0 < \phi < 2\pi$$

The mass of the hemisphere

$$M = \rho\frac{2}{3}\pi R^3 \tag{4}$$

Using (2), (3) and (4) in (1)

Fig. 1.26



$$Z_{\mathrm{CM}}\frac{\int_0^R r^3 \mathrm{d}r \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\,\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi}{\frac{2\pi R^3}{3}} = \frac{3R}{8}$$

1.46 The mass of any portion of the disc will be proportional to its surface area. The area of the original disc is πR^2, that corresponding to the hole is $\frac{1}{4}\pi R^2$ and that of the remaining portion is $\pi R^2 - \frac{\pi R^2}{4} = \frac{3}{4}\pi R^2$.

Let the centre of the original disc be at O, Fig. 1.10. The hole touches the circumference of the disc at A, the centre of the hole being at C. When this hole is cut, let the centre of mass of the remaining part be at G, such that

$$\mathrm{OG} = x \text{ or } \mathrm{AG} = \mathrm{AO} + \mathrm{OG} = R + x$$

If we put back the cut portion of the hole and fill it up then the centre of the mass of this small disc (C) and that of the remaining portion (G) must be located at the centre of the original disc at O

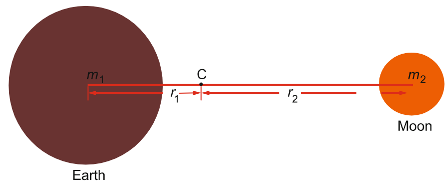
$$\mathrm{AO} = R = \frac{\mathrm{AC}\pi(R^2/4) + \mathrm{AG}\frac{3\pi}{4}R^2}{\pi R^2/4 + 3\pi R^2/4} = \frac{R}{8} + \frac{3}{4}(R + x)$$

$$\therefore \quad x = \frac{R}{6}$$

Thus the C:M of the remaining portion of the disc is located at distance $R/6$ from O on the left side.

1.47 Let m_1 be the mass of the earth and m_2 that of the moon. Let the centre of mass of the earth–moon system be located at distance r_1 from the centre of the earth and at distance r_2 from the centre of the moon, so that $r = r_1 + r_2$ is the distance between the centres of earth and moon, Fig. 1.27. Taking the origin at the centre of mass

Fig. 1.27



$$\frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = 0$$

$$m_1 r_1 - m_2 r_2 = 0$$

$$r_1 = \frac{m_2 r_2}{m_1} = \frac{m_2(r - r_1)}{81 m_2} = \frac{60R - r_1}{81}$$

$$r_1 = 0.7317R = 0.7317 \times 6400 = 4683\,\text{km}$$

along the line joining the earth and moon; thus, the centre of mass of the earth–moon system lies within the earth.

1.48 Let the centre of mass be located at a distance r_c from the carbon atom and at r_0 from the oxygen atom along the line joining carbon and oxygen atoms. If r is the distance between the two atoms, m_c and m_o the mass of carbon and oxygen atoms, respectively

$$m_c r_c = m_o r_o = m_o(r - r_c)$$

$$r_c = \frac{m_o r}{m_o + m_o} = \frac{16 \times 1.13}{12 + 16} = 0.646\,\text{Å}$$

1.49 Let C be the centroid of the equilateral triangle formed by the three H atoms in the *xy*-plane, Fig. 1.28. The N–atom lies vertically above C, along the *z*-axis. The distance r_{CN} between C and N is

$$r_{CN} - \sqrt{r^2_{NH_3} - r^2_{CH_3}}$$

$$r_{CN} = \frac{r^2_{H_1H_2}}{\sqrt{3}} = \frac{1.628}{1.732} = 0.94\,\text{Å}$$

$$r_{CN} = \sqrt{(1.014)^2 - (0.94)^2} = 0.38\,\text{Å}$$

Now, the centre of mass of the three H atoms $3m_H$ lies at C. The centre of mass of the NH_3 molecule must lie along the line of symmetry joining N and C and is located below N atom at a distance

