MAT1071

MATHEMATICS I

1	Functions:Domain of a Function,Functions and Graphs, Even-Odd Functions, Symmetry, Operations on Functions (Sum, difference, multiplication, division and powers),Composite Functions, Piecewise Functions, Polynomials and Rational Functions, Trigonometric Functions	Textbook 1 (Chapter 1) $\frac{0}{0}$, $\frac{\infty}{\infty}$, ...
2	Limits and Contiunity: Limit of a Function and Limit Laws, The Sandwich (The Squeeze theorem), The Precise Definition of a Limit, One-sided Limits, , Limits İnvolving İnfinity, Infinity Limits	Textbook 1 (Chapter 2)
3	Contiunity at a Point, Continuous Functions, The İntermediate Value Theorem Types of Discontiunity, Differentiation:Tangents ,Normal Lines , The Derivative at a Point, The Derivate as a Function, Onesided Derivatives	Textbook 1 (Chapter 3)
4	Differentiable on an Interval, Differentiation Rules, High order Derivatives, Derivatives of Trigonometric Functions, The Chain Rule, Implicit Differentiation, Linearization and Differentials, Increasing Functions and Decrasing Functions	Textbook 1 (Chapter 3)
5	Transcendental Functions:Inverse Functions and Their Derivatives,Logarithms and Exponential Functions and Their Derivatives, Logarithmic Differentiation, Inverse Trigonometric Functions and Their Derivatives, Hyperbolic Functions and Their Derivatives,Inverse Hyperbolic Functions and Their Derivatives	Textbook 1 (Chapter 7)
6	Indeterminate Forms and L'Hospitals Rule, Extrem Values of Functions, Critical Points,	Textbook 1 (Chapter 7)
7	Rolle's Theorem, The Mean Value Theorem, The First Derivative Test for Local Extrema, Concavity , The Second Derivative Test for Concavity, Point of İnflection, The Second Derivative Test for Local Extrema	Textbook 1 (Chapter 4)
8	Midterm 1	Textbook 1 (Chapter 4)
9	Asymptotes of Graphs, Curve Sketching, Antiderivatives, Indefinite Integrals, Integral Tables	Textbook 1 (Chapter 6)
10	Integration:Area and Estimating with Finite Sums, Sigma Notation and Limits of Finite Sums, Riemann Sums, Definite İntegral, Properties of Definite İntegral, Area Under the Graph of a nonnegative Function, Average Value of Continuous Functions	Textbook 1 (Chapter 5)
11	Mean Value Theorem fo Definite İntegrals, The Fundamental Theorem of Calculus: Fundamental Theorem Part 1, Fundamental Theorem Part 2, Techniques of Integration: Integration by Substitution, Integration by Parts, Trigonometric Integrals, Reduction Formulas	Textbook 1 (Chapter 8)
12	Quizz 1, Trigonometric Substitutions, Tan (θ/2) subtitutions, Integrations of Rational Functions by Partial Fractions	Textbook 1 (Chapter 8)
13	Applications of definite integrals:Area between two curves, Volumes Using Cross-sections, The Disk Method, the Washer Method, The Ccylindrical Shell method, Arch Length, Areas of Surfuces of Revolution	Textbook 1 (Chapter 6)
14	Improper Integrals, Improper Integrals of Type 1 and Type 2	Textbook 1 (Chapter 8)
15	Improper Integrals,Improper Integrals of Type I and Type II	Textbook 1 (Chapter 8)

Evaluation System

FUNCTIONS

In calculus, a symbolic way to say "y is a function of x" is by writing

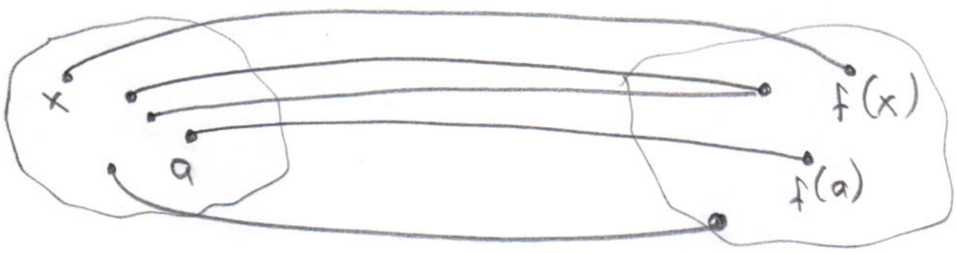
$$y = f(x)$$

In this notation, x is called the **independent variable**, represents the input value of f and y is called **dependent variable**, represents the corresponding output value of f at x.

Definition: A function is a rule that assigns each element of the set D to an **unique element** in Y.

$$f : D \to Y \quad , \quad x \to f(x)$$

The set D of all possible input values is called the **domain set** of the function. The set $R(f)$ of all values of $f(x)$ as x varies throughout D is called the **range set** of the function. The range may not include every element in the set Y.



