11.98 Both gravitational field and electric field obey inverse square law. Using this analogy show that the differential Gauss' law for gravitation is given by $\nabla \cdot g = -\rho/G$, where ρ is the mass density.

11.3 Solutions

11.3.1 Electric Field and Potential

11.1 **(a)**

(i) $F = \dfrac{q_1 q_2}{4\pi\varepsilon_0 r^2} = \dfrac{(8 \times 1.6 \times 10^{-19})(-4 \times 1.6 \times 10^{-19})}{4\pi \times 8.85 \times 10^{-12} \times (0.2)^2}$
$= -1.8432 \times 10^{-25}\,\text{N}$

(ii) For the position of zero electric field the forces due to the two charges must be equal in magnitude but oppositely directed. Clearly the neutral point must be on the x-axis. On the left of Q_1, the forces will be oppositely directed but cannot be equal as $|Q_1| > |Q_2|$. Between Q_1 and Q_2, the forces are exerted in the same direction. On the right of Q_2 conditions are favourable for a null point. Let the zero electric field be situated at a distance x from Q_2 on the right.

$$\frac{8e}{(x+0.2)^2} - \frac{4e}{x^2} = 0$$

whence $x = 0.4828$ on the right of Q_2.

(b) $E = \dfrac{e}{4\pi\varepsilon_0 r^2} = \dfrac{1.6 \times 10^{-19}}{4\pi \times 8.85 \times 10^{-12} \times (5.3 \times 10^{-11})^2}$
$= 5.12 \times 10^{18}\,\text{N/C}$
Force $F = Ee = 5.12 \times 10^{18} \times 1.6 \times 10^{-19} = 81.92\,\text{N}$

11.2 **(a)** As the electric field is downwards, the force on the positive charge will be downwards and the force on the negative charge will be upwards.

(i) $q = +8\,\mu\text{C}$

$F_q = qE = +8 \times 10^{-6} \times 300 = 2.4 \times 10^{-3}\,\text{N}$

$F_g = mg = 0.6 \times 10^{-3} \times 9.8 = 5.88 \times 10^{-3}\,\text{N}$

$\therefore$ Tension in the thread

$$T = F_g + F_q = 5.88 \times 10^{-3} + 2.4 \times 10^{-3} = 8.28 \times 10^{-3}\,\text{N}$$

(ii) $q = -8\,\mu\text{C}$

$F_q = qE = -2.4 \times 10^{-3}\,\text{N}$

Tension in the thread

$T = 5.88 \times 10^{-3} - 2.4 \times 10^{-3} = 3.48 \times 10^{-3}\,\text{N}$

(b) As the electric field is in the negative x-direction, point b will be at a higher potential than a.

(i) Therefore $V_\text{b} - V_\text{a}$ will be positive

(ii) $E = \dfrac{V_\text{b} - V_\text{a}}{d} = \dfrac{10^5}{(6-2)} = 2.5 \times 10^4\,\text{N/C}$

11.3 The total charge on the circular loop $Q = 2\pi R\lambda$; the distance of the point $P(0, 0, Z)$ from the loop is $r = (Z^2 + R^2)^{1/2}$. Therefore, the electric potential P will be

$$V = \frac{Q}{4\pi\varepsilon_0 r} = \frac{\lambda R}{2\varepsilon_0 (Z^2 + R^2)^{1/2}}$$
$$E = -\frac{\partial V}{\partial z} = \frac{\lambda R Z}{2\varepsilon_0 (Z^2 + R^2)^{3/2}}$$

11.4 **(a)** $F = \dfrac{q}{4\pi\varepsilon_0 r^2}$ $(Q = 1)$

$$V = -\int F\,\mathrm{d}r = -\frac{q}{4\pi\varepsilon_0}\int \frac{\mathrm{d}r}{r^2} = \frac{q}{4\pi\varepsilon_0 r} + C$$

When $r = \infty$, $V = 0$. Therefore $C = 0$

(b) $U = \dfrac{q^2}{4\pi\varepsilon_0}\left[\dfrac{1}{d} - \dfrac{1}{d} + \dfrac{1}{d} - \dfrac{1}{d} - \dfrac{\sqrt{2}}{d} - \dfrac{\sqrt{2}}{d}\right]$

For six pairs of charges

$$U = -\frac{q^2}{\sqrt{2}\pi\varepsilon_0 d}$$

The potential energy does not depend on the order in which the charges are assembled.

(c) Consider the forces on the top left-hand charge due to the three other charges.

$$E_x = E_y = -\frac{q^2}{8\,\pi\varepsilon_0 d^2}$$
$$\therefore \quad E = \sqrt{E_x^2 + E_y^2} = \sqrt{2}\,\frac{q^2}{8\,\pi\varepsilon_0 d^2} \neq 0$$

Therefore the charge is not in equilibrium. Same thing is true for the other three charges.

11.5 **(i)** $V_1 = \dfrac{q}{4\pi\varepsilon_0 r} = \dfrac{2 \times 10^{-15} \times 9 \times 10^9}{10^{-3}} = 0.018\,\text{V}$

(ii) If n droplets each of radius r coalesce to form a large drop of radius R, then assuming that the droplets are incompressible, the volume does not change. Therefore

$$\frac{4}{3}\pi R^3 = n\,\frac{4}{3}\pi r^3$$

$$\text{or} \qquad R = n^{1/3} r \tag{1}$$

As charge is conserved

$$Q = nq \tag{2}$$

where Q and q are the charges on the drop and droplet, respectively. Then the potential of the droplet is

$$V_2 = \frac{Q}{4\pi\varepsilon_0 R} = \frac{nq}{4\pi\varepsilon_0 n^{1/3} r} = \frac{n^{2/3} q}{4\pi\varepsilon_0 r}$$
$$= n^{2/3}\,V_1 = 2^{2/3}\,V_1 = 0.0286\,\text{V}$$

where $n = 2$, and $V_1 = 0.018\,\text{V}$ by (i).

11.6 Electric force

$$F = qE = 2 \times 10^{-8} \times 20,000 = 4 \times 10^{-4}$$

Gravitational force $mg = 80 \times 10^{-6} \times 9.8 = 7.84 \times 10^{-4}\,\text{N}$
Balancing the horizontal and vertical components of forces, Fig. 11.19

$$T \sin\theta = F \tag{1}$$
$$T \cos\theta = mg \tag{2}$$

where T is the tension in the thread.

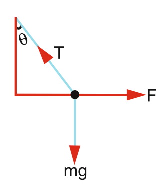


Fig. 11.19

$$\tan\theta = \frac{F}{mg} = \frac{4\times10^{-4}}{7.84\times10^{-4}} = 0.51$$
$$\theta = 27^\circ$$

Squaring (1) and (2) and adding and extracting the square root

$$T = \sqrt{F^2 + (mg)^2} = \sqrt{(4\times10^{-4})^2 + (7.84\times10^{-4})^2} = 8.8\times10^{-4}\,\text{N}$$

11.7 Electric potential

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{1} + \frac{q}{2} + \frac{q}{4} + \frac{q}{8} + \cdots\right)$$
$$= \frac{q}{4\pi\varepsilon_0}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right)$$

The terms in brackets form a geometric progression of infinite terms whose sum is

$$S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$$
$$\therefore \quad V = \frac{q}{2\pi\varepsilon_0}$$

Electric field

$$E = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{1} + \frac{q}{2^2} + \frac{q}{4^2} + \frac{q}{8^2} + \cdots\right]$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{\left(1-\frac{1}{4}\right)} = \frac{q}{3\pi\varepsilon_0}$$

11.8 $$V = \frac{1}{4\pi\varepsilon_0}\left(q - \frac{q}{2} + \frac{q}{4} - \frac{q}{8} + \frac{q}{16} - \frac{q}{32} + \cdots\right)$$
$$= \frac{q}{4\pi\varepsilon_0}\left(1 + \frac{1}{4} + \frac{1}{16} + \cdots\right) - \frac{q}{8\pi\varepsilon_0}\left(1 + \frac{1}{4} + \frac{1}{16} + \cdots\right)$$
$$= \frac{1}{4\pi\varepsilon_0}\left(q - \frac{q}{2}\right)\left(1 + \frac{1}{4} + \frac{1}{16} + \cdots\right)$$
$$= \frac{q}{8\pi\varepsilon_0}\frac{1}{1-\frac{1}{4}} = \frac{q}{6\pi\varepsilon_0}$$

$$E = \frac{1}{4\pi\varepsilon_0}\left(q - \frac{q}{2^2} + \frac{q}{4^2} - \frac{q}{8^2} + \frac{q}{16^2}\cdots\right)$$
$$= \frac{q}{4\pi\varepsilon_0}\left(1 + \frac{1}{4^2} + \frac{1}{16^2} + \cdots\right) - \frac{q}{16\pi\varepsilon_0}\left[1 + \frac{1}{4^2} + \frac{1}{16^2} + \cdots\right]$$
$$= \frac{1}{4\pi\varepsilon_0}\left(q - \frac{q}{4}\right)\left(1 + \frac{1}{4^2} + \frac{1}{16^2} + \cdots\right)$$
$$= \frac{3q}{16\pi\varepsilon_0}\frac{1}{1 - \frac{1}{16}} = \frac{q}{5\pi\varepsilon_0}$$

11.9 By prob. (11.3), $E = \frac{Qx}{4\pi\varepsilon_0\left(x^2 + R^2\right)^{3/2}}$, where we have replaced z by x and substituted $\lambda = Q/2\pi R$. As $x << R$ and $F = qE$,

$$F = \frac{Qqx}{4\pi\varepsilon_0 R^3} = -kx, \text{ where } q \text{ is negative and } \quad k = \frac{Qq}{4\pi\varepsilon_0 R^3}$$
$$= \frac{10^{-5} \times 10^{-6} \times 9 \times 10^9}{1^3} = 0.09$$

Thus, the motion of the negatively charged particle is approximately simple harmonic with angular frequency

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.09}{0.9 \times 10^{-3}}} = 10$$
$$T = \frac{2\pi}{\omega} = \frac{2\pi}{10} = 0.628\,\mathrm{s}$$

11.10 Let a be the side of the equilateral triangle. The forces on the charge q placed at C due to the charges at A and B are repulsive and represented by CE and CD, respectively, each given by $q^2/4\pi\varepsilon_0 a^2$. The resultant of these two forces is given by CP, the diagonal of the parallelogram CDPE, Fig. 11.20

$$\mathrm{CP} = 2\mathrm{CD}\,\cos 30° = \frac{2q^2}{4\pi\varepsilon_0 a^2}\frac{\sqrt{3}}{2} = \frac{\sqrt{3}q^2}{4\pi\varepsilon_0 a^2}$$

The force on q at C due to Q at the centre of the triangle is

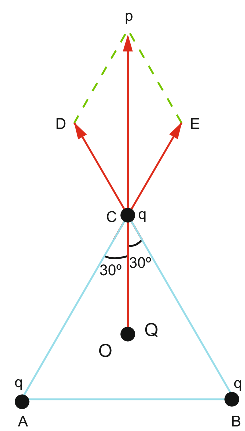
$$\frac{\mathrm{Q}q}{4\pi\varepsilon_0(\mathrm{OC})^2} = \frac{3\,\mathrm{Q}q}{4\pi\varepsilon_0 a^2}$$

i. If $Q = -q$, this force will be attractive and will be directed along CO. As the attractive force due to $-q$ is greater than the combined repulsive force due to charges $+q$ at A and B, the charge at C will be attracted towards O. Same is true for the charges placed at A and B.

ii. For equilibrium, the attractive force must balance the repulsive force:

$$\frac{\sqrt{3}q^2}{4\pi\varepsilon_0 a^2} - \frac{3Qq}{4\pi\varepsilon_0 a^2} = 0 \rightarrow Q = -q/\sqrt{3}$$

Fig. 11.20



11.11 Referring to Fig. 11.21, the Coulomb force between the spheres in air is

$$F = \frac{q^2}{4\pi\varepsilon_0 x^2}$$

In liquid $F' = \dfrac{q^2}{4\pi\varepsilon_0\, K\, x^2}$; weight of the sphere in air $= mg$

Apparent weight of the sphere in liquid $= mg\left(1 - \dfrac{\rho'}{\rho}\right)$, where ρ is the density of the material of sphere and ρ' that of the liquid. For equilibrium the vertical and horizontal components of the force must separately balance. If T is the tension in the string when the sphere is in air and T' when it is in the liquid,

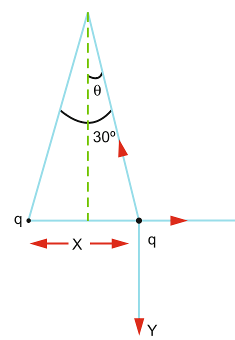
$$T \sin\theta = F, T \cos\theta = mg$$

$$T' \sin\theta = F',\ T' \cos\theta = mg\left(1 - \frac{\rho'}{\rho}\right)$$

$$\tan\theta = \frac{F}{mg} = \frac{F'}{mg\left(1 - \frac{\rho'}{\rho}\right)}$$

$$\frac{F}{F'} = K = \frac{1}{1 - \frac{\rho'}{\rho}} = \frac{\rho}{\rho - \rho'} = \frac{1.6}{1.6 - 0.8} = 2$$

Fig. 11.21



11.12 From the geometry of Fig. 11.22, AB = 0.1 m, OB = OC = $0.05\sqrt{2}$ m, AC = $(0.1)\sqrt{2}$ m.

Potential energy of q at B is $U(\mathrm{B}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{q_1}{\mathrm{AB}} + \frac{q_2}{\mathrm{OB}}\right]$

Potential energy of q at C is $U(\mathrm{C}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{q_1}{\mathrm{AC}} + \frac{q_2}{\mathrm{OC}}\right]$

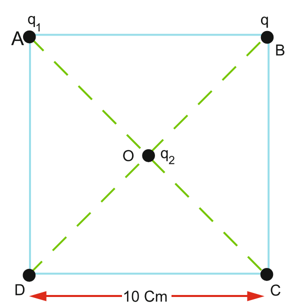


Fig. 11.22

Work done in carrying the charge q from C to B will be

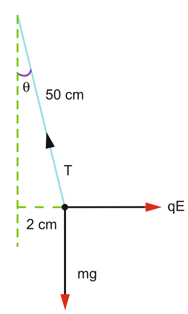
$$W_{\mathrm{CB}} = U(\mathrm{B}) - U(\mathrm{C}) = \frac{qq_1}{4\pi\varepsilon_0}\left[\frac{1}{\mathrm{AB}} - \frac{1}{\mathrm{AC}}\right]$$
$$= 5 \times 10^{-9} \times 6 \times 10^{-8} \times 9 \times 10^{9}\left(\frac{1}{0.1} - \frac{1}{(0.1)\sqrt{2}}\right)$$
$$= 7.9 \times 10^{-6}\,\mathrm{J}$$

11.13 The problem is similar to prob. (11.6), Fig. 11.23. For equilibrium $T\sin\theta = qE$, $T\cos\theta = mg$, $\tan\theta = \frac{qE}{mg} \sim \frac{2}{50} = 0.04$

$$\therefore \qquad E = \frac{0.04\,mg}{q} = \frac{(0.04)(0.5 \times 10^{-3})(9.8)}{3 \times 10^{-10}} = 6.53 \times 10^{5}\,\mathrm{N/C}$$

which is directed away from the equilibrium position.

