

MAT1071

MATHEMATICS I

1	Functions:Domain of a Function, Functions and Graphs, Even-Odd Functions, Symmetry, Operations on Functions (Sum, difference, multiplication, division and powers), Composite Functions, Piecewise Functions, Polynomials and Rational Functions, Trigonometric Functions	Textbook 1 (Chapter 1)
2	Limits and Continuity: Limit of a Function and Limit Laws, The Sandwich (The Squeeze theorem), The Precise Definition of a Limit, One-sided Limits, , Limits Involving Infinity, Infinity Limits	Textbook 1 (Chapter 2)
3	Continuity at a Point, Continuous Functions, The Intermediate Value Theorem Types of Discontinuity, Differentiation:Tangents ,Normal Lines , The Derivative at a Point, The Derivative as a Function, Onesided Derivatives	Textbook 1 (Chapter 3)
4	Differentiable on an Interval, Differentiation Rules, High order Derivatives, Derivatives of Trigonometric Functions, The Chain Rule, Implicit Differentiation, Linearization and Differentials, Increasing Functions and Decreasing Functions	Textbook 1 (Chapter 3)
5	Transcendental Functions:Inverse Functions and Their Derivatives, Logarithms and Exponential Functions and Their Derivatives, Logarithmic Differentiation, Inverse Trigonometric Functions and Their Derivatives, Hyperbolic Functions and Their Derivatives, Inverse Hyperbolic Functions and Their Derivatives	Textbook 1 (Chapter 7)
6	Indeterminate Forms and L'Hospital's Rule, Extrem Values of Functions, Critical Points,	Textbook 1 (Chapter 7)
7	Rolle's Theorem, The Mean Value Theorem, The First Derivative Test for Local Extrema, Concavity , The Second Derivative Test for Concavity, Point of Inflection, The Second Derivative Test for Local Extrema	Textbook 1 (Chapter 4)
8	Midterm 1	Textbook 1 (Chapter 4)
9	Asymptotes of Graphs, Curve Sketching, Antiderivatives, Indefinite Integrals, Integral Tables	Textbook 1 (Chapter 6)
10	Integration:Area and Estimating with Finite Sums, Sigma Notation and Limits of Finite Sums, Riemann Sums, Definite Integral, Properties of Definite Integral, Area Under the Graph of a nonnegative Function, Average Value of Continuous Functions	Textbook 1 (Chapter 5)
11	Mean Value Theorem for Definite Integrals, The Fundamental Theorem of Calculus: Fundamental Theorem Part 1, Fundamental Theorem Part 2, Techniques of Integration: Integration by Substitution, Integration by Parts, Trigonometric Integrals, Reduction Formulas	Textbook 1 (Chapter 8)
12	Quiz 1, Trigonometric Substitutions, Tan ($\theta/2$) substitutions, Integrations of Rational Functions by Partial Fractions	Textbook 1 (Chapter 8)
13	Applications of definite integrals:Area between two curves, Volumes Using Cross-sections, The Disk Method, the Washer Method, The Cylindrical Shell method, Arch Length, Areas of Surfaces of Revolution	Textbook 1 (Chapter 6)
14	Improper Integrals, Improper Integrals of Type 1 and Type 2	Textbook 1 (Chapter 8)
15	Improper Integrals, Improper Integrals of Type I and Type II	Textbook 1 (Chapter 8)

Evaluation System

FUNCTIONS

In calculus, a symbolic way to say "y is a function of x" is by writing

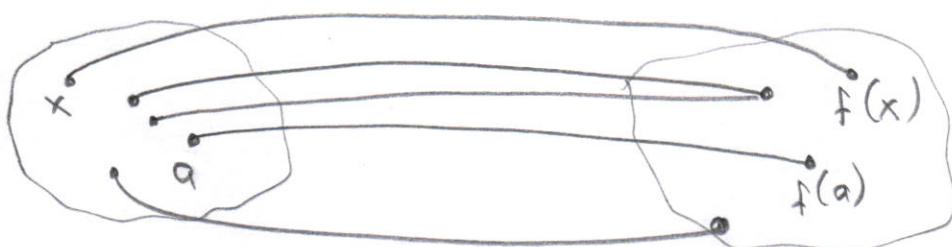
$$y = f(x)$$

In this notation, x is called the **independent variable**, represents the input value of f and y is called **dependent variable**, represents the corresponding output value of f at x.

Definition: A function is a rule that assigns each element of the set D to an **unique element** in Y.

$$f: D \rightarrow Y, \quad x \rightarrow f(x)$$

The set D of all possible input values is called the **domain set** of the function. The set R(f) of all values of $f(x)$ as x varies throughout D is called the **range set** of the function. The range may not include every element in the set Y.



