Concourse 18.02 Problem Set 2 – Fall 2016

due Monday, September 26

(1) Read the Matrices and Linear Algebra Notes, the Supplement on Solving Systems of Linear Equations via Row Reduction;

(2) Read sections 10.4 (Parametric Curves), 12.5 (Curves and Motions in Space), 12.6 (Curvature and Acceleration), and the Supplementary Notes (1F, 1G, 1H, 1J).

Part I problems (to be turned in)

1. [SN-1F/8b] If $\mathbf{A}\begin{bmatrix}2\\0\\0\end{bmatrix}=\begin{bmatrix}-2\\0\\4\end{bmatrix}$, $\mathbf{A}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}3\\0\\3\end{bmatrix}$, and $\mathbf{A}\begin{bmatrix}0\\2\\1\end{bmatrix}=\begin{bmatrix}7\\1\\1\end{bmatrix}$, what is the 3×3 matrix $\mathbf{A}$?

2. [SN-1G/2] (a) Solve the equation $\mathbf{Ax}=\mathbf{b}$ by finding $\mathbf{A}^{-1}$: $\mathbf{A}=\begin{bmatrix}4&3\\3&2\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix}-1\\1\end{bmatrix}$.

   (b) Solve the equation $\mathbf{Ax}=\mathbf{b}$ by finding $\mathbf{A}^{-1}$: $\mathbf{A}=\begin{bmatrix}4&3\\3&2\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix}2\\3\end{bmatrix}$.

3. [10.4/9] Given the parametric equations $\{x=5\cos t,\ y=3\sin t\}$, eliminate the parameter and then sketch the curve.

4. [12.5/33] Given the acceleration vector $\mathbf{a}(t)=(\cos t)\mathbf{i}+(\sin t)\mathbf{j}$, the initial position vector $\mathbf{r}_0=\mathbf{r}(0)=\mathbf{j}$, and the initial velocity vector $\mathbf{v}_0=\mathbf{v}(0)=-\mathbf{i}+5\mathbf{k}$ of a particle moving in *xyz*-space, find its position vector $\mathbf{r}(t)$ at time *t*.

5.[12.5/58] Suppose that a point moved on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ with position vector $\mathbf{r}(t)=(a\cos\omega t)\mathbf{i}+(b\sin\omega t)\mathbf{j}$ (ω is constant). Prove that the acceleration vector $\mathbf{a}(t)$ satisfies the equation $\mathbf{a}(t)=c\mathbf{r}(t)$, where *c* is a negative constant. To what sort of external force $\mathbf{F}(t)$ does this motion correspond?

6. [12.6/3] Find the length of the curve given parametrically by $\{x=6e^t\cos t,\ y=6e^t\sin t,\ z=17e^t\}$ from $t=0$ to $t=1$.

7. [SN 1I/3c] Describe the motion given by the position vector $\mathbf{r}=(t^2+1)\mathbf{i}+(t^3)\mathbf{j}$ as *t* goes from $-\infty$ to ∞. Give the *xy*-equation of the curve along which P travels, and tell what part of the curve is actually traced out by P.

8. [SN 1I/5] A string is wound clockwise around the circle of radius a centered at the origin *O*; the initial position of the end *P* of the string is $(a,0)$. Unwind the string, always pulling it taut (so it stays tangent to the circle). Write parametric equations for the motion of *P*. (Use vectors; express the position vector $\overrightarrow{OP}$ as a vector function of one variable.)

9. [SN-1J/2] Let $\overrightarrow{OP}=\dfrac{1}{1+t^2}\mathbf{i}+\dfrac{t}{1+t^2}\mathbf{j}$ be the position vector for a motion.

   a) Calculate $\mathbf{v}$, the speed $|ds/dt|$, and the unit tangent vector $\mathbf{T}$.

   b) At what point in the speed greatest? smallest?

   c) Find the *xy*-equation of the curve along which the point *P* is moving, and describe it geometrically.

10. [SN-1J/5a] Suppose a point moves with constant speed. Show that its velocity vector and acceleration vector are perpendicular.

