MATH3975 Assignment 1: Solutions

1. **Single-period market model [6 marks]** Consider a single-period market model $\mathcal{M} = (B, S)$ on a finite sample space $\Omega = \{\omega_1, \omega_2, \omega_3\}$. We assume that the money market account B equals $B_0 = 1$ and $B_1 = 4$ and the stock price S satisfies $S_0 = 2.5$ and $S_1 = (18, 10, 2)$. The real-world probability $\mathbb{P}$ is such that $\mathbb{P}(\omega_i) = p_i > 0$ for $i = 1, 2, 3$.

   (a) Find the class $\mathbb{M}$ of all martingale measures for the model $\mathcal{M}$. Is the market model $\mathcal{M}$ arbitrage-free? Is this market model complete?

   Answer: [1 mark] We need to solve: $q_1 + q_2 + q_3 = 1,\ 0 < q_i < 1$ and (since $1 + r = 4$)

   $$\mathbb{E}_{\mathbb{Q}}(S_1) = 18q_1 + 10q_2 + 2q_3 = (1 + r)S_0 = 10$$

   or, equivalently,

   $$\mathbb{E}_{\mathbb{Q}}(S_1 - (1 + r)S_0) = 8q_1 - 8q_3 = 0.$$

   Let $q_2 = \alpha$. Then $q_1 = q_3 = \frac{1-\alpha}{2}$ where $0 < \alpha < 1$. Hence

   $$\mathbb{M} = \Big\{(q_1, q_2, q_3) \,\big|\, q_1 = q_3 = \frac{1 - \alpha}{2},\ q_2 = \alpha,\ 0 < \alpha < 1\Big\}.$$

   The market model $\mathcal{M}$ is arbitrage-free since $\mathbb{M} \neq \emptyset$. Moreover, it is incomplete since the uniqueness of a martingale measure fails to hold.

   (b) Find the replicating strategy $(\varphi_0^0, \varphi_0^1)$ for the claim $X = (5, 1, -3)$ and compute the arbitrage price $\pi_0(X)$ at time 0 through replication.

   Answer: [1 mark] *First solution.* The wealth process of a portfolio $(\varphi_0^0, \varphi_0^1)$ satisfies for $t = 0, 1$

   $$V_0(\varphi) = \varphi_0^0 B_0 + \varphi_0^1 S_0, \quad V_1(\varphi) = \varphi_0^0 B_1 + \varphi_0^1 S_1.$$

   Replication of a claim X means that $V_1(\varphi)(\omega_i) = X(\omega_i)$ for $i = 1, 2, 3$. Hence to find a replicating strategy for X, we need to solve the following equations

   $$\begin{aligned} 4\varphi_0^0 + 18\varphi_0^1 &= 5, \\ 4\varphi_0^0 + 10\varphi_0^1 &= 1, \\ 4\varphi_0^0 + 2\varphi_0^1 &= -3. \end{aligned}$$

   We obtain $(\varphi_0^0, \varphi_0^1) = (-1, 0.5)$ and thus $\pi_0(X) = \varphi_0^0 B_0 + \varphi_0^1 S_0 = -1 + 0.5(2.5) = 0.25$. Hence at time 0 we buy 0.25 shares of stock. For this purpose, after receiving 0.25 units of cash from the buyer of the claim X, we borrow one unit of cash in the money market.

   Second solution. Alternatively, one may use a portfolio $(x, \varphi) \in \mathbb{R}^2$ and represent the wealth as follows: $V_0(x, \varphi) = x$ and

   $$V_1(x, \varphi) = (x - \varphi S_0)(1 + r) + \varphi S_1 = x(1 + r) + \varphi\big(S_1 - S_0(1 + r)\big) = xB_1 + \varphi\big(S_1 - S_0 B_1\big).$$

   Then we solve the following equations

   $$\begin{aligned} 4x + 8\varphi &= 5, \\ 4x + 0\varphi &= 1, \\ 4x - 8\varphi &= -3. \end{aligned}$$

   From the second equation, we obtain $\pi_0(X) = x = 0.25$ and thus $\varphi = 0.5$.

