ω_0 being the resonance frequency.

$$x = A\cos(\omega t - \varepsilon) \tag{6.66}$$

$$\tan\varepsilon = \frac{2b\omega}{\omega_0^2 - \omega^2} \tag{6.67}$$

Mechanical impedance

$$Z_m = \sqrt{(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2} \tag{6.68}$$

$$A = \frac{p}{Z_m} \tag{6.69}$$

$$Q = \frac{\omega_0}{2b} \tag{6.70}$$

Power

$$W = \frac{F^2 - \sin\varepsilon}{2Z_m} \tag{6.71}$$

6.2 Problems

6.2.1 Simple Harmonic Motion (SHM)

6.1 The total energy of a particle executing SHM of period 2π s is 0.256 J. The displacement of the particle at $\pi/4$ s is $8\sqrt{2}$ cm. Calculate the amplitude of motion and mass of the particle.

6.2 A particle makes SHM along a straight line and its velocity when passing through points 3 and 4 cm from the centre of its path is 16 and 12 cm/s, respectively. Find **(a)** the amplitude; **(b)** the time period of motion.

[Northern Universities of UK]

6.3 A small bob of mass 50 g oscillates as a simple pendulum, with amplitude 5 cm and period 2 s. Find the velocity of the bob and the tension in the supporting thread when velocity of the bob is maximum.

[University of Aberystwyth, Wales]

6.4 A particle performs SHM with a period of 16 s. At time $t = 2$ s, the particle passes through the origin while at $t = 4$ s, its velocity is 4 m/s. Show that the amplitude of the motion is $32\sqrt{2}/\pi$.

[University of Dublin]

6.5 Show that given a small vertical displacement from its equilibrium position a floating body subsequently performs simple harmonic motion of period $2\pi\sqrt{V/Ag}$ where V is the volume of displaced liquid and A is the area of the plane of floatation. Ignore the viscous forces.

6.6 Imagine a tunnel bored along the diameter of the earth assumed to have constant density. A box is thrown into the tunnel (chute). **(a)** Show that the box executes SHM inside the tunnel about the centre of the earth. **(b)** Find the time period of oscillations.

6.7 A particle which executes SHM along a straight line has its motion represented by $x = 4\sin(\pi t/3 + \pi/6)$. Find **(a)** the amplitude; **(b)** time period; **(c)** frequency; **(d)** phase difference; **(e)** velocity; **(f)** acceleration, at $t = 1$ s, x being in cm.

6.8 **(a)** At what distance from the equilibrium position is the kinetic energy equal to the potential energy for a SHM?
(b) In SHM if the displacement is one-half of the amplitude show that the kinetic energy and potential energy are in the ratio 3:1.

6.9 A mass M attached to a spring oscillates with a period 2 s. If the mass is increased by 2 kg, the period increases by 1 s. Assuming that Hooke's law is obeyed, find the initial mass M.

6.10 A particle vibrates with SHM along a straight line, its greatest acceleration is $5\pi^2$ cm/s^2, and when its distance from the equilibrium is 4 cm the velocity of the particle is 3π cm/s. Find the amplitude and the period of oscillation of the particle.

6.11 If the maximum acceleration of a SHM is α and the maximum velocity is β, show that the amplitude of vibration is given by β^2/α and the period of oscillation by $2\pi\beta/\alpha$.

6.12 If the tension along the string of a simple pendulum at the lowest position is 1% higher than the weight of the bob, show that the angular amplitude of the pendulum is 0.1 rad.

6.13 A particle executes SHM and is located at $x = a$, b and c at time t_0, $2t_0$ and $3t_0$, respectively. Show that the frequency of oscillation is $\dfrac{1}{2\pi t_0}\cos^{-1}\dfrac{a+c}{2b}$.

6.14 A 4 kg mass at the end of a spring moves with SHM on a horizontal frictionless table with period 2 s and amplitude 2 m. Determine **(a)** the spring constant; **(b)** maximum force exerted on the spring.

