pitched whistle at 1.5 m/s. Find the beat frequency that he hears. Assume the sound velocity of 330 m/s.

8.82 A tuning fork of frequency 300 c/s gives 2 beats/s with another fork of unknown frequency. On loading the unknown fork the beats increase to 5/s, while transferring the load to the fork of known frequency increases the number of beats per second to 9. Calculate the frequency of the unknown fork (unloaded) assuming the load produces the same frequency change in each fork.

[University of Newcastle]

8.2.10 Waves in Pipes

8.83 An open organ pipe sounding its fundamental note is in tune with a fork of frequency 439 cycles/s. How much must the pipe be shortened or lengthened in order that 2 beats/s shall be heard when it sounded with the fork? Assume the speed of sound is 342 m/s.

[University of Durham]

8.84 A light pointer fixed to one prong of a tuning fork touches a vertical plate. The fork is set vibrating and the plate is allowed to fall freely. Eight complete oscillations are counted when the plate falls through 10 cm. What is the frequency of the tuning fork?

[Indian Institute of Technology 1997]

8.85 Air in a tube closed at one end vibrates in resonance with tuning fork whose frequencies are 210 and 350 vibrations/s, when the temperature is 20°C. Explain how this is possible and find the effective length of the tube. Assume that the velocity in air at 0°C is 33, 150 cm/s.

[University of London]

8.86 An open organ pipe is suddenly closed with the result that the second overtone of the closed pipe is found to be higher in frequency by 100 vibrations/s than the first overtone of the original pipe. What is the fundamental frequency of the open pipe?

[University of Bristol]

8.3 Solutions

8.3.1 Vibrating Strings

8.1 The one-dimensional wave equation is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \tag{1}$$

Given function is $y = A\sqrt{x + vt}$

$$\frac{\partial y}{\partial x} = \frac{A}{2\sqrt{x + vt}}$$

$$\frac{\partial^2 y}{\partial x^2} = -\frac{A}{4(x + vt)^{3/2}} \tag{2}$$

$$\frac{\partial y}{\partial t} = \frac{Av}{2\sqrt{(x + vt)}}$$

$$\frac{\partial^2 y}{\partial t^2} = -\frac{A}{4}\frac{v^2}{(x + vt)^{3/2}} \tag{3}$$

Equation (1) is satisfied with the use of (2) and (3).

8.2 The wave equation is

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2} \tag{1}$$

$$y = 2A \sin\left(\frac{n\pi x}{L}\right)\cos(2\pi f t) \quad \text{(standing wave)}$$

$$\frac{\partial y}{\partial x} = \frac{2An\pi}{L}\cos\left(\frac{n\pi x}{L}\right)\cos(2\pi f t)$$

$$\frac{\partial^2 y}{\partial x^2} = -\left(\frac{2An^2\pi^2}{L^2}\right)\sin\left(\frac{n\pi x}{L}\right)\cos(2\pi f t) = -\frac{n^2\pi^2 y}{L^2} \tag{2}$$

$$\frac{\partial y}{\partial t} = -4\pi f A \sin\left(\frac{n\pi x}{L}\right)\sin(2\pi f t)$$

$$\frac{\partial^2 y}{\partial t^2} = -8\pi^2 f^2 A \sin\left(\frac{n\pi x}{L}\right)\cos(2\pi f t) = -4\pi^2 f^2 y$$

$$\text{but} \quad \sqrt{\frac{F}{\mu}} = v = f\lambda$$

$$\therefore \quad \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2} = -\frac{4\pi^2 f^2 y}{v^2} = -\frac{4\pi^2 y}{\lambda^2} = -\frac{n^2\pi^2 y}{L^2} \tag{3}$$

$$\because \quad L = \frac{n\lambda}{2}$$

Thus

$$\frac{\mu}{F}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

8.3 Let the string AB of length L be plucked at the point C, distant d from the end A and be raised through height h, Fig. 8.4.

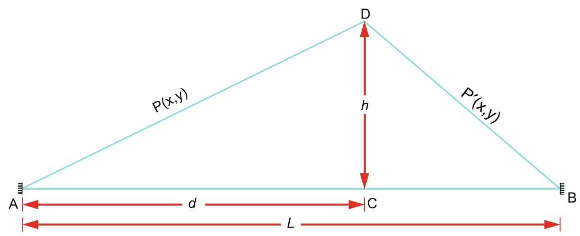


Fig. 8.4

The general form of the displacement at any point x and time t is given by the Fourier expansion

$$y = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi vt}{L}\right) \quad (1)$$

The coefficient a_n is obtained from

$$a_\mathrm{n} = \frac{2}{L}\int_0^L y_0 \sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \quad (2)$$

where $y_0 = y(x, 0)$.
We break the integral into two parts, one from 0 to d and the other from d to L. In the interval from 0 to d the equation of the initial configuration of the string for a typical point $p(x, y)$ is

$$\frac{y}{x} = \frac{h}{d} \quad \text{or} \quad y = \frac{hx}{d} \quad \text{for} \quad o < x < d$$

and in the interval d to L, the equation for $P'(x, y)$ is

$$\frac{y}{L - x} = \frac{h}{L - d} \quad \text{or} \quad y = \frac{h(L - x)}{L - d} \text{ for } d < x < L$$

so that by substituting (1) into (2) with $t = 0$

$$a_\mathrm{n} = \frac{2}{L}\left[\int_0^d \frac{hx}{d} \sin\left(\frac{\pi nx}{L}\right)\mathrm{d}x + \int_d^L \frac{h(L - x)}{L - d} \sin\frac{\pi nx}{L}\mathrm{d}x\right] \quad (3)$$

Integrating by parts

$$a_{\mathrm{n}} = \frac{2hL^2}{n^2\pi^2 d(L-d)} \sin\left(\frac{n\pi d}{L}\right) \tag{4}$$

Here $d = \frac{1}{2}L$, so that (4) becomes

$$a_{\mathrm{n}} = \frac{8h}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

If n is an even integer then the corresponding a_{n} is zero. If n is an odd integer, then the sine term alternates in sign as $\sin\frac{\pi}{2} = 1$, $\sin\frac{3\pi}{2} = -1$, $\sin\frac{5\pi}{2} = 1\ldots$, so that we may write

$$a_{\mathrm{n}} = \frac{8h}{\pi^2 n^2}(-1)^{(n-1)/2} \tag{5}$$

Using (5) in (1)

$$y = \frac{8h}{\pi^2}\left[\sin\frac{\pi x}{L}\cos\frac{\pi \upsilon t}{L} - \frac{1}{9}\sin\frac{3\pi x}{L}\cos\frac{3\pi \upsilon t}{L} + \frac{1}{25}\sin\frac{5\pi x}{L}\cos\frac{5\pi \upsilon t}{L} - \frac{1}{49}\sin\frac{7\pi x}{L}\cos\frac{7\pi \upsilon t}{L} + \cdots\right]$$

Note that the even harmonics are absent. Since the intensity of a wave is proportional to the square of its amplitude, then for the sound emitted by the string, the fundamental would have an intensity of 81 times the third harmonic and 625 times the fifth harmonic, etc.

Formula (4) shows that the nth harmonic will be absent if $\sin\left(\frac{n\pi d}{L}\right) = 0$. $a_{\mathrm{n}} = 0$ if $d = L/n,\ 2L/n,\ 3L/n$, i.e. nd/L is any integer or whenever there is any node of the nth harmonic situated at D, Fig. 8.4. If the string is divided into n equal parts and is plucked at any dividing point, the nth harmonic will disappear from the resultant vibration. In particular, any force applied at the midpoint of the string cannot produce even harmonics. Further after the application of force at the midpoint of the string, if this point be lightly touched the string ceases to vibrate. This is because odd harmonics cannot be sustained with a node at the midpoint, and the even harmonics are already absent for reasons discussed above.

8.4
$$\begin{aligned} y &= y_1 + y_2 \\ &= A\sin(kx - \omega t) + 3A\sin(kx + \omega t) \\ &= [A\sin(kx - \omega t) + A\sin(kx + \omega t)] + 2A\sin(kx + \omega t) \\ &= 2A\sin kx\cos\omega t + 2A\sin(kx + \omega t) \end{aligned}$$

where we have used the identity

$$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

Thus the resultant wave = standing wave + travelling wave in the negative direction.
The amplitudes are (a) $2A$ (b) $2A$

8.5 **(a)** The wave number $k = \frac{2\pi}{\lambda} = \frac{2\pi}{2} = \pi/\text{m}$

Frequency $f = \frac{v}{\lambda} = \frac{8}{2} = 4\,\text{Hz}$

Angular frequency $\omega = 2\pi f = (2\pi)(4) = 8\pi$ rad/s

(b) $y = A\sin(kx - \omega \text{t}) = A\sin\pi(x - 8\text{t})$

8.6 Let $y = A\sin(kx - \omega\text{t} + \phi)$

At $x = 0$, $t = 0$, the wave has the maximum displacement and $y = A$:

$$A = A\sin(0 - 0 + \phi)$$

$$\text{or} \quad \sin\phi = 1 \rightarrow \phi = \frac{\pi}{2}$$

$$\therefore \quad y = A\sin\left(kx - \omega t + \frac{1}{2}\pi\right) = A\cos(kx - \omega\text{t})$$

$$\therefore \quad y = 0.2\cos(3x - 20t)$$

8.7 $y = 2A\sin kx\cos\omega\text{t}$ (standing wave)

$$\frac{\partial y}{\partial t} = -2A\omega\sin kx\sin\omega\text{t}$$

Acceleration, $a = \frac{\partial^2 y}{\partial t^2} = -\omega^2 2A\sin kx\cos\omega t = -\omega^2 y$.

