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PORTFOLIO OPTIMIZATION WITH PERFORMANCE RATIOS

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We consider the portfolio selection problem of maximizing a performance measure in a continuous-time diffusion model. The performance measure is the ratio of the overperformance to the underperformance of a portfolio relative to a benchmark. Following a strategy from fractional programming, we analyze the problem by solving a family of related problems, where the objective functions are the numerator of the original problem minus the denominator multiplied by a penalty parameter. These auxiliary problems can be solved using the martingale method for stochastic control. The existence of solution is discussed in a general setting and explicit solutions are derived when both the reward and the penalty functions are power functions.

Keywords: Performance ratio; portfolio optimization; stochastic control; martingale method.

1. Introduction

The mean-variance model of Markowitz (1952) is popular both in academia and in practice. Closely related to the mean-variance model is the performance ratio known as the Sharpe ratio (Sharpe 1966), which measures performance as the expected excess return of an investment above the risk-free interest rate divided by the standard deviation of its returns. Since these seminal works, a large literature on performance ratios has developed; see, for example, Prigent (2007).

A performance measure that has been popular recently, particularly in the evaluation of alternative investments, is the Omega measure, introduced by

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Keating & Shadwick (2002). This measure shares the basic structure of most performance measures, consisting of a measure of reward divided by a measure of risk. In the case of the Omega, the reward and risk are defined with respect to an exogenously specified benchmark return. Unlike many performance measures, such as the Sharpe ratio, Sortino ratio (Sortino & Price 1994) or the Kappa ratios of Kaplan & Knowles (2004), the Omega does not require the assumption of the existence of higher moments to be well-defined.[a]

A number of recent papers have investigated portfolio selection problems using the Omega measure as the objective function. For example, Mausser *et al.* (2006) employ a technique from fractional linear programming to transform the portfolio selection problem into a linear program. The transformation only works when the optimal Omega is greater than 1. Kapsos *et al.* (2014) also introduce a transformation technique by changing the original problem to a family of linear programming problems or a linear fractional programming problem. Avouyi-Dovi *et al.* (2004) apply a threshold accepting algorithm to solve the Omega optimization problem. Kane *et al.* (2009) use the multi-level coordinate splitting method to optimize the Omega.

The above-mentioned literature considers optimizing the Omega measure in a discrete time framework, typically on a finite sample space. Bernard *et al.* (2019) show that the Omega optimization problem is unbounded in a continuous time financial model. We present a related result in the classical diffusion-based Merton (1969) framework in Sec. 2. In order to reflect different attitudes towards reward and risk, we modify the Omega ratio to include a utility function for overperformance and a penalty function for underperformance in the definition of the performance ratio. With this modified definition, we consider the portfolio selection problem of maximizing the performance ratio and structuring the optimal trading strategy.

Difficulties arise as the objective function of our problem is a ratio, and is neither concave nor convex. Following classical methods in fractional programming (Dinkelbach 1967), as well as more recent work on the continuous time mean-variance stochastic control problem by Zhou & Li (2000), we transform the original portfolio selection problem to a family of solvable ones, where one of the reformulated problems recovers the solution to the original problem. More specifically, we optimize the ratio by considering a family of "linearized" problems in which the objective function is the numerator of the original problem minus the denominator multiplied by a penalty parameter. To solve the transformed problems, we apply the martingale approach and convex duality methods; see Karatzas & Shreve (1998) for more details). As the objective in each "linearized" maximization problem is still not concave, we apply the concavification technique used in Carpenter (2000), He & Kou (2018) and Lin *et al.* (2017).

The remainder of this paper is structured as follows. Section 2 presents the formulation of the portfolio selection problem, rules out ill-posed problems, introduces

[a]A finite mean, and a positive probability of underperforming the benchmark are necessary.

the linearized problems and discusses properties of their optimal values as a function of the penalty parameter. In Sec. 3, we solve the linearized problems using Lagrangian duality and the pointwise optimization technique. Section 4 presents the explicit optimal solutions of the original portfolio selection problem for power penalty and utility functions, and provides some numerical examples. Further sensitivity analysis of the optimal solutions with respect to the model parameters is presented in Sec. 5. Section 6 concludes the paper.

