第八周练习题

一、求：$\lim\limits_{n\to\infty}(1+\frac{1}{n^2})(1+\frac{2}{n^2})\cdots(1+\frac{n}{n^2})$

解：(1).$(1+\frac{1}{n^2})(1+\frac{2}{n^2})\cdots(1+\frac{n}{n^2})=\mathrm{e}^{\sum\limits_{k=1}^{n}\ln(1+\frac{k}{n^2})}$

$$(2).u_k=\ln(1+\frac{k}{n^2})=\frac{k}{n^2}-\frac{k^2}{2n^4}+o(\frac{1}{n^2})$$

$$(3).\sum_{k=1}^{n}\frac{k}{n^2}-\frac{1}{2n}+o(\frac{1}{n})\le\sum_{k=1}^{n}u_k\le\sum_{k=1}^{n}\frac{k}{n^2}-\frac{1}{2n^3}+o(\frac{1}{n})$$

$$(4).\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2}=\int_0^1 x\mathrm{d}x=\frac{1}{2}$$

$$(5).\lim_{n\to\infty}(1+\frac{1}{n^2})(1+\frac{2}{n^2})\cdots(1+\frac{n}{n^2})=\mathrm{e}^{\frac{1}{2}}$$

二、已知$f(x)$在$[0,+\infty)$上有二阶连续导数,$f(0)=f'(0)=0.f''(x)>0$。若$g(x)$是曲线

$y=f(x)$过切点$(x,f(x))$的切线在x轴上的截距。求：$\lim\limits_{x\to0^+}\dfrac{\int_0^{g(x)}f(t)\mathrm{d}t}{\int_0^x f(t)\mathrm{d}t}$

解：(1). $g(x)=-\dfrac{f(x)}{f'(x)}+x,f(x):\ \dfrac{1}{2}f''(0)x^2,f'(x):\ f''(0)x,g(x):\ \dfrac{1}{2}x$

$$(2).\lim_{x\to0^+}\frac{\int_0^{g(x)}f(t)dt}{\int_0^x f(t)dt}=\lim_{x\to0^+}\frac{f[g(x)]g'(x)}{f(x)}=\lim_{x\to0^+}\frac{\frac{1}{8}x^2f''(0)\frac{f(x)f''(x)}{[f'(x)]^2}}{f(x)}$$

$$=\frac{1}{8}\lim_{x\to0^+}\frac{x^2}{[f'(x)]^2}[f''(0)]^2=\frac{1}{8}[\lim_{x\to0^+}\frac{1}{f''(x)}\ f''(0)]^2=\frac{1}{8}$$

三、已知曲线Γ：$\begin{cases}x^2=3y,\\2xy=9z\end{cases}$。求证：曲线$\Gamma$的任一切线与一定向量成一定角。

证：Γ：$\begin{cases}y=\dfrac{x^2}{3},\\z=\dfrac{2x^3}{27}\end{cases}\Rightarrow\begin{cases}y'=\dfrac{2x}{3},\\z'=\dfrac{2x^2}{9}\end{cases}\Rightarrow$切线方向$\vec{v}=(1,\dfrac{2x}{3},\dfrac{2x^2}{9})\Rightarrow|\vec{v}|=1+\dfrac{2x^2}{9}$；

取定向量$\vec{u}=(1,0,1)$。则$\cos\langle\vec{u},\vec{v}\rangle=\dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}=\dfrac{\sqrt{2}}{2}\Rightarrow\langle\vec{u},\vec{v}\rangle=\dfrac{\pi}{4}$。

即：曲线Γ的任一切线与一定向量$\vec{u}=(1,0,1)$成向量45°定角

四、已知可微函数$f(x)$满足:$|f'(x)|\le|f(x)|$且$f(a)=0$. 求证：$f(x)\equiv 0$.

证：（1）。$x\in[a,a+1]$。

$$|f(x)|=|f(x)-f(a)|=|f'(x_1)|(x-a)\le|f(x_1)|(x-a),\ (a<x_1<x\le 1)$$

同理：$|f(x_1)|<|f(x_2)|(x_1-a),\ (a<x_2<x_1<x\le 1)$，

$|f(x_2)|<|f(x_3)|(x_2-a)<|f(x_3)|(x_1-a),\ (a<x_3<x_2<x_1<x\le 1)$

$\therefore|f(x)|\le|f(x_n)|(x-a)(x_1-a)^{n-1},\ (a<x_n<x_1<x\le 1)$

因$f(x)$在$[a,a+1]$连续，所以$f(x)$在$[a,a+1]$有界，所以存在M，使得$|f(x)|\le M$

$\therefore|f(x)|\le M(x-a)(x_1-a)^{n-1}$，又$(x_1-a)<1\therefore\lim\limits_{n\to\infty}(x_1-a)^{n-1}=0$，

所以$f(x)\equiv 0, x\in[a,a+1]$

（2）由$f(a+1)=0,$同理可得$f(x)\equiv 0, x\in[a+1,a+2]$。

$x\in[a+2,a+3],\mathrm{L}\ [a+n,a+n+1], f(x)\equiv 0$

因此$x\in[a,+\infty), f(x)\equiv 0$

（3）同理可证$x\in(-\infty,a], f(x)\equiv 0$.

