Eliminating v between the two equations and noting that

$$\cos\theta = 1 - h/R$$

we find $h = R/3$.

(iv) A motorcyclist goes around in a vertical circle inside a spherical cage. Find the minimum speed at the top so that he may successfully complete the circular ride.
Here we equate the reaction on the cage to the total weight of the rider plus motorcycle

$$mg = mv^2/R$$
$$\text{or } v = \sqrt{gR}$$

(v) *Loop-the-Loop* is a track which consists of a frictionless slide connected to a vertical loop of radius R, Fig. 3.1. Let a particle start at a height h on the slide and acquire a velocity v at the bottom of the loop.
If $v < \sqrt{2gR}$, the particle will not be able to climb up beyond the point B. It will oscillate in the lower semicircle about the point D.

$$\text{If } \sqrt{2gR} < v < \sqrt{5gR}$$

the particle will be able to climb up the arc BC and leave at some point E and describe a parabolic path. If $v > \sqrt{5gR}$, the particle will be able to execute a complete circle. This corresponds to a height $h = 2.5R$.

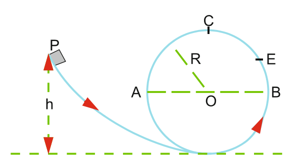


Fig. 3.1 Loop-the-loop

3.2 Problems

3.2.1 Motion in a Horizontal Plane

3.1 Show that a particle with coordinates $x = a\cos t$, $y = a\sin t$ and $z = t$ traces a path in time which is a helix.

[Adapted from Hyderabad Central University 1988]

3.2 A particle of mass m is moving in a circular path of constant radius r such that its centripetal acceleration a varies with time t as $a = k^2 r t^2$, where k is a constant. Show that the power delivered to the particle by the forces acting on it is $mk^4 r^2 t^5/3$

[Adapted from Indian Institute of Technology 1994]

3.3 A particle is moving in a plane with constant radial velocity of magnitude $\dot{r} = 5\,\text{m/s}$ and a constant angular velocity of magnitude $\dot{\theta} = 4\,\text{rad/s}$. Determine the magnitude of the velocity when the particle is 3 m from the origin.

3.4 A point moves along a circle of radius 40 cm with a constant tangential acceleration of $10\,\text{cm/s}^2$. What time is needed after the motion begins for the normal acceleration of the point to be equal to the tangential acceleration?

3.5 A point moves along a circle of radius 4 cm. The distance x is related to time t by $x = ct^3$, where $c = 0.3\,\text{cm/s}^3$. Find the normal and tangential acceleration of the point at the instant when its linear velocity is $v = 0.4\,\text{m/s}$.

3.6 **(a)** Using the unit vectors $\hat{i}$ and $\hat{j}$ write down an expression for the position vector in the polar form. **(b)** Show that the acceleration is directed towards the centre of the circular motion.

3.7 Find the angular acceleration of a wheel if the vector of the total acceleration of a point on the rim forms an angle 30° with the direction of linear velocity of the point in 1.0 s after uniformly accelerated motion begins.

3.8 A wheel rotates with a constant angular acceleration $\alpha = 3\,\text{rad/s}^2$. At time $t = 1.0\,\text{s}$ after the motion begins the total acceleration of the wheel becomes $a = 12\sqrt{10}\,\text{cm/s}^2$. Determine the radius of the wheel.

3.9 A car travels around a horizontal bend of radius R at constant speed V.

(i) If the road surface has a coefficient of friction μ_s, what is the maximum speed, V_{max}, at which the car can travel without sliding?

(ii) Given $\mu_s = 0.85$ and $R = 150\,\text{m}$, what is V_{max}?

(iii) What is the magnitude and direction of the car's acceleration at this speed?

(iv) If $\mu_s = 0$, at what angle would the bend need to be banked in order for the car to still be able to round it at the same maximum speed found in part (ii)?

[University of Durham 2000]

3.10 The conical pendulum consists of a bob of mass m attached to the end of an inflexible light string tied to a fixed point O and swung around so that it describes a circle in a horizontal plane; while revolving the string generates a conical surface around the vertical axis ON, the height of the cone being ON $= H$, the projection of OP on the vertical axis (Fig. 3.2). Show that the angular velocity of the bob is given by $\omega = \sqrt{g/H}$, where g is the acceleration due to gravity.

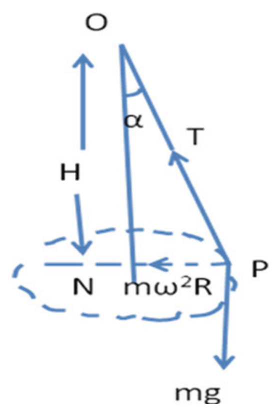


Fig. 3.2 Conical pendulum

3.11 If the number of steady revolutions per minute of a conical pendulum is increased from 70 to 80, what would be the difference in the level of the bob?