D = Domain set

Y = Set containing the range

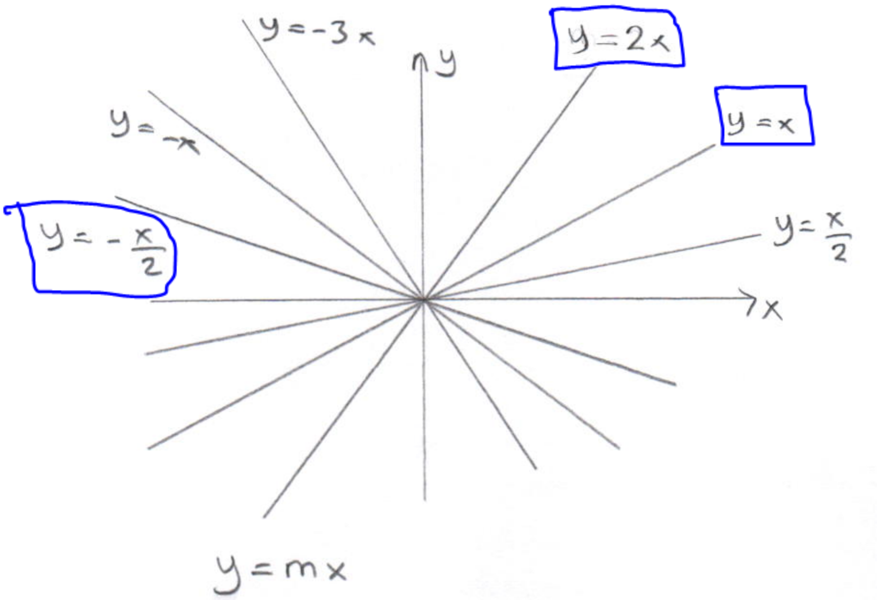
Domain and Ranges of Some Functions

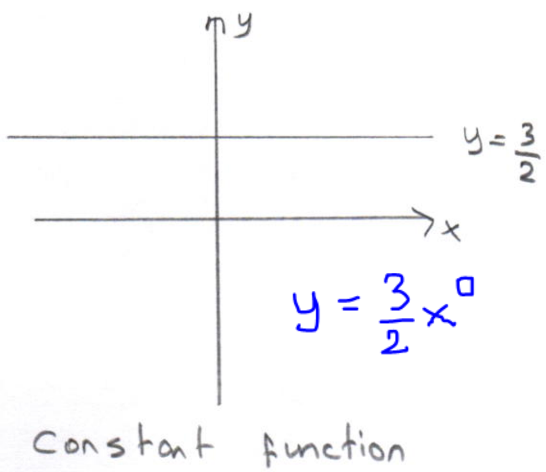
Function	Domain	Range
1. $y = x^2$	$\mathbb{R}$	$[0, \infty)$
2. $y = \frac{1}{x}$	$\mathbb{R} - \{0\}$	$\mathbb{R} - \{0\}$
3. $y = \sqrt[n]{x}$ → n odd	$\mathbb{R}$	$\mathbb{R}$
→ n even	$[0, \infty)$	$[0, \infty)$
4. $y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$
5. $\sin x$ / (or) $\cos x$	$\mathbb{R}$	$[-1, 1]$
6. $\arcsin x / \arccos x$	$[-1, 1]$	$\mathbb{R}$
7. $\ln x$	$(0, \infty)$	$\mathbb{R}$
8. e^x	$\mathbb{R}$	$(0, \infty)$

Function Types

1. Linear Functions

A function $f(x) = \underline{\underline{m}}x + \underline{\underline{b}}$ for constants m and b is called a linear function. (line)

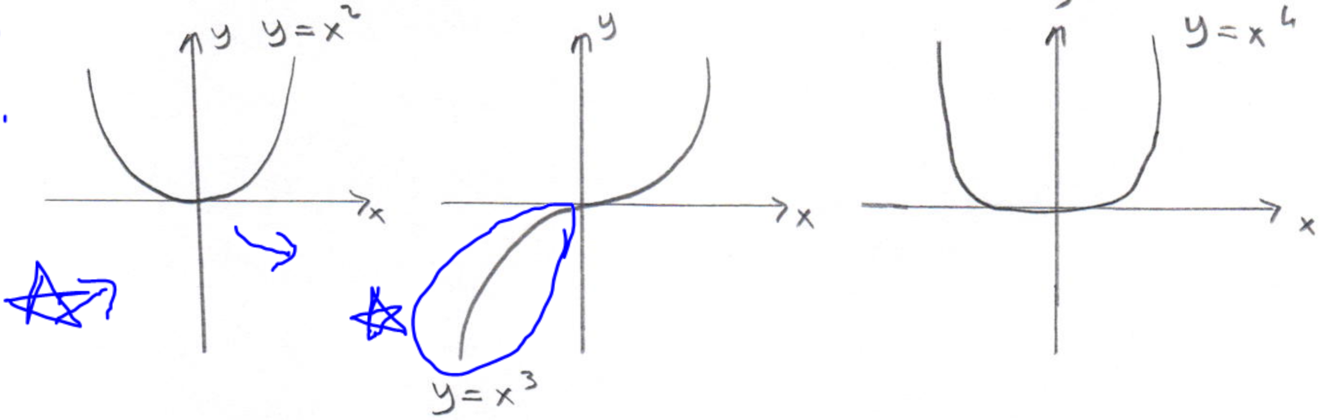


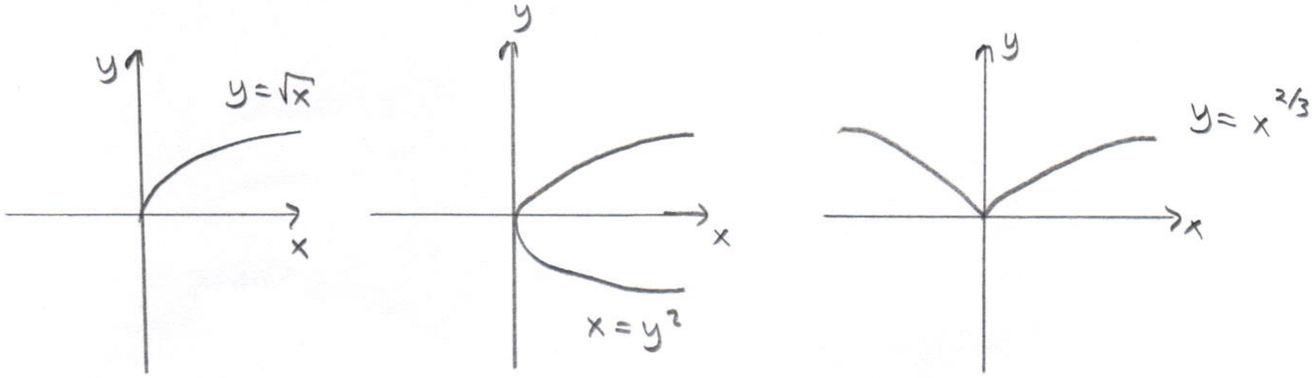


2. Power Functions $\quad x^a \quad a \in \mathbb{Z}^+ \cup \{0\}$

A function $f(x) = x^a$ where a is a constant is called a **power function**.

$\Rightarrow$ polyn.





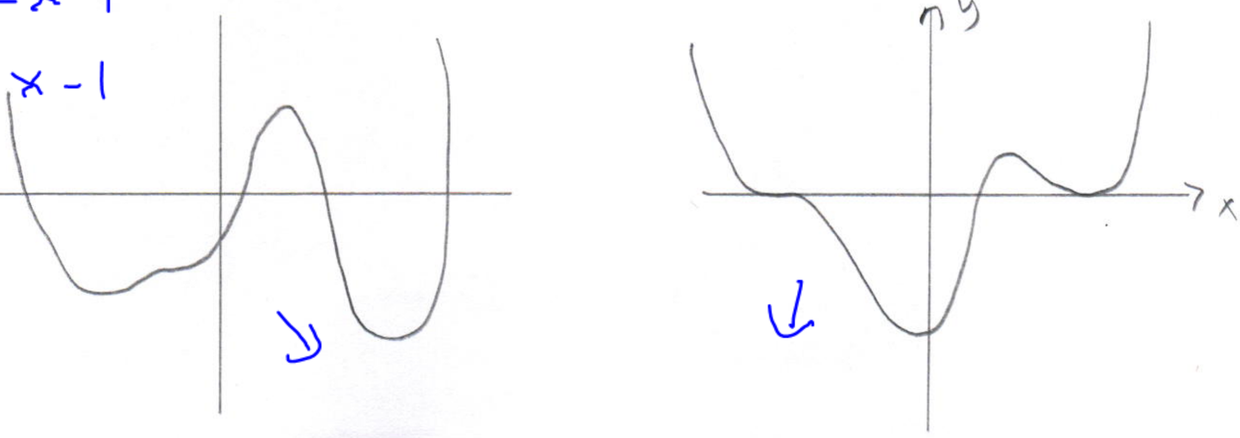
3. Polynomials

A function $P(x)$ is a **polynomial** if $P(x) = a_n x^n \pm \dots \pm a_1 x \pm a_0$ where n is a nonnegative integer and a_i are real constants (coefficients of the polynomial)