This is the defining equation for the SHM.

8.8 $f_N = \dfrac{Nv}{2L}$

$$f_1 = \frac{1 \times 120}{2 \times 2} = 30\text{ Hz}$$

$$f_2 = \frac{2 \times 120}{2 \times 2} = 60\text{ Hz}$$

$$f_3 = \frac{3 \times 120}{2 \times 2} = 90\text{ Hz}$$

$$f_4 = \frac{4 \times 120}{2 \times 2} = 120\text{ Hz}$$

8.9 $f_1 = \frac{1}{2L_1}\sqrt{\frac{F}{\mu_1}}$

$f_2 = \frac{1}{2L_2}\sqrt{\frac{F}{\mu_2}}$

$\therefore \frac{\mu_2}{\mu_1} = \frac{(L_1 f_1)^2}{(L_2 f_2)^2} = \frac{(0.05 \times 4800)^2}{(2.0 \times 32)^2} \simeq 14$

8.10 $y = 5 \sin \pi(0.02x - 4.00t) = 5 \sin 2\pi(0.01x - 2.00t)$ (given equation) (1)

$y = A \sin 2\pi \left(\frac{x}{\lambda} - ft\right)$ (standard equation) (2)

Comparing (1) and (2)

$A = 5\,\text{cm},\ f = 2\,\text{Hz}\ \frac{1}{\lambda} = 0.01$ or $\lambda = 100\,\text{cm}$

$v = f\lambda = 2 \times 100 = 200\,\text{cm/s}$

8.11 $y = 4 \sin \frac{1}{2}\pi x \cos 20\pi t$ (standing wave) (1)

$y = 2A \sin kx \cos \omega t$ (standard equation) (2)

Comparing (1) and (2)

(a) $2A = 4$ or $A = 2\,\text{cm},\ k = \frac{\pi}{2},\quad \omega = 20\pi$

$v = \frac{\omega}{k} = \frac{20\pi}{\pi/2} = 40\,\text{cm/s}$

(b) $\lambda = \frac{2\pi}{k} = \frac{2\pi}{\pi/2} = 4\,\text{cm}$

Distance between nodes $= \frac{\lambda}{2} = \frac{4}{2} = 2\,\text{cm}$

(c) $\frac{\partial y}{\partial t} = -(4)(20\pi) \sin \frac{1}{2}\pi x \sin 20\pi \text{t}$

$\frac{\partial y}{\partial t}\Big|_{x=1.0,\ t=9/4} = -80\pi \sin \frac{\pi}{2} \sin 45\pi = 0$

8.12 The wave is of the form

$y = A \sin(kx - \omega t + \phi)$

(a) $\omega = 2\pi f = (2\pi)(250) = 500\pi$ rad/s

$k = \frac{\omega}{v} = \frac{500\pi}{375} = \frac{4\pi}{3}\text{m}^{-1}$

$$\phi = 60^\circ = \frac{\pi}{3}\text{ rad}$$
$$x = \frac{\phi}{k} = \frac{\pi/3}{4\pi/3} = 0.25\,\text{m}$$

(b) $\phi = \omega t = (500\pi)(10^{-3}) = \frac{\pi}{2}\text{ rad} = 90^\circ$

8.13 $y_1 = A_1 \sin(kx - \omega t)$

$$y_2 = A_2 \sin\left(kx - \omega t + \frac{\pi}{2}\right) = A_2 \cos(kx - \omega t)$$
$$y = y_1 + y_2$$
$$= A_1 \sin(kx - \omega t) + A_2 \cos(kx - \omega t)$$
$$= \sqrt{A_1^2 + A_2^2}\left[\frac{A_1}{\sqrt{A_1^2 + A_2^2}} \sin(kx - \omega t) + \frac{A_2}{\sqrt{A_1^2 + A_2^2}} \cos(kx - \omega t)\right]$$

Put $\frac{A_1}{\sqrt{A_1^2 + A_2^2}} = \cos\alpha$. Then $\frac{A_2}{\sqrt{A_1^2 + A_2^2}} = \sin\alpha$

$$\therefore \quad y = \sqrt{A_1^2 + A_2^2}[\sin(kx - \omega t)\cos\alpha + \cos(kx - \omega t)\sin\alpha]$$
$$= \sqrt{A_1^2 + A_2^2}\sin(kx - \omega t + \alpha)$$

which has the amplitude $A = \sqrt{A_1^2 + A_2^2} = \sqrt{6^2 + 8^2} = 10\,\text{cm}$.

Graphical Method

This method was outlined in prob. (6.50). The waves are represented as vectors, the magnitudes being proportional to the amplitudes, the orientation according to the phase difference. Here the vectors OA and AB are laid in the head-to-tail fashion, Fig. 8.5. The amplitude of the resultant wave is given by OB which is found to be 10 cm from the right angle triangle OAB

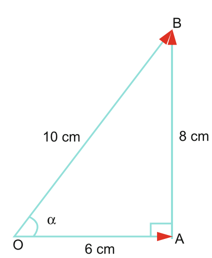


Fig. 8.5

8.14 **(a)** $y = A\ln(x + v\mathrm{t})$

$$\frac{\partial y}{\partial x} = \frac{A}{x + v\mathrm{t}}, \quad \frac{\partial^2 y}{\partial x^2} = -\frac{A}{(x + v\mathrm{t})^2}$$
$$\frac{\partial y}{\partial \mathrm{t}} = \frac{Av}{x + v\mathrm{t}}, \quad \frac{\partial^2 y}{\partial \mathrm{t}^2} = -\frac{Av^2}{(x + v\mathrm{t})^2}$$
$$\therefore \quad \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = -\frac{A}{(x + v\mathrm{t})^2} = \frac{\partial^2 y}{\partial x^2}$$

Thus the wave equation is satisfied.

(b) $y = A\cos(x + v\mathrm{t})$

$$\frac{\partial y}{\partial x} = -A\sin(x + v\mathrm{t}) \qquad \frac{\partial^2 y}{\partial x^2} = -A\cos(x + v\mathrm{t})$$
$$\frac{\partial y}{\partial t} = -vA\sin(x + v\mathrm{t})$$
$$\frac{\partial^2 y}{\partial t^2} = -v^2 A\cos(x + v\mathrm{t})$$
$$\therefore \quad \frac{1}{v^2}\frac{\partial^2 y}{\partial \mathrm{t}^2} = -A\cos(x + v\mathrm{t}) = \frac{\partial^2 y}{\partial x^2}$$

Thus the wave equation is satisfied.

8.15 **(a)** By prob. (8.3)

$$y = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi vt}{L}\right) \tag{1}$$
$$a_n = \frac{2hL^2}{n^2\pi^2 d(L - d)}\sin\left(\frac{n\pi d}{L}\right) \tag{2}$$

Here $d = \dfrac{L}{3}$ and (2) becomes

$$a_n = \frac{9h}{n^2\pi^2}\sin\frac{n\pi}{3} \tag{3}$$

Inserting (3) in (1)

$$\therefore \quad y = \frac{3^{5/2}h}{2\pi^2}\left[\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi v\mathrm{t}}{L}\right) + \frac{1}{4}\sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{2\pi v\mathrm{t}}{L}\right) - \frac{1}{16}\sin\left(\frac{4\pi x}{L}\right)\cos\left(\frac{4v\mathrm{t}}{L}\right)\dots\right] \tag{4}$$

(b) For $n = 3$, 6 or 9, the sine term in (3) becomes zero. Therefore, the third, sixth and ninth harmonics will be absent.

8.16 General equation for a progressive wave in the negative x-direction is

$$y = A\sin(kx + \omega t)$$
$$\omega = 2\pi f = 2\pi \times 170 = 340\pi \text{ rad/s}$$
$$k = \frac{\omega}{v} = \frac{340\pi}{340} = \pi/\text{m}$$
$$\therefore \quad y = 0.01\sin\pi(x + 340t)$$

8.17 **(a)** $y_1 = A\sin(kx - \omega t)$

$y_2 = A\sin(kx + \omega t)$

$y = y_1 + y_2 = 2A\sin kx\cos\omega t$

where we have used the identity stated in prob. (8.4).