五、已知$f(x)$是周期为2π的连续函数，$a_0,a_n,b_n\ (n=1,2,3,\cdots)$为其傅里叶系数，

$A_0,A_n,B_n\ (n=1,2,3,\cdots)$为$F(x)=\dfrac{1}{\pi}\int_{-\pi}^{\pi}f(t)f(x+t)\mathrm{d}t$的傅里叶系数，

(1).试用a_0,a_n,b_n表示A_0,A_n,B_n；(2)求证：$\dfrac{1}{\pi}\int_{-\pi}^{\pi}f^2(t)\mathrm{d}t=\dfrac{a_0^2}{2}+\sum\limits_{n=1}^{\infty}\left(a_n^2+b_n^2\right)$.

解；（1）$F(x)=\dfrac{1}{\pi}\int_{-\pi}^{\pi}f(t)f(x+t)\mathrm{d}t$的周期为$2\pi$，且为偶函数，所以$B_n=0$；

（2）

$$A_0=\frac{1}{\pi}\int_{-\pi}^{\pi}F(x)\mathrm{d}x=\frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)f(x+t)\mathrm{d}t\right]dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)f(x+t)\mathrm{d}x\right]\mathrm{d}t$$

$$\overset{令x+t=u}{=}\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\left[\frac{1}{\pi}\int_{-\pi+t}^{\pi+t}f(u)\mathrm{d}u\right]\mathrm{d}t=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\mathrm{d}t\left(\frac{1}{\pi}\int_{-\pi}^{\pi}f(u)\mathrm{d}u\right)=a_0^2$$

（3）。$A_n=\frac{1}{\pi}\int_{-\pi}^{\pi}F(x)\cos nx\mathrm{d}x=\frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)f(x+t)\mathrm{d}t\right]\cos x\mathrm{d}x$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)f(x+t)\cos nx\mathrm{d}x\right]\mathrm{d}t$$

$$\overset{x+t=u}{=}\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\left[\frac{1}{\pi}\int_{-\pi+t}^{\pi+t}f(u)\cos n(u-t)\mathrm{d}u\right]\mathrm{d}t$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\left[\frac{1}{\pi}\int_{-\pi+t}^{\pi+t}f(u)\left(\cos nu\cos nt+\sin nu\sin nt\right)\mathrm{d}u\right]\mathrm{d}t$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos nt\mathrm{d}t\left(\frac{1}{\pi}\int_{-\pi}^{\pi}f(u)\cos nu\mathrm{d}u\right)+\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin nt\mathrm{d}t\left(\frac{1}{\pi}\int_{-\pi}^{\pi}f(u)\sin nu\mathrm{d}u\right)$$

$$=a_n^2+b_n^2$$

（4）$F(x)$是周期为2π的连续函数，所以

$$\frac{A_0}{2}+\sum_{n=1}^{\infty}A_n\cos nx=F(x)\Rightarrow\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)\cos nx=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)f(x+t)\mathrm{d}t$$

令$x=0$得：$\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)=\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(t)\mathrm{d}t$

六、证明：$\sum_{k=1}^{n}\frac{1}{k^2}\sqrt{\arctan\frac{1}{k^2+k+1}}\le\sqrt{\frac{\pi}{3}},(n\ge 1)$.

证明：由柯西不等式，得

$$\sum_{k=1}^{n}\frac{1}{k^2}\sqrt{\arctan\frac{1}{k^2+k+1}}\le(\sum_{k=1}^{n}\frac{1}{k^4})^{\frac{1}{2}}(\sum_{k=1}^{n}\arctan\frac{1}{k^2+k+1})^{\frac{1}{2}}.$$

因为$\arctan\frac{1}{k^2+k+1}=\arctan\frac{k+1-k}{1+k(k+1)}=\arctan(k+1)-\arctan k$，故

$$\sum_{k-1}^{n}\arctan\frac{1}{k^2+k+1}=\arctan(n+1)-\arctan 1=\arctan(n+1)-\frac{\pi}{4}.$$

注意到，$\sum\limits_{k=1}^{n}\frac{1}{k^4}=1+\sum\limits_{k=2}^{n}\frac{1}{k^4}\le 1+\int_1^n\frac{\mathrm{d}x}{x^4}=\frac{4}{3}-\frac{1}{3n^3}$. 因此，

$$\sum_{k=1}^{n}\frac{1}{k^2}\sqrt{\arctan\frac{1}{k^2+k+1}}\le(\frac{4}{3}-\frac{1}{3n^3})^{\frac{1}{2}}(\arctan(n+1)-\frac{\pi}{4})^{\frac{1}{2}}$$

$$\le(\frac{4}{3})^{\frac{1}{2}}(\frac{\pi}{2}-\frac{\pi}{4})^{\frac{1}{2}}=\sqrt{\frac{\pi}{3}}.$$

七、设 $0<x<\frac{\pi}{2}$，证明：$\frac{1}{\sin^2 x}\le\frac{1}{x^2}+1-\frac{4}{\pi^2}$.