$y = x^2 - x - 1$

$y = \frac{1}{x} + x - 1$

$\cancel{x^{-1}}$



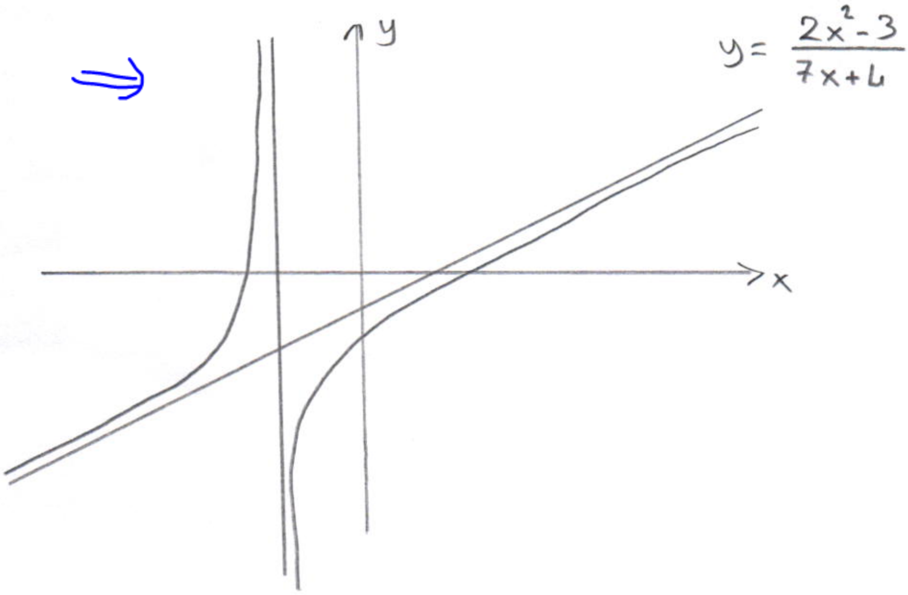
4. Rational Functions

A **rational function** is a quotient or ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)} \rightarrow \text{polyn.}$$

where $P(x)$ and $Q(x)$ are polynomials. The domain of a rational function is the set of all real x for which $Q(x) \neq 0$.

$\Rightarrow$



5. Algebraic Functions

An **algebraic function** is a function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division and taking roots). Rational functions are special cases of algebraic functions.

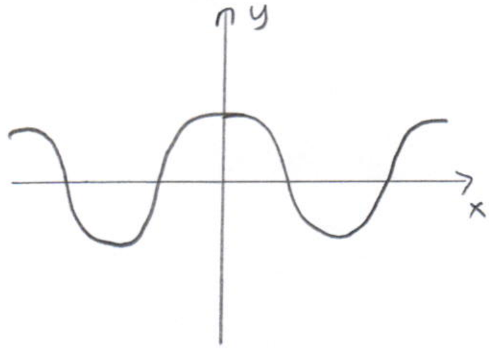
$+$ $-$ $\times$ $\div$

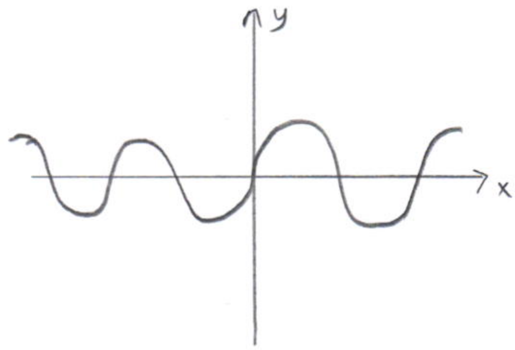
$(\,)^3, \sqrt[3]{\,}, \sqrt[2]{\,}.$

$y = x^3 + x \Rightarrow$ algeb.

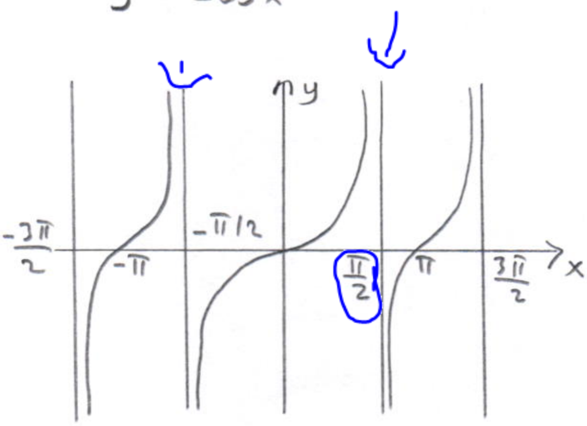
$y = x - \sqrt[3]{x}$

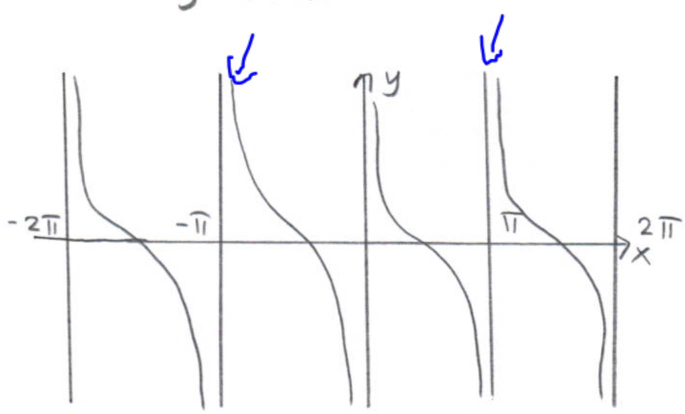
6. Trigonometric Functions





$y = \cos x$ $\qquad$ $y = \sin x$

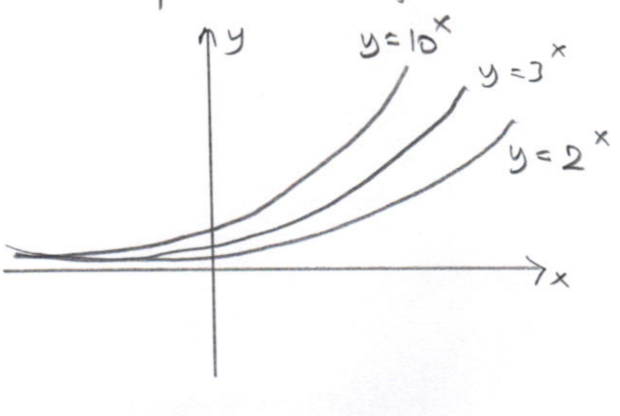


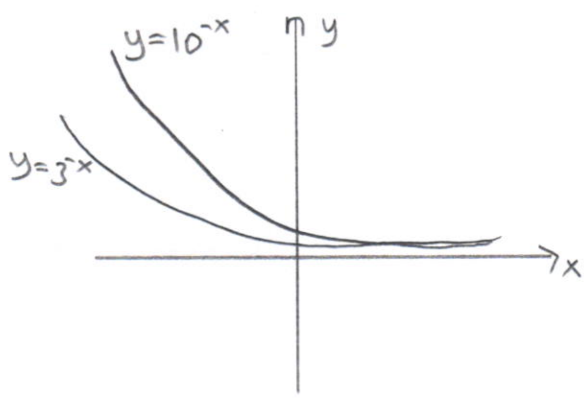


$y = \tan x = \dfrac{\sin x}{\cos x} \Rightarrow x = \dfrac{\pi}{2}$ $\qquad$ $y = \cot x = \dfrac{\cos x}{\sin x}$

