MATH3075/3975 Financial Mathematics

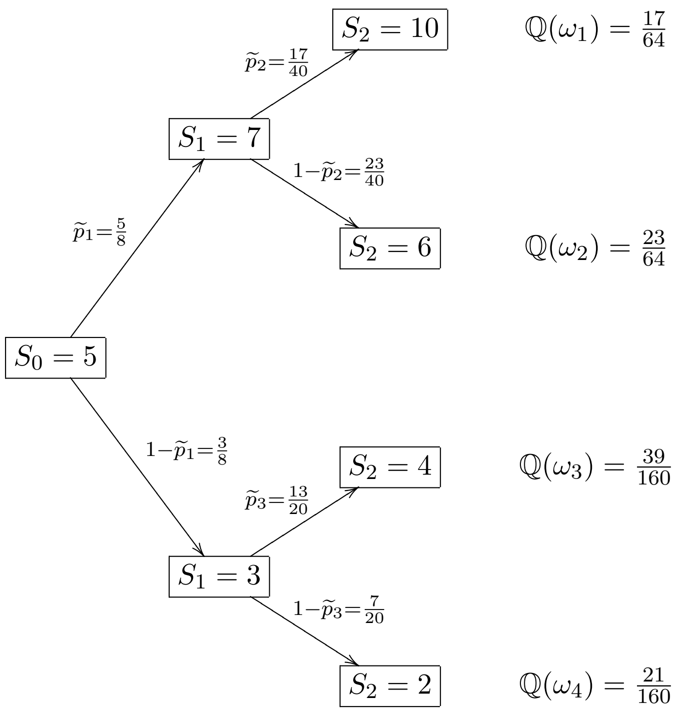
Tutorial 8: Solutions

Exercise 1 (a) We first focus on the conditional risk-neutral probabilities. To compute them, we consider the three embedded single-period two-state models. In each of these models, we may use the single-period formula

$$\widetilde{p} = \frac{1+r-d}{u-d}.$$

Since $r = 0.1$, we obtain

$$\widetilde{p}_1 = \frac{\frac{11}{10} - \frac{3}{5}}{\frac{7}{5} - \frac{3}{5}} = \frac{5}{8}, \quad \widetilde{p}_2 = \frac{\frac{11}{10} - \frac{6}{7}}{\frac{10}{7} - \frac{6}{7}} = \frac{17}{40}, \quad \widetilde{p}_3 = \frac{\frac{11}{10} - \frac{2}{3}}{\frac{4}{3} - \frac{2}{3}} = \frac{13}{20}.$$



We conclude that the unique risk-neutral probability measure $\mathbb{Q}$ for the model $\mathcal{M} = (B, S)$ satisfies

$$\mathbb{Q} = (q_1, q_2, q_3, q_4) = \left(\frac{17}{64}, \frac{23}{64}, \frac{39}{160}, \frac{21}{160}\right).$$

(b) We start by noting that the digital call option has the following payoff at time $T = 2$

$$X = h(S_2) = \big(X(\omega_1), X(\omega_2), X(\omega_3), X(\omega_4)\big) = (1, 0, 0, 0).$$

To compute the replicating strategy for the claim X, we proceed by the backward induction.

- We first consider the replicating portfolio for X at time 1 on the event

$$A_1 = \{S_1 = 7\} = \{\omega_1, \omega_2\}.$$

Let $\widetilde{\varphi}^0$ stand for the amount of cash in the savings account and let φ^1 be the number of shares held. Then we need to solve the linear equations

$$\begin{cases} 1.1\widetilde{\varphi}^0 + 10\varphi^1 = 1, \\ 1.1\widetilde{\varphi}^0 + 6\varphi^1 = 0. \end{cases}$$

We find that $(\widetilde{\varphi}^0, \varphi^1) = \left(-\frac{15}{11}, \frac{1}{4}\right)$. The value of this portfolio equals, on the event A_1,

$$V_1(\varphi) = -\tfrac{15}{11} + \tfrac{1}{4}\,7 = \tfrac{17}{44}.$$

- We now consider the replicating portfolio for X at time 1 on the event

$$A_2 = \{S_1 = 3\} = \{\omega_3, \omega_4\}.$$

By solving the linear equations

$$\begin{cases} 1.1\widetilde{\varphi}^0 + 4\varphi^1 = 0, \\ 1.1\widetilde{\varphi}^0 + 2\varphi^1 = 0, \end{cases}$$

we get $\widetilde{\varphi}^0 = \varphi^1 = 0$ and thus the value of the portfolio equals $V_1(\varphi) = 0$ on the event $\omega \in A_2$.