(b) The nodes are formed when $kx = n\pi$ or $\frac{2\pi}{\lambda}x = n\pi$

or $\quad x = \frac{n\lambda}{2}$

$x = 0,\ \frac{\lambda}{2}, \lambda, \ldots$

The antinodes are formed when $kx = \frac{n\pi}{2}$ or $x = \frac{n\lambda}{4}$

$x = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}\ldots$

8.18 $y_1 = A\sin(kx - \omega t)$

$y_2 = A\sin(kx - \omega t + \delta)$

$$y = y_1 + y_2 = A[\sin(kx - \omega t) + \sin(kx - \omega t + \delta)]$$
$$= 2A\cos\frac{1}{2}\delta\sin\left(kx - \omega t + \frac{\delta}{2}\right)$$

Thus the amplitude of the resultant wave is $2A\cos\frac{1}{2}\delta$.

For $A = 6$ cm and $\delta = \frac{\pi}{2}$, the amplitude of the resultant wave will be $2 \times 6\cos\frac{\pi}{4}$ or $6\sqrt{2}$ cm.

For $2A\cos\frac{1}{2}\delta = 6$

$$\cos\frac{1}{2}\delta = \frac{6}{2A} = \frac{6}{2 \times 6} = \frac{1}{2} = \cos\frac{\pi}{3}$$
$$\therefore \quad \frac{1}{2}\delta = \frac{\pi}{3} \quad \text{or} \quad \delta = \frac{2\pi}{3}$$

If two sound waves with slightly different frequencies are produced then beats are heard. These consist of regular swelling and fading of the sound. In one set of waves compressions and rarefactions will be spaced further apart, in another they will be close enough. At some instant, two compressions arrive together at the ear of the listener and the sound is loud. At a later time, the compression of one wave arrives with the rarefaction of the other and the sound will be faint. Beats are thus caused due to interference of sound waves of neighbouring frequencies in time. The beat frequency is equal to the difference $f_1 \sim f_2$ for the two component waves. Beats between two tones can be detected by the ear up to a frequency of about 7/s.

8.19 Consider an infinitesimal element of length dx of the string of linear mass density μ. The mass element $\mu \mathrm{d}x$ will execute SHM with amplitude A. The maximum kinetic energy will be $\frac{1}{2}(\mu \mathrm{d}x)\omega^2 A^2$.
Energy transmitted across the string per second, i.e. power

$$P = \frac{1}{2}\left(\mu \frac{\mathrm{d}x}{\mathrm{d}t}\right)\omega^2 A^2 = \frac{1}{2}\mu v \omega^2 A^2$$

8.20 Let the fork of frequency f be in unison with 99 cm of the string. Then

$$f = \frac{1}{2 \times 99}\sqrt{\frac{F}{\mu}} \tag{1}$$

When the length of the string was 100 cm the frequency must have been less by 4 beats. Thus

$$f - 4 = \frac{1}{2 \times 100}\sqrt{\frac{F}{\mu}} \tag{2}$$

Dividing (1) by (2) and solving

$$\frac{f}{f-4} = \frac{100}{99}$$

We get $f = 400/\mathrm{s}$.

8.21
$$y(x,t) = \frac{0.10}{(2x-t)^2 + 4}$$

$$\therefore \quad y(0,0) = \frac{0.10}{4} = 0.025$$

$$\text{Let} \quad y(x,t) = 0.025 = \frac{0.10}{4 + (2x-t)^2}$$

Solving we find

$v = \dfrac{x}{t} = 0.5$ m/s along the $+\,x -$ direction.

Now, $y(-x, t) = \dfrac{0.10}{4 + (2x + t)^2} \neq y(x, t)$

Therefore, the pulse is not symmetric.

8.22 **(a)** $f = \dfrac{N}{2l}\sqrt{\dfrac{F}{\mu}} \quad (N = 1)$

$$\mu = \frac{F}{4f^2L^2} = \frac{300}{(4)(660)^2(0.6)^2} = 4.78 \times 10^{-4}\ \text{kg/m}$$

(b) The frequencies of the first two harmonics are $f_2 = 2f = 1320$ Hz and $f_3 = 3f = 1980$ Hz.

(c) For open pipe length is

$$L = \frac{\lambda}{2} = \frac{v}{2f} = \frac{340}{2 \times 660} = 0.2576\ \text{m}$$

8.23 **(a)** First harmonic – second harmonic (Fig. 8.6)

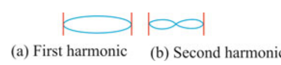


Fig. 8.6

$$v = \sqrt{\frac{F}{\mu}}, \quad \lambda = \frac{2L}{N}$$

$$f_N = \frac{v}{\lambda_N}$$

$$f_N = \frac{N}{2L}\sqrt{\frac{F}{\mu}}, \quad N = 1, 2, 3, \ldots$$

(b) The standard equation for the standing wave is

$$y(x, t) = 2A \sin kx \cos \omega t \tag{1}$$

Given equation is

$$y(x, t) = 0.024 \sin(62.8x) \cos(471t) \tag{2}$$

Comparison shows that

$k = 62.8 \quad$ and $\quad \omega = 471$

Wave velocity $v = \frac{\omega}{k} = \frac{471}{62.8} = 7.5$ m/s

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{62.8} = 0.1\,\text{m}$$

Distance between nodes $= \frac{\lambda}{2} = \frac{0.1}{2} = 0.05\,\text{m}$

8.24 $y = A\sin(kx + \omega t)$

(i) $y = 8.2 \times 10^{-2}\sin(22x + 100t)$ (negative x-direction)

(ii) $y = 8.2 \times 10^{-2}\sin(100t - 22x)$ (positive x-direction)

(iii) $\lambda = \frac{2\pi}{k} = \frac{2\pi}{22} = 0.2856\,\text{m}$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{100} = 0.0628\,\text{m}$$
$$v = \frac{\omega}{k} = \frac{100}{22} = 4.545\,\text{m/s}$$

(iv) $y = 8.2 \times 10^{-2} \times \sin(22 \times 3.2 + 100 \times 2.5)$
$= 8.2 \times 10^{-2} \times \sin(51 \times 2\pi) = 0$

8.25 $\left[\frac{F}{\mu}\right]^{1/2} = \left[\frac{MLT^{-2}}{ML^{-1}}\right]^{1/2} = \left[LT^{-1}\right] = [v]$

8.26 Let the travelling wave be represented by

$$y = A\sin(kx - \omega t)$$

Then $\frac{\partial y}{\partial x} = kA\cos(kx - \omega t)$ (1)

$$\frac{\partial y}{\partial t} = -\omega A\cos(kx - \omega t)$$

$= -vkA\cos(kx - \omega t) = -v\frac{\partial y}{\partial x}$ (2)

Combining (1) and (2), $\frac{\partial y}{\partial x} = -\frac{\partial y}{\partial t}/v$.

8.27 **(a)** Let a long string of linear density μ be stretched by a force F. Assume that the damping is negligible. Take the x-axis in the direction of the undisplaced string and y-axis in the direction perpendicular to it. If θ is the angle between the tangent to the string and the x-axis, the tension in the horizontal direction (x-axis) would be $T\cos\theta$ and in the vertical direction (y-axis) it would be $T\sin\theta$. Assuming that θ is very small, $\cos\theta \simeq 1$ and consequently the x-component of the tension remains constant. We are therefore concerned only with the y-component of the tension.

Referring to Fig. 8.1 note that the forces across Δx the element of length of the string make angles θ_1 and θ_2 with the x-axis. Let $\theta_2 = \theta$ and $\theta_1 = \theta + \mathrm{d}\theta$. To find the equation of motion of this element subject to these forces, the difference in tension acting across Δx in the y-direction is

$$\begin{aligned}
&\mathrm{d}F_y = F\{\sin(\theta + \mathrm{d}\theta) - \sin\theta\}\\
&\qquad = F\{\sin\theta\cos(\mathrm{d}\theta) + \cos\theta\sin(\mathrm{d}\theta) - \sin\theta\}\\
&\text{but } \cos(\mathrm{d}\theta) \simeq 1 \quad \text{and} \quad \sin(\mathrm{d}\theta) \simeq \mathrm{d}\theta,\ \text{since } \mathrm{d}\theta \text{ is small :}\\
&\therefore \quad \mathrm{d}F_y = F\cos\theta\,\mathrm{d}\theta = F\mathrm{d}(\sin\theta)
\end{aligned}$$

In the small angle approximation

$$\sin\theta \simeq \tan\theta = \frac{\partial y}{\partial x}$$

this last quantity being the gradient of the curve

$$\therefore \quad \mathrm{d}F_y = F\frac{\partial}{\partial x}\frac{\partial}{\partial y}\mathrm{d}x = F\left(\frac{\partial^2 y}{\partial x^2}\right)\mathrm{d}x \tag{1}$$

The mass of the element Δx is $\mu\,\mathrm{d}x$, and its acceleration in the y-direction is $\mathrm{d}^2y/\mathrm{d}t^2$. Hence by Newton's second law of motion

$$\mu\mathrm{d}x\frac{\partial^2 y}{\partial t^2} = F\left(\frac{\partial^2 y}{\partial x^2}\right)\mathrm{d}x$$

$$\text{or} \quad \frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2} \tag{2}$$

(b) Let $y(x - vt)$ be a solution of (2)

$$\frac{\partial y}{\partial \mathrm{t}}(x - vt) = y'(x - vt)\frac{\partial}{\partial \mathrm{t}}(x - vt) = -vy'(x - vt)$$

where y' is another function of $(x - vt)$ defined by $y'(x - vt) = \dfrac{\mathrm{d}y(x - vt)}{\mathrm{d}(x - vt)}$.

