MATH3075/3975 Financial Derivatives

9: Sensitivities aka the Greeks

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Sensitivities of options

The most important sensitivities of options are:

Delta = Δ

Gamma = Γ

Theta = θ

Vega = υ

Rho = ρ

Black-Scholes formulae for call and put options

$$c = SN(d_+) - Ke^{-r(T-t)}N(d_-)$$

$$p = Ke^{-r(T-t)}N(-d_-) - SN(-d_+)$$

Notice that c and p as real-valued functions of several variables:

$$c = c(S, T-t, r, \sigma, K)$$

$$p = p(S, T-t, r, \sigma, K)$$

Once prices c and p are calculated, what if the values of variables (or parameters) change?

Measures of sensitivity

The question reads: how sensitive a portfolio's value is to small changes in any of the variables that contribute to its marked-to-market value?

These variables are: $S, T-t, r, \sigma$ and K.

Each one of the Greeks indicates the change in the value of the position as a result of a small change in the corresponding variable. Formally, the Greeks are simply partial derivatives of functions c and p.

Delta Δ

In mathematical terms: the delta of an option is the first order derivative of the option's price with respect to S. As such, the delta carries the units of the stock price.

For standard call and put options, we have:

$$\text{for a call: } \Delta(c) = \partial c/\partial S$$

$$\text{for a put: } \Delta(p) = \partial p/\partial S$$

Notice that: $\Delta(p) = \Delta(c) - 1$

$$\Delta(S) = \partial S/\partial S = 1$$

Black-Scholes delta

The deltas of a call and a put on the same stock with identical strike price K and time to expiration must satisfy the following relationship:

$$\Delta(p) = \Delta(c) - 1$$

Using the Black and Scholes formula, we obtain:

$$\Delta(c) = N(d_+) \Rightarrow 0 < \Delta(c) < 1$$

$$\Delta(p) = N(d_+) - 1 \Rightarrow -1 < \Delta(p) < 0$$

Gamma Γ

The gamma of an option measures the change in delta when the price of the underlying asset S changes.

Formally, gamma is the second order derivative of the price of an option with respect to the stock price:

$$\Gamma(c) = \partial\Delta(c)/\partial S = \partial^2 c/\partial S^2$$

$$\Gamma(p) = \partial\Delta(p)/\partial S = \partial^2 p/\partial S^2$$

In particular,

$$\Gamma(c) = \Gamma(p)$$

$$\Gamma(S) = 0.$$

Gamma Γ

Gamma of an option is the change in its delta when the price of the stock changes. Let us assume that:

$$\Gamma = 0.0181, \quad \Delta(c) = 0.6151, \quad \Delta(p) = -0.3849.$$

Then, if the stock price increases from $100 to $101:

$\Delta(c)$ increases to 0.6332

$\Delta(p)$ increases to -0.3668

and if the stock price decreases to $99:

$\Delta(c)$ decreases to 0.5970

$\Delta(p)$ decreases to -0.4030

Black-Scholes gamma

The gammas of a put and a call are equal.

In the Black-Scholes model, we know that

$$\Delta(c) = n(d_+) \quad \text{and} \quad \Delta(p) = n(d_+) - 1$$

and thus the partial derivatives of these deltas with respect to S are equal and strictly positive.

The equality of gammas can also be deduced from the put-call parity and thus it holds in any market model.

Gamma Γ

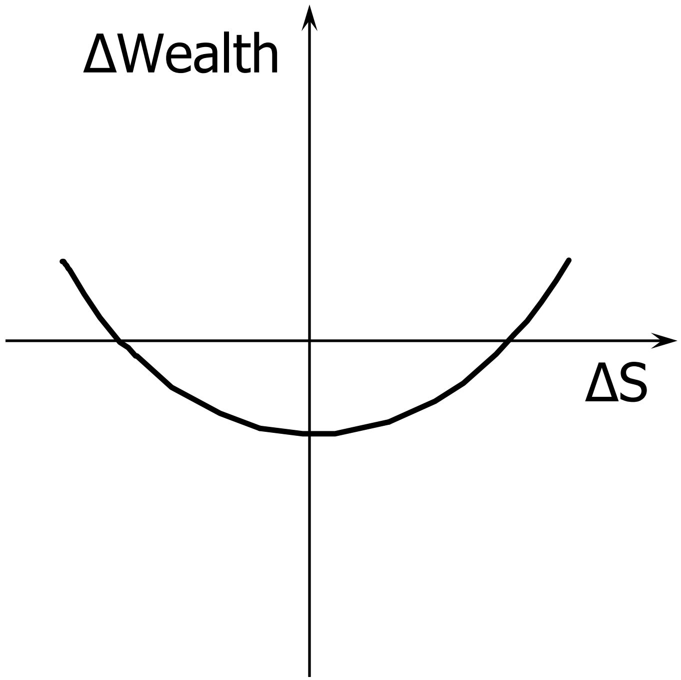
The gamma of any portfolio of assets and options is the change of the portfolio's delta, which is due to a small change in the underlying asset price.