2. Model Formulation and Preliminary Analysis

2.1. *Financial market model*

We assume that an agent, with initial wealth $x_0 > 0$, invests capital in a risk-free bond B and p risky assets with price processes as follows:

$$\begin{cases} dB_t = rB_t dt, \\ dS_t^{(i)} = S_t^{(i)} \left[\mu^{(i)} dt + \sum_{j=1}^{p} \sigma_{ij} dW_t^{(j)} \right], \quad i = 1, \ldots, p, \end{cases} \tag{2.1}$$

where $r > 0$ is the risk-free rate, $\mu^{(i)} > r$ is the expected return rate of the risky asset i, for $i = 1, \ldots, p$, and we let $\mu = (\mu^{(1)}, \ldots, \mu^{(p)})^\top$ be the vector of expected returns of the risky assets. $\sigma = \{\sigma_{ij}\}_{1 \le i,j \le p}$ is the corresponding volatility matrix, which is invertible with inverse σ^{-1}. $W \equiv \{W_t, t \ge 0\} := \{(W_t^{(1)}, \ldots, W_t^{(p)})^\top, t \ge 0\}$ is a standard Brownian motion valued on $\mathbb{R}^p$ under the physical measure $\mathbb{P}$ defined over a probability space $(\Omega, \mathcal{F})$. We use $\mathbb{F} := \{\mathcal{F}_t, t \in [0, T]\}$ to denote the $\mathbb{P}$-augmentation of the natural filtration generated by the Brownian motion W.

We consider a finite investment time horizon $[0, T]$ with $T > 0$. Let $\pi_t := (\pi_t^{(1)}, \ldots, \pi_t^{(p)})^\top$, where $\pi_t^{(i)}$ denotes the dollar amount of capital invested in the ith risky asset at time t, for $t \ge 0$ and $i = 1, \ldots, p$. With the trading strategy $\pi := \{\pi_t, 0 \le t \le T\}$, the portfolio value process, denoted by X_t^π, evolves according to the following stochastic differential equation (SDE):

c(t)

$$dX_t^\pi = [rX_t^\pi + \pi_t^\top (\mu - r\mathbf{1})]dt + \pi_t^\top \sigma dW_t, \quad t \ge 0, \tag{2.2}$$

where $\mathbf{1}$ denotes the p-dimensional column vector with each element equal to 1. It is natural to assume that the trading strategy π is $\mathbb{F}$-progressively measurable and satisfies $\int_0^T \|\pi_t\|^2 dt < \infty$ a.s. so that a unique strong solution exists for the SDE (2.2), where $\|\cdot\|$ denotes the usual L^2-norm and thus $\|\pi_t\|^2 = \sum_{i=1}^p (\pi_t^i)^2$.

Definition 2.1. A trading strategy $\pi := \{\pi_t, 0 \le t \le T\}$ is called admissible with initial wealth $x_0 > 0$ if it belongs to the following set:

$$\mathcal{A}(x_0) := \{\pi \in \mathcal{S} : \pi_t \in \mathbb{R}^p, X_0^\pi = x_0 \text{ and } X_t^\pi \ge 0, \text{ a.s., } \forall\, 0 \le t \le T\}, \tag{2.3}$$

where $\mathcal{S}$ denotes the set of $\mathbb{F}$-progressively measurable processes π such that $\int_0^T \|\pi_t\|^2 dt < \infty$ a.s.

We consider the market price of risk defined as

$$\zeta \equiv (\zeta_1, \ldots, \zeta_p)^\top := \sigma^{-1}(\mu - r\mathbf{1}) \tag{2.4}$$

and the state-price density process given by

H_t

$$\xi_t := \exp\left\{-\left(r + \frac{\|\zeta\|^2}{2}\right) t - \zeta^\top W_t\right\}. \tag{2.5}$$

We also employ the notation

$$\xi_{t,s} = \xi_t^{-1}\xi_s = \exp\left[-\left(r + \frac{\|\zeta\|^2}{2}\right)(s-t) - \zeta^\top (W_s - W_t)\right], \quad t \leq s. \tag{2.6}$$

Note that $\xi_t = \xi_{0,t}$ and $\xi_{t,s}$ is independent of $\mathcal{F}_t$ under $\mathbb{P}$.