7. Exponential Functions

Functions of the form $f(x) = a^x$ where the base $a > 0$ is a positive constant and $a \neq 1$ are called exponential functions. All exponential functions have domain $\mathbb{R}$ and range $(0, \infty)$. So an exponential function never assumes the value 0.

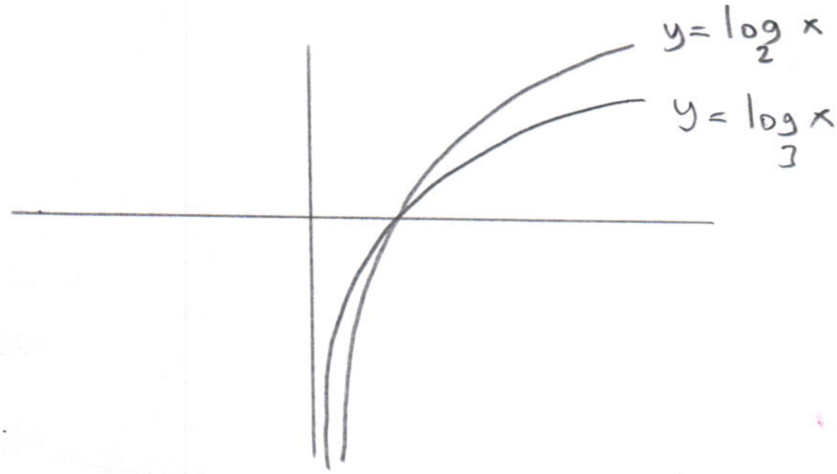




$f(x)=e^x$ is called natural exponential function.

8. Logarithmic Functions

$f(x)=\log_a x$ where the base $a \neq 1$ is a positive constant is called logarithmic function. This is the inverse function of the exponential function.



$f(x)=\ln x$ is called natural logaritmic function.

9. Transcendental Functions

These are functions that are NOT algebraic. They include

- Trigonometric
- Inverse trigonometric
- Exponential
- Logarithmic etc...

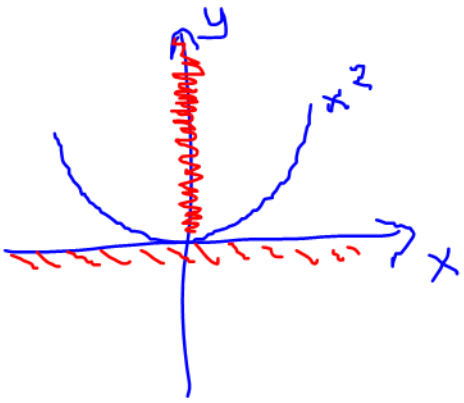
Even and Odd Functions

If $f(-x)=f(x)$, then $f(x)$ is an even function.
If $f(-x)=-f(x)$, then $f(x)$ is an odd function.

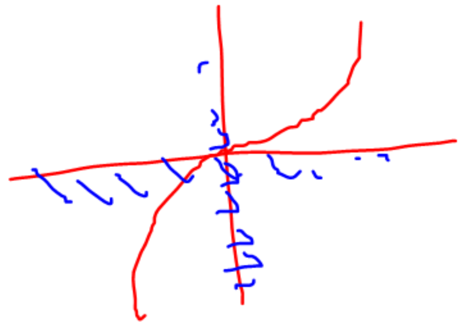
$\boxed{y: \mathbb{R} \to \mathbb{R}}$

Domain and Ranges of Some Functions

$y = x^2$ $\qquad y: \mathbb{R} \to \mathbb{R}^+ \cup \{0\} = [0, \infty)$



$y = x^3$



$y: \mathbb{R} \to \mathbb{R}$

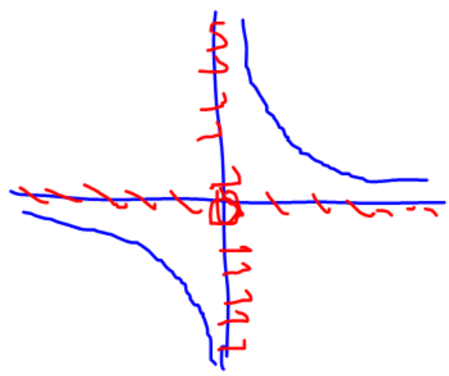
$xy = 1$

$y = \frac{1}{x}$ $\quad 1 \to 1 \cdot x^0$

$x = 0$

$y: \mathbb{R} - \{0\} \to \mathbb{R} - \{0\}$

$\frac{P(x)}{Q(x)}$ > domain: $\mathbb{R}$



$\frac{e^x+e^{-x}}{2}$ $\frac{e^x-e^{-x}}{2}$

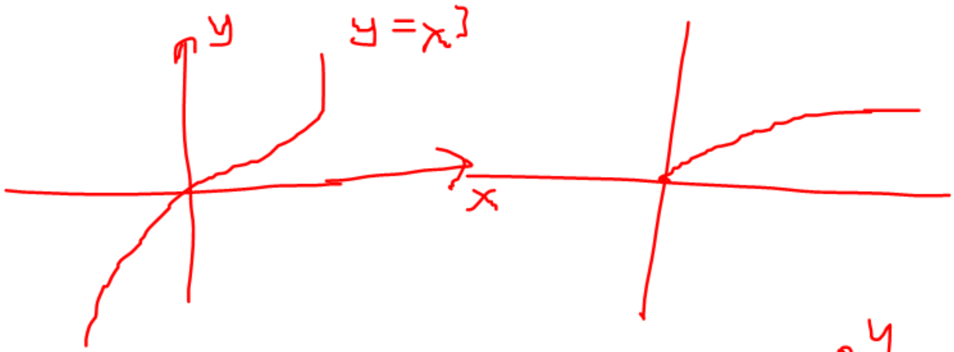
$y=\sqrt[n]{x}$ $\to$ n odd $\mathbb{R}\to\mathbb{R}$

