6.60 A damped oscillator loses 3% of its energy in each cycle. **(a)** How many cycles elapse before half its original energy is dissipated? **(b)** What is the Q factor?

6.61 A damped oscillator has frequency which is 9/10 of its natural frequency. By what factor is its amplitude decreased in each cycle?

6.62 Show that for small damping $\omega' \approx (1 - r^2/8mk)\omega_0$ where ω_0 is the natural angular frequency, ω' the damped angular frequency, r the resistance constant, k the spring constant and m the particle mass.

6.63 Show that the time elapsed between successive maximum displacements of a damped harmonic oscillator is constant and equal to $4\pi m/\sqrt{4km - r^2}$, where m is the mass of the vibrating body, k is the spring constant, $2b = r/m$, r being the resistance constant.

6.64 A dead weight attached to a light spring extends it by 9.8 cm. It is then slightly pulled down and released. Assuming that the logarithmic decrement is equal to 3.1, find the period of oscillation.

6.65 The position of a particle moving along x-axis is determined by the equation $d^2x/dt^2 + 2dx/dt + 8x = 16\cos 2t$.

(a) What is the natural frequency of the vibrator?
(b) What is the frequency of the driving force?

6.66 Show that the time $t_{1/2}$ for the energy to decrease to half its initial value is related to the time constant by $t_{1/2} = t_c \ln 2$.

6.67 The amplitude of a swing drops by a factor $1/e$ in 8 periods when no energy is pumped into the swing. Find the Q factor.

6.3 Solutions

6.3.1 Simple Harmonic Motion (SHM)

6.1 $x = A\sin\omega t$ (SHM)

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1\ \text{rad/s}$$

$$8\sqrt{2} = A\sin\left(\frac{1\cdot\pi}{4}\right)$$

$$A = 16\ \text{cm} = 0.16\ \text{m}$$

$$E = \frac{1}{2}mA^2\omega^2$$

$$\therefore\quad m = \frac{2E}{A^2\omega^2} = \frac{2\times 0.256}{(0.16)^2\times 1^2} = 20.0\ \text{kg}$$

6.2 **(a)** $$v = \omega\sqrt{A^2 - x^2} \tag{1}$$

$$16 = \omega\sqrt{A^2 - 3^2} \tag{2}$$

$$12 = \omega\sqrt{A^2 - 4^2} \tag{3}$$

Solving (2) and (3) A = 5 cm and $\omega = 4\,\text{rad/s}$

(b) Therefore $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{4} = 1.57\,\text{s}$

6.3 $x = A\sin\omega t$

$$v = \frac{dx}{dt} = \omega A\cos\omega t$$

$$v_{\max} = A\omega = \frac{2\pi A}{T} = \frac{2\pi \times 5}{2} = 5\pi\ \text{cm/s}$$

At the equilibrium position the weight of the bob and the tension act in the same direction

$$\text{Tension} = mg + \frac{mv_{\max}^2}{L}$$

Now the length of the simple pendulum is calculated from its period T.

$$L = \frac{gT^2}{4\pi^2} = \frac{980 \times 2^2}{4\pi^2} = 99.29\,\text{cm}$$

$$\text{Tension} = m\left(1 + \frac{v_{\max}^2}{gL}\right)g = 50\left(1 + \frac{25\pi^2}{980 \times 99.29}\right)g$$

$$= 50.13\ \text{g dynes} = 50.13\ \text{g wt}$$

6.4 The general equation of SHM is

$$x = A\sin(\omega t + \varepsilon)$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{16} = \frac{\pi}{8}$$

When $t = 2\,\text{s}$, $x = 0$.

$$0 = A\sin\left(\frac{\pi}{8} \times 2 + \varepsilon\right)$$

Since $A \neq 0$, $\sin\left(\dfrac{\pi}{4} + \varepsilon\right) = 0$

$$\therefore \quad \frac{\pi}{4} + \varepsilon = 0 \quad \varepsilon = -\frac{\pi}{4}$$

Now $v = \dfrac{dx}{dt} = A\omega \cos(\omega t + \varepsilon)$

When $t = 4$, $v = 4$.

$$\therefore \quad 4 = \frac{A\pi}{8} \cos\left(\frac{\pi}{8} 4 - \frac{\pi}{4}\right)$$

$$\therefore \quad A = \frac{32\sqrt{2}}{\pi}$$

6.5 Let the body with uniform cross-section A be immersed to a depth h in a liquid of density D. Volume of the liquid displaced is $V = Ah$. Weight of the liquid displaced is equal to VDg or $AhDg$. According to Archimedes principle, the weight of the liquid displaced is equal to the weight of the floating body Mg.

$Mg = Ahdg$ or $\mathrm{M} = AhD$

The body occupies a certain equilibrium position. Let the body be further depressed by a small amount x. The body now experiences an additional upward thrust in the direction of the equilibrium position. When the body is released it moves up with acceleration

$$a = -\frac{AxDg}{M} = -\frac{AxDg}{AhD} = -\frac{gx}{h} = -\omega^2 x$$

with $\quad \omega^2 = \dfrac{g}{h}$

Time period $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{h}{g}} = 2\pi\sqrt{\dfrac{V}{Ag}}$

6.6 The acceleration due to gravity g at a depth d from the surface is given by

$$g = g_0\left(1 - \frac{d}{R}\right) \tag{1}$$

where g_0 is the value of g at the surface of the earth of radius R.

$$\text{Writing } x = R - d \tag{2}$$

Equation (1) becomes $g = g_0 \dfrac{x}{R}$ (3)

where x measures the distance from the centre. The acceleration g points opposite to the displacement x. We can therefore write

$$a = g = -\frac{g_0 x}{R} = -\omega^2 x \tag{4}$$

with $\omega^2 = \dfrac{g_0}{R}$

Equation (4) shows that the box performs SHM. The period is calculated from

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R}{g_0}} = 2\pi\sqrt{\frac{6.4 \times 10^6}{9.8}} = 5074\,\text{s or } 84.6\,\text{min}$$

6.7 Standard equation for SHM is

$x = A\sin(\omega t + \varepsilon)$

$x = 4\sin\left(\dfrac{\pi t}{3} + \dfrac{\pi}{6}\right)$

(a) A = 4 cm

(b) $\omega = \dfrac{\pi}{3}$. Therefore $T = \dfrac{2\pi}{\omega} = 6\,\text{s}$

(c) $f = \dfrac{1}{T} = \dfrac{1}{6}/\,\text{s}$

(d) $\varepsilon = \dfrac{\pi}{6}$

(e) $v = \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{4\pi}{3}\cos\left(\dfrac{\pi t}{3} + \dfrac{\pi}{6}\right) = \dfrac{4\pi}{3}\cos\left(\dfrac{\pi}{3} \times 1 + \dfrac{\pi}{6}\right) = 0$

(f) $a = \dfrac{\mathrm{d}v}{\mathrm{d}t} = -\dfrac{4\pi^2}{9}\sin\left(\dfrac{\pi}{3} \times 1 + \dfrac{\pi}{6}\right) = -\dfrac{4\pi^2}{9}$

6.8 **(a)** $K = \dfrac{1}{2}m\omega^2(A^2 - x^2) \quad U = \dfrac{1}{2}m\omega^2 x^2 \quad K = U$

$$\therefore \quad \frac{1}{2}m\omega^2(A^2 - x^2) = \frac{1}{2}m\omega^2 x^2$$

$$\therefore \quad x = \frac{A}{\sqrt{2}}$$

(b) $K = \dfrac{1}{2}m\omega^2\left(A^2 - \dfrac{A^2}{4}\right) = \dfrac{1}{2}m\omega^2 \dfrac{3}{4}A^2$

$$U = \frac{1}{2}m\omega^2\frac{A^2}{4}$$
$$\therefore \quad K : U = 3 : 1$$

6.9 $$T = 2\pi\sqrt{\frac{M}{k}} \tag{1}$$
$$2 = 2\pi\sqrt{\frac{M}{k}} \tag{2}$$
$$3 = 2\pi\sqrt{\frac{M+2}{k}} \tag{3}$$

Dividing (2) by (3) and solving for M, we get $M = 1.6\,\text{kg}$.