Fig. 11.23



11.14 The electric field $E = V/d$ where V is the PD and d is the distance of separation of plates. The electric force on the droplet is $F = qE = qV/d$. If the upper plate is negative then the condition for equilibrium against gravitational force acting downwards is

$$\frac{qV}{d} = mg$$
$$V = \frac{mgd}{q} = \frac{10^{-14} \times 9.8 \times 0.01}{3.2 \times 10^{-19}} = 3062.5\,\mathrm{V}$$

If the polarity of the plates is reversed, both the electric and gravitational forces would act down. The net force would become

$$F' = qE + mg = 2mg$$

$$\text{Acceleration } a = \frac{F'}{m} = 2g = 2 \times 9.8 = 19.6\,\text{m/s}^2$$

11.15 Let q be the charge on each body. Electric force = gravitational force

$$\frac{q^2}{4\pi\varepsilon_0 r^2} = \frac{GMm}{r^2}$$

$$q = \sqrt{4\pi\varepsilon_0 GMm}$$

$$= \left[\frac{6.67 \times 10^{-11} \times 6 \times 10^{24} \times 7.4 \times 10^{22}}{9 \times 10^9}\right]^{1/2} = 5.736 \times 10^{13}\,\text{C}$$

11.16 Energy $W = qV$

$$q = \frac{W}{V} = \frac{10^{-5}}{5 \times 10^6} = 2 \times 10^{-12}\,\text{C}$$

$$\text{Number of electrons flowed out} = \frac{q}{e} = \frac{2 \times 10^{-12}}{1.6 \times 10^{-19}} = 1.25 \times 10^7$$

11.17 Consider an element dx of the rod at distance x from the point P on the axis of the rod. In the length dx the charge is $\mathrm{d}q = \lambda \mathrm{d}x$, Fig. 11.24.
The field at P due to dq will be

$$\mathrm{d}E = \frac{\lambda\,\mathrm{d}x}{4\pi\varepsilon_0 x^2}$$

The total electric field will be

$$E = \int \mathrm{d}E = \int_{0.1}^{0.35} \frac{\lambda\,\mathrm{d}x}{4\pi\varepsilon_0 x^2} = \frac{\lambda}{4\pi\varepsilon_0} \cdot \frac{1}{x}\Big|_{0.35}^{0.1}$$

$$= 200 \times 10^{-6} \times 9 \times 10^9 \times \left(\frac{1}{0.1} - \frac{1}{0.35}\right) = 1.286 \times 10^7\,\text{N/C}$$

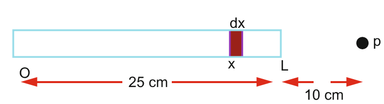


Fig. 11.24

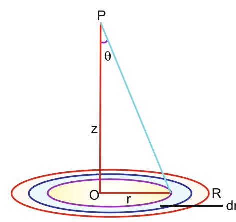
11.18 Let p be the field point on the axis of the disc at distance z from the origin. Consider a ring of radius r and width dr. The charge on the ring is $\mathrm{d}q = 2\pi r\,\mathrm{d}r\,\sigma$ where σ is the charge density (charge per unit area), Fig. 11.25.

The electric field at P can be resolved into a component along the z-axis and perpendicular to it. The perpendicular components when added become zero for reasons of symmetry. The components along the z-axis are added

$$\mathrm{d}E_{||} = \frac{\mathrm{d}q\ \cos\theta}{4\pi\varepsilon_0(z^2 + r^2)} = \frac{2\pi\ r\ \mathrm{d}r\ \sigma}{4\pi\varepsilon_0(z^2 + r^2)} \cdot \frac{z}{(z^2 + r^2)^{1/2}}$$

$$E = \int \mathrm{d}E_{||} = \frac{\sigma}{2\varepsilon_0} z \int_0^R \frac{r\mathrm{d}r}{(z^2 + r^2)^{3/2}} = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{(z^2 + R^2)^{1/2}}\right)$$

Fig. 11.25



11.19 Equations of motion are

$$eE + mg - 6\pi\eta v_1 \mathrm{r} = 0 \quad \text{(downward field)}$$
$$eE - mg - 6\pi\eta v_2 \mathrm{r} = 0 \quad \text{(upward field)}$$

Adding and solving for e

$$e = \frac{3\pi\eta}{E}(v_1 + v_2)r$$

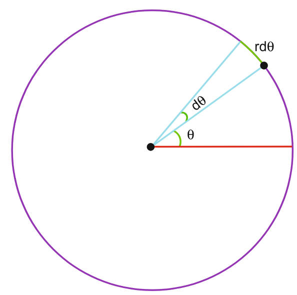
11.20 Consider an element of the circular wire ds (Fig. 11.26). Then $\mathrm{d}q = \lambda \mathrm{d}s$.

$$\text{Now } \mathrm{d}s = r\,\mathrm{d}\theta$$

$$\therefore \quad \mathrm{d}q = (\lambda\cos^2\theta)\,(r\mathrm{d}\theta)$$

$$\therefore \quad q = \int \mathrm{d}q = \lambda_0 r \int_0^{2\pi} \cos^2\theta \mathrm{d}\theta = \pi\lambda_0 r$$

Fig. 11.26



11.21 Electric force $F_e = \dfrac{q^2}{4\pi\varepsilon_0 r^2}$ (11.48)

Gravitational force $F_g = \dfrac{GMm}{r^2}$ (11.49)

$$\frac{F_e}{F_g} = \frac{q^2}{4\pi\varepsilon_0}\frac{1}{GMm} = \frac{(1.6\times10^{-19})^2(9\times10^9)}{(6.67\times10^{-11})(1.66\times10^{-27})(9.1\times10^{-31})} \quad (11.50)$$

$$= 2.29\times10^{39} \quad (11.51)$$

The distance is immaterial. Note that the gravitational force at the atomic and sub-atomic levels is small simply because the masses are small.

11.22 $q_1 + q_2 = 15\,\mu\text{C}$ (1)

$$F = \frac{q_1q_2}{4\pi\varepsilon_0 r^2} = \frac{9\times10^9}{(0.3)^2}q_1q_2 = 5.4$$

or $\quad q_1\, q_2 = 54\ (\mu\text{C})^2$ (2)

Solving (1) and (2), $q_1 = 6\,\mu\text{C}$, $q_2 = 9\,\mu\text{C}$.

11.23 At P, the electric field due to $+q$ is $\dfrac{q}{4\pi\varepsilon_0(a/\sqrt{2})^2}$ or $\dfrac{2q}{4\pi\varepsilon_0(a)^2}$ and points towards $+2q$. The field due to $+2q$ is $\dfrac{4q}{4\pi\varepsilon_0 a^2}$ and points towards $+q$. The resultant field due to the pair $(q, 2q)$ is $\dfrac{4q}{4\pi\varepsilon_0 a^2} - \dfrac{2q}{4\pi\varepsilon_0 a^2}$ or $E_1 = \dfrac{2q}{4\pi\varepsilon_0 a^2}$ towards $+q$.

Similarly, the resultant filed due to the pair of charges $(-q, -2q)$ will be $E_2 = -\dfrac{2q}{4\pi\varepsilon_0 a^2}$ towards $-2q$ or $+\dfrac{2q}{4\pi\varepsilon_0 a^2}$ towards $-q$.

Now $E_1 = E_2$ in magnitude and act at right angles (from the geometry of the diagram). The overall field will then be $E = \sqrt{2}\,E_1 = \dfrac{2\sqrt{2}q}{4\pi\varepsilon_0 a^2}$ along positive y-axis.

11.24 Consider an infinitesimal length dx at distance x from O, the centre of the rod. The charge on dx will be d$q = q\,(\mathrm{d}x/L)$. The field at P due to dq shown by an arrow can be resolved into x- and y-components. The x-component of the field will be cancelled by a symmetric charge on the negative side at equal distance. The $y-$components of the field will be added up, Fig. 11.27.

$$\mathrm{d}E_y = \mathrm{d}E\,\cos\theta = \frac{q\mathrm{d}x}{4\pi\varepsilon_0 L(x^2+y^2)}\frac{y}{(x^2+y^2)^{1/2}}$$

$$\therefore \quad E = \int \mathrm{d}E_y = \frac{qy}{4\pi\varepsilon_0 L}\int \frac{\mathrm{d}x}{(x^2+y^2)^{3/2}}$$

Put $x = y\tan\theta$, $\mathrm{d}x = y\sec^2\theta\mathrm{d}\theta$

$$E = \frac{q}{4\pi\varepsilon_0 Ly}\int_{-\alpha}^{\alpha}\cos\theta\mathrm{d}\theta = \frac{q}{2\pi\varepsilon_0 Ly}\sin\alpha = \frac{q}{2\pi\varepsilon_0(4y^2+L^2)^{1/2}}$$

where we have put $\sin\alpha = \dfrac{L/2}{(y^2+L^2/4)^{1/2}}$.

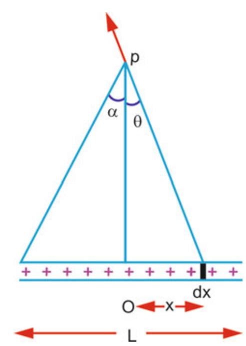


Fig. 11.27

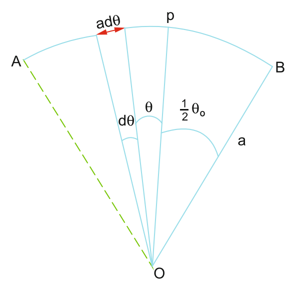
11.25 Consider an element of angle between θ and $\theta+\mathrm{d}\theta$. Let OP be the bisector of angle θ_0 subtended by the arc AB at the centre O. The charge on the element of the arc $a\mathrm{d}\theta$ will be $q\,\dfrac{\mathrm{d}\theta}{\theta_0}$. The electric field at O due to this element of arc can be resolved $E_{||}$ along PO and $E_{\perp}$ perpendicular to it.

The perpendicular components will be cancelled for reasons of symmetry while the parallel components get added up, Fig. 11.28.

$$\mathrm{d}E = \mathrm{d}E_{\parallel} = \frac{q\,\mathrm{d}\theta}{4\pi\varepsilon_0\theta_0 a^2}\ \cos\theta$$

$$E = \int \mathrm{d}E_{\parallel} = \frac{q}{4\pi\varepsilon_0\theta_0 a^2}\int_{-\theta_0/2}^{\theta_0/2}\cos\theta\,\mathrm{d}\theta = \frac{q}{2\pi\varepsilon_0 a^2}\frac{\sin(\theta_0/2)}{\theta_0}$$

Fig. 11.28



11.26 The electric field at distance z from the centre of the ring on the axis of the ring is given by prob. (11.3)

$$E = \frac{\lambda r}{2\varepsilon_0}\frac{z}{(z^2 + r^2)^{3/2}}$$

The maximum field is obtained by setting $\dfrac{\partial E}{\partial z} = 0$.

This gives $(z^2 + r^2)^{1/2}(r^2 - 2z^2) = 0$.

Since the first factor cannot be zero for any real value of z, the second factor gives $z = r/\sqrt{2}$.