- Finally consider the replicating portfolio for $V_1(\varphi)$ at time 0. We now solve the equations

$$\begin{cases} 1.1\widetilde{\varphi}^0 + 7\varphi^1 = \frac{17}{44}, \\ 1.1\widetilde{\varphi}^0 + 3\varphi^1 = 0, \end{cases}$$

and we obtain $(\widetilde{\varphi}^0, \varphi^1) = \left(-\frac{255}{968}, \frac{17}{176}\right)$. The value of this portfolio equals, for all $\omega \in \Omega$,

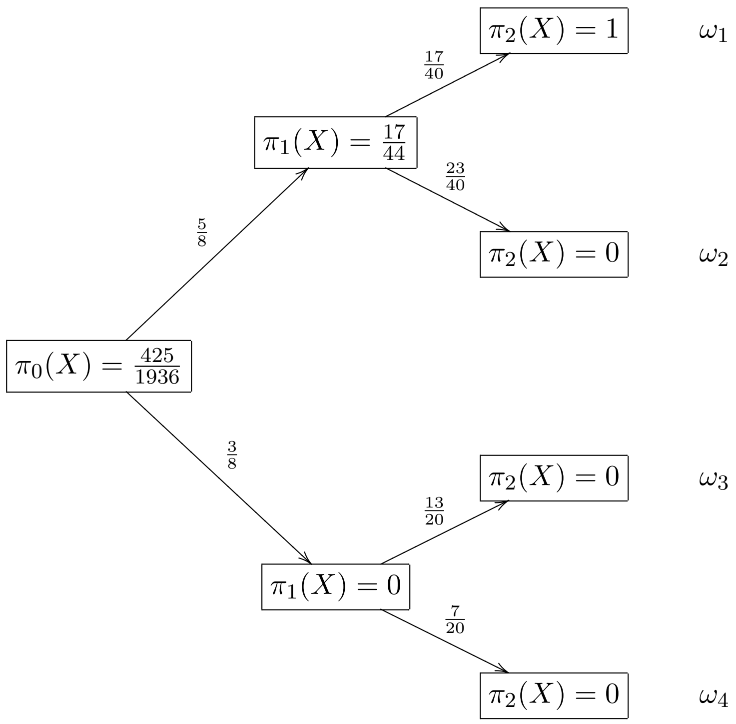
$$V_0(\varphi) = -\tfrac{255}{968} + \tfrac{17}{176}\,5 = \tfrac{425}{1936}.$$

To summarise, the replicating strategy for the claim X and its wealth process are given by:

	$t=0$	$t=1$	
		ω_1, ω_2	ω_4, ω_4
φ_t^0	$-\frac{255}{968}$	$-\frac{150}{121}$	0
φ_t^1	$\frac{17}{176}$	$\frac{1}{4}$	0
$V_t(\varphi)$	$\frac{425}{1936}$	$\frac{17}{44}$	0

This means that we first buy $\frac{17}{176}$ shares of the stock at time 0. If the price of the stock rises during the first period, then we adjust our portfolio at time 1 by purchasing, in addition, $\frac{1}{4} - \frac{17}{176} = \frac{27}{176}$ shares of the stock. However, if the price of the stock declines during the first period, then we sell all $\frac{17}{176}$ shares purchased at time 0 and we pay back our debt with interest; we then end up with null portfolio at time 1 (and thus, obviously, also at time 2).