6.10 $$a_{\max} = \omega^2 A$$
$$5\pi^2 = \omega^2 A \tag{1}$$
$$v = \omega\sqrt{A^2 - x^2}$$
$$3\pi = \omega\sqrt{A^2 - 16} \tag{2}$$

Solving (1) and (2), we get $A = 5\,\text{cm}$ and $T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2\,\text{s}$.

6.11 $$\alpha = \omega^2 A \tag{1}$$
$$\beta = \omega A \tag{2}$$
$$\therefore \quad \beta^2 = \omega^2 A^2 = \alpha A$$
$$\text{or} \quad A = \frac{\beta^2}{\alpha}$$

Dividing (2) by (1)

$$\frac{\beta}{\alpha} = \frac{1}{\omega}$$
$$\text{or} \quad T = \frac{2\pi}{\omega} = \frac{2\pi\beta}{\alpha}$$

6.12 By problem $\frac{mg + mv^2/L}{mg} = 1.01$

$$\therefore \quad \frac{v^2}{gL} = 0.01$$

Conservation of energy gives

$$\frac{1}{2}mv^2 = mgh = mgL(1 - \cos\theta) \simeq mgL\frac{\theta^2}{2} \quad \text{for small } \theta$$

$$\theta^2 = \frac{v^2}{gL} = 0.01$$
$$\therefore \quad \theta = \sqrt{0.01} = 0.1 \text{ rad}$$

6.13 $a = A \sin \omega t_0$

$$b = A \sin 2\omega t_0$$
$$c = A \sin 3\omega t_0$$
$$a + c = 2A \sin 2\omega t_0 \cos \omega t_0$$
$$\frac{a+c}{2b} = \cos \omega t_0$$
$$\omega = \frac{1}{t_0} \cos^{-1}\left(\frac{a+c}{2b}\right)$$
$$f = \frac{1}{2\pi t_0} \cos^{-1}\left(\frac{a+c}{2b}\right)$$

6.14 **(a)** $\omega = \sqrt{\frac{k}{m}}$

$$k = m\omega^2 = \frac{4\pi^2 m}{T^2} = \frac{4\pi^2 \times 4}{2^2} = 39.478\,\text{N/m}$$

(b) $F_{\text{max}} = m\omega^2 A = kA = 39.478 \times 2 = 78.96\,\text{N}$

6.15 $x = a \sin \omega t$

$$y = b \cos \omega t$$
$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \sin^2 \omega t + \cos^2 \omega t = 1$$

Thus the path of the particle is an ellipse.

6.16 **(a)** To show that $\nabla \times \boldsymbol{F} = 0$.

$$\nabla \times \boldsymbol{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -Kx & 0 & 0 \end{vmatrix} = 0$$

(b) $U = -\int F \mathrm{d}x = -\int (K\ ix)\,(-\hat{i}\,\mathrm{d}x) = \frac{1}{2}Kx^2$

6.17 **(a)** $F = kx$

$$\therefore \quad k = \frac{F}{x} = \frac{2 \times 9.8}{5 \times 10^{-2}} = 392\,\text{N/m}$$

(b) 10 cm

(c) $f = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}} = \dfrac{1}{2\pi}\sqrt{\dfrac{392}{2 \times 9.8}} = 0.712/\,\text{s}$

6.18 Let x_0 be the extension of the spring. Deformation energy = gravitational potential energy

$$\frac{1}{2}kx_0^2 = mgh + mgx_0$$

Rearranging

$$x_0^2 - \frac{2mg}{k}x_0 - mgh = 0$$

The quadratic equation has the solutions

$$x_{01} = \frac{mg}{k} + \sqrt{\frac{m^2g^2}{k^2} + \frac{2mgh}{k}}$$

$$x_{02} = \frac{mg}{k} - \sqrt{\frac{m^2g^2}{k^2} + \frac{2mgh}{k}}$$

The equilibrium position is depressed by $x_0 = \dfrac{mg}{k}$ below the initial position. The amplitude of the oscillations as measured from the equilibrium position is equal to $\sqrt{\dfrac{m^2g^2}{k^2} + \dfrac{2mgh}{k}}$.

6.19 It is reasonable to assume that the probability density $\dfrac{\mathrm{d}p(x)}{\mathrm{d}x}$ for finding the particle is proportional to the time spent at a given point and is therefore inversely proportional to its speed v.

$$\frac{\mathrm{d}p(x)}{\mathrm{d}x} = \frac{C}{v} \tag{1}$$

where C = constant of proportionality.

But $v = \omega\sqrt{A^2 - x^2}$ (2)

The probability density

$$\frac{\mathrm{d}p(x)}{\mathrm{d}x} = \frac{C}{\omega\sqrt{A^2 - x^2}} \tag{3}$$

C can be found by normalization of distribution

$$\int_{-A}^{A} \mathrm{d}p(x) = \frac{C}{\omega} \int_{-A}^{A} \frac{\mathrm{d}x}{\sqrt{A^2 - x^2}} = 1$$

$$\text{or} \quad \frac{C\pi}{\omega} = 1 \to \frac{C}{\omega} = \frac{1}{\pi}$$

$$\therefore \quad \frac{\mathrm{d}p(x)}{\mathrm{d}x} = \frac{1}{\pi\sqrt{A^2 - x^2}}$$

6.20 $U = \frac{1}{2}kx^2$

Using the result of prob. (6.19)

$$\langle U \rangle = \int U \,\mathrm{d}p(x) = \int_{-A}^{A} \frac{1}{2}kx^2 \frac{\mathrm{d}x}{\pi\sqrt{A^2 - x^2}}$$

Put $x = A\sin\theta, \quad \mathrm{d}x = A\cos\theta \,\mathrm{d}\theta$

$$\langle U \rangle = \left(\frac{kA^2}{2\pi}\right) \int_{-\pi/2}^{\pi/2} \sin^2\theta \,\mathrm{d}\theta = \frac{1}{4}kA^2$$

Also, $\langle K \rangle = \langle E - U \rangle = \frac{1}{2}kA^2 - \frac{1}{4}kA^2 = \frac{1}{4}kA^2$

