5.48 Find the law of force for the orbit $r = a \sin n\theta$.

5.49 Find the law of force to the pole when the orbit described by the cardioid $r = a(1 - \cos\theta)$.

5.50 In prob. (5.49) prove that if Q be the force at the apse and v the velocity, $3v^2 = 4aQ$.

5.51 A particle moves in a plane under an attractive force varying as the inverse cube of the distance. Find the equation of the orbit distinguishing three cases which may arise.

5.52 Show that the central force necessary to make a particle describe the lemniscate $r^2 = a^2 \cos 2\theta$ is inversely proportional to r^7.

5.53 Show that if a particle describes a circular orbit under the influence of an attractive central force directed towards a point on the circle, then the force varies as the inverse fifth power of distance.

5.54 If the sun's mass suddenly decreased to half its value, show that the earth's orbit assumed to be originally circular would become parabolic.

5.3 Solutions

5.3.1 Field and Potential

5.1 $F = \dfrac{GM_1M_2}{r^2}$

If R is the radius of either sphere, the distance between the centre of the spheres in contact is $r = 2R$:

$$M_1 = M_2 = M = \frac{4}{3}\pi R^3 \rho$$

$$F = \frac{GM^2}{4R^2} = \frac{4\pi^2 G R^4 \rho^2}{9}$$

$$= \frac{4\pi^2}{9} \times 6.67 \times 10^{-11} \times (0.2)^4 (11300)^2 = 5.98 \times 10^{-5}\,\mathrm{N}$$

5.2 As the mass of A and B are identical and the distance PA = PB, the magnitude of the force $F_{\mathrm{PA}} = F_{\mathrm{PB}}$. Resolve these forces in the horizontal and vertical direction. The horizontal components being in opposite direction get cancelled. The vertical components get added up.

$$F_{\mathrm{PA}} = F_{\mathrm{PB}} = G\frac{(0.01)(0.26)}{(0.1)^2} = 0.26G$$

Each vertical component $= 0.26\,G \times \dfrac{6}{10} = 0.156\,G$

Therefore $F_{\text{Net}} = 2\times 0.156\,G = 2\times 0.156\times 6.67\times 10^{-11}\,\text{N} = 2.08\times 10^{-1}\,\text{N}$

5.3 At distance r, $F_{\text{M}} = F_{\text{m}} = \dfrac{GMm}{r^2}$

Acceleration of mass m $\quad a_{\text{m}} = \dfrac{F_{\text{m}}}{m} = \dfrac{GM}{r^2}$

Acceleration of mass M $\quad a_{\text{M}} = \dfrac{F_{\text{m}}}{M} = \dfrac{Gm}{r^2}$

$$a_{\text{rel}} = a_{\text{m}} + a_{\text{M}} = \frac{G(M+m)}{r^2}$$

$$a_{\text{rel}} = \frac{\text{d}v_{\text{rel}}}{\text{d}t} = v_{\text{rel}}\frac{\text{d}v_{\text{rel}}}{\text{d}r} = \frac{G(M+m)}{r^2}$$

Integrating $\displaystyle\int v_{\text{rel}}\,\text{d}v_{\text{rel}} = \frac{v_{\text{rel}}^2}{2} = G(M+m)\int_d^\infty \frac{\text{d}r}{r^2} = \frac{G(M+m)}{d}$

$$\therefore \qquad v_{\text{rel}} = \sqrt{\frac{2G(M+m)}{d}}$$

5.4 Gravitational force $F = -\dfrac{GMm}{x^2}$

where M and m are the masses of the sun and the earth which are a distance x apart.

Earth's acceleration

$$a = \frac{\text{d}v}{\text{d}t} = \frac{F}{m} = -\frac{GM}{x^2}$$

$$\therefore \quad \frac{\text{d}v}{\text{d}t} = \frac{v\text{d}v}{\text{d}x} = -\frac{GM}{x^2}$$

$$v\text{d}v = -GM\frac{\text{d}x}{x^2}$$

Integrating $\displaystyle\int v\text{d}v = \frac{v^2}{2} = -Gm\int \frac{\text{d}x}{x^2} + C$

where $C =$ constant

$$\frac{v^2}{2} = \frac{GM}{x} + C$$

Initially $v = 0,\ x = r$

$$\therefore \quad C = -\frac{GM}{r}$$

$$\therefore \quad v = \frac{dx}{dt} = \sqrt{2GM}\sqrt{\frac{1}{x} - \frac{1}{r}}$$

$$dt = \frac{1}{\sqrt{2GM}} \frac{dx}{\sqrt{\frac{1}{x} - \frac{1}{r}}}$$

Integrating

$$t = \int dt = \frac{1}{\sqrt{2GM}} \int_0^r \frac{dx}{\sqrt{\frac{1}{x} - \frac{1}{r}}}$$

Put $x = r\cos^2\theta$, $dx = -2r\ \sin\theta\cos\theta d\theta$

$$t = -2\sqrt{\frac{r^3}{2GM}} \int_{\pi/2}^{0} \cos^2\theta d\theta = 2\sqrt{\frac{r^3}{2GM}}\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = \frac{\pi}{2\sqrt{2}}\sqrt{\frac{r^3}{GM}}$$

But the period of earth's orbit is

$$T = 2\pi\sqrt{\frac{r^3}{GM}}$$

$$\therefore \quad t = \frac{T}{4\sqrt{2}} = \frac{365}{4\sqrt{2}} = 64.53 \text{ days}$$

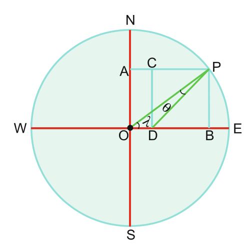


Fig. 5.9

5.5 If the earth were at rest, then the gravitational force on a body of mass at P would be in the direction PO, i.e. towards the centre of the earth, Fig. 5.9.

However, due to the rotation of the earth about the polar axis NS, a part of the gravitational force is used up to provide the necessary centripetal force to enable the mass m at P in the latitude λ to describe a circular radius PA $= r = R\cos\lambda$, where PO $= R$ is the earth's radius. This is equal to $m\omega^2 r$, or $m\omega^2 R\cos\lambda$ towards the centre and is represented by CA, ω being the angular velocity of earth's diurnal rotation. In the absence of rotation the gravitational force mg acts radially towards the centre O and is represented by PO. Resolve this into two mutually perpendicular components, one along PA given by mg $\cos\lambda$ and the other along PB given by mg $\sin\lambda$ and is represented by PB. Drop CD perpendicular on the EW-axis. Then the resultant force mg' is given by PD both in magnitude and direction. A plumb line at P will make a small angle $\theta(\mathrm{O}\hat{P}D)$ with line PO.

$$mg' = \sqrt{(mg\cos\lambda - m\omega^2 R\cos\lambda)^2 + (mg\sin\lambda)^2}$$
$$= m\sqrt{g^2 - 2gR\omega^2\cos^2\lambda + \omega^4 R^2\cos^2\lambda} \tag{1}$$

The third term in the radical is much smaller than the second term and is neglected.