The arbitrage price process of the digital call option coincides with the wealth process $V(\varphi)$, so it is given by:



(c) We now search for the arbitrage price process of an *Asian option* with the following payoff at time 1

$$Y = \left(\tfrac{1}{3}\big(S_0 + S_1 + S_2\big) - 4\right)^+ = \big(Y(\omega_1), Y(\omega_2), Y(\omega_3), Y(\omega_4)\big) = \big(\tfrac{10}{3}, 2, 0, 0\big).$$

To compute the arbitrage price of the Asian option, we may argue that the model is complete and thus any contingent claim can be replicated. Hence the unique arbitrage price for the Asian option can be computed by a direct application of the risk-neutral valuation formula

$$\pi_t(Y) = B_t\,\mathbb{E}_{\mathbb{Q}}\left(\frac{Y}{B_T}\,\Big|\,\mathcal{F}_t\right)$$

where $\mathbb{Q}$ is the unique risk-neutral probability measure, which was found in part (a). In fact, it is better to rely on the backward induction, that is, the following relationship between $\pi_t(Y)$ and $\pi_{t+1}(Y)$

$$\pi_t(Y) = B_t\,\mathbb{E}_{\mathbb{Q}}\left(\frac{\pi_{t+1}(Y)}{B_{t+1}}\,\Big|\,\mathcal{F}_t\right)$$

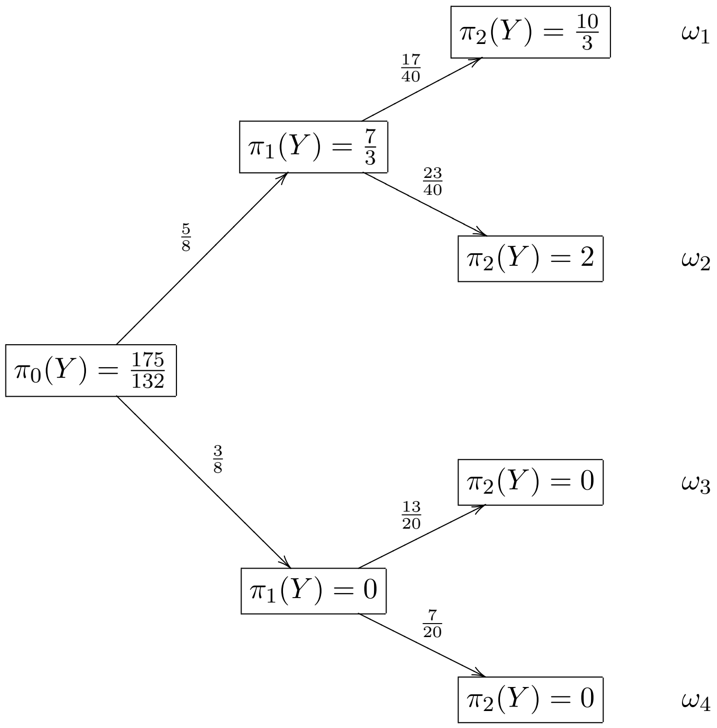
Within the two-period setup, it suffices to compute first the conditional expectation

$$\pi_1(Y) = (1+r)^{-1}\,\mathbb{E}_{\mathbb{Q}}(Y \,|\, \mathcal{F}_1^S) = (1+r)^{-1}\,\mathbb{E}_{\mathbb{Q}}(Y \,|\, S_1)$$

and subsequently the expected value

$$\pi_0(Y) = (1+r)^{-1}\,\mathbb{E}_{\mathbb{Q}}(\pi_1(Y)).$$

Using the conditional risk-neutral probabilities computed in part (a), we obtain the following representation for the arbitrage price of the claim Y