6.21 $K_{\text{trans}} + K_{\text{rot}} + U = \text{constant}$

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}kx^2 = \text{constant}$$

But $I = \frac{1}{2}mR^2$ and $\omega = \frac{v}{R}$

$$\therefore \quad \frac{3}{4}m\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \frac{1}{2}kx^2 = 0 \text{ constant}$$

Differentiating

$$\frac{3}{2}m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\frac{\mathrm{d}x}{\mathrm{d}t} + kx\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$

Cancelling $\mathrm{d}x/\mathrm{d}t$ throughout and simplifying

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \left(\frac{2k}{3m}\right)x = 0$$

This is the equation for SHM

with $\omega^2 = \left(\frac{2k}{3m}\right)$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3m}{2k}}$$

6.22 The time period of the pendulums is

$$T_1 = 2\pi\sqrt{\frac{60}{g}} \quad (1)$$

$$T_2 = 2\pi\sqrt{\frac{63}{g}} \quad (2)$$

Let the time be t in which the longer length pendulum makes n oscillations while the shorter one makes $(n+1)$ oscillations. Then

$$t = (n+1)T_1 = nT_2 \quad (3)$$

Using (1) and (2) in (3), we find $n = 40.5$ and $t = 64.49$ s.

6.23 Let g_0 be the acceleration due to gravity on the ground and g at height above the ground. Then

$$g = \frac{g_0 R^2}{(R+h)^2}$$

At the ground, $T_0 = 2\pi\sqrt{\frac{L}{g_0}}$. At height h, $T = 2\pi\sqrt{\frac{L}{g}}$

$$T = T_0\sqrt{\frac{g_0}{g}} = T_0\left(1 + \frac{h}{R}\right) = 2\left(1 + \frac{320}{6.4 \times 10^6}\right) = 2.0001\,\text{s}$$

Time lost in one oscillation on the top of the tower $= 2.0001 - 2.0000 = 0.0001$ s. Number of oscillations in a day for the pendulum which beats seconds on the ground

$$= \frac{86400}{2.0} = 43{,}200$$

Therefore, time lost in 43,200 oscillations

$= 42{,}300 \times 0.0001 = 4.32\,\text{s}$

6.24 $$g = g_0\left(1 - \frac{d}{R}\right) \qquad (1)$$

where g and g_0 are the acceleration due to gravity at depth d and surface, respectively, and R is the radius of the earth.

$$T = T_0\sqrt{\frac{g_0}{g}} = T_0\left(1 - \frac{d}{R}\right)^{-1/2} = T_0\left(1 + \frac{d}{2R}\right)$$

Time registered for the whole day will be proportional to the time period. Thus

$$\frac{T}{T_0} = \frac{t}{t_0} = 1 + \frac{d}{2R}$$
$$\frac{86{,}400}{86{,}400 - 300} = 1 + \frac{d}{2R}$$

Substituting $R = 6400\,\text{km}$, we find $d = 44.6\,\text{km}$.

6.25 **(a)** Let the liquid level in the left limb be depressed by x, so that it is elevated by the same height in the right limb (Fig. 6.17). If ρ is the density of the liquid, A the cross-section of the tube, M the total mass, and m the mass of liquid corresponding to the length $2x$, which provides the unbalanced force,

$$\frac{M\mathrm{d}^2x}{\mathrm{d}t^2} = -mg = -(2xA\rho)g$$
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{2A\rho g}{M}x = -\frac{2A\rho g x}{hA\rho} = -\frac{2gx}{h} = -\omega^2 x$$

This is the equation of SHM.

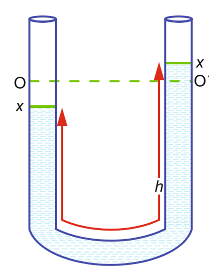


Fig. 6.17

(b) The time period is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{h}{2g}}$$

6.26 **(a)** Let the gas at pressure P and volume V be compressed by a small length x, the new pressure being p' and new volume V' (Fig. 6.18) under isothermal conditions.

$$P'V' = PV$$

$$\text{or} \quad P'(l - x)A = PlA$$

where A is the cross-sectional area.

$$P' = \frac{Pl}{l - x} - P\left(1 - \frac{x}{l}\right)^{-1} \simeq P\left(1 + \frac{x}{l}\right)$$

where we have expanded binomially up to two terms since $x << l$. The change in pressure is

$$\Delta P = P' - P = \frac{Px}{h}$$

The unbalanced force

$$F = -\Delta PA = -\frac{APx}{l}$$

and the acceleration

$$a = \frac{F}{m} = -\frac{APx}{ml} = -\omega^2 x$$

which is the equation for SHM.

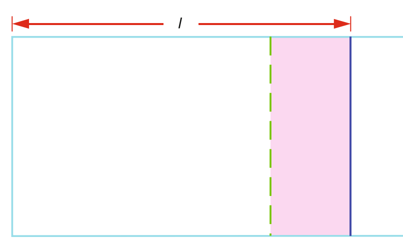


Fig. 6.18

(b) The time period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{ml}{AP}}$$

6.27 $y = 8\sin\left(\frac{2\pi t}{T} + \phi\right)$

At $t = 0$; $\quad 4 = 8\sin\phi$

$\therefore \quad \phi = 30^\circ = \frac{\pi}{6}$

$y = 8\sin\left(\frac{2\pi \times 6}{24} + \frac{\pi}{6}\right) = 8\sin 120 = 4\sqrt{3}\,\text{cm}$

6.28 Time period of a loaded spring

$$T = 2\pi\sqrt{\frac{M + \frac{m}{3}}{k}} \tag{1}$$

where M is the suspended mass, m is the mass of the spring and k is the spring constant

$$0.89 = 2\pi\sqrt{\frac{1.5 + \frac{m}{3}}{k}} \tag{2}$$

$$1.13 = 2\pi\sqrt{\frac{2.5 + \frac{m}{3}}{k}} \tag{3}$$

Dividing the two equations and solving for m, we get $m = 0.39\,\text{kg}$.

6.29 **(a)** $k_A > k_B$

Let the springs be stretched by the same amount. Then the work done on the two springs will be

$$W_A = \frac{1}{2}k_A x^2 \quad W_B = \frac{1}{2}k_B x^2$$

$$\frac{W_A}{W_B} = \frac{k_A}{k_B}$$

Thus $W_A > W_B$, i.e. when two springs are stretched by the same amount, more work will be done on the stiffer spring.

(b) Let the two springs be stretched by equal force. Thus the work done

$$W_A = \frac{1}{2}k_A x^2 = \frac{1}{2}k_A\left(\frac{F}{k_A}\right)^2 = \frac{1}{2}\frac{F^2}{k_A}$$

$$W_B = \frac{1}{2}\frac{F^2}{k_B}$$

$$\therefore \quad \frac{W_A}{W_B} = \frac{k_B}{k_A}$$

Thus when two springs are stretched by the same force, less work will be done on the stiffer spring.

