Lecture 3

Probability and Probability Distributions

Probability

- Example: If we toss a coin 10 times and get 10 heads in a row;
- Question: Do you believe it is a fair coin?
- Answer: No.
- Reason: If the coin is fair, the chance to have 10 heads in a row is less than 0.1% (According to probability theory).
- Tool and foundation of statistics; Evaluate reliability of statistical conclusions…

Basic Concepts

- An **experiment** is the process by which an observation (or measurement) is obtained.
- **Experiment: Record an age**
- **Experiment: Toss a die**
- **Experiment: Record an opinion (yes, no)**
- **Experiment: Toss two coins**

Basic Concepts

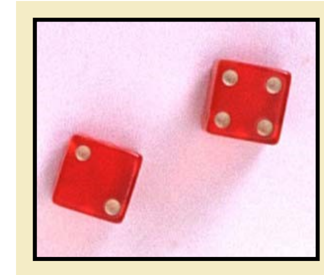
- A **simple event** is the outcome that is observed on a single repetition of the experiment.
  - The basic element to which probability is applied.
  - One and only one simple event can occur when the experiment is performed.

- A **simple event** is denoted by E with a subscript.

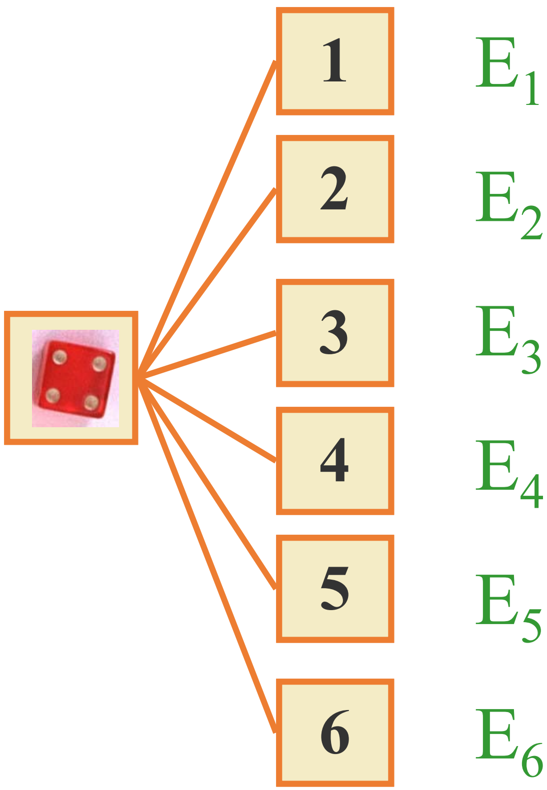
Basic Concepts

- Each simple event will be assigned a probability, measuring "how often" it occurs.
- The set of all simple events of an experiment is called the **sample space,** usually denoted by **S.**

Example



- **The die toss:**
- Simple events:

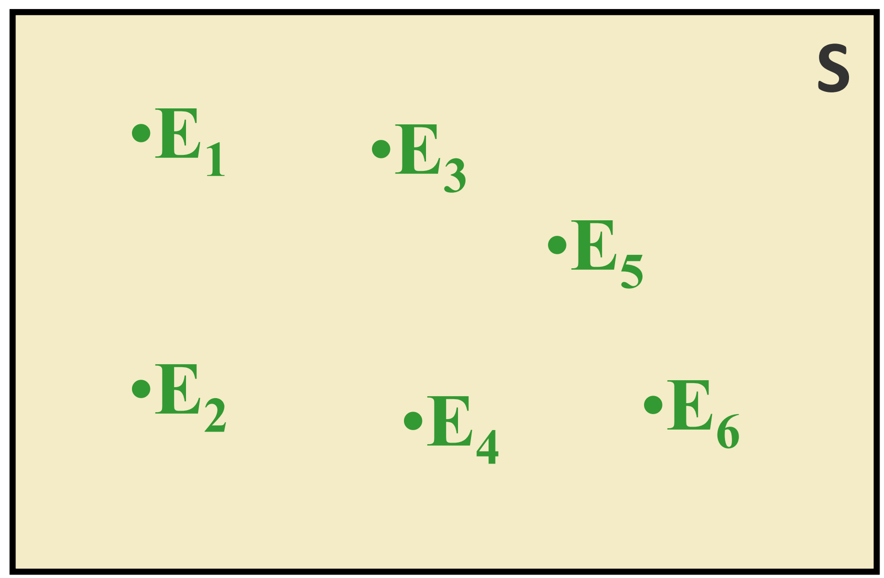


Sample space:

$S = \{E_1, E_2, E_3, E_4, E_5, E_6\}$

(or $S = \{1, 2, 3, 4, 5, 6\}$)

Venn Diagram



Example

- **Record a person's blood type:**
- Simple events: Sample space:

E_1	A
E_2	AB
E_3	B
E_4	O

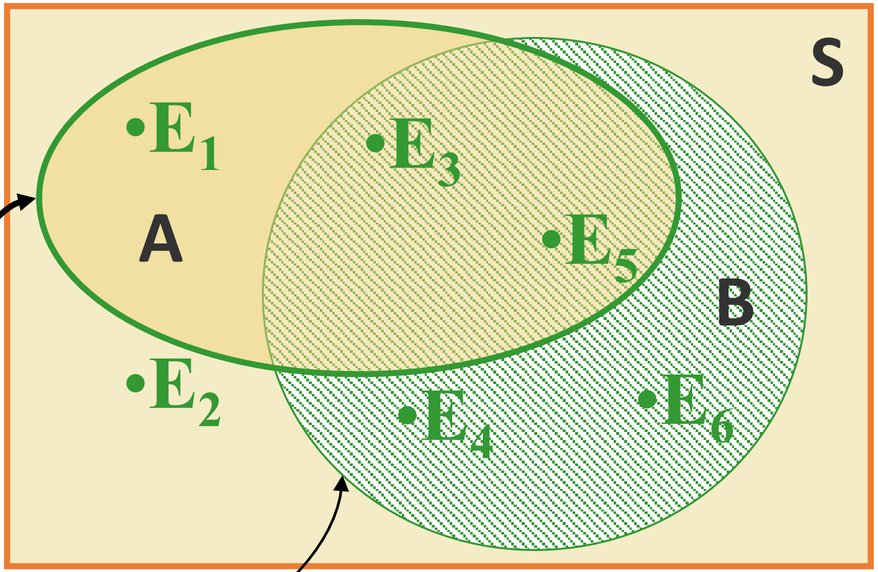
$$\mathbf{S = \{E_1, E_2, E_3, E_4\}}$$

$$\mathbf{S = \{A, B, AB, O\}}$$

Basic Concepts

- An **event** is a collection of one or more **simple events.**

- **The die toss:**
  - –A: an odd number
  - –B: a number > 2



$A = \{E_1, E_3, E_5\}$

$B = \{E_3, E_4, E_5, E_6\}$

Basic Concepts

- Two events are **mutually exclusive** if, when one event occurs, the other cannot, and vice versa.