As the second derivative with respect to the price S, the gamma is the sensitivity of the portfolio's value to larger changes of the underlying asset price.

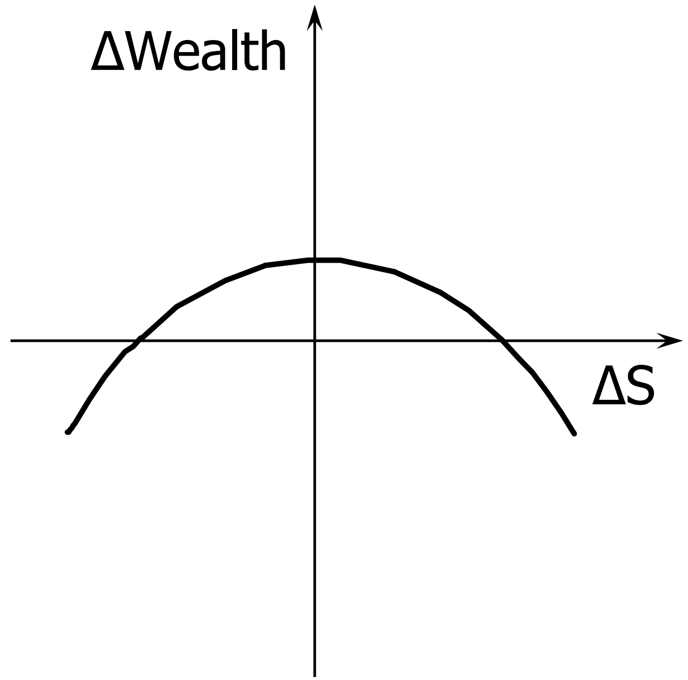
Notice that gamma may be either positive or negative (price convexity or concavity).

Interpretation of gamma

The sign of the gamma affects the second order term in Taylor's expansion of portfolio's wealth.



Positive gamma



Negative gamma

Theta θ

Let us denote $\tau = T - t$. Theta measures are given by:

$$\theta(c) = \partial c/\partial \tau \qquad \theta(p) = \partial p/\partial \tau$$

Thetas are positive but the they are always reported as negative values. The negative sign indicates that as time passes, the time to expiration decreases and so does the option's value. This loss of the option's value is labeled the option's time decay. Prices of calls and puts decrease when the time to expiration decreases.

Notice that for the stock we have: $\theta(S) = 0$.

Theta θ

Theta gives the sensitivity of the option's price to a small change in the time to expiration:

$$\theta(c) = \partial c/\partial \tau \qquad \theta(p) = \partial p/\partial \tau$$

Theta is given in terms is $ per 1 year.

Assume, for instance, that $\theta(c) = -\$12.2607$.

If time to expiration increases (decreases) by one year, then the call price increases (decreases) by $12.2607.

This means $12.2607/365 = $0.0335 per day.

Vega υ

Vega measures the sensitivity of the option's market price to small changes in the volatility of the underlying asset's return:

$$\upsilon(c) = \partial c/\partial\sigma \qquad \upsilon(p) = \partial p/\partial\sigma$$

Vega is given in terms of $ per 1% change in σ.

For instance, if $\upsilon = 0.2684$ then the price of the option will increase when the volatility of the underlying stock price increases.

Observe that for the stock we have $\upsilon(c)=0$ since the function S satisfies $\partial S/\partial\sigma = 0$.

Rho ρ

Rho measures the sensitivity of the price of an option to small changes in the rate of interest r, that is,

$$\rho(c) = \partial c/\partial r \qquad \rho(p) = \partial p/\partial r$$

Rho is given in terms of $ per 1% change of interest rate.

Notice that for the underlying stock, we have $\rho(S) = 0$

since the function S satisfies $\partial S/\partial r = 0$.

Rho ρ

Rho is a measure of sensitivity of the option's price to small changes in the rate of interest.

For example, we may have the following values of rho:

call option	put option
$\rho = 0.2525$	$\rho = -\ 0.2214$

Rho of the call option (respectively, put) is positive (respectively, negative). This feature can be explained by funding costs of hedging. Cash is borrowed to hedge a call and cash is lent when hedging a put.