11.27 Consider a circular strip symmetric about z-axis of radius r and width $a\mathrm{d}\theta$ (Fig. 11.29). The charge on the strip is

$$\mathrm{d}q = q\frac{2\pi r\,a\mathrm{d}\theta}{2\pi\,a^2} = \frac{qr\,\mathrm{d}\theta}{a} = q\ \sin\theta\,\mathrm{d}\theta$$

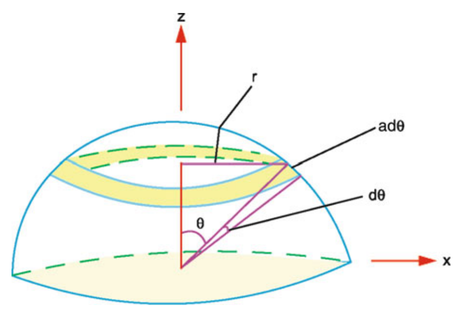
(a) At the centre of the hemisphere, the x-component of the field will be cancelled for reasons of symmetry. The entire field will be contributed by the z-component alone.

$$\mathrm{d}E = \mathrm{d}E_z = \frac{q \sin\theta \mathrm{d}\theta\ \cos\theta}{4\pi\varepsilon_0 a^2}$$

$$\therefore \quad E = \int \mathrm{d}E_z = \frac{q}{4\pi\varepsilon_0 a^2} \int_0^{\pi/2} \sin\theta\cos\theta \mathrm{d}\theta = \frac{q}{8\pi\varepsilon_0 a^2}$$

(b) $\mathrm{d}V = \dfrac{q \sin\theta \mathrm{d}\theta}{4\pi\,\varepsilon_0 a}$; $\quad V = \displaystyle\int dv = \frac{q}{4\pi\varepsilon_0 a} \int_0^{\pi/2} \sin\theta \mathrm{d}\theta = \frac{q}{4\pi\,\varepsilon_0 a}$

Fig. 11.29



11.28 The x-component of the field due to front charges will get cancelled and the y-component is added up to

$$\frac{2q}{4\pi\varepsilon_0} \frac{a/2}{\left[\left(r - \frac{a}{2}\right)^2 + \frac{a^2}{4}\right]^{1/2}}$$

along the negative y-axis, Fig. 11.30.
Similarly the field due to the other two charges will be
$\dfrac{2q \cdot a/2}{4\pi\varepsilon_0 \left[\left(r + \frac{a}{2}\right)^2 + \frac{a^2}{4}\right]^{3/2}}$ along the positive y-axis.
Neglecting terms of the order of a^2, the net field will be

$$E = \frac{qa}{4\pi\varepsilon_0 r^3} \left[\left(1 - \frac{a}{r}\right)^{-3/2} - \left(1 + \frac{a}{r}\right)^{-3/2}\right]$$

Using the binomial expansion up to retaining terms linear in a,

$$E = \frac{3qa^2}{4\pi\varepsilon_0 r^4}. \text{ Then } E\ \alpha\ \frac{1}{r^4}$$

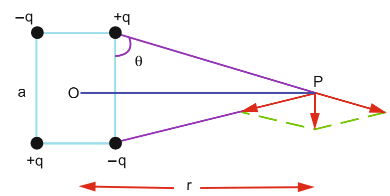


Fig. 11.30

11.29 The electric force F_e = the gravitational force

$$\frac{q^2}{4\pi\varepsilon_0 r^2} = \frac{Gm^2}{r^2}$$

$$\frac{q}{m} = (4\pi\varepsilon_0 G)^{1/2} = \left(\frac{6.67 \times 10^{-11}}{9 \times 10^9}\right)^{1/2} = 8.65 \times 10^{-9}\ \mathrm{C/kg}$$

11.30 Consider the equilibrium of one of the spheres, Fig. 11.31. If T is the tension in the string then

$$T\cos\theta = mg$$

$$T\ \sin\theta = \frac{q^2}{4\pi\varepsilon_0 x^2}$$

$$\therefore \quad \tan\theta \simeq \theta = \frac{x}{2L} = \frac{q^2}{4\pi\varepsilon_0 mgx^2}$$

$$\therefore \quad q = \left(\frac{2\pi\varepsilon_0 mg}{L}\right)^{1/2} x^{3/2}$$

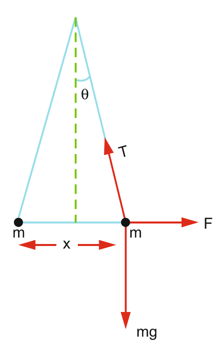


Fig. 11.31

$$\therefore \quad \frac{dq}{dx} = \frac{3}{2}\sqrt{\frac{2\pi\varepsilon_0 mg}{L}}\sqrt{x}$$

$$\frac{dq}{dt} = \frac{dq}{dx}\frac{dx}{dt} = \frac{dq}{dx}v = \frac{dq}{dx}\frac{a}{\sqrt{x}} = \frac{3}{2}a\sqrt{\frac{2\pi\varepsilon_0 mg}{L}}$$

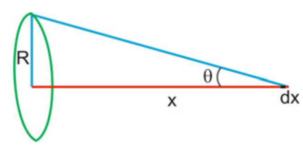
11.31 Consider an element of length dx of the thread at distance x from the centre of the ring. The force between the ring and the element dx can be resolved into x- and y-components, Fig. 11.32. The y-component will get cancelled for reasons of symmetry. The field is entirely contributed by the x-component. The charge in length dx is λdx. The electric force between the wire and the ring is given by

$$F = F_x = \int \frac{1}{4\pi\varepsilon_0}\frac{q\,\lambda\,dx\cos\theta}{(R^2+x^2)} = \frac{q\lambda}{4\pi\varepsilon_0}\int_0^\infty \frac{x\,dx}{(x^2+R^2)^{3/2}}$$

Put $x = R\cot\theta$, $dx = -R\operatorname{cosec}^2\theta\,d\theta$

$$F = -\frac{q\lambda}{4\pi\varepsilon_0 R}\int_{\pi/2}^{0}\cos\theta\,d\theta = \frac{q\lambda}{4\pi\varepsilon_0 R}$$

Fig. 11.32



11.32 Consider an element of wire dx at distance x from O, Fig. 11.33. The charge in dx will be λdx. The x-component of the electric field will be

$$E_x = \int E\sin\theta = \int \frac{\lambda dx\ \sin\theta}{4\pi\varepsilon_0(x^2+y^2)} = \frac{\lambda}{4\pi\varepsilon_0}\int_0^\infty \frac{x\,dx}{(x^2+y^2)^{3/2}}$$

Put $\quad x^2+y^2 = z^2,\ x\,dx = z\,dz$

where $y =$ constant.

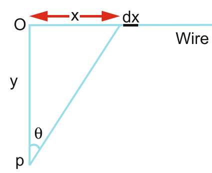
$$E_x = \frac{\lambda}{4\pi\varepsilon_0}\int_y^\infty \frac{dz}{z^2} = \frac{\lambda}{4\pi\varepsilon_0 y}$$

Similarly, $\quad E_y = \int E\ \cos\theta = \dfrac{\lambda}{4\pi\varepsilon_0 y}$

$$E = \sqrt{E_x^2 + E_y^2} = \frac{\sqrt{2}\,\lambda}{4\pi\varepsilon_0 y}$$
$$\tan\alpha = \frac{E_y}{E_x} = 1 \rightarrow \alpha = 45°$$

Thus $\vec{E}$ makes an angle 45° with the y-axis.

Fig. 11.33



11.33 $\phi = Cxy$

$$E_x = -\frac{\partial\phi}{\partial x} = -cy, \quad E_y = \frac{\partial\phi}{\partial y} = -cx$$
$$\therefore \quad \vec{E} = -c(y\hat{i} + x\hat{j})$$

11.34 Let the charges Q and $-2Q$ be located on the x-axis at distance x on the opposite side of the y-axis. Let the point $P(x, y)$ be at distance r_1 from Q and at r_2 from $-2Q$, Fig. 11.34. By problem

$$\frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r_1} - \frac{2Q}{r_2}\right] = 0$$

or $\quad r_2 = 2r_1 \qquad (1)$

Writing $\quad r_1^2 = (x+a)^2 + y^2 \qquad (2)$

and $\quad r_2^2 = (a-x)^2 + y^2 \qquad (3)$

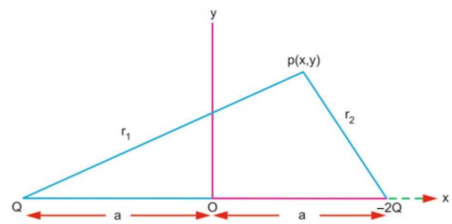
and eliminating r_1 and r_2 in (1), (2) and (3) and simplifying

$$3x^2 + 3y^2 + 10xa + 3a^2 = 0$$

or $\quad \left(x + \frac{5}{3}a\right)^2 + y^2 = \frac{16}{9}a^2$

which is the equation to a circle.

Fig. 11.34



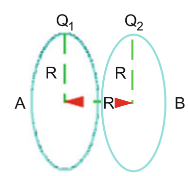
11.35 Let the charge q be taken from the centre of ring A to the centre of ring B, Fig. 11.35

$$\text{At A,}\quad U_A = \frac{qQ_1}{4\pi\varepsilon_0 R} + \frac{q\,Q_2}{4\pi\varepsilon_0(\sqrt{2}\,R)}$$

$$\text{At B,}\quad U_B = \frac{qQ_2}{4\pi\varepsilon_0 R} + \frac{q\,Q_1}{4\pi\varepsilon_0(\sqrt{2}\,R)}$$

$$\text{Work done,}\quad W = U_B - U_A = \frac{(\sqrt{2}-1)\,q\,(Q_2 - Q_1)}{4\sqrt{2}\pi\varepsilon_0 R}$$

Fig. 11.35



11.36 **(a)** Consider an infinitesimal length of the rod, at distance y from the origin, Fig. 11.36. The charge in dy will be λdy. The distance of P_1 from dy will be $2a - y$. The potential at P_1 is

$$V_1 = \frac{1}{4\pi\varepsilon_0}\int_{-a}^{a}\frac{\lambda\,dy}{(2a-y)} = \frac{\lambda}{4\pi\varepsilon_0}\ln 3$$

The potential at P_2 is

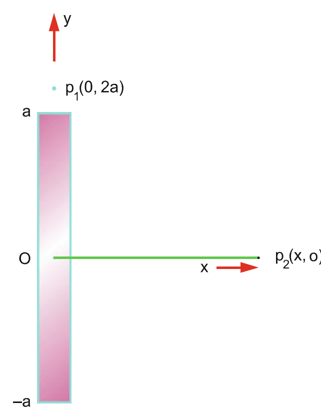
$$V_2 = \frac{1}{4\pi\varepsilon_0}\int_{-a}^{a}\frac{\lambda dy}{\sqrt{y^2+x^2}} = \frac{2\lambda}{4\pi\varepsilon_0}\ln\left(\frac{a+\sqrt{a^2+x^2}}{x}\right)$$

By problem $V_2 = V_1$

$$\therefore \quad \left(\frac{a + \sqrt{a^2 + x^2}}{x}\right)^2 = 3 \rightarrow x = \sqrt{3}\,\mathrm{a}$$

$$V_1 = V_2 = \lambda \times 9 \times 10^9 \times \ln 3$$
$$= 9.89 \times 10^9\ \mathrm{V}$$

Fig. 11.36



11.37 $U = U_{Qq} + U_{Qq} + U = \dfrac{1}{4\pi\varepsilon_0}\left[\dfrac{Qq}{a} + \dfrac{Qq}{a} + \dfrac{q^2}{\sqrt{2}a}\right] = 0$ by problem. Therefore $Q = -\dfrac{q}{2\sqrt{2}}\cdot$

11.38 Work done $W = U_{12} + U_{23} + U_{34} + U_{41} + U_{13} + U_{24}$ where charges l and 3 are positive and 2 and 4 negative assuming that the potential energy is zero for infinite separation of charges.

$$W = \frac{q^2}{4\pi\varepsilon_0}\left[-\frac{1}{a} - \frac{1}{a} - \frac{1}{a} - \frac{1}{a} + \frac{1}{\sqrt{2}a} + \frac{1}{\sqrt{2}a}\right]$$
$$= \frac{-q^2}{4\pi\varepsilon_0 a}\left(4 - \sqrt{2}\right)$$

11.39 **(a)** The charge density is given by $\rho = \dfrac{3q}{4\pi R^3}$. Consider a shell of radius r and thickness dr concentric with the sphere. The volume of the shell is $4\pi r^2\mathrm{d}r$ and the charge in it will be $\mathrm{d}q = 4\pi r^2 \mathrm{d}r\rho$, Fig. 11.37. The charge of the sphere of radius r is $\mathrm{d}q' = \dfrac{4}{3}\pi r^3 \rho$ and may be considered to be concentrated at the centre. The interaction energy between the shell and the sphere of radius r will be

$$\mathrm{d}U = \frac{1}{4\pi\varepsilon_0}\frac{(\mathrm{d}q)(\mathrm{d}q')}{r} = \frac{1}{4\pi\varepsilon_0}\frac{(4\pi r^2 \mathrm{d}r\rho)\left(\frac{4}{3}\pi r^3\rho\right)}{r} = \frac{4\pi\rho^2 r^4 \mathrm{d}r}{3\varepsilon_0}$$

Total interaction energy

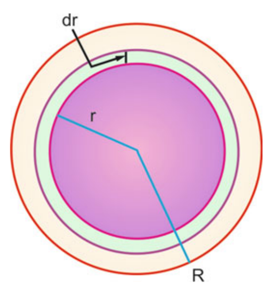
$$U = \int \mathrm{d}U = \frac{4\pi\rho^2}{3\varepsilon_0}\int_0^R r^4 \mathrm{d}r = \frac{4\pi\rho^2 R^5}{15\varepsilon_0} = \frac{3q^2}{20\pi\varepsilon_0 R}$$

where we have substituted the value of ρ.