The second derivative with respect to time gives

$$\frac{\partial^2 y(x - vt)}{\partial t^2} = v^2 y''(x - vt) \tag{3}$$

where $y''(x - vt)$ is yet another function of $(x - vt)$ defined by

$$y''(x - vt) = \frac{\mathrm{d}y'(x - vt)}{\mathrm{d}(x - vt)} = \frac{\mathrm{d}^2 y'(x - vt)}{\mathrm{d}(x - vt)^2}$$

proceeding along similar lines, differentiation of the function $y(x - vt)$ with respect to x yields

$$\frac{\partial y(x - vt)}{\partial x} = y'(x - vt)\frac{\partial}{\partial x}(x - vt) = y''(x - vt)$$
$$\frac{\partial^2 y(x - vt)}{\partial x^2} = f''(x - vt) \tag{4}$$

where $y(x - vt)$ and $y''(x - vt)$ are the same functions of $(x - vt)$ as in (3). Substitution of (3) and (4) into (2) shows that $y(x - vt)$ is a solution, provided we set

$$v^2 = \frac{F}{\mu}$$

8.28 **(a)** The incident wave has the form

$$y_1 = A_1 \sin(\omega t - k_1 x) \tag{1}$$

The reflected wave has the form

$$y_2 = A_2 \sin(\omega t + k_1 x) \tag{2}$$

The transmitted wave has the form

$$y_3 = A_3 \sin(\omega t - k_2 x) \tag{3}$$

The boundary conditions at the boundary ($x = 0$) are that the displacement and its first derivative be single valued:

$$y_1 |_{x=0} + y_2 |_{x=0} = y_3 |_{x=0}$$
$$A_1 + A_2 = A_3 \tag{4}$$
$$\frac{\partial y_1}{\partial x} |_{x=0} + \frac{\partial y_2}{\partial x} |_{x=0} = \frac{\partial y_3}{\partial x} |_{x=0}$$
$$-k_1 A_1 + k_1 A_2 = -k_2 A_3 \tag{5}$$

Solving (4) and (5)

$$A_2 = \frac{(k_1 - k_2)A_1}{k_1 + k_2}; \quad A_3 = \frac{2k_1 A_1}{k_1 + k_2} \tag{6}$$

(b) A_2 is negative when $k_2 > k_1$ or $\frac{\omega}{v_2} > \frac{\omega}{v_1}$ or $\mu_2 > \mu_1$, i.e. wire 2 has greater linear density than wire 1.

8.29 $\frac{5}{2}\lambda = 15\,\text{cm}$

$\therefore \quad \lambda = 6\,\text{cm}$

$f = \frac{v}{\lambda} = \frac{2400}{6} = 400\,\text{Hz}$

$\omega = 2\pi f = 2512\,\text{rad/s}$

$k = \frac{2\pi}{\lambda} = 1.047/\text{cm}$

$A = 6\,\text{cm}$

$y = A\sin(\omega t - kx) = 6\sin(2512t - 1.047x)\,\text{cm}$

8.30 $P = \frac{1}{2}\omega^2 A^2 \mu v = 2\pi^2 f^2 A^2 \mu v$

$= 2\pi^2 \times (400)^2 (0.06)^2 (2.5 \times 10^{-4})(24) = 68.2\,\text{W}$

8.31 **(a)** The amplitude of any point of the plucked string at time t may be written as

$$y = \sum_{n=1}^{\infty} a_n \cos\omega_n t \ \sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\omega_n t \ \sin\left(\frac{n\pi x}{L}\right) \tag{1}$$

The kinetic energy of vibration of an element of length of string dx in the nth mode is given by

$$\begin{aligned} \mathrm{d}K_n &= \frac{1}{2}(\mu \mathrm{d}x)(\dot{y})^2 \\ &= \frac{1}{2}\mu\omega_n^2(-a_n \sin\omega_n t + b_n \cos\omega_n t)^2 \sin^2(k_n x)\mathrm{d}x \end{aligned} \tag{2}$$

where we have used the value of velocity $\dot{y}$ by differentiating (1) for the nth mode with respect to t.
The potential energy of an element of string of length dx is

$$\begin{aligned} \mathrm{d}U_n &= \frac{1}{2}ky^2\mathrm{d}x \\ &= \frac{1}{2}\mu\omega_n^2(a_n \cos\omega_n t + b_n \sin\omega_n t)^2 \sin^2 k_n x \,\mathrm{d}x \end{aligned} \tag{3}$$

where we have used (1).
Adding (2) and (3), the total energy

$$\mathrm{d}E_n = \mathrm{d}K_n + \mathrm{d}U_n = \frac{1}{2}\mu\omega_n^2(a_n^2 + b_n^2)\sin^2 k_n x \,\mathrm{d}x \tag{4}$$

The total energy of the entire string is obtained by integrating from 0 to L

$$E_n = \int \mathrm{d}E_n = \frac{1}{2}\mu\omega_n^2(a_n^2 + b_n^2)\int_0^L \sin^2(k_n x)\mathrm{d}x$$

$$\text{Now} \quad \int_0^L \sin^2(k_n x)\mathrm{d}x = \int_0^L \sin^2\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \frac{L}{2}$$

$$\therefore \quad E_n = \frac{1}{4}\mu L\omega_n^2(a_n^2 + b_n^2) = \frac{1}{4}M\omega_n^2(a_n^2 + b_n^2)$$

where M is the total mass of the string.

(b) For the string plucked at the centre $a_n = \dfrac{8h}{n^2\pi^2}$ (see prob. 8.3). Further $\omega_n = k_n v = \dfrac{n\pi v}{L}$ and $b_n = 0$. Thus the energy of vibration

$$E_n = \frac{M}{4}\left[\frac{n\pi v}{L}\right]^2\left[\frac{8h}{n^2\pi^2}\right]^2 = \frac{16Mh^2v^2}{n^2\pi^2L^2}$$

$$\therefore \quad \frac{E_1}{E_3} = \frac{9}{1}$$

8.3.2 Waves in Solids

8.32 **(a)** For the rod clamped at one end and free at the other (fixed–free)

$$f_n = \frac{n}{4L}\sqrt{\frac{\gamma}{\rho}} \quad (n = 1, 3, 5, \ldots)$$

$$f_1 = \frac{1}{4L}\sqrt{\frac{\gamma}{\rho}} = \frac{1}{4 \times 0.25}\sqrt{\frac{2 \times 10^{11}}{7860}} = 5044\,\text{Hz}$$

(b) **(i)** For the rod free at both ends (free–free)

$$f_n = \frac{n}{2L}\sqrt{\frac{Y}{\rho}} \quad (n = 1, 2, 3, \ldots)$$

(ii) For the rod clamped at the midpoint

$$f_n = \frac{n}{2L}\sqrt{\frac{Y}{\rho}} \quad (n = 1, 3, 5, \ldots)$$

(iii) For the bar clamped at both ends (fixed–fixed)

$$f_n = \frac{n}{2L}\sqrt{\frac{Y}{\rho}} \quad (n = 1, 2, 3, \ldots)$$

For the case (a) (bar clamped at one end only) the frequency of the fundamental is half that of a similar free–free (case (b) (i)) or fixed–fixed (case (b) (iii)) bar and only the odd-numbered harmonic overtones are present. This is to be expected since the effect of clamping a free–free bar at its centre is to suppress all its even harmonics.

For case (b) (ii) rod clamped at the midpoint only odd partials are present similar to case (a) (fixed–free) and differs from case (b) (i) (free–free) where all the partials are present. However the fundamental has the same frequency in cases (b) (ii) and (b) (i).

8.33 **(a)** $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

Now $k = \frac{F}{x}$ and $Y = \frac{F/A}{x/L} = \frac{FL}{xA} = \frac{kL}{A}$

$$\therefore \quad f = \frac{1}{2\pi}\sqrt{\frac{AY}{mL}} = \frac{1}{2\pi}\sqrt{\frac{1\times 10^{-5}\times 2\times 10^{11}}{2\times 1.0}} = 159\,\text{c/s}$$

(b) For the given system

$$kL\tan(kL) = \frac{M}{m}$$

Mass of the bar, $M = \rho AL = 7800 \times 1 \times 10^{-5} \times 1.0 = 0.078\,\text{kg}$. The frequency condition becomes

$$kL\tan(kL) = \frac{0.078}{2.0} = 0.039$$

The solution to the above equation is

$$kL = 0.196$$

$$\therefore \quad \frac{\omega L}{v} = 0.196$$

$$\text{or} \quad f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\frac{0.196}{L}\sqrt{\frac{Y}{\rho}} = \frac{0.196}{2\pi\times 1.0}\sqrt{\frac{2\times 10^{11}}{7800}} = 158\,\text{c/s}$$

Observe that the results of (b) are nearly the same as those for (a), showing thereby for small values of kL, the mass loaded system approximates that of a simple harmonic oscillator with the mass fixed at the end.

