MATH3075/3975
Financial Derivatives

School of Mathematics and Statistics
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Tutorial sheet 4

Background: Section 2.2 – Single-Period Market Models.

Exercise 1 Consider the elementary market model $\mathcal{M} = (B, S)$ on a sample space $\Omega = \{\omega_1, \omega_2\}$ with $\mathbb{P}(\omega_1) = p \in (0, 1)$. We assume that $S_0 > 0$ and $0 < d < 1 + r < u$.

(a) Find the probability measure $\widehat{\mathbb{P}}$ such that $\mathbb{E}_{\widehat{\mathbb{P}}}(\widehat{B}_T) = \widehat{B}_0$ where the process $\widehat{B}$ is defined by $\widehat{B}_t = B_t/S_t$ for $t = 0, 1$. Compute the Radon-Nikodym density L of $\widehat{\mathbb{P}}$ with respect to the martingale measure $\widetilde{\mathbb{P}}$ and show directly that $\mathbb{E}_{\widetilde{\mathbb{P}}}(L) = 1$.

(b) Let $X = g(S_T)$ be any contingent claim. Show that the price $\pi_0(X)$ satisfies

$$\pi_0(X) = S_0 \, \mathbb{E}_{\widehat{\mathbb{P}}}\left(\frac{X}{S_T}\right).$$

(c) Consider the put option with the payoff $P_T(K) = (K - S_T)^+$ for some $K > 0$. Show that the price $P_0(K)$ admits the following representation

$$P_0(K) = K(1 + r)^{-1} \, \widetilde{\mathbb{P}}(S_T < K) - S_0 \, \widehat{\mathbb{P}}(S_T < K).$$

Find an analogous representation for the price $C_0(K)$ of the call option with strike K.

(d) Show that the extended model $\mathcal{M}^e = (B, S, P(K))$ is arbitrage-free, in the sense of Definition 2.2.3 from the course notes. Here $P(K) = (P_0(K), P_T(K))$ is the price process of the put option for some fixed strike $K > 0$.

(e) Let a strike K such that $S_0 d < K < S_0 u$ be fixed. Consider the modified market model $\mathcal{N} = (B, P(K))$ where $P(K)$ is now traded at time 0 at the price $P_0(K)$. Does the price of an arbitrary claim X computed in $\mathcal{N} = (B, P(K))$ coincides with its arbitrage price computed in the original model $\mathcal{M} = (B, S)$? In particular, find the arbitrage price at time 0 for the claim $X = S_T$ in the model $\mathcal{N}$.

Exercise 2 Verify the equality (see Section 2.2)

$$\widehat{V}_t := \frac{V_t}{B_t} = \left(x - \sum_{j=1}^{n} \phi^j S_0^j \right) + \sum_{j=1}^{n} \phi^j \widehat{S}_t^j \tag{1}$$

for $t \in \{0, 1\}$ with $B_0 = 1$ and $B_1 = 1 + r$, and derive the equality

$$\widehat{G}_1(x, \phi) = \sum_{j=1}^{n} \phi^j \Delta \widehat{S}_1^j = \sum_{j=1}^{n} \phi^j (\widehat{S}_1^j - \widehat{S}_0^j) \tag{2}$$

where $\widehat{G}_1(x, \phi) := \widehat{V}_1(x, \phi) - \widehat{V}_0(x, \phi)$.

Exercise 3 Consider the market model $\mathcal{M} = (B, S)$ with $k = 3$, $n = 1$, $r = \frac{1}{9}$, $S_0 = 5$ and the random stock price S_1 given by the table

	ω_1	ω_2	ω_3
S_1	$\frac{60}{9}$	$\frac{40}{9}$	$\frac{30}{9}$

Find the class $\mathbb{M}$ of all risk-neutral probability measures for this market model by making use of Definition 2.2.4.

Exercise 4 We consider the market model $\mathcal{M} = (B, S^1, S^2)$ introduced in Example 2.2.1 in the course notes but with $k = 4$ and the stock prices in state ω_4 given by $S_1^1(\omega_4) = \frac{20}{9}$ and $S_1^2(\omega_4) = \frac{120}{9}$. The interest rate equals $r = \frac{1}{9}$. Stock prices at time $t = 0$ are given by $S_0^1 = 5$ and $S_0^2 = 10$, respectively, and stock prices at time $t = 1$ are given in the following table

	ω_1	ω_2	ω_3	ω_4
S_1^1	$\frac{60}{9}$	$\frac{60}{9}$	$\frac{40}{9}$	$\frac{20}{9}$
S_1^2	$\frac{40}{3}$	$\frac{80}{9}$	$\frac{80}{9}$	$\frac{120}{9}$

(a) Compute explicitly the random variables $V_1(x, \phi)$, $G_1(x, \phi)$, $\widehat{V}_1(x, \phi)$ and $\widehat{G}_1(x, \phi)$.

(b) Does $G_1(x, \phi)$ (or $\widehat{G}_1(x, \phi)$) depend on the initial endowment x?

Exercise 5 (MATH3975) Consider again the market model $\mathcal{M} = (B, S^1, S^2)$ introduced in Exercise 4.

(a) Give an explicit representation for the linear space $\mathbb{W} \subset \mathbb{R}^4$.

(b) Find explicitly the linear space $\mathbb{W}^\perp \subset \mathbb{R}^4$.

(c) Is the market model $\mathcal{M} = (B, S^1, S^2)$ arbitrage free?

(d) Find the class $\mathbb{M}$ of all risk-neutral probability measures for $\mathcal{M}$ using the equality $\mathbb{M} = \mathbb{W}^\perp \cap \mathcal{P}^+$.

Exercise 6 (MATH3975) Give a proof of Proposition 2.2.1.