Chapter 2: First order equations

Michele Coti Zelati[♯]

ABSTRACT. These notes follow closely the book of S. Salsa [1].

CONTENTS

1. Linear transport equations

We start by consider the simplest possible first order partial differential equation. Consider a fluid, water, say, flowing at a constant rate $a > 0$ along a horizontal pipe of fixed cross section in the positive x direction. A substance, say a pollutant, is suspended in the water. Calling $\rho(t, x)$ its concentration at position x and at time t, the amount of pollutant in the interval $[0, b]$ at the time t is

$$M = \int_0^b \rho(t, x)\mathrm{d}x. \tag{1.1}$$

At the later time $t + h$, the same molecules of pollutant have moved to the right by $a \cdot h$. Hence

$$M = \int_0^b \rho(t, x)\mathrm{d}x = \int_{ah}^{b+ah} \rho(t + h, x)\mathrm{d}x. \tag{1.2}$$

Differentiating with respect to b, we get

$$\rho(t, b) = \rho(t + h, b + ah). \tag{1.3}$$

[♯] *m.coti-zelati@imperial.ac.uk*, Department of Mathematics, Imperial College London, London, SW7 2AZ, UK.

Differentiating with respect to h and putting $h = 0$, we get

$$0 = \partial_t \rho + a\partial_x \rho, \tag{1.4}$$

which is the first PDE that we will study.

1.1. Pure transport. Consider the *pure transport* equation

$$\partial_t \rho + a\partial_x \rho = 0. \tag{1.5}$$

Introducing the vector

$$\boldsymbol{a} = \begin{pmatrix} a \\ 1 \end{pmatrix}, \tag{1.6}$$

we immediately see from (1.5) that

$$\nabla \rho \cdot \boldsymbol{a} = a\partial_x \rho + \partial_t \rho = 0. \tag{1.7}$$

Therefore $\nabla \rho$ and $\boldsymbol{a}$ are orthogonal. However, $\nabla \rho$ is also orthogonal to the level lines of ρ, along which ρ is constant. Therefore the level lines of ρ are the straight lines parallel to $\boldsymbol{a}$, of equation

$$x = at + x_0, \tag{1.8}$$

for an arbitrary $x_0 \in \mathbb{R}$. These straight lines are called *characteristics* (see Figure 1).

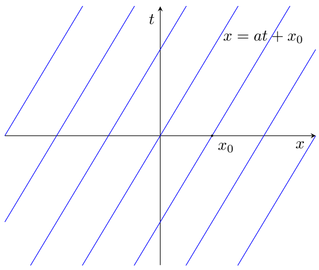


FIGURE 1. The characteristic lines in the (x, t) plane.

Computing ρ along the characteristics and using the transport equation (1.5), we rightfully find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\rho(t, at + x_0)\right] = (\partial_t \rho + a\partial_x \rho)(t, at + x_0) = 0. \tag{1.9}$$

Hence, ρ is constant along the characteristics. In particular, given an initial condition

$$\rho(0, x) = g(x), \qquad x \in \mathbb{R}, \tag{1.10}$$

for an assigned function g, the above calculation tells us that

$$\rho(t, at + x_0) = \rho(0, x_0) = g(x_0). \tag{1.11}$$

Using (1.8), or simply changing variables, we can then write the solution to the (1.5) with initial condition (1.10) as

$$\rho(t, x) = g(x - at). \tag{1.12}$$

The solution (1.12) represents a *traveling wave*, moving with speed a in the positive x-direction (see Figure 2).

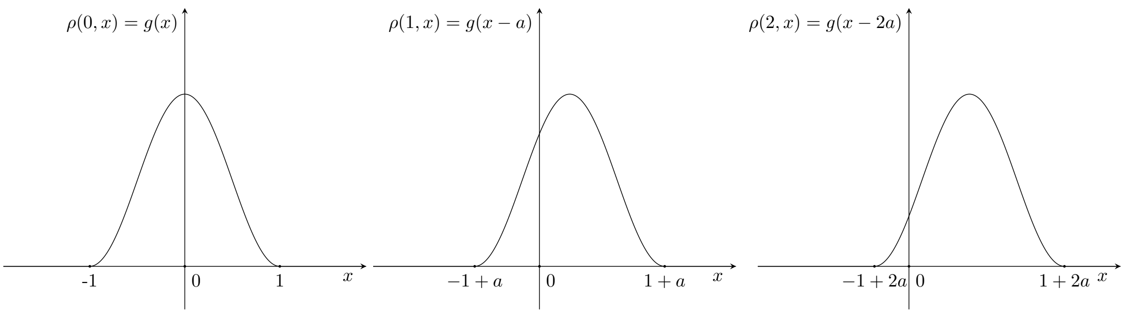


FIGURE 2. The solution of the pure transport equation, at times $t = 0, 1, 2$. It is simply a translation of the initial datum, to the right since $a > 0$.

1.2. Distributed source. Given $f = f(t,x)$, consider the initial value problem

$$\begin{cases} \partial_t \rho + a\partial_x \rho = f(t,x), & t > 0,\ x \in \mathbb{R}, \\ \rho(0,x) = g(x), & x \in \mathbb{R}. \end{cases} \tag{1.13}$$

The function f represents the intensity of an external distributed source along the channel. Using the characteristics and arguing as in (1.9), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\rho(t, at + x_0)\right] = f(t, at + x_0). \tag{1.14}$$

Integrating on $(0,t)$, we find

$$\rho(t, at + x_0) = g(x_0) + \int_0^t f(s, as + x_0)\mathrm{d}s. \tag{1.15}$$

Therefore, the solution to (1.13) is

$$\rho(t,x) = g(x - at) + \int_0^t f(s, x - a(t-s))\mathrm{d}s. \tag{1.16}$$

1.3. Damped traveling waves. Suppose that the density ρ decays at the rate $-\delta\rho$, for some $\delta > 0$. This could be due to biological decomposition if the density ρ models the concentration of a chemical substance in water. Without external sources, the mathematical model to study becomes

$$\begin{cases} \partial_t \rho + a\partial_x \rho = -\delta\rho, & t > 0,\ x \in \mathbb{R}, \\ \rho(0,x) = g(x), & x \in \mathbb{R}. \end{cases} \tag{1.17}$$

Again, from an analogous computation as in (1.9), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\rho(t, at + x_0)\right] = -\delta\rho(t, at + x_0). \tag{1.18}$$

This is exactly an ODE of the type $y' = -\delta y$, where $y(t) = \rho(t, at + x_0)$. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{\delta t}\rho(t, at + x_0)\right] = 0, \tag{1.19}$$

and integrating we obtain

$$\mathrm{e}^{\delta t}\rho(t, at + x_0) = g(x_0). \tag{1.20}$$

Thus, the solution to (1.17) is

$$\rho(t,x) = \mathrm{e}^{-\delta t} g(x - at). \tag{1.21}$$

Notice that this a traveling wave like (1.12), except that the amplitude decays in time (if, for example, g is a bounded function). For this reason, it is called damped traveling wave (see Figure 3).

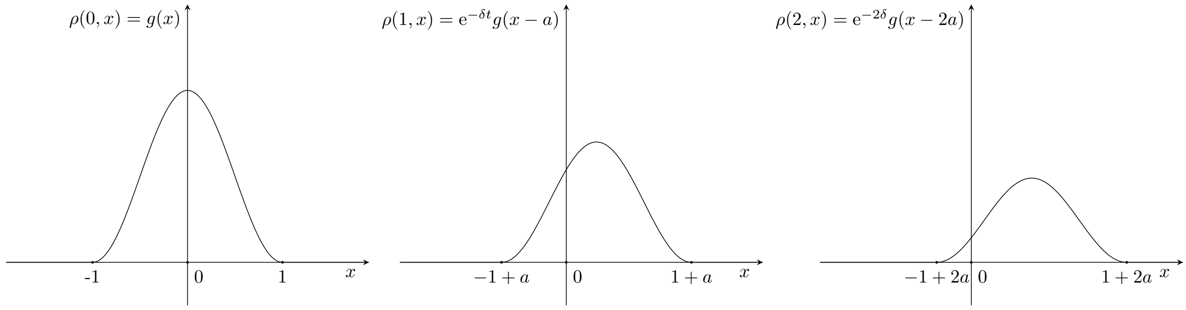


FIGURE 3. The solution of (1.17), at times $t = 0, 1, 2$. It is still a translation of the initial datum, but with decreasing amplitude.