D = Domain set

Y = Set containing
the range

Domain and Ranges of Some Functions

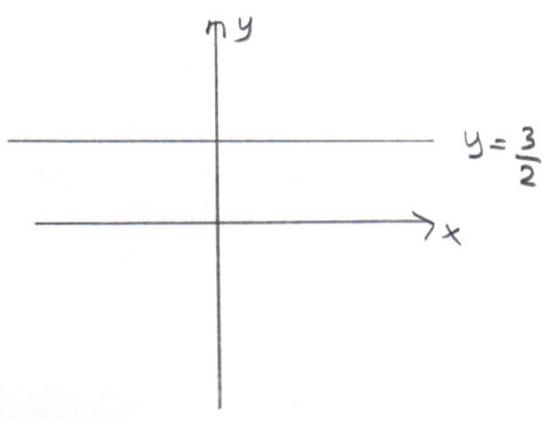
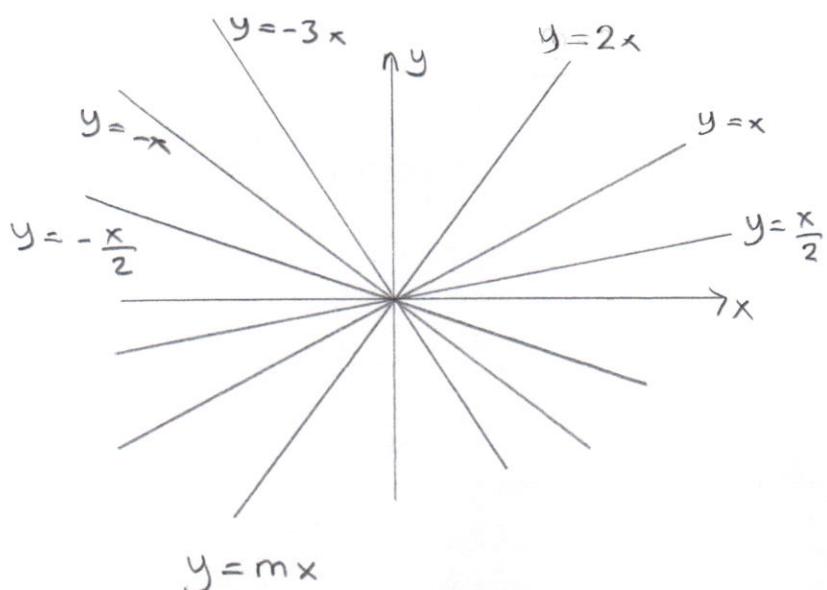
<u>Function</u>	<u>Domain</u>	<u>Range</u>
1. $y = x^2$	\mathbb{R}	$[0, \infty)$
2. $y = \frac{1}{x}$	$\mathbb{R} - \{0\}$	$\mathbb{R} - \{0\}$
3. $y = \sqrt[n]{x} \rightarrow$ n odd \mathbb{R} n even $[0, \infty)$		\mathbb{R} $[0, \infty)$
4. $y = \sqrt{1-x^2}$	$[-1, 1]$	$[0, 1]$
5. $\sin x / \cos x$ (or)	\mathbb{R}	$[-1, 1]$
6. $\arcsin x / \arccos x$	$[-1, 1]$	\mathbb{R}
7. $\ln x$	$(0, \infty)$	\mathbb{R}
8. e^x	\mathbb{R}	$(0, \infty)$

Function Types

1. Linear Functions

A function $f(x) = mx + b$ for constants m and b

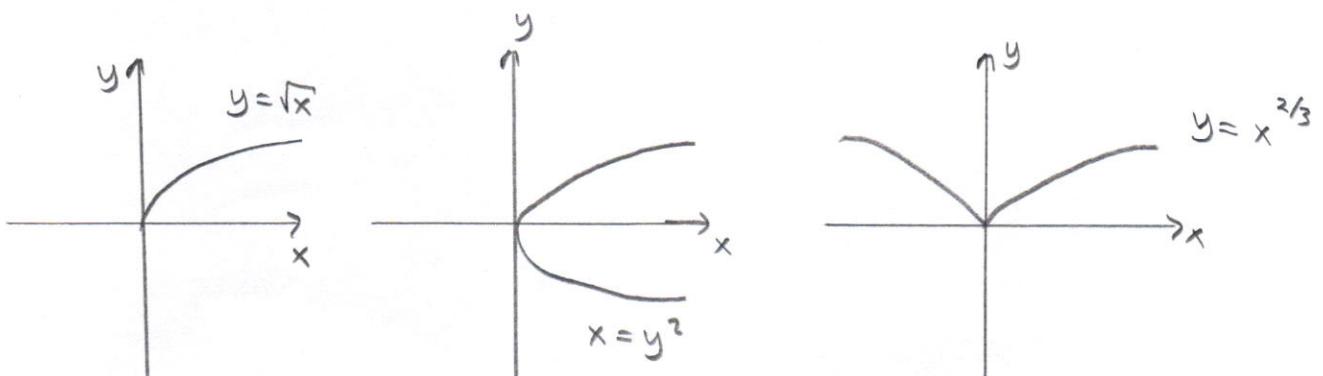
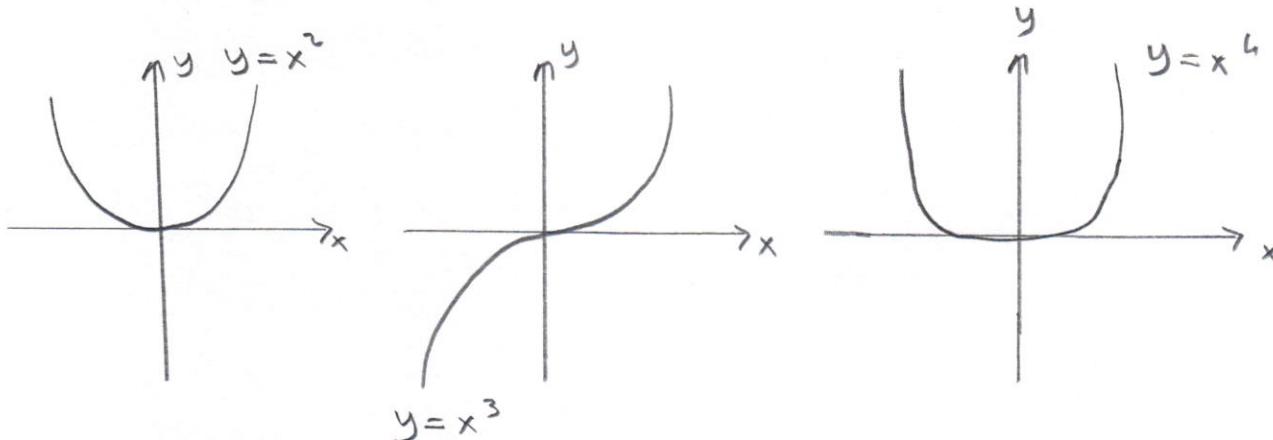
is called a linear function.