Summary of the Greeks

Position	Delta	Gamma	Vega	Theta	Rho
LONG STOCK	1	0	0	0	0
SHORT STOCK	-1	0	0	0	0
LONG CALL	+	+	+	-	+
SHORT CALL	-	-	-	+	-
LONG PUT	-	+	+	-	-
SHORT PUT	+	-	-	+	+

Sensitivities of portfolios

1. A portfolio is a combination of traded primary assets and financial derivatives (e.g., call and put options).
2. All the sensitivity measures are partial derivatives of portfolio's value with respect to relevant variables.
3. The derivative of any linear combination of functions is the combination of the derivatives of these functions. Hence the sensitivities of a portfolio of securities are linear combinations of sensitivities of components.

Greeks based dynamic strategies

Greeks based trading strategies are opened and later maintained in order to get a desired level of sensitivity.

Mostly, these dynamic strategies are set to attain zero sensitivities at any given date and thus they need to be adjusted from time to time.

1. Delta-neutral portfolios
2. Delta-gamma-neutral portfolios
3. Delta-gamma-vega-rho-neutral portfolios

Delta-neutral portfolios

Suppose that an issuer wrote n(c) calls and wishes to protect the value of his/her portfolio against possible adverse movement of the underlying asset price.

He uses shares of the underlying asset S in a quantity that guarantees that a small change of the price of S will not have any impact on his/her position.

A portfolio is said to be delta-neutral if Δ(portfolio) = 0.

Delta-neutral call-stock portfolio

$$V(\text{portfolio}) = Sn(S) + cn(c)$$

$$\Delta(\text{portfolio}) = \Delta(S)n(S) + \Delta(c)n(c)$$

$$\Delta(\text{portfolio}) = 0 \quad \Leftrightarrow \quad n(S) + \Delta(c)n(c) = 0$$

Hence $n(S) = - n(c)\Delta(c)$.

The negative sign indicates that the calls and the shares of the underlying stock must be held in opposite direction: short calls and long shares (long calls and short shares).

Example A: a risk-neutral call-stock portfolio

We sold 10 CBOE calls with delta 0.54.

Each CBOE call covers 100 shares. Recall that

$$n(S) = - n(c)\Delta(c)$$

$$\Delta(c) = 0.54 \text{ and } n(c) = -1000$$

$$n(s) = -[-1{,}000(0.54)] = 540$$

The delta-neutral portfolio consists of:

10 short CBOE calls and 540 long shares.

Black-Scholes numerical example

Let S=100, K=100, r=8%, τ= 180 days and σ=0.3.

	Call	Put
Price	\$10.3044	\$6.4360

The Greeks:

Delta = Δ	0.6151	-0.3849
Theta = θ	-0.0335	-0.0125
Gamma = Γ	0.0181	0.0181
Vega = υ	0.2684	0.2684
Rho = ρ	0.2525	-0.2215

Black-Scholes numerical example

Let the call price be \$10.3044 and delta $\Delta(c) = 0.6151$.

With 100 CBOE short calls, we obtain

$$n(S) = -\Delta(c)n(c)$$

$$n(c) = -10{,}000$$

$$n(S) = -(0.6151)[-10{,}000] = 6{,}151 \text{ (shares)}$$

The value of this portfolio of calls/shares is:

$$V = -10{,}000(\$10.3044) + 6{,}151(\$100)$$

$$V = \$512{,}056$$

Suppose that the stock price rises by $1

$$S_{UP} = 100 + 1 = \$101/\text{share}$$

$$V = -10{,}000(\$10.3044 + \$0.6151) + 6{,}151(\$101)$$

$$V = -\ 10{,}000(\$10.3044) + 6{,}151(\$100)$$

$$-10{,}000(\$0.6151) + \$1(6{,}151)$$

$$V = \$512{,}056 - \$6{,}151 + \$6{,}151$$

$$V = \$512{,}056$$

Suppose that the stock price falls by $1

$$S_{DOWN} = 100 - 1 = \$99/\text{share}$$

$$V = -\ 10{,}000(\$10.3044 - \$.6151) + 6{,}151(\$99)$$

$$V = -\ 10{,}000(\$10.3044) + 6{,}151(\$100)$$

$$-\ 10{,}000(-\ \$.6151) - \$1(6{,}151)$$

$$V = \$512{,}056 + \$6{,}151 - \$6{,}151$$

$$V = \$512{,}056$$

Portfolio's P&L

Price/share:	+$1	-$1
Shares	+$6,151	- $6,151
Calls	- $6,151	+$6,151
Portfolio	$0	$0