$$\therefore \quad g' \simeq (g^2 - 2gR\omega^2\cos^2\lambda)^{1/2}$$
$$= g\left(1 - \frac{2R}{g}\omega^2\cos^2\lambda\right)^{1/2}$$
$$\simeq g\left(1 - \frac{R}{g}\omega^2\cos^2\lambda\right)^{1/2} \tag{2}$$

where we have expanded binomially and retained only the first two terms.
Now in ΔOPD

$$\frac{\mathrm{PD}}{\sin \mathrm{P}\hat{O}D} = \frac{\mathrm{OD}}{\sin\theta} \tag{3}$$

$$\text{or} \quad \frac{g - R\omega^2\cos^2\lambda}{\sin\lambda} = \frac{\omega^2 R\cos\lambda}{\sin\theta} \tag{4}$$

$$\sin\theta \simeq \theta = \frac{\omega^2 R\cos\lambda\sin\lambda}{g - R\omega^2\cos^2\lambda} \tag{5}$$

$\theta \simeq \dfrac{\omega^2}{g} R\cos\lambda\sin\lambda$ ($\because$ the second term in the denominator of (5) is much smaller than the first term)

$$\simeq \frac{2\pi^2 R}{gT^2}\sin 2\lambda$$

(a) θ will be maximum when $\sin 2\lambda$ is maximum, i.e. $2\lambda = 90°$ or $\lambda = 45°$.
(b) At the poles $\lambda = 90°$ and so $\theta = 0°$.
(c) At the equator $\lambda = 0°$ and so $\theta = 0°$.

5.6 Consider a spherical shell of radius r and thickness dr concentric with the sphere of radius R. If ρ is the density, then

$$\rho = \frac{3M}{4\pi R^3} \tag{1}$$

The mass of the shell $= 4\pi r^2 \mathrm{d}r\rho$.
The mass of the sphere of radius r which is equal to $4\pi r^3/3$ may be considered to be concentrated at the centre.
The gravitational potential energy between the spherical shell and the sphere of radius r is

$$\mathrm{d}U = -\frac{G(4\pi r^2 \mathrm{d}r\rho)\left(\frac{4\pi}{3}r^3\rho\right)}{r} = -\frac{16\pi^2 G\rho^2 r^4 \mathrm{d}r}{3} \tag{2}$$

The total gravitational energy of the earth

$$U = \int \mathrm{d}U = -\frac{16\pi^2 G\rho^2}{3}\int_0^R r^4 \mathrm{d}r = -\frac{16\pi^2 G\rho^2 R^5}{15}$$
$$= -\frac{3}{5}\frac{GM^2}{R} \tag{3}$$

where we have used (1).

$$U = -\frac{6.67 \times 10^{-11} \times 0.6 \times (6 \times 10^{24})^2}{6.4 \times 10^6} = 2.25 \times 10^{32}\mathrm{J}$$

5.7 If g_0 is the gravity at the earth's surface, g_h at height h and g_d at depth d, then

$$g_\mathrm{h} = g_0 \frac{R^2}{(R+h)^2} \tag{1}$$

$$g_\mathrm{d} = g_0\left(1 - \frac{d}{R}\right) \tag{2}$$

By problem, $g_\mathrm{d} = g_\mathrm{h}$ at $h = d$, (3)

From (1) and (2) we get

$$d^2 + dR - R^2 = 0$$

$$d = \frac{\left(\sqrt{5} - 1\right)}{2} R = 0.118R = 0.118 \times 6400 = 755\,\text{km}$$

5.8 Weight $mg = \dfrac{GMm}{R^2}$

$$M = \frac{4}{3}\pi R^3 \rho$$

$$\therefore \quad \rho = \frac{3g}{4\pi G R} = \frac{3 \times 980}{4\pi \times 6.67 \times 10^{-8} \times 6.38 \times 10^8} = 5.5\,\text{g/cm}^3$$

5.9 Let M_s and M_E be the masses of the sun and earth, respectively. Let the body of mass m be at distance x from the centre of the earth and d the distance between the centres of the sun and the earth. The forces are balanced if

$$\frac{G_\text{m} M_\text{E}}{x^2} = \frac{G_\text{m} M_\text{s}}{(d - x)^2}$$

Given that $M_\text{s} = 3.24 \times 10^5 M_\text{E}$

$$x = \frac{d}{570.2} = \frac{9.3 \times 10^7}{570.2} = 1.631 \times 10^5\,\text{km}$$

5.10 By problem (5.5) $g' = g - R\omega^2 \cos^2 \lambda$
Set $\quad \lambda = 0,\ \omega = 7.27 \times 10^{-5}\,\text{rad/s},\ R = 6.4 \times 10^8\,\text{cm}$

$$\Delta g = g - g' = R\omega^2 = 6.4 \times 10^8 \times (7.27 \times 10^{-5})^2 = 3.38\,\text{cm/s}^2$$

5.11 Figure 5.10 shows the cross-section of a solid sphere of mass M and radius 'a' with constant density ρ, its centre being at O. It is required to find the potential $V(r)$ at the point P, at distance r from the centre. The contribution to $V(r)$ comes from two regions, one V_1 from mass lying within the sphere of radius r and the other V_2 from the region outside it. Thus

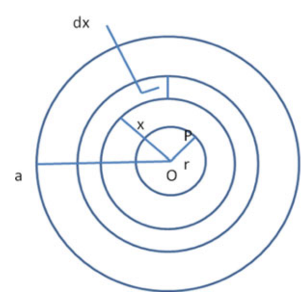


Fig. 5.10

$$V(r) = V_1 + V_2 \tag{1}$$

The potential V_1 at P is the same as due to the mass of the sphere of radius r concentrated at the centre O and is given by

$$V_1 = -G\frac{4\pi r^3}{3}\frac{\rho}{r} = -\frac{4}{3}\pi G r^2 \rho \tag{2}$$

For the mass outside r, consider a typical shell at distance x from the centre O and of thickness dx.
Volume of the shell $= 4\pi x^2\,\mathrm{d}x$

Mass of the shell $= (4\pi x^2\,\mathrm{d}x)\rho$

Potential due to this shell at the centre or at any point inside the shell, including at P, will be

$$\mathrm{d}V_2 = -\frac{4\pi\rho x^2\,\mathrm{d}x}{x} = -4\pi G\rho x\,\mathrm{d}x \tag{3}$$

Potential V_2 at P due to the outer shells $(x > r)$ is obtained by integrating (3) between the limits r and a.

$$V_2 = \int \mathrm{d}V_2 = -4\pi G\rho \int_r^a x\,\mathrm{d}x = -2\pi G\rho(a^2 - r^2) \tag{4}$$

Using (2) and (4) in (1) and using $\rho = \frac{3M}{4\pi a^3}$

$$V(r) = -\frac{GM}{2a}\left(3 - \frac{r^2}{a^2}\right) \tag{5}$$

The potential (5) is that of a simple harmonic oscillator as the force

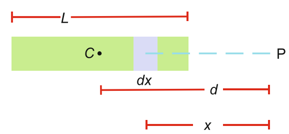
$$F = -\frac{\mathrm{d}V}{\mathrm{d}r} = -\frac{GMr}{a^3}$$

i.e. the force is opposite and proportional to the distance.