8.34 The frequency condition for this system is

$$kL\tan(kL) = \frac{M}{m}$$

Expanding tan (kL) by series

$$kL\left[kL+\frac{(kL)^3}{3}+2\frac{(kL)^5}{15}+\cdots\right]=\frac{M}{m}$$

If $kL < 0.2$, we may retain only the first term within the brackets:

$$k^2L^2=\frac{M}{m}$$

$$\frac{\omega^2L^2}{v^2}=\frac{M}{m}$$

$$f=\frac{\omega}{2\pi}=\frac{1}{2\pi}\frac{v}{L}\sqrt{\frac{M}{m}}=\frac{1}{2\pi L}\sqrt{\frac{Y}{\rho}\frac{M}{m}}$$

But $Y=\dfrac{kL}{A}$ and $M=AL\rho$

$$\therefore \quad f=\frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

8.3.3 *Waves in Liquids*

8.35 **(a)** $v^2=\dfrac{g}{k}\tanh(kh)=\dfrac{g\lambda}{2\pi}\tanh\left(\dfrac{2\pi}{\lambda}\dfrac{\lambda}{4}\right)=\dfrac{g\lambda}{2\pi}\tan h\left(\dfrac{\pi}{2}\right)$

$$=\frac{9.8}{2\pi}\times 0.917\,\lambda$$

$$v=1.2\sqrt{\lambda}\text{ m/s}$$

(b) $v=\sqrt{\dfrac{g}{k}}=\sqrt{\dfrac{g\lambda}{2\pi}}=1.25\sqrt{\lambda}\text{ m/s}$

(c) $v=\sqrt{gh}=\sqrt{\dfrac{g\lambda}{4}}=\sqrt{\dfrac{9.8\lambda}{4}}=1.56\sqrt{\lambda}\text{ m/s}$

8.36 The fractional error introduced by the use of the formula $v=\sqrt{gh}$ is

$$\frac{\sqrt{gh}-\sqrt{\dfrac{g}{k}\tanh(kh)}}{\sqrt{\dfrac{g}{k}\tanh(kh)}}=0.01$$

Put $kh = x$, then $\dfrac{\tanh x}{x} = 0.96$

This gives the solution $x = 0.25$ or $h = \dfrac{x}{k} = \dfrac{0.25\lambda}{2\pi} = 0.04\lambda$.

8.37 Surface conditions are modified by the surface tension S. For the capillary waves

$$v^2 = \left(\frac{2\pi S}{\rho\lambda} + \frac{g\lambda}{2\pi}\right)\tanh\left(\frac{2\pi h}{\lambda}\right) \tag{1}$$

If $h >> \lambda$, $\tanh\left(\dfrac{2\pi h}{\lambda}\right) \to 1$, and

$$v^2 = \frac{2\pi S}{\rho\lambda} + \frac{g\lambda}{2\pi} \tag{2}$$

Substituting $\lambda = 0.366\,\text{cm}$, $\rho = 1.0\,\text{g/cm}^3$, $g = 980\,\text{cm/s}^2$ and $v = f\lambda = 100 \times 0.366 = 36.6\,\text{cm/s}$ in (2) we find $S = 74.7$ dynes/cm.

8.38 For capillary waves when $h >> \lambda$

$$v^2 = \frac{2\pi S}{\rho\lambda} + \frac{g\lambda}{2\pi} \tag{1}$$

The minimum value of the wavelength λ_m can be found out by minimizing (1):

$$\frac{\partial(v^2)}{\partial\lambda} = -\frac{2\pi \mathrm{S}}{\rho\lambda_m^2} + \frac{g}{2\pi} = 0$$

$$\lambda_m = 2\pi\sqrt{\frac{S}{g\rho}} \tag{2}$$

Ignoring the second term in the right-hand side of (1) and using (2)

$$v = \left(\frac{gs}{\rho}\right)^{1/4}$$

For mercury and water

$$v_1 : v_2 = \left(\frac{S_1}{\rho_1}\right)^{1/4} : \left(\frac{S_2}{\rho_2}\right)^{1/4}$$

$$= \left(\frac{544}{13.56}\right)^{1/4} : \left(\frac{74}{1}\right)^{1/4} = 0.858 : 1$$

8.39 $$\frac{\omega^2}{k^2} = \frac{F}{\mu} + \alpha k^2 \tag{1}$$

The phase velocity $v_p = \frac{\omega}{k} = \sqrt{\frac{F}{\mu} + \alpha k^2} = \sqrt{\frac{F}{\mu}}\left[1 + \frac{\alpha \mu k^2}{F}\right]^{1/2}$

$$= \sqrt{\frac{F}{\mu}}\left[1 + \alpha\frac{k^2\mu}{2F} + \cdots\right] \text{ (for small } \alpha) \tag{2}$$

Since $k = 2\pi/\lambda$, v_p increases as λ decreases.
The group velocity is given by

$$v_g = v_p + \frac{k\, \mathrm{d}v_p}{\mathrm{d}k} = v_p + \alpha k^2\sqrt{\frac{\mu}{F}}$$

8.40 **(a)** $\omega = \frac{S}{\rho}k^{3/2}$

$$v_p = \frac{\omega}{k} = \frac{S}{\rho}\sqrt{k}$$

(b) $v_g = v_p + \frac{k\, \mathrm{d}v_p}{\mathrm{d}k}$

$$= \frac{S}{\rho}\sqrt{k} + \frac{kS}{2\rho\sqrt{k}} = \frac{3}{2}\frac{S}{\rho}\sqrt{k}$$

(c) From (a) and (b) $v_g > v_p$

8.41 **(a)** $$\omega^2 = \left(gk + \frac{S}{\rho}k^3\right)\tanh(kh) \tag{1}$$

If $kh << 1$, then tanh $(kh) = kh$ and (1) becomes

$$\omega^2 = \left(gk + \frac{S}{\rho}k^3\right)kh \tag{2}$$

If the second term in the brackets is smaller than the first one $\left(\frac{S}{\rho}k^2 << g\right)$

$$\omega^2 = ghk^2$$

$$\therefore \quad \omega = k\sqrt{gh} \tag{3}$$

$$v_p = \frac{\omega}{k} = \sqrt{gh} \tag{4}$$

$$v_g = \frac{d\omega}{dk} = \sqrt{gh} \tag{5}$$

$$\therefore \quad v_g = v_p$$

(b) $kh >> 1$, $\tanh(kh) = 1$ and (1) becomes

$$\omega^2 = gk + \frac{S}{\rho}k^3$$

$$\frac{\omega^2}{k^2} = \frac{g}{k} + \frac{S}{\rho}k$$

$$v_p = \frac{\omega}{k} = \sqrt{\frac{g}{k} + \frac{Sk}{\rho}} \tag{6}$$

$$v_g = v_p + k\frac{dv_p}{dk}$$

$$= v_p + \frac{k}{2}\left(\frac{S}{\rho} - \frac{g}{k^2}\right)\left(\frac{g}{k} + \frac{Sk}{\rho}\right)^{-1/2}$$

Using (6)

$$\frac{v_g}{v_p} = \frac{\frac{g}{2k} + \frac{3}{2}\frac{ks}{\rho}}{\frac{g}{k} + \frac{ks}{\rho}} \tag{7}$$

For short wavelengths k is larger, the first term in both the numerator and denominator will be smaller and $v_g = \frac{3}{2}v_p$, while for long wavelengths, k is smaller and $v_g = \frac{1}{2}v_p$.

8.42 With reference to prob. (8.41) for small ripples, λ is small and k is large so that the second term in (6) dominates over the first term in the radical.

$$v_p = \sqrt{\frac{Sk}{\rho}} = \sqrt{\frac{2\pi\,S}{\lambda\rho}} = \sqrt{\frac{2\pi \times 0.075}{0.01 \times 1000}} = 0.217\,\text{m/s}$$

$$v_g = \frac{3}{2}v_p = 0.325\,\text{m/s}$$

For large waves, first term in (6) is important:

$$v_{\mathrm{p}} = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}} = \sqrt{\frac{9.8 \times 1.0}{2\pi}} = 1.25\,\mathrm{m/s}$$

$$v_{\mathrm{g}} = \frac{1}{2}v_{\mathrm{p}} = 0.625\,\mathrm{m/s}$$

8.43 $E^2 = c^2 p^2 + m^2 c^4$

$$\hbar^2\omega^2 = c^2\hbar^2 k^2 + m^2 c^4$$

$$\therefore \quad \omega = \sqrt{c^2k^2 + \frac{m^2c^4}{\hbar^2}}$$

$$v_{\mathrm{p}} = \frac{\omega}{k} = \sqrt{c^2 + \frac{m^2c^4}{\hbar^2k^2}}$$

$$v_{\mathrm{g}} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{c^2k}{\sqrt{c^2k^2 + \frac{m^2c^4}{\hbar^2}}} = \frac{c^2}{\sqrt{c^2 + \frac{m^2c^4}{\hbar^2k^2}}}$$

$$\therefore \quad v_{\mathrm{p}}v_{\mathrm{g}} = c^2$$

8.44 $$v_{\mathrm{p}} = \sqrt{\frac{g\lambda}{2\pi} + \frac{2\pi S}{\rho\lambda}} \tag{1}$$