$\to$ n even $[0,\infty)\to[0,\infty)$

$y=\sqrt{x} \Rightarrow x\geq 0 \Rightarrow y:[0,\infty)\to[0,\infty)$

$y=\sqrt[3]{x} \Rightarrow y:\mathbb{R}\to\mathbb{R}$

$\boxed{x=y^3}$



$y=x^3$

$y=\sqrt{1-x^2} \Rightarrow$

$1-x^2\geq 0$

$(1-x)(1+x)\geq 0$

	-1		1	
$1-x$	+	+	0	−
$1+x$	−	0 +		+
$1-x^2$	−	+		−

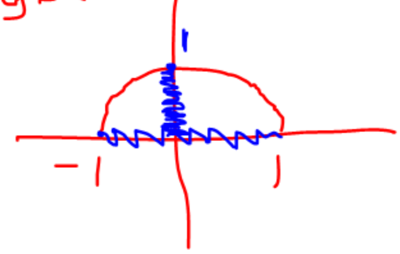
Domain: $\{x\in\mathbb{R} \mid (1-x^2)\geq 0\}$

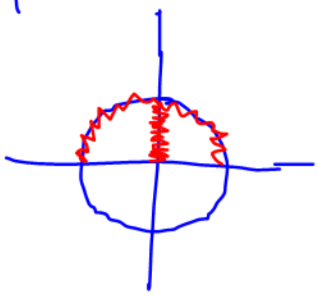
$: [-1,1] \to [0,1]$

$x^2+y^2=1$

$\Downarrow$

$y=\sqrt{1-x^2}$



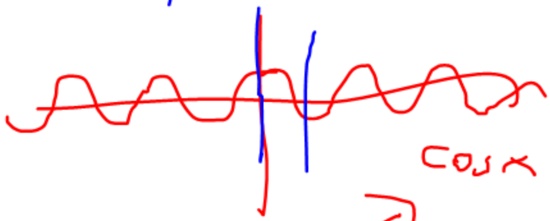


$y=\sqrt{1-x^2}$

$1-x^2$

(or)
sin x / cos x

$y: \mathbb{R} \to [-1, 1]$



$y: \mathbb{R} \to [-1, 1]$

$[-1, 1] \to \mathbb{R}$

$\arcsin x : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\arccos x : [-1, 1] \to [0, \pi]$

arcsin x / arccos x

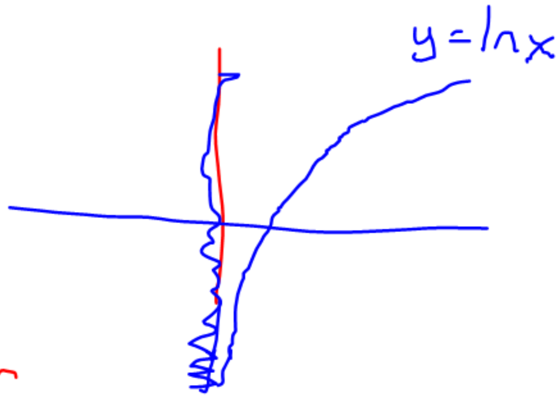
$y = \sin x$

$\arcsin y = x$

$\arcsin x : [-1, 1] \to \mathbb{R}$

$\arccos x : \to$

$\ln x = \log_e x$



$\ln x : \boxed{(0,\infty)} \to \mathbb{R}$

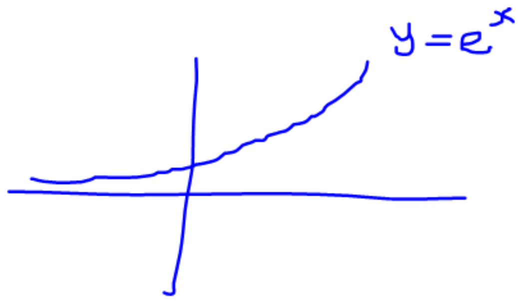
$\boxed{x > 0}$

$\ln(x-1)$

$\boxed{x-1>0}$

$x>1 \Rightarrow (1,\infty)$

$y = e^x$



$y: \mathbb{R} \to \mathbb{R}^+$

Examples Find the domain of the functions.

1, $f(x) = \sqrt{4 - x^2}$

$4 - x^2 \geq 0 \qquad x^2 \leq 4 \Rightarrow -2 \leq x \leq 2$

$\rightarrow$ Domain $= [-2, 2]$

$D(f) = \{x \in \mathbb{R} \mid 4 - x^2 \geq 0\}$

$f(x) = \dfrac{1}{\sqrt{4 - x^2}}$

$D: (-2, 2)$

2. $f(x) = \cancel{\ln}\left(\dfrac{x^2 - 3x + 2}{x + 1}\right)$

$\ln(x) : \mathbb{R}^+ \rightarrow \mathbb{R}$

$x > 0$

$\dfrac{x^2 - 3x + 2}{x + 1} > 0$ and $x + 1 \neq 0$

$\dfrac{(x-2)(x-1)}{x+1} > 0$ and $\boxed{x \neq -1}$

$x^2 - 3x + 2$

$-2 \quad -1$

	-1	1	2	
$x-2$	$-$	$-$	$-$ 0	$+$
$x-1$	$-$	$-$	0 $+$	$+$
$x+1$	$-$ 0	$+$	$+$	$+$
	$-$	$+$	$-$	$+$

	1	2	
$1-x$	$+$ 0	$-$	$-$
$2-x$	$+$	$+$ 0	$-$

$D(f): (-1, 1) \cup (2, \infty)$

3. $f(x) = \sqrt{\ln \frac{5x - x^2}{4}}$

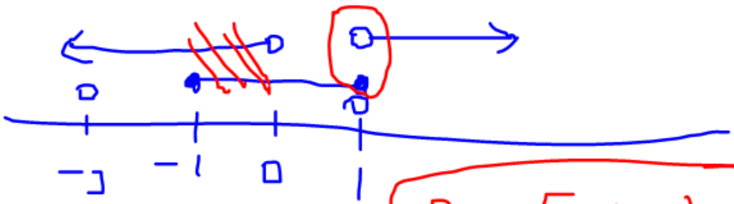
4. $f(x) = \arcsin\left(\log \frac{x}{10}\right)$

α

5. $f(x) = \dfrac{\sqrt{|2x+5|-3}}{x^2+1}$

6. $f(x) = \dfrac{1}{\cos^2 x - \frac{1}{4}}$

7. $f(x) = \log\left[1 - \log(x^2 - 5x + 16)\right]$



$\boxed{D = [-1, 0)}$

8. $f(x) = \dfrac{\sqrt{1-x^2} \cdot \ln(x^2 - x)}{x^2 + 2x - 3}$

$+3 \quad -1$

$(x+3)(x-1) \neq 0$

$1 - x^2 \geq 0$ and $x^2 - x > 0$ and $x^2 + 2x - 3 \neq 0$

$-1 \leq x \leq 1$ and $x(x-1) > 0$ and $\boxed{x \neq -3,\ x \neq 1}$

	0	1	
x	$-$	$+$	$+$
$x-1$	$-$	$-$	$+$
	$+$	$-$	$+$

2 (9 → 11 tues.