Constant function

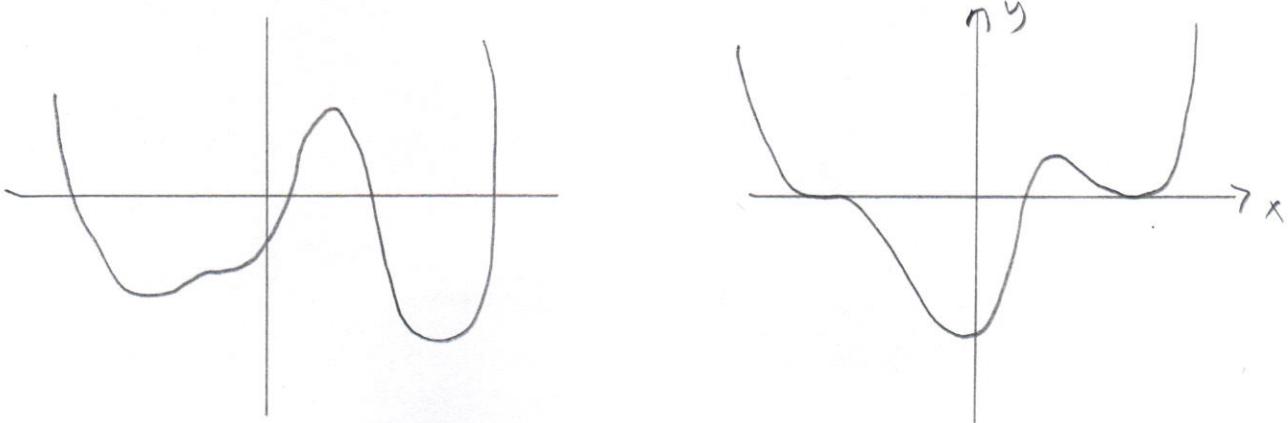
2. Power Functions

A function $f(x) = x^a$ where a is a constant is called a **power function**.



3. Polynomials

A function $P(x)$ is a **polynomial** if $P(x) = a_n x^n + \dots + a_1 x + a_0$ where n is a nonnegative integer and a_i are real constants (coefficients of the polynomial)

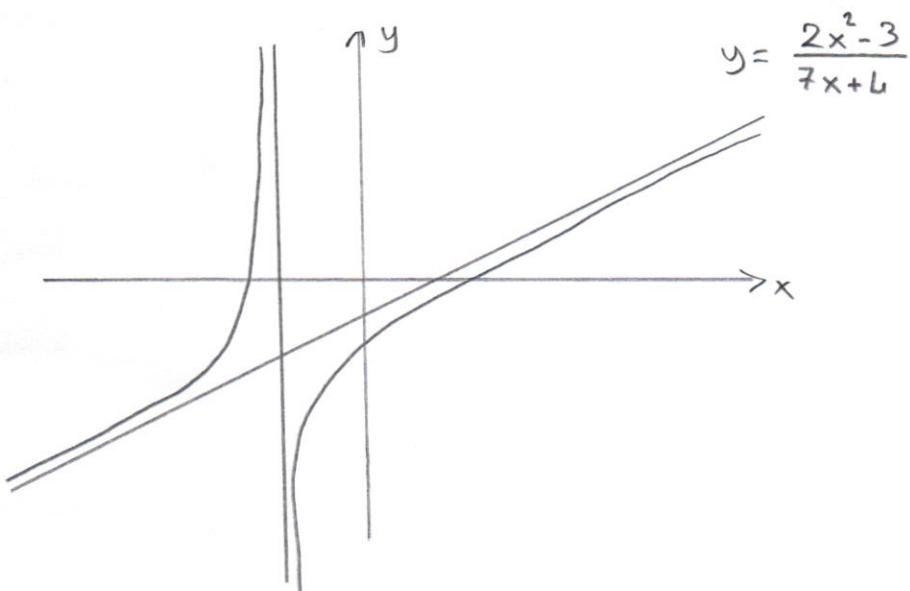


4. Rational Functions

A rational function is a quotient or ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

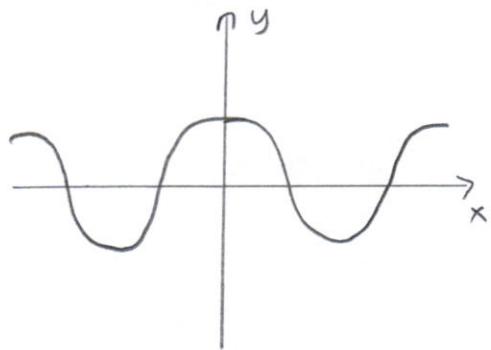
where $P(x)$ and $Q(x)$ are polynomials. The domain of a rational function is the set of all real x for which $Q(x) \neq 0$.