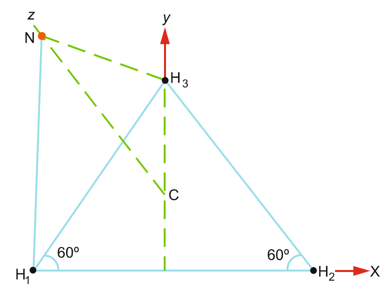


Fig. 1.28 Centre of mass of NH_3 molecule

$$Z_{\mathrm{CM}} = \frac{3m_{\mathrm{H}}}{3m_{\mathrm{H}} + m_N} \times r_{CN} = \frac{3m_{\mathrm{H}}}{3m_{\mathrm{H}} + 14m_{\mathrm{H}}} \times 0.38 = 0.067\,\text{Å}$$

1.50 Take the origin at A at the left end of the boat, Fig. 1.29. Let the boy of mass m be initially at B, the other end of the boat. The boat of mass M and length L has its centre of mass at C. Let the centre of mass of the boat + boy system be located at G, at a distance x from the origin. Obviously AC = 1.5 m:

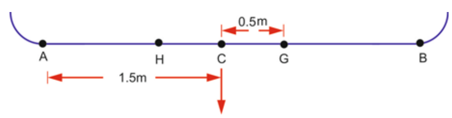


Fig. 1.29

$$\mathrm{AG} = x = \frac{M\mathrm{AC} + m\mathrm{AB}}{M + m}$$
$$= \frac{100 \times 1.5 + 50 \times 3}{100 + 50} = 2\,\mathrm{m}$$

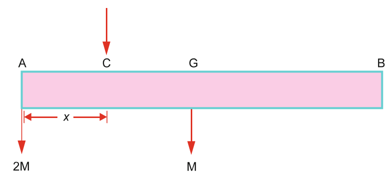
Thus CG = AG − AC

= 2.0 − 1.5 = 0.5 m

When the boy reaches A, from symmetry the CM of boat + boy system would have moved to H by a distance of 0.5 m on the left side of C. Now, in the absence of external forces, the centre of mass should not move, and so to restore the original position of the CM the boat moves towards right so that the point H is brought back to the original mark G. Since HG = 0.5 + 0.5 = 1.0, the boat in the mean time moves through 1.0 m toward right.

1.51 If the rod is to move with pure translation without rotation, then it should be struck at C, the centre of mass of the loaded rod. Let C be located at distance x from A so that
$GC = \frac{1}{2}L - x$, Fig. 1.30. Let M be the mass of the rod and $2M$ be attached at A. Take torques about C

Fig. 1.30



$$2Mx = M\left(\frac{L}{2} - x\right) \quad \therefore x = \frac{L}{6}$$

Thus the rod should be struck at a distance $\frac{L}{6}$ from the loaded end.

1.52 Volume of the cone, $V = \frac{1}{3}\pi R^2 h$ where R is the radius of the base and h is its height, Fig. 1.31. The volume element at a depth z below the apex is $\mathrm{d}V = \pi r^2 \mathrm{d}z$, the mass element $\mathrm{d}m = \rho \mathrm{d}V = \pi r^2 \mathrm{d}z f$

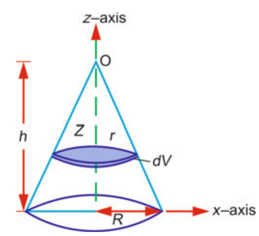


Fig. 1.31

$$\mathrm{d}m = \rho \mathrm{d}v = \rho \pi r^2 \mathrm{d}z$$
$$\frac{z}{r} = \frac{h}{R} \quad \therefore \mathrm{d}z = \frac{h}{R}\mathrm{d}r$$

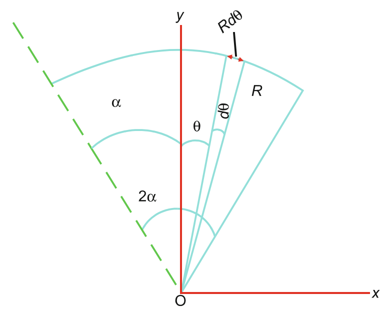
For reasons of symmetry, the centre of mass must lie on the axis of the cone. Take the origin at O, the apex of the cone:

$$Z_{\mathrm{CM}} = \frac{\int Z\,\mathrm{d}m}{\int \mathrm{d}m} = \frac{\int_0^R \left(\frac{hr}{R}\right)\rho\pi r^2 \left(\frac{h}{r}\,\mathrm{d}r\right)}{\frac{1}{3}\pi R^2 h\rho} = \frac{3h}{4}$$

Thus the CM is located at a height $h - \frac{3}{4}h = \frac{1}{4}h$ above the centre of the base of the cone.