$\boxed{12 \to 15}$

9 : 30 → 11 : 30

9 : 45 : 11 : 45

9 : 45 = 11 : 45 tuesday

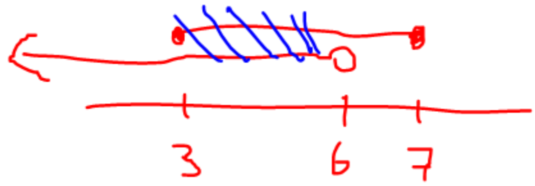
⇒ 12 : 00 – 15 : 00 saturday

$\log(6-x) = \ln(\cancel{6}-x)$

$\mathbb{R} \to [-1, 1]$

arcsin: $\boxed{[-1, 1]}$

9. $\underbrace{\arcsin\left(\frac{x-5}{2}\right)}+\overbrace{\ln(6-x)}$



$\underbrace{-1\le\frac{x-5}{2}\le 1}$ and $\underbrace{6-x>0}$

$-2\le x-5\le 2$

$\boxed{3\le x\le 7}$ and $\boxed{x<6}$

Domain: $[3,6)$

$\ln x>0$

$e^{\ln x} > e^{0}$

$x>e^0=1$

10. $\underbrace{\arcsin(1-x)}+\overbrace{\ln(\ln x)}$

$e^{\ln x}>e^0$

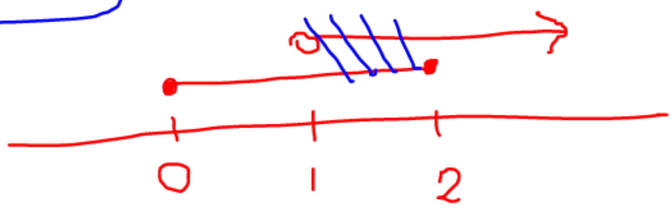
$\boxed{x>1}$

$-1\le 1-x\le 1$ and $\ln x>0$ and $\cancel{x>0}$

$-2\le -x\le 0$

$\boxed{0\le x\le 2}$ and $x>1$

log



$D(f):(1,2]$

11. $f(x) = \dfrac{1}{x^2-4} + \dfrac{\sqrt{x-2}}{x^2+1} + \sqrt{x-1}$

LIMITS

Before giving an informal definition of limit let's look at behavior of a function near a point.

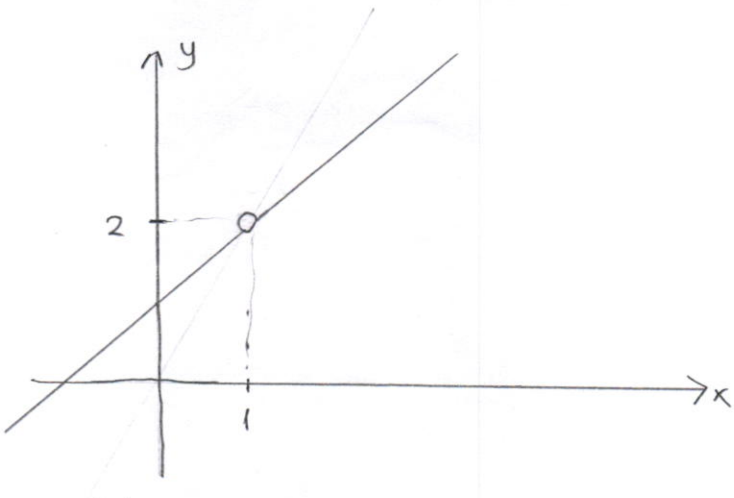
Example. What is the behavior of the function

$$f(x) = \frac{x^2-1}{x-1} \qquad \text{near } x=1\,?$$

$f(x)$ is defined for all real numbers x, except $x=1$.
For any $x \neq 1$, we can write $f(x)$ as

$$f(x) = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{(x-1)} = (x+1)$$

The graph of $f(x)$ is a line; but there is a hole at the point $(1,2)$



$f(1)$ is not defined, but we can make the value of $f(x)$ as close as we want to 2 by choosing x close enough to 1.

Therefore, we can say that $f(x)$ approaches arbitrarily close to 2 as x approaches 1. We can write as

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$$

An Informal Definition of Limit

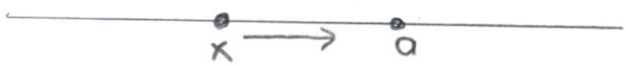
If $f(x)$ is defined for all x near a and if we can guarantee that $f(x)$ is as close as we want to L by taking x close enough to a; but not equal to a, we say that $f(x)$ approaches to the limit L as x approaches to a.

$$\lim_{x \to a} f(x) = L$$

An Informal Definition of Left and Right Limit

If $f(x)$ is defined on some interval (b,a) extending to the left of $x=a$ and if we can guarantee that $f(x)$ is as close as we want to L by taking x to the left of a and close enough to a, we say that $f(x)$ has **left limit** L at $x=a$ and we write

$$\lim_{x \to a^-} f(x) = L$$

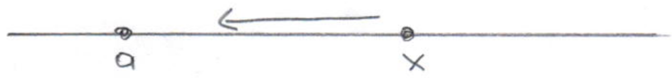


$x \to a^-$ means x approaches a from the left

P.S: $\lim_{x \to a^-} f(x) = \lim_{h \to 0^+} f(a-h)$

If $f(x)$ is defined on some interval (a,b) extending to the right of $x=a$ and if we can guarantee that $f(x)$ is as close as we want to L by taking x to the right of a and close enough to a, we say that $f(x)$ has right limit L at $x=a$ and we write

$$\lim_{x \to a^+} f(x) = L$$



$x \to a^+$ means x approaches a from the right

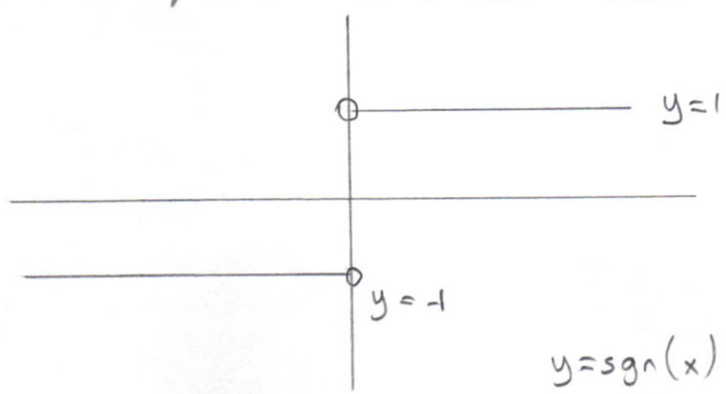
P.S: $\lim_{x \to a^+} f(x) = \lim_{h \to 0^+} f(a+h)$

Example. what is the limit of the signum function

$$sgn(x) = \frac{x}{|x|}$$

$$\lim_{x \to 0^+} sgn(x) = \frac{x}{x} = 1 \qquad \lim_{x \to 0^-} \frac{x}{-x} = -1$$

You can easily see that sgn function has left and right limits; but does not have limit !!!.