2.2. *Performance ratios and problem formulation*

The performance ratio considered in this paper is similar to the Omega measure introduced by Keating & Shadwick (2002). Given a benchmark return level l, the Omega for a random return R is defined as:

$$\Omega_l(R) = \frac{\mathbb{E}[(R-l)_+]}{\mathbb{E}[(l-R)_+]}, \tag{2.7}$$

d

where $(x)_+ := \max\{x, 0\}$ for $x \in \mathbb{R}$. Considering Omega as a performance measure for optimization of the portfolio with value process X_t^π defined in Eq. (2.2) leads to the problem

$$\max_{\pi \in \mathcal{A}(x_0)} \left\{\Omega_L(X_T^\pi) := \frac{\mathbb{E}[(X_T^\pi - L)_+]}{\mathbb{E}[(L - X_T^\pi)_+]}\right\} \tag{2.8}$$

for a given constant benchmark $L \in \mathbb{R}$. It is noted that the Omega ratio was originally defined in terms of returns, whereas the formulation in (2.8) is specified in terms of terminal wealth. For simple returns, we have

收益率

$$R_T^\pi = \frac{X_T^\pi}{X_0} - 1, \tag{2.9}$$

thus,

$$\Omega_L(X_T^\pi) = \frac{\mathbb{E}[(R_T^\pi - \tilde{L})_+]}{\mathbb{E}[(\tilde{L} - R_T^\pi)_+]}. \tag{2.10}$$

The optimization problems are equivalent. For log-returns, this is not the case; see Lin (2018).

As we will see shortly in Proposition 2.2, optimizing the Omega ratio in Eq. (2.8) is not well-posed due to the linear growth of its numerator. Consequently, we introduce two weighting functions and consider performance measures of the form

$$R(X_T) = \frac{\mathbb{E}\{U[(X_T - L)_+]\}}{\mathbb{E}\{D[(L - X_T)_+]\}}, \tag{2.11}$$

where $U : \mathbb{R}_+ \mapsto \mathbb{R}$ and $D : \mathbb{R}_+ \mapsto \mathbb{R}$ are two strictly increasing measurable functions. The numerator $\mathbb{E}\{U[(X_T - L)_+]\}$ measures the benefit from exceeding the benchmark wealth L, while the denominator $\mathbb{E}\{D[(L - X_T)_+]\}$ penalizes shortfalls. 不足
For this reason, we refer to U as the *reward function* and D as the *penalty function* throughout the paper. In order to ensure well-posedness, we insist on U being
凹 strictly concave. No such requirement is necessary for D to avoid ill-posedness of the optimization problem; however, we focus on the cases when D is either concave or convex. As we shall see, optimization problems maximizing $\mathbb{E}[g_\lambda(X_T)]$, where $g_\lambda(x) = U(x - L)_+ - \lambda D(L - x)_+$, $\lambda \geq 0$ are closely related to the problem of maximizing (2.11). When D is convex, then g_λ behaves much like a traditional concave utility function. In this case, the investor is risk-averse, and the convex function D penalizes larger underperformance at an increasing rate. On the other hand, when D is concave, then g_λ takes the form of the *S-shaped* utility functions that appear in cumulative prospect theory (see, e.g. Tversky & Kahneman (1992)). The investor is risk-averse when considering gains (compared to the benchmark wealth level L) and risk-seeking when considering losses.

We formulate the agent's portfolio selection problem as X_T^π

$$\begin{cases} \sup\limits_{\pi \in \mathcal{A}(x_0)}, & \dfrac{\mathbb{E}\{U[(X_T^\pi - L)_+]\}}{\mathbb{E}\{D[(L - X_T^\pi)_+]\}}, \\ \text{subject to}, & \mathbb{E}[\xi_T X_T^\pi] \leq x_0. \end{cases} \tag{2.12}$$

Hereafter, we assume that the threshold $L > 0$. The budget constraint $\mathbb{E}[\xi_T X_T^\pi] \leq x_0$ restricts the initial portfolio value to cost no more than x_0. Indeed, we apply Itô's formula in conjunction with Eqs. (2.2) and (2.5) to obtain:

$$\xi_t X_t^\pi = x_0 + \int_0^t \xi_s(\pi_s^\top \sigma - \zeta^\top X_s^\pi) dW_s, \quad t \in [0, T]. \tag{2.13}$$

The right-hand side in the above equation is a non-negative local martingale and thus a super-martingale, which implies $\mathbb{E}[\xi_T X_T^\pi] \leq \mathbb{E}[\xi_0 X_0^\pi] = x_0$; see Proposition 1.1.7 in (Pham 2009) or Chap. 1, Problem 5.19 in (Karatzas & Shreve 1991).