(b) $U = \dfrac{3}{5}\dfrac{\times 9\times 10^9 \times (92\times 1.6\times 10^{-19})^2}{1.5\times (238)^{1/3}\times 10^{-15}} = 1.259\times 10^{-11}\,\mathrm{J}$

$= 78.7\,\mathrm{MeV}$

Fig. 11.37



11.40 **(i)** $E = \dfrac{Q}{4\pi\varepsilon_0}\left[\dfrac{2}{x^2} - \dfrac{1}{(x+d)^2} - \dfrac{1}{(x-d)^2}\right]$

$= -\dfrac{2Q\,d^2(3x^2-d^2)}{4\pi\varepsilon_0 x^2(x^2-d^2)^2}$

(ii) For $x >> d$, $3x^2 - d^2 \simeq 3x^2$ and $x^2 - d^2 \simeq x^2$

$E = -\dfrac{6Q\,d^2}{4\pi\varepsilon_0 x^4}$

(iii) $E = -\dfrac{6\times 9\times 10^9\times 2\times 10^{-6}\times (10^{-4})^2}{(0.2)^2} = 0.675\,\mathrm{N/C}$

11.41 **(a)** $q = CV = 4\pi\varepsilon_0 r\, V = \dfrac{5 \times 10^6 \times 10^{-3}}{9 \times 10^9} = 5.5 \times 10^{-7}\,\text{C}$

(b) $V = \dfrac{q}{4\pi\varepsilon r} = \dfrac{5.5 \times 10^{-7} \times 9 \times 10^9}{1 \times 10^{-3}} = 1.65 \times 10^6\,\text{V}$

11.42 Initial potential energy $U_1 = -\dfrac{qe}{4\pi\varepsilon_0 r_1}$

Final potential energy $U_2 = -\dfrac{qe}{4\pi\varepsilon_0 r_2}$

$$\Delta U = U_1 - U_2 = \frac{qe}{4\pi\varepsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

By work–energy theorem, gain in kinetic energy = loss in potential energy.

$$\frac{1}{2}mv^2 = \frac{qe}{4\pi\varepsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

$$v = \left[\frac{2qe}{4\pi\varepsilon_0 m}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)\right]^{1/2}$$

$$= \left[\frac{2 \times 2 \times 10^{-9} \times 1.6 \times 10^{-19} \times 9 \times 10^9}{9.1 \times 10^{-31}}\left(\frac{1}{0.18} - \frac{1}{1.2}\right)\right]^{1/2}$$

$$= 5.467 \times 10^6\,\text{m/s}$$

11.43 $$V(r) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{2}{r} - \frac{1}{\sqrt{r^2+d^2}} - \frac{1}{\sqrt{r^2+d^2}}\right]$$

$$= \frac{2Q}{4\pi\varepsilon_0}\left[\frac{1}{r} - \frac{1}{r}\left(1 + \frac{d^2}{r^2}\right)^{-1/2}\right] \simeq \frac{Qd^2}{4\pi\varepsilon_0 r^3}$$

Thus $V(r)\,\alpha\,\dfrac{1}{r^3}$

11.44 **(i)** $F = qE = (1.6 \times 10^{-19})(2 \times 10^3) = 3.2 \times 10^{-16}\,\text{N}$

(ii) Acceleration $a = \dfrac{F}{m} = \dfrac{3.2 \times 10^{-16}}{9.1 \times 10^{-31}} = 3.516 \times 10^{14}\,\text{m/s}^2$

$$v = \sqrt{2as} = \sqrt{2 \times 3.516 \times 10^{14} \times 0.015}$$

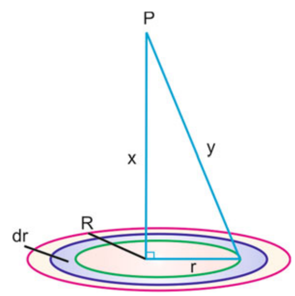
$$= 3.25 \times 10^6\,\text{m/s}$$

(iii) Outside the plates there is no force on the electron as there is no electric field.

11.45 $$V(r) = \frac{Q}{4\pi\varepsilon_0 r} \tag{1}$$

For continuous distribution of charge, each element $\mathrm{d}q$ can be treated as point charge so that the contribution $\mathrm{d}V$ to the potential can be written according to (1), Fig. 11.38:

Fig. 11.38



$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{r} \tag{2}$$

For the potential due to the entire distribution of all the elements, (2) is integrated:

$$V = \int dV = \frac{1}{4\pi\varepsilon_0}\int \frac{dQ}{r} \tag{3}$$

In case of uniform charge distribution we can write $dQ = \lambda\, ds$, $dQ = \sigma dA$ or $dQ = \rho dV$ depending on the geometry of the problem. Here λ is the linear charge density, σ is the surface charge density and ρ is the volume charge density.

(a) $q = \pi R^2 \sigma$

(b) Consider a ring of radius r and width dr concentric with the disc of radius R. The charge on the ring is $dq = \sigma 2\pi r dr$. The potential at P, at a distance x on the axis of the disc, will be

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{y} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma 2\pi r\, dr}{\sqrt{r^2 + x^2}} \tag{4}$$

Potential due to the disc will be

$$V = \int dV = \frac{\sigma}{2\varepsilon_0}\int_0^R \frac{r\, dr}{\sqrt{r^2 + x^2}} \tag{5}$$

Put $r^2 + x^2 = y^2$, $r\, dr = y\, dy$, then (5) becomes

$$V = \frac{\sigma}{2\varepsilon_0}\int dy = \frac{\sigma}{2\varepsilon_0} y \Big|_x^{\sqrt{x^2+R^2}} = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{x^2 + R^2} - x\right] \tag{6}$$

(c) If $x >> R$, (6) can be expanded binomially,
$V \rightarrow \frac{\sigma}{2\varepsilon_0}\left[x\left(1+\frac{R^2}{x^2}\right)^{1/2} - x\right] = \frac{\sigma R^2}{4\varepsilon_0 x} = \frac{q}{4\pi\varepsilon_0 x}$, an expression which is appropriate for the point charge. This result is reasonable since at very large distances the disc appears as a point.

11.46 For circular motion of electron, the speed

$$v = \frac{2\pi r}{T} \tag{1}$$

For a stable orbit, the centripetal force = electric force.

$$\frac{mv^2}{r} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} \tag{2}$$

Eliminating v between (1) and (2)

$$T^2 = 16\pi^3\varepsilon_0 m r^3$$

or $T^2 \,\alpha\, r^3$ (Kepler's third law)

11.47 Figure 11.39 shows the forces on charge 2 due to the charges 1, 3 and 4. The forces

$$F_{12} = F_{32} = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{a^2}$$

$$F_{42} = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{\left(\sqrt{2}a\right)^2} = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{(2a^2)}$$

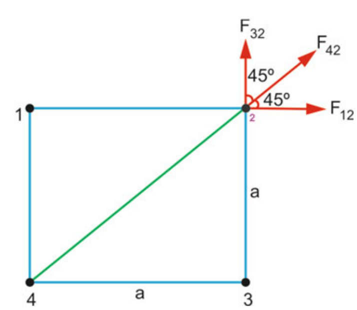


Fig. 11.39

Now $|\boldsymbol{F}_{32} + \boldsymbol{F}_{12}| = \sqrt{2}\, F_{12}$($\because$ $\boldsymbol{F}_{12}$ and $\boldsymbol{F}_{32}$ act at right angle and are equal in magnitude)

Further, $\boldsymbol{F}_{42}$ acts in the same direction as the combined force of $\boldsymbol{F}_{32}$ and $\boldsymbol{F}_{12}$.

$$\therefore \qquad |\boldsymbol{F}_{12} + \boldsymbol{F}_{32} + \boldsymbol{F}_{42}| = \frac{Q^2}{4\pi\varepsilon_0 a^2}\left(\sqrt{2} + \frac{1}{2}\right) = \frac{1.914\, Q^2}{4\pi\varepsilon_0 a^2}$$

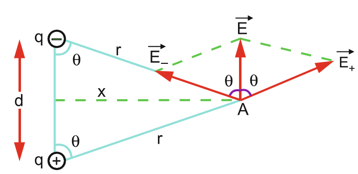
11.48 **(a)** A dipole consists of two equal and opposite charges. To find the electric field at A, on the perpendicular bisector of the dipole, at distance x. As the point A is equidistant from the two charges, the magnitudes E_+ and E_- are equal. The net electric field $\boldsymbol{E}$ at A is given by the vector addition of $\boldsymbol{E}_+$ and $\boldsymbol{E}_-$ (Fig. 11.40)

$$\boldsymbol{E} = \boldsymbol{E}_+ + \boldsymbol{E}_- \qquad (1)$$

$$E_+ = E_- = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{q}{[x^2 + (d/2)^2]} \qquad (2)$$

Since both E_+ and E_- are equal and equally inclined to the y-axis, their x-components gets cancelled and the combined field is contributed by the y-component alone.

Fig. 11.40



$$E = E_y = E_+\cos\theta + E_-\cos\theta = 2E_+\cos\theta$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{qd}{[x^2 + (d/2)^2]^{3/2}} \qquad (3)$$

(b) For $x >> d/2$, $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{p}{x^3}$, where $p = qd$ is the dipole. Thus the electric field at large distance varies inversely as the third power of distance, which is much more rapid than the inverse square dependence for point charge.

(c) Potential energy $U = -\boldsymbol{P}\cdot\boldsymbol{E} = -pE\cos\theta$, for parallel alignment, $\theta = 0$, and

$$U_1 = -pE = -(6\times 10^{-32})(3\times 10^{6}) = -1.8\times 10^{-25}\ \text{J}$$

For antiparallel arrangement, $\theta = 180°$ and $U_2 = +PE = +1.8 \times 10^{-25}$ J.
Therefore the difference in the potential energy $\Delta U = U_2 - U_1 = 3.6 \times 10^{-25}$ J.

11.3.2 Gauss' Law

11.49 **(a)** If ϕ_E is the electric flux, $\boldsymbol{E}$ the electric field, q the charge enclosed and d$\boldsymbol{A}$ the element of area then $q = \varepsilon_0 \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A}$. The integration is to be carried over the entire surface. The circle on the integral sign indicates that the surface of integration is a closed surface.

(b) Figure 11.41 shows a portion of a thin non-conducting infinite sheet of charge of constant charge density σ (charge per unit area). To calculate the electric field at points close to the sheet construct a Gaussian surface in the form of a closed cylinder of cross-sectional area A, piercing the plane of the sheet, Fig. 11.41. From symmetry, it is obvious that $\boldsymbol{E}$ points are at right angle to the end caps, away from the plane, and are positive at both the end caps. There is no contribution to the flux from the curved wall of the cylinder as $\boldsymbol{E}$ does not pierce. By Gauss law

$$\varepsilon_0 \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A} = q$$

$$\varepsilon_0 (EA + EA) = q$$

where σA is the enclosed charge. Thus $E = \sigma / 2\varepsilon_0$.

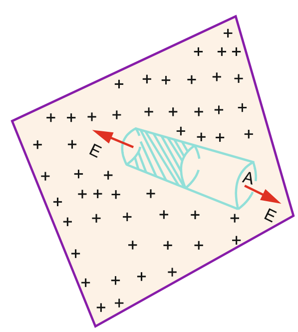


Fig. 11.41

For a non-conducting sheet the field

$$E = \frac{\sigma}{2\varepsilon_0} \tag{1}$$

The electric force acting on the sphere is

$$F = qE = \frac{q\sigma}{2\varepsilon_0} \tag{2}$$

(c) The sphere is held in equilibrium under the joint action of three forces:

(1) Weight acting vertically down,
(2) Electric force F acting horizontally, and
(3) Tension in the thread acting along the thread at an angle θ with the vertical.

From Fig. 11.42, $F/mg = \tan\theta$ (3)
Combining (2) and (3)

$$\sigma = \frac{2\varepsilon_0 mg\ \tan\theta}{q} = \frac{2 \times 8.9 \times 10^{-12} \times 2 \times 10^{-6} \times 9.8 \times \tan 10^\circ}{5 \times 10^{-8}}$$

$$= 2.15 \times 10^{-11}\ \mathrm{C/m^2}$$

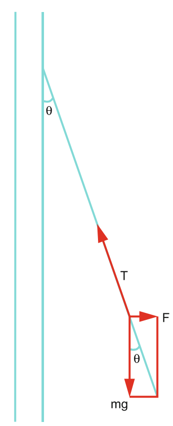


Fig. 11.42

11.50 **(a)** Integral form of Gauss' law:

$$\int \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = \frac{\Sigma Q}{\varepsilon_0}$$

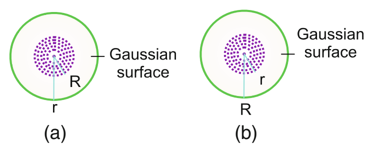
Differential form: $\nabla \cdot \boldsymbol{E} = \dfrac{\rho}{\varepsilon_0}$

(b) $r > R$. Construct a Gaussian surface in the form of a sphere of radius $r > R$, concentric with the charged sphere of radius R, Fig. 11.43a.
By Gauss' law

$$\int \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A} = (E)\, 4\pi\, r^2 = \frac{Q}{\varepsilon_0}$$

$$\therefore \quad E = \frac{Q}{4\pi\varepsilon_0 r^2} \tag{1}$$

Fig. 11.43



(c) $r < R$. Construct a Gaussian surface in the form of a sphere of radius $r < R$, concentric with the charged sphere of radius R, Fig. 11.43b. Let charge q' reside inside the Gaussian surface. Then by Gauss' law

$$\varepsilon_0 \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A} = (\varepsilon_0 E)\,(4\pi r^2) = q'$$

$$E = \frac{q'}{4\pi\varepsilon_0 r^2} \tag{2}$$

Now the charge outside the sphere of radius r does not contribute to the electric field. Assuming that ρ is constant throughout the charge distribution,

$$\frac{q'}{q} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}$$

$$or \qquad q' = q\,\frac{r^3}{R^3} \tag{3}$$

Using (3) in (2), $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{qr}{R^3}$ (4)

Thus for $r < R$, E varies linearly with r. Note that at $r = R$, both (1) and (4) give the same value as they should.