$y = sgn(x)$

Relation Between One-Sided and Two-Sided Limits

A function $f(x)$ has limit L at $x=a$ if and only if it has both left and right limits there and these one-sided limits are both equal to L:

and these one-sided limits are both equal to L

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$$

Example. If $f(x) = \dfrac{|x-2|}{x^2+x-6}$, find $\lim\limits_{x\to 2^+} f(x)$, $\lim\limits_{x\to 2^-} f(x)$

and $\lim\limits_{x\to 2} f(x)$.

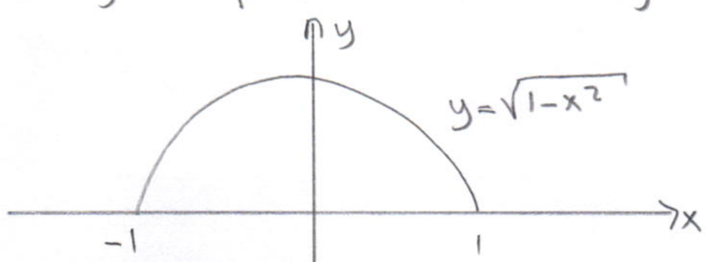
$$\lim_{x\to 2^+} \frac{(x-2)}{(x-2)(x+3)} = \lim_{x\to 2^+} \frac{1}{x+3} = \frac{1}{5}$$

$$\lim_{x\to 2^-} \frac{-(x-2)}{(x-2)(x+3)} = \lim_{x\to 2^-} \frac{-1}{x+3} = -\frac{1}{5}$$

Since $\lim\limits_{x\to 2^-} f(x) \neq \lim\limits_{x\to 2^+} f(x)$ $\lim\limits_{x\to 2} f(x)$ does not exist!

Example. what one-sided limits goes $g(x)=\sqrt{1-x^2}$ have at $x=-1$ and $x=1$?

The domain of $g(x)$ is $[-1,1]$, so $g(x)$ is defined only to the right of $x=-1$ and only to the left $x=1$.



$$\lim_{x \to -1^+} g(x) = 0 \quad \text{and} \quad \lim_{x \to 1^-} g(x) = 0$$

$g(x)$ has no left limit or limit at $x=-1$

$g(x)$ has no right limit or limit at $x=1$

Example. $f(x) = \begin{cases} x-1 & \text{if } x \le -1 \\ x^2+1 & \text{if } -1 < x \le 0 \\ (x+\pi)^2 & \text{if } x > 0 \end{cases}$

Find the indicated limits.

a) $\lim\limits_{x \to -1^-} f(x)$ b) $\lim\limits_{x \to -1^+} f(x)$ c) $\lim\limits_{x \to 0^+} f(x)$ d) $\lim\limits_{x \to 0^-} f(x)$

a) $\lim\limits_{x \to -1^-} (x-1) = -2$

c) $\lim\limits_{x \to 0^+} (x+\pi)^2 = \pi^2$

b) $\lim\limits_{x \to -1^+} x^2+1 = 2$

d) $\lim\limits_{x \to 0^-} x^2+1 = 1$

We can see that $f(x)$ does not have limits at $x=-1$ and $x=0$.

Rules for Calculating Limits

If $\lim\limits_{x \to a} f(x) = L$, $\lim\limits_{x \to a} g(x) = M$ and k is a constant,

then

1. $\lim\limits_{x \to a} [f(x) \mp g(x)] = L \mp M$

2. $\lim_{x \to a} f(x)\, g(x) = L \cdot M$

3. $\lim_{x \to a} k f(x) = k \cdot L$

4. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$

5. If m is an integer and n is a positive integer, then

$\lim_{x \to a} [f(x)]^{m/n} = L^{m/n}$ provided $L > 0$ if n is even

and $L \neq 0$ if $m < 0$

6. If $f(x) \leq g(x)$ on an interval containing a in its interior, then

$$L \leq M$$

If $f(x) \leq g(x)$

7. If $P(x)$ is a polynomial and a is any real number, then

$\lim_{x \to a} P(x) = P(a)$

8. If $P(x)$ and $Q(x)$ are polynomials and $Q(a) \neq 0$, then

$$\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}$$

The Sandwich Theorem

Suppose that $f(x) \leq g(x) \leq h(x)$ holds for all x in some open interval containing a, except possibly at $x = a$ itself. Suppose also that

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \Rightarrow \lim_{x \to a} g(x) = L$$

Example. Using the sandwich theorem, show that $\lim_{x \to 0} x^2 \cos \frac{1}{x} = 0$.

we know that $-1 \leq \cos \frac{1}{x} \leq 1$

$$\Rightarrow \underbrace{-x^2}_{f(x)} \leq \underbrace{x^2 \cos \frac{1}{x}}_{g(x)} \leq \underbrace{x^2}_{h(x)}$$

Since $\lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0$, then $\lim_{x \to 0} x^2 \cos \frac{1}{x} = 0$.

Formal Definition of Limit

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself.

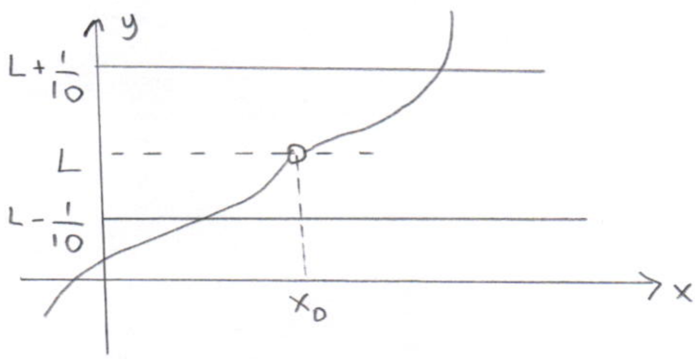
We say that the limit of $f(x)$ as x approaches x_0 is the number L, and write

$$\lim_{x \to x_0} f(x) = L,$$

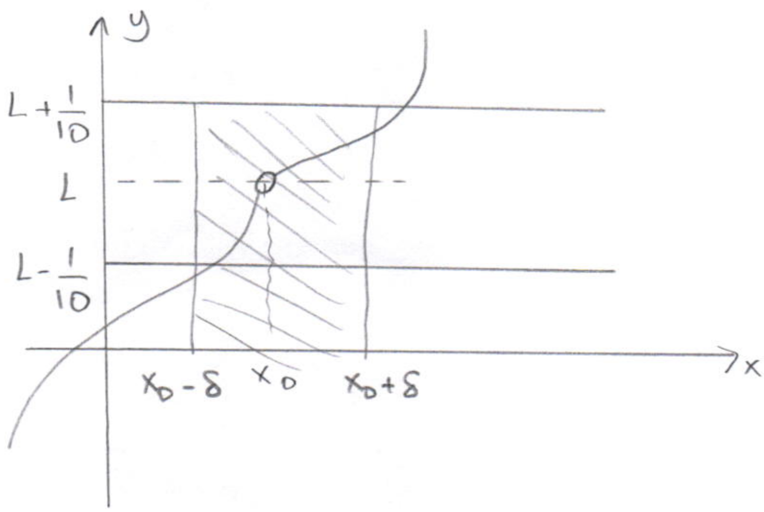
if, for evey number $\varepsilon > 0$, there exists a corresponding number $\delta = \delta(\varepsilon) > 0$ such that for all x,