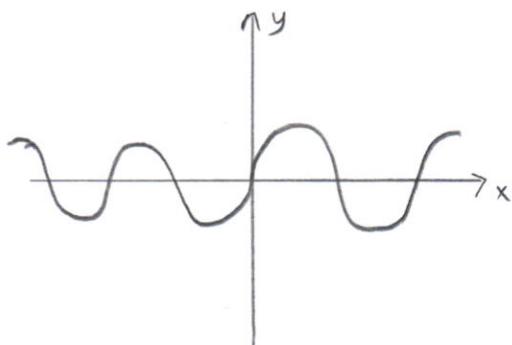
5. Algebraic Functions

An algebraic function is a function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division and taking roots). Rational functions are special cases of algebraic functions.

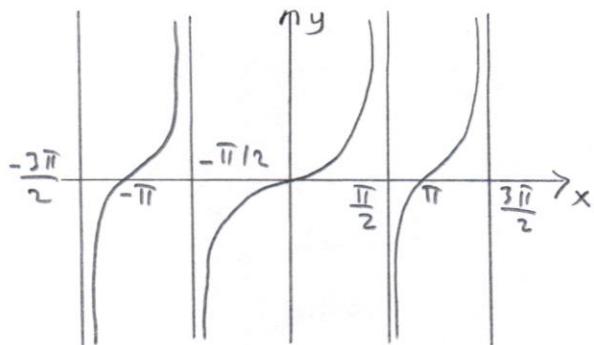
6. Trigonometric Functions



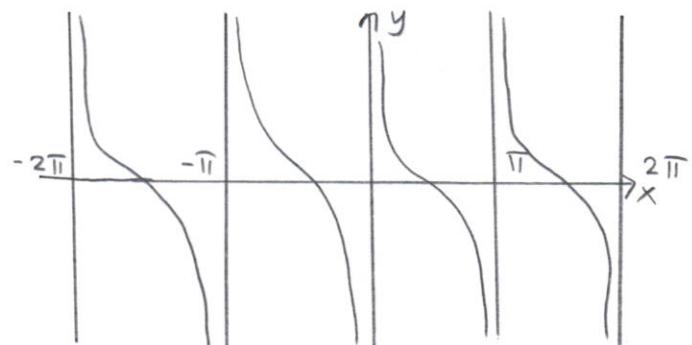
$$y = \cos x$$



$$y = \sin x$$



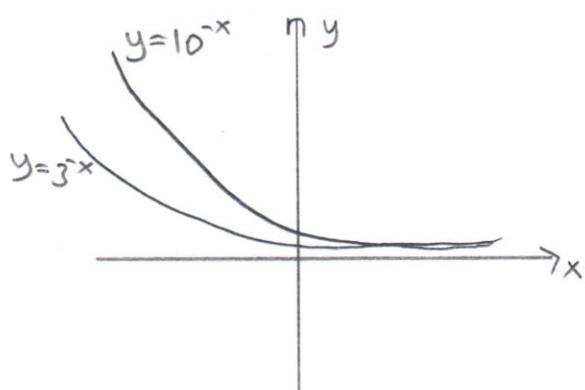
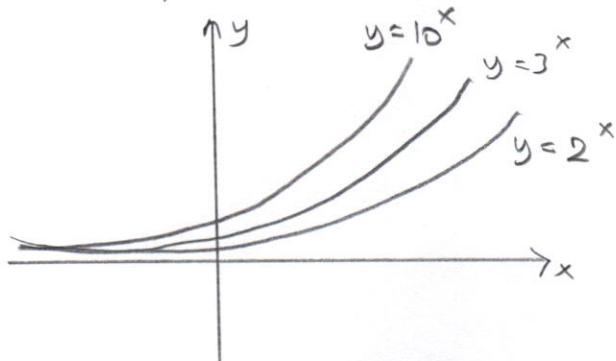
$$y = \tan x$$



$$y = \cot x$$

7. Exponential Functions

Functions of the form $f(x) = a^x$ where the base $a > 0$ is a positive constant and $a \neq 1$ are called **exponential functions**. All exponential functions have domain \mathbb{R} and range $(0, \infty)$. So an exponential function never assumes the value 0.

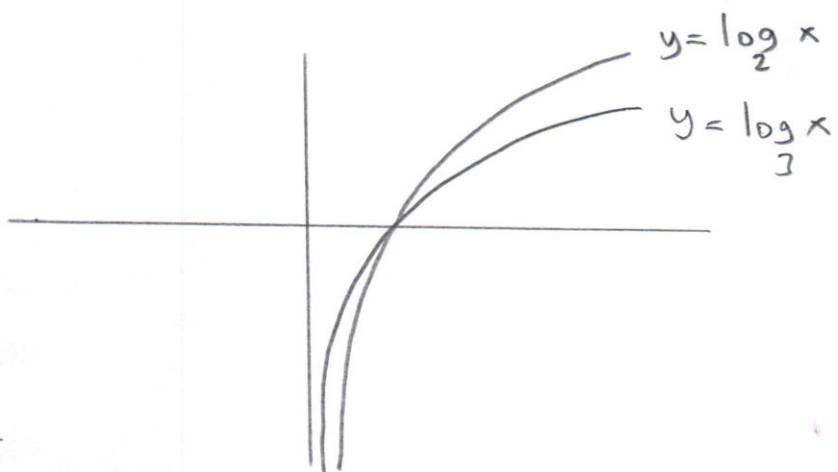


$f(x) = e^x$ is called natural exponential function.