5.12 Consider a length element dx of a thin rod of length L, at distance x from P (Fig. 5.11). The mass element is $(M/L)\,\mathrm{d}x$. The potential at P due to this mass length will be

$$\mathrm{d}V = -\frac{GM}{L}\frac{\mathrm{d}x}{x}$$

Fig. 5.11



The potential at p from the entire rod is given by

$$V = \int \mathrm{d}V = -\frac{GM}{L}\int_{d-\frac{L}{2}}^{d+\frac{L}{2}} \frac{\mathrm{d}x}{x} = -\frac{GM}{L}\ln\frac{2d+L}{2d-L}$$

5.13 The linear speed of an object on the equator

$$v = \omega R = (7.27 \times 10^{-5})(6.4 \times 10^{6}) = 465.3\,\mathrm{m/s}$$

The orbital velocity of a surface satellite is

$$v_0 = \sqrt{gr} = \sqrt{9.8 \times 6.4 \times 10^{6}} = 7920\,\mathrm{m/s}$$

When launched in the westerly direction the launching speed v_0 will be added to v as the earth rotates from west to east, while in the easterly direction it will be subtracted.

$$\frac{\text{westerly launching speed}}{\text{easterly launching speed}} = \frac{7920 + 465}{7920 - 465} = 1.125$$

or 11%.

5.14 Consider an element of arc of length $\mathrm{d}s = R\,\mathrm{d}\theta$, Fig. 5.12. The corresponding mass element $\mathrm{d}m = \lambda \mathrm{d}s = \lambda R\,\mathrm{d}\theta$.

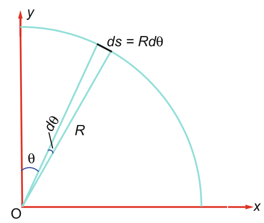
The intensity at the origin where λ is the linear density (mass per unit length) due to $\mathrm{d}m$ will be

$$\frac{G\lambda R\,\mathrm{d}\theta}{R^2} \text{ or } \frac{G\lambda\,\mathrm{d}\theta}{R}$$

The x-component of intensity at the origin due to $\mathrm{d}m$ will be

$$\mathrm{d}E_x = \frac{G\lambda}{R}\mathrm{d}\theta \sin\theta$$

Fig. 5.12



Therefore, the x-component of intensity due to the quarter of circle at the origin will be

$$E_x = \frac{G\lambda}{R}\int_0^{\pi/2} \sin\theta \, \mathrm{d}\theta = \frac{G\lambda}{R}$$

Similarly, the y-component of intensity due to the quarter of circle at the origin will be

$$E_y = \frac{G\lambda}{R}\int_0^{\pi/2} \cos\theta \, \mathrm{d}\theta = \frac{G\lambda}{R}$$

$$\therefore \quad E = \sqrt{E_x^2 + E_y^2} = \sqrt{2}\frac{G\lambda}{R}$$

5.15 $g_\mathrm{B} = \dfrac{F_\mathrm{B}}{M} = \dfrac{GmM}{d^2M} = \dfrac{Gm}{d^2}$

$$g_\mathrm{C} = \frac{F_\mathrm{c}}{M} = \frac{GmM}{(d+R)^2M} = \frac{Gm}{(d+R)^2}$$

$$\Delta g = g_\mathrm{B} - g_\mathrm{C} = \frac{Gm}{d^2} - \frac{Gm}{(d+R)^2} = \frac{Gm(2Rd + R^2)}{d^2(d+R)^2}$$

Since $d >> R$, $\Delta g \approx \dfrac{2GmR}{d^3}$

5.16 By problem (5.6) the gravitational energy is given by

$$U = -\frac{3}{5}\frac{GM^2}{R} \tag{1}$$

The volume of the star

$$V = \frac{4\pi R^3}{3} \tag{2}$$

$$\therefore \quad R = \left(\frac{3V}{4\pi}\right)^{1/3} \tag{3}$$

$$\therefore \quad U = -\frac{3}{5}\left(\frac{4\pi}{3V}\right)^{1/3} GM^2 \tag{4}$$

$$P = -\frac{\partial U}{\partial V} = -\frac{1}{5}\left(\frac{4\pi}{3}\right)\frac{GM^2}{V^{4/3}}$$

$$\therefore \quad P \propto V^{-4/3}$$

5.17 Consider a line element dx at A distance x from O, Fig. 5.13. The field point P is at a distance R from the infinite line. Let PA $= r$. The x-component of gravitational field at P due to this line element will get cancelled by a symmetric line element on the other side. However, the y-component will add up. If λ is the linear mass density, the corresponding mass element is λdx

$$\mathrm{d}E = \mathrm{d}E_y = -\frac{G\lambda \mathrm{d}x \sin\theta}{r^2} \tag{1}$$

$$\text{Now,} \quad r^2 = x^2 + R^2 \tag{2}$$

$$x = R\cot\theta \tag{3}$$

$$\therefore \quad r^2 = R^2 \operatorname{cosec}^2\theta \tag{4}$$

$$dx = R \operatorname{cosec}^2\theta \mathrm{d}\theta \tag{5}$$

Using (4) and (5) in (1)

$$\mathrm{d}E = -\frac{G\lambda}{R}\sin\theta \mathrm{d}\theta$$

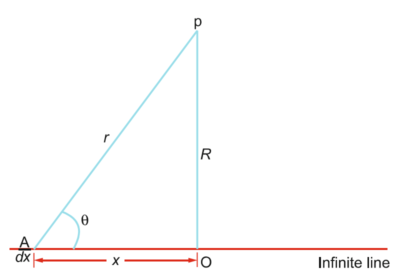


Fig. 5.13

Integrating from 0 to $\pi/2$ for the contribution from the line elements on the left-hand side of O and doubling the result for taking into account contributions on the right-hand side

$$E = -\frac{2\lambda}{R}\int_0^{\pi/2} \sin\theta \mathrm{d}\theta = -\frac{2G\lambda}{R}$$

5.18 Let the neutral point be located at distance x from the earth's centre on the line joining the centres of the earth and moon. If M_e and M_m are the masses of the earth and the moon, respectively, and m the mass of the body placed at the neutral point, then the force exerted by M_e and M_m must be equal and opposite to that of M_m on m.

$$\frac{GM_\mathrm{e}m}{x^2} = \frac{GM_\mathrm{m}m}{(d-x)^2}$$

$$\therefore \quad \frac{M_\mathrm{e}}{M_\mathrm{m}} = 81 = \frac{x^2}{(d-x)^2}$$

Since $d > x$, there is only one solution

$$\frac{x}{d-x} = +9$$

$$\text{or} \quad x = \frac{9}{10}d$$

5.19 For a homogeneous sphere of mass M the potential for $r \leq R$ is given by $V(r) = -\frac{1}{2}\frac{GM}{R}\left(3 - \frac{r^2}{R^2}\right)$. At the centre of the sphere $V(0) = -\frac{3}{2}\frac{GM}{R}$. For a hemisphere at the centre of the base $V(0) = -\frac{3}{4}\frac{GM}{R}$. The work done to move a particle of mass m to infinity will be $\frac{3}{4}\frac{GMm}{R}$.