11.51 The electric field on the surface of the original sphere is

$$E_{\mathrm{R}} = \frac{q}{4\pi\varepsilon_0 R^2}$$

After creating the cavity, the charge in the remaining part of the sphere will be

$$q' = \left[\frac{\frac{4}{3}\pi R^3 \rho - \frac{4}{3}\pi \left(\frac{R}{2}\right)^3 \rho}{\frac{4}{3}\pi R^3 \rho}\right] q = \frac{7}{8} q$$

$\therefore$ The electric field on the surface is now

$$E'_{\mathrm{R}} = \frac{q'}{4\pi\varepsilon_0 R^2} = \frac{7}{8}\frac{q}{4\pi\varepsilon_0 R^2} = \frac{7}{8} E$$

11.52 **(a)** $E = \dfrac{q}{4\pi\varepsilon_0 r}\dfrac{r}{R^3}$

$$V(r) = -\int E \mathrm{d}r = -\frac{q}{4\pi\varepsilon_0 R^3}\int r\,\mathrm{d}r = -\frac{qr^2}{8\pi\varepsilon_0 R^3} + C$$

Now the potential at the surface ($r = R$) is

$$V(R) = \frac{q}{4\pi\varepsilon_0 R} = -\frac{q}{8\pi\varepsilon_0 R} + C$$

$$\therefore \quad C = \frac{3q}{8\pi\varepsilon_0 R}$$

$$\therefore \quad V(r) = \frac{q}{8\pi\varepsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)$$

(b) At the centre $r = 0$. From the result of ***(a)***

$$V(0) = \frac{3q}{8\pi\varepsilon_0 R} = \frac{3}{2} V(R)$$

11.53 **(a)** $E = 0$ when $r < b$ as no charge exists in this region.

(b) Region $b < r < a$: Consider a Gaussian surface, a sphere of radius r, where $b < r < a$. Charge at distance a between b and r only will contribute to the field. The charge residing in the shell of radii b and r is

$$Q' = \frac{\frac{4}{3}\pi\,(r^3 - b^3)Q}{\frac{4\pi}{3}\,(a^3 - b^3)} = \frac{(r^3 - b^3)}{(a^3 - b^3)}Q$$

$$E = \frac{Q'}{4\pi\varepsilon_0 r^2} = \frac{Q'}{4\pi\varepsilon_0 r^2}\,\frac{(r^3 - b^3)}{(a^3 - b^3)}$$

(c) Region $r > a$: $E = \dfrac{Q}{4\pi\varepsilon_0 r^2}$

11.54 **(a)** Construct a Gaussian surface in the form of a cylinder of radius $r > R$ and height h. By Gauss' law

$$(\varepsilon_0 E)\,(2\pi r h) = Q = \rho\pi\,R^2 h$$

$$\therefore \qquad E = \frac{1}{2}\frac{\rho}{\varepsilon_0}\frac{R^2}{r}$$

(b) Construct a Gaussian surface in the form of a cylinder of height h and radius $r < R$ coaxial with the cylinder of radius R and height h. By Gauss' law

$$(\varepsilon_0 E)\,(2\pi r h) = Q' = \rho\pi\,R^2 h$$

$$\therefore \qquad E = \frac{1}{2}\frac{\rho}{\varepsilon_0}\frac{R^2}{r} \quad (r < R)$$

11.55 **(a)** $E = \dfrac{q}{4\,\pi\varepsilon_0 r^2}$

$$\therefore \qquad q = 4\,\pi\varepsilon_0 r^2 E = \frac{1}{9\times 10^9}\times(0.5)^2\times 800 = 2.22\times 10^{-8}\,\text{C}$$

(b) $E = \dfrac{\sigma}{\varepsilon_0}$

$$\therefore \qquad \sigma = E\varepsilon_0 = 120\times 8.85\times 10^{-12} = 1.062\times 10^{-9}\,\text{C/m}^2$$

(c) $q = \varepsilon_0\phi$

$$\phi = EA = 120\times(100\times 75) = 9\times 10^5\,\text{N m}^2/\text{C}$$

11.56 **(a)** $E_0 \oint \boldsymbol{E}\cdot \text{d}\boldsymbol{A} = q \qquad \text{(Gauss' law)} \qquad (1)$

Consider an isolated positive point charge. Construct a Gaussian surface, a sphere of radius r centred at the charge. At every point on the spherical surface, the field is perpendicular to the surface. As both $\boldsymbol{E}$ and d$\boldsymbol{A}$ are directed radially outwards, the angle θ between them is zero so that the dot product $\boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A}$ reduces to $E\mathrm{d}A$ and (1) can be written as

$$\varepsilon_0 \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A} = \varepsilon_0 \int E\mathrm{d}A = q \tag{2}$$

As E is constant, it can be factored out of the integral:

$$\varepsilon_0 E \int \mathrm{d}A = (\varepsilon_0 E)\,(4\pi\, r^2) = q$$

$$\therefore \quad E = \frac{q}{4\pi\varepsilon_0 r^2} \qquad \text{(Coulomb's law)} \tag{3}$$

(b) **(i)** $r < R$

By prob. (11.50), $E = \dfrac{Qr}{4\pi\varepsilon_0 R^3}$

$$\phi = EA = \frac{Q\,r}{4\pi\varepsilon_0 R^3} 4\pi\, r^2 = \frac{Q\,r^3}{\varepsilon_0\, R^3}$$

(ii) $r > R$

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

$$\phi = EA = \frac{Q}{4\pi\varepsilon_0 r^2} 4\pi r^2 = \frac{Q}{\varepsilon_0}$$

11.57 $\phi = \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A}, \qquad q = \varepsilon_0 A$

(a) Construct a Gaussian surface, a sphere of radius r_1 concentric with the spherical shells. Since no charge is enclosed by the Gaussian surface with $r < R_1$, $E = 0$.

(b) Here the net charge enclosed by the Gaussian surface is Q_1. As $\boldsymbol{E}$ is normal to the spherical surface by Gauss' law

$$(\varepsilon_0 E)\,(4\pi\, r^2) = Q_1 \quad \text{or} \quad E = \frac{Q_1}{4\pi\varepsilon_0 r^2}$$

(c) Here the net charge enclosed by the Gaussian surface is $Q_1 + Q_2$ and $\boldsymbol{E}$ is normal to the spherical surface (Fig. 11.44). By Gauss' law

$$(\varepsilon_0 E)(4\pi r^2) = Q_1 + Q_2 \quad \text{or} \quad E = \frac{Q_1 + Q_2}{4\pi\varepsilon_0 r^2}$$

In (c) if $Q_1 + Q_2 = 0$, then $Q_1/Q_2 = -1$.

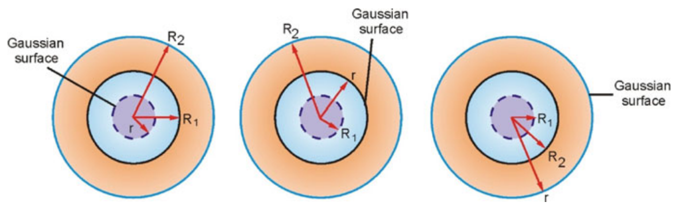


Fig. 11.44

11.58 Let the charges be $q_1 = q_2$ on the two spheres before contact.

$$\text{Force } F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} = \frac{q_1^2}{4\pi\varepsilon_0 r^2} \tag{1}$$

When the spheres are brought into contact and separated they are at a common potential. Let the new charges be q_1' and q_2', respectively

$$q_1' + q_2' = q_1 + q_2 = 2q_1 \qquad \text{(from charge conservation)} \tag{2}$$

$$\text{Also} \quad V = \frac{q_1'}{4\pi\varepsilon_0 R_1} = \frac{q_2'}{4\pi\varepsilon_0 R_2} \tag{3}$$

$$\text{or} \quad \frac{q_1'}{q_2'} = \frac{R_1}{R_2} \tag{4}$$

Solving (2) and (4)

$$q_1' = \frac{2q_1 R_1}{R_1 + R_2}, \quad q_2' = \frac{2q_1 R_2}{R_1 + R_2} \tag{5}$$

$$\text{After separation, force } F' = \frac{q_1' q_2'}{4\pi\varepsilon_0 r^2} \tag{6}$$

$$\frac{F}{F'} = \frac{q_1^2}{q_1' q_2'} = \frac{(r_1 + r_2)^2}{4r_1 r_2} = \frac{(1+3)^2}{4 \times 1 \times 3} = \frac{4}{3} \tag{7}$$

11.59 $V = \dfrac{q}{4\pi\varepsilon_0 r}$

$$\therefore \quad q = 4\pi\varepsilon_0 r\, V = \frac{(5)(2 \times 10^4)}{9 \times 10^9} = 1.11 \times 10^{-5}\,\text{C}$$

11.60 **(a)** $V = \dfrac{q}{4\pi\varepsilon_0 a}$

$$\therefore \text{ Capacitance } C = \frac{q}{V} = 4\pi\varepsilon_0 a$$

(b) $q_1 = 4\pi\varepsilon_0 a V_1, \quad q_2 = 4\pi\varepsilon_0 a V_2$

$$U_1 = \frac{1}{2} q_1 V_1 = \frac{1}{2} \frac{q_1^2}{4\pi\varepsilon_0 a}, U_2 = \frac{1}{2} \frac{q_2^2}{4\pi\varepsilon_0 a}$$

Initially total energy $U = U_1 + U_2 = \dfrac{\left(q_1^2 + q_2^2\right)}{8\pi\varepsilon_0 a}$ (1)

When the spheres are connected they reach a common potential and when disconnected let the charges be q_1' and q_2'.

$$q_1' = 4\pi\varepsilon_0 a V, \quad q_2' = 4\pi\varepsilon_0 a V$$

$$\therefore \quad q_2' = q_1'$$

Final energy in the spheres

$$U_1' = \frac{1}{2} q_1' V = \frac{q_1'^2}{8\pi\varepsilon_0 a}, \; U_2' = \frac{q_2'^2}{8\pi\varepsilon_0 a}$$

Total final energy $U' = U_1' + U_2' = \dfrac{q_1'^2 + q_2'^2}{8\pi\varepsilon_0 a}$ (2)

$q_1 + q_2 = q_1' + q_2' = 2q_1'$ (charge conservation) (3)

$$\Delta U = U_1 - U_2 = \frac{1}{8\pi\varepsilon_0 a} \left(q_1^2 + q_2^2 - 2q_1'^2\right) \tag{4}$$

Eliminating q_1' between (3) and (4) and simplifying

$$\Delta U = \frac{1}{16\pi\varepsilon_0 a} (q_1 - q_2)^2 \tag{5}$$

This energy is dissipated in Joule heating of the wire.

11.61 When the spheres are brought into contact they reach a common potential, say V. If the charges on them are now Q_1' and Q_2'

$$V = \frac{Q_1'}{4\pi\varepsilon_0 R_1} = \frac{Q_2'}{4\pi\varepsilon_0 R_2} \tag{1}$$

$$\sigma_1' = \frac{Q_1'}{4\pi R_1^2} = \frac{\varepsilon_0 V}{R_1} \tag{2}$$

where we have used (1).

Similarly

$$\sigma_2' = \frac{\varepsilon_0 V}{R_2} \tag{3}$$

Thus $\sigma' \,\alpha\, \dfrac{1}{R}$.

11.62 **(a)** $V = Er = 5 \times 10^6 \times 0.1 = 5 \times 10^5\,\text{V}$

(b) Pressure $p = \dfrac{1}{2}\dfrac{\sigma^2}{\varepsilon_0} = \dfrac{1}{2\varepsilon_0}(E\varepsilon_0)^2 = \dfrac{1}{2}\varepsilon_0 E^2$

$$= \frac{1}{2} \times 8.85 \times 10^{-12} \times (5 \times 10^6)^2 = 110.6\,\text{N/m}^2$$

(c) If q is the charge and C the capacitance then the electrostatic energy

$$U = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}\frac{q^2}{4\pi\varepsilon_0 r}$$

Now $q = 4\pi\varepsilon_0 r^2 E = \dfrac{(0.1)^2 \times 5 \times 10^6}{9 \times 10^9} = 5.55 \times 10^{-6}\,\text{C}$

$$\therefore \qquad U = \frac{1}{2} \times 9 \times \frac{10^9}{0.1} \times (5.55 \times 10^{-6})^2 = 1.386\,\text{J}$$

11.63 Work done in the isobaric expansion (constant pressure) is

$$W = P\,\Delta\upsilon = \frac{4\pi}{3}\left(R_2^3 - R_1^3\right) p \tag{1}$$

where we have written υ for volume.
Increase in electrostatic energy

$$\Delta U = \frac{1}{2}\frac{q^2}{C_1} - \frac{1}{2}\frac{q^2}{C_2} = \frac{q^2}{2}\,\frac{1}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{2}$$

where C is the capacitance of the spherical bubble. Equating (1) and (2) and simplifying, we obtain

$$q = \left[\frac{32}{3}\pi^2\varepsilon_0 p\, R_1 R_2 \left(R_1^2 + R_1 R_2 + R_2^2\right)\right]^{1/2} \tag{3}$$

11.64 $$q = \int_{r_1}^{r_2} \rho(r)\,\mathrm{d}\upsilon = \int \frac{A}{r} \cdot 4\pi r^2 \mathrm{d}r = 2\pi\, A\left(r_2^2 - r_1^2\right) \tag{1}$$

Region (i), $0 \le r < r_1$, $\rho(r) = 0$

As no charge is enclosed, $E = 0$ (2)

Region (ii), $r_1 \le r \le r_2$, $\rho(r) = \dfrac{A}{r}$

By Gauss' law

$$\varepsilon_0 \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \int_{r_1}^{r} \rho(r)\mathrm{d}\upsilon = \int_{r_1}^{r} \frac{A}{r} 4\pi r^2 \mathrm{d}r = 2\pi\, A\left(r^2 - r_1^2\right)$$

$$\therefore \quad (\varepsilon_0 E)(4\pi r^2) = 2\pi\, A\left(r_2^2 - r_1^2\right)$$

$$\therefore \quad E = \frac{A}{2\varepsilon_0 r^2}\left(r^2 - r_1^2\right) \tag{3}$$

Region (iii), $r > r_2$, $\rho(r) = 0$

$$\varepsilon_0 \int \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = q = 2\pi\, A\left(r_2^2 - r_1^2\right) \text{ by (1)}$$

$$\therefore \quad (\varepsilon_0 E)(4\pi r^2) = 2\pi\, A\left(r_2^2 - r_1^2\right)$$

$$\therefore \quad E = \frac{A}{2\varepsilon_0 r^2}\left(r_2 - r_1^2\right) \tag{4}$$

11.65 **(a)** In order to show that the electric field is conservative, it is sufficient to establish the existence of a potential. Now, if potential V exists, it must be such that

$$\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = -\mathrm{d}V$$

where $\boldsymbol{F} = f(r)\boldsymbol{e}_r$ is the central force and $\boldsymbol{e}_r$ is the unit vector along the radius vector $\boldsymbol{r}$

$$\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = f(r)\boldsymbol{e}_r \cdot \mathrm{d}\boldsymbol{r} = f(r)\,\mathrm{d}r$$

$$\therefore \quad -\mathrm{d}V = f(r)\,\mathrm{d}r$$

$$\text{or} \quad V = -\int f(r)\mathrm{d}r$$

We conclude that the field is conservative and V represents the potential given by the above relation.