8. Logarithmic Functions

$f(x) = \log_a x$ where the base $a \neq 1$ is a

positive constant is called logarithmic function.
This is the inverse function of the exponential
function.



$f(x) = \ln x$ is called natural logarithmic function.

9. Transcendental Functions

These are functions that are NOT algebraic.

They include

- Trigonometric
- Exponential
- Inverse trigonometric
- Logarithmic etc...

Even and Odd Functions

If $f(-x) = f(x)$, then $f(x)$ is an even function.

If $f(-x) = -f(x)$, then $f(x)$ is an odd function.

Domain and Ranges of Some Functions

$$y = x^2$$

$$y = \frac{1}{x}$$

$$y = \sqrt[n]{x}$$

$$y = \sqrt{1-x^2}$$

$\sin x / \cos x$
(or)

$\arcsin x / \arccos x$

$\ln x$

e^x

Examples Find the domain of the functions.

1. $f(x) = \sqrt{4 - x^2}$

2. $f(x) = \ln \left(\frac{x^2 - 3x + 2}{x+1} \right)$

$$3. f(x) = \sqrt{\ln \frac{5x-x^2}{4}}$$

$$4. f(x) = \arcsin \left(\log \frac{x}{10} \right)$$

$$5. f(x) = \frac{\sqrt{|2x+5|-3}}{x^2+1}$$

$$6. f(x) = \frac{1}{\cos^2 x - \frac{1}{4}}$$

$$7. f(x) = \log [1 - \log (x^2 - 5x + 16)]$$

$$8. f(x) = \frac{\sqrt{1-x^2} \ln(x^2-x)}{x^2+2x-3}$$

$$9. \arcsin\left(\frac{x-5}{2}\right) + \log(6-x)$$

$$10. \arcsin(1-x) + \ln(|\ln x|)$$

$$11. f(x) = \frac{1}{x^2 - 4} + \frac{\sqrt{x-2}}{x^2 + 1} + \sqrt{x-1}$$

LIMITS

Before giving an informal definition of limit let's look at behavior of a function near a point.

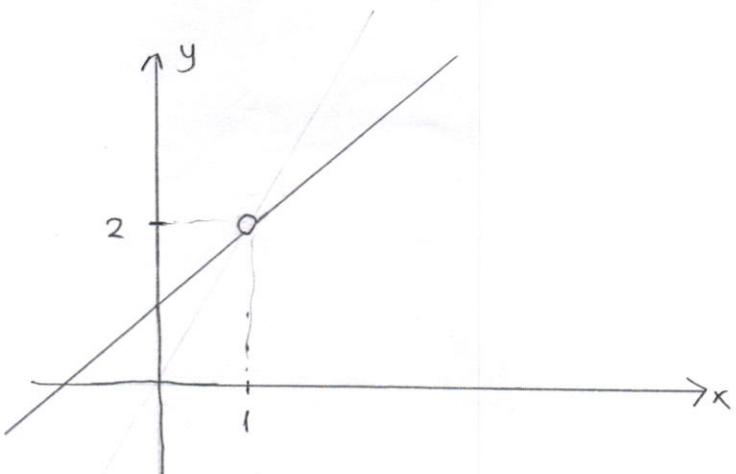
Example. What is the behavior of the function

$$f(x) = \frac{x^2 - 1}{x - 1} \quad \text{near } x=1?$$

$f(x)$ is defined for all real numbers x , except $x=1$. For any $x \neq 1$, we can write $f(x)$ as

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{(x-1)} = (x+1)$$

The graph of $f(x)$ is a line; but there is a hole at the point $(1, 2)$



$f(1)$ is not defined, but we can make the value of $f(x)$ as close as we want to 2 by choosing x close enough to 1.

Therefore, we can say that $f(x)$ approaches arbitrarily

close to 2 as x approaches 1. We can write as

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2$$

An Informal Definition of Limit

If $f(x)$ is defined for all x near a and if we can guarantee that $f(x)$ is as close as we want to L by taking x close enough to a ; but not equal to a , we say that $f(x)$ approaches to the limit L as x approaches to a .

$$\lim_{x \rightarrow a} f(x) = L$$

An Informal Definition of Left and Right Limit

If $f(x)$ is defined on some interval (b, a) extending to the left of $x=a$ and if we can guarantee that $f(x)$ is as close as we want to L by taking x to the left of a and close enough to a , we say that $f(x)$ has **left limit L** at $x=a$ and we write

$$\lim_{x \rightarrow a^-} f(x) = L$$

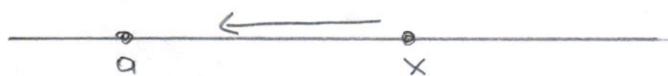


$x \rightarrow a^-$ means x approaches a from the left

P.S: $\lim_{x \rightarrow a^-} f(x) = \lim_{h \rightarrow 0^+} f(a-h)$

If $f(x)$ is defined on some interval (a, b) extending to the right of $x=a$ and if we can guarantee that $f(x)$ is as close as we want to L by taking x to the right of a and close enough to a , we say that $f(x)$ has **right limit** L at $x=a$ and we write