1.53 Take the origin at O, Fig. 1.32. Let the mass of the wire be M. Consider mass element dm at angles θ and $\theta + \mathrm{d}\theta$

Fig. 1.32



$$\mathrm{d}m = \frac{M R\,\mathrm{d}\theta}{2\alpha R} = \frac{M\,\mathrm{d}\theta}{2\alpha} \tag{1}$$

From symmetry the CM of the wire must be on the y-axis.
The y-coordinate of dm is $y = R\sin\theta$

$$y_{\mathrm{CM}} = \frac{1}{M}\int y_{dm} = \int_{90-\alpha}^{90+\alpha} \frac{R\sin\theta\,\mathrm{d}\theta}{2\alpha} = \frac{R\sin\alpha}{\alpha}$$

Note that the results of prob. (1.43) follow for $\alpha = \frac{1}{2}\pi$.

1.54 $V_{\mathrm{CM}} = \dfrac{\Sigma m_i v_i}{\Sigma m_i} = \dfrac{4mv_0 + (m)(0)}{5m} = \dfrac{4v_0}{5}$

1.55 $\rho = cx(c = \text{constant});\ \mathrm{d}m = \rho\,\mathrm{d}x = cx\,\mathrm{d}x$

$$x_{\mathrm{CM}} = \frac{\int x\,\mathrm{d}m}{\int \mathrm{d}m} = \frac{\int_0^L xcx\,\mathrm{d}x}{\int_0^L cx\,\mathrm{d}x} = \frac{2}{3}L$$

1.56
$$\begin{aligned} x_{\mathrm{CM}} &= \frac{\Sigma m_i x_i}{\Sigma m_i} = \frac{mL + (2m)(2L) + (3m)(3L) + \cdots + (nm)(nL)}{m + 2m + 3m + \cdots + nm} \\ &= \frac{(1+4+9+\cdots+n^2)L}{1+2+3+\cdots+n} = \frac{(\text{sum of squares of natural numbers})L}{\text{sum of natural numbers}} \\ &= \frac{n(n+1)(2n+1)L/6}{n(n+1)/2} = (2n+1)\frac{L}{3} \end{aligned}$$

1.57 The diagram is the same as for prob. (1.44)

$$y = r\sin\theta$$
$$dA = r\,d\theta dr$$
$$dm = r\,d\theta dr\rho = r d\theta dr c r^2 = cr^3\,dr\,d\theta$$

$$\text{Total mass} \quad M = \int dm = c\int_0^R r^3 dr \int_0^\pi d\theta = \frac{\pi c R^4}{4} \tag{1}$$

$$y_{CM} = \frac{1}{M}\int y\,dm = \frac{1}{M}\int\int (r\sin\theta)cr^3\,dr d\theta$$
$$= \frac{C}{M}\int_0^R r^4 dr \int_0^\pi \sin\theta\,d\theta$$
$$= \frac{C}{M}\frac{2}{5}R^5 = \frac{8a}{5\pi} \tag{2}$$

where we have used (1).

1.58 The CM of the two H atoms will be at G the midpoint joining the atoms, Fig. 1.33. The bisector of $\widehat{\text{HOH}}$

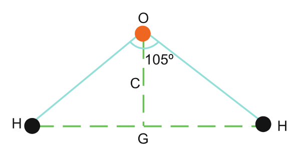


Fig. 1.33

$$\text{OG} = (\text{OH})\cos\left(\frac{105°}{2}\right) = 1.77 \times 0.06088 = 1.0775\,\text{Å}$$

Let the CM of the O atom and the two H atoms be located at C at distance y_{CM} from O on the bisector of angle $\hat{\text{HOH}}$

$$y_{CM} = \frac{2M_H}{M_0} \times OG = \frac{2\times 1}{16} \times 1.0775 = 0.1349\,\text{Å}$$

1.59 The CM coordinates of three individual laminas are

$$\text{CM}(1) = \left(\frac{a}{2}, \frac{a}{2}\right), \text{CM}(2) = \left(\frac{3a}{2}, \frac{a}{2}\right), \text{CM}(3) = \left(\frac{3a}{2}, \frac{3a}{2}\right)$$