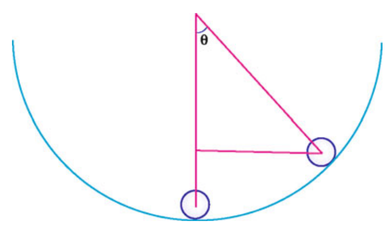


Fig. 6.19

6.30 $K_{\text{trans}} + K_{\text{rot}} + U = C = \text{constant}$

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mg(R-r)(1-\cos\theta) = C$$

$$\text{Now} \quad I = \frac{1}{2}mr^2 \quad \omega = \frac{v}{r}$$

$$\frac{3}{4}m\left(\frac{dx}{dt}\right)^2 + mg(R-r)\frac{\theta^2}{2} = C$$

Differentiating with respect to time

$$\frac{3}{2}m\frac{d^2x}{dt^2}\frac{dx}{dt} + mg(R-r)\theta\frac{d\theta}{dt} = 0$$

Now $x = (R-r)\theta$

$$\therefore \quad \frac{3}{2}\frac{d^2x}{dt^2}(R-r)\frac{d\theta}{dt} + gx\frac{d\theta}{dt} = 0$$

Cancelling $\frac{d\theta}{dt}$ throughout

$$\frac{d^2x}{dt^2} + \frac{2}{3}\frac{gx}{(R-r)} = 0$$

which is the equation for SHM, with

$$\omega^2 = \frac{2}{3}\frac{g}{R-r}$$
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3(R-r)}{2g}}$$

6.3.2 Physical Pendulums

6.31 If α is the angular acceleration, the torque τ is given by

$$\tau = I\alpha = I\frac{d^2\phi}{dt^2} \tag{1}$$

The restoring torque for an angular displacement ϕ is

$$\tau = -MgD\sin\phi \tag{2}$$

which arises due to the tangential component of the weight. Equating the two torques for small ϕ,

$$I\frac{d^2\phi}{dt^2} = -MgD\sin\phi = -MgD\,\phi$$
$$\text{or} \quad \frac{d^2\phi}{dt^2} + \frac{MgD}{I}\phi = 0 \tag{3}$$

which is the equation for SHM with

$$\omega^2 = \frac{MgD}{I} \tag{4}$$
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{MgD}}$$

6.32 Equation for the oscillatory motion is obtained by putting $I = \frac{1}{3}ML^2$ and $D = \frac{L}{2}$ in (3) of prob. (6.31).

$$\frac{d^2\theta}{dt^2} + \frac{MgD}{I}\theta = 0 \tag{3}$$

$$\frac{d^2\theta}{dt^2} + \frac{3}{2}\frac{g}{L}\theta = 0$$

$$\omega^2 = \frac{3}{2}\frac{g}{L}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2L}{3g}} \tag{4}$$

For a simple pendulum

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{5}$$

Comparing (4) and (5), the equivalent length of a simple pendulum is $l = \frac{2}{3}L$.

6.33 From the results of prob. (6.31) the time period of a physical pendulum is given by

$$T = 2\pi\sqrt{\frac{I}{MgD}} \tag{1}$$

where I is the moment of inertia about the pivot A, Fig. 6.9.

Now $I = I_C + MD^2$ and $I_C = Mk^2$ (2)

where k is the radius of gyration. Formula (1) then becomes

$$T = 2\pi\sqrt{\frac{k^2 + D^2}{gD}} \tag{3}$$

and the length of the simple equivalent pendulum is $D + \frac{k^2}{D}$.

If a point B be taken on AG such that AB $= D + \frac{k^2}{D}$, A and B are known as the centres of suspension and oscillation, respectively. Here G is the centre of mass (CM) of the physical pendulum.

Suppose now the body is suspended at B, then the time of oscillation is obtained by substituting $\frac{k^2}{D}$ for D in the expression

$$2\pi\sqrt{\frac{k^2+D^2}{gD}} \text{ and is therefore } 2\pi\sqrt{\frac{k^2+\frac{k^4}{D^2}}{g\frac{k^2}{D}}} \text{ i.e. } 2\pi\sqrt{\frac{D^2+k^2}{gD}}$$

Thus the centres of suspension and oscillation are convertible, for if the body be suspended from either it will make small vibrations in the same time as a simple pendulum whose length L is the distance between these centres.

$$T = 2\pi\sqrt{\frac{L}{g}} \quad \text{or} \quad g = \frac{4\pi^2 L}{T^2}$$

6.34 $$\omega = \sqrt{\frac{mgd}{I}} \tag{1}$$

$$d = \frac{4R}{3\pi} \tag{2}$$

the distance of the point of suspension from the centre of mass

$$I = \frac{mR^2}{2} \tag{3}$$

Substituting (2) and (3) in (1) and simplifying

$$\omega = \sqrt{\frac{8g}{3\pi R}}$$

6.35 $$T = 2\pi\sqrt{\frac{I}{mgd}}$$

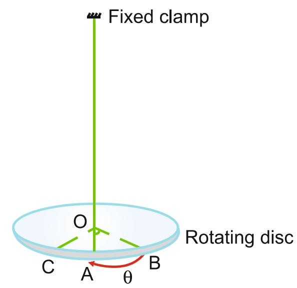
$$T_1 = 2\pi\sqrt{\frac{mr^2+mr^2}{mgr}} = 2\pi\sqrt{\frac{2r}{g}}$$

$$T_2 = 2\pi\sqrt{\frac{\frac{1}{2}mr^2+mr^2}{mgr}} = 2\pi\sqrt{\frac{3}{2}\frac{r}{g}}$$

$$\therefore \quad \frac{T_1}{T_2} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$$

6.36 In Fig. 6.20 OA is the reference line or the disc in the equilibrium position. If the disc is rotated in the horizontal plane so that the reference line occupies the line OB, the wire would have twisted through an angle θ. The twisted wire will exert a restoring torque on the disc causing the reference line to move to

Fig. 6.20



its original position. For small twists the restoring torque will be proportional to the angular displacement in accordance with Hooke's law.

$$\tau = -C\theta \tag{1}$$

where C is known as torsional constant. If I is the moment of inertia of the disc about its axis, α the angular acceleration, the torque τ is given by

$$\tau = I\alpha = I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \tag{2}$$

Comparing (1) and (2)

$$I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -C\theta$$

$$\text{or} \quad \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{C}{I}\theta = 0 \tag{3}$$

which is the equation for angular SHM with $\omega^2 = \dfrac{C}{I}$. Time period for small oscillations is given by

$$T = 2\pi\sqrt{\frac{I}{C}} \tag{4}$$

6.37 Total kinetic energy of the system

$$K = K(\text{mass}) + K(\text{pulley}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2$$

Replacing x by $r\theta$ and $\dot{x}$ by $r\dot{\theta}$

$$K = \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}(mr^2 + I)\dot{\theta}^2$$

Potential energy of the spring

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kr^2\theta^2$$

Total energy

$$E = K + U = \frac{1}{2}(mr^2 + I)\dot{\theta}^2 + \frac{1}{2}kr^2\theta^2 = \text{constant}$$

Differentiating with respect to time

$$\frac{\mathrm{d}E}{\mathrm{d}t} = (mr^2 + I)\dot{\theta}\cdot\ddot{\theta} + kr^2\theta\cdot\dot{\theta} = 0$$