3.12 A central wheel can rotate about its central axis, which is vertical. From a point on the rim hangs a simple pendulum. When the wheel is caused to rotate uniformly, the angle of inclination of the pendulum to the vertical is θ_0. If the radius of the wheel is R cm and the length of the pendulum is 1 cm, obtain an expression for the number of rotations of the wheel per second.

[University of Newcastle]

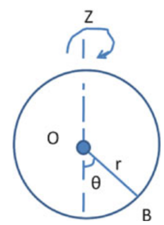
3.13 A coin is placed at a distance r from the centre of a gramophone record rotating with angular frequency $\omega = 2\pi f$. Find the maximum frequency for which the coin will not slip if μ is the coefficient of friction.

3.14 A particle of mass m is attached to a spring of initial length L_0 and spring constant k and rotated in a horizontal plane with an angular velocity ω. What is the new length of the spring and the tension in the spring?

3.15 A hollow cylinder drum of radius r is placed with its axis vertical. It is rotated about an axis passing through its centre and perpendicular to the face and a coin is placed on the inside surface of the drum. If the coefficient of friction is μ, what is the frequency of rotation so that the coin does not fall down?

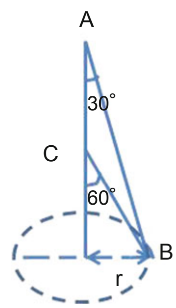
3.16 A bead B is threaded on a smooth circular wire frame of radius r, the radius vector $\vec{r}$ making an angle θ with the negative z-axis (see Fig. 3.3). If the frame is rotated with angular velocity ω about the z-axis then show that the bead will be in equilibrium if $\omega = \sqrt{\dfrac{g}{r\cos\theta}}$.

Fig. 3.3



3.17 A wire bent in the form ABC passes through a ring B as in Fig. 3.4. The ring rotates with constant speed in a horizontal circle of radius r. Show that the speed of rotation is $\sqrt{gr}$ if the wires are to maintain the form.

Fig. 3.4



3.18 A small cube placed on the inside of a funnel rotates about a vertical axis at a constant rate of f rev/s. The wall of the funnel makes an angle θ with the horizontal (Fig. 3.5). If the coefficient of static friction is μ and the centre of the cube is at a distance r from the axis of rotation, show that the largest frequency for which the block will not move with respect to the funnel is

$$f_{\max} = \frac{1}{2\pi}\sqrt{\frac{g(\sin\theta + \mu\cos\theta)}{r(\cos\theta - \mu\sin\theta)}}$$

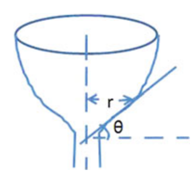


Fig. 3.5

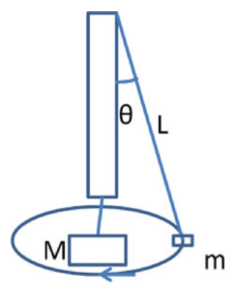
3.19 In prob. (3.18), show that the minimum frequency for which the block will not move with respect to the funnel will be

$$f_{\min} = \frac{1}{2\pi}\sqrt{\frac{g(\sin\theta - \mu\cos\theta)}{r(\cos\theta + \mu\sin\theta)}}$$

3.20 A large mass M and a small mass m hang at the two ends of a string that passes through a smooth tube as in Fig. 3.6. The mass m moves around in a circular path which lies in a horizontal plane. The length of the string from the mass m to the top of the tube is L, and θ is the angle this length makes with the vertical. What should be the frequency of rotation of the mass m so that the mass M remains stationary?

[Indian Institute of Technology 1978]

Fig. 3.6



3.21 An object is being weighed on a spring balance going around a curve of radius 100 m at a speed of 7 m/s. The object has a weight of 50 kg wt. What reading is registered on the spring balance?

3.22 A railway carriage has its centre of gravity at a height of 1 m above the rails, which are 1.5 m apart. Find the maximum safe speed at which it could travel round the unbanked curve of radius 100 m.

3.23 A curve on a highway has a radius of curvature r. The curved road is banked at θ with the horizontal. If the coefficient of static friction is μ,

(a) Obtain an expression for the maximum speed v with which a car can go over the curve without skidding.
(b) Find v if $r = 100\,\text{m}$, $\theta = 30°$, $g = 9.8\,\text{m/s}^2$, $\mu = 0.25$

3.24 Determine the linear velocity of rotation of points on the earth's surface at latitude of 60°.

3.25 With what speed an aeroplane on the equator must fly towards west so that the passenger in the plane may see the sun motionless?

3.2.2 Motion in a Vertical Plane

3.26 A particle is placed at the highest point of a smooth sphere of radius R and is given an infinitesimal displacement. At what point will it leave the sphere?
[University of Cambridge]

3.27 A small sphere is attached to a fixed point by a string of length 30 cm and whirls round in a vertical circle under the action of gravity at such a speed that the tension in the string when the sphere is at the lowest point is three times the tension when the sphere is at its highest point. Find the speed of the sphere at the highest point.
[University of Cambridge]

3.28 A light rigid rod of length L has a mass m attached to its end, forming a simple pendulum. The pendulum is put in the horizontal position and released from rest. Show that the tension in the suspension will be equal to the magnitude of weight at an angle $\theta = \cos^{-1}(1/3)$ with the vertical.