(b) Initial electrostatic energy

$$U_1 = \frac{1}{2}qV = \frac{1}{2}CV^2 = \frac{1}{2}4\pi\varepsilon_0 r_1 V^2$$
$$= \frac{1}{2} \times \frac{1}{9 \times 10^9} \times 0.01 \times 10^2 = 5.55 \times 10^{-11}\,\text{J}$$

Final electrostatic energy

$$U_2 = \frac{1}{2}4\pi\varepsilon_0 r_2 V^2 = \frac{1}{2} \times \frac{1}{9 \times 10^9} \times 0.001 \times 10^2$$
$$= 5.55 \times 10^{-12}\,\text{J}$$
$$\therefore \quad \text{Energy decrease} = U_1 - U_2 = (5.55 - 0.555) \times 10^{-11} = 5 \times 10^{-11}\,\text{J}$$

11.66 Construct a Gaussian cylindrical surface of radius r and length L coaxial with the cylinder.

(i) $r > a$. The charge enclosed is $q = \pi a^2 L\rho$. By Gauss' law

$$\varepsilon_0 \oint E \cdot \mathrm{d}A = q$$
$$\therefore \quad \varepsilon_0 E\,(2\pi r L) = \pi a^2 L\rho$$
$$\therefore \quad E = \frac{a^2\rho}{2\varepsilon_0 r}$$

(ii) $0 < r < a$. By Gauss' law

$$\varepsilon_0 \oint E \cdot \mathrm{d}A = q'$$
$$(\varepsilon_0 E)(2\pi r L) = \pi(a^2 - r^2)L\rho$$
$$\therefore \quad E = \frac{(a^2 - r^2)\rho}{2\varepsilon_0 r}$$

Centripetal force = Electric force

$$\frac{mv^2}{r} = Ee = \frac{a^2\rho e}{2\varepsilon_0 r}$$
$$\therefore \quad v = \sqrt{\frac{a^2\rho e}{2\varepsilon_0 m}}$$

11.67 For an infinite non-conducting charge sheet $E = \sigma/2\varepsilon_0$

$$\therefore \qquad \sigma = 2E\varepsilon_0 = 2 \times 200 \times 8.85 \times 10^{-12} = 3.54 \times 10^{-9}\,\mathrm{C/m^2}$$

The electric field is independent of the distance.
For an infinite conducting sheet

$$\sigma = E\varepsilon_0 = 200 \times 8.85 \times 10^{-12} = 1.77 \times 10^{-9}\,\mathrm{C/m^2}$$

11.3.3 Capacitors

11.68 The field strength E_0 between the plates of a parallel plate capacitor in vacuum is

$$E_0 = \frac{V}{d} = \frac{\sigma}{\varepsilon_0} \tag{1}$$

where V is the applied voltage, σ the charge density (charge per unit area) and ε_0 the permittivity in vacuum. Now $\sigma = q/A$. Therefore the capacitance in air or vacuum will be

$$C_0 = \frac{q}{V} = \frac{A\varepsilon_0}{d} \tag{2}$$

With the introduction of the dielectric slab the electric field in the slab will be E_0/K, and the potential across the capacitor becomes

$$V = E_0(d - K) + \frac{E_0 t}{K} = E_0\left[(d - t) + \frac{t}{K}\right]$$

$$= \frac{q}{A\varepsilon_0}\left[(d - t) + \frac{t}{K}\right]$$

$$\therefore \quad C = \frac{q}{V} = \frac{\varepsilon_0 A}{d - t\left(1 - \dfrac{1}{K}\right)} \tag{3}$$

If a metal of thickness t is to be introduced, the effective distance between the capacitor plates is reduced and the capacitance becomes

$$C = \frac{\varepsilon_0 A}{d - t} \tag{4}$$

a result which is obtained by putting $K = \infty$ in (3).

11.69 $C_1 + C_2 = 9 \qquad \text{(parallel)} \qquad (1)$

$$\frac{C_1 C_2}{C_1 + C_2} = 2 \qquad \text{(series)} \qquad (2)$$

$$(C_1 - C_2)^2 = (C_1 + C_2)^2 - 4C_1C_2 = (C_1 + C_2)^2 - 8(C_1 + C_2) = 9$$

where we have used (1)

$$\therefore \qquad C_1 - C_2 = 3 \qquad (3)$$

Solving (1) and (3), $C_1 = 6\,\mu\text{F}$ and $C_2 = 3\,\mu\text{F}$.

11.70 **(a)** $U_1 = \frac{1}{2}C_1V^2 = \frac{1}{2} \times 2 \times 10^{-6} \times (100)^2 = 0.01\,\text{J}$

$$U_2 = \frac{1}{2}C_2V^2 = \frac{1}{2} \times 4 \times 10^{-6} \times (100)^2 = 0.02\,\text{J}$$

(b) $C = \frac{C_1C_2}{C_1 + C_2} = \frac{2 \times 10^{-6} \times 4 \times 10^{-6}}{(2+4) \times 10^{-6}} = \frac{4}{3} \times 10^{-6}\,\text{F}$

$$U = \frac{1}{2}CV^2 = \frac{1}{2} \times \frac{4}{3} \times 10^{-6} \times (100)^2 = 0.0067\,\text{F}$$

(c) $C = C_1 + C_2 = (2+4) \times 10^{-6} = 6 \times 10^{-6}\,\text{F}$

$$U = \frac{1}{2}CV^2 = \frac{1}{2} \times 6 \times 10^{-6} \times (100)^2 = 0.03\,\text{J}$$

11.71 $C_0 = \frac{\varepsilon_0 A}{d} = \frac{8.85 \times 10^{-12} \times 1}{0.01} = 8.85 \times 10^{-10}\,\text{F}$

$$U_0 = \frac{1}{2}\frac{Q^2}{C_0} = \frac{1}{2} \times \frac{(10^{-6})^2}{8.85 \times 10^{-10}} = 5.65 \times 10^{-4}\,\text{J}$$

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{Q^2}{C_0K} = \frac{U_0}{K} = \frac{5.65 \times 10^{-4}}{2} = 2.83 \times 10^{-4}\,\text{J}$$

The energy is decreased by $\Delta U = U_0 - U = (5.65 - 2.83) \times 10^{-4} = 2.82 \times 10^{-4}\,\text{J}$, that is, by a factor of 2.

11.72 $C = \frac{\varepsilon_0 K A}{d}$

$$\therefore \quad K = \frac{Cd}{\varepsilon_0 A} = \frac{0.1 \times 10^{-6} \times 0.001}{8.85 \times 10^{-12} \times 1.0} = 11.3$$

11.73 $Q = C_1V = 5 \times 10^{-6} \times 250 = 1.25 \times 10^{-3}\,\text{C}$
For the parallel connection

$$C = C_1 + C_2 = 5 \times 10^{-6} + 20 \times 10^{-6} = 25 \times 10^{-6}\,\text{F}$$

The resulting voltage

$$V' = \frac{Q}{C} = \frac{1.25 \times 10^{-3}}{25 \times 10^{-6}} = 50\,\text{V}$$

11.74 The combination of $2\,\mu\text{F}$ and $2\,\mu\text{F}$ in parallel is equivalent to $4\,\mu\text{F}$. This in series with $8\,\mu\text{F}$ gives a combined capacitance of $8/3\,\mu\text{F}$; $12\,\mu\text{F}$ and $6\,\mu\text{F}$ in series gives an equivalent capacitance of $4\,\mu\text{F}$. $4\,\mu\text{F}$ with $4\,\mu\text{F}$ in parallel gives $8\,\mu\text{F}$ which in series with $1\,\mu\text{F}$ yields $8/9\,\mu\text{F}$.

Combination of $8/9\,\mu\text{F}$ and $8/3\,\mu\text{F}$ in parallel gives $32/9\,\mu\text{F}$.

Effective value of C with $32/9\,\mu\text{F}$ in series gives

$$\frac{\frac{32}{9}C}{C + \frac{32}{9}} = 1, \quad \text{by problem}$$

$$\therefore \quad C = 1\frac{9}{23}\,\mu\text{F}$$

11.75 Combined capacitance C for the three capacitors in series:

$$C = \frac{C_1C_2C_3}{C_1C_2 + C_2C_3 + C_3C_1} = \frac{(4 \times 3 \times 2) \times 10^{-18}}{(4 \times 3 + 3 \times 2 + 2 \times 4) \times 10^{-12}}$$
$$= 0.923 \times 10^{-6}\,\text{F}$$

(a)
$$q = CV = 0.923 \times 10^{-6} \times 260 = 240 \times 10^{-6}\,\text{C}$$
$$\therefore \quad q_1 = q_2 = q_3 = 240 \times 10^{-6}\,\text{C}$$

(b) $V_1 = \dfrac{q}{C_1} = \dfrac{240 \times 10^{-6}}{4 \times 10^{-6}} = 60\,\text{V}$

$$V_2 = \frac{q}{C_2} = \frac{240 \times 10^{-6}}{3 \times 10^{-6}} = 80\,\text{V}$$

$$V_3 = \frac{q}{C_3} = \frac{240}{2} = 120\,\text{V}$$

(c) $W_1 = \dfrac{1}{2}C_1V_1^2 = \dfrac{1}{2} \times 4 \times 10^{-6} \times (60)^2 = 0.0072\,\text{J}$

$$W_2 = \frac{1}{2}C_2V_2^2 = \frac{1}{2} \times 3 \times 10^{-6} \times (80)^2 = 0.0096\,\text{J}$$

$$W_3 = \frac{1}{2}C_3V_3^2 = \frac{1}{2} \times 2 \times 10^{-6} \times (120)^2 = 0.0144\,\text{J}$$

11.76 Charge on the first capacitor

$$q_1 = C_1 V = 1 \times 10^{-6} \times 12 = 12 \times 10^{-6}\,\mathrm{C}$$

Charge on the second capacitor

$$q_2 = C_2 V = 2 \times 10^{-6} \times 12 = 24 \times 10^{-6}\,\mathrm{C}$$

Capacitance for the parallel combination

(a) $C = C_1 + C_2 = (1 + 2) \times 10^{-6} = 3 \times 10^{-6}\,\mathrm{F}$

$$q = q_1 + q_2 = (12 + 24) \times 10^{-6} = 36 \times 10^{-6}\,\mathrm{C}$$

$$V = \frac{q}{C} = \frac{36 \times 10^{-6}}{3 \times 10^{-6}} = 12\,\mathrm{V}$$

(b) $q' = q_2 - q_1 = (24 - 12) \times 10^{-6} = 12 \times 10^{-6}\,\mathrm{C}$

$$V = \frac{q'}{C} = \frac{12 \times 10^{-6}}{3 \times 10^{-6}} = 4\,\mathrm{V}$$

11.77 **(a)** If the positive end of a capacitor of capacitance C_1, charged to potential difference V_1, is connected in parallel to the positive end of the capacitor of capacitance C_2 charged to potential difference V_2, then conservation of charge gives the equation

$$(C_1 + C_2)V = C_1 V_1 + C_2 V_2 \tag{1}$$

$$\therefore \quad V = \frac{C_1 V_1 + C_2 V_2}{C_1 + C_2} \quad \text{(common potential)} \tag{2}$$

The energy loss

$$\begin{aligned} \Delta W &= \frac{1}{2} C_1 V_1^2 + \frac{1}{2} C_2 V_2^2 - \frac{1}{2}(C_1 + C_2)\,V^2 \\ &= \frac{1}{2}\,\frac{C_1 C_2}{C_1 + C_2}\,(V_1 - V_2)^2 \end{aligned} \tag{3}$$

where we have used (2).