- **Experiment: Toss a die**
  - A: observe an odd number
  - B: observe a number greater than 2
  - C: observe a 6
  - D: observe a 3

Not Mutually Exclusive

Mutually Exclusive

A and C?
A and D?
B and C?

The Probability of an Event

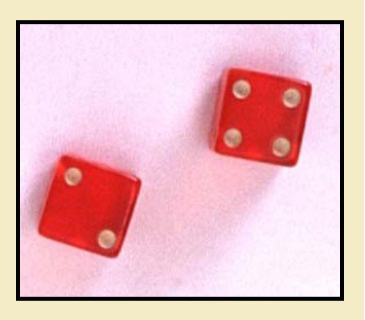
- The probability of an event A measures "how often" we think A will occur. We write **P(A).**
- Suppose that an experiment is performed *n* times. The relative frequency for an event A is

$$\frac{\text{Number of times A occurs}}{n} = \frac{f}{n}$$

- If we let *n* get infinitely large,

$$\boxed{P(A) = \lim_{n \to \infty} \frac{f}{n}}$$

The Probability of an Event



- P(A) must be between 0 and 1.
  - If event A can never occur, P(A) = 0.
  - If event A always occurs, P(A) =1.
- The sum of the probabilities for all simple events in S equals 1. P(S)=1.

The **probability of an event A** can be found by adding the probabilities of all the simple events in A.

Finding Probabilities

- Probabilities can be found using
  - Estimates from empirical studies
  - Common sense estimates based on equally likely events.

- **Examples:**
  - –Toss a fair coin. P(Head) = 1/2
  - –10% of the U.S. population has red hair. Select a person at random. P(Red hair) = .10

Example

- Toss a fair coin twice. What is the probability of observing at least one head (event A)? Exactly one Head (event B)?

1st Coin	2nd Coin	E_i	$P(E_i)$
H	H	HH	1/4
	T	HT	1/4
T	H	TH	1/4
	T	TT	1/4

Tree Diagram

P(at least 1 head)= P(A)

$= P(E_1) + P(E_2) + P(E_3)$

$= 1/4 + 1/4 + 1/4 = 3/4$

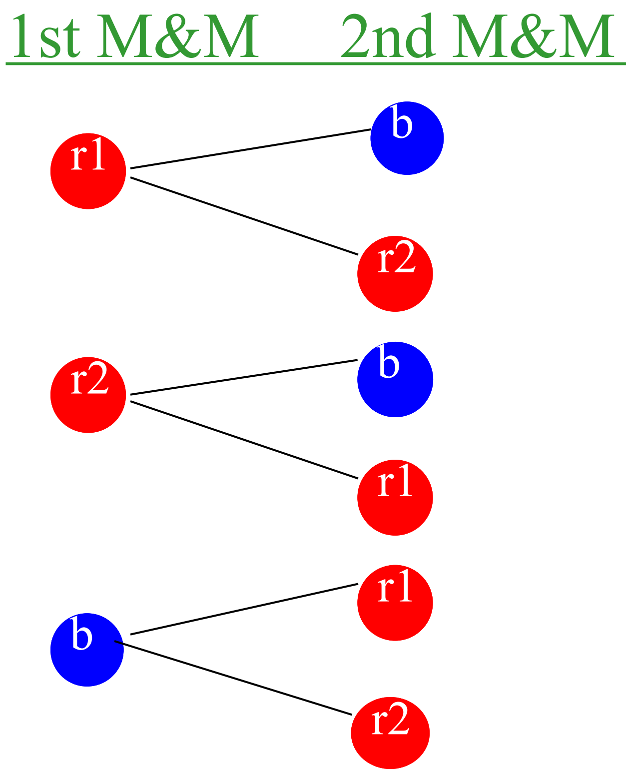
P(exactly 1 head)=P(B)

$= P(E_2) + P(E_3)$

$= 1/4 + 1/4 = 1/2$

Example

- **A bowl contains three M&Ms®, two reds, one blue. A child selects two M&Ms at random. Probability of observing exactly two reds?**



E_i	$P(E_i)$
r1b	**1/6**
r1r2	**1/6**
r2b	**1/6**
r2r1	**1/6**
br1	**1/6**
br2	**1/6**

P(exactly two reds)

= P(r1r2) + P(r2r1)

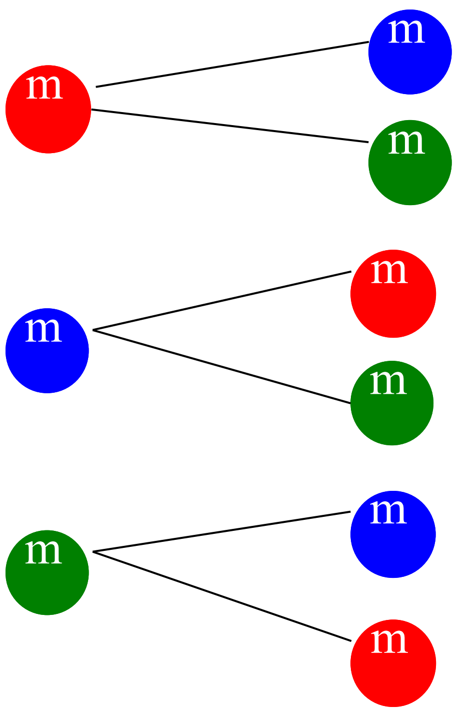
= 1/6 +1/6

= 1/3

Example

- **A bowl contains three M&Ms®, one red, one blue and one green. A child takes two M&Ms randomly one at a time. What is the probability that at least one is red?**

*1*st 2nd



E_i	$P(E_i)$
RB	**1/6**
RG	**1/6**
BR	**1/6**
BG	**1/6**
GB	**1/6**
GR	**1/6**

$$P(\text{at least 1 red}) = P(E_1) + P(E_2) + P(E_3) + P(E_6) = 4/6 = 2/3$$

Example

- Toss a fair coin 3 times. What is the probability of observing at least two heads (event A)? Exactly two Heads (event B)?