2. The continuity equation

A possible multi-dimensional generalization is as follows. Let $\Omega \subset \mathbb{R}^d$ be a smooth domain, and let $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x}) : \mathbb{R} \times \Omega \to \mathbb{R}^d$ a smooth vector field. Consider the Cauchy problem

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, & t > 0, \ \boldsymbol{x} \in \Omega, \\ \rho(0, \boldsymbol{x}) = g(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \end{cases} \tag{2.1}$$

for a given regular initial datum. This is called *continuity* equation in the physics literature, and describes the evolution of ρ from the *Eulerian* point of view, namely in which the unknowns are measured at a stationary position $(t, \boldsymbol{x})$ in space time. Using the ideas of the previous section, we would like to find a solution to (2.1). As before, one should think as ρ to be a density that is transported by a fluid with velocity $\boldsymbol{u}$.

2.1. Particle trajectories. We may first consider the change in a quantity as experienced by a particle that is traveling with the fluid. This is the *Lagrangian* description. Given an initial configuration of particles, labeled by $\boldsymbol{a} \in \Omega$, the unknowns are the *particle trajectories* at times $t > 0$:

$$\boldsymbol{X}(t, \boldsymbol{a}) = (X_1, \ldots, X_d)(t, \boldsymbol{a}) \colon [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d. \tag{2.2}$$

The map

$$\boldsymbol{X}(t, \cdot) \colon \Omega \to \Omega, \qquad \boldsymbol{a} \mapsto \boldsymbol{X}(t, \boldsymbol{a}) \tag{2.3}$$

is called the *flow map* associated to the velocity field $\boldsymbol{u}$. Under suitable conditions this map is an (volume-preserving) isomorphism of Ω. We denote the inverse of $\boldsymbol{X}$ by

$$\boldsymbol{A}(t, \boldsymbol{x}) = (A_1, \ldots, A_d)(t, \boldsymbol{x}) \colon [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \tag{2.4}$$

which obeys

$$\boldsymbol{A}(t, \boldsymbol{X}(t, \boldsymbol{a})) = \boldsymbol{a}, \qquad \boldsymbol{X}(t, \boldsymbol{A}(t, \boldsymbol{x})) = \boldsymbol{x} \tag{2.5}$$

for all $\boldsymbol{a}, \boldsymbol{x} \in \mathbb{R}^d$. The map $\boldsymbol{A}$ is sometimes called the *back-to-labels* map, and we sometimes write $\boldsymbol{A} = \boldsymbol{X}^{-1}$.

The connection between the Eulerian and the Lagrangian description of fluid flow is given by the fact that the particle trajectories $\boldsymbol{X}$ move on the integral curves of the velocity field $\boldsymbol{u}$, that is they obey the ODE

$$\partial_t \boldsymbol{X}(t, \boldsymbol{a}) = \boldsymbol{u}(t, \boldsymbol{X}(t, \boldsymbol{a})) \tag{2.6}$$

with initial conditions

$$\boldsymbol{X}(0, \boldsymbol{a}) = \boldsymbol{a}. \tag{2.7}$$

Geometrically, this means that the velocity field $\boldsymbol{u}$, evaluated at position $(t, x) = (t, \boldsymbol{X}(t, \boldsymbol{a}))$, is tangent to the curve described by the motion of the particle $\boldsymbol{a}$, namely $\{\boldsymbol{X}(\tau, \boldsymbol{a})\}_{\tau \in I}$, at the point $\boldsymbol{X}(t, \boldsymbol{a})$.

The instantaneous rate of change of any function f with respect to time t, at the fixed position $\boldsymbol{x}$, is the partial time derivative $\partial_t f$. However, in many cases in fluid mechanics it is more natural to measure the rate of change along the flow, denoted by the *convective derivative*

$$\partial_t \left(f(t, \boldsymbol{X}(t, \boldsymbol{a})) \right) = (D_t f)(t, \boldsymbol{X}(t, \boldsymbol{a})). \tag{2.8}$$

From the chain rule and (2.6), assuming all of the functions involved are C^1, we observe that

$$D_t f = \partial_t f + \boldsymbol{u} \cdot \nabla f, \tag{2.9}$$

which is precisely measuring the change in time of f as experienced by a particle moving along the integral curves of $\boldsymbol{u}$. Notice that (2.1) can be rewritten as

$$\begin{cases} D_t \rho = -\rho \nabla \cdot \boldsymbol{u}, & t > 0,\ \boldsymbol{x} \in \Omega, \\ \rho(0, \boldsymbol{x}) = g(\boldsymbol{x}), & \boldsymbol{x} \in \Omega. \end{cases} \tag{2.10}$$

The quantity $\nabla \cdot \boldsymbol{u}$ plays an important role, as we shall see in the next sections.

It is useful to get comfortable with the change of variables given by $\boldsymbol{X}$. Let V be a volume element in the fluid, and denote by

$$V(t) = \boldsymbol{X}(t, V) = \{ \boldsymbol{X}(t, \boldsymbol{a}) \colon \boldsymbol{a} \in V \}. \tag{2.11}$$

Recall the change of variables formula

$$\int_{V(t)} f(t, \boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_V f(t, \boldsymbol{X}(t, \boldsymbol{a})) \det(\nabla_{\boldsymbol{a}} \boldsymbol{X})(t, \boldsymbol{a}) \mathrm{d}\boldsymbol{a} = \int_V f(t, \boldsymbol{X}(t, \boldsymbol{a})) J(t, \boldsymbol{a}) \mathrm{d}\boldsymbol{a}, \tag{2.12}$$

where we have denoted by $J(t, \boldsymbol{a})$ the determinant of Jacobian $\nabla_{\boldsymbol{a}} \boldsymbol{X}$ associated to the map $\boldsymbol{a} \mapsto \boldsymbol{X}(t, \boldsymbol{a})$, that is

$$J(t, \boldsymbol{a}) = \det(\nabla_{\boldsymbol{a}} \boldsymbol{X})(t, \boldsymbol{a}). \tag{2.13}$$

One of the most useful properties of J is stated in the following lemma.

LEMMA 2.1. *Assume that the vector field* $\mathbf{u}$ *is* C^1*, let* $\boldsymbol{X}$ *be defined by* (2.6)–(2.7)*, and* J *be given by* (2.13)*. Then we have*

$$\partial_t J(t, \boldsymbol{a}) = J(t, \boldsymbol{a}) (\nabla \cdot \boldsymbol{u}(t, \boldsymbol{X}(t, \boldsymbol{a}))) \tag{2.14}$$

pointwise in $(t, \boldsymbol{a})$.

A consequence of Lemma 2.1 is that we may compute the rate of change of the average of a quantity f over a domain $V(t)$ transported by the fluid.

LEMMA 2.2. *Let* $f \colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ *be a* C^1 *function, and assume that the velocity field* $\mathbf{u}$ *defining the flow map* $\boldsymbol{X}(t, \cdot)$ *is also* C^1*. Then we have that*

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\int_{V(t)} f(t, \boldsymbol{x}) \mathrm{d}\boldsymbol{x} \right) = \int_{V(t)} \left(\partial_t f + \nabla \cdot (f \boldsymbol{u}) \right)(t, \boldsymbol{x}) \mathrm{d}\boldsymbol{x} \tag{2.15}$$

for every $t > 0$ *and every fluid element* V.