5.20 Let the point P be at distance r from the centre of the shell such that $a < r < b$. The gravitational field at P will be effective only from matter within the sphere of radius r. The mass within the shell of radii a and r is $\frac{4\pi}{3}(r^3 - a^3)\rho$. Assume that this mass is concentrated at the centre. Then the gravitational field at a point distance r from the centre will be

$$g(r) = -\frac{4\pi}{3}\frac{(r^3 - a^3)}{r^2}\rho G$$

$$\text{But} \quad \rho = \frac{3M}{4\pi(b^3 - a^3)}$$

$$\therefore \quad g(r) = -\frac{GM(r^3 - a^3)}{r^2(b^3 - a^3)}$$

5.21 Let the disc be located in the xy-plane with its centre at the origin. P is a point on the z-axis at distance z from the origin. Consider a ring of radii r and $r + \mathrm{d}r$ concentric with the disc, Fig. 5.14. The mass of the ring will be

$$\mathrm{d}m = 2\pi r \mathrm{d}r \sigma \tag{1}$$

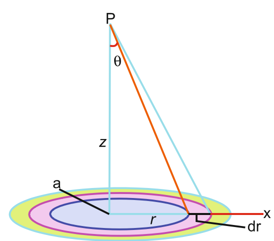
The horizontal component of the field at P will be zero because for each point on the ring there will be another point symmetrically located on the ring which will produce an opposite effect. The vertical component of the field at P will be

$$\mathrm{d}g_z = -\frac{G \times 2\pi r \,\mathrm{d}r\, \sigma \cos\theta}{(r^2 + z^2)} \tag{2}$$

$$\text{But} \quad \cos\theta = \frac{z}{\sqrt{r^2 + z^2}} \tag{3}$$

$$g = g_z = \int \mathrm{d}g_z = -2\pi\sigma G \int_0^a \frac{zr\,\mathrm{d}r}{(r^2 + z^2)^{3/2}} \tag{4}$$

Fig. 5.14



Put $r = z\tan\theta$, $\mathrm{d}r = z\sec^2\theta \mathrm{d}\theta$

$$g = -2\pi G = \int_0^{z/\sqrt{a^2+z^2}} \sin\theta d\theta$$

$$= -2\pi\sigma G\left[1 - \frac{z}{\sqrt{z^2 + a^2}}\right]$$

5.22 Initially the particle is located at a distance $2R$ from the centre of the spherical shell and is at rest. Its potential energy is $-GMm/2R$. When the particle

arrives at the opening the potential energy will be $-GMm/R$ and kinetic energy $\frac{1}{2}mv^2$.
Kinetic energy gained = potential energy lost

$$\frac{1}{2}mv^2 = -\frac{GMm}{2R} - \left(-\frac{GMm}{R}\right) = \frac{1}{2}\frac{GMm}{R}$$
$$\therefore \quad v = \sqrt{\frac{GM}{R}}$$

After passing through the opening the particle traverses a force-free region inside the shell. Thus, within the shell its velocity remains unaltered. Therefore, it hits the point C with velocity $v = \sqrt{\frac{GM}{R}}$.

5.3.2 Rockets and Satellites

5.23 Energy conservation gives

$$\frac{1}{2}mv^2 - \frac{GmM}{R} = -\frac{GmM}{r} + 0$$

where r is the distance from the earth's centre.
Using $v = \sqrt{\frac{GM}{2R}}$, we find $r = \frac{4}{3}R$
Maximum height attained

$$h = r - R = \frac{R}{3}$$

5.24 In Fig. 5.15, total energy at P = total energy at O.

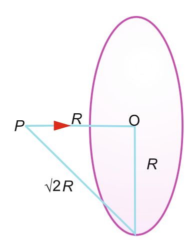


Fig. 5.15

$$-\frac{GmM}{\sqrt{2}R} = \frac{1}{2}mv^2 - \frac{GmM}{R}$$

$$\therefore \quad v = \sqrt{(2-\sqrt{2})\frac{GM}{R}}$$

5.25 The potential energy of the satellite on the earth's surface is

$$U(R) = -\frac{GMm}{R} \tag{1}$$

where M and m are the mass of the earth and the satellite, respectively, and R is the earth's radius.
The potential energy at a height $h = 0.5R$ above the earth's surface will be

$$U(R+h) = -\frac{GMm}{R+h} = -\frac{GMm}{1.5R} \tag{2}$$

Gain in potential energy

$$\Delta U = -\frac{GMm}{1.5R} - \left[-\frac{GMm}{R}\right] = \frac{GMm}{3R} \tag{3}$$

Thus the work done W_1 in taking the satellite from the earth's surface to a height $h = 0.5\,r$

$$W_1 = \frac{GMm}{3R} \tag{4}$$

Extra work W_2 required to put the satellite in the orbit at an attitude $h = 0.5R$ is equal to the extra energy that must be supplied:

$$W_2 = \frac{1}{2}mv_0 = \frac{1}{2}m\left[\frac{GM}{R+h}\right] = \frac{GMm}{3R} \tag{5}$$

where v_0 is the satellite's orbital velocity.
Thus from (4) and (5), $\dfrac{W_1}{W_2} = 1.0$.

5.26 The initial angular momentum of the asteroid about the centre of the planet is $L = mv_0d$.
At the turning point the velocity $\boldsymbol{v}$ of the asteroid will be perpendicular to the radial vector. Therefore the angular momentum $L' = mvR$ if the asteroid is to just graze the planet. Conservation of angular momentum requires that $L' = L$. Therefore

$$mvR = mv_0 d \quad (1)$$

$$\text{or} \quad v = \frac{v_0 d}{R} \quad (2)$$

Energy conservation requires

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 - \frac{GMm}{R} \quad (3)$$

$$\text{or} \quad v^2 = v_0^2 + \frac{2GM}{R} \quad (4)$$

Eliminating v between (2) and (4) the minimum value of v_0 is obtained.

$$v_0 = \sqrt{\frac{2GMR}{d^2 - R^2}}$$

5.27 According to Kepler's third law

$$T^2 \propto r^3$$

(i) $$\frac{T_E^2}{r_E^3} = \frac{(365.3)^2}{(1.5 \times 10^{11})^3} = 3.9539 \times 10^{-29} \quad \text{days}^2/\text{m}^3$$

$$\frac{T_V^2}{r_V^2} = \frac{(224.7)^2}{(1.08 \times 10^{11})^3} = 4.0081 \times 10^{-29} \quad \text{days}^2/\text{m}^3$$

Thus Kepler's third law is verified

(ii) $$T = 2\pi\sqrt{\frac{r^3}{GM}} \quad (1)$$

where M is the mass of the parent body.

$$M = \frac{4\pi^2}{G}\frac{r^3}{T^2} \quad (2)$$

From (i) the mean value, $\left\langle \frac{T^2}{r^3} \right\rangle = 3.981 \times 10^{-29}$ days^2/m^3 $= 2.972 \times 10^{-19}$ s^2/m^3

$$M = \frac{4\pi^2}{6.67 \times 10^{-11}} \times \frac{1}{2.972 \times 10^{-19}} = 1.99 \times 10^{30}\,\text{kg}$$

5.28 At the perihelion (nearest point from the focus) the velocity (v_p) is maximum and at the aphelion (farthest point) the velocity (v_A) is minimum. At both these points the velocity is perpendicular to the radius vector. Since the angular momentum is constant

$$m v_A r_A = m v_p r_p$$
$$\text{or} \quad r_A = \frac{v_p r_p}{v_A} \tag{1}$$

where $r_A = r_{max}$ and $r_p = r_{min}$
The eccentricity

$$\varepsilon = \frac{r_{max} - r_{min}}{r_{max} + r_{min}} = \frac{r_A - r_p}{r_A + r_p} = \frac{v_p - v_A}{v_p + v_A} \tag{2}$$

where we have used (1)

$$\varepsilon = \frac{30.0 - 29.2}{30.0 + 29.2} = 0.0135$$

A small value of eccentricity indicates that the orbit is very nearly circular.