Simple Events	Probabilities
HHH	1/8
HHT	1/8
HTH	1/8
HTT	1/8
THH	1/8
THT	1/8
TTH	1/8
TTT	1/8

A={HHH, HHT, HTH, THH}

P(at least 2 heads)=P(A)

= P(HHH)+P(HHT)+P(HTH)+P(THH)

= 1/8 + 1/8 +1/8 + 1/8 = 1/2

B={HHT,HTH,THH}

P(Exactly 2 heads)= P(B)

= P(HHT) + P(HTH) + P(THH)

= 1/8 + 1/8 + 1/8 = 3/8

Example

A: at least two heads; B: exactly two heads;
C: at least two tails; D: exactly one tail.
Questions: A and C mutually exclusive? B and D?

<u>Simple Events</u>

HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

A={HHH,HHT,HTH,THH}

Mutually Exclusive

C={HTT,THT,TTH,TTT}

B={HHT,HTH,THH}

Not Mutually Exclusive

D={HHT,HTH,THH}

Example

- Toss a fair coin twice. What is the probability of observing at least one head (Event A)?

S={HH, HT, TH, TT}

A={HH, HT, TH}

1st Coin	2nd Coin	E_i	$P(E_i)$
H	H	HH	1/4
	T	HT	1/4
T	H	TH	1/4
	T	TT	1/4

P(at least 1 head)

= P(A)

= P(HH) + P(HT) + P(TH)

= 1/4 + 1/4 + 1/4 = 3/4

Example

- A bowl contains three M&Ms®, two reds, one blue. A child selects two M&Ms at random. What is the probability that exactly two reds (Event A)?

1st M&M	2nd M&M	E_i	$P(E_i)$
r1	b	**r1b**	**1/6**
	r2	**r1r2**	**1/6**
r2	b	**r2b**	**1/6**
	r1	**r2r1**	**1/6**
b	r1	**br1**	**1/6**
	r2	**br2**	**1/6**

A={r1r2, r2r1}

$$P(A) = P(r1r2) + P(r2r1) = 1/6 + 1/6 = 2/6 = 1/3$$

Counting Rules

- If the simple events in an experiment are **equally likely,** we can calculate

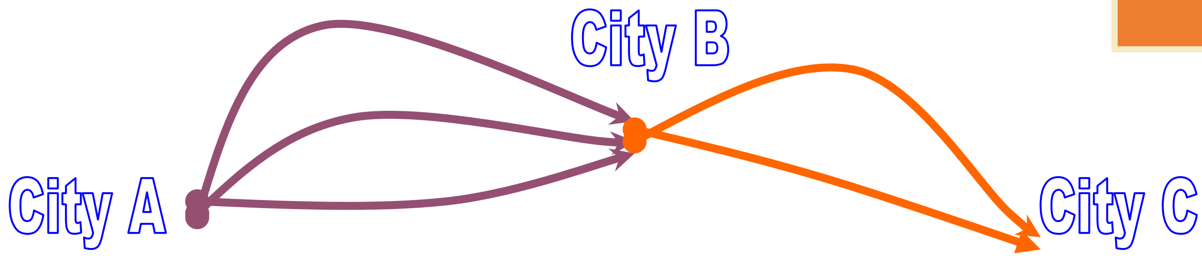
$$P(A) = \frac{\# A}{\# S} = \frac{\text{number of simple events in A}}{\text{total number of simple events}}$$

- We can use **counting rules** to find $\#A$ and $\#S$.

Counting

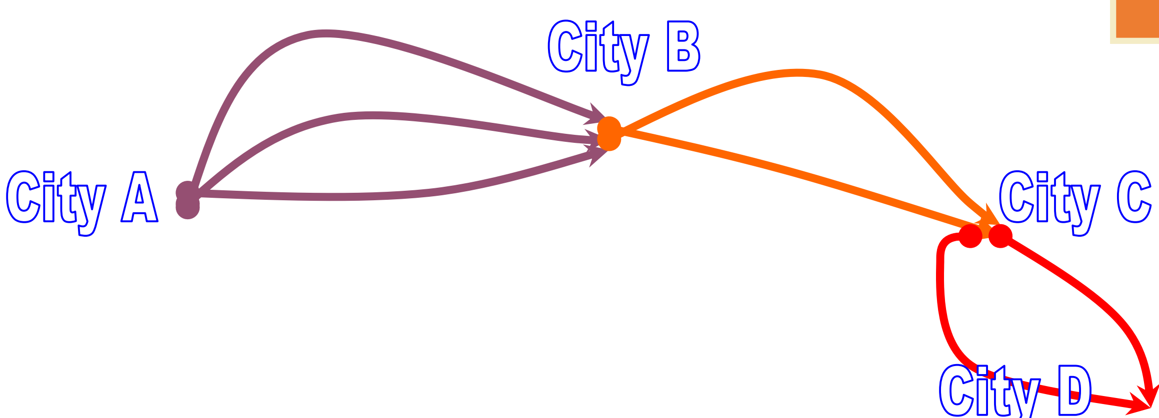
- **How many ways from A to C?**

$$3 \times 2 = 6$$



- **How many ways from A to D?**

$$3 \times 2 \times 2 = 12$$



The *mn* Rule

- For a two-stage experiment,

  m ways to accomplish the first stage

  n ways to accomplish the second stage

  then there are ***mn*** ways to accomplish the whole experiment.

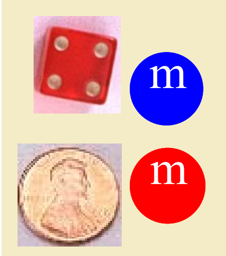
- For a ***k***-stage experiment, number of ways equal to

$$n_1 n_2 n_3 \ldots n_k$$

Example: Toss two coins. The total number of simple events is:

$$\boxed{2 \times 2 = 4}$$

Examples



Example: Toss three coins. The total number of simple events is:

$$2 \times 2 \times 2 = 8$$

Example: Toss two dice. The total number of simple events is:

$$6 \times 6 = 36$$

Example: Two M&Ms are drawn in order from a dish containing four candies. The total number of simple events is:

$$4 \times 3 = 12$$

Permutations

Example: How many 3-digit lock passwords can we make by using 3 different numbers among 1, 2, 3, 4 and 5?

The order of the choice is important!

$$5(4)(3) = 60$$

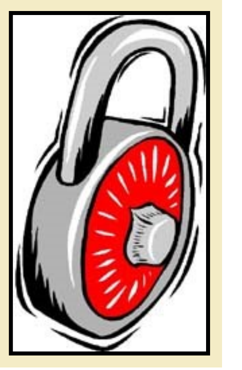
- ***n*** distinct objects, take ***r*** objects at a time and arrange them ***in order***. The number of different ways you can take and arrange is

$$P_r^n = \frac{n!}{(n-r)!}$$

where $n! = n(n-1)(n-2)\ldots(2)(1)$ and $0! \equiv 1$.

$$P_3^5 = \frac{5!}{(5-3)!} = \frac{5(4)(3)(2)(1)}{2(1)} = 60$$

Example



Example: A lock consists of five parts and can be assembled in any order. A quality control engineer wants to test each order for efficiency of assembly. How many orders are there?