Exercise 2 (a) We first show explicitly that the contingent claim X is path-dependent. We have: $S_0 = 80$, $S_1 = (S_1^u, S_1^d) = (104,\ 88)$ and

$$S_2 = (S_2^{uu}, S_2^{ud} = S_2^{du}, S_2^{dd}) = (135.2,\ 114.4,\ 96.8).$$

Hence for $\omega_2 = (u, d)$ we get

$$X(\omega_2) = \big(S_2(\omega_2) - S_1(\omega_2)\big)\mathbb{1}_{\{S_2(\omega_2) - S_1(\omega_2) > 20\}} = \big(114.4 - 104\big)\mathbb{1}_{\{10.4 > 20\}} = 0$$

and for $\omega_3 = (d, u)$ we get

$$X(\omega_3) = \big(S_2(\omega_3) - S_1(\omega_3)\big)\mathbb{1}_{\{S_2(\omega_3) - S_1(\omega_3) > 20\}} = \big(114.4 - 88\big)\mathbb{1}_{\{26.4 > 20\}} = 26.4.$$

Since $X(\omega_2) \neq X(\omega_3)$, we conclude that the claim is path-dependent.

(b) We will now compute the arbitrage price of the claim X using the risk-neutral valuation formula

$$\pi_t(X) = B_t\, \mathbb{E}_{\widetilde{\mathbb{P}}}\big(X B_T^{-1} \,|\, \mathcal{F}_t\big), \quad t = 0, 1, 2,$$

where $\widetilde{\mathbb{P}}$ is the unique equivalent martingale measure for the model $\mathcal{M} = (B, S)$. We have $u = \frac{104}{80} = 1.3$ and $d = \frac{88}{80} = 1.1$. Consequently,

$$\widetilde{p} = \frac{1 + r - d}{u - d} = \frac{1.2 - 1.1}{1.3 - 1.1} = \frac{0.1}{0.2} = \frac{1}{2}.$$

The claim X can be represented as follows:

$$X = (X(\omega_1), X(\omega_2), X(\omega_3), X(\omega_4)) = (31.2,\ 0,\ 26.4,\ 0).$$

Hence

$$\begin{aligned}
\pi_1^u(X) &= \pi_1(X)(\omega_1) = \pi_1(X)(\omega_2) = \frac{1}{1.2}\cdot\frac{1}{2}\cdot 31.2 = 13,\\
\pi_1^d(X) &= \pi_1(X)(\omega_3) = \pi_1(X)(\omega_4) = \frac{1}{1.2}\cdot\frac{1}{2}\cdot 26.4 = 11,\\
\pi_0(X) &= \frac{1}{1.2}\cdot\frac{1}{2}\cdot(13+11) = 10.
\end{aligned}$$

(c) We now find the replicating portfolio (φ^0, φ^1) for the claim X and check that the equality $V_t(\varphi) = \pi_t(X)$ is satisfied for $t = 0, 1, 2$. For $\omega \in \{\omega_1, \omega_2\}$, we need to solve

$$\begin{aligned}
1.2\,\widetilde{\varphi}_1^0 + 135.2\,\varphi_1^1 &= 31.2,\\
1.2\,\widetilde{\varphi}_1^0 + 114.4\,\varphi_1^1 &= 0.
\end{aligned}$$

We get $(\widetilde{\varphi}_1^0, \varphi_1^1) = (-143,\, 1.5)$ and thus $V_1(\varphi)(\omega) = 13$ for $\omega \in \{\omega_1, \omega_2\}$. For $\omega \in \{\omega_3, \omega_4\}$, we need to solve

$$\begin{aligned}
1.2\,\widetilde{\varphi}_1^0 + 114.4\,\varphi_1^1 &= 26.4,\\
1.2\,\widetilde{\varphi}_1^0 + 96.8\,\varphi_1^1 &= 0.
\end{aligned}$$

Here $(\widetilde{\varphi}_1^0, \varphi_1^1) = (-121,\, 1.5)$ and thus $V_1(\varphi)(\omega) = 11$ for $\omega \in \{\omega_3, \omega_4\}$. At $t = 0$, we solve

$$\begin{aligned}
1.2\,\varphi_0^0 + 104\,\varphi_0^1 &= 13,\\
1.2\,\varphi_0^0 + 88\,\varphi_0^1 &= 11.
\end{aligned}$$

Hence $(\varphi_0^0, \varphi_0^1) = (0, 0.125)$. We check that $V_0(\varphi) = 0.125 \cdot 80 = 10$.