The CM coordinates of the system of these three laminas will be

$$x_{CM} = \frac{m\frac{a}{2} + m\frac{3a}{2} + m\frac{3a}{2}}{m+m+m} = \frac{7a}{6} \quad y_{CM} = \frac{m\frac{a}{2} + m\frac{a}{2} + m\frac{3a}{2}}{m+m+m} = \frac{5a}{6}$$

1.3.6 Equilibrium

1.60 $U(x) = k(2x^3 - 5x^2 + 4x)$ (1)

$$\frac{\mathrm{d}U(x)}{\mathrm{d}x} = k(6x^2 - 10x + 4) \quad (2)$$

$$\therefore \quad \frac{\mathrm{d}U(x)}{\mathrm{d}x}\Big|_{x=1} = k(6x^2 - 10x + 4)\Big|_{x=1} = 0$$

which is the condition for maximum or minimum. For stable equilibrium position of the particle it should be a minimum. To this end we differentiate (2) again:

$$\frac{\mathrm{d}^2U(x)}{\mathrm{d}x^2} = k(12x - 10)$$

$$\therefore \quad \frac{\mathrm{d}^2U(x)}{\mathrm{d}x^2}\Big|_{x=1} = +2k$$

This is positive because k is positive, and so it is minimum corresponding to a stable equilibrium.

1.61 $U(x) = k(x^2 - 4xl)$ (1)

$$\frac{\mathrm{d}U(x)}{\mathrm{d}x} = 2k(x - 2l) \quad (2)$$

$$\text{At } x = 2l,\ \frac{\mathrm{d}U(x)}{\mathrm{d}x} = 0 \quad (3)$$

Differentiating (2) again

$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2} = 2k$$

which is positive. Hence it is a minimum corresponding to a stable equilibrium. Force

$$F = -\frac{\mathrm{d}U}{\mathrm{d}x} = -2k(x - 2l)$$

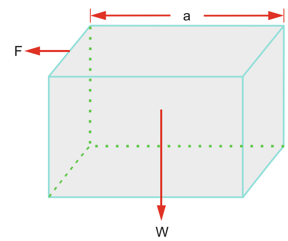
Put $X = x - 2l$, $\ddot{X} = \ddot{x}$

acceleration $\ddot{X} = \dfrac{F}{m} = -\dfrac{2k}{m}X = -\omega^2 X$

$$\therefore f = \frac{1}{2\pi}\sqrt{\frac{2k}{m}}$$

1.62 Let 'a' be the side of the cube and a force F be applied on the top surface of the cube, Fig. 1.34. Take torques about the left-hand side of the edge. The condition that the cube would topple is

Fig. 1.34



Counterclockwise torque > clockwise torque

$$Fa > W\frac{a}{2}$$

or

$$F > 0.5W \tag{1}$$

Condition for sliding is

$$F > \mu W \tag{2}$$

Comparing (1) and (2), we conclude that the cube will topple if $\mu > 0.5$ and will slide if $\mu < 0.5$.

1.63 In Fig. 1.35 let the ladder AB have length L, its weight mg acting at G, the CM of the ladder (middle point). The weight mg produces a clockwise torque τ_1 about B:

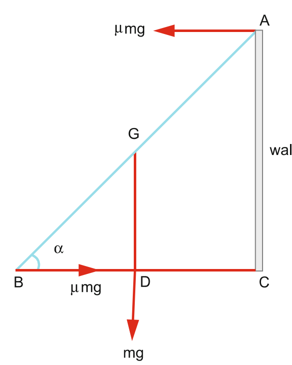


Fig. 1.35

$$\tau_1 = (mg)(\mathrm{BD}) = (mg)\left(\frac{\mathrm{BD}}{\mathrm{BG}}\mathrm{BG}\right) = mg\frac{L}{2}\cos\alpha \tag{1}$$

The friction with the ground, which acts toward right produces a counterclockwise torque τ_2:

$$\tau_2 = (\mu\, mg)\mathrm{AC} = \mu mg\frac{\mathrm{AC}}{\mathrm{AB}}\mathrm{AB} = \mu mgL\sin\alpha \tag{2}$$

For limiting equilibrium $\tau_1 = \tau_2$

$$\therefore \quad mg\frac{L}{2}\cos\alpha = \mu mgL\sin\alpha$$

$$\therefore \quad \mu = \frac{1}{2}\cot\alpha$$