The order of the choice is important!

$$P_5^5 = \frac{5!}{0!} = 5(4)(3)(2)(1) = 120$$

Example

- How many ways to select a student committee of 3 members: chair, vice chair, and secretary out of 8 students?

The order of the choice is important! ---- Permutation

$$P_3^8 = \frac{8!}{(8-3)!}$$

$$= \frac{(8)(7)(6)(5)(4)(3)(2)(1)}{5(4)(3)(2)(1)}$$

$$= 8(7)(6) = 336$$

Combinations

- ***n*** distinct objects, select ***r*** objects at a time *without regard to the order*. The number of different ways you can select is

$$C_r^n = \frac{n!}{r!(n-r)!}$$

Example: Three members of a 5-person committee must be chosen to form a subcommittee. How many different subcommittees could be formed?

The order of the choice is not important!

$$C_3^5 = \frac{5!}{3!(5-3)!} = \frac{5(4)(3)(2)1}{3(2)(1)(2)1} = \frac{5(4)}{(2)1} = 10$$

Example

- How many ways to select a student committee of 3 members out of 8 students?
- (Don’t assign chair, vice chair and secretary).

The order of the choice is **NOT** important! → **Combination**

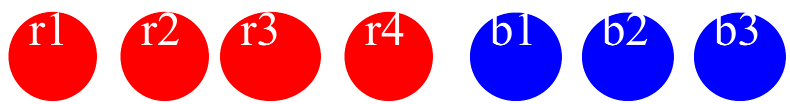
$$C_3^8 = \frac{8!}{3!(8-3)!}$$

$$= \frac{8(7)(6)(5)(4)(3)(2)(1)}{[3(2)(1)][5(4)(3)(2)(1)]}$$

$$= \frac{8(7)(6)}{3(2)(1)} = 56$$

Question

- A box contains 7 M&Ms®, 4 reds and 3 blues. A child selects three M&Ms at random.
- What is the probability that exactly one is red (**Event A**) ?



- Simple Events and sample space **S**:

  {r1r2r3, r1r2b1, r2b1b2…… }

- Simple events in event **A**:

  {r1b1b2, r1b2b3, r2b1b2……}

Solution

- Choose 3 MMs out of 7. (Total number of ways, i.e. size of sample space S)

The order of the choice is not important!

$$C_3^7 = \frac{7!}{3!4!} = 35$$

- Event A: one red, two blues

$$P(A) = \frac{\# A}{\# S} = \frac{12}{35}$$

Choose one red

$$C_1^4 = \frac{4!}{1!3!} = 4$$

Choose Two Blues

$$C_2^3 = \frac{3!}{2!1!} = 3$$

$4 \times 3 = 12$ ways to choose 1 red and 2 blues (mn Rule)

Question

A box contains 6 M&Ms, 3 reds and 3 blues. A child selects three M&Ms at random. What is the probability that exactly one is Blue (Event A)? What is the probability that exactly 2 is Red (Event B)?

Probability of Sample Space =$C_3^6 = 20$
Event A: One Blue, two Red
One Blue Probability = $C_1^3 = 3$
Two red Probability = C_2^3=3
One Blue, two Red = 3x3 = 9
P(A) = 9/20

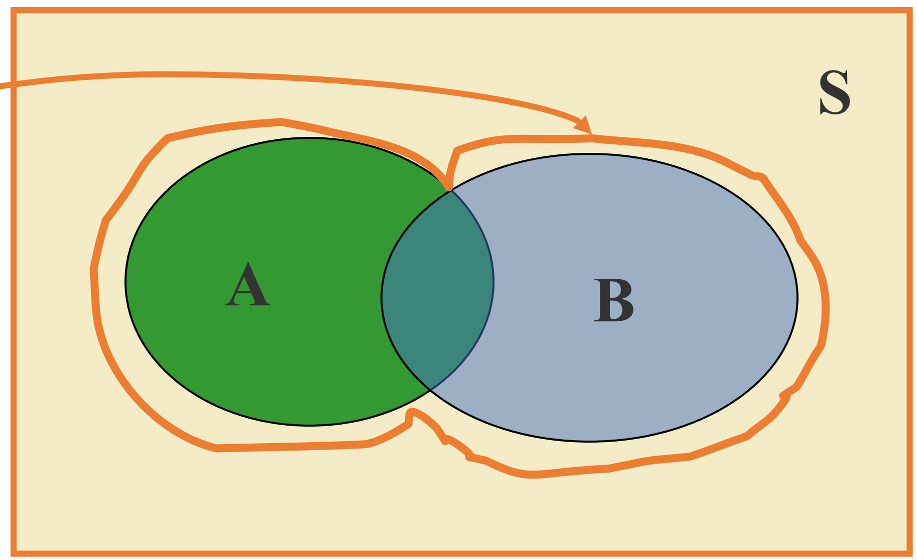
Probability of Sample Space =$C_3^6 = 20$
Event B: Two Red, One Blue
Two Red Probability = $C_2^3 = 3$
One Blue Probability = C_1^3=3
One Blue, two Red = 3x3 = 9
P(A) = 9/20

Event Relations - Union

The **union** of two events, A and B, is the event that either A **or** B **or both** occur when the experiment is performed. We write

$$\mathbf{A \cup B}$$

$$A \cup B$$

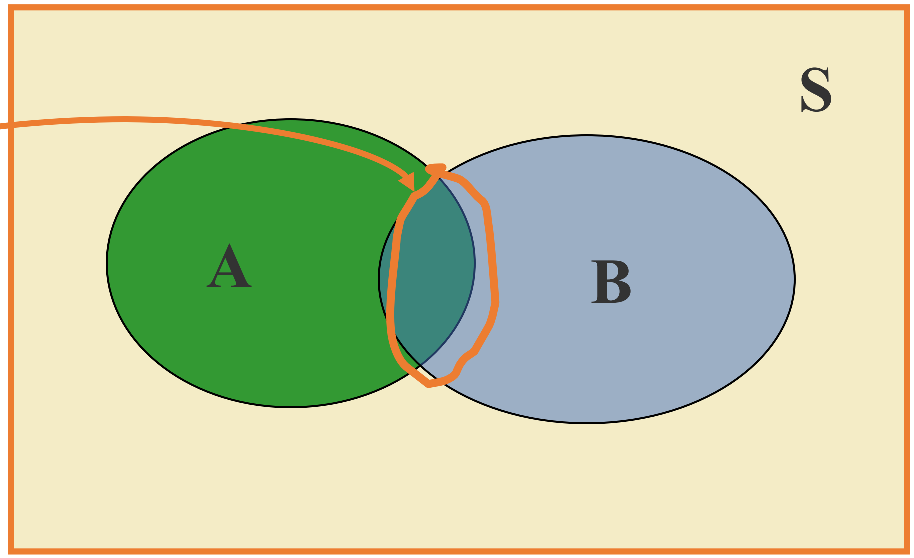


Event Relations-Intersection

The **intersection** of two events, **A** and **B**, is the event that both A **and** B occur.