Substituting $v_{\mathrm{p}} = 30\,\mathrm{cm/s}$, $g = 980\,\mathrm{cm/s^2}$, $S = 75$ dynes/cm and $\rho = 1\,\mathrm{g/cm^3}$, on simplification (1) reduces to the quadratic equation in λ:

$$\lambda^2 - 5.767\lambda + 1.153 = 0$$

The two roots are $\lambda_1 = 5.56\,\mathrm{cm}$ and $\lambda_2 = 0.207\,\mathrm{cm}$.
In determining surface tension it is preferable to use the shorter wavelength because the surface effect will dominate over gravity:

8.45 $$v_{\mathrm{p}} = \frac{\omega}{k} = \sqrt{\frac{g}{k}}$$

$$v_{\mathrm{g}} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{1}{2}\sqrt{\frac{g}{k}}$$

$$\therefore \quad v_{\mathrm{g}} = \frac{1}{2}v_{\mathrm{p}}$$

8.46 $\omega = \sqrt{\frac{\gamma RT}{M}}k$

$$v_p = \frac{\omega}{k} = \sqrt{\frac{\gamma RT}{M}}$$

$$v_g = \frac{d\omega}{dk} = \sqrt{\frac{\gamma RT}{M}}$$

$$\therefore \quad v_g = v_p$$

8.47 $v_p^2 = \frac{g}{k} + \frac{Sk}{\rho}$ (1)

Maximize (1)

$$\frac{d(v_p^2)}{dk} = -\frac{g}{k^2} + \frac{S}{\rho} = 0$$

$$\therefore \quad k = \frac{2\pi}{\lambda} = \sqrt{\frac{g\rho}{S}}$$

$$\therefore \quad \lambda_{\min} = 2\pi\sqrt{\frac{S}{g\rho}}$$

8.3.4 Sound Waves

8.48 Let the displacement be represented by

$$y = A\cos(kx - \omega t) \tag{1}$$

$$\frac{\partial y}{\partial x} = -kA\sin(kx - \omega t)$$

$$\text{but } P = -B\frac{\partial y}{\partial x} = BkA\sin(kx - \omega t)$$

where B is the bulk modulus:

$$B = v^2\rho_0$$

$$P = [k\rho_0 v^2 A]\sin(kx - \omega t) \tag{2}$$

P represents the change from standard pressure P_0. The term in square bracket represents maximum change in pressure and is called the pressure amplitude $P_{\max}$. Then

$$P = P_{max} \sin(kx - \omega t) \tag{3}$$

$$\text{where } P_{max} = k\rho_0 v^2 A \tag{4}$$

If the displacement wave is represented by the cosine function, (1), then the pressure wave is represented by the sine function, (3). Here the displacement wave is 90° out of phase with the pressure wave.

8.49 $I = \frac{1}{2} P_{max}^2 / \rho_0 v$

$$\therefore \quad P_{max} = \sqrt{2I\rho_0 v} = \sqrt{2 \times 10^{-12} \times 1.29 \times 331} = 2.92 \times 10^{-5}\,\text{N/m}^2$$

8.50 $IL = 10 \log \frac{I}{I_0}$

$$60 = 10 \log \frac{I}{10^{-12}}$$

$$\log I + \log 10^{12} = 6 \quad \log I = -6$$

$$\therefore \quad I = 10^{-6}\,\text{W/m}^2 = 1\,\mu\,\text{W/m}^2$$

8.51 $I = \frac{\text{Power}}{4\pi r^2} = \frac{4}{4\pi \times 25^2} = 5.093 \times 10^{-4}\,\text{W/m}^2$

$$IL = 10 \log \frac{I}{I_0} = 10 \log \frac{5.093 \times 10^{-4}}{10^{-12}} = 10 \log(5.093 \times 10^8)$$

$$= 10[\log 5.093 + 8] = 87\,\text{dB}$$

8.52 $A = \frac{P_{max}}{k\rho_0 v^2} = \frac{P_{max}}{2\pi\rho_0 f v}$

where we have substituted $k = \frac{2\pi}{\lambda}$ and $v = f\lambda$:

$$\therefore \quad A = \frac{29}{2\pi \times 1.22 \times 2000 \times 331} = 5.7 \times 10^{-6}\,\text{m}$$

8.53 $I = \frac{P_{max}^2}{2\rho_0 v}$

By problem, $P_{max}(\text{air}) = P_{max}(\text{water})$

$$\therefore \quad \frac{I_{\text{Water}}}{I_{\text{Air}}} = \frac{\rho_A v_A}{\rho_W v_W} = \frac{1.293 \times 330}{1000 \times 1450} = 2.94 \times 10^{-4}$$

8.54 $P = 2.4 \sin \pi (x - 330t) = 2.4 \sin 2\pi \left(\frac{1}{2}x - 165t\right)$

$P = P_{\max} \sin 2\pi \left(\frac{x}{\lambda} - ft\right)$ (standard expression)

On comparing the two expressions we find

(a) $2.4\,\mathrm{N/m^2}$, (b) $165\,\mathrm{Hz}$, (c) $2.0\,\mathrm{m}$, (d) $v = f\lambda = 165 \times 2 = 330\,\mathrm{m/s}$

8.55 $I = 2\pi^2 \rho_0 A^2 f^2 v$

$$\therefore \quad A = \frac{1}{\pi f}\sqrt{\frac{I}{2\rho_0 v}} = \frac{1}{1200\pi}\sqrt{\frac{2 \times 10^{-6}}{2 \times 1.293 \times 330}} = 1.28 \times 10^{-4}\,\mathrm{m}$$

8.56 $P_{\mathrm{e}} = 1$ microbar $= 10^{-6}$ bar $= 0.1\,\mathrm{N/m^2}$

$$I = \frac{P_{\mathrm{e}}^2}{\rho_0 v} = \frac{(0.1)^2}{1.293 \times 330} = 2.34 \times 10^{-5}\,\mathrm{W/m^2}$$

$$IL = 10 \log \frac{I}{I_0} = 10 \log \left(\frac{2.34 \times 10^{-5}}{10^{-12}}\right)$$

$$= 10(7 + \log 2.34) = 73.7\,\mathrm{dB} \simeq 74\,\mathrm{dB}$$

8.57 $\mathrm{IL} = 10 \log \frac{I}{I_0}$

$$70 = 10 \log \frac{I}{10^{-12}} = 10\,[\log I + 12]$$

$\log I = -5 \quad I = 10^{-5}\mathrm{W/m^2}$

Energy density

$$E = \frac{I}{v} = \frac{10^{-5}}{331} = 3 \times 10^{-8}\,\mathrm{J/m^3}$$

Effective pressure

$$P_{\mathrm{e}} = \sqrt{I\rho_0 v} = \sqrt{10^{-5} \times 1.293 \times 331} = 0.0654\,\mathrm{N/m^2}$$

8.58 $P_{\max} = \sqrt{2I\rho_0 v} = \sqrt{2 \times 1 \times 1.293 \times 331} = 29.26\,\mathrm{N/m^2}$

8.59 $v = \sqrt{\frac{\gamma RT}{M}} = \sqrt{\frac{1.4 \times 8.317 \times 273}{2.016 \times 10^{-3}}} = 1256\,\mathrm{m/s}$

8.60 $v = \sqrt{\dfrac{\gamma P}{\rho}}$ (Laplace formula)

$$\therefore \quad \frac{v_H}{v_O} = \sqrt{\frac{\rho_O}{\rho_H}} = \sqrt{16} = 4$$

$$\therefore \quad v_H = 4v_O = 4 \times 317 = 1268\,\text{m/s}$$

8.61 $\text{IL} = 10 \log\left(\dfrac{I_2}{I_1}\right) = 10 \log\left(\dfrac{10}{0.4}\right) = 14\,\text{dB}$

8.62 $\text{IL} = 10 \log \dfrac{I_2}{I_1} = 6$

$$\log \frac{I_2}{I_1} = 0.6$$

$$\therefore \quad \frac{I_2}{I_1} = 3.98 \text{ or } 4$$

8.63 The threshold of hearing intensity is taken as $10^{-12}\,\text{W/m}^2$. Let r be the distance from the source at which the sound can be audible

$$I = \frac{\text{power}}{4\pi r^2} = 10^{-12}$$

$$\therefore \quad r = \sqrt{\frac{\text{power}}{4\pi \times 10^{-12}}} = \sqrt{\frac{0.009}{4\pi \times 10^{-12}}} = 2.677 \times 10^4\,\text{m} = 26.8\,\text{km}$$