Cancelling $\dot{\theta}$

$$\ddot{\theta} + \frac{kr^2\theta}{mr^2 + I} = 0$$

which is the equation for angular SHM with

$\omega^2 = \dfrac{kr^2}{mr^2 + I}$. Therefore

$$\omega = \sqrt{\frac{kr^2}{mr^2 + I}}$$

6.38 Let at any instant the centre of the cylinder be displaced by x towards right. Then the spring at C is compressed by x while the spring at P is elongated by $2x$. If $v = \dot{x}$ is the velocity of the centre of mass of the cylinder and $\omega = \dot{\theta}$ its angular velocity, the total energy in the displaced position will be

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_C\dot{\theta}^2 + \frac{1}{2}k_1x^2 + \frac{1}{2}k_2(2x)^2 \tag{1}$$

Substituting $x = r\theta$, $\dot{x} = r\dot{\theta}$, and $I_C = \dfrac{1}{2}mr^2$, where r is the radius of the cylinder, (1) becomes

$$E = \frac{3}{4}mr^2\dot{\theta}^2 + \frac{1}{2}r^2(k_1 + 4k_2)\theta^2 = \text{constant}$$
$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{3}{2}mr^2\dot{\theta}\ddot{\theta} + r^2(k_1 + 4k_2)\theta\dot{\theta} = 0$$
$$\therefore\quad \ddot{\theta} + \frac{2}{3m}(k_1 + 4k_2)\theta = 0$$

which is the equation for angular SHM with $\omega^2 = \frac{2}{3m}(k_1 + 4k_2)$.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3m}{2(k_1 + 4k_2)}}$$

6.39 $U(x) = \frac{a}{x^2} - \frac{b}{x}$

Equilibrium position is obtained by minimizing the function $U(x)$.

$$\frac{dU}{dx} = -\frac{2a}{x^3} + \frac{b}{x^2} = 0$$
$$x = x_0 = \frac{2a}{b}$$

Measuring distances from the equilibrium position and replacing x by $x + \frac{2a}{b}$

$$F = -\frac{dU}{dx} = \frac{2a}{x^3} - \frac{b}{x^2}$$
$$F = \frac{2a}{(x + 2a/b)^3} - \frac{b}{(x + 2a/b)^2}$$
$$= \frac{2a}{(2a/b)^3}\left(1 + \frac{bx}{2a}\right)^{-3} - \frac{b}{(2a/b)^2}\left(1 + \frac{bx}{2a}\right)^{-2}$$

Since the quantity $bx/2a$ is assumed to be small, use binomial expansion retaining terms up to linear in x.

$$F = -\frac{b^4 x}{8a^3}$$

Acceleration $a = \frac{F}{m} = -\frac{b^4 x}{8a^3 m} = -\omega^2 x$

where $\omega = \sqrt{\frac{b^4}{8a^3 m}}$

$$T = \frac{2\pi}{\omega} = 4\pi\sqrt{\frac{2ma^2}{b^4}}$$

6.3.3 Coupled Systems of Masses and Springs

6.40 Let spring 1 undergo an extension x_1 due to force F. Then $x_1 = \dfrac{F}{k_1}$. Similarly, for spring 2, $x_2 = \dfrac{F}{k_2}$.
The force is the same in each spring, but the total displacement x is the sum of individual displacements:

$$x = x_1 + x_2 = \frac{F}{k_1} + \frac{F}{k_2}$$

$$k_{\text{eq}} = \frac{F}{x} = \frac{F}{x_1 + x_2} = \frac{F}{\dfrac{F}{k_1} + \dfrac{F}{k_2}} = \frac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2}} = \frac{k_1 k_2}{k_1 + k_2}$$

$$\therefore \quad T = 2\pi\sqrt{\frac{m}{k_{\text{eq}}}} = 2\pi\sqrt{\frac{(k_1 + k_2)m}{k_1 k_2}}$$

6.41 The displacement is the same for both the springs and the total force is the sum of individual forces.

$$F_1 = k_1 x, \;\; F_2 = k_2 x$$
$$F = F_1 + F_2 = (k_1 + k_2)x$$
$$k_{\text{eq}} = \frac{F}{x} = k_1 + k_2$$
$$T = 2\pi\sqrt{\frac{m}{k_{\text{eq}}}} = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

6.42 Let the centre of mass be displaced by x. Then the net force

$$F = -k_1 x - k_2 x = -(k_1 + k_2)x$$

Acceleration $a = \dfrac{F}{m} = -(k_1 + k_2)\dfrac{x}{m} = -\omega^2 x$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

6.43 Spring constant of the wire is given by

$$k' = \frac{YA}{L} \tag{1}$$

Since the spring and the wire are in series, the effective spring constant k_{eff} is given by

$$k_{\text{eff}} = \frac{k'k}{k+k} \tag{2}$$

The time period of oscillations is given by

$$T = 2\pi\sqrt{\frac{m}{k_{\text{eff}}}} \tag{3}$$

Combining (1), (2) and (3)

$$T = 2\pi\sqrt{\frac{m(YA+kL)}{YAk}}$$

6.44 In Fig. 6.15, C is the point of contact around which the masses M and m rotate. As it is the instantaneous centre of zero velocity, the equation of motion is of the form $\Sigma\tau_c = I_c\ddot{\theta}$, where I_c is the moment of inertia of masses M and m with respect to point C. Now

$$I_c = \left(\frac{1}{2}MR^2 + MR^2\right) + md^2 \tag{1}$$

where $d^2 = L^2 + R^2 - 2RL\cos\theta$. (2)

For small oscillations, $\sin\theta \simeq \theta$, $\cos\theta \simeq 1$ and

$$I_c = \frac{3MR^2}{2} + m(L-d)^2 \tag{3}$$

Therefore the equation of motion become

$$\left[\frac{3MR^2}{2} + m(L-d)^2\right]\ddot{\theta} = -mgL\sin\theta = -mgL\theta$$

$$\text{or} \quad \ddot{\theta} + \frac{mgL}{3MR^2/2 + m(L-d)^2}\theta = 0$$

$$\therefore \quad \omega = \sqrt{\frac{mgL}{3MR^2/2 + m(L-d)^2}}\ \text{rad/s}$$

6.45 Figure 6.21 shows the semicircular disc tilted through an angle θ compared to the equilibrium position (b). G is the centre of mass such that $a = \text{OG} = \dfrac{4r}{3\pi}$, where r is the radius.