We write **A ∩ B.**

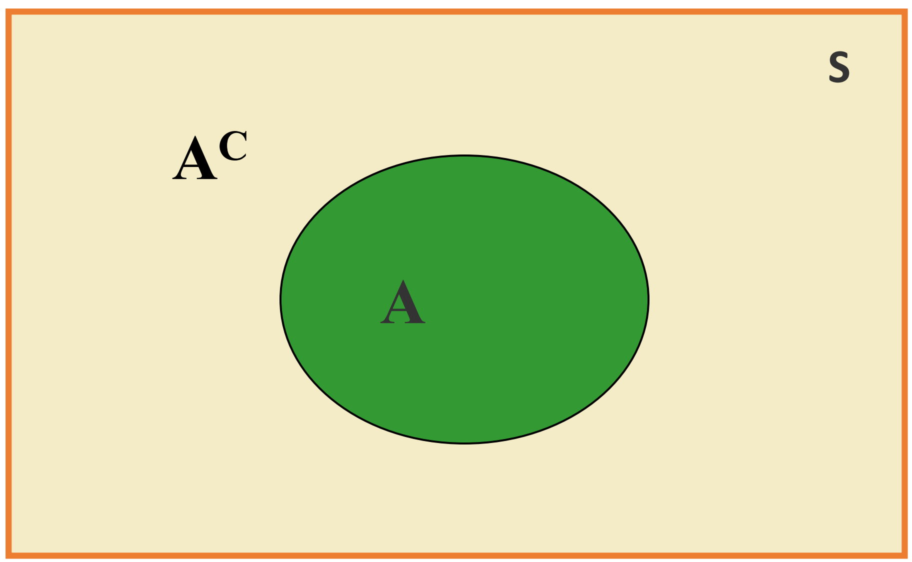
$A \cap B$



- If A and B are **mutually exclusive,** then P(**A ∩ B) = 0.**

Event Relations - Complement

The **complement** of an event **A** consists of all outcomes of the experiment that do not result in event A. We write **A^C (The event that event A doesn't occur).**



Example

Select a student from a college

- **A:** student is colorblind
- **B:** student is female
- **C:** student is male

Mutually exclusive and $B = C^C$

- What is the relationship between events **B** and **C**?
- A^C: Student is not colorblind
- $B \cap C$: Student is both male and female = $\varnothing$
- $B \cup C$: Student is either male or female = all students = S

Example

Toss a coin twice

- **A:** At least one head {HH, HT, TH};
- **B:** Exact one head {HT, TH};
- **C:** At least one tail {HT, TH, TT}.

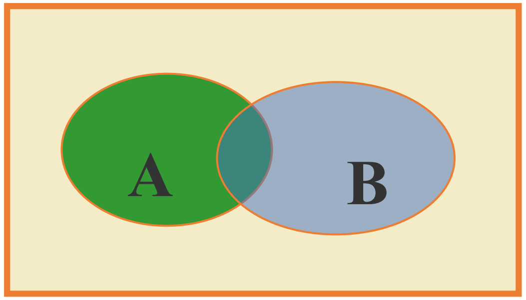
- A^C: {TT} No head
- $A \cap B$: {HT, TH} Exact one head
- $A \cup C$: {HH, HT, TH, TT}=S -- Sample space

Probabilities for Unions

The Additive Rule for Unions:

- For any two events, **A** and **B**, the probability of their union, **P(A ∪ B)**, is

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$



Example: Additive Rule

Example: Suppose that there were 1000 students in a college, and that they could be classified as follows:

A: Colorblind

P(A) = 42/1000=.042

B: Male

P(B) = 510/1000=.51

	Male (B)	Female
Colorblind (A)	40	2
Not Colorblind	470	488

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 42/1000 + 510/1000 - 40/1000$

$= 512/1000 = .512$

Check: $P(A \cup B)$

$= (40 + 2 + 470)/1000$

$= .512$

A Special Case

When two events A and B are **mutually exclusive,** $P(A \cap B) = 0$ and **$P(A \cup B) = P(A) + P(B)$.**

A: male and colorblind

$P(A) = 40/1000$

B: female and colorblind

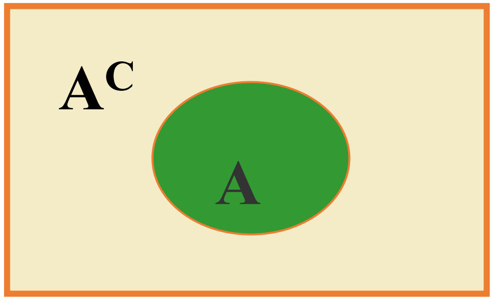
$P(B) = 2/1000$

	Male	Female
Colorblind	40	2
Not Colorblind	470	488

A and B are mutually exclusive, so that

$P(A \cup B) = P(A) + P(B)$
$= 40/1000 + 2/1000$
$= 42/1000 = .042$

Probabilities for Complements



- We know that for any event **A:**

  $P(A \cap A^C) = 0$

- Since either **A** or $\mathbf{A^C}$ must occur,

  $P(A \cup A^C) = 1$

- so that $P(A \cup A^C) = P(A) + P(A^C) = 1$

$$\boxed{P(A^C) = 1 - P(A)}$$

Example

Select a student at random from the college. Define:

	Male	Female
Colorblind	40	2
Not Colorblind	470	488

A: male

P(A) = 510/1000=.51

B: female

A and B are complementary, so that

P(B) = 1- P(A)
= 1- .51
=.49

Example

- Toss a fair coin twice. Define
  - A: head on second toss
  - B: head on first toss
  - If B occurred, what is probability that A occurred?
  - If B didn't occur, what is probability that A occurred?

HH	1/4
HT	1/4
TH	1/4
TT	1/4

P(A given B occurred) = ½

P(A given B did not occur) = ½

P(A) does not change, whether B happens or not… → A and B are independent!