PROOF OF LEMMA 2.2. By using the change of variables formula, the convective derivative identity (2.8), and Lemma 2.1 we deduce

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{V(t)} f(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{V} f(t,\boldsymbol{X}(t,\boldsymbol{a}))J(t,\boldsymbol{a})\mathrm{d}\boldsymbol{a}\right)\\
&= \int_V (D_t f)(t,X(t,\boldsymbol{a}))J(t,\boldsymbol{a})\mathrm{d}\boldsymbol{a} + \int_V f(t,\boldsymbol{X}(t,\boldsymbol{a}))\partial_t J(t,\boldsymbol{a})\mathrm{d}\boldsymbol{a}\\
&= \int_V (\partial_t f + \boldsymbol{u}\cdot\nabla f + (\nabla\cdot\boldsymbol{u})f)(t,\boldsymbol{X}(t,\boldsymbol{a}))J(t,\boldsymbol{a})\mathrm{d}\boldsymbol{a}\\
&= \int_V (\partial_t f + \nabla\cdot(f\boldsymbol{u}))(t,\boldsymbol{X}(t,\boldsymbol{a}))J(t,\boldsymbol{a})\mathrm{d}\boldsymbol{a}\\
&= \int_{V(t)} (\partial_t f + \nabla\cdot(f\boldsymbol{u}))(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x} \qquad (2.16)
\end{aligned}
$$

which concludes the proof. □

REMARK 2.3 (Conservation of mass). Let V be a volume element in the fluid. If we think of the solution ρ to (2.1) as the density of a concentration, its total mass in this volume element is given by

$$
m(t,V) = \int_V \rho(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \qquad (2.17)
$$

Setting $f=\rho$ in Lemma 2.2 and using (2.1) gives

$$
\frac{\mathrm{d}}{\mathrm{d}t} m(t,V(t)) = 0, \qquad (2.18)
$$

namely, *mass is neither created nor destroyed* in a volume element moving with the fluid.

2.2. Incompressible flows. We say that the velocity field $\boldsymbol{u}$ is *incompressible* if the flow map $\boldsymbol{X}(t,\cdot)$ is *volume-preserving*, meaning that

$$
|V| = |V(t)| \qquad (2.19)
$$

for all $V\subset\Omega$, and all $t\geq 0$. Here and throughout we denote by $|V|$ the Lebesgue measure of a set V. As it turns out, assumption (2.19) is equivalent to

$$
\nabla\cdot\boldsymbol{u} = 0 \qquad (2.20)
$$

namely that the velocity field is *divergence free*. This is a consequence of Lemma 2.2.

PROPOSITION 2.4. *The velocity field $\boldsymbol{u}$ is incompressible if and only if $\nabla\cdot\boldsymbol{u}\equiv 0$ for every $\boldsymbol{x}\in\Omega$ and every $t\geq 0$.*

PROOF. Take $f=1$ in Lemma 2.2. Then

$$
\frac{\mathrm{d}}{\mathrm{d}t}|V(t)| = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{V(t)} 1\mathrm{d}\boldsymbol{x}\right) = \int_{V(t)} (\nabla\cdot\boldsymbol{u})(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x} \qquad (2.21)
$$

for every $t\geq 0$ and every open set $V\subset\Omega$. If $\nabla\cdot\boldsymbol{u}\equiv 0$, then $|V(t)|$ does not change in time, and therefore it is equal to its initial value $|V|$. On the other hand, if $|V(t)|$ is constant in time, then

$$
\int_{V(t)} (\nabla\cdot\boldsymbol{u})(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0, \qquad (2.22)
$$

for every $t\geq 0$ and every open set $V\subset\Omega$. Since $\boldsymbol{X}(t,\cdot)$ is a bijection, it follows that

$$
\int_W (\nabla\cdot\boldsymbol{u})(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0, \qquad (2.23)
$$

for every open set $W\subset\Omega$ and all $t\geq 0$, from which $\nabla\cdot\boldsymbol{u}\equiv 0$ immediately follows. □

Another important property of incompressible flows follows from Lemma 2.1.

COROLLARY 2.5. *Under the assumption of Lemma 2.1, further assume that $\boldsymbol{u}$ is divergence free, i.e. that (2.20) holds. Then we have that*

$$J(t,\boldsymbol{a}) = 1 \tag{2.24}$$

for all $\boldsymbol{a} \in \Omega$ and $t > 0$.

PROOF OF COROLLARY 2.5. From (2.7) it follows that $J(0,\cdot) = \det(I) = 1$ identically. Solving the differential equation (2.14) we arrive at

$$J(t,\boldsymbol{a}) = J(0,\boldsymbol{a}) \exp\left(\int_0^t (\nabla \cdot \boldsymbol{u})(s, \boldsymbol{X}(s,\boldsymbol{a}))\mathrm{d}s\right) = J(0,\boldsymbol{a}) = 1 \tag{2.25}$$

by using that $\nabla \cdot \boldsymbol{u} = 0$ identically. □

Now, the solution to the continuity equation (2.1) is easily computed if we take into account the equivalent formulation (2.10), the incompressibility condition (2.20) and (2.8). Indeed,

$$D_t\rho = 0 \qquad \Rightarrow \qquad \rho(t,\boldsymbol{X}(t,\boldsymbol{a})) = \rho(0,\boldsymbol{X}(0,\boldsymbol{a})) = \rho(0,\boldsymbol{a}) = g(\boldsymbol{a}), \tag{2.26}$$

for every $\boldsymbol{a} \in \Omega$. Hence,

$$\rho(t,\boldsymbol{x}) = g(\boldsymbol{A}(t,\boldsymbol{x})). \tag{2.27}$$

It is clear at this point the knowing when the map $\boldsymbol{X}$ is invertible (with inverse $\boldsymbol{A}$) is crucial to be able to define such solution. A possible condition for this to happen is that $\boldsymbol{u}$ is C^1.

2.3. Compressible flows. If $\nabla \cdot \boldsymbol{u} \neq 0$, then again from (2.8) and (2.10) we have

$$\partial_t\left(\rho(t,\boldsymbol{X}(t,\boldsymbol{a}))\right) = -\rho(t,\boldsymbol{X}(t,\boldsymbol{a}))\nabla\cdot\boldsymbol{u}(t,\boldsymbol{X}(t,\boldsymbol{a})). \tag{2.28}$$

Hence,

$$\begin{aligned}\rho(t,\boldsymbol{X}(t,\boldsymbol{a})) &= \rho(0,\boldsymbol{X}(0,\boldsymbol{a}))\exp\left[-\int_0^t \nabla\cdot\boldsymbol{u}(s,\boldsymbol{X}(s,\boldsymbol{a}))\mathrm{d}s\right]\\ &= g(\boldsymbol{a})\exp\left[-\int_0^t \nabla\cdot\boldsymbol{u}(s,\boldsymbol{X}(s,\boldsymbol{a}))\mathrm{d}s\right]\end{aligned} \tag{2.29}$$

From Lemma 2.1, we have that

$$J(t,\boldsymbol{a}) = \exp\left[\int_0^t \nabla\cdot\boldsymbol{u}(s,\boldsymbol{X}(s,\boldsymbol{a}))\mathrm{d}s\right], \tag{2.30}$$

we can write

$$\rho(t,\boldsymbol{X}(t,\boldsymbol{a})) = \frac{g(\boldsymbol{a})}{J(t,\boldsymbol{a})}, \tag{2.31}$$

and therefore

$$\rho(t,\boldsymbol{x}) = \frac{g(\boldsymbol{A}(t,\boldsymbol{x}))}{J(t,\boldsymbol{A}(t,\boldsymbol{x}))}. \tag{2.32}$$

Once again, this formula reduces to (2.27) in the incompressible case.