(d) We first show that in any CRR model

$$\mathbb{E}_{\widetilde{\mathbb{P}}}(S_2 - S_1) = r(1+r)S_0.$$

From the definition of $\widetilde{\mathbb{P}}$, we obtain

$$\mathbb{E}_{\widetilde{\mathbb{P}}}(S_1) = (1+r)S_0, \quad \mathbb{E}_{\widetilde{\mathbb{P}}}(S_2) = (1+r)^2 S_0.$$

Hence

$$\mathbb{E}_{\widetilde{\mathbb{P}}}(S_2 - S_1) = \mathbb{E}_{\widetilde{\mathbb{P}}}(S_2) - \mathbb{E}_{\widetilde{\mathbb{P}}}(S_1) = (1+r)^2 S_0 - (1+r)S_0 = r(1+r)S_0.$$

Let us now consider the claim Y with maturity $T = 2$ given by

$$Y = \big(S_2 - S_1\big)\mathbb{1}_{\{S_2 - S_1 \le 20\}}.$$

We wish to find the price of Y at time 0 using the additivity of arbitrage prices and the fact that $X + Y = S_2 - S_1$. We have

$$\pi_0(Y) = \mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{S_2 - S_1}{(1+r)^2}\right) - \pi_0(X) = \frac{r}{(1+r)}\,S_0 - \pi_0(X) = \frac{0.2}{1.2}\,80 - 10 = \frac{10}{3}.$$

It remains to double-check this result by computing

$$\pi_0(Y) = B_0\,\mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{Y}{B_2}\right).$$

We observe that $Y = (0, 10.4, 0, 8.8)$ and thus we obtain

$$\pi_0(Y) = (1+r)^{-2}\,\mathbb{E}_{\widetilde{\mathbb{P}}}(Y) = (1.2)^{-2}\,0.25(10.4+8.8) = \frac{10}{3}.$$

(e) We need to find the unique probability measure $\widehat{\mathbb{P}}$ on $(\Omega, \mathcal{F}_2)$ such that the process $\widehat{B}_t = B_t/S_t$ for $t = 0, 1, 2$ is a martingale under $\widehat{\mathbb{P}}$ with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0,1,2}$. We may use the following equalities

$$\widehat{\mathbb{P}}(\omega_1, \omega_2) = \widehat{p} = \frac{\widetilde{p}u}{1+r} = \frac{13}{24}, \quad \widehat{\mathbb{P}}(\omega_3, \omega_4) = 1 - \widehat{p} = \frac{11}{24}.$$

Hence

$$\widehat{\mathbb{P}} = \left(\left(\frac{13}{24}\right)^2, \frac{13 \cdot 11}{(24)^2}, \frac{11 \cdot 13}{(24)^2}, \left(\frac{11}{24}\right)^2 \right).$$

We have

$$S_0\,\mathbb{E}_{\widehat{\mathbb{P}}}\left(\frac{Y}{S_2}\right) = 80\left(\frac{10.4}{114.4}\frac{13 \cdot 11}{(24)^2} + \frac{8.8}{96.8}\left(\frac{11}{24}\right)^2\right) = \frac{10}{3}.$$

Observe that this result is consistent with the computation performed in part (d).

Exercise 3 (MATH3975) (a) It is assumed that the process X is $\mathbb{F}$-adapted has independent increments with respect to $\mathbb{F}$ under $\mathbb{P}$. We observe that since the process X is $\mathbb{F}$-adapted, the process Y is $\mathbb{F}$-adapted as well since if X_t is $\mathcal{F}_t$-measurable, then $X_t + c$ is $\mathcal{F}_t$-measurable for any constant c. It thus suffices to consider the dates t and $t+1$ for an arbitrary $t = 0, 1, \dots$ and to show that $\mathbb{E}_{\mathbb{P}}(Y_{t+1} \,|\, \mathcal{F}_t) = Y_t$ or, equivalently, $\mathbb{E}_{\mathbb{P}}(Y_{t+1} - Y_t \,|\, \mathcal{F}_t) = 0$.