5.29 **(a)** For circular orbit,

$$T = \frac{2\pi a}{v}$$
$$T = 14.4 \text{ days} = 1.244 \times 10^6 \text{s}$$
$$2a = \frac{vT}{\pi} = \frac{2.2 \times 10^5 \times 1.244 \times 10^6}{3.1416} = 8.7 \times 10^{10} \text{ m}$$

(b) Since the velocity of each component is the same, the masses of the components are identical.

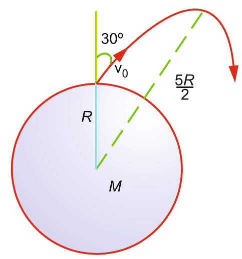
$$v^2 = \frac{G(M+m)}{a} = \frac{2GM}{a} \quad (\because m = M)$$
$$\therefore \quad M = \frac{av^2}{2G} = \frac{(4.35 \times 10^{10})(2.2 \times 10^5)^2}{2 \times 6.67 \times 10^{-11}} = 1.58 \times 10^{31} \text{ kg}$$

5.30 At the surface, the component of velocity of the satellite perpendicular to the radius R is

$$v_0 \sin 30° = \frac{v_0}{2} \quad \text{(Fig. 5.16)}$$

Therefore, the angular momentum at the surface $= \dfrac{m v_0 R}{2}$

Fig. 5.16



At the apogee (farthest point), the velocity of the satellite is perpendicular to the radius vector. Therefore, the angular momentum at the apogee $= (mv)(5R/2)$.
Conservation of angular momentum gives

$$\frac{5}{2}mvR = \frac{m}{2}v_0 R$$
$$\text{or} \quad v = \frac{v_0}{5} \tag{1}$$

The kinetic energy at the surface $K_0 = \frac{1}{2}mv_0^2$ and potential energy $U_0 = -\frac{GMm}{R}$.
Therefore, the total mechanical energy at the surface is

$$E_0 = \frac{1}{2}mv_0^2 - \frac{GMm}{R} \tag{2}$$

At the apogee kinetic energy $K = \frac{1}{2}mv^2$ and the potential energy $U = -\frac{2GMm}{5R}$.
Therefore, the total mechanical energy at the apogee is

$$E = \frac{1}{2}mv^2 - \frac{2}{5}\frac{GMm}{R} \tag{3}$$

Conservation of total energy requires that $E = E_0$. Eliminating v in (3) with the aid of (1) and simplifying we get

$$v_0 = \sqrt{\frac{5GM}{4R}}$$

5.31 For an elliptic orbit

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$

$$\therefore \quad v_{\max} = \sqrt{GM\left(\frac{2}{r_{\min}} - \frac{1}{a}\right)} = \sqrt{\frac{GM(2a - r_{\min})}{ar_{\min}}} = \sqrt{\frac{GMr_{\max}}{ar_{\min}}} \qquad (1)$$

as $\quad r_{\max} + r_{\min} = 2a$

$$v_{\min} = \sqrt{GM\left(\frac{2}{r_{\max}} - \frac{1}{a}\right)} = \sqrt{\frac{GM(2a - r_{\max})}{ar_{\max}}} = \sqrt{\frac{GMr_{\min}}{ar_{\max}}} \qquad (2)$$

Multiplying (1) and (2)

$$v_{\max}v_{\min} = \frac{GM}{a}$$

or $\quad \sqrt{v_{\max}v_{\min}} = \sqrt{\frac{GM}{a}} = a\sqrt{\frac{GM}{a^3}} = \frac{2\pi a}{T}$

as $\quad T = 2\pi\sqrt{\frac{a^3}{GM}}$

$$\therefore \quad a = \frac{T}{2\pi}\sqrt{v_{\max}v_{\min}}$$

5.32 $$T = 2\pi\sqrt{\frac{r^3}{GM}} \qquad (1)$$

But $r = a + b + c$ (2)

Combining (1) and (2)

$$M = \frac{4\pi^2(a + b + c)^3}{GT^2}$$

5.33 $L = |r \times p| = rp \sin\theta$

L per unit mass $= rv\sin\theta$

$= (1.75 \times 1.5 \times 10^{11})(3 \times 10^4)\sin 30^\circ$

$= 3.9375 \times 10^{15}\ \text{m}^2/\text{s}$

When the comet is closest to the sun its velocity will be perpendicular to the radius vector. The angular momentum $L' = r'v'$. Angular momentum conservation requires

$L' = L$

$$\therefore \quad v' = \frac{L'}{r'} = \frac{L}{r} = \frac{3.9375 \times 10^{15}}{0.39 \times 1.5 \times 10^{11}} = 6.73 \times 10^4 \,\text{m/s} = 67.3\,\text{km/s}$$

Total energy per unit mass

$$E = \frac{1}{2}v^2 - \frac{GM}{r} = \frac{1}{2}(3 \times 10^4)^2 - \frac{6.67 \times 10^{-11} \times 2 \times 10^{30}}{1.75 \times 10^{11}} = -3.12 \times 10^8 \text{J}$$

a negative quantity. Therefore the orbit is bound.

5.34 **(a)** The centripetal force is provided by the gravitational force.

$$\frac{GM_\text{E}M_\text{S}}{R^2} = \frac{M_\text{E}v^2}{R}$$

$$\text{or} \quad GM_\text{S} = v^2R \tag{1}$$

(b) Total energy of the comet when it is closest to the sun

$$E = \frac{1}{2}M_\text{C}(2v)^2 - \frac{GM_\text{C}M_\text{S}}{R/2} \tag{2}$$

Using (1) in (2) we find $E = 0$.

(c) At the distance of the closest approach, the comet's velocity is perpendicular to the radius vector. Therefore the angular momentum

$$L = M_\text{C}(2v)\left(\frac{R}{2}\right) = M_\text{C}vR \tag{3}$$

Let v_t be the comet's velocity which is tangential to the earth's orbit at P. Then the angular momentum at P will be

$$L' = M_\text{c}v_\text{t}R \tag{4}$$

Angular momentum conservation gives

$$M_\text{C}v_\text{t}R = M_\text{C}vR \tag{5}$$

$$\text{or} \quad v_\text{t} = v \tag{6}$$

(d) The total energy of the comet at P is

$$E' = \frac{1}{2}M_\text{C}(v')^2 - \frac{GM_\text{S}M_\text{C}}{R} = 0 \tag{7}$$

where v' is the comet's velocity at P, because $E' = E = 0$, by energy conservation.

Using (1) in (7) we find

$$v' = \sqrt{2}v \tag{8}$$

If θ is the angle between $\boldsymbol{v}'$ and the radius vector $\boldsymbol{R}$ angular momentum conservation gives

$$M_C vR = M_C v' R\sin\theta = M_C\sqrt{2}vR\sin\theta$$

$$\text{or} \quad \sin\theta = \frac{1}{\sqrt{2}} \quad \theta = 45^\circ$$

5.35 At both perigee and apogee the velocity of the satellite is perpendicular to the radius vector. In order to show that the angular momentum is conserved we must show that

$$mv_p r_p = mv_A r_A$$

$$\text{or} \quad v_p r_p = v_A r_A$$

where m is the mass of the satellite.

$$v_p r_p = 10.25 \times 6570 = 67342.5$$

$$v_A r_A = 1.594 \times 42250 = 67346.5$$

The data are therefore consistent with the conservation of angular momentum.