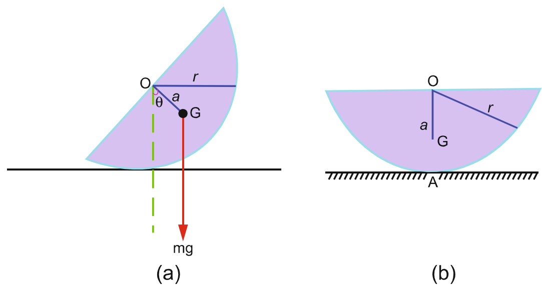


Fig. 6.21

We use the energy method.

$$K\,(\max) = \frac{1}{2} I_A \omega^2$$
$$= \frac{1}{2}(I_G + \overline{GA}^2)\omega^2$$
$$= \frac{1}{2}[I_0 - ma^2 + m(r-a)^2]\omega^2 = \frac{1}{2}\left[\frac{1}{2}mr^2 + mr(r-2a)\right]\omega^2$$
$$= mr\left(\frac{3}{4}r - a\right)\omega^2$$

$$K_{\max} = U_{\max}$$
$$mr\left(\frac{3}{4}r - a\right)\omega^2 = mga(1-\cos\theta)$$

But $a = \dfrac{4r}{3\pi}$

$$\omega = 4\sqrt{\frac{(1-\cos\theta)g}{(9\pi - 16)r}}$$

6.46 Referring to Fig. 6.16, take torques about the two hinged points P and Q.

$$mb^2\ddot{\theta}_1 = -mgb\theta_1 - kb^2(\theta_1 - \theta_2)$$

The left side gives the net torque which is the product of moment of inertia about P and the angular acceleration. The first term on the right side gives the torque of the force mg, which is force times the perpendicular distance from the vertical through P. The second term on the right side is the torque produced by the spring which is $k(x_1 - x_2)$ times the perpendicular distance from P, that is, $k(x_1 - x_2)b$ or $k(\theta_1 - \theta_2)b^2$. The second equation of motion can be similarly written. Thus, the two equations of motion are

$$mb\ddot{\theta}_1 + mg\theta_1 + kb(\theta_1 - \theta_2) = 0 \tag{1}$$

$$mb\ddot{\theta}_2 + mg\theta_2 + kb(\theta_2 - \theta_1) = 0 \tag{2}$$

The harmonic solutions are

$$\theta_1 = A\sin\omega t, \quad \theta_2 = B\sin\omega t \tag{3}$$

$$\ddot{\theta}_1 = -A\omega^2\sin\omega t, \quad \ddot{\theta}_2 = -B\omega^2\sin\omega t \tag{4}$$

Substituting (3) and (4) in (1) and (2) and simplifying

$$(mg + kb - mb\omega^2)A - kb\,B = 0 \tag{5}$$

$$-kbA + (mg + kb - mb\omega^2)B = 0 \tag{6}$$

The frequency equation is obtained by equating to zero the determinant formed by the coefficients of A and B.

$$\begin{vmatrix} \left(mg + kb - mb\omega^2\right) & -kb \\ -kb & \left(mg + kb - mb\omega^2\right) \end{vmatrix} = 0$$

Expanding the determinant and solving for ω we obtain

$$\omega_1 = \sqrt{\frac{g}{b}}, \quad \omega_2 = \sqrt{\frac{g}{b} + \frac{2k}{m}}$$

6.47 In prob. (6.46) equations of motion (1) and (2) can be re-written in terms of Cartesian coordinates x_1 and x_2 since $x_1 = b\theta_1$ and $x_2 = b\theta_2$.

$$m\ddot{x}_1 + \frac{mgx_1}{b} + k(x_1 - x_2) = 0 \tag{1}$$

$$m\ddot{x}_2 + \frac{mgx_2}{b} + k(x_2 - x_1) = 0 \tag{2}$$

It is possible to make linear combinations of x_1 and x_2 such that a combination involves but a single frequency. These new coordinates X_1 and X_2, called normal coordinates, vary harmonically with but a single frequency. No energy transfer occurs from one normal coordinate to another. They are completely independent.

$$x_1 = \frac{X_1 + X_2}{2}, \quad x_2 = \frac{X_1 - X_2}{2} \tag{3}$$

Substituting (3) in (1) and (2)

$$\frac{m}{2}(\ddot{X}_1 + \ddot{X}_2) + \frac{mg}{2b}(X_1 + X_2) + kX_2 = 0 \tag{4}$$

$$\frac{m}{2}(\ddot{X}_1 - \ddot{X}_2) + \frac{mg}{2b}(X_1 - X_2) - kX_2 = 0 \tag{5}$$

Adding (4) and (5)

$$m\ddot{X}_1 + \frac{mg}{b}X_1 = 0 \tag{6}$$

which is a linear equation in X_1 alone with constant coefficients.
Subtracting (5) from (4), we obtain

$$m\ddot{X}_2 + \left(\frac{mg}{b} + 2k\right)X_2 = 0 \tag{7}$$

This is again a linear equation in X_2 as the single dependent variable. Since the coefficients of X_1 and X_2 are positive, both (6) and (7) are differential equations of simple harmonic motion having frequencies $\omega_1 = \sqrt{\frac{g}{b}}$ and $\omega_2 = \sqrt{\frac{g}{b} + \frac{2k}{m}}$. Thus when equations of motion are expressed in normal coordinates, the equations are linear with constant coefficients and each contains only one dependent variable.
We now calculate the energy in normal coordinates. The potential energy arises due to the energy stored in the spring and due to the position of the body.

$$V = \frac{1}{2}k(x_1 - x_2)^2 + mgb(1 - \cos\theta_1) + mgb(1 - \cos\theta_2) \tag{8}$$

Now $b(1 - \cos\theta_1) = b\frac{\theta_1^2}{2} = \frac{x_1^2}{2b}$

Similarly $b(1 - \cos\theta_2) = \frac{x_2^2}{2b}$

Hence $V = \frac{k}{2}(x_1 - x_2)^2 + \frac{mgx_1^2}{2b} + \frac{mgx_2^2}{2b}$ (9)

Kinetic energy $T = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right)$ (10)

Although there is no cross-product term in (10) for the kinetic energy, there is one in the potential energy of the spring in (9). The presence of the cross-product term means coupling between the components of the vibrating system. However, in normal coordinates the cross-product terms are avoided. Using (3) in (9) and (10)

$$V = \frac{mg}{4b} X_1^2 + \left(\frac{mg}{4b} + \frac{k}{2} \right) X_2^2 \tag{11}$$

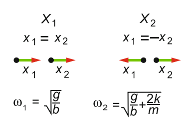
$$T = \frac{m}{4} (\dot{X}_1^2 + \dot{X}_2^2) \tag{12}$$

Thus the cross terms have now disappeared. The potential energy V is now expressed as a sum of squares of normal coordinates multiplied by constant coefficients and kinetic energy. T is expressed in the form of a sum of squares of the time derivatives of the normal coordination.

We can now describe the mode of oscillation associated with a given normal coordinate. Suppose $X_2 = 0$, then $0 = x_1 - x_2$, which implies $x_1 = x_2$. The mode X_1 is shown in Fig. 6.22, where the particles oscillate in phase with frequency $\omega_1 = \sqrt{g/b}$ which is identical for a simple pendulum of length b. Here the spring plays no role because it remains unstretched throughout the motion.

If we put $X_1 = 0$, then we get $x_1 = -x_2$. Here the pendulums are out of phase. The X_2 mode is also illustrated in Fig. 6.22, the associated frequency being $\omega_2 = \sqrt{\dfrac{g}{b} + \dfrac{2k}{m}}$. Note that $\omega_2 > \omega_1$, because greater potential energy is now available due to the spring.