3. The method of characteristics

In this section we study equations of the form

$$a(x,y,u)\partial_x u + b(x,y,u)\partial_y u = c(x,y,u), \tag{3.1}$$

for $x, y \in \Omega \subset \mathbb{R}^2$ and with $a, b, c : \mathbb{R}^3 \to \mathbb{R}$ continuously differentiable functions. Notice that this is a first order quasilinear equation. The ideas involved are generalizations of what we have seen in Section 1. As before, solutions to (3.1) can be constructed by taking advantage of the following geometric interpretation: let (x_0, y_0, z_0) be a point on the graph of u, namely, $z_0 = u(x_0, y_0)$. The tangent plane to the graph u at (x_0, y_0, z_0) has equation

$$\partial_x u(x_0,y_0)(x-x_0) + \partial_y u(x_0,y_0)(y-y_0) - (z-z_0) = 0. \tag{3.2}$$

Hence, the vector

$$\boldsymbol{n}_0 = (\partial_x u(x_0, y_0), \partial_y u(x_0, y_0), -1) \tag{3.3}$$

is normal to the plane defined by (3.2), but also normal to the vector

$$\boldsymbol{v}_0 = (a(x_0, y_0, z_0), b(x_0, y_0, z_0), c(x_0, y_0, z_0)), \tag{3.4}$$

by (3.1). Thus, $\boldsymbol{v}_0$ is tangent to the graph of u. In other words, (3.1) says that, at every point (x, y, z), the graph of any solution is tangent to the vector field

$$\boldsymbol{v} = (a(x, y, z), b(x, y, z), c(x, y, z)). \tag{3.5}$$

In this case we say that the graph of a solution is an integral surface of the vector field $\boldsymbol{v}$. The idea is to construct integral surfaces of $\boldsymbol{v}$ as union of curves of tangent to $\boldsymbol{v}$ at every point. These curves are called *characteristics*, and take the form

$$\frac{\mathrm{d}x}{\mathrm{d}s} = a(x, y, z), \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = b(x, y, z), \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = c(x, y, z). \tag{3.6}$$

Note that $z(s)$ gives the value of u along a characteristic, that is

$$z(s) = u(x(s), y(s)). \tag{3.7}$$

Indeed, differentiating (3.7) and using (3.1) and (3.6), we have

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}s} &= \partial_x u(x(s), y(s))\frac{\mathrm{d}x}{\mathrm{d}s} + \partial_y u(x(s), y(s))\frac{\mathrm{d}y}{\mathrm{d}s} \\ &= a(x(s), y(s), z(s))\partial_x u(x(s), y(s)) + b(x(s), y(s), z(s))\partial_y u(x(s), y(s)) \\ &= c(x(s), y(s), z(s)).\end{aligned} \tag{3.8}$$

Thus, along a characteristic the partial differential equation (3.1) degenerates into an ordinary differential equation.

3.1. The Cauchy problem for first order quasilinear equations. Let $I \subset \mathbb{R}$ be an interval containing 0, and $\gamma : I \to \mathbb{R}^2$ be a smooth curve in the (x, y)-plane, parametrized as

$$I \ni \tau \mapsto \gamma(\tau) = (\gamma_1(\tau), \gamma_2(\tau)) \in \mathbb{R}^2. \tag{3.9}$$

Our goal is to find a solution to the Cauchy problem

$$\begin{cases} a(x, y, u)\partial_x u + b(x, y, u)\partial_y u = c(x, y, u), & (x, y) \in \mathbb{R}^2, \\ u(\gamma_1(\tau), \gamma_2(\tau)) = g(\tau), & \tau \in I. \end{cases} \tag{3.10}$$

We assume that γ_1, γ_2 and g are continuously differentiable in I. The data are often assigned in the form of initial values, where $\gamma_1(\tau) = \tau$ and $\gamma_2(\tau) = 0$, so that

$$u(\tau, 0) = g(\tau) \tag{3.11}$$

and y plays the role of time (compare with (1.13)), but it does not always have to be this way. Writing down the characteristic system (3.6), we look at the system

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s} &= a(x, y, z), \qquad x(0) = \gamma_1(\tau), \\ \frac{\mathrm{d}y}{\mathrm{d}s} &= b(x, y, z), \qquad y(0) = \gamma_2(\tau), \\ \frac{\mathrm{d}z}{\mathrm{d}s} &= c(x, y, z), \qquad z(0) = g(\tau), \end{aligned} \tag{3.12}$$

where $\tau \in I$ is a parameter. Invoking the Cauchy-Lipschitz theorem for ODEs, (3.12) has a unique solution in a neighborhood of $s = 0$, for every $\tau \in I$, which we denote by

$$x = X(s, \tau), \qquad y = Y(s, \tau), \qquad z = Z(s, \tau). \tag{3.13}$$

We now need to check that the above (X, Y, Z) define a function $u(x, y)$, so that this u is a solution to (3.12). To check invertibility, let us think in a neighborhood of $s = 0$, and for a fixed $\tau_0 \in I$ set

$$X(0, \tau_0) = \gamma_1(\tau_0) = x_0, \qquad Y(0, \tau_0) = \gamma_2(\tau_0) = y_0, \qquad Z(0, \tau_0) = g(\tau_0) = z_0. \tag{3.14}$$

The goal is to invert the first two equations in (3.13) and find $s = S(x, y)$ and $\tau = T(x, y)$ of class C^1 in a neighborhood of (x_0, y_0) such that

$$S(x_0, y_0) = 0, \qquad T(x_0, y_0) = \tau_0. \tag{3.15}$$

Then, from the third equation $z = Z(s, \tau)$, reasoning as in (3.7), we get

$$z = Z(S(x, y), T(x, y)) = u(x, y). \tag{3.16}$$

From the inverse function theorem, this is possible if

$$\begin{vmatrix} \partial_s X(0, \tau_0) & \partial_\tau X(0, \tau_0) \\ \partial_s Y(0, \tau_0) & \partial_\tau Y(0, \tau_0) \end{vmatrix} \neq 0. \tag{3.17}$$

From (3.12) and (3.13), this is equivalent to

$$\begin{vmatrix} a(x_0, y_0, z_0) & \gamma_1'(\tau_0) \\ b(x_0, y_0, z_0) & \gamma_2'(\tau_0) \end{vmatrix} = a(x_0, y_0, z_0)\gamma_2'(\tau_0) - b(x_0, y_0, z_0)\gamma_1'(\tau_0) \neq 0. \tag{3.18}$$

The above means that the vectors $(a(x_0, y_0, z_0), b(x_0, y_0, z_0))$ and $(\gamma_1'(\tau_0), \gamma_2'(\tau_0))$ are not parallel. Therefore, if condition (3.18) holds, then (3.16) is a well defined function of class C^1. A precise statement of what we have just discussed is contained in the following result.

THEOREM 3.1. *Let a, b, c be C^1-functions in a neighborhood of $(x_0, y_0, u(x_0, y_0))$ and assume that γ_1, γ_2, g are C^1-functions in I. If* (3.18) *holds, then, in a neighborhood of (x_0, y_0), there exists a unique C^1-solution $u = u(x, y)$ of the Cauchy problem* (3.10)*. Moreover, u is defined by the parametric equations* (3.13)*.*

Let us summarize the steps to solve the Cauchy problem (3.10).