5.36 **(a)**

$$v_0 = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)} \tag{1}$$

$$GM = (6.67 \times 10^{-11})(6 \times 10^{24}) = 4 \times 10^{14}$$

$$r = R = 6.4 \times 10^6$$

$$a = 8 \times 10^7\ m$$

$$v_0 = 1.095 \times 10^4\ \text{m/s} = 10.095\ \text{km/s}$$

(b)

$$\varepsilon = \sqrt{1 + \frac{2EJ^2}{G^2M^2m^3}} \tag{2}$$

$$J = mRv_0 \sin 45^\circ = \frac{mRv_0}{\sqrt{2}} \tag{3}$$

$$E = -\frac{GMm}{2a} \tag{4}$$

Combining (1), (2), (3) and (4)

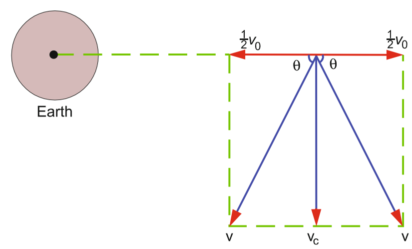
$$\varepsilon = \sqrt{1 - \frac{R}{a} + \frac{1}{2}\frac{R^2}{a^2}} \tag{5}$$

Now $\quad \dfrac{R}{a} = \dfrac{6400}{80000} = 0.08$

$\therefore \quad \varepsilon = 0.96$

5.37 The resultant velocity v of each fragment is obtained by combining the velocities $\dfrac{1}{2}v_0$ and v_0 vectorially, Fig. 5.17.

Fig. 5.17



$$v = \sqrt{\left(\frac{1}{2}v_0\right)^2 + v_0^2} = \frac{1}{2}\sqrt{5}v_0$$

Kinetic energy of each fragment

$$K = \frac{1}{2}\left(\frac{m}{2}\right)\left(\frac{\sqrt{5}}{2}v_0\right)^2 = \frac{5}{16}mv_0^2 = \frac{5}{16}m\frac{GM}{r}$$

Potential energy of each fragment $U = -\dfrac{GM\left(\frac{1}{2}m\right)}{r}$

$\therefore \quad$ Total energy $E = K + U = \dfrac{5GMm}{16r} - \dfrac{1}{2}\dfrac{GMm}{r} = -\dfrac{3}{16}\dfrac{GMm}{r}$

If $\boldsymbol{v}$ makes on angle θ with the radius vector $\boldsymbol{r}$, then $v \sin\theta = v_0$. The angular momentum of either fragment about the centre of the earth is

$$J = \frac{1}{2}mv_0 r = \frac{mr}{2}\sqrt{\frac{GM}{r}} = \frac{1}{2}m\sqrt{GMr}$$

5.38 Velocity at the nearer apse is given by

$$v^2 = GM\left[\frac{2}{a(1-\varepsilon)} - \frac{1}{a}\right] = \frac{GM}{a}\left(\frac{1+\varepsilon}{1-\varepsilon}\right) \tag{1}$$

as there is no instantaneous change of velocity. If a_1 is the semi-major axis for the new orbit

$$v^2 = GM\left[\frac{2}{a(1-\varepsilon)} - \frac{1}{a_1}\right] \tag{2}$$

As the nearer and farther apses are inter-changed

$$a_1(1-\varepsilon_1) = a(1+\varepsilon) \tag{3}$$

Equating the right-hand side of (1) and (2) and eliminating a_1 from (3) and solving for ε_1 we get

$$\varepsilon_1 = \frac{\varepsilon(3+\varepsilon)}{1-\varepsilon}$$

5.39 **(a)** $$\frac{1}{T}\int \frac{\mathrm{d}t}{r} = \frac{1}{T}\int \frac{\mathrm{d}\theta}{r\dot{\theta}} \tag{1}$$

Now, $J = mr^2\dot{\theta}$ (constant) (2)

$$\therefore \quad \frac{1}{T}\int \frac{\mathrm{d}t}{r} = \frac{m}{TJ}\int r\,\mathrm{d}\theta \tag{3}$$

$$r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta} \tag{4}$$

Using (4) in (3)

$$\frac{1}{T}\int \frac{\mathrm{d}t}{r} = \frac{ma(1-\varepsilon^2)}{TJ}\int_0^{2\pi} \frac{\mathrm{d}\theta}{1+\varepsilon\cos\theta}$$

$$= \frac{ma(1-\varepsilon^2)}{TJ}\frac{2\pi}{\sqrt{1-\varepsilon^2}} \tag{5}$$

where we have used the integral $\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$

Further $T = \frac{2\pi ma^2\sqrt{1-\varepsilon^2}}{J}$ (6)

Using (6) in (5)

$$\frac{1}{T}\int\frac{\mathrm{d}t}{r}=\frac{1}{a} \tag{7}$$

(b) $$\frac{1}{T}\int v^2\mathrm{d}t=\frac{GM}{T}\int\left(\frac{2}{r}-\frac{1}{a}\right)\mathrm{d}t$$
$$=2GM\frac{1}{T}\int\frac{\mathrm{d}t}{r}-\frac{GM}{Ta}\int\mathrm{d}t=\frac{2GM}{a}-\frac{GM}{a}=\frac{GM}{a}$$

where we have used (7) and put $\int \mathrm{d}t = T$.

5.40 The distance between the focus and the end of minor axis is a. Let the new semi-major axis be a_1. Since the instantaneous velocity does not change

$$GM\left(\frac{2}{a}-\frac{1}{a}\right)=G(M+m)\left(\frac{2}{a}-\frac{1}{a_1}\right)$$
$$\text{or}\quad a_1=\frac{a\left(1+\frac{m}{M}\right)}{1+\frac{2m}{M}}\approx a\left(1+\frac{m}{M}\right)\left(1-\frac{2m}{M}\right)$$
$$a_1=a\left(1-\frac{m}{M}\right) \tag{1}$$

The new time period

$$T_1=\frac{2\pi a_1^{3/2}}{\sqrt{G(M+m)}}=\frac{2\pi a^{3/2}}{\sqrt{GM}}\left(1-\frac{m}{M}\right)^{3/2}\left(1+\frac{m}{M}\right)^{-1/2}$$
$$\approx T\left(1-\frac{3m}{2M}\right)\left(1-\frac{m}{2M}\right)\approx T\left(1-\frac{2m}{M}\right)$$

where we have used binomial expansion and the value of the old time period.

5.41 *Case 1: Apse is farther*
It is sufficient to show that the total energy is zero.

$$r_1=a(1+\varepsilon)=a(1+0.5)=1.5a$$
$$v_1^2=GM\left(\frac{2}{r_1}-\frac{1}{a}\right)=GM\left(\frac{2}{1.5a}-\frac{1}{a}\right)=\frac{GM}{3a}$$

New velocity $v_1'=2v_1$.
New kinetic energy

$$K_1'=\frac{1}{2}m(v_1')^2=\frac{1}{2}m(2v_1)^2=\frac{2GMm}{3a}$$

The potential energy is unaltered and is therefore

$$U_1 = -\frac{GMm}{r_1} = -\frac{2GMm}{3a}$$

Total energy $E_1' = K' + U_1 = \frac{2GMm}{3a} - \frac{2GMm}{3a} = 0$

Case 2: Apse is nearer

It is sufficient to show that the total energy is positive.

$$r_2 = a(1-\varepsilon) = a(1-0.5) = 0.5a$$

$$v_2^2 = GM\left(\frac{2}{r_2} - \frac{1}{a}\right) = GM\left(\frac{2}{0.5a} - \frac{1}{a}\right) = \frac{3GM}{a}$$

New velocity $v_2' = 2v_2$.