(c) Compute the arbitrage price $\pi_0(X)$ using the risk-neutral valuation formula with an arbitrary martingale measure $\mathbb{Q}$ from $\mathbb{M}$.

Answer: [1 mark] For any $0 < q_2 = \alpha < 1$ and $q_1 = q_3 = \frac{1-\alpha}{2}$, the risk-neutral valuation formula yields

$$\pi_0(X) = \mathbb{E}_{\mathbb{Q}}(X/B_1) = \frac{1}{4}\Big(5\,\frac{1-\alpha}{2} + \alpha - 3\,\frac{1-\alpha}{2}\Big) = 0.25.$$

(d) Show directly that the contingent claim $Y = (Y(\omega_1), Y(\omega_2), Y(\omega_3)) = (10, 8, -2)$ is not attainable and find the range of arbitrage prices for Y using the class $\mathbb{M}$ of martingale measures.

Answer: [1 mark] We need to solve equations

$$\begin{aligned} 4\varphi_0^0 + 18\varphi_0^1 &= 10, \\ 4\varphi_0^0 + 10\varphi_0^1 &= 8, \\ 4\varphi_0^0 + 2\varphi_0^1 &= -2. \end{aligned}$$

The strategy $(\varphi_0^0, \varphi_0^1) = \big(\frac{11}{8}, \frac{1}{4}\big)$ solves the first two equations, but fails to satisfy the last. We now compute the range of prices for Y consistent with the no-arbitrage principle. We have

$$\pi_0(Y) = \mathbb{E}_{\mathbb{Q}}(Y/B_1) = \frac{1}{4}\Big(10\,\frac{1-\alpha}{2} + 8\alpha - 2\,\frac{1-\alpha}{2}\Big) = 1 + \alpha.$$

Since $\alpha \in (0,1)$, it is clear the range of prices $\pi_0(Y)$ consistent with the no-arbitrage principle is the open interval $(1, 2)$.

(e) For the contingent claim $Z = (20, 16, -4)$, find the minimal initial endowment $\bar{x}$ for which there exists a portfolio $(\bar{x}, \bar{\varphi})$ with $V_0(\bar{x}, \bar{\varphi}) = \bar{x}$ and such that the inequality $V_1(\bar{x}, \bar{\varphi})(\omega_i) \geq Z(\omega_i)$ holds for $i = 1, 2, 3$.

Answer: [1 mark] Recall that $V_1(x, \varphi) = (x - \varphi S_0)(1 + r) + \varphi S_1$. We thus need to find the minimal x for which the following inequalities are satisfied for some $\varphi \in \mathbb{R}$:

$$\begin{aligned} 4x + \varphi(18 - 10) &\geq 20, \\ 4x + \varphi(10 - 10) &\geq 16, \\ 4x + \varphi(2 - 10) &\geq -4. \end{aligned}$$

Equivalently, we search for the minimal x such that there exists $\varphi \in \mathbb{R}$ so that

$$x \geq -2\varphi + 5, \quad x \geq 4, \quad x \geq 2\varphi - 1.$$

It is easy to check that the solution $\bar{x} = 4$ is attained for every $\bar{\varphi} \in [0.5,\, 2.5]$.

(f) Can we interpret the number $\bar{x}$ as an arbitrage price for Z? Can we complete the market by assuming that Z is an additional primary asset traded at time 0 at the initial price equal to 3?