$$\lim_{x \rightarrow a^+} f(x) = L$$



$x \rightarrow a^+$ means x approaches a from the right

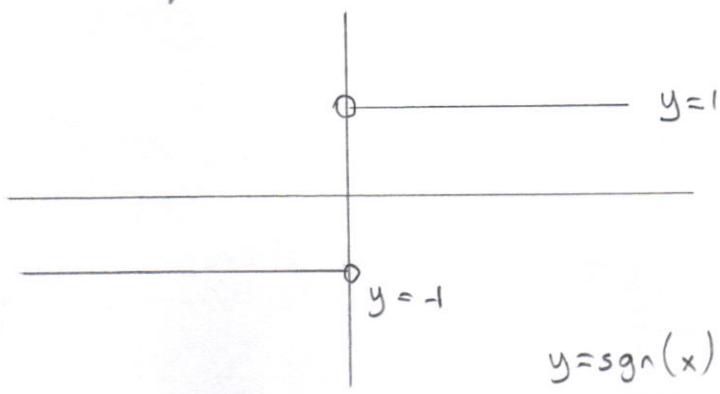
P.S: $\lim_{x \rightarrow a^+} f(x) = \lim_{h \rightarrow 0^+} f(a+h)$

Example. what is the limit of the signum function

$$\operatorname{sgn}(x) = \frac{x}{|x|}$$

$$\lim_{x \rightarrow 0^+} \operatorname{sgn}(x) = \frac{x}{x} = 1 \quad \lim_{x \rightarrow 0^-} \frac{x}{-x} = -1$$

You can easily see that sgn function has left and right limits; but **does not have limit !!!**



Relation Between One-Sided and Two-Sided Limits

A function $f(x)$ has limit L at $x=a$ if and only if it has both left and right limits there and these one-sided limits are both equal to L : and these one-sided limits are both equal to L

$$\lim_{x \rightarrow a} f(x) = L \Leftrightarrow \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = L$$

Example. If $f(x) = \frac{|x-2|}{x^2+x-6}$, find $\lim_{x \rightarrow 2^+} f(x)$, $\lim_{x \rightarrow 2^-} f(x)$

and $\lim_{x \rightarrow 2} f(x)$.

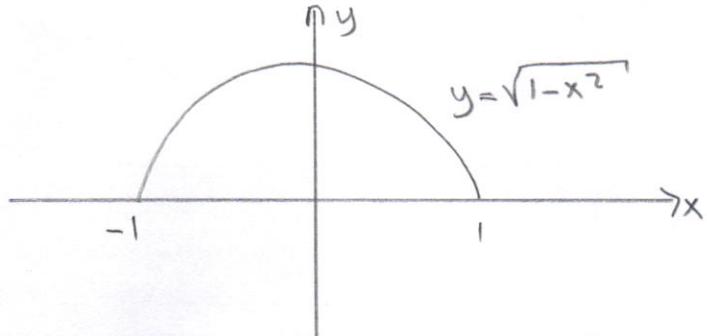
$$\lim_{x \rightarrow 2^+} \frac{(x-2)}{(x-2)(x+3)} = \lim_{x \rightarrow 2^+} \frac{1}{x+3} = \frac{1}{5}$$

$$\lim_{x \rightarrow 2^-} \frac{-(x-2)}{(x-2)(x+3)} = \lim_{x \rightarrow 2^-} \frac{-1}{x+3} = -\frac{1}{5}$$

Since $\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$ $\lim_{x \rightarrow 2} f(x)$ does not exist!

Example. what one-sided limits goes $g(x) = \sqrt{1-x^2}$ have at $x=-1$ and $x=1$?

The domain of $g(x)$ is $[-1, 1]$, so $g(x)$ is defined only to the right of $x=-1$ and only to the left $x=1$.



$$\lim_{x \rightarrow -1^+} g(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 1^-} g(x) < 0$$

$g(x)$ has no left limit or limit at $x = -1$

$g(x)$ has no right limit or limit at $x = 1$

Example. $f(x) = \begin{cases} x-1 & \text{if } x \leq -1 \\ x^2+1 & \text{if } -1 < x \leq 0 \\ (x+\pi)^2 & \text{if } x > 0 \end{cases}$

Find the indicated limits.

a) $\lim_{x \rightarrow -1^-} f(x)$ b) $\lim_{x \rightarrow -1^+} f(x)$ c) $\lim_{x \rightarrow 0^+} f(x)$ d) $\lim_{x \rightarrow 0^-} f(x)$

a) $\lim_{x \rightarrow -1^-} (x-1) = -2$ c) $\lim_{x \rightarrow 0^+} (x+\pi)^2 = \pi^2$

b) $\lim_{x \rightarrow -1^+} x^2+1 = 2$ d) $\lim_{x \rightarrow 0^-} x^2+1 = 1$

We can see that $f(x)$ does not have limits at $x = -1$ and $x = 0$.