6.15 A particle moves in the *xy*-plane according to the equations $x = a \sin \omega t$; $y = b \cos \omega t$. Determine the path of the particle.

6.16 (**a**) Prove that the force $\boldsymbol{F} = -kx\hat{i}$ acting in a SHO is conservative. (**b**) Find the potential energy of an SHO.

6.17 A 2 kg weight placed on a vertical spring stretches it 5 cm. The weight is pulled down a distance of 10 cm and released. Find **(a)** the spring constant; **(b)** the amplitude; **(c)** the frequency of oscillations.

6.18 A mass m is dropped from a height h on to a scale-pan of negligible weight, suspended from a spring of spring constant k. The collision may be considered to be completely inelastic in that the mass sticks to the pan and the pan begins to oscillate. Find the amplitude of the pan's oscillations.

6.19 A particle executes SHM along the x-axis according to the law $x = A \sin \omega t$. Find the probability $\mathrm{d}p(x)$ of finding the particle between x and $x + \mathrm{d}x$.

6.20 Using the probability density distribution for the SHO, calculate the mean potential energy and the mean kinetic energy over an oscillation.

6.21 A cylinder of mass m is allowed to roll on a smooth horizontal table with a spring of spring constant k attached to it so that it executes SHM about the equilibrium position. Find the time period of oscillations.

6.22 Two simple pendulums of length 60 and 63 cm, respectively, hang vertically one in front of the other. If they are set in motion simultaneously, find the time taken for one to gain a complete oscillation on the other.

[Northern Universities of UK]

6.23 A pendulum that beats seconds and gives correct time on ground at a certain place is moved to the top of a tower 320 m high. How much time will the pendulum lose in 1 day? Assume earth's radius to be 6400 km.

6.24 Taking the earth's radius as 6400 km and assuming that the value of g inside the earth is proportional to the distance from the earth's centre, at what depth below the earth's surface would a pendulum which beats seconds at the earth's surface lose 5 min in a day?

[University of London]

6.25 A U-tube is filled with a liquid, the total length of the liquid column being h. If the liquid on one side is slightly depressed by blowing gently down, the levels of the liquid will oscillate about the equilibrium position before finally coming to rest. **(a)** Show that the oscillations are SHM. **(b)** Find the period of oscillations.

6.26 A gas of mass m is enclosed in a cylinder of cross-section A by means of a frictionless piston. The gas occupies a length l in the equilibrium position and is at pressure P. **(a)** If the piston is slightly depressed, show that it will execute SHM. **(b)** Find the period of oscillations (assume isothermal conditions).

6.27 A SHM is given by $y = 8\sin\left(\frac{2\pi t}{\tau} + \varphi\right)$, the time period being 24 s. At $t = 0$, the displacement is 4 cm. Find the displacement at $t = 6$ s.

6.28 In a vertical spring-mass system, the period of oscillation is 0.89 s when the mass is 1.5 kg and the period becomes 1.13 s when a mass of 1.0 kg is added. Calculate the mass of the spring.

6.29 Consider two springs A and B with spring constants k_A and k_B, respectively, A being stiffer than B, that is, $k_A > k_B$. Show that

(a) when two springs are stretched by the same amount, more work will be done on the stiffer spring.
(b) when two springs are stretched by the same force, less work will be done on the stiffer spring.

6.30 A solid uniform cylinder of radius r rolls without sliding along the inside surface of a hollow cylinder of radius R, performing small oscillations. Determine the time period.

6.2.2 Physical Pendulums

6.31 Consider the rigid plane object of weight Mg shown in Fig. 6.7, pivoted about a point at a distance D from its centre of mass and displaced from equilibrium by a small angle φ. Such a system is called a physical pendulum. Show that the oscillatory motion of the object is simple harmonic with a period given by $T = 2\pi\sqrt{\frac{I}{MgD}}$ where I is the moment of inertia about the pivot point.