11. [SN-1J/6] For the helical motion $\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + bt\,\mathbf{k}$,

a) calculate **v**, **a**, **T**, $|ds/dt|$

b) show that **v** and **a** are perpendicular; explain using **1J-5**.

12. [text 12.5/42] The angular momentum $\mathbf{L}(t)$ and torque $\boldsymbol{\tau}(t)$ of a moving particle of mass m with position vector $\mathbf{r}(t)$, velocity vector $\mathbf{v}(t) = \mathbf{r}'(t)$, momentum vector $\mathbf{p}(t) = m\mathbf{v}(t)$, and applied force $\mathbf{F}(t) = \mathbf{p}'(t) = m\mathbf{a}(t)$ are defined to be:

$$\mathbf{L}(t) = \mathbf{r}(t) \times \mathbf{p}(t) = \mathbf{r}(t) \times m\mathbf{v}(t), \qquad \boldsymbol{\tau}(t) = \mathbf{r}(t) \times \mathbf{F}(t) = \mathbf{r}(t) \times m\mathbf{a}(t)$$

Prove that $\mathbf{L}'(t) = \boldsymbol{\tau}(t)$. It follows that $\mathbf{L}(t)$ must be a constant vector if $\boldsymbol{\tau} \equiv \mathbf{0}$; this is the law of conservation of angular momentum.

Part II problems (to be turned in)

Problem 1: a) If $\mathbf{A}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\3\\1\end{bmatrix}$, $\mathbf{A}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\-1\end{bmatrix}$, what is the 3×2 matrix **A**?

b) Using the matrix **A** from part (a), is there a vector $\mathbf{x} = \begin{bmatrix}x\\y\end{bmatrix}$ such that $\mathbf{Ax} = \begin{bmatrix}8\\16\\12\end{bmatrix}$? Explain.

c) Using the matrix **A** from part (a), is there a vector $\mathbf{x} = \begin{bmatrix}x\\y\end{bmatrix}$ such that $\mathbf{Ax} = \begin{bmatrix}9\\16\\12\end{bmatrix}$? Explain.

Problem 2. "A rooster is worth five coins, a hen three coins, and 3 chicks one coin. With 100 coins we buy 100 of them. How many roosters, hens, and chicks can we buy?"
(From the *Mathematical Manual* by Zhang Qiujian, Chapter 3, Problem 38; 5th century A.D. This famous *Hundred Fowl Problem* has reappeared in countless variations in Indian, Arabic, and European texts.)

Problem 3. Express the set of all points satisfying the following system of equations <u>parametrically in terms of one or more independent parameters</u>.

(a) $\{(x,y,z):\ x - 2y + 3z = 4\}$ (b) $\left\{\begin{array}{l} x + 4y + z = 4 \\ 4x + 13y + 7z = 1 \\ 7x + 22y + 13z = -2 \end{array}\right\}$ (c) $\left\{\begin{array}{r} x + y - z + 3w = 3 \\ y - 3z - w = 1 \\ 3x - y + 9z + 13w = 5 \\ 3x + y + 3z + 11w = 7 \end{array}\right\}$

Problem 4: Suppose we know that when the three planes P_1, P_2 and P_3 in $\mathbf{R}^3$ intersect in pairs, we get three lines L_1, L_2, and L_3 which are *distinct* and *parallel*.

a) Sketch a picture of this situation.

b) Show that the three normals to P_1, P_2 and P_3 all lie in one plane, using a geometric argument.

c) Show that the three normals to P_1, P_2 and P_3 all lie in one plane, using an algebraic argument. (Note that the three planes clearly do *not* all intersect at one point.)

Problem 5: A circular disk of radius 2 has a dot marked at a point half-way between the center and the circumference. Denote this point by *P*. Suppose that the disk is tangent to the *x*-axis with the center initially at (0, 2) and *P* initially at (0, 1), and that it starts to roll to the right on the *x*-axis at unit speed. Let C be the curve traced out by the point *P*.

a) Make a sketch of what you think the curve C will look like.

b) Use vectors to find the parametric equations for $\overrightarrow{OP}$ as a function of time *t*.