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon .$$

What is the meaning of limit geometrically?



$$|f(x) - L| < \varepsilon = \frac{1}{10}$$



$$|x - x_0| < \delta_{1/10}$$

Example. Show that $\lim\limits_{x \to 1} 5x - 3 = 2$

Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that for all x

$0 < |x - 1| < \delta$ implies $|f(x) - L| < \varepsilon$

$$|f(x) - L| = |5x - 3 - 2| = |5(x - 1)| < 5 . \delta = \varepsilon \Rightarrow \boxed{\delta = \frac{\varepsilon}{5}}$$

Example. Show that $\lim_{x \to 2} x^2 - 2x - 1 = -1$.

Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that for all x

$0 < |x-2| < \delta$ implies $|x^2 - 2x - 1 - (-1)| < \varepsilon$

$|x^2 - 2x| = |x||x-2| < |x| . \delta$

$\left(0 < |x-2| < \delta \Rightarrow -\delta < x-2 < \delta \Rightarrow 2-\delta < x < \delta + 2 \right)$

$< (2+\delta)\delta = \varepsilon \Rightarrow \delta^2 + 2\delta - \varepsilon = 0$

$\delta_{1,2} = \dfrac{-2 \mp \sqrt{4+4\varepsilon}}{2} = -1 \mp \sqrt{1+\varepsilon}$ $\rightarrow \delta_1 = -1 + \sqrt{1+\varepsilon}$, $\rightarrow \delta_2 = -1 - \sqrt{1-\delta}$

Since $\delta_2 < 0$, only δ_1 possess the limit.

Example. Show that $\lim_{x \to 2} \sqrt{x+2} = 4$

Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that for all x

$0 < |x-2| < \delta$ implies $|\sqrt{x+2} - 2| < \varepsilon$

Here we have to find a relationship between

$|x-2|$ and $|\sqrt{x+2}-2|$

$$x-2=(\sqrt{x+2}-2)(\sqrt{x+2}+2)$$

$$|x-2|=|\sqrt{x+2}-2|\,|\sqrt{x+2}+2|$$

Since $\sqrt{x+2}+2>1$, $|x-2|>\sqrt{x+2}-2$

$$|\sqrt{x+2}-2|<|x-2|<\delta=\varepsilon$$

Limits at Infinity of Rational Functions

To determine the limit of a rational function as $x\to\mp\infty$, we face three posibilities:

1. If degree of numerator is the same with the degree of denominator

Example. Find the limit

$$\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2}=\lim_{x\to\infty}\frac{\cancel{x^2}\left(5+\overset{0}{\frac{8}{x}}-\overset{0}{\frac{3}{x^2}}\right)}{\cancel{x^2}\left(3+\underset{0}{\frac{2}{x^2}}\right)}=\frac{5}{3}$$

2. If degree of numerator is less than the degree of denominator.

Example. Find the limit

$$\lim_{x\to-\infty}\frac{11x+2}{2x^3-1}=\lim_{x\to-\infty}\frac{\cancel{x}\left(11+\overset{0}{\frac{2}{x}}\right)}{\cancel{x}\left(2x^2-\underset{0}{\frac{1}{x}}\right)}=0$$

3. If degree of numerator is greater than the degree of denominator

Example. Find the limit

$$\lim_{x\to\infty} \frac{2x^3+4}{2x-1} = \lim_{x\to\infty} \frac{\cancel{x}\left(2x^2+\overset{0}{\left(\frac{4}{x}\right)}\right)}{\cancel{x}\left(2-\underset{0}{\left(\frac{1}{x}\right)}\right)} = \infty$$

Some Notes About Limits

1. $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$; $\lim\limits_{x\to\infty} \frac{\sin x}{x} = \frac{1}{\infty} \cdot \overset{-1\le a\le 1}{\sin\infty} = 0 \cdot a = 0$

2. $\lim\limits_{x\to 0} \frac{\cos x - 1}{x} = 0$; $\lim\limits_{x\to 0} \frac{\tan x}{x} = \lim\limits_{x\to 0} \frac{\arctan x}{x} = 1$

3. $\lim\limits_{x\to\infty} e^x = \infty$; $\lim\limits_{x\to -\infty} e^x = 0$

4. $\lim\limits_{x\to\infty} \ln x = \infty$; $\lim\limits_{x\to 0^+} \ln x = -\infty$; $\lim\limits_{x\to 0} \frac{\ln(x+1)}{x} = 1$

5. $0 < a < 1$ $\lim\limits_{x\to\infty} a^x = 0$; $\lim\limits_{x\to -\infty} a^x = \infty$

6. $a > 1$ $\lim\limits_{x\to\infty} a^x = \infty$; $\lim\limits_{x\to -\infty} a^x = 0$

7. $\lim\limits_{x\to 0} \frac{e^{g(x)} - 1}{g(x)} = 1$, where $\lim\limits_{x\to 0} g(x) = 0$

8. $\lim\limits_{x\to 0} (1+ax)^{1/x} = e^a$; $\lim\limits_{x\to\infty} \left(1+\frac{a}{x}\right)^x = e^a$

Limit Examples

1. $$\lim_{x \to 1} \frac{\sqrt{2x^2-1}-1}{x-1} = ?$$

2. $$\lim_{x \to 0} \frac{\sqrt{x+1}-1}{\sqrt{x}} = ?$$

3. $$\lim_{x \to \infty} \sqrt{x^2+2x} - x = ?$$

4. $\lim\limits_{x \to 8} \dfrac{\sqrt{x} - 2\sqrt{2}}{\sqrt[3]{x} - 2} = ?$

5. $\lim\limits_{x \to 0^+} \dfrac{1 - \cos x}{x^2} = ?$

6. $\lim\limits_{x \to 0} \dfrac{\sin x^2 + x \sin 3x}{x \sin x} = ?$

7. $\lim\limits_{x \to 0} \dfrac{1-\sqrt{2-\cos x}}{\sin x} = ?$

8. $\lim\limits_{x \to 0} \dfrac{e^{ax}-e^{bx}}{x} = ?$

10. $\lim\limits_{x \to \infty} \left(\dfrac{x+1}{x-1}\right)^{2x-1}$

16. $\lim\limits_{x \to \infty} x^2 \left(1 - \cos \frac{1}{x}\right) = ?$

19. $\lim\limits_{x \to 0} \frac{1 - \cos x^2}{x^2 \sin x^2} = ?$

20. $\lim\limits_{x \to 0^-} \frac{\sqrt{1 - \cos 2x}}{\sin 2x} = ?$