(b) If the positive end is joined to the negative end, the common potential difference will be

$$V = \frac{C_1 V_1 - C_2 V_2}{C_1 + C_2} \tag{4}$$

and the energy loss will be

$$\Delta W = \frac{1}{2}\,\frac{C_1 C_2}{C_1 + C_2}(V_1 + V_2)^2 \tag{5}$$

11.78 **(a)** Battery remains connected

(i) $V' = V$; potential remains unchanged.
(ii) $E' = E$; electric field is unchanged.
(iii) $q' = Kq$; charge is increased by a factor K. The additional charge $(K - 1)q$ is moved from the negative to the positive plate by the battery, as the dielectric slab is inserted.
(iv) $C' = KC$; capacitance is increased by a factor K.
(v) $U' = \frac{1}{2}q'V' = \frac{1}{2}KqV = KU$

Energy is increased by a factor K.

(b) The battery is disconnected

(i) $V' = \frac{V}{K}$; potential is decreased by a factor K
(ii) $E' = \frac{E}{K}$; electric field is decreased by a factor K. Both (i) and (ii) follow from the fact that $q' = q$ so that $C'V' = CV$ and $V' = \frac{CV}{C'} = \frac{V}{K}$. Same reasoning holds good for E'.
(iii) $q' = q$; charge remains unchanged as there is no path for charge transfer.
(iv) $C' = KC$; capacitance is increased by a factor K.
(v) $U' = \frac{1}{2}q'V' = \frac{1}{2}\frac{qV}{K} = \frac{U}{K}$

The energy is lowered by a factor K.

11.79 **(a)** The battery remains connected

(i) $V' = V$; potential remains unchanged.
(ii) $E' < E$; the electric field is decreased since $E = V/d$, and V is constant.
(iii) $C' < C$; capacitance is reduced since $C \propto 1/d$.
(iv) $q' < q$; the charge is reduced since $q = CV$, with C decreasing and V remaining constant. Some charge is transferred from the capacitor to the charging battery.
(v) $U' < U$; the energy is decreased since $U = \frac{1}{2}qV$, with q decreasing and V remaining constant.

(b) Battery is disconnected

(i) $V' > V$, the potential increases because $q = CV$, with C decreasing and q remaining constant.
(ii) $E' = E$, the electric field is constant because $q = CV = \frac{\varepsilon_0 AV}{d} = \varepsilon_0 AE$, with q remaining constant.
(iii) $C' < C$; the capacitance is decreased since $C \propto 1/d$.

(iv) $q' = q$; the charge remains constant.

(v) $U' > U$; energy increases because $U = \frac{1}{2} qV$, with V increasing and q remaining constant.

11.80 As the plates carry equal but opposite charges, the force of attraction, which is conservative, is given by

$$F = -\frac{\mathrm{d}U}{\mathrm{d}x}$$

But $U = \frac{1}{2}\frac{q^2}{C}$

For the parallel plate capacitor,

$$C = \frac{\varepsilon_0 A}{x}$$

where x is the distance of separation. Combining the above equations

$$F = -\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\frac{q^2 x}{\varepsilon_0 A}\right] = -\frac{q^2}{2\varepsilon_0 A} = -\frac{1}{2}\frac{\varepsilon_0 A V^2}{d^2}$$

where we have put $x = d$.

11.81 **(a)** As the drops are assumed to be incompressible, the volume does not change.

$$\frac{4}{3}\pi R^3 = n\frac{4}{3}\pi r^3$$
$$\therefore \quad R = n^{1/3} r$$

(b) $C' = 4\pi\varepsilon_0 R = 4\pi\varepsilon_0 r\, n^{1/3}$

$\therefore \quad C' = n^{1/3} C$

(c) $Q = nq$ (charge conservation)

$$V' = \frac{Q}{4\pi\varepsilon_0 R} = \frac{nq}{4\pi\varepsilon_0 n^{1/3} r} = \frac{n^{2/3} q}{4\pi\varepsilon_0 r} = n^{2/3}\, V$$

(d) $\sigma' = \frac{Q}{4\pi R^2} = \frac{nq}{4\pi r^2 n^{2/3}} = n^{1/3}\,\sigma$

(e) $U' = \frac{1}{2} QV' = \frac{1}{2} nqn^{2/3} V = n^{5/3} U$

11.82 The electric field for a cylindrical capacitor is

$$E = \frac{q}{2\pi\varepsilon_0 l r} \tag{1}$$

where l is the length and r the radius. The energy density (energy/unit volume)

$$u = \frac{1}{2}\,\varepsilon_0\,E^2 = \frac{q^2}{8\pi^2\varepsilon_0 l^2 r^2} \tag{2}$$

where we have used (1).
The energy stored between the coaxial cylinders of length l and radii R and a is

$$U = \int u\mathrm{d}v = \int_a^R u\,(2\pi r l)\,\mathrm{d}r \tag{3}$$

where $\mathrm{d}v = (2\pi r\mathrm{d}r)l$ is the volume element. Using (2) in (3)

$$U = \frac{q^2}{4\pi\varepsilon_0 l}\int_a^R \frac{\mathrm{d}r}{r} = \frac{q^2}{4\pi\varepsilon_0 l}\,\ln\,\frac{R}{a}$$

Similarly, the energy stored between the coaxial cylinders of radii b and a is

$$U_0 = \frac{q^2}{4\pi\varepsilon_0 l}\,\ln\,\frac{b}{a}$$

$$\therefore\quad \frac{U}{U_0} = \frac{\ln(R/a)}{\ln(b/a)}$$

Set $\quad \dfrac{U}{U_0} = \dfrac{1}{2}$

$$\therefore\quad \frac{\ln(R/a)}{\ln(b/a)} = \frac{1}{2} \rightarrow \ln\,\frac{b}{a} = 2\,\ln\,\frac{R}{a} = \ln\,\frac{R^2}{a^2}$$

or $\quad \dfrac{b}{a} = \dfrac{R^2}{a^2} \rightarrow R = \sqrt{ab}$

11.83 The charge on C_1 is

$$q_1 = C_1 V_1 = 3\times 10^{-6}\times 4000 = 0.012\,\mathrm{C}$$

The charge on C_2 is

$$q_2 = C_2 V_2 = 6 \times 10^{-6} \times 3000 = 0.018\,\mathrm{C}$$

The combined capacitance

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{3 \times 6}{3 + 6} \times 10^{-6} = 2 \times 10^{-6}\,\mathrm{F}$$

Take the lower charge to find the maximum voltage V:

$$V = \frac{q_1}{C} = \frac{0.012}{2 \times 10^{-6}} = 6000\,\mathrm{V}$$

11.84 If the dielectric is present, Gauss' law gives

$$\varepsilon_0 \oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = q - q' = \frac{q}{K} \tag{1}$$

where $-q'$ is the induced surface charge, q is the free charge and K is the dielectric constant. Construct a Gaussian surface in the form of a coaxial cylinder of radius r and length l, closed by end caps. Applying (1),

$$\varepsilon_0 E\,(2\pi r l) = \frac{q}{K}$$

$$\text{or} \quad E = \frac{q}{2\pi\varepsilon_0 r l k} \tag{2}$$

In (1) the integral is contributed only by the curved surface and not the end caps. The potential difference between the central rod and the surrounding tube is given by

$$V = -\int_a^b \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} = \int_a^b E\,\mathrm{d}r = \int_a^b \frac{q}{2\pi\varepsilon_0 l K}\,\frac{\mathrm{d}r}{r} = \frac{q}{2\pi\varepsilon_0 l K}\,\ln\frac{b}{a}$$

The capacitance is given by

$$C = \frac{q}{V} = \frac{2\pi\varepsilon_0 l K}{\ln(b/a)}$$

11.85 The field at point P is caused entirely by the charge Q on the inner sphere, Fig. 11.45, and has the value

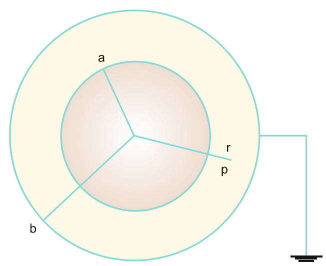
$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

The potential difference between the two spheres is given by

$$V = -\int_b^a \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} = -\frac{Q}{4\pi\varepsilon_0}\int_b^a \frac{\mathrm{d}r}{r^2} = \frac{Q\,(b-a)}{4\pi\varepsilon_0 ab}$$

whence $C = \dfrac{Q}{V} = \dfrac{4\pi\varepsilon_0 ab}{b-a}$

Fig. 11.45



11.86 By prob. (11.85)

$$C = \frac{4\pi\varepsilon_0 ab}{b-a} \tag{1}$$

Let $b = a + \Delta$ where Δ is a small quantity. Then (1) can be written as

$$C = \frac{4\pi\varepsilon_0 a(a + \Delta\, a)}{\Delta a} \simeq \frac{4\pi\varepsilon_0 a^2}{\Delta a} \tag{2}$$

Now the surface area $A = 4\pi a^2$ and $\Delta a = d$, the distance between the surfaces, so that

$$C \simeq \frac{\varepsilon_0 A}{d} \qquad \text{(parallel plate capacitor)}$$

11.87 **(i)** $q = q_0\, \mathrm{e}^{-t/RC}$

$$\mathrm{e}^{-t/RC} = \frac{q}{q_0} = \frac{90}{100}$$

$$\therefore \qquad t = RC \ln\frac{10}{9} = 1 \times 10^6 \times 10 \times 10^{-6} \times 0.1056 = 1.056\,\mathrm{s}$$

(ii) At one time constant $t = RC$

$$V = V_0\,e^{-1} = \frac{120}{2.718} = 44.15\,\mathrm{V}$$
$$U = \frac{1}{2}\,CV^2 = \frac{1}{2} \times 10 \times 10^{-6} \times (44.15)^2 = 0.097\,\mathrm{J}$$

(iii) $H = i^2\,Rt = \dfrac{V^2}{R}t = V^2C = (44.15)^2 \times 10 \times 10^{-6} = 0.0195\,\mathrm{J}$

11.88 Equilibrium energy $U_0 = \dfrac{1}{2}CV_0^2$
Energy at time t

$$U = \frac{1}{2}\,CV^2 = \frac{1}{2}\,CV_0^2\,\left(1 - e^{-t/RC}\right)^2 = U_0\left(1 - e^{-t/RC}\right)^2$$
$$\frac{U}{U_0} = \frac{1}{2} = \left(1 - e^{-t/RC}\right)^2$$

Solving $t = 1.228RC$.
Thus after 1.228 time constants the energy stored in the capacitor will reach half of its equilibrium value.

11.89 Let the capacitor be divided into differential strips which are practically parallel. Consider a strip at distance x of length a perpendicular to the plane of paper and of width dx in the plane of paper, the area of the strip being $\mathrm{d}A = a\mathrm{d}x$, Fig. 11.46. At the distance x, the separation of the plates is seen to be $t = D + x\theta$. The capacitance due to the differential strip facing each plate is

$$\mathrm{d}C = \frac{\varepsilon_0 \mathrm{d}A}{D} = \frac{\varepsilon_0 a \mathrm{d}x}{D + x\,\theta}$$

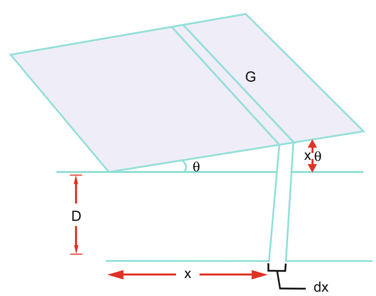


Fig. 11.46

The capacitance is given by

$$C = \int \mathrm{d}C = \int_0^a \frac{\varepsilon_0 a\,\mathrm{d}x}{D + x\,\theta} = \varepsilon_0 a \int_0^a \frac{\mathrm{d}x}{D + x\,\theta} = \frac{\varepsilon_0 a}{D} \int_0^a \left(1 + \frac{x\,\theta}{D}\right)^{-1} \mathrm{d}x$$

$$= \frac{\varepsilon_0 a}{D} \int_0^a \left(1 - \frac{x\theta}{D} + \cdots\right) \mathrm{d}x \simeq \frac{\varepsilon_0 a}{D} \left(x - \frac{x^2\theta}{2D}\right)\Bigg|_0^a = \frac{\varepsilon_0 a^2}{D} \left(1 - \frac{a\theta}{2D}\right)$$

Note that for $\theta = 0$, capacitance reduces to that for the parallel plate capacitor.