Fig. 6.22



6.48 $y = A \cos 6\pi t \sin 90\pi$

Now $\sin C + \sin D = 2 \sin \dfrac{1}{2}(C + D) \cos \dfrac{1}{2}(C - D)$

Comparing the two equations we get

$$\frac{C + D}{2} = 90\pi \quad \frac{C - D}{2} = 6\pi$$

$$\therefore \quad C = 96\pi \text{ and } D = 84\pi$$

$$\omega_1 = 2\pi f_1 = 96\pi \quad \text{or} \quad f_1 = 48\,\text{Hz}$$

$$\omega_2 = 2\pi f_2 = 84\pi \quad \text{or} \quad f_2 = 42\,\text{Hz}$$

Thus the frequency of the component vibrations are 48 Hz and 42 Hz. The beat frequency is $f_1 - f_2 = 48 - 42 = 6$ beats/s.

6.49 The frequency is given by

$$f = \frac{1}{2\pi} \sqrt{\frac{k}{\mu}}$$

where μ is the reduced mass given by

$$\mu = \frac{m_H m_{Cl}}{m_H + m_{Cl}} = \frac{10. \times 36.46}{1.0 + 36.46}$$
$$= 0.9733\,\text{amu} = 0.9733 \times 1.66 \times 10^{-27}\,\text{kg} = 1.6157 \times 10^{-27}\,\text{kg}$$
$$f = \frac{1}{2\pi}\sqrt{\frac{480}{1.6157 \times 10^{-27}}} = 8.68 \times 10^{13}\,\text{Hz}$$

6.50 Each vibration is plotted as a vector of magnitude which is proportional to the amplitude of the vibration and in a direction which is determined by the phase angle. Each phase angle is measured with respect to the x-axis. The vectors are placed in the head-to-tail fashion and the resultant is obtained by the vector joining the tail of the first vector with the head of the last vector, Fig. 6.23. $y_1 = \text{OA} = 1$ unit, parallel to x-axis in the positive direction, $y_2 = \text{AB} = \frac{1}{2}$ unit parallel to y-axis and $y_3 = \text{BC} = \frac{1}{3}$ unit parallel to the x-axis in the negative direction.

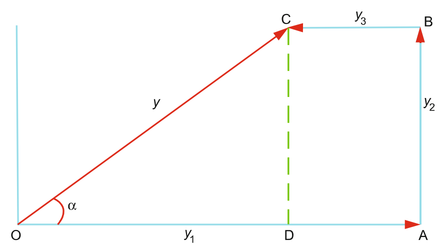


Fig. 6.23

The resultant is given by OC both in magnitude and in direction. From the geometry of the diagram

$$y = \text{OC} = \sqrt{\text{OD}^2 + \text{DC}^2} = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{2}\right)^2} = 5/6$$
$$\alpha = \tan^{-1}(\text{CD}/\text{OD}) = \tan^{-1}\left(\frac{1/2}{2/3}\right) = \tan^{-1}(3/4) = 37^\circ$$

6.3.4 Damped Vibrations

6.51 The logarithmic decrement Δ is given by

$$\Delta = bT' \qquad (1)$$

where $T' = \frac{2\pi}{\omega'}$ is the time period for damped vibration and $b = \sqrt{\omega_0^2 - \omega'^2}$, where ω_0 and ω' are the angular frequencies for natural and damped vibrations, respectively.

$$\Delta = 2\pi\sqrt{\frac{\omega_0^2}{\omega'^2} - 1} = 2\pi\sqrt{\frac{f^2}{f'^2} - 1} = 2\pi\sqrt{\left(\frac{20}{16}\right)^2 - 1} = \frac{3\pi}{2}$$

6.52 The equation for damped oscillations is $4\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{r\mathrm{d}x}{\mathrm{d}t} + 32x = 0$
Dividing the equation by 4

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{r}{4}\frac{\mathrm{d}x}{\mathrm{d}t} + 8x = 0$$

Comparing the equation with the standard equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{r}{m}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{k}{m}x = 0$$

$$m = 4, \quad \frac{k}{m} = 8 \rightarrow k = 32$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{8} = 2\sqrt{2}$$

The quantity $b = \frac{r}{2m}$ represents the decay rate of oscillation where r is the resistance constant.

(a) The motion will be underdamped if

$$b < \omega_0 \text{ or } \frac{r}{2m} < \sqrt{\frac{k}{m}} \text{ or } r < 2\sqrt{km}$$

$$\text{i.e.} \quad r < 2\sqrt{32 \times 4} \text{ or } r < 16\sqrt{2}$$

(b) The motion is overdamped if $r > 16\sqrt{2}$.
(c) The motion is critically damped if $r = 16\sqrt{2}$.

6.53 **(a)** $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{20}{4}} = 2.23\,\text{rad/s}$

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{2.23} = 2.8\,\text{s}$$

(b) $\omega' = \dfrac{2\pi}{T'} = \dfrac{2\pi}{10} = 0.628\,\text{rad/s}$

$b = \sqrt{\omega_0^2 - \omega'^2} = \sqrt{2.236^2 - 0.628^2} = 2.146$

$\dfrac{r}{2m} = b$ or $r = 2mb = 2 \times 4 \times 2.146 = 17.17\,\text{Ns/m}$

(c) $\Delta = bT' = 2.146 \times 10 = 21.46$

6.54 $\dfrac{d^2x}{dt^2} + \dfrac{2dx}{dt} + 5x = 0$

Let $x = e^{\lambda t}$. The characteristic equation then becomes $\lambda^2 + 2\lambda + 5 = 0$ with the roots $\lambda = -1 \pm 2i$

$x = Ae^{-(1-2i)t} + Be^{-(1+2i)t}$

or $\quad x = e^{-t}[C\cos 2t + D\sin 2t]$

where A, B, C and D are constants.

C and D can be determined from initial conditions. At $t = 0$, $x = 5$. Therefore $C = 5$.

Also $\dfrac{dx}{dt} = -e^{-t}(C\cos 2t + D\sin 2t) + e^{-t}(-2C\sin 2t + 2D\cos 2t)$

At $t = 0$, $\dfrac{dx}{dt} = -3$

$\therefore \quad -3 = -C + 2D = -5 + 2D$

$\therefore \quad D = 1$

The complete solution is

$x = e^{-t}(5\cos 2t + \sin 2t)$

6.55 $F = mg = kx$

$k = \dfrac{mg}{x} = \dfrac{(1.0)(9.8)}{0.2} = 49\,\text{N/m}$

Equation of motion is

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = 0 \qquad (1)$$

Substituting $m = 1.0$, $r = 14$, $k = 49$, (1) becomes

$$\frac{d^2x}{dt^2} + 14\frac{dx}{dt} + 49x = 0 \tag{2}$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{49}{1}} = 7\,\text{rad/s}$$

$$b = \frac{r}{2m} = \frac{14}{2 \times 1} = 7$$

(a) Therefore the motion is critically damped.
(b) For critically damped motion, the equation is