Answer: [1 mark] The number $\bar{x}$ cannot be interpreted as an arbitrage price for Z since if Z is sold at the price $\bar{x}$, then arbitrage opportunities arise for the seller of Z if we take an arbitrary $\bar{\varphi}$ from the interval $[0.5,\, 2.5]$. For instance, for $(\bar{x}, \bar{\varphi}) = (4, 1)$ the seller's (random) profit at time 1 equals $(4, 0, 12)$. Furthermore, since $Z = 2Y$ we deduce from part (d) that the range of prices $\pi_0(Z)$ consistent with the no-arbitrage principle is the open interval $(2, 4)$. Since 3 belongs to the interval $(2, 4)$, the extended market is arbitrage-free.

The unique martingale measure for the extended market model can be found from the equality

$$\pi_0(Z) = \mathbb{E}_{\mathbb{Q}}(Z/B_1) = \frac{1}{4}\Big(20\,\frac{1-\alpha}{2} + 16\alpha - 4\,\frac{1-\alpha}{2}\Big) = 2(1+\alpha) = 3,$$

which yields $\alpha = 0.5$. Hence we deduce from part (a) that the unique martingale measure for the extended market equals $(q_1, q_2, q_3) = (0, 25, 0.5, 0.25)$. We conclude that the extended market model is arbitrage-free and complete and thus any contingent claim can be replicated in the model where B, S and Z are traded assets.

2. **Static hedging with options [4 marks]** We consider a path-independent European claim $X = g(S_T)$ with maturity T and we assume that the payoff function $g : \mathbb{R}_+ \to \mathbb{R}$ is twice continuously differentiable.

   (a) Using the integration by parts formula, show that for arbitrary $x, y \in \mathbb{R}_+$

$$g(x) - g(y) = g'(y)(x-y) + \int_0^y (z-x)^+ g''(z)\,dz + \int_y^\infty (x-z)^+ g''(z)\,dz. \qquad \textbf{(1)}$$

   Answer: [1 mark] We first observe that the equality is clearly true when $x = y$. Let us assume that $0 \le x < y$ so that

$$\int_y^\infty (x-z)^+ g''(z)\,dz = 0.$$

   Since

$$zg''(z)\,dz = d\big(zg'(z)\big) - g'(z)\,dz$$

   we obtain

$$\int_0^y (z-x)^+ g''(z)\,dz = \int_x^y (z-x) g''(z)\,dz = \int_x^y zg''(z)\,dz - x\int_x^y g''(z)\,dz$$
$$= yg'(y) - xg'(x) - (g(y) - g(x)) - x(g'(y) - g'(x)) = g(x) - g(y) - g'(y)(x-y)$$

   which shows that (1) is valid.
   The derivation of (1) for $x > y \ge 0$ is similar. It suffices to observe that we now have that

$$\int_0^y (z-x)^+ g''(z)\,dz = 0$$

   and

$$\int_y^\infty (x-z)^+ g''(z)\,dz = \int_y^x (x-z) g''(z)\,dz = x\int_y^x g''(z)\,dz - \int_y^x zg''(z)\,dz$$
$$= x(g'(x) - g'(y)) + yg'(y) - xg'(x) + g(x) - g(y) = g(x) - g(y) - g'(y)(x-y)$$

   so that equality (1) is valid.

   (b) We assume that call and put options are traded in an arbitrage-free market model $\mathcal{M}$ at unique prices $C_0(K)$ and $P_0(K)$ for all $K > 0$. Using the risk-neutral valuation formula and equality (1), show that the arbitrage price $\pi_0(X)$ of the claim X in $\mathcal{M}$ admits the following representation, for any $L \ge 0$,

$$\pi_0(X) = g(L)B(0,T) + g'(L)(C_0(L) - P_0(L)) + \int_0^L P_0(K)g''(K)\,dK + \int_L^\infty C_0(K)g''(K)\,dK$$

   where $B(0,T)$ is the price at time 0 of the zero-coupon bond with maturity T.

   Answer: [3 marks] It suffices to use the risk-neutral valuation formula under any martingale measure $\mathbb{Q}$ from the class $\mathbb{M}$ of all martingale measures for the market model $\mathcal{M}$. We observe that the claim can be replicated in $\mathcal{M}$ and thus its arbitrage price is unique and can be computed using the risk-neutral valuation formula with any martingale measure $\mathbb{Q}$ for the market model $\mathcal{M}$.