New kinetic energy $K_2' = \frac{1}{2}m\left(v_2'\right)^2 = \frac{1}{2}m(2v_2)^2 = \frac{6GMm}{a}$

Potential energy is unaltered and is given by

$$U_2 = -\frac{GMm}{r_2} = -\frac{GMm}{0.5a} = -\frac{2GMm}{a}$$

Total energy $E_2 = K_2' + U_2 = \frac{6GMm}{a} - \frac{2GMm}{a} = +\frac{4GMm}{a}$,

which is a positive quantity.

5.42 The velocity of the particle in the orbit is given by

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

When the particle is at one extremity of the minor axis, $r = a$

$$v^2 = GM\left(\frac{2}{a} - \frac{1}{a}\right) = \frac{GM}{a}$$

Let the new axes be $2a_1$ and $2b_1$. By problem the force is increased by half, but the velocity at $r = a$ is unaltered.

$$v^2 = 1.5\,GM\left(\frac{2}{a} - \frac{1}{a_1}\right) = \frac{GM}{a}$$

$$\therefore \quad 2a_1 = \frac{3a}{2}$$

As v is unaltered in both magnitude and direction, the semi-latus rectum $l = \frac{b^2}{a} = a(1-\varepsilon^2)$. The constant $h^2 = $ (GM) (semi-latus rectum) is unchanged.

$$\therefore \quad GM\frac{b^2}{a} = \frac{3}{2}GM\frac{b_1^2}{a_1}$$

$$\therefore \quad b_1^2 = \frac{2b^2}{3}\frac{a_1}{a} = \frac{2}{3}\cdot\frac{3}{4}b^2$$

$$\therefore \quad 2b_1 = \sqrt{2}b$$

5.43 **(a)** The forces acting on the satellite are gravitational force and centripetal force.

(b) Equating the centripetal force and gravitational force

$$\frac{mv^2}{R} = mg$$

$$\therefore \quad v = \sqrt{gR} = \frac{2\pi R}{T}$$

$$\therefore \quad T = 2\pi\sqrt{\frac{R}{g}} = 2\pi\sqrt{\frac{R^3}{GM}} \tag{1}$$

(c) The geocentric satellite must fly in the equatorial plane so that its centripetal force is entirely used up by the gravitational force. Second, it must fly at the right altitude so that its time period is equal to that of the diurnal rotation of the earth.

(d) 24 h.

(e) Using (1)

$$r = \left[\frac{T^2 GM}{4\pi^2}\right]^{1/3}$$

Using $T = 86,400$ s, $G = 6.67 \times 10^{-11}$ kg^{-1} m^3/s^2, $M = 6.4 \times 10^{24}$ kg, we find $r = 4.23 \times 10^7$ m or 42,300 km.

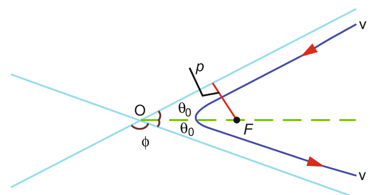
5.44 At both perigee and apogee $\boldsymbol{v}$ is perpendicular to $\boldsymbol{r}$. Angular momentum conservation gives $mv_A r_A = mv_p r_p$

$$r_A = 2a - r_p = 4r - r = 3r$$

$$v_A = \frac{vr}{3r} = \frac{v}{3}$$

5.45 The orbit of the small body will be a hyperbola with the heavy body at the focus F, Fig. 5.18.

Fig. 5.18



$$r = \frac{a(\varepsilon^2 - 1)}{\varepsilon \cos\theta - 1} \tag{1}$$

As $r \to \infty$, the denominator on the right-hand side of (1) becomes zero and the limiting angle θ_0 is given by

$$\cos\theta_0 = \frac{1}{\varepsilon}$$

or $\quad \cot\theta_0 = \frac{1}{\sqrt{\varepsilon^2 - 1}}$

The complete angle of deviation

$$\phi = \pi - 2\theta_0$$

or $\quad \frac{\phi}{2} = \frac{\pi}{2} - \theta_0$

$$\tan\frac{\phi}{2} = \cot\theta_0 = \frac{1}{\sqrt{\varepsilon^2 - 1}}$$

But $\quad \varepsilon = \sqrt{1 + \frac{2Eh^2}{G^2M^2}}$

where $h = pv$ and $E = \frac{1}{2}v^2$

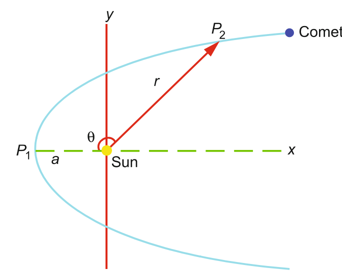
$$\therefore \quad \tan\frac{\phi}{2} = \frac{1}{\sqrt{\varepsilon^2 - 1}} = \frac{GM}{h\sqrt{2E}} = \frac{GM}{pv^2}$$

5.46 In Fig. 5.19

$$r = \frac{2a}{1 + \cos\theta}$$

$r^2\dot{\theta} = h \quad$ (constant, law of areas)

Fig. 5.19



$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{r^2}{h}$$

Time taken for the object to move from P_1 to P_2 (Fig. 5.19) is given by

$$t = \int \mathrm{d}t = \int \frac{r^2 \mathrm{d}\theta}{h} = \frac{4a^2}{h} \int_0^\theta \frac{\mathrm{d}\theta}{(1+\cos\theta)^2}$$

$$= \frac{a^2}{h} \int_0^\theta \sec^4\left(\frac{\theta}{2}\right) \mathrm{d}\theta = \frac{2a^2}{h} \int_0^\theta \left(1 + \tan^2\frac{\theta}{2}\right) \mathrm{d}\left(\tan\frac{1}{2}\theta\right)$$

$$= \frac{2a^2}{h}\left(\tan\frac{1}{2}\theta + \frac{1}{3}\tan^3\frac{1}{2}\theta\right)$$

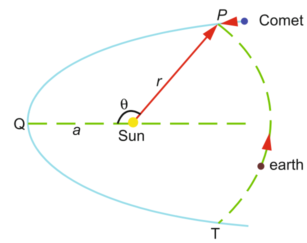
But $\quad h = \sqrt{GM \times \text{semi - latus rectum}} = \sqrt{2aGM}$

$$\therefore \quad t = \sqrt{\frac{2a^3}{GM}}\left(\tan\frac{1}{2}\theta + \frac{1}{3}\tan^3\frac{1}{2}\theta\right)$$

5.47 Required time for traversing the arc PQT is obtained by the formula derived in problem (5.46), Fig. 5.20

$$t_0 = 2t = 2\sqrt{\frac{2a^3}{GM}}\left(\tan\frac{1}{2}\theta + \frac{1}{3}\tan^3\frac{1}{2}\theta\right) \tag{1}$$

Fig. 5.20



For parabola

$$r = \frac{2a}{1 + \cos\theta} \tag{2}$$

$$\text{or} \quad \cos\theta = \frac{2a}{r} - 1 \tag{3}$$

$$\therefore \quad \tan\frac{\theta}{2} = \sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}} = \sqrt{\frac{R}{a} - 1} \tag{4}$$

where we have put $r = R$, the radius of earth's orbit. Using (4) in (1)

$$t_0 = \frac{2}{3}\sqrt{\frac{2}{GM}}(2a + R)\sqrt{R - a} \tag{5}$$

t_0 is maximized by setting $\frac{\mathrm{d}t_0}{\mathrm{d}a} = 0$. This gives

$$a = \frac{R}{2} \tag{6}$$

Using (6) in (5) gives

$$t_0(\max) = \frac{4}{3}\sqrt{\frac{R^3}{GM}} = \frac{2}{3\pi}2\pi\sqrt{\frac{R^3}{GM}} = \frac{2}{3\pi}T$$

where $T = 2\pi\sqrt{\frac{R^3}{GM}} = 1$ year is the time period of the earth.