Conditional Probabilities

The probability that A occurs, given that event B has occurred is called the **conditional probability** of A given B and is defined as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \text{ if } P(B) \neq 0$$

"given"

Probabilities for Intersections

In the previous example, we found $P(A \cap B)$ directly from the table. Sometimes this is impractical or impossible. The rule for calculating $P(A \cap B)$ depends on the idea of **independent and dependent events.**

Two events, **A** and **B**, are said to be **independent** if and only if the probability that event **A** occurs is not changed by occurrence of event **B**, or vice versa.

Example 1

- Toss a fair coin twice. Define
  - A: head on second toss
  - B: head on first toss
  - $A \cap B$: head on both first and second

HH	1/4
HT	1/4
TH	1/4
TT	1/4

$P(A|B) = P(A \cap B)/P(B) = (1/4)/(1/2) = 1/2$

$P(A|\text{not } B) = 1/2$

P(A) does not change, whether B happens or not…

A and B are independent!

Defining Independence

- We can redefine independence in terms of conditional probabilities:

Two events A and B are **independent** if and only if

$$\mathbf{P(A|B) = P(A)} \quad \text{or} \quad \mathbf{P(B|A) = P(B)}$$

Otherwise, they are **dependent.**

- Once you've decided whether or not two events are independent, you can use the following rule to calculate their intersection.

Multiplicative Rule for Intersections

- For any two events, **A** and **B**, the probability that both **A** and **B** occur is

$$\boxed{\begin{aligned} P(A \cap B) &= P(A)\, P(B \text{ given that } A \text{ occurred}) \\ &= P(A)P(B|A) \end{aligned}}$$

- If the events **A** and **B** are **independent**, then the probability that both **A** and **B** occur is

$$\boxed{P(A \cap B) = P(A)\, P(B)}$$

Example 1

In a certain population, 10% of the people can be classified as being high risk for a heart attack. Three people are randomly selected from this population. What is the probability that exactly one of the three is high risk?

Define H: high risk N: not high risk

P(exactly one high risk) = P(HNN) + P(NHN) + P(NNH)

= P(H)P(N)P(N) + P(N)P(H)P(N) + P(N)P(N)P(H)

$= (.1)(.9)(.9) + (.9)(.1)(.9) + (.9)(.9)(.1) = 3(.1)(.9)^2 = .243$

Example 2

Suppose we have additional information in the previous example. We know that only **49%** of the population are female. Also, of the female patients, **8%** are high risk. A single person is selected at random. What is the probability that it is a high risk female?

Define H: high risk F: female

From the example, $P(F) = .49$ and $P(H|F) = .08$.
Use the Multiplicative Rule:

$$P(\text{high risk female}) = P(H \cap F)$$

$$= P(F)P(H|F) = .49(.08) = .0392$$

Example 3

At a large university, the probability that a student takes calculus and is on the merit list is 0.042. The probability that a student is on the merit list is 0.21. Find the probability that a student takes calculus, given that he or she is on the merit list.

Example 3

At a large university, the probability that a student takes calculus and is on the merit list is 0.042. The probability that a student is on the merit list is 0.21. Find the probability that a student takes calculus, given that he or she is on the merit list.

P(calculus| merit list) = P(calculus)/P(merit list)
=0.042/0.21 =0.2

Example 4

The probability that a student owns a car is 0.65 and the probability that a student owns a computer is 0.82. If the probability that a student owns both is 0.55, what is the probability that a randomly selected student owns a car or computer? What is the probability that a randomly selected student does not own a car or computer?

Example 4

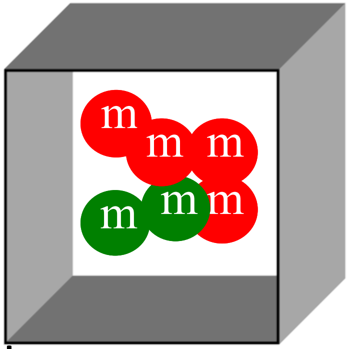
The probability that a student owns a car is 0.65 and the probability that a student owns a computer is 0.82. If the probability that a student owns both is 0.55, what is the probability that a randomly selected student owns a car or computer? What is the probability that a randomly selected student does not own a car or computer

P(car or computer) = P(car) + P(computer) – P(both)

=0.56+.82-.55=.92

P(does not own) = 1-P(car or computer) = 1-.92=0.08

Example 3



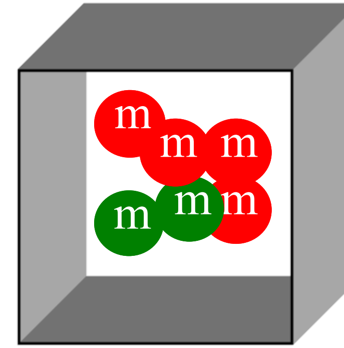
2 green and 4 red M&Ms are in a box; Two of them are selected at random.

A: First is green;

B: Second is red.

- Find **P(A$\cap$B).**

Method 1



- Choose 2 MMs out of 6. Order is recorded. (Total number of ways, i.e. size of sample space S)

The order of the choice is important! Permutation

$$P_2^6 = \frac{6!}{(6-2)!} = \frac{6!}{4!} = 6(5) = 30$$

- Event **A∩B:** First green, second red

First green

$$C_1^2 = 2$$

Second Red

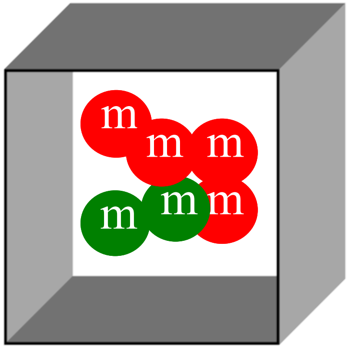
$$C_1^4 = 4$$

$2 \times 4 = 8$ ways to choose first green and second red

(mn Rule)

$$P(A \cap B) = \frac{\# A \cap B}{\# S} = \frac{8}{30}$$

Method 2



A: First is green;

B: Second is red;
A∩B: First green, second red

$P(A)$ 2/6

$P(B|A)$ P(Second red | First green)=4/5

$$P(A \cap B) = P(A)P(B|A)$$

$$P(A \cap B) = 2/6(4/5)=8/30$$

Example 2

A box contains 6 red marbles and 4 black marbles. Two marbles are drawn *with replacement* from the box. What is the probability that both of the marbles are black?

Method 2

A box contains 6 red marbles and 4 black marbles. Two marbles are drawn *with replacement* from the box. What is the probability that both of the marbles are black?