The portfolio consisting of 100 short calls and 6,151 long shares is delta-neutral:

Delta-neutral put-stock portfolios

$$V(\text{portfolio}) = Sn(S) + pn(p)$$

$$\Delta(\text{portfolio}) = \Delta(S)n(S) + \Delta(p)n(p)$$

$$\Delta(\text{portfolio}) = 0 \quad \Leftrightarrow \quad n(S) + \Delta(p)n(p) = 0$$

Hence $n(S) = - n(p)\Delta(p)$

Since the put option delta is negative, then the negative sign means in fact that the puts and the underlying asset must be held in the same direction (e.g., long puts and long shares).

Example B: a put-stock portfolio

We bought 10 CBOE puts with delta -0.70.

Each put covers 100 shares. We have

$$n(S) = - n(p)\Delta(p)$$

$$\Delta(p) = - 0.70 \text{ and } n(p) = 1000$$

$$n(S) = -1{,}000(-0.70) = 700$$

Hence the delta-neutral portfolio consists of:

10 long puts and 700 long shares.

Black-Scholes numerical example

Put price is \$6.4360 and delta $\Delta(p) = -0.3849$.

The delta-neutral position with 100 long CBOE puts requires the holding of $n(S)$ shares where

$$n(S) = -\Delta(p)n(p)$$

$$n(S) = -(-0.3849)[10{,}000] = 3{,}849 \text{ (shares)}$$

The value of this portfolio is:

$$V = 10{,}000(\$6.4360) + 3{,}849(\$100)$$

$$V = \$449{,}260$$

Suppose that the stock price rises by $1

$$S_{UP} = 100 + 1 = \$101/\text{share}$$

$$V = 10{,}000(\$6.4360 - \$0.3849) + 3{,}849(\$101)$$

$$V = 10{,}000(\$6.4360) - 3{,}849(\$100)$$

$$- 10{,}000(\$0.3849) + 3{,}849(\$1)$$

$$V = \$449{,}260 - \$3{,}849 + \$3{,}849$$

$$V = \$449{,}260$$

Suppose that the stock price falls by $1

$$S_{DOWN} = 100 - 1 = \$99/\text{share}$$

$$V = 10{,}000(\$6.4360 + \$0.3849) + 3{,}849(\$99)$$

$$V = 10{,}000(\$6.4360) + 3{,}849(\$100)$$
$$+ 10{,}000(\$0.3849) - \$1(3{,}849)$$

$$V = \$449{,}260 + \$3{,}849 - \$3{,}849$$

$$V = \$449{,}260$$

Portfolio's P&L

Price/share:	+$1	-$1
Shares	+$3,849	-$3,849
Calls	+(-$3,849)	-(-$3,849)
Portfolio	$0	$0

The portfolio consisting of 100 long puts and 3,849 long shares is delta-neutral.

Delta-neutral calls, puts and the stock portfolio

Consider a portfolio of calls, puts and shares

$$V(\text{portfolio}) = Sn(S) + cn(c) + pn(p)$$

$$\Delta(\text{portfolio}) = \Delta(S)n(S) + \Delta(c)n(c) + \Delta(p)n(p)$$

Since $\Delta S = 1$, the delta-neutral portfolio requires

$$n(S) = -\Delta(c)n(c) - \Delta(p)n(p)$$

Special case: long share, long call and short put

From the put-call parity we deduce that the portfolio:

long share of the underlying stock, long put option and short call option is always delta-neutral since

$$\Delta(\text{portfolio})$$

$$= 1+ \Delta(p)n(p) +\Delta(c)n(c)$$

$$= 1 + [\Delta(c) -1] + \Delta(c) = 0$$

Example C

We short 20 calls and 20 puts whose deltas are 0.70 and -0.30, respectively. Every call and every put covers 100 shares.

How many shares of the underlying stock we must purchase in order to create a delta-neutral position? The answer is:

$$n(S) = -\Delta(c)n(c) + [-\Delta(p)]n(p)$$

$$n(S) = -(0.70)(-2{,}000) – (-0.30)(-2{,}000)$$

$$n(S) = 800$$

Example C: Portfolio's P&L

Price/share	+$1	-$1
Shares	+$800	-$800
Calls	-$1,400	+$1,400
Puts	+$600	-$600
Portfolio	$0	$0

The portfolio consisting of 20 short calls, 20 short put options and 800 long shares is delta-neutral.