Since the random variable $X_{t+1} - X_t$ is independent of the σ-field $\mathcal{F}_t$, it is clear that for any $c \in \mathbb{R}$ the random variable $X_{t+1} - X_t + c$ is independent of $\mathcal{F}_t$ as well. Therefore, for any $t = 0, 1, \dots,$

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(Y_{t+1} - Y_t \,|\, \mathcal{F}_t) &= \mathbb{E}_{\mathbb{P}}\big(X_{t+1} - X_t - \mathbb{E}_{\mathbb{P}}(X_{t+1}) + \mathbb{E}_{\mathbb{P}}(X_t) \,|\, \mathcal{F}_t\big) \\
&= \mathbb{E}_{\mathbb{P}}\big(X_{t+1} - X_t - \mathbb{E}_{\mathbb{P}}(X_{t+1}) + \mathbb{E}_{\mathbb{P}}(X_t)\big) \\
&= \mathbb{E}_{\mathbb{P}}(X_{t+1}) - \mathbb{E}_{\mathbb{P}}(X_t) - \mathbb{E}_{\mathbb{P}}(X_{t+1}) + \mathbb{E}_{\mathbb{P}}(X_t) = 0.
\end{aligned}$$

This proves that the process Y is a martingale under $\mathbb{P}$ with respect to the filtration $\mathbb{F}$.

(b1) Recall that $A_0 = 0$ and for all $t = 0, 1, \dots$

$$A_{t+1} - A_t = \mathbb{E}_{\mathbb{P}}\big(X_{t+1} - X_t \,|\, \mathcal{F}_t\big). \tag{1}$$

We wish to show the process $\widetilde{Y}$ satisfying $\widetilde{Y}_t = X_t - A_t$ for $t = 0, 1, \dots$ is a martingale under $\mathbb{P}$ with respect to $\mathbb{F}$. We have, for all $t = 0, 1, \dots,$

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\big(\widetilde{Y}_{t+1} - \widetilde{Y}_t \,|\, \mathcal{F}_t\big) &= \mathbb{E}_{\mathbb{P}}\big(X_{t+1} - X_t - (A_{t+1} - A_t) \,|\, \mathcal{F}_t\big) \\
&= \mathbb{E}_{\mathbb{P}}\big(X_{t+1} - X_t \,|\, \mathcal{F}_t\big) - \big(A_{t+1} - A_t\big) \overset{(1)}{=} 0.
\end{aligned}$$

Hence the process $\widetilde{Y}$ is a martingale. It is worth noting that for every $t = 0, 1, \dots$ the random variable A_{t+1} is $\mathcal{F}_t$-measurable (this can be proven by induction). Moreover, if X is a process of independent increments then $A_t = \mathbb{E}_{\mathbb{P}}(X_t) - \mathbb{E}_{\mathbb{P}}(X_0)$. Hence this case can be seen as a natural extension of part (a).

(b2) On the one hand, we assume that $\widehat{Y}$ is a martingale under $\mathbb{P}$ with respect to $\mathbb{F}$ so that $\mathbb{E}_{\mathbb{P}}\big(\widehat{Y}_{t+1} - \widehat{Y}_t \,|\, \mathcal{F}_t\big) = 0$ for all $t = 0, 1, \dots$. On the other hand, $\widehat{Y}_t = X_t - \widehat{A}_t$ for all $t = 0, 1, \dots$ and thus

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\big(\widehat{Y}_{t+1} - \widehat{Y}_t \,|\, \mathcal{F}_t\big) &= \mathbb{E}_{\mathbb{P}}\big(X_{t+1} - X_t - (\widehat{A}_{t+1} - \widehat{A}_t) \,|\, \mathcal{F}_t\big) \\ &= \mathbb{E}_{\mathbb{P}}\big(X_{t+1} - X_t \,|\, \mathcal{F}_t\big) - \big(\widehat{A}_{t+1} - \widehat{A}_t\big)\end{aligned}$$

where the second equality holds since $\widehat{A}$ is assumed to be $\mathbb{F}$-predictable. We conclude that for all $t = 0, 1, \dots$

$$\widehat{A}_{t+1} - \widehat{A}_t = \mathbb{E}_{\mathbb{P}}\big(X_{t+1} - X_t \,|\, \mathcal{F}_t\big) = A_{t+1} - A_t.$$

Since $A_0 = \widehat{A}_0 = 0$, we conclude that $A_t = \widehat{A}_t$ for all $t = 0, 1, \dots$.