Let A = the event that the first marble is black; and
let B = the event that the second marble is black. We know the following:

- In the beginning, there are 10 marbles in the box, 4 of which are black. Therefore, $P(A) = 4/10$.

- After the first selection, we replace the selected marble; so there are still 10 marbles in the box, 4 of which are black. Therefore, $P(B|A) = 4/10$.

Therefore, based on the rule of multiplication:

$$P(A \cap B) = P(A)\, P(B|A)$$

$$P(A \cap B) = (4/10)*(4/10) = 16/100 = 0.16$$

Probability Rules & Relations of Events

<u>Complement Event</u> $$\boxed{P(A^c) = 1 - P(A)}$$

<u>Additive Rule</u> $$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

<u>Multiplicative Rule</u> $$\boxed{P(A \cap B) = P(A)P(B \mid A)}$$

<u>Conditional probability</u> $$\boxed{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$$

<u>Mutually Exclusive Events</u> $$\boxed{P(A \cap B) = 0}$$
$$\boxed{P(A \cup B) = P(A) + P(B)}$$

<u>Independent Events</u> $$\boxed{P(A \cap B) = P(A)P(B)}$$
$$\boxed{P(A \mid B) = P(A)}$$
$$\boxed{P(A \cap B \cap C) = P(A)P(B)P(C)}$$

Random Variables

- A **random variable** is a function corresponding to the outcome of an experiment, is a chance or random event.
- Random variables can be **discrete** or **continuous.**

- **Examples:**
  - ✓ x = SAT score for a randomly selected student
  - ✓ x = number of people in a room at a randomly selected time of day
  - ✓ x = weight of a fish drawn at random

Probability Distributions for Discrete Random Variables

Probability distribution of a discrete random variable *x* , is a graph, table or formula that gives

- possible values of *x*
- probability *p*(*x*) associated with each value *x*.

We must have

$$0 \le p(x) \le 1 \text{ and } \sum p(x) = 1$$

Example 1

- Toss a fair coin once,
- Define ***x* = number of heads.**
- Find distribution of ***x***

		x
H	1/2	1
T	1/2	0

$$P(x = 0) = 1/2$$
$$P(x = 1) = 1/2$$

x	$p(x)$
0	1/2
1	1/2

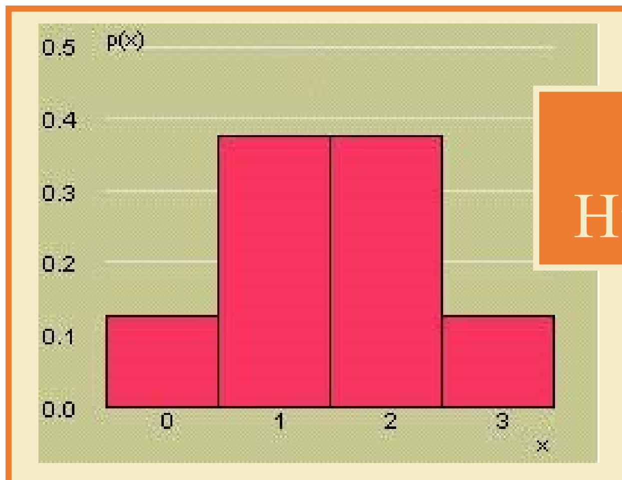
Example 2

- Toss a fair coin three times and define ***x* = number of heads.**

		x
HHH	1/8	3
HHT	1/8	2
HTH	1/8	2
THH	1/8	2
HTT	1/8	1
THT	1/8	1
TTH	1/8	1
TTT	1/8	0

$P(x = 0) = 1/8$
$P(x = 1) = 3/8$
$P(x = 2) = 3/8$
$P(x = 3) = 1/8$

x	$p(x)$
0	1/8
1	3/8
2	3/8
3	1/8



Probability Histogram for x

Expected Value of Random Variable

- Let x be a discrete random variable with probability distribution $p(x)$. Then the **expected value**, denoted by $E(x)$, is defined by

$$\boxed{\begin{array}{l}\text{Expected Value (Expectation, Mean } \mu\text{)} \\ E(x) = \mu = \sum x p(x)\end{array}}$$

Example

- Toss a fair coin 3 times and record *x* the number of heads.

x	$p(x)$	$xp(x)$
0	1/8	0(1/8)=0
1	3/8	1(3/8)=0.375
2	3/8	2(3/8)=0.75
3	1/8	3(1/8)=0.375
		Total 1.5

$$E(x) = \sum xp(x)$$
$$= 1.5$$

Mean & Standard Deviation

- Let x be a discrete random variable with probability distribution $p(x)$. Then the mean, variance and standard deviation of x are given as

$$\boxed{\begin{aligned}&\text{Mean}: \mu = \sum xp(x)\\ &\text{Variance}: \sigma^2 = \sum (x-\mu)^2 p(x)\\ &\text{Standard deviation}: \sigma = \sqrt{\sigma^2}\end{aligned}}$$

$$\boxed{\text{Variance}: \sigma^2 = \sum x^2 p(x) - \mu^2}$$

Example

- Toss a fair coin 3 times and record x the number of heads.

- Find variance by the definition formula.

x	$p(x)$	$(x-\mu)^2 p(x)$
0	**1/8**	$(0-1.5)^2(1/8)=.28125$
1	**3/8**	$(1-1.5)^2(3/8)=.09375$
2	**3/8**	$(2-1.5)^2(3/8)=.09375$
3	**1/8**	$(3-1.5)^2(1/8)=.28125$
		Total .75

$$\mu = E(x) = \sum xp(x) = 1.5$$

$$\sigma^2 = \sum (x-\mu)^2 p(x)$$
$$= 0.75$$

Example

- Toss a fair coin 3 times and record *x* the number of heads.
- Find the variance by the computational formula.

$$\mu = E(x) = \sum xp(x) = 1.5$$

x	$p(x)$	$x^2p(x)$
0	1/8	$0^2(1/8)=0$
1	3/8	$1^2(3/8)=0.375$
2	3/8	$2^2(3/8)=1.5$
3	1/8	$3^2(1/8)=1.125$
		Total 3

$$\sigma^2 = \sum x^2 p(x) - \mu^2$$
$$= 3 - 1.5^2$$
$$= 0.75$$

Example

A fair coin is tossed twice. Let x be the number of heads that are observed.

1. Construct the probability distribution of x.
2. Find the probability that at least one head is observed.

HH – ¼ - 2
HT-1/4- 1
TH – ¼ - 1
TT -1/4 - 0

X	P(x)
0	¼
1	½
2	1/4

X>=1, X=1 and X=2, P(1) + P(2) = .75

Example

A pair of fair dice is rolled. Let X denote the sum of the number of dots on the top faces.