8.64 $P_{\max} = k\rho_0 v^2 A = \dfrac{2\pi}{\lambda}\rho_0 v(f\lambda)A = 2\pi\rho_0 v f A$

$$\therefore \quad A = \frac{P_{\max}}{2\pi\rho_0 v f} = \frac{2 \times 10^{-5}}{2\pi \times 1.22 \times 331 \times 1000} = 7.9 \times 10^{-12}\,\text{m}$$

8.65 **(a)** $A = \dfrac{P_{\max}}{2\pi\rho_0 f v}$

$$\frac{A_1}{A_2} = \frac{\rho_2}{\rho_1}\frac{v_2}{v_1} = \frac{4}{3} \times \frac{2}{3} = \frac{8}{9}$$

(b) $I = \dfrac{1}{2}\dfrac{p_{\max}^2}{\rho_0 v}$

$$\frac{I_1}{I_2} = \frac{\rho_2 v_2}{\rho_1 v_1} = \frac{4}{3} \times \frac{2}{3} = \frac{8}{9}$$

(c) $E = \dfrac{I}{v}$

$$\frac{E_1}{E_2} = \frac{I_1}{I_2}\frac{v_2}{v_1} = \frac{8}{9} \times \frac{2}{3} = \frac{16}{27}$$

8.66 **(a)** $\lambda = \dfrac{v}{f}$

$$\therefore \quad \frac{\lambda_A}{\lambda_W} = \frac{v_A}{v_W} = \frac{331}{1450} = 0.228 \quad (\because f_A = f_W)$$

(b) $P_{max} = \sqrt{2I\rho_0 v}$

$$\therefore \quad \frac{P_A}{P_W} = \sqrt{\frac{\rho_A}{\rho_W}\frac{v_A}{v_W}} = \sqrt{\frac{1.293}{1000} \times \frac{331}{1450}} = 0.0172 \quad (\because I_A = I_W)$$

(c) $A = \sqrt{\dfrac{2I}{\rho v \omega^2}}$

$$\frac{A_A}{A_W} = \sqrt{\frac{\rho_W}{\rho_A}\frac{v_W}{v_A}} = \sqrt{\frac{1000}{1.293} \times \frac{1450}{331}} = 33.88 \quad (\because I_A = I_W \text{ and } f_A = f_w)$$

8.67 Characteristic impedance of a gas

$$Z = \rho_0 v \tag{1}$$

Now $v = \sqrt{\dfrac{B}{\rho_0}}$ or $\rho_0 = \dfrac{B}{v^2}$

$$\therefore \quad Z = \frac{B}{v} = B\sqrt{\frac{M}{\gamma RT}} \tag{2}$$

Thus $Z \propto \dfrac{1}{\sqrt{T}}$

(a) At 0°C, $v = 331$ m/s, $\rho_0 = 1.293\,\text{kg/m}^3$

$$Z = \rho_0 v = 1.293 \times 331 = 428\,\text{rayl}$$

(b) $Z \propto \dfrac{1}{\sqrt{T}}$

$$\therefore \quad Z(80°\text{C}) = Z(0°\text{C}) \times \sqrt{\frac{273}{273 + 80}} = 428 \times 0.879 = 376\,\text{rayl}$$

8.68 **(a)** $I = \frac{\text{Power}}{\text{area}} = \frac{50}{\pi(0.25)^2} = 255\,\text{W/m}^2$

(b) $P_{\max} = \sqrt{2I\rho_0 v} = \sqrt{2 \times 255 \times 10^3 \times 1450} = 2.72 \times 10^4\,\text{N/m}^2$

(c) $A = \sqrt{\frac{2I}{\rho_0 v 4\pi^2 f^2}} = \sqrt{\frac{2 \times 255}{1000 \times 1450 \times 4\pi^2 \times (25 \times 10^3)^2}}$
$= 1.19 \times 10^{-7}\,\text{m}$

(d) $U_{\max} = A\omega = 1.19 \times 10^{-7} \times 2\pi \times 25 \times 10^3 = 0.019\,\text{m}$

(e) $S_{\max} = \frac{2\pi A}{\lambda} = \frac{2\pi A f}{v} = \frac{2\pi \times 1.19 \times 10^{-7} \times 25 \times 10^3}{1450}$
$= 1.29 \times 10^{-5}$

8.69 Consider sound waves of finite amplitude. Now, the bulk modulus is constant only for infinitesimal volume changes:

$$B = -V\frac{\mathrm{d}p}{\mathrm{d}V} \tag{1}$$

where the acoustic pressure p has been replaced by the pressure change $\mathrm{d}p$. Now, $V\rho = \text{mass} = \text{constant}$

$$\therefore \quad V\frac{\mathrm{d}\rho}{\mathrm{d}P} + \rho\frac{\mathrm{d}V}{\mathrm{d}P} = 0$$

$$\text{or} \quad -V\frac{\mathrm{d}P}{\mathrm{d}V} = \rho\frac{\mathrm{d}P}{\mathrm{d}\rho} = B$$

where we have used (1)

$$\therefore \quad v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{\mathrm{d}P}{\mathrm{d}\rho}} \tag{2}$$

When a second wave passes through a gas the changes in volume are assumed to be adiabatic so that

$$\frac{P}{\rho^\gamma} = C = \text{constant} \tag{3}$$

where γ is the ratio of specific heats of the gas at constant pressure to that at constant volume. Differentiation of (3) gives

$$-\gamma P\rho^{\gamma-1}\mathrm{d}\rho + \rho^\gamma \mathrm{d}P = 0$$

$$\text{or} \quad \frac{\mathrm{d}P}{\mathrm{d}\rho} = \frac{\gamma P}{\rho} \tag{4}$$

Using (4) in (2), we get

$$v = \sqrt{\frac{\gamma P_0}{\rho_0}} \qquad \text{(Laplace formula)} \tag{5}$$

where P_0 and ρ_0 refer to equilibrium conditions of pressure and density. The velocity v_0 at 0°C can be found out by substituting $\gamma = 1.4$, $P_0 = 1.013 \times 10^5\,\mathrm{N/m^2}$ and $\rho_0 = 1.293\,\mathrm{kg/m^3}$. We find $v_0 = 331.2\,\mathrm{m/s}$, in good agreement with the experiment.

With the assumption of isothermal changes we would have obtained the formula $v = \sqrt{P_0/\rho_0}$, (Newton's treatment) which gives a value of 20% lower.

8.70 **(a)** $v|_{t=20} = 1403 + 5 \times 20 - 0.06 \times (20)^2 + 0.0003 \times (20)^3 = 1481.4\,\mathrm{m/s}$

(b) $\frac{dv}{dt} = 5 - 0.12t + 0.0009t^2$

$$\frac{dv}{dt}\Big|_{t=20} = 5 - 0.12 \times 20 + 0.0009 \times (20)^2 = 2.62\,\mathrm{m/s/°C}$$

8.3.5 Doppler Effect

8.71 $v_s = 72\,\mathrm{km/h} = 72 \times \frac{5}{18}\,\mathrm{m/s} = 20\,\mathrm{m/s}$

$$f_0 = \frac{v f_s}{v + v_s} \quad \text{(direct)}$$

$$f_0' = \frac{v f_s}{v - v_s} \quad \text{(reflected from the wall of the rock)}$$

$$\frac{f_0'}{f_0} = \frac{v + v_s}{v - v_s} = \frac{340 + 20}{340 - 20} = \frac{9}{8}$$

8.72 $v_0 = v_s = 90\,\mathrm{km/h} = 90 \times \frac{5}{18} = 25\,\mathrm{m/s}$

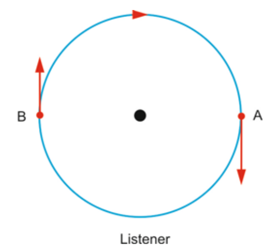
$$v = 350\,\mathrm{m/s}$$

$$f' = \frac{f(v + v_0)}{v - v_s} = \frac{520(350 + 25)}{350 - 25} = 600\,\mathrm{Hz}.$$

8.73 $f' = \frac{f(v - v_0)}{v + v_s} = 520\frac{(330 - 25)}{330 + 25} = 446.8\,\mathrm{Hz}.$

8.74 Let the whistle rotate clockwise, Fig. 8.7. At point A the linear velocity of the whistle will be towards the distant listener and at B away from the listener.