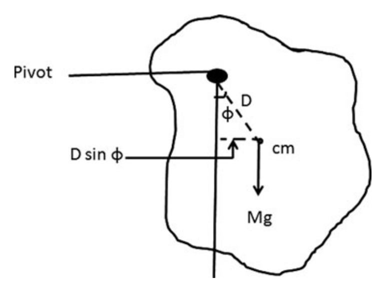
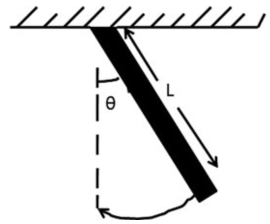


Fig. 6.7

6.32 A thin, uniform rod of mass M and length L swings from one of its ends as a physical pendulum (see Fig. 6.8). Given that the moment of inertia of a

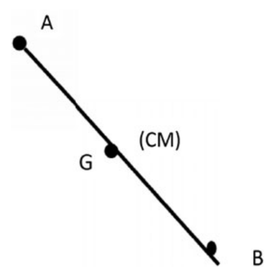
Fig. 6.8



uniform rod about one end is $I = \frac{1}{3}ML^2$, obtain an equation for the period of the oscillatory motion for small angles. What would be the length l of a simple pendulum that has the same period as the swinging rod?

6.33 The physical pendulum has two possible pivot points A and B, distance L apart, such that the period of oscillations is the same (Fig. 6.9). Show that the acceleration due to gravity at the pendulum's location is given by $g = 4\pi^2 L/T^2$.

Fig. 6.9



6.34 A semi-circular homogeneous disc of radius R and mass m is pivoted freely about the centre. If slightly tilted through a small angle and released, find the angular frequency of oscillations.

6.35 A ring is suspended on a nail. It can oscillate in its plane with time period T_1 or it can oscillate back and forth in a direction perpendicular to the plane of the ring with time period T_2. Find the ratio T_1/T_2.

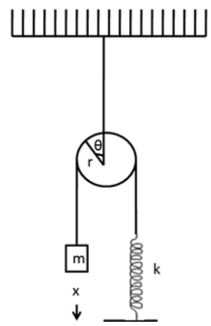
6.36 A torsional oscillator consists of a flat metal disc suspended by a wire. For small angular displacements show that time period is given by

$$T = 2\pi\sqrt{\frac{I}{C}}$$

where I is the moment of inertia about its axis and C is known as torsional constant given by $\tau = -C\theta$, where τ is the torque.

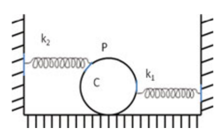
6.37 In the arrangement shown in Fig. 6.10, the radius of the pulley is r, its moment of inertia about the rotation axis is I and k is the spring constant. Assuming that the mass of the thread and the spring is negligible and that the thread does not slide over the frictionless pulley, calculate the angular frequency of small oscillations.

Fig. 6.10



6.38 Two unstretched springs with spring constants k_1 and k_2 are attached to a solid cylinder of mass m as in Fig. 6.11. When the cylinder is slightly displaced and released it will perform small oscillations about the equilibrium position. Assuming that the cylinder rolls without sliding, find the time period.

Fig. 6.11



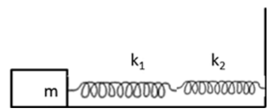
6.39 A particle of mass m is located in a one-dimensional potential field $U(x) = \dfrac{a}{x^2} - \dfrac{b}{x}$ where a and b are positive constants. Show that the period of small oscillations that the particle performs about the equilibrium position will be $T = 4\pi\sqrt{\dfrac{2a^3m}{b^4}}$

[Osmania University 1999]

6.2.3 Coupled Systems of Masses and Springs

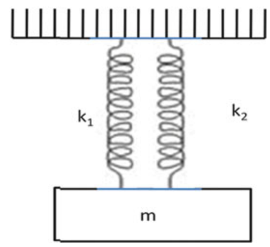
6.40 Two springs of constants k_1 and k_2 are connected in series, Fig. 6.12. Calculate the effective spring constant.