2.3. *Optimal payoff problem*

In problem (2.12), we consider maximizing a performance ratio over all admissible trading strategies. Each admissible trading strategy produces a nonnegative terminal wealth, and the objective function only depends on this terminal wealth. Furthermore, it is well-known from the theory of derivatives pricing (e.g. Karatzas & Shreve (1998)) that a large class of nonnegative terminal payoffs can be replicated through admissible trading strategies. Consequently, in relation to (2.12), it is natural to consider the following problem, which we refer to as the optimal payoff problem:

考虑随机变量问题

$$\begin{cases} \sup\limits_{Z\in\mathcal{M}_+}, & \dfrac{\mathbb{E}\{U[(Z-L)_+]\}}{\mathbb{E}\{D[(L-Z)_+]\}}, \\ \text{subject to,} & \mathbb{E}[\xi_T Z] \le x_0, \end{cases} \tag{2.14}$$

where $\mathcal{M}_+$ denotes the set of non-negative $\mathcal{F}_T$-measurable random variables. We denote the feasible set of the above problem by $\mathcal{C}(x_0)$:

$$\mathcal{C}(x_0) = \{Z \in \mathcal{M}_+ \,|\, \mathbb{E}[\xi_T Z] \le x_0\} = \{Z \in \mathcal{M}_+ \,|\, \mathbb{E}^{\mathbb{Q}}[Z] \le x_0 e^{rT}\}, \tag{2.15}$$

where $\mathbb{Q}$ is defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{rT}\xi_T. \tag{2.16}$$

Note that $\mathbb{Q}(Z < L) > 0$ and therefore $\mathbb{P}(Z < L) > 0$ for all $Z \in \mathcal{C}(x_0)$ whenever $x_0 < e^{-rT}L$.

The following proposition reveals the relationship between the portfolio optimization problem (2.12) and the optimal payoff problem (2.14).

Proposition 2.1. *Suppose there exists $\Lambda_T \in \mathcal{M}_+$ such that $\mathbb{E}[\xi_T\Lambda_T] = x_0$. Then there exists a process $\pi := \{\pi_t, 0 \le t \le T\} \in \mathcal{A}(x_0)$ satisfying $X_T^\pi = \Lambda_T$ a.s.*

Proof. The result is a multidimensional generalization of Proposition 2.1 in Lin *et al.* (2017). The proof can be obtained in parallel and is therefore omitted. □

Under certain conditions an optimal solution can be obtained for the optimal payoff problem (2.14) such that the constraint is binding (see Proposition 3.2), and from such a solution, we can construct an optimal trading strategy for the portfolio optimization problem (2.12) by invoking Proposition 2.1. Let Z^* be a solution to (2.14) satisfying $\mathbb{E}[\xi_T Z^*] = x_0$. Define

$$Y_t^* := \xi_t^{-1}\mathbb{E}[\xi_T Z^* \,|\, \mathcal{F}_t], \quad 0 \le t \le T. \tag{2.17}$$

Then it is easy to verify that $\{\xi_t Y_t^*, t \ge 0\}$ is a martingale relative to the filtration $\mathbb{F}$, and by the martingale representation theorem (see Karatzas & Shreve (1991)), there exists an $\mathbb{R}^p$-valued $\mathbb{F}$-progressively measurable process $\{\theta_t^*, 0 \le t \le T\}$ satisfying $\int_0^T \|\theta_t^*\|^2 dt < \infty$, a.s., and

$$\xi_t Y_t^* = x_0 + \int_0^t (\theta_s^*)^\top dW_s, \quad 0 \le t \le T. \tag{2.18}$$

We further denote

$$(\pi_t^*)^\top = \xi_t^{-1}(\theta_t^*)^\top\sigma^{-1} + Y_t^*\zeta^\top\sigma^{-1}, \quad 0 \le t \le T. \tag{2.19}$$

By standard arguments it can be proved that $\pi^* := \{\pi_t^*, 0 \leq t \leq T\} \in \mathcal{A}(x_0)$ given in (2.19) solves the portfolio optimization problem (2.12), and the optimal portfolio value at time t, $0 \leq t \leq T$, is given by $X_t^{\pi^*} = Y_t^*$, where Y_t^* is defined in equation (2.17).