Step 1. Determine the solution

$$x = X(s, \tau), \qquad y = Y(s, \tau), \qquad z = Z(s, \tau) \tag{3.19}$$

of the characteristic system

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s} &= a(x, y, z), \qquad x(0) = \gamma_1(\tau), \\ \frac{\mathrm{d}y}{\mathrm{d}s} &= b(x, y, z), \qquad y(0) = \gamma_2(\tau), \\ \frac{\mathrm{d}z}{\mathrm{d}s} &= c(x, y, z), \qquad z(0) = g(\tau). \end{aligned} \tag{3.20}$$

Step 2. For each $\tau_0 \in I$, check that

$$\begin{vmatrix} \partial_s X(0, \tau_0) & \partial_s Y(0, \tau_0) \\ \gamma_1'(\tau_0) & \gamma_2'(\tau_0) \end{vmatrix} \neq 0. \tag{3.21}$$

3.2. The Cauchy problem for first order linear equations. In the special case in which a, b and c do not depend on u, (3.10) becomes the linear equation

$$\begin{cases} a(x, y)\partial_x u + b(x, y)\partial_y u = c(x, y), & (x, y) \in \mathbb{R}^2, \\ u(\gamma_1(\tau), \gamma_2(\tau)) = g(\tau), & \tau \in I. \end{cases} \tag{3.22}$$

The characteristic system (3.20) then becomes

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s} &= a(x, y), \qquad x(0) = \gamma_1(\tau), \\ \frac{\mathrm{d}y}{\mathrm{d}s} &= b(x, y), \qquad y(0) = \gamma_2(\tau), \\ \frac{\mathrm{d}z}{\mathrm{d}s} &= c(x, y), \qquad z(0) = g(\tau). \end{aligned} \tag{3.23}$$

It is important to notice that once we solve for x, y, then z is simply obtained by integration. In other words, the equation for z is *decoupled* from the other two. Thus, once we determine

$$x = X(s, \tau), \qquad y = Y(s, \tau), \tag{3.24}$$

then

$$z = Z(s,\tau) = g(\tau) + \int_0^s c(X(\sigma,\tau), Y(\sigma,\tau))\mathrm{d}\sigma. \tag{3.25}$$

Thus

$$u(X(s,\tau), Y(s,\tau)) = g(\tau) + \int_0^s c(X(\sigma,\tau), Y(\sigma,\tau))\mathrm{d}\sigma. \tag{3.26}$$

which gives a well defined function of class C^1. At this point, one still has to check (3.21) and invert (3.24).

EXAMPLE 3.2. We want to solve the Cauchy problem

$$\begin{cases} -y\partial_x u + x\partial_y u = 4xy, & (x,y) \in \mathbb{R}^2, \\ u(x,0) = g(x), & x > 0. \end{cases} \tag{3.27}$$

In this case, $\gamma(\tau) = (\tau, 0)$, for $\tau \in (0,\infty)$. First, we have to solve the characteristic system, which in this case reads

$$\frac{\mathrm{d}x}{\mathrm{d}s} = -y, \qquad x(0) = \tau, \tag{3.28}$$

$$\frac{\mathrm{d}y}{\mathrm{d}s} = x, \qquad y(0) = 0, \tag{3.29}$$

$$\frac{\mathrm{d}z}{\mathrm{d}s} = 4xy, \qquad z(0) = g(\tau). \tag{3.30}$$

From the first two equations we obtain

$$x = X(s,\tau) = \tau\cos(s), \qquad y = Y(s,\tau) = \tau\sin(s), \tag{3.31}$$

so that from (3.26) we obtain

$$u(X(s,\tau), Y(s,\tau)) = Z(s,\tau) = g(\tau) + 4\tau^2\int_0^s \cos(\sigma)\sin(\sigma)\mathrm{d}\sigma = g(\tau) + 2\tau^2\sin^2(s). \tag{3.32}$$

Now, (3.31) is a familiar change of coordinate, the polar coordinates! One can easily check (3.21) and thus (since $x > 0$)

$$s = S(x,y) = \arctan\left(\frac{y}{x}\right), \qquad \tau = T(x,y) = \sqrt{x^2+y^2}. \tag{3.33}$$

From (3.32) we then find that

$$u(x,y) = g\left(\sqrt{x^2+y^2}\right) + 2y^2 \tag{3.34}$$

is the unique solution to (3.27).

4. Scalar conservation laws

The nonlinear one-dimensional generalization of (1.5) is the *scalar conservation law*

$$\partial_t\rho + \partial_x[q(\rho)] = 0, \tag{4.1}$$

where $q(\rho)$ is a nonlinear function. If q is differentiable, the above equation can be equivalently written as

$$\partial_t\rho + q'(\rho)\partial_x\rho = 0. \tag{4.2}$$

Both formulations are useful. We can naturally put (4.2) in the framework analyzed in the previous section (see (3.10)) to study the Cauchy problem

$$\begin{cases} \partial_t\rho + q'(\rho)\partial_x\rho = 0, & x \in \mathbb{R},\ t > 0, \\ \rho(0,x) = g(x), & x \in \mathbb{R}. \end{cases} \tag{4.3}$$

In particular, from (3.20) we can write down the characteristic system as

$$\frac{\mathrm{d}x}{\mathrm{d}s} = q'(z), \qquad x(0) = \tau, \tag{4.4}$$

$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1, \qquad t(0) = 0, \tag{4.5}$$

$$\frac{\mathrm{d}z}{\mathrm{d}s} = 0, \qquad z(0) = g(\tau). \tag{4.6}$$

Solving the system gives

$$x = X(s,\tau) = \tau + q'(g(\tau))s, \qquad t = T(s,\tau) = s, \qquad z = Z(s,\tau) = g(\tau). \tag{4.7}$$

In particular, we easily obtain the formula

$$x = \tau + q'(g(\tau))t. \tag{4.8}$$

Thus, the characteristics are straight lines with slope $q'(g(\tau))$. Different values of τ give, in general, different values of the slope. We are thus facing again a problem of *invertibility* of the coordinate system. A typical situation in which this is problematic is depicted in Figure 4. For certain initial data, it may well happen that two characteristics intersect, hence producing a discontinuity in the solution.

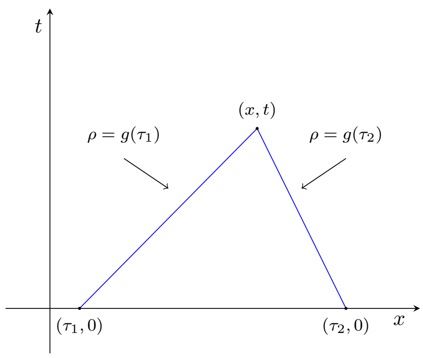


FIGURE 4. Intersection of characteristics. At the point (x,t), the solution ρ has a discontinuity due to the intersection of characteristics.

The purpose now is to understand how to deal with this and other issues that can arise in conservation laws.

4.1. Existence of classical solutions. We begin by giving a precise definition of solutions to (4.22), commonly referred to as classical

DEFINITION 4.1. Let $T > 0$ be given. A *classical solution* ρ to (4.3) is a function in $C^1([0,T) \times \mathbb{R})$ such that $\rho(0,x) = g(x)$ and that satisfies (4.3) for every $(t,x) \in (0,T) \times \mathbb{R}$. If $T = \infty$, then ρ is a global-in-time classical solution.

The following is an existence and uniqueness theorem for classical solutions to (4.3).

THEOREM 4.2. *Suppose that $q \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$, and assume that there exists $M > 0$ such that*

$$\sup_{r\in\mathbb{R}} |q''(r)| \le M, \qquad \sup_{r\in\mathbb{R}} |g'(r)| \le M. \tag{4.9}$$

Then there exists $T = T(M) > 0$ such that there exists a unique classical solution ρ to (4.3) defined on $[0,T] \times \mathbb{R}$. If further

$$q''(r)g'(s) \ge 0, \qquad \forall r, s \in \mathbb{R}, \tag{4.10}$$

then the solution is global in time, namely, it is defined for $(t,x) \in [0,\infty) \times \mathbb{R}$.

PROOF. From (4.8), we have that the solution to (4.3) is given by

$$\rho(t,x) = g\left(x - q'(g(\tau))t\right). \tag{4.11}$$

Since $\rho(t,x) = g(\tau)$ along the characteristic based at $(\tau, 0)$, from (4.11) we obtain that ρ is implicitly defined by the equation

$$G(t,x,\rho) = \rho - g\left(x - q'(\rho)t\right) = 0. \tag{4.12}$$

Since g and ρ are regular enough, the Implicit Function Theorem implies that equation (4.12) defines ρ as a function of (t,x), as long as

$$\partial_\rho G(t,x,\rho) = 1 + tq''(\rho)g'\left(x - q'(\rho)t\right) \neq 0. \tag{4.13}$$

Now, if (4.9) holds, then the above condition holds whenever $t < M^{-2}$, proving the first part of the theorem. If furthermore (4.10) is satisfied, then (4.13) is satisfied for every $t \geq 0$, thereby concluding the proof.