(c1) We need to prove that: a game X is fair $\Leftrightarrow$ X is a martingale.

($\Leftarrow$) Let us first assume that the process X is a martingale under $\mathbb{P}$ with respect to $\mathbb{F}$. We fix $t \in \{1, 2, \dots, T\}$ and we consider an arbitrary gambling strategy H. Using the properties of conditional expectation and the fact that H is $\mathbb{F}$-adapted, we obtain

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}(G_t) &= \mathbb{E}_{\mathbb{P}}\Big(\sum_{u=0}^{t-1} H_u(X_{u+1} - X_u)\Big) = \sum_{u=0}^{t-1} \mathbb{E}_{\mathbb{P}}\Big(\mathbb{E}_{\mathbb{P}}\big(H_u(X_{u+1} - X_u) \,|\, \mathcal{F}_u\big)\Big) \\ &= \sum_{u=0}^{t-1} \mathbb{E}_{\mathbb{P}}\Big(H_u\, \mathbb{E}_{\mathbb{P}}\big(X_{u+1} - X_u \,|\, \mathcal{F}_u\big)\Big) = \sum_{u=0}^{t-1} \mathbb{E}_{\mathbb{P}}(H_u \cdot 0) = 0\end{aligned}$$

since the martingale property of X means that $\mathbb{E}_{\mathbb{P}}\big(X_{u+1} - X_u \,|\, \mathcal{F}_u\big) = 0$ for every $u = 0, 1, \dots, t-1$

($\Rightarrow$) We now assume that the game is fair. Let us fix t and let us consider a gambling strategy, which equals zero for all $u \neq t$ and $H_t = \mathbb{1}_A$ for an arbitrary event A from $\mathcal{F}_t$. Since the game is fair, we have that, for any event $A \in \mathcal{F}_t$,

$$\mathbb{E}_{\mathbb{P}}(G_t) = \mathbb{E}_{\mathbb{P}}\big(H_t(X_{t+1} - X_t)\big) = \mathbb{E}_{\mathbb{P}}\big(\mathbb{1}_A(X_{t+1} - X_t)\big) = 0.$$

From the definition of conditional expectation, the last equality in turn implies that

$$\mathbb{E}_{\mathbb{P}}\big(X_{t+1} - X_t \,|\, \mathcal{F}_t\big) = 0.$$

Since the date t was arbitrary, we conclude that X is a martingale under $\mathbb{P}$ with respect to $\mathbb{F}$.

(c2) Let A be an $\mathbb{F}$-predictable process such that

$$A_{t+1} - A_t = \mathbb{E}_{\mathbb{P}}(X_{t+1} - X_t \,|\, \mathcal{F}_t).$$

We now assume that a player is required to 'pay' at time t an upfront fee $A_{t+1} - A_t$ per one unit of the bet (note that $A_{t+1} - A_t$ is not necessarily positive). Then the cumulative profits/losses generated by a gambling strategy H by time t are given by the following expression

$$G_t = \sum_{u=0}^{t-1} H_u\big(X_{u+1} - X_u - (A_{u+1} - A_u)\big) = \sum_{u=0}^{t-1} H_u\big(\widetilde{Y}_{u+1} - \widetilde{Y}_u\big)$$

where we denote $\widetilde{Y} = X - A$. From part (b1), we know that the process $\widetilde{Y}$ is a martingale and thus, by part (c1), we obtain $\mathbb{E}_{\mathbb{P}}(G_t) = 0$ for all $t = 1, 2, \dots, T$. Put another way, if the fee $A_{t+1} - A_t$ is charged at time t per one unit of the bet then a general game X becomes a fair game $\widetilde{Y}$.