$$x = x_0 e^{-bt}(1 + bt) \tag{3}$$

With $b = 7$ and $x_0 = 1.5$, (3) becomes

$$x = 1.5\,e^{-7t}(1 + 7t)$$

6.56 $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{150}{60}} = 5$

Damping force $f_r = r \cdot v$

$$\text{or} \quad r = \frac{f_r}{v} = \frac{80}{2} = 40$$

$$b = \frac{r}{2m} = \frac{40}{2 \times 6} = 3.33\,\text{rad/s}$$

$$\omega(\text{res}) = \sqrt{\omega_0^2 - 2b^2} = \sqrt{5^2 - 2 \times (3.33)^2} = 1.66\,\text{rad/s}$$

$$f(\text{res}) = \frac{\omega(\text{res})}{2\pi} = 0.265\,\text{vib/s}$$

6.57 Equation of motion is

$$\frac{2d^2x}{dt^2} + 1.5\frac{dx}{dt} + 40x = 12\cos 4t$$

Dividing throughout by 2

$$\frac{d^2x}{dt^2} + 0.75\frac{dx}{dt} + 20x = 6\cos 4t$$

Comparing this with the standard equation

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = p\cos\omega t$$
$$b = 0.375;\ \omega_0 = \sqrt{20},\ p = 6,\ \omega = 4$$
$$Z_M = \sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4b^2\omega^2} = \sqrt{(20-16)^2 + 4\times 0.375^2 \times 4^2} = 5$$

(a) $A = \dfrac{p}{Z_m} = \dfrac{6}{5} = 1.2$

(b) $\tan\varepsilon = \dfrac{2b\omega}{\omega_0^2 - \omega^2} = \dfrac{2\times 0.375\times 4}{(20-16)} = 0.75 \to \varepsilon = 37°$

(c) $Q = \dfrac{\omega_0 m}{r} = \dfrac{\omega_0}{2b} = \dfrac{\sqrt{20}}{2\times 0.375} = 5.96$

(d) $F = pm = 6\times 2 = 12$

$$W = \frac{F^2}{2Z_m}\sin\varepsilon = \frac{12^2}{2\times 5}\sin 37° = 8.64\,\text{W}$$

6.58 $Q = \dfrac{2\pi t_c}{T} = 2\pi t_c f = 2\pi \times 2\times 100 = 1256$

6.59 **(a)** Energy is proportional to the square of amplitude

$$E = \text{const.}A^2$$
$$\frac{dE}{E} = \frac{2dA}{A} = \frac{2\times 5}{100} = 10\%$$

(b) $E = E_0 e^{-t/t_c}$

$$\therefore \quad \frac{E}{E_0} = \frac{A^2}{A_0^2} = e^{-t/t_c}$$
$$\therefore \quad \frac{A}{A_0} = \frac{95}{100} = e^{-t/2t_c}$$
$$\frac{t}{2t_c} = \ln\left(\frac{100}{95}\right) = 0.05126$$
$$t_c = \frac{3}{2\times 0.05126} = 29.26\,\text{s}$$

(c) $Q = \dfrac{2\pi t_c}{T} = \dfrac{(2\pi)(29.26)}{3.0} = 61.25$

6.60 **(a)** $E = E_0 e^{-t/t_c}$

$$\therefore \quad \frac{t}{t_c} = \ln\left(\frac{E_0}{E}\right) = \ln 2 = 0.693$$

Put $\quad t = nT$

$$\therefore \quad n = 0.693\frac{t_c}{T}$$

But $\quad -\dfrac{\Delta E}{E} = \dfrac{3}{100} = \dfrac{T}{t_c}$

$$\therefore \quad n = 0.693 \times \frac{100}{3} = 23.1$$

(b) $Q = \dfrac{2\pi t_c}{T} = 2\pi \times \dfrac{100}{3} = 209.3$

6.61 $\omega' = \omega_0\sqrt{1 - \dfrac{1}{4Q^2}} = \dfrac{9\omega_0}{10}$

$\therefore \quad Q = 1.147$

$Q = \dfrac{2\pi t_c}{T}$

or $\dfrac{T}{2t_c} = \dfrac{\pi}{Q} = \dfrac{3.14}{1.147} = 2.737$

$\dfrac{A}{A_0} = e^{-T/2t_c} = e^{-2.737} = 0.065$

6.62 $\omega'^2 = \omega_0^2 - b^2$ (1)

where $b = \dfrac{r}{2m}$ (2)

$$\omega' = \omega_0\left(1 - \frac{b^2}{\omega_0^2}\right)^{1/2} \approx \omega_0\left(1 - \frac{b^2}{2\omega_0^2}\right) \quad (3)$$

where we have expanded the radical binomially, assuming that $b/\omega_0 << 1$.

Now $\quad \omega_0^2 = \dfrac{k}{m}$ (4)

$$\therefore \quad \frac{b^2}{2\omega_0^2} = \frac{r^2}{8mk} \quad (5)$$

Substituting (5) in (3)

$$\omega' = \omega_0\left(1 - \frac{r^2}{8mk}\right) \quad \text{(for small damping)}$$

6.63 The time elapsed between successive maximum displacements of a damped harmonic oscillator is represented by T', the period.

$$T' = \frac{2\pi}{\omega'} = \frac{2\pi}{\sqrt{\omega_0^2 - b^2}} = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{r^2}{4m^2}}} = \frac{4\pi m}{\sqrt{4km - r^2}} = \text{constant}$$

6.64 Force $= mg = kx$

$$\therefore \quad \frac{k}{m} = \frac{g}{x} = \frac{980}{9.8} = 100$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{100} = 10\,\text{rad/s}$$

$$\Delta = bT' = \frac{2\pi b}{\sqrt{\omega_0^2 - b^2}} \tag{1}$$

Substituting $\Delta = 3.1$ and $\omega_0 = 10$ in (1), $b = 4.428$

$$T' = \frac{2\pi}{\sqrt{\omega_0^2 - b^2}} = \frac{2\pi}{\sqrt{10^2 - (4.428)^2}} = 0.7\,\text{s}$$

6.65 $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{2\mathrm{d}x}{\mathrm{d}t} + 8x = 16\cos 2t \tag{1}$$

This is the equation for the forced oscillations, the standard equation being

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + r\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F\cos\omega t \tag{2}$$

Comparing (1) and (2) we find

$$m = 1\,\text{kg},\ r = 2,\ k = 8,\ F = 16\,N,\ \omega = 2$$

(a) $\omega_0 = 2\pi f_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{1}} = 2\sqrt{2}$

$$\therefore \quad f_0 = \frac{2\sqrt{2}}{2\pi} = \frac{\sqrt{2}}{\pi}/\text{s}$$

(b) $\omega = 2\pi f = 2$

$$\therefore \quad f = \frac{2}{2\pi} = \frac{1}{\pi}/\text{s}$$

6.66 $E(t) = E_0 e^{-t/t_c}$

$$\therefore \quad \frac{E(t_{1/2})}{E_0} = \frac{1}{2} = e^{-t_{1/2}/t_c}$$

or $t_{1/2} = t_c \ln 2$

6.67 $A(t) = A_0 e^{-t/2t_c}$

$$\frac{A(t)}{A_0} = \frac{1}{e}$$

If $t = 2t_c = 8T$

$$\therefore \quad t_c = 4T$$

$$Q = 2\pi \frac{t_c}{T} = 2\pi \times 4 = 25.1$$