□

EXAMPLE 4.3 (Burgers equation). The simplest nonlinearity that one can think of is a quadratic one, when

$$q(\rho) = \frac{\rho^2}{2}. \tag{4.14}$$

From (4.2), this gives the Cauchy problem for the so called *Burgers equation*

$$\begin{cases} \partial_t \rho + \rho \partial_x \rho = 0, & x \in \mathbb{R},\ t > 0, \\ \rho(0,x) = g(x), & x \in \mathbb{R}. \end{cases} \tag{4.15}$$

We claim that a global classical solution exists if and only if g is increasing. Since $q(\rho) = \rho^2/2$ in this case, if g is increasing then (4.10) is satisfied, and therefore Theorem 4.2 guarantees that we have a global solution. On the other hand, suppose that g is not increasing. Then there are $\tau_1 < \tau_2$ such that $g(\tau_1) > g(\tau_2)$. From (4.8), we have the two characteristics

$$x = \tau_1 + g(\tau_1)t, \qquad x = \tau_2 + g(\tau_2)t, \tag{4.16}$$

which intersect at

$$t = \frac{\tau_2 - \tau_1}{g(\tau_1) - g(\tau_2)} > 0, \tag{4.17}$$

which is equivalent to the fact the solution has a discontinuity, and hence cannot be a classical solution. Notice that Theorem 4.2 still guarantees the existence of a classical solution for small times (less than some $T > 0$). In particular, from (4.12), for $t < T$ we know that

$$\rho = g\left(x - \rho t\right). \tag{4.18}$$

Differentiating with respect to x we find

$$\partial_x \rho = g'\left(x - \rho t\right)\left(1 - t\partial_x \rho\right) \qquad \Rightarrow \qquad \partial_x \rho = \frac{g'\left(x - \rho t\right)}{1 + g'\left(x - \rho t\right)t}. \tag{4.19}$$

Now, assume that $g' < 0$ is bounded below and attains its minimum at some point $\tau_s \in \mathbb{R}$. Let

$$t_s = -[g'(\tau_s)]^{-1} > 0, \qquad x_s = \tau_s + g(\tau_s)t_s. \tag{4.20}$$

By construction, (t_s, x_s) is a point on the characteristic $x = \tau_s + g(\tau_s)t$ emanating from $(\tau_s, 0)$. From (4.19), we then take a limit as $(t,x) \to (t_s, x_s)$ *on the characteristic* $x = \tau_s + g(\tau_s)t$ and find

$$\lim_{(t,x)\to(t_s,x_s)} \partial_x \rho(t,x) = \lim_{\tau \to \tau_s} \frac{g'\left(\tau\right)}{1 - g'\left(\tau\right)[g'(\tau_s)]^{-1}} = \infty. \tag{4.21}$$

Hence (t_s, x_s) corresponds to the time and location of a *shock*, at which the solution ceases to be a C^1 function.

4.2. Weak solutions. We have seen that the method of characteristics is not sufficient, in general, to determine the solution of an initial value problem for all times $t > 0$. This is a common theme in PDEs. If we are too greedy about the regularity requirements in the definition of a solution to a problem, we may not be able to find one for all times. We introduce a more flexible definition of solutions for the Cauchy problem

$$\begin{cases} \partial_t \rho + \partial_x [q(\rho)] = 0, & x \in \mathbb{R},\ t > 0, \\ \rho(0,x) = g(x), & x \in \mathbb{R}. \end{cases} \tag{4.22}$$

as follows. Let v be a smooth function in $[0,\infty) \times \mathbb{R}$, with compact support. We call v a *test function*. Multiply the differential equation by v and integrate on $[0,\infty) \times \mathbb{R}$, to get

$$\int_0^\infty \int_{\mathbb{R}} (\partial_t \rho + \partial_x [q(\rho)])\, v \mathrm{d}x \mathrm{d}t = 0. \tag{4.23}$$

An integration by parts in both t and x and the fact that v has compact support yields

$$\int_0^\infty \int_{\mathbb{R}} (\rho \partial_t v + q(\rho)\partial_x v)\, \mathrm{d}x \mathrm{d}t = \int_{\mathbb{R}} g(x) v(0,x) \mathrm{d}x. \tag{4.24}$$

We have obtained an integral equation, valid for every test function v, in which *no derivative* on ρ appears. On the other hand, suppose that a smooth function ρ satisfies (4.24) for every test function v. Integrating by parts in the reverse order, we arrive to the equation

$$\int_0^\infty \int_{\mathbb{R}} (\partial_t \rho + \partial_x [q(\rho)])\, v \mathrm{d}x \mathrm{d}t + \int_{\mathbb{R}} [g(x) - \rho(0,x)]\, v(0,x) \mathrm{d}x = 0. \tag{4.25}$$

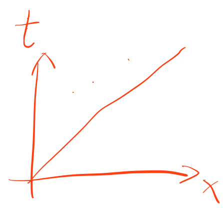
Choosing v vanishing for $t = 0$, the second integral is zero and the arbitrariness of v implies that ρ satisfies the differential equation in (4.22). Choosing now v non vanishing for $t = 0$, and using that $\partial_t \rho + \partial_x [q(\rho)] = 0$ we get that $\rho(0,x) = g(x)$, and therefore ρ is the classical solution to (4.22). It is then natural to introduce the following notion of solution.

DEFINITION 4.4. A function ρ, bounded in $[0,\infty) \times \mathbb{R}$ is a *weak solution* to (4.22) if equation (4.24) holds for every test function v in $[0,\infty) \times \mathbb{R}$ with compact support.

The only requirement on ρ is being a bounded function, so in particular it is allowed for ρ to be discontinuous. This will help us dealing with shocks. However, a possible drawback in enlarging the class of solutions is that we may lose their uniqueness.

EXAMPLE 4.5 (Rarefaction waves for Burgers). Consider the Burgers equation (4.15) with initial datum

$$g(x) = \begin{cases} 0, & x \le 0, \\ 1, & x > 0. \end{cases} \tag{4.26}$$



The characteristics are the straight lines

$$x = \tau + g(\tau) t. \tag{4.27}$$

Therefore, $\rho(t,x) \equiv 0$ for $x \le 0$, and $\rho(t,x) \equiv 1$ for $x > t$. Since the region $S = \{(t,x) : 0 < x \le t\}$ is not covered by the characteristics, we connect the states 0 and 1 through a *rarefaction wave* (see Figure 5a). This mean that we seek a solution $\rho(t,x) = h(x/t)$, for some function h to be determined. Using (4.15), we can write an equation for h as

$$-\frac{x}{t^2} h' + \frac{1}{t} h h' = 0 \qquad \Rightarrow \qquad h(x/t) = \frac{x}{t}. \tag{4.28}$$

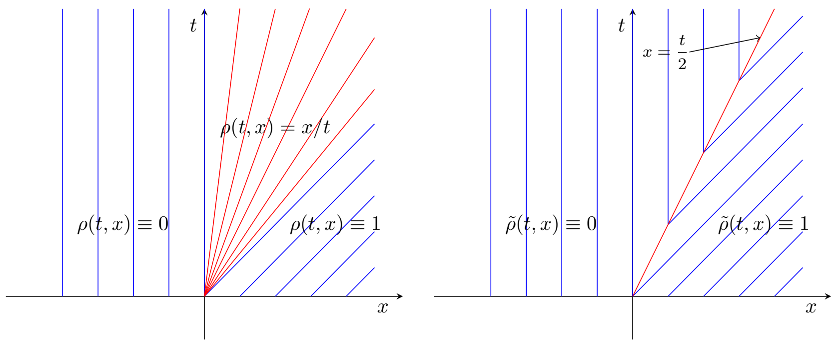
Thus, the weak solution constructed can be written as

$$\rho(t,x) = \begin{cases} 0, & x \le 0, \\ x/t, & 0 < x < t, \\ 1, & x \ge t. \end{cases} \tag{4.29}$$

However, ρ is not the unique weak solution, but there exists also a shock wave solution (see Figure 5b). It is not hard to check that

$$\tilde{\rho}(t,x) = \begin{cases} 0, & x < t/2, \\ 1, & x > t/2, \end{cases} \tag{4.30}$$

is another weak solution. As we shall see, this shock wave has to be considered not physically acceptable. What happens is that we have more than one way to "fill in" the region S that is not covered by the characteristics, as depicted in Figure 5.