Example D: Straddle

Long straddle: long 15 puts and long 15 calls with the same underlying stock, strike and time to expiration.

Let us assume that:

$$\Delta(c) = 0.64 \quad \text{and} \quad \Delta(p) = -0.36.$$

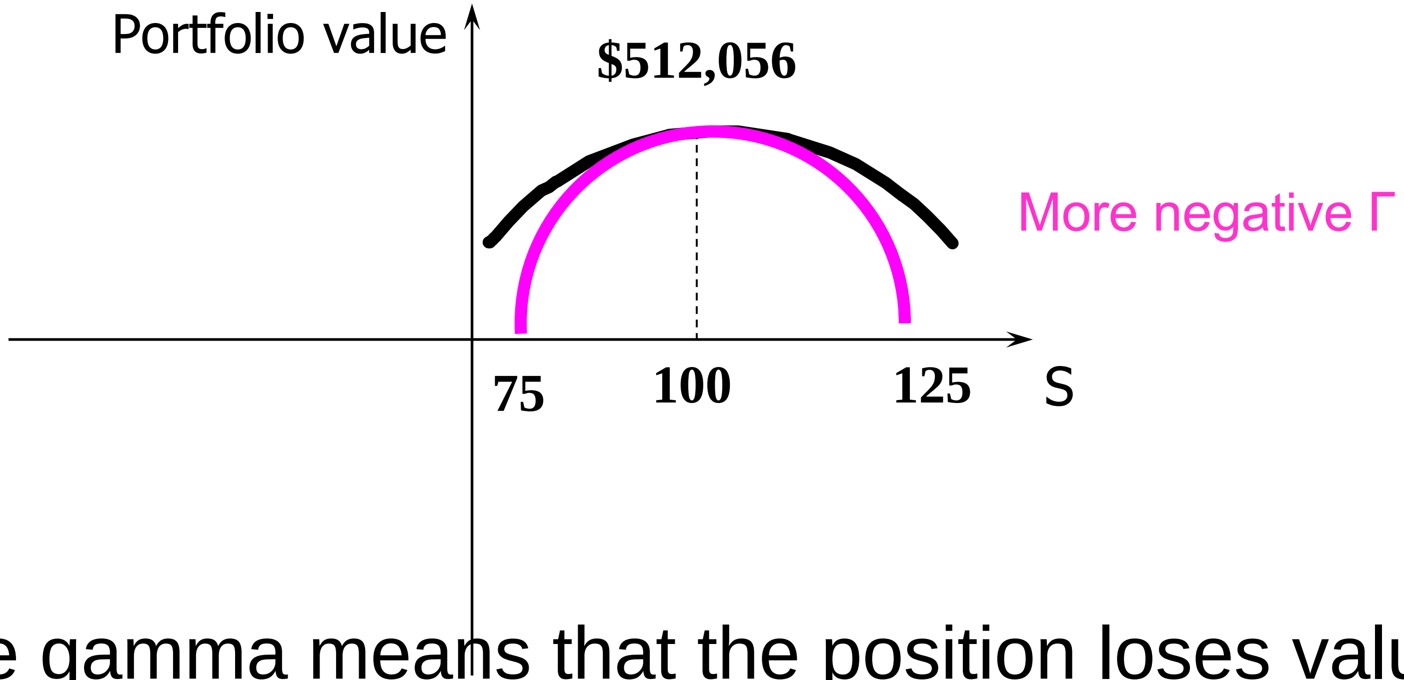
Hence

$$\Delta(\text{straddle}) = 15[0.64+(-0.36)] = \$4.20$$

Conclusion: we may short 4.20 shares of the stock to delta-neutralize this straddle.

Interpretation of gamma

Suppose that the delta-neutral portfolio with 100 short calls and 6,151 long shares has Γ = -$181.



Negative gamma means that the position loses value when the stock price moves more and more away from its initial value.

Example E: Delta-gamma neutral portfolio

Assume that: $\Delta(c) = 0.70$, $\Delta(p) = -0.30$ and $\Gamma = 0.2345$. If we are holding a long call and a short put, then:

$$\Delta = 0.70 - (-0.30) = 1.00$$

$$\Gamma = 0.2345 - 0.2345 = 0$$

Holding the underlying asset long, a long put and a short call yields a portfolio with:

$$\Delta = 1.00 - 0.70 + (-0.30) = 0$$

$$\Gamma = 0 + 0.2345 - 0.2345 = 0$$

We thus see that the portfolio is delta-gamma-neutral.