Fig. 8.7



Maximum frequency will be heard when the whistle will be at A and minimum when it is at B:

$$v_s = \omega r = 15 \times 2 = 30\,\text{m/s}$$
$$f_{\max} = \frac{v f_s}{v - v_s} = \frac{330 \times 540}{330 - 30} = 594\,\text{Hz}$$
$$f_{\min} = \frac{v f_s}{v - v_s} = \frac{330 \times 540}{330 + 30} = 495\,\text{Hz}$$

8.75 **(a)** The frequency of the rod is fixed, and so also for the air and the gas. The distance between successive heaps of cork dust is equal to the distance between two neighbouring nodes which is $\frac{1}{2}\lambda$, Fig. 8.8:

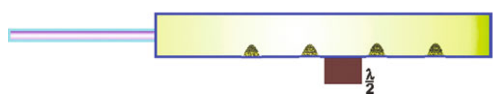


Fig. 8.8

$$v_{\text{air}} = f\lambda_{\text{air}}$$
$$v_{\text{gas}} = f\lambda_{\text{gas}}$$
$$\therefore \quad v_{\text{gas}} = v_{\text{air}}\frac{\lambda_{\text{gas}}}{\lambda_{\text{air}}} = 330 \times \frac{10}{8} = 425\,\text{m/s}$$

(b) $$v_{\text{rod}} = f\lambda_{\text{rod}} = \frac{v_{\text{air}}}{\lambda_{\text{air}}} \cdot \lambda_{\text{rod}} = \left(\frac{340}{2 \times 0.08}\right)(2L_{\text{rod}})$$
$$= \left(\frac{340}{0.16}\right)(2 \times 1.2) = 5100\,\text{m/s}$$

8.76 **(i)** $v = f\lambda$

$$\therefore \quad \lambda = \frac{v}{f} = \frac{340}{514} = 0.66\,\text{m}$$

(ii) $\lambda' = \dfrac{v - v_s}{f} = \dfrac{340 - 15}{514} = 0.63\,\text{m}$

(iii) $\lambda' = \dfrac{v + v_s}{f} = \dfrac{340 + 15}{514} = 0.69\,\text{m}$

8.3.6 Shock Wave

8.77 **(a)** If an object flies with a supersonic speed (speed greater than that of sound) a shock wave is emitted, a booming sound. In the two-dimensional drawing, Fig. 8.9, the wave fronts CB and DB represent the V-shaped wave. In three dimensions the bunching of the wave fronts actually forms a cone called the Mach cone. The shock wave lies on the surface of the cone.

(b) The half-angle θ of the cone called the Mach cone is given by

$$\sin\theta = \frac{vt}{v_s t} = \frac{v}{v_s} = \frac{1}{2}$$

$$\text{The Mach number} = \frac{v_s}{v_p} = 2$$

(c) The Mach angle $\theta = 30°$

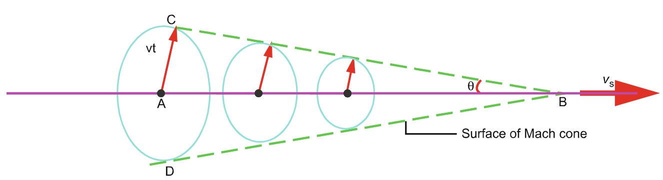


Fig. 8.9 Shock wave

8.3.7 Reverberation

8.78 If V is the volume, S the surface area and K the absorption coefficient then the reverberation time t_R is given by

$$t_R = \frac{0.16V}{\sum_i k_i S_i} \quad \text{(Sabine Law)}$$

$$V = 10 \times 18 \times 4 = 720\,\text{m}^3$$

$$\Sigma K_i S_i = 10 \times 18 \times 0.6$$
$$+ 10 \times 18 \times 0.02 + (18 \times 4 + 10 \times 4) \times 2 \times 0.03 + 50 \times 0.5$$
$$= 143.3$$
$$t_R = \frac{0.16 \times 720}{143.3} = 0.8\,\text{s}$$

8.3.8 Echo

8.79 The drum rate, that is, frequency when the echo is inaudible, is 40/min or 2/3 per second. Therefore, the time period of drum beats $t_1 = \frac{3}{2}$ s. Time for the echo, $t_1 = \frac{2x}{v}$, where x is the initial distance from the mountain and v is the sound velocity.

Thus $$\frac{2x}{v} = \frac{3}{2} \tag{1}$$

On moving 90 m towards the mountain ht is $x - 90$ m from the mountain, the drum rate is 60/min or 1/s and again the echo is not heard.
Thus

$$\frac{2(x-90)}{v} = 1 \tag{2}$$

Solving (1) and (2) we get

$x = 270\,\text{m}$ and $v = 360\,\text{m/s}$.

8.80 **(a)** If the width of the valley is d m and the rifle shot is fired at a distance x from one of the mountains the echoes will be heard in time t_1 and t_2 s:

$$t_1 = \frac{2x}{v} \tag{1}$$
$$t_2 = 2\frac{(d-x)}{v} \tag{2}$$

Adding (1) and (2)

$$t_1 + t_2 = 2 + 4 = \frac{2d}{v} = \frac{2d}{360}$$

Therefore $d = 1080\,\text{m}$.

(b) Solving (1) with $t_1 = 2$ s and $v = 360$ m we find $x = 360$ m and therefore $d - x = 720$ m. Subsequent echoes will be heard after 6, 8, 10,... s.

8.3.9 Beat Frequency

8.81 When the man moves towards the source

$$f_0 = \frac{(v + v_0) f_s}{v}$$

When the man moves away from the source

$$f_0' = \frac{(v - v_0) f_s'}{v}$$

$$\therefore \quad f_0 - f_0' = \frac{v_0}{v}(f_s + f_s') + f_s - f_s'$$

$$= \frac{1.5}{330}(548 + 552) + 548 - 552 = 1\text{ s}$$

$\therefore$ Beat frequency $= 1/\text{s}$.

8.82 Suppose the frequency of the unknown fork (unloaded) is n. Then $n = 300 \pm 2$
Case (*i*) Suppose $n = 300 - 2 = 298$
Let the frequency of the loaded unknown fork be n_1 and the loaded known fork be n_2:

$$300 - n_1 = 5 \tag{1}$$

$$298 - n_2 = 9 \tag{2}$$

Also frequency changes in both the forks are the same

$$\therefore \quad 300 - n_2 = 298 - n_1$$

$$\text{or} \quad n_2 - n_1 = 2 \tag{3}$$

Subtracting (2) from (1)

$$n_2 - n_1 = -6 \tag{4}$$

Obviously (3) and (4) are inconsistent.
Case (*ii*) Suppose $n = 300 + 2 = 302$

$$300 - n_1 = 5 \tag{5}$$

$$302 - n_2 = 9 \tag{6}$$

$$\text{also } 300 - n_2 = 302 - n_1$$

$$\text{or} \quad n_1 - n_2 = 2 \tag{7}$$

Subtracting (5) from (6)

$$n_1 - n_2 = 2 \tag{8}$$

Thus (7) and (8) are consistent. Therefore, correct solution is $n = 302$.

8.3.10 Waves in Pipes

8.83 $L = \dfrac{v}{2f} = \dfrac{342}{2 \times 439} = 0.3895\,\mathrm{m}$

The new frequency with the changed length L_1 is

$$f_1 = 439 + 2 = 441$$

$$f_2 = 439 - 2 = 437$$

$$L_1 = \frac{v}{2f_1} = \frac{342}{2 \times 441} = 0.3877\,\mathrm{m}$$

$$L_2 = \frac{v}{2f_2} = \frac{342}{2 \times 437} = 0.3913\,\mathrm{m}$$

The pipe must be shortened by $0.3895 - 0.3877 = 0.0018\,\mathrm{m}$ or 1.8 mm or lengthened by $0.3913 - 0.3895 = 0.0018\,\mathrm{m} = 1.8\,\mathrm{mm}$, so that 2 beats/s may be heard when it is sounded with the fork.

8.84 Time taken for the plate to fall is

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 10}{980}} = \frac{1}{7}\,\mathrm{s}$$

Time period

$$T = \frac{t}{8} = \frac{1}{7 \times 8} = \frac{1}{56}\,\mathrm{s}$$

Frequency, $f = \dfrac{1}{T} = 56\,\mathrm{Hz}$

8.85 $v = v_\mathrm{t} = v_0\sqrt{\dfrac{t + 273}{273}} = 33150\sqrt{\dfrac{20 + 273}{273}} = 34343\,\mathrm{cm/s}$

$$L = (2N + 1)\frac{\lambda}{4} = \frac{(2N + 1)v}{4f_1} = \frac{(2N + 1) \times 34,343}{4 \times 210}\ (N = 0, 1, 2)\ldots \quad (1)$$

Also $L = (2M + 1)\dfrac{v}{4f_2} = \dfrac{(2M + 1) \times 34343}{4 \times 350}\ (M = 0, 1, 2\ldots)$ (2)

Equating right-hand sides of (1) and (2) and simplifying

$$\frac{(2N + 1)}{(2M + 1)} = \frac{3}{5} \quad (3)$$

The choice of $N = 1$ and $M = 2$ satisfies (3). Using $N = 1$ in (1) or M in (2) gives $L = 122.65\,\mathrm{cm}$.

Resonance with tuning forks of different frequencies is possible because resonance for $f = 210$ occurs with the first overtone of the tube and for $f = 350$ it occurs with the second overtone.

8.86 $f = \dfrac{v}{2L}$ (open pipe, fundamental)

$f_1 = \dfrac{v}{L}$ (open pipe, first overtone)

$f_2 = \dfrac{5v}{4L}$ (closed pipe, second overtone)

$f_2 - f_1 = 100$ (by problem)

$$\frac{5v}{4L} - \frac{v}{L} = \frac{v}{4L} = 100$$

$\therefore$ Fundamental frequency of open pipe, $f = \dfrac{v}{2L} = 200\,\text{Hz}$.