We fix $y > 0$ and we observe that the following equality holds for random variables

$$g(S_T) = g(y) + g'(y)(S_T - y) + \int_0^y (z - S_T)^+ g''(z)\, dz + \int_y^\infty (S_T - z)^+ g''(z)\, dz.$$

We multiply both sides by B_T^{-1} and take the expectation under a martingale measure $\mathbb{Q}$

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\big(B_T^{-1} g(S_T)\big) &= \mathbb{E}_{\mathbb{Q}}\big(g(y)B_T^{-1}\big) + \mathbb{E}_{\mathbb{Q}}\big(g'(y)(B_T^{-1}S_T - B_T^{-1}y)\big) \\ &\quad + \mathbb{E}_{\mathbb{Q}}\Big(\int_0^y B_T^{-1}(z - S_T)^+ g''(z)\, dz\Big) + \mathbb{E}_{\mathbb{Q}}\Big(\int_y^\infty B_T^{-1}(S_T - z)^+ g''(z)\, dz\Big).\end{aligned}$$

Consequently,

$$\begin{aligned}\pi_0(X) &= g(y)B(0,T) + g'(y)(S_0 - yB(0,T)) \\ &\quad + \int_0^y \mathbb{E}_{\mathbb{Q}}\big(B_T^{-1}(z - S_T)^+\big) g''(z)\, dz + \int_y^\infty \mathbb{E}_{\mathbb{Q}}\big(B_T^{-1}(S_T - z)^+\big) g''(z)\, dz.\end{aligned}$$

The put-call parity yields $C_0(y) - P_0(y) = S_0 - yB(0,T)$ and thus, by changing the notation $y \mapsto L$ and $z \mapsto K$, we conclude that

$$\pi_0(X) = g(L)B(0,T) + g'(L)\big(C_0(L) - P_0(L)\big) + \int_0^L P_0(K)g''(K)\, dK + \int_L^\infty C_0(K)g''(K)\, dK.$$

(c) Let us take $g(x) = ax + b$ for some real numbers a and b. Using the equality established in part (b) and the put-call parity relationship, show that the price of the claim $X = g(S_T)$ at time 0 equals $aS_0 + bB(0,T)$.

Answer: [1 mark] If we choose L such that $C_0(L) = P_0(L)$ (that is, $L = S_0/B(0,T)$) then for any twice continuously differentiable function $g : \mathbb{R}_+ \to \mathbb{R}$, we get

$$\pi_0(X) = g(L)B(0,T) + \int_0^L P_0(K)g''(K)\, dK + \int_L^\infty C_0(K)g''(K)\, dK.$$

In particular, if $g(x) = ax + b$ then $g'' = 0$ and thus since $L = S_0/B(0,T)$

$$\pi_0(X) = \pi_0(aS_T + b) = g(L)B(0,T) = (aL + b)B(0,T) = aS_0 + bB(0,T).$$

(d) Apply the formula derived in part (b) with $L = 0$ to the claim $X = g(S_T) = S_T^2$.

Answer: [1 mark] From part (b), we know that for any $L \geq 0$

$$\pi_0(X) = g(L)B(0,T) + g'(L)\big(C_0(L) - P_0(L)\big) + \int_0^L P_0(K)g''(K)\, dK + \int_L^\infty C_0(K)g''(K)\, dK$$

so that for $L = 0$ (notice that $P_0(0) = 0$ and $C_0(0) = S_0$)

$$\pi_0(X) = g(0)B(0,T) + g'(0)S_0(0) + \int_0^\infty C_0(K)g''(K)\, dK.$$

If we take $g(x) = x^2$ then we obtain

$$\pi_0(S_T^2) = 2\int_0^\infty C_0(K)\, dK.$$