11.90 **(a)** The equivalent capacitance of C_1 and C_2 in parallel is $C_{12} = 8 + 4 = 12\,\mu\mathrm{F}$.
The combined capacitance of C_{12} and C_3 in series is $C = \frac{C_3 C_{12}}{C_3 + C_{12}} = \frac{3 \times 12}{3 + 12} = 2.4\,\mu\mathrm{F}$.
Applied charge $q = CV = 2.4 \times 100 = 240\,\mu\mathrm{C}$. Therefore charge on C_3 will be $q_3 = 240\,\mu\mathrm{C}$. PD across C_3 will be $V_3 = \frac{q_3}{C_3} = \frac{240}{3} = 80\,\mathrm{V}$. The PD across C_1 and C_2 will be equal.

$$V_1 = V_2 = (V - V_3) = (100 - 80) = 20\,\mathrm{V}$$

(b) Now $\frac{q_1}{C_1} = \frac{q_2}{C_2} \qquad (\because\ V_1 = V_2)$

$$\therefore \qquad q_1 = \frac{C_1 q_2}{C_2} = \frac{8}{4} q_2 = 2q_2$$

Also $q_1 + q_2 = 240$

$$\therefore \qquad q_1 = 160\,\mu\mathrm{C} \text{ and } q_2 = 80\,\mu\mathrm{C}$$

(c) $U_1 = \frac{1}{2} C_1 V_1^2 = \frac{1}{2} \times 8 \times 10^{-6} \times 20^2 = 0.0016\,\mathrm{J}$

$$U_2 = \frac{1}{2} C_2 V_2^2 = \frac{1}{2} \times 4 \times 10^{-6} \times 20^2 = 0.0008\,\mathrm{J}$$

$$U_3 = \frac{1}{2}C_3V_3^2 = \frac{1}{2} \times 3 \times 10^{-6} \times 80^2 = 0.0096\,\text{J}$$

Note that $U_1 + U_2 + U_3 = 0.012\,\text{J} = U = \frac{1}{2}CV^2$

11.91 **(a)** The combination of C_1 and C_2 in series yields $C_{12} = \frac{8 \times 4}{8 + 4} = 2.667 \times 10^{-6}\,\text{F}$.

$$C = C_{12} + C_3 = (2.667 + 3.0) \times 10^{-6} = 5.667 \times 10^{-6}\,\text{F}$$

$$q = CV = 5.667 \times 10^{-6} \times 100 = 5.667 \times 10^{-4}\,\text{C}$$

$$q_3 = C_3V = 3 \times 10^{-6} \times 100 = 3 \times 10^{-4}\,\text{C}$$

$$q_1 = q_2 = q - q_3 = (5.667 - 3.0) \times 10^{-4} = 2.667 \times 10^{-4}\,\text{C}$$

(b) $V_3 = 100\,\text{V}$

$$V_1 = \frac{q_1}{C_1} = \frac{2.667 \times 10^{-4}}{8 \times 10^{-6}} = 33.33\,\text{V}$$

$$V_2 = \frac{q_2}{C_2} = \frac{2.667 \times 10^{-4}}{4 \times 10^{-6}} = 66.66\,\text{V}$$

(c) $U_1 = \frac{1}{2}C_1V_1^2 = \frac{1}{2} \times 8 \times 10^{-6} \times (33.33)^2 = 0.00444\,\text{J}$

$$U_2 = \frac{1}{2}C_2V_2^2 = \frac{1}{2} \times 4 \times 10^{-6} \times (66.66)^2 = 0.00889\,\text{J}$$

$$U_3 = \frac{1}{2}C_3V_3^2 = \frac{1}{2} \times 3 \times 10^{-6} \times 100^2 = 0.015\,\text{J}$$

Note that $U_1 + U_2 + U_3 = U = \frac{1}{2}CV^2$, as it should.

11.92 Let the effective capacitance between points a and b be C. Apply a potential difference V between a and b and let C be charged to q, Fig. 11.47.
Let the charge across C_1 and C_5 be q_1 and q_5, respectively; the charges across various capacitors are shown in Fig. 11.47.

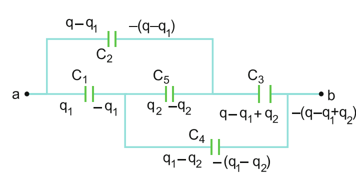


Fig. 11.47

The potential drop across C_1 plus that across C_4 must be equal to the potential drop across C_2 plus that across C_3.

$$V_1 + V_4 = V_2 + V_3 = V \tag{1}$$

$$\therefore \quad \frac{q_1}{C_1} + \frac{q_1 - q_2}{C_4} = \frac{q - q_1}{C_2} + \frac{q - q_1 + q_2}{C_3} \tag{2}$$

By problem $C_1 = C_2 = C_3 = C_4$.

$$\therefore \quad 2q_1 - q_2 = C_1 V \tag{3}$$

$$2q - 2q_1 + q_2 = C_1 V \tag{4}$$

Adding (3) and (4)

$$2q = 2C_1 V \to C = \frac{q}{V} = C_1 = 2\,\mu\text{F}$$

11.93 Applying the loop theorem to the circuit, traversing clockwise from the negative terminal of the battery we have the equation

$$\xi - iR - \frac{q}{C} = 0 \tag{1}$$

where ξ is the emf of the battery and the second and the third terms represent the potential drop across the resistor and the capacitor.

$$\text{Now} \qquad i = \frac{\mathrm{d}q}{\mathrm{d}t} \tag{2}$$

Using (2) in (1)

$$R\,\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = \xi \tag{3}$$

This differential equation describes the time variation of the charge on the capacitor. Re-arranging (3)

$$\frac{\mathrm{d}q}{C\xi - q} = \frac{\mathrm{d}t}{RC} \tag{4}$$

$$\text{Integrating (4)} \qquad -\ln(C\xi - q) = \frac{t}{RC} + A \tag{5}$$

where A is the constant of integration which can be determined from the initial condition.
At $t = 0$, $q = 0$ since the capacitor was uncharged.

$$\therefore \qquad A = -\ln \xi C \tag{6}$$

Using (6) in (5) and re-arranging

$$q = C\xi(1 - \mathrm{e}^{-t/RC})$$

(a) $i = \dfrac{\mathrm{d}q}{\mathrm{d}t} = \dfrac{\xi}{R}e^{-t/RC}$

(b) $P = i\xi = \dfrac{\xi^2}{R}e^{-t/RC}$

(c) $H = i^2R = \dfrac{\xi^2}{R}e^{-2t/RC}$

(d) $\dfrac{\mathrm{d}U}{\mathrm{d}t} = \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{1}{2}\dfrac{q^2}{e}\right) = \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\left[C^2\xi^2\left(1 - e^{-t/RC}\right)^2\right] = \dfrac{C}{R}\xi^2 e^{-t/RC}$

11.94 **(i)** By prob. (11.93)

$$q = C\xi(1 - \mathrm{e}^{-t/RC})$$
$$\therefore \quad i = \frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\xi}{R}\,\mathrm{e}^{-t/RC}$$
$$R = (1200 + 600) \times 10^3 = 1.8 \times 10^6\,\Omega$$
$$RC = 1.8 \times 10^6 \times 2.5 \times 10^{-6} = 4.5$$
$$\text{At} \quad t = 0,\ i = \frac{\xi}{R} = \frac{50}{1.8 \times 10^6} = 27.8 \times 10^{-6}\,\mathrm{A} = 27.8\,\mu\mathrm{A}$$

(ii) At $t = \infty, i = 0$

(iii) $i = -\dfrac{\xi\,\mathrm{e}^{-t/RC}}{R} = -\dfrac{50 \times \mathrm{e}^{-t/4.5}}{1.8 \times 10^6} = -27.8\,\mathrm{e}^{-0.222t}\,\mu\mathrm{A}$

11.95 **(i)** Time constant, $RC = 200 \times 10^3 \times 500 \times 10^{-6} = 100$
When the switch is closed there is no emf in the circuit, and (3) in prob. (11.93) reduces to

(ii) $$R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = 0 \tag{1}$$

$$\text{or} \quad \frac{\mathrm{d}q}{q} = -\frac{\mathrm{d}t}{RC} \tag{2}$$

Integrating, $\ln q = -t/RC + A$
where A is the constant of integration. When $t = 0$, $q = q_0$. Therefore $A = \ln q_0$.

$$\therefore \quad \ln\frac{q}{q_0} = -\frac{t}{RC}$$

$$\therefore \quad q = q_0 e^{-t/RC} \tag{3}$$

$$\frac{q}{q_0} = \frac{1}{2} = e^{-t/RC}$$

$$\therefore \quad t = RC\ln 2 = 100\ln 2 = 69.3\,\text{s}.$$

(iii) In (3) put $t = 0$. Then

$$q = q_0 = C\xi = 500 \times 10^{-6} \times 200 \times 10^3 = 100\,\text{C}$$

(iv) Differentiating (3) with respect to time

$$i = \frac{dq}{dt} = -\frac{\xi}{R}\,e^{-t/RC} \tag{4}$$

The negative sign shows that the current in the discharging process flows opposite to that in the charging process. At $t = 0$

$$i = -\frac{\xi}{R} = -\frac{900}{200 \times 10^3} = -4.5 \times 10^{-3}\,\text{A}$$

(v) From (4) $V = iR = -\xi e^{-t/RC}$

At $\quad t = 25\,\text{s},\ V = -900\,e^{-25/100} = -701\,\text{V}.$

11.96 **(i)** By (4), prob. (11.50) the electric field inside the sphere is given by

$$\boldsymbol{E} = \frac{q\boldsymbol{r}}{4\pi\varepsilon_0 R^3} \tag{1}$$

$$\therefore \quad \operatorname{div}\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0 R^3}\operatorname{div}\boldsymbol{r} \tag{2}$$

$$\text{Now} \quad \operatorname{div}\boldsymbol{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(\hat{i}x + \hat{j}y + \hat{k}z)$$

$$= \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 1 + 1 + 1 = 3$$

$$\therefore \quad \operatorname{div}\boldsymbol{r} = \frac{3q}{4\pi\varepsilon_0 R^3} = \frac{3 \times 10^{-9} \times 9 \times 10^9}{1^3} = 27$$

(ii) $F = QE = \dfrac{Qqr}{4\pi\varepsilon_0 R^3}$

$$= 1.6 \times 10^{-19} \times 10^{-9} \times 9 \times 10^9 \times \frac{0.8}{1^3} = 1.152 \times 10^{-18}\,\text{N}$$

(iii) By prob. (11.52), $V(r) = \dfrac{q}{8\pi\varepsilon_0 R}\left(3 - \dfrac{r^2}{R^2}\right)$. The potential energy of the proton at r will be $U\,(r) = QV(r) = \dfrac{Qq}{8\pi\varepsilon_0 R}\left(3 - \dfrac{r^2}{R^2}\right)$.

$$\therefore \qquad U(r = 0.8\,\text{m}) = \frac{1.6 \times 10^{-19} \times 10^{-9} \times 9 \times 10^{9}}{2 \times 1.0}\left(3 - \left(\frac{0.8}{1.0}\right)^2\right)$$
$$= 1.7 \times 10^{-19}\,\text{J}$$

Since $U(r = \infty) = 0$, work done $= 1.7 \times 10^{-19}$ J.

11.97 **(a)** If the dielectric is present, Gauss' law gives

$$\varepsilon_0 \oint \boldsymbol{E} \cdot \text{d}\boldsymbol{s} = \varepsilon_0 E A = q - q' \qquad \text{(Integral form)}$$

$$\text{or} \quad E = \frac{1}{\varepsilon_0 A}\,(q - q')$$

where q is the free charge and $-q'$ the induced charge.

$$\nabla \cdot \boldsymbol{E} = \rho/\varepsilon \qquad \text{(Differential form)}$$

(b) The displacement vector

$$D = \frac{q}{A} \tag{1}$$

where A is the area

$$E = \frac{E_0}{K} = \frac{q}{K\varepsilon_0 A} \tag{2}$$

Combining (1) and (2)

$$D = K\varepsilon_0 E \tag{3}$$

As $\boldsymbol{E}$ is uniform in a parallel plate capacitor, $\boldsymbol{D}$ will be also uniform via (3)

(c) By Gauss law

$$\varepsilon_0 \oint K(x)\,\boldsymbol{E}(x) \cdot \text{d}\boldsymbol{s} = q = \varepsilon_0 \oint \boldsymbol{E}_0 \cdot \text{d}\boldsymbol{s}$$

$$\therefore \quad E(x) = \frac{E_0}{K(x)} \tag{4}$$

$$E_0 = \frac{q}{\varepsilon_0 A} = \frac{\sigma A}{\varepsilon_0 A} = \frac{\sigma}{\varepsilon_0}\ (\because\ \sigma = q/A) \tag{5}$$

(d) $$V = \int E(x)\mathrm{d}x = E_0 \int_0^d \frac{\mathrm{d}x}{ax+b} \tag{6}$$

Put $y = ax + b$, $\mathrm{d}x = \mathrm{d}y/a$. Then (6) becomes

$$V = \frac{E_0}{a} \int_b^{ad+b} \frac{\mathrm{d}y}{y} = \frac{E_0}{a}\ \ln\ y|_b^{ad+b} = \frac{\sigma_0}{\varepsilon_0 a}\ \ln\left(1 + \frac{ad}{b}\right) \tag{7}$$

where we have used (5).

The capacitance $$C = \frac{q}{V} = \frac{A\varepsilon_0 a}{\ln\left(1 + \dfrac{ad}{b}\right)} \tag{8}$$

where we have used (7) and $q = \sigma A$.

(e) Vacuum polarization charge density

$$P(x) = \varepsilon_0(k(x) - 1)E = \frac{\varepsilon_0 E_0}{k(x)}(k(x) - 1)$$
$$= \varepsilon_0 E_0\left[1 - \frac{1}{ax+b}\right] \tag{9}$$

11.98 Newton's law of gravitation is

$$\boldsymbol{F} = -\frac{GMm}{r^2}\hat{e}_r = m\boldsymbol{g}$$

The Gauss' law for gravitation may be written as

$$\oint \boldsymbol{g} \cdot \mathrm{d}\boldsymbol{s} = -\frac{GM}{r^2}4\pi r^2 = -4\pi GM$$

The divergence theorem gives

$$\oint \nabla \cdot \boldsymbol{g}\, \mathrm{d}^3 r = \oint \boldsymbol{g} \cdot \mathrm{d}\boldsymbol{s}$$
$$\therefore \quad \nabla \cdot g\frac{4}{3}\pi r^3 = -4\pi GM$$
$$\therefore \quad \nabla \cdot g = -4\pi\ G\rho m$$

This is analogous to the law for electric field

$$\nabla \cdot E = \frac{\rho_q}{\varepsilon_0}$$