(A) The rarefaction wave solution ρ (B) The shock wave solution $\tilde{\rho}$

FIGURE 5. An example of non-uniqueness of solutions to the Burgers equation with initial datum (4.26).

4.3. The Rankine-Hugoniot condition. The appearance of shocks is hidden in the definition of weak solutions. Consider an open set V, contained in the half-plane $t > 0$, partitioned into two disjoint domains V_+ and V_- by a smooth (shock) curve Γ of equation $x = \sigma(t)$, as in Figure 6.

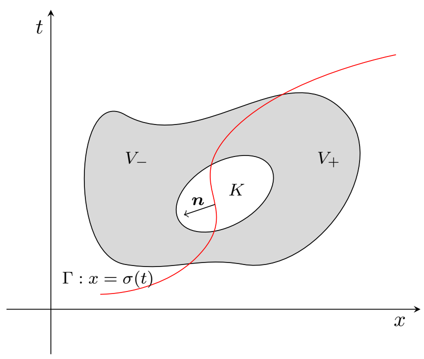


FIGURE 6. A shock curve dividing a domain V

Now, assuming that ρ is a classical solution on both sides of Γ, with continuous derivatives up to Γ, we take a test v function supported in a compact set $K \subset V$ that intersects Γ. Since $v(0,x) = 0$, we compute from the definition of weak solution in (4.24) that

$$\begin{aligned} 0 &= \int_0^\infty \int_{\mathbb{R}} (\rho \partial_t v + q(\rho)\partial_x v) \, \mathrm{d}x\mathrm{d}t \\ &= \int_{V_+} (\rho \partial_t v + q(\rho)\partial_x v) \, \mathrm{d}x\mathrm{d}t + \int_{V_-} (\rho \partial_t v + q(\rho)\partial_x v) \, \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{4.31}$$

Integrating by parts and observing that $v = 0$ on $\partial V_+ \setminus \Gamma$, we have

$$\begin{aligned}\int_{V_+} (\rho\partial_t v + q(\rho)\partial_x v)\,\mathrm{d}x\mathrm{d}t &= -\int_{V_+} (\partial_t\rho + \partial_x q(\rho))\,v\mathrm{d}x\mathrm{d}t + \int_\Gamma (\rho_+ n_2 + q(\rho_+)n_1)\,v\mathrm{d}\gamma \\ &= \int_\Gamma (\rho_+ n_2 + q(\rho_+)n_1)\,v\mathrm{d}\gamma, \end{aligned} \tag{4.32}$$

where ρ_+ denotes the value of ρ on Γ from the V_+ side, $\boldsymbol{n} = (n_1, n_2)$ is the outward unit normal vector on ∂V_+, and $\mathrm{d}\gamma$ denotes the arc length on Γ. Similarly, since $\boldsymbol{n}$ is inward with respect to V_-, we find

$$\int_{V_-} (\rho\partial_t v + q(\rho)\partial_x v)\,\mathrm{d}x\mathrm{d}t = -\int_\Gamma (\rho_- n_2 + q(\rho_-)n_1)\,v\mathrm{d}\gamma, \tag{4.33}$$

where ρ_- denotes the value of ρ on Γ from the V_- side. Thus, plugging (4.32)-(4.33) back into (4.31) and using that v is arbitrary, we find

$$(\rho_+ - \rho_-)n_2 + (q(\rho_+) - q(\rho_-))n_1 = 0, \qquad \text{on } \Gamma. \tag{4.34}$$

Of course, if ρ is continuous on Γ, the above is automatically satisfied since $\rho_+ = \rho_-$. Otherwise, since Γ is given by the equation $x = \sigma(t)$, writing explicitly that

$$\boldsymbol{n} = (n_1, n_2) = \frac{1}{\sqrt{1 + |\sigma'(t)|^2}}(-1, \sigma'(t)), \tag{4.35}$$

we find that

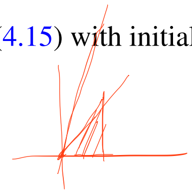
$$\sigma' = \frac{q(\rho_+(t,\sigma)) - q(\rho_-(t,\sigma))}{\rho_+(t,\sigma) - \rho_-(t,\sigma)}, \tag{4.36}$$

known as the *Rankine-Hugoniot condition* for the shock Γ. Therefore, to find the equation of a shock, one in general has to solve a nonlinear ODE of the type above. In general, functions constructed by connecting classical solutions and rarefaction waves in a continuous way are weak solutions. The same is true for shock waves satisfying the Rankine-Hugoniot condition. However, the shock wave solution of Example 4.5 does satisfy the Rankine-Hugoniot condition (double-check this!), and so uniqueness of solutions is not guaranteed even if we require (4.36) to hold. We now look at another example.

EXAMPLE 4.6 (Shock for Burgers). Consider the Burgers equation (4.15) with initial datum

$x = \tau + g(\tau)\, t$

$$g(x) = \begin{cases} 1, & x \le 0, \\ 0, & x > 0. \end{cases} \tag{4.37}$$



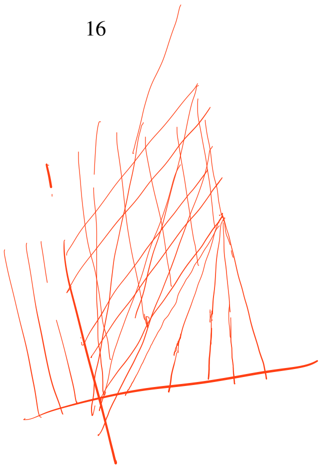
As in Example 4.5, we know that the characteristics are straight line (see Figure 7), but they start intersecting right away at the point $(t,x) = (0,0)$, forming a shock. Since $\rho_+ \equiv 0$ to the right of the shock and $\rho_- \equiv 1$ to the left, the Rankine-Hugoniot condition (4.36) tells us that

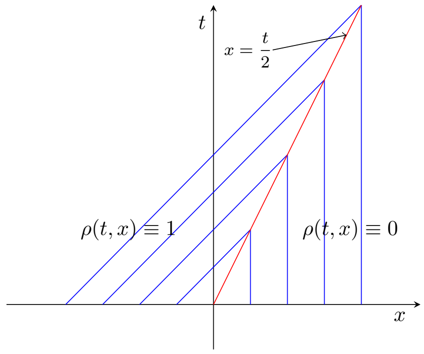
$$\sigma' = 2, \qquad \sigma(0) = 0 \qquad \Rightarrow \qquad x = \sigma(t) = \frac{t}{2}. \tag{4.38}$$

The initial condition for the shock reflects the fact that characteristics intersect starting at the point $(t,x) = (0,0)$. Hence a solution is given by

$$\rho(t,x) = \begin{cases} 1, & x < t/2, \\ 0, & x > t/2. \end{cases} \tag{4.39}$$

Notice that the shock line is the same as the one found in Example 4.5 for $\tilde\rho$. However, the important difference is that here characteristics go *into* a shock, while in Figure 5b characteristics are emanating *from* a shock.





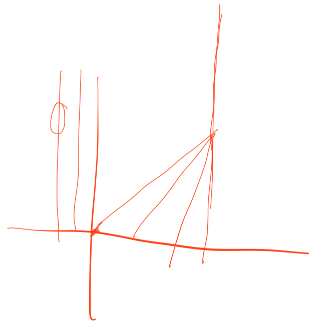
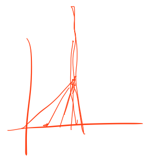


FIGURE 7. Characteristics for the Burgers equation with initial datum (4.37) and the corresponding shock line.