Thus $t_0(\max) = \frac{2}{3\pi}$ years.

5.48 $$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 \tag{1}$$

$$r = a \sin n\theta \tag{2}$$

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = n^2a^2(1 - \sin^2 n\theta) = n^2a^2\left(1 - \frac{r^2}{a^2}\right) \tag{3}$$

Using (3) in (1)

$$\frac{1}{p^2} = \frac{n^2a^2}{r^4} + \frac{1-n^2}{r^2}$$

Differentiating

$$-\frac{2}{p^3}\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{4n^2a^2}{r^5} - \frac{2(1-n^2)}{r^3}$$

or $$\frac{1}{p^3}\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{2n^2a^2}{r^5} + \frac{1-n^2}{r^3}$$

Force per unit mass

$$f = -\frac{h^2}{p^3}\frac{\mathrm{d}p}{\mathrm{d}r} = -h^2\left(\frac{2n^2a^2}{r^5} + \frac{1-n^2}{r^3}\right)$$

5.49 $$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 \tag{1}$$

$$r = a(1 - \cos\theta)$$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = a \sin\theta$$

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = a^2\sin^2\theta = a^2\left[1 - \left(1 - \frac{r}{a}\right)^2\right] = 2ra - r^2 \tag{2}$$

Using (2) in (1)

$$\frac{1}{P^2} = \frac{2a}{r^3}$$

or $$p^2 = \frac{r^3}{2a} \tag{3}$$

Force per unit mass

$$f = -\frac{h^2}{p^3}\frac{\mathrm{d}p}{\mathrm{d}r} \tag{4}$$

Differentiating (3)

$$2p\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{3r^2}{2a} \tag{5}$$

Using (5) in (4)

$$f = -\frac{3}{4}\frac{h^2r^2}{ap^4} = -\frac{3ah^2}{r^4} \tag{6}$$

where we have used (3). Thus the force is proportional to the inverse fourth power of distance.

5.50 If $u = \dfrac{1}{r}$, then at the apse

$$\frac{\mathrm{d}u}{\mathrm{d}\theta} = 0$$

$$\text{or} \quad -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta} = 0$$

$$\therefore \quad -\frac{1}{r^2}a\sin\theta = 0$$

from which either $\sin\theta = 0$ or r is infinite, the latter case being inadmissible so long the particle is moving along the cardioid.

Thus $\theta = \pi$ or 0

When $\theta = \pi,\ r = 2a$

and $Q = \dfrac{3ah^2}{r^4} = \dfrac{3ah^2}{16a^4} = \dfrac{3h^2}{16a^3}$

Also $p^2 = \dfrac{r^2}{2a} = \dfrac{8a^3}{2a} = 4a^2$

and $v^2 = \dfrac{h^2}{p^2} = \dfrac{h^2}{4a^2}$

Thus $4a\ Q = \dfrac{3h^2}{4a^2} = 3v^2$

When $\theta = 0$, $r = 0$ and $p = 0$ and the particle is moving with infinite velocity along the axis of the cardioid and continues to move in a straight line.

5.51 Let the force $f = -\dfrac{k}{r^3} = -ku^3$

where $u = \dfrac{1}{r}$

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = -\frac{f}{h^2u^2} = \frac{ku}{h^2}$$

Case (i): $\dfrac{k}{h^2} > 1$

Let $\dfrac{k}{h^2} - 1 = n^2$

$$\frac{d^2u}{d\theta^2} - n^2u = 0$$

which has the solution $u = Ae^{n\theta} + Be^{-n\theta}$, where the constants A and B depend on the initial conditions of projection. If these are such that either A or B is zero then the path is an equiangular spiral

Case (ii): $\dfrac{k}{h^2} = 1$, the equation becomes $\dfrac{d^2u}{d\theta^2} = 0$, whose solution is $v = A\theta + B$, a curve known as the reciprocal spiral curve.

Case (iii): $\dfrac{k}{h^2} < 1$. Let $1 - \dfrac{k}{h^2} = n^2$, the equation becomes $\dfrac{d^2u}{d\theta^2} + n^2u = 0$ whose solution is $u = A\cos n\theta + B\sin n\theta$, a curve with infinite branches.

5.52
$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \tag{1}$$
$$r^2 = a^2\cos^2\theta \tag{2}$$
$$r\frac{dr}{d\theta} = -a^2\sin^2\theta \tag{3}$$
$$\therefore \quad \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{a^4}{r^6}\sin^2 2\theta = \frac{a^4}{r^6}\left(1 - \frac{r^4}{a^4}\right) = \frac{a^4}{r^6} - \frac{1}{r^2} \tag{4}$$

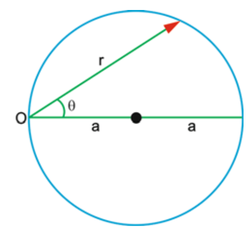
From (1) and (4)

$$\frac{1}{p^2} = \frac{a^4}{r^6}$$
$$\text{or} \quad p = \frac{r^3}{a^2} \tag{5}$$
$$\therefore \quad \frac{dp}{dr} = \frac{3r^2}{a^2} \tag{6}$$
$$f = -\frac{h^2}{p^3}\frac{dp}{dr} = -\frac{3h^2a^4}{r^7}$$

where we have used (5) and (6).

5.53 The polar equation of a circle with the origin on the circumference is $r = 2a\cos\theta$ where a is the radius of the circle, Fig. 5.21.

Fig. 5.21



$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 \tag{1}$$

$$r = 2a\cos\theta \tag{2}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -2a\sin\theta \tag{3}$$

$$\therefore \quad \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = 4a^2(1-\cos^2\theta) = 4a^2 - r^2 \tag{4}$$

$$\therefore \quad \frac{1}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = \frac{4a^2}{r^4} - \frac{1}{r^2} \tag{5}$$

Using (5) in (1) and simplifying

$$p = \frac{r^2}{2a} \tag{6}$$

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{r}{a} \tag{7}$$

$$f = -\frac{h^2}{p^3}\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{h^2a^4}{r^5}$$

5.54 Initially the earth's orbit is circular and its kinetic energy would be equal to the modulus of potential energy

$$\frac{1}{2}mv_0^2 = \frac{1}{2}\frac{mGM}{r} \tag{1}$$

Suddenly, sun's mass becomes half and the earth is placed with a new quantity of potential energy, its instantaneous value of kinetic energy remaining unaltered.

New total energy = new potential energy + kinetic energy

$$= -G\left(\frac{M}{2}\right)\frac{m}{r} + \frac{1}{2}mv_0^2$$
$$= -\frac{GMm}{2r} + \frac{1}{2}\frac{GMm}{r} = 0$$

As the total energy $E = 0$ the earth's orbit becomes parabolic.