Rules for Calculating Limits

If $\lim_{x \rightarrow a} f(x) = L$, $\lim_{x \rightarrow a} g(x) = M$ and k is a constant,

then

L. $\lim_{x \rightarrow a} [f(x) \mp g(x)] = L \mp M$

$$2. \lim_{x \rightarrow a} f(x)g(x) = L \cdot M$$

$$3. \lim_{x \rightarrow a} kf(x) = kL$$

$$4. \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{M} \quad \text{if } M \neq 0$$

5. If m is an integer and n is a positive integer, then
 $\lim_{x \rightarrow a} [f(x)]^{m/n} = L^{m/n}$ provided $L > 0$ if n is even
 and $L \neq 0$ if $m < 0$

6. If $f(x) \leq g(x)$ on an interval containing a in its interior, then

$$L \leq M$$

$$\text{If } f(x) \leq g(x)$$

7. If $P(x)$ is a polynomial and a is any real number, then

$$\lim_{x \rightarrow a} P(x) = P(a)$$

8. If $P(x)$ and $Q(x)$ are polynomials and $Q(a) \neq 0$, then

$$\lim_{x \rightarrow a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}$$

The Sandwich Theorem

Suppose that $f(x) \leq g(x) \leq h(x)$ holds for all x in some open interval containing a , except possibly at $x=a$ itself. Suppose also that

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L \Rightarrow \lim_{x \rightarrow a} g(x) = L$$

Example. Using the sandwich theorem, show that

$$\lim_{x \rightarrow 0} x^2 \cos \frac{1}{x} = 0.$$

we know that $-1 \leq \cos \frac{1}{x} \leq 1$

$$\underline{\frac{-x^2}{f(x)}} \leq \underline{\frac{x^2 \cos \frac{1}{x}}{g(x)}} \leq \overline{h(x)}$$

$$\text{Since } \lim_{x \rightarrow 0} -x^2 = \lim_{x \rightarrow 0} x^2 = 0, \text{ then } \lim_{x \rightarrow 0} x^2 \cos \frac{1}{x} = 0.$$

Formal Definition of Limit

Let $f(x)$ be defined on an open interval about x_0 , except possibly at x_0 itself.

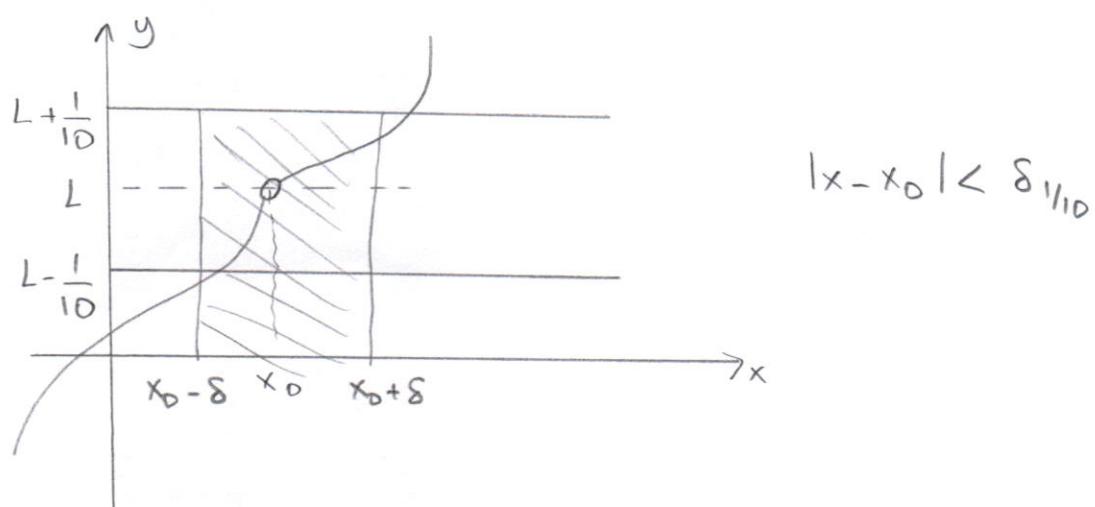
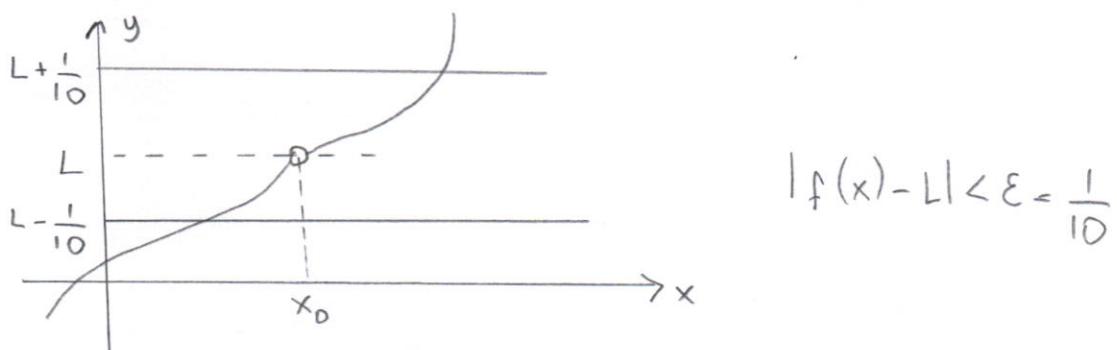
We say that the limit of $f(x)$ as x approaches x_0 is the number L , and write

$$\lim_{x \rightarrow x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta = \delta(\epsilon) > 0$ such that for all x ,

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

What is the meaning of limit geometrically?



Example. Show that $\lim_{x \rightarrow 1} 5x - 3 = 2$

Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x

$$0 < |x - 1| < \delta \text{ implies } |f(x) - L| < \epsilon$$

$$|f(x) - L| = |5x - 3 - 2| = |5(x - 1)| < 5 \cdot \delta = \epsilon \Rightarrow \boxed{\delta = \frac{\epsilon}{5}}$$

Example. Show that $\lim_{x \rightarrow 2} x^2 - 2x - 1 = -1$.

Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x

$$0 < |x - 2| < \delta \text{ implies } |x^2 - 2x - 1 - (-1)| < \epsilon$$

$$|x^2 - 2x| = |x||x - 2| < |x|\cdot\delta$$

$$(0 < |x - 2| < \delta \Rightarrow -\delta < x - 2 < \delta \Rightarrow 2 - \delta < x < \delta + 2)$$

$$< (2 + \delta)\delta = \epsilon \Rightarrow \delta^2 + 2\delta - \epsilon = 0$$

$$\delta_{1,2} = \frac{-2 \pm \sqrt{4 + 4\epsilon}}{2} = -1 \pm \sqrt{1 + \epsilon} \quad \begin{cases} \delta_1 = -1 + \sqrt{1 + \epsilon} \\ \delta_2 = -1 - \sqrt{1 + \epsilon} \end{cases}$$

Since $\delta_2 < 0$, only δ_1 possess the limit.

Example. Show that $\lim_{x \rightarrow 2} \sqrt{x+2} = b$

Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x

$$0 < |x - 2| < \delta \text{ implies } |\sqrt{x+2} - b| < \epsilon$$

Here we have to find a relationship between

$$|x-2| \text{ and } |\sqrt{x+2} - 2|$$

$$x-2 = (\sqrt{x+2} - 2)(\sqrt{x+2} + 2)$$

$$|x-2| = |\sqrt{x+2} - 2| |\sqrt{x+2} + 2|$$

$$\text{Since } \sqrt{x+2} + 2 > 1, \quad |x-2| > \sqrt{x+2} - 2$$

$$|\sqrt{x+2} - 2| < |x-2| < \delta = \varepsilon$$

Limits at infinity of Rational Functions

To determine the limit of a rational function as $x \rightarrow \pm\infty$, we face three possibilities:

1. If degree of numerator is the same with the degree of denominator

Example. Find the limit

$$\lim_{x \rightarrow \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} = \lim_{x \rightarrow \infty} \frac{x^2 \left(5 + \frac{8}{x} - \frac{3}{x^2}\right)}{x^2 \left(3 + \frac{2}{x^2}\right)} = \frac{5}{3}$$

2. If degree of numerator is less than the degree of denominator.

Example. Find the limit

$$\lim_{x \rightarrow -\infty} \frac{11x+2}{2x^3 - 1} = \lim_{x \rightarrow -\infty} \frac{x \left(11 + \frac{2}{x}\right)}{x \left(2x^2 - \frac{1}{x}\right)} = 0$$

3. If degree of numerator is greater than the degree of denominator

Example. Find the limit

$$\lim_{x \rightarrow \infty} \frac{2x^3 + 4}{2x - 1} = \lim_{x \rightarrow \infty} \frac{x(2x^2 + \frac{4}{x})}{x(2 - \frac{1}{x})} = \frac{\infty}{\infty}$$

Some Notes About Limits

1. $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 ; \lim_{x \rightarrow \infty} \frac{\sin x}{x} = \frac{1}{\infty} \cdot \text{sin } \infty = 0 \cdot 0 = 0$

2. $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0 ; \lim_{x \rightarrow 0} \frac{\tan x}{x} = \lim_{x \rightarrow 0} \frac{\arctan x}{x} = 1$

3. $\lim_{x \rightarrow \infty} e^x = \infty ; \lim_{x \rightarrow -\infty} e^x = 0$

4. $\lim_{x \rightarrow \infty} \ln x = \infty ; \lim_{x \rightarrow 0^+} \ln x = -\infty ; \lim_{x \rightarrow 0} \frac{\ln(x+1)}{x} = 1$

5. $0 < a < 1 \quad \lim_{x \rightarrow \infty} a^x = 0 ; \lim_{x \rightarrow -\infty} a^x = \infty$

6. $a > 1 \quad \lim_{x \rightarrow \infty} a^x = \infty ; \lim_{x \rightarrow -\infty} a^x = 0$

7. $\lim_{x \rightarrow 0} \frac{e^{g(x)} - 1}{g(x)} = 1 , \text{ where } \lim_{x \rightarrow 0} g(x) = 0$

8. $\lim_{x \rightarrow 0} (1+ax)^{1/x} = e^a ; \lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^x = e^a$

Limit Examples

$$1. \lim_{x \rightarrow 1} \frac{\sqrt{2x^2-1} - 1}{x-1} = ?$$

$$2. \lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{\sqrt{x}} = ?$$

$$3. \lim_{x \rightarrow \infty} \sqrt{x^2 + 2x} - x = ?$$

$$4. \lim_{x \rightarrow 8} \frac{\sqrt{x} - 2\sqrt{2}}{\sqrt[3]{x} - 2} = ?$$

$$5. \lim_{x \rightarrow 0^+} \frac{1 - \cos x}{x^2} = ?$$

$$6. \lim_{x \rightarrow 0} \frac{\sin x^2 + x \sin 3x}{x \sin x} = ?$$

$$7. \lim_{x \rightarrow 0} \frac{1 - \sqrt{2 - \cos x}}{\sin x} = ?$$

$$8. \lim_{x \rightarrow 0} \frac{e^{ax} - e^{bx}}{x} = ?$$

$$10. \lim_{x \rightarrow \infty} \left(\frac{x+1}{x-1} \right)^{2x-1}$$

$$16. \lim_{x \rightarrow \infty} x^2 \left(1 - \cos \frac{1}{x} \right) = ?$$

$$19. \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2 \sin x} = ?$$

$$20. \lim_{x \rightarrow 0^-} \frac{\sqrt{1 - \cos 2x}}{\sin 2x} = ?$$