4.4. The entropy condition. As we saw in Example 4.5, weak solutions to conservation laws satisfying the Rankine-Hugoniot condition may not be unique. To recover uniqueness, we introduce the following concept to select the "physically relevant" solution among the possible weak solutions.

Assume that we are given a weak solution ρ to the conservation law (4.3), and suppose that ρ is discontinuous along a shock Γ of equation $x = \sigma(t)$, with left and right limits ρ_- and ρ_+, respectively (see Section 4.3). The shock is called *entropic* if

$$\rho_+(t, \sigma(t)) < \sigma'(t) < \rho_-(t, \sigma(t)), \tag{4.40}$$



for every t for which the shock is defined. In words, the slope of a shock curve is less than the slope of the left-characteristics and greater than the slope of the right-characteristics. Roughly, the characteristics hit forward in time the shock line, so that it is not possible to go back in time along characteristics and hit a shock line (see Figure 7), expressing a sort of irreversibility after a shock. The above considerations lead us to select the entropy solutions as the only physically meaningful ones. On the other hand, if the characteristics hit a shock curve backward in time, the shock wave is to be considered non-physical (see Figure 5b). We state the following theorem, without proof.

THEOREM 4.7. *If $q \in C^2(\mathbb{R})$ is convex (or concave) and g is bounded, there exists a unique entropy solution of the problem*

$$\begin{cases} \partial_t \rho + \partial_x [q(\rho)] = 0, & x \in \mathbb{R},\ t > 0, \\ \rho(0, x) = g(x), & x \in \mathbb{R}. \end{cases} \tag{4.41}$$

4.5. Traffic dynamics. An intense traffic on a highway can be considered as a fluid flow and described by means of macroscopic variables such as the density of cars ρ, their average speed v and their flux q. These three functions are linked by the simple convection relation

$$q = v\rho. \tag{4.42}$$

To construct a model for the evolution of ρ, we assume that there is only one lane and overtaking is not allowed, there are no exit or entrance gates, and the average speed is not constant and depends on the density alone, namely $v = v(\rho)$. Since we expect that the speed decreases as the density increases, we can assume that $v'(\rho) \le 0$. Moreover, we may think that there is a maximum velocity $v_m > 0$, given by a speed limit, and that traffic slows down and stops at the maximum density $\rho_m > 0$. The simplest model consistent with the above considerations gives

$$v(\rho) = v_m \left(1 - \frac{\rho}{\rho_m}\right) \qquad \Rightarrow \qquad q(\rho) = v_m \rho \left(1 - \frac{\rho}{\rho_m}\right). \tag{4.43}$$

Since

$$q'(\rho) = v_m \left(1 - \frac{2\rho}{\rho_m}\right), \qquad q''(\rho) = -\frac{2v_m}{\rho_m} < 0, \tag{4.44}$$

so q is strictly concave. Writing down the conservation law (4.41) explicitly, we find

$$\begin{cases} \partial_t \rho + v_m \left(1 - \dfrac{2\rho}{\rho_m}\right) \partial_x \rho = 0, & x \in \mathbb{R},\ t > 0, \\ \rho(0,x) = g(x), & x \in \mathbb{R}. \end{cases} \tag{4.45}$$

The associated characteristics are given as in (4.8) by the straight lines

$$x = \tau + q'(g(\tau))t, \qquad \tau \in \mathbb{R}. \tag{4.46}$$

We now analyze a few possible situations.

EXAMPLE 4.8 (The green light problem). Suppose that traffic is standing at a red light, placed at $x = 0$, while the road ahead is empty. Accordingly, the initial density profile is

$$g(x) = \begin{cases} \rho_m, & x \le 0, \\ 0, & x > 0. \end{cases} \tag{4.47}$$

At time $t = 0$ the traffic light turns green and we want to describe the car flow evolution for $t > 0$. At the beginning, only the cars nearer to the light start moving while most remain standing. Since

$$q'(g(\tau)) = \begin{cases} -v_m, & \tau \le 0, \\ v_m, & \tau > 0, \end{cases} \tag{4.48}$$

and the characteristics (see Figure 8) are the straight lines

$$x = \tau - v_m t, \qquad \tau \le 0, \tag{4.49}$$
$$x = \tau + v_m t, \qquad \tau > 0. \tag{4.50}$$

We are therefore in the presence of a rarefaction wave. We therefore look for a solution of the form $\rho(t,x) = h(x/t)$, where, from (4.45), h needs to satisfy

$$-\frac{x}{t^2} h' + \frac{1}{t} v_m \left(1 - \frac{2h}{\rho_m}\right) h' = 0 \qquad \Rightarrow \qquad h(x/t) = \frac{\rho_m}{2}\left(1 - \frac{x}{v_m t}\right). \tag{4.51}$$

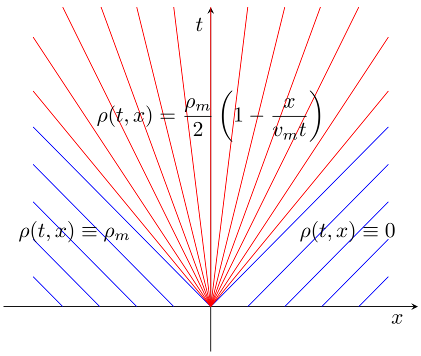


FIGURE 8. The rarefaction wave in the green light problem.

Therefore, the unique entropy solution is given by

$$\rho(t,x) = \begin{cases} \rho_m, & x \le -v_m t, \\ \dfrac{\rho_m}{2}\left(1 - \dfrac{x}{v_m t}\right), & -v_m t < x < v_m t, \\ 0, & x \ge v_m t. \end{cases} \tag{4.52}$$

EXAMPLE 4.9 (Traffic jam ahead). Suppose that the initial density profile is

$$g(x) = \begin{cases} \rho_m/8, & x \le 0, \\ \rho_m, & x > 0. \end{cases} \tag{4.53}$$

For $x > 0$, the density is maximal and therefore the traffic is bumper-to-bumper. The cars on the left move with speed $v = \frac{7}{8}v_m$ so that we expect congestion propagating back into the traffic. The characteristics are

$$x = \tau + \frac{3}{4}v_m t, \qquad \tau \leq 0, \tag{4.54}$$

$$x = \tau - v_m t, \qquad \tau > 0. \tag{4.55}$$

Since they intersect (see Figure 9), we are in presence of a shock. From the Rankine-Hugoniot condition (4.36), we find

$$\sigma' = -\frac{v_m}{8}, \qquad \sigma(0) = 0 \qquad \Rightarrow \qquad x = \sigma(t) = -\frac{v_m}{8}t. \tag{4.56}$$

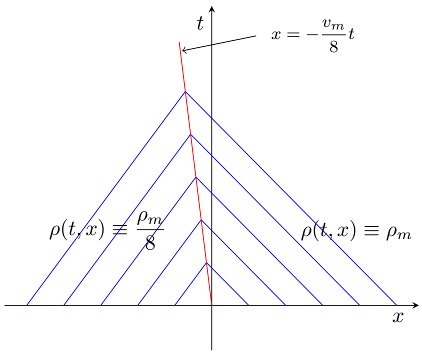


FIGURE 9. The shock wave in the traffic jam problem.

Note that the slope of the shock is negative: the shock propagates back with speed $-\frac{1}{8v_m}$, as it is revealed by the braking of the cars, slowing down because of a traffic jam ahead. The unique entropy solution is given by

$$\rho(t,x) = \begin{cases} \frac{\rho_m}{8}, & x < -\frac{v_m}{8}t, \\ \rho_m, & x > \frac{v_m}{8}t. \end{cases} \tag{4.57}$$

References

[1] S. Salsa, *Partial differential equations in action*, Third, Unitext, vol. 99, Springer, 2016. From modelling to theory, La Matematica per il 3+2.