Fig. 6.12



6.41 A mass m is connected to two springs of constants k_1 and k_2 in parallel, Fig. 6.13. Calculate the effective (equivalent) spring constant.

Fig. 6.13



6.42 A mass m is placed on a frictionless horizontal table and is connected to fixed points A and B by two springs of negligible mass and of equal natural length with spring constants k_1 and k_2, Fig. 6.14. The mass is displaced along x-axis and released. Calculate the period of oscillation.

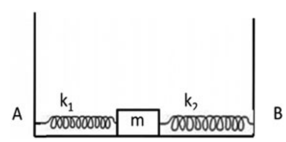


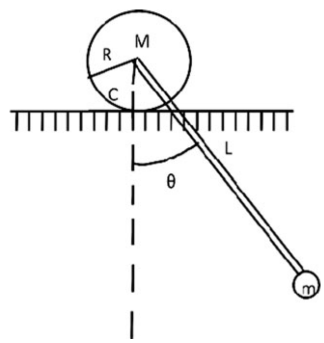
Fig. 6.14

6.43 One end of a long metallic wire of length L is tied to the ceiling. The other end is tied to a massless spring of spring constant k. A mass m hangs freely from the free end of the spring. The area of cross-section and the Young's modulus of the wire are A and Y respectively. The mass is displaced down and released. Show that it will oscillate with time period $T = 2\pi\sqrt{\dfrac{m(YA + kL)}{YAk}}$.

[Adapted from Indian Institute of Technology 1993]

6.44 The mass m is attached to one end of a weightless stiff rod which is rigidly connected to the centre of a uniform cylinder of radius R, Fig. 6.15. Assuming that the cylinder rolls without slipping, calculate the natural frequency of oscillation of the system.

Fig. 6.15



6.45 Find the natural frequency of a semi-circular disc of mass m and radius r which rolls from side to side without slipping.

6.46 Determine the eigenfrequencies and describe the normal mode motion for two pendula of equal lengths b and equal masses m connected by a spring of force constant k as shown in Fig. 6.16. The spring is unstretched in the equilibrium position.

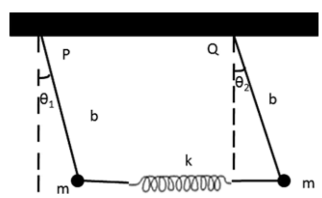


Fig. 6.16

6.47 In prob. (6.46) express the equations of motion and the energy in terms of normal coordinates. What are the characteristics of normal coordinates?

6.48 The superposition of two harmonic oscillations in the same direction leads to the resultant displacement $y = A\cos 6\pi t \sin 90\pi$, where t is expressed in seconds. Find the frequency of the component vibrations and the beat frequency.

6.49 Find the fundamental frequency of vibration of the HCl molecule. The masses of H and Cl may be assumed to be 1.0 and 36.46 amu.

$1\,\text{amu} = 1.66 \times 10^{-27}\,\text{kg}$ and $k = 480\,\text{N/m}$

6.50 Find the resultant of the vibrations $y_1 = \cos \omega t$, $y_2 = {}^1\!/_2 \cos(\omega t + \pi/2)$ and $y_3 = \dfrac{1}{3}\cos(\omega t + \pi)$, acting in the same straight line.

6.2.4 Damped Vibrations

6.51 A mass attached to a spring vibrates with a natural frequency of 20 c/s while its frequency for damped vibrations is 16 c/s. Determine the logarithmic decrement.

6.52 The equation of motion for a damped oscillator is given by

$$4\mathrm{d}^2x/\mathrm{d}t^2 + r\mathrm{d}x/\mathrm{d}t + 32x = 0$$

For what range of values for the damping constant will the motion be **(a)** underdamped; **(b)** overdamped; **(c)** critically damped?