证明：令 $f(x)=\frac{1}{\sin^2 x}-\frac{1}{x^2}$，因为

$$\lim_{x\to 0}f(x)=\lim_{x\to 0}(\frac{1}{\sin^2 x}-\frac{1}{x^2})=\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^4}$$

$$=\lim_{x\to 0}\frac{x+\sin x}{x}\frac{x-\sin x}{x^3}=2\lim_{x\to 0}\frac{x-\sin x}{x^3}$$

$$=\frac{2}{3}\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{3}.$$

$$\lim_{x\to\frac{\pi}{2}}f(x)=\lim_{x\to\frac{\pi}{2}}(\frac{1}{\sin^2 x}-\frac{1}{x^2})=1-\frac{4}{\pi^2}.$$

补充定义 $f(0)=\frac{1}{3}$，$f(\frac{\pi}{2})=1-\frac{4}{\pi^2}$，则函数 $f(x)$ 在 $[0,\frac{\pi}{2}]$ 上连续.

$$f'(x)=\frac{2}{x^3}-\frac{2\cos x}{\sin^3 x}=2(\frac{1}{x^3}-\frac{\sqrt[3]{\cos x}}{\sin x}).$$

记 $g(x)=\frac{\sin x}{\sqrt[3]{\cos x}}-x$，$x\in(0,\frac{\pi}{2})$. 因为

$$g'(x)=\cos^{\frac{2}{3}}x+\frac{1}{3}\sin^2 x\cdot\cos^{-\frac{4}{3}}x-1,$$

$$g''(x)=\frac{4}{9}\sin^3 x\cdot\cos^{-\frac{7}{3}}x>0$$

由 $g'(0)=0$，可得当 $0<x<\frac{\pi}{2}$ 时，$g'(x)>g'(0)=0$.

同理，因为 $g(0)=0$，则当 $0<x<\frac{\pi}{2}$ 时，$g(x)>g(0)=0$.

从而，当 $0<x<\frac{\pi}{2}$ 时，$f'(x)>0$.

事实上，$f'(\frac{\pi}{2})=\frac{16}{\pi^3}>0$.

故当$0<x<\frac{\pi}{2}$时，$f(x)<f(\frac{\pi}{2})=1-\frac{4}{\pi^2}$，即

$$\frac{1}{\sin^2 x}\le\frac{1}{x^2}+1-\frac{4}{\pi^2}.$$

八、设$a>0$，$f(x)$在$[0,a]$上连续可导，试证：

$$|f(0)|\le\frac{1}{a}\int_0^a|f(x)|\mathrm{d}x+\int_0^a|f'(x)|\mathrm{d}x.$$

证明一：因为$f(x)$在$[0,a]$上连续，由积分中值定理，存在$\xi\in[0,a]$，使得

$$\int_0^a f(x)\mathrm{d}x=f(\xi)a.$$

又因为$f(\xi)-f(0)=\int_0^\xi f'(x)\mathrm{d}x$，所以

$$\begin{aligned}|f(0)|&\le|f(\xi)|+\int_0^\xi|f'(x)|\mathrm{d}x\\&\le\frac{1}{a}\int_0^a f(x)\mathrm{d}x+\int_0^a|f'(x)|\mathrm{d}x\end{aligned}$$

证明二：$f(0)=f(x)-\int_0^x f'(x)\mathrm{d}x$，所以，

$$|f(0)|\le|f(x)|+\int_0^a|f'(x)|\mathrm{d}x.$$

两边从 0 到a积分，得

$$a|f(0)|\le\int_0^a|f(x)|\mathrm{d}x+a\int_0^a|f'(x)|\mathrm{d}x,$$

故 $$|f(0)|\le\frac{1}{a}\int_0^a|f(x)|\mathrm{d}x+\int_0^a|f'(x)|\mathrm{d}x.$$

证明三：因为$|f(x)|$在$[0,a]$上连续，故$f(x)$在$[0,a]$上取得最小值. 因此，存在$c\in[0,a]$，使得$|f(c)|=\min\limits_{x\in[0,a]}|f(x)|$.

于是，$$\begin{aligned}\frac{1}{a}\int_0^a|f(x)|\mathrm{d}x+\int_0^a|f'(x)|\mathrm{d}x&\ge|f(c)|+\int_0^c|f'(x)|\mathrm{d}x\\&\ge|f(c)|+\left|\int_0^c f'(x)\mathrm{d}x\right|\\&\ge|f(c)|+|f(c)-f(0)|\\&\ge|f(c)-[f(c)-f(0)|=|f(0)|.\end{aligned}$$