1. Construct the probability distribution of X for a paid of fair dice.
2. Find $P(X \geq 9)$.
3. Find the probability that X takes an even value.

The sample space of equally likely outcomes is

11 21 31 41 51 61 12 22 32 42 52 62 13
23 33 43 53 63 14 24 34 44 54 64 15 25
35 45 55 65 16 26 36 46 56 66 11 12 13
14 15 16 21 22 23 24 25 26 31 32 33 34 35 36
41 42 43 44 45 46 51 52 53 54 55 56 61 62 63
64 65 66

where the first digit is die 1 and the second number is die 2.

Example

A pair of fair dice is rolled. Let x denote the sum of the number of dots on the top faces.
1.Construct the probability distribution of x for a paid of fair dice.
2.Find $P(X\geq 9)$.
3.Find the probability that x takes an even value.

a. The possible values for X are the numbers 2 through 12.

X=2 is the event {11}, so P(2)=1/36.
X=3 is the event {12,21}, so P(3)=2/36. Continuing this way we obtain the following table

x	2	3	4	5	6	7	8	9	10	11	12
$P(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

This table is the probability distribution of X.

Example

A pair of fair dice is rolled. Let X denote the sum of the number of dots on the top faces.

1. Construct the probability distribution of X for a paid of fair dice.
2. Find $P(X \geq 9)$.
3. Find the probability that X takes an even value.

2. The event $X \geq 9$ is the union of the mutually exclusive events $X=9$, $X=10$, $X=11$, and $X=12$. Thus

$$\begin{aligned} P(X \geq 9) &= P(9) + P(10) + P(11) + P(12) \\ &= \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} \\ &= \frac{10}{36} \\ &= 0.2\bar{7} \end{aligned}$$

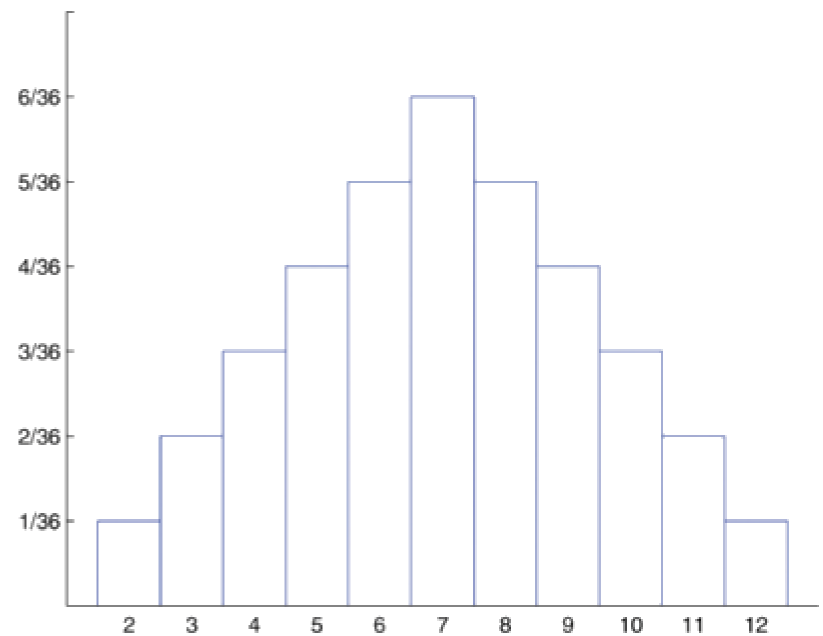
Example

A pair of fair dice is rolled. Let x denote the sum of the number of dots on the top faces.

1. Construct the probability distribution of x for a paid of fair dice.
2. Find $P(X \geq 9)$.
3. Find the probability that x takes an even value.

Before we immediately jump to the conclusion that the probability that X takes an even value must be 0.5, note that X takes six different even values but only five different odd values. We compute

$$\begin{aligned} P(X \text{ is even}) &= P(2) + P(4) + P(6) + P(8) + P(10) + P(12) \\ &= \frac{1}{36} + \frac{3}{36} + \frac{5}{36} + \frac{5}{36} + \frac{3}{36} + \frac{1}{36} \\ &= \frac{18}{36} \\ &= 0.5 \end{aligned}$$



Example

Find the mean of the discrete random variable X whose probability distribution is

x	-2	1	2	3.5
$P(x)$	0.21	0.34	0.24	0.21

Mean = (-2)* 0.21+ 1*(0.34) +2 * 0.24 + 3.5*0.21
=1.135

Key Concepts

I. Experiments and the Sample Space

1. Experiments, events, mutually exclusive events, simple events
2. The sample space
3. Venn diagrams, tree diagrams, probability tables

II. Probabilities

1. Relative frequency definition of probability
2. Properties of probabilities
    a. Each probability lies between 0 and 1.
    b. Sum of all simple-event probabilities equals 1.
3. $P(A)$, the sum of the probabilities for all simple events in A

Key Concepts

III. Counting Rules

1. *mn* Rule, extended *mn* Rule
2. Permutations: $$P_r^n = \frac{n!}{(n-r)!}$$
3. Combinations: $$C_r^n = \frac{n!}{r!(n-r)!}$$

IV. Event Relations

1. Unions and intersections
2. Events
   - a. Disjoint or mutually exclusive: $$P(A \cap B) = 0$$ $$P(A \cup B) = P(A) + P(B)$$
   - b. Complementary: $$P(A^c) = 1 - P(A)$$

Key Concepts

3. Conditional probability:

$$\boxed{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$$

4. Independent events

$$\boxed{P(A \cap B) = P(A)P(B)}$$

$$\boxed{P(A \mid B) = P(A)}$$

5. Additive Rule of Probability:

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

6. Multiplicative Rule of Probability:

$$\boxed{P(A \cap B \cap C) = P(A)P(B)P(C)}$$

$$\boxed{P(A \cap B) = P(A)P(B \mid A)}$$

Key Concepts

V. Discrete Random Variables and Probability Distributions

1. Random variables, discrete and continuous

2. Properties of probability distributions

$$\boxed{0 \le p(x) \le 1 \text{ and } \sum p(x) = 1}$$

3. Mean or expected value of a discrete random variable: $\boxed{\text{Mean}: \mu = \sum x p(x)}$

4. Variance and standard deviation of a discrete random variable:

$$\boxed{\begin{array}{l} \text{Variance}: \sigma^2 = \sum (x-\mu)^2 p(x) = \sum x^2 p(x) - \mu^2 \\ \text{Standard deviation}: \sigma = \sqrt{\sigma^2} \end{array}}$$