6.53 A mass of 4 kg attached to the lower end of a vertical spring of constant 20 N/m oscillates with a period of 10 s. Find **(a)** the natural period; **(b)** the damping constant; **(c)** the logarithmic decrement.

6.54 Solve the equation of motion for the damped oscillator $\mathrm{d}^2x/\mathrm{d}t^2 + 2\mathrm{d}x/\mathrm{d}t + 5x = 0$, subject to the condition $x = 5$, $\mathrm{d}x/\mathrm{d}t = -3$ at $t = 0$.

6.55 A 1 kg weight attached to a vertical spring stretches it 0.2 m. The weight is then pulled down 1.5 m and released. **(a)** Is the motion underdamped, overdamped or critically damped? **(b)** Find the position of the weight at any time if a damping force numerically equal to 14 times the instantaneous speed is acting.

6.56 A periodic force acts on a 6 kg mass suspended from the lower end of a vertical spring of constant 150 N/m. The damping force is proportional to the instantaneous speed of the mass and is 80 N when $v = 2\,\text{m/s}$. find the resonance frequency.

6.57 The equation of motion for forced oscillations is $2\,\mathrm{d}^2x/\mathrm{d}t^2 + 1.5\mathrm{d}x/\mathrm{d}t + 40x = 12\cos 4t$. Find **(a)** amplitude; **(b)** phase lag; **(c)** Q factor; **(d)** power dissipation.

6.58 An electric bell has a frequency 100 Hz. If its time constant is 2 s, determine the Q factor for the bell.

6.59 An oscillator has a time period of 3 s. Its amplitude decreases by 5% each cycle **(a)** By how much does its energy decrease in each cycle? **(b)** Find the time constant **(c)** Find the Q factor.

6.60 A damped oscillator loses 3% of its energy in each cycle. **(a)** How many cycles elapse before half its original energy is dissipated? **(b)** What is the Q factor?

6.61 A damped oscillator has frequency which is 9/10 of its natural frequency. By what factor is its amplitude decreased in each cycle?

6.62 Show that for small damping $\omega' \approx (1 - r^2/8mk)\omega_0$ where ω_0 is the natural angular frequency, ω' the damped angular frequency, r the resistance constant, k the spring constant and m the particle mass.

6.63 Show that the time elapsed between successive maximum displacements of a damped harmonic oscillator is constant and equal to $4\pi m/\sqrt{4km - r^2}$, where m is the mass of the vibrating body, k is the spring constant, $2b = r/m$, r being the resistance constant.

6.64 A dead weight attached to a light spring extends it by 9.8 cm. It is then slightly pulled down and released. Assuming that the logarithmic decrement is equal to 3.1, find the period of oscillation.

6.65 The position of a particle moving along x-axis is determined by the equation $d^2x/dt^2 + 2dx/dt + 8x = 16\cos 2t$.

(a) What is the natural frequency of the vibrator?
(b) What is the frequency of the driving force?

6.66 Show that the time $t_{1/2}$ for the energy to decrease to half its initial value is related to the time constant by $t_{1/2} = t_c \ln 2$.

6.67 The amplitude of a swing drops by a factor $1/e$ in 8 periods when no energy is pumped into the swing. Find the Q factor.

6.3 Solutions

6.3.1 Simple Harmonic Motion (SHM)

6.1 $x = A\sin\omega t$ (SHM)

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1\,\text{rad/s}$$

$$8\sqrt{2} = A\sin\left(\frac{1\cdot\pi}{4}\right)$$

$$A = 16\,\text{cm} = 0.16\,\text{m}$$

$$E = \frac{1}{2}mA^2\omega^2$$

$$\therefore \quad m = \frac{2E}{A^2\omega^2} = \frac{2\times 0.256}{(0.16)^2\times 1^2} = 20.0\,\text{kg}$$