3.29 In a hollow sphere of diameter 20 m in a circus, a motorcyclist rides with sufficient speed in the vertical plane to prevent him from sliding down. If the coefficient of friction is 0.8, find the minimum speed of the motorcyclist.

3.30 The bob of a pendulum of mass m and length L is displaced 90° from the vertical and gently released. What should be the minimum strength of the string in order that it may not break upon passing through the lowest point?

3.31 The bob of a simple pendulum of length L is deflected through a small arc s from the equilibrium position and released. Show that when it passes through the equilibrium position its velocity will be $s\sqrt{g/L}$, where g is the acceleration due to gravity.

3.32 A simple pendulum of length 1.0 metre with a bob of mass m swings with an angular amplitude of 60°. What would be the tension in the string when its angular displacement is 45°?

3.33 The bob of a pendulum is displaced through an angle θ with the vertical line and is gently released so that it begins to swing in a vertical circle. When it passes through the lowest point, the string experiences a tension equal to double the weight of the bob. Determine θ.

3.2.3 Loop-the-Loop

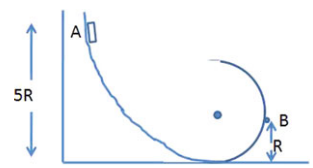
3.34 The bob of a simple pendulum of length 1.0 m has a velocity of 6 m/s when it is at the lowest point. At what height above the centre of the vertical circle will the bob leave the path?

3.35 A block of 2 g when released on an inclined plane describes a circle of radius 12 cm in the vertical plane on reaching the bottom. What is the minimum height of the incline?

3.36 A particle slides down an incline from rest and enters the loop-the-loop. If the particle starts from a point that is level with the highest point on the circular track then find the point where the particle leaves the circular groove above the lowest point.

3.37 A small block of mass m slides along the frictionless loop-the-loop track as in Fig. 3.7. If it starts at A at height $h = 5R$ from the bottom of the track then show that the resultant force acting on the track at B at height R will be $\sqrt{65}$ mg.

Fig. 3.7



3.38 In prob. (3.37), the block is released from a height h above the bottom of the loop such that the force it exerts against the track at the top of the loop is equal to its weight. Show that $h = 3R$.

3.39 A particle of mass m is moving in a vertical circle of radius R. When m is at the lowest position, its speed is $0.8944\sqrt{5gR}$. The particle will move up the track to some point p at which it will lose contact with the track and travel along a path shown by the dotted line (Fig. 3.8). Show that the angular position of θ will be 30°.

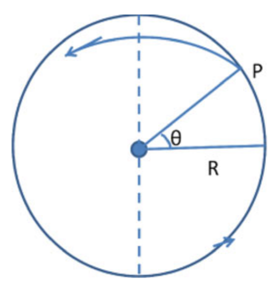


Fig. 3.8

3.40 A block is allowed to slide down a frictionless track freely under gravity. The track ends in a circular loop of radius R. Show that the minimum height from which the block must start is $2.5R$ so that it completes the circular track.

3.41 A nail is located at a certain distance vertically below the point of suspension of a simple pendulum. The pendulum bob is released from a position where the string makes an angle 60° with the vertical. Calculate the distance of the nail from the point of suspension such that the bob will just perform revolutions with the nail as centre. Assume the length of the pendulum to be 1 m.
[Indian Institute of Technology 1975]

3.42 A test tube of mass 10 g closed with a cork of mass 1 g contains some ether. When the test tube is heated, the cork flies out under the pressure of the ether gas. The test tube is suspended by a weightless rigid bar of length 5 cm. What is the minimum velocity with which the cork would fly out of the test tube so that the test tube describes a full vertical circle about the point of suspension? Neglect the mass of ether.
[Indian Institute of Technology 1969]

3.43 A car travels at a constant speed of 14.0 m/s round a level circular bend of radius 45 m. What is the minimum coefficient of static friction between the tyres and the road in order for the car to go round the bend without skidding?
[University of Manchester 2008]

3.3 Solutions

3.3.1 Motion in a Horizontal Plane

3.1 $x = a\cos t$ (1)

$y = a\sin t$ (2)

$z = t$ (3)

Squaring (1) and (2) and adding

$$x^2 + y^2 = a^2(\cos^2 t + \sin^2 t) = a^2$$

which is the equation of a circle.
Since $z = t$, the circular path drifts along the z-axis so that the path is a helix.

3.2 $a = \dfrac{dv}{dt} = k^2rt^2$

$$v = \int dv = k^2r\int t^2 dt = \frac{k^2r\,t^3}{3}$$

Power, $P = Fv = mav = mk^2r\,t^2\dfrac{k^2r\,t^3}{3} = \dfrac{mk^4r^2t^5}{3}$