2.4. *Ill-posedness of the portfolio selection problem for some performance measures*

Without additional assumptions, problem (2.12) may be unbounded. In this section, we study such cases and establish the framework that we will use to study (2.12) in the remainder of the paper. We begin by making the following assumption:

(H1) U and D are strictly increasing and twice differentiable with $U(0) = 0$ and $D(0) = 0$.

The monotonicity of both U and D in (H1) is natural from the interpretation of $\mathbb{E}\{U[(X_T - L)_+]\}$ and $\mathbb{E}\{D[(L - X_T)_+]\}$ as the reward for outperformance and penalty for underperformance, respectively. The condition $U(0) = D(0) = 0$ mimics the definition of the Omega measure in the sense that outperformance and underperformance are both zero if the portfolio value is exactly equal to the benchmark.

Under the above assumption, it clearly only makes sense to consider the optimization problem when $x_0 < e^{-rT}L$, as otherwise investing all wealth in the risk-free asset (setting $\pi_t \equiv 0$) leads to zero underperformance and a zero denominator in the performance measure. The following proposition specifies another situation in which problem (2.12) is unbounded.

Proposition 2.2. *Suppose that $x_0 < e^{-rT}L$, and* (H1) *holds. If the reward function U is a convex function, then problem* (2.12) *is unbounded.*

Proof. Note that since U is convex and strictly increasing, $\lim_{y\to\infty} U(y) = \infty$. Jin & Zhou (2008) show how to construct a sequence of positive random variables Z_n such that $\mathbb{E}[\xi_T Z_n] = x_0$ and $\mathbb{E}[Z_n] \to \infty$. Applying Jensen's inequality then implies that

$$\mathbb{E}\{U[(Z_n - L)_+]\} \geq U(\mathbb{E}[(Z_n - L)_+]) \geq U(\mathbb{E}[Z_n] - L) \to \infty. \tag{2.20}$$

Problem (2.14) is thus unbounded since for any $Z \in \mathcal{M}_+$,

$$\frac{\mathbb{E}\{U[(Z - L)_+]\}}{\mathbb{E}\{D[(L - Z)_+]\}} \geq \frac{1}{D(L)}\mathbb{E}\{U[(Z - L)_+]\}. \tag{2.21}$$

By Proposition 2.1, for any integer $n > 1$, we can construct a trading strategy π to attain $X_T^\pi = Z_n$ a.s., and thus problem (2.12) is also unbounded. □

Proposition 2.2 excludes convex reward functions U for problem (2.12) to be well-posed. We consider concave reward functions instead, and impose the following two specific conditions on U:

(H2) The reward function U satisfies the Inada condition, i.e.

$$\lim_{x\searrow 0} U'(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} U'(x) = 0;$$

(H3) The reward function U is strictly concave, with $U''(z) < 0$ for all $z \in (0, \infty)$.

The Inada condition given in (H2) is a common technical assumption in the literature on utility maximization problems. In the sequel, we allow the penalty function D to be either concave or strictly convex, with certain mild conditions. A convex penalty function places more severe penalties on extreme events, and reflects a greater aversion to large losses.

2.5. *Linearization of the optimal payoff problem*

Since the optimal payoff problem (2.14) involves a non-convex objective function, it is difficult to solve directly. In order to reformulate it into a tractable problem, we set up the following family of linearized problems[b] parameterized by $\lambda \geq 0$:

$$v(\lambda; x_0) = \sup_{Z\in\mathcal{C}(x_0)} \mathbb{E}\{U[(Z-L)_+]\} - \lambda\mathbb{E}\{D[(L-Z)_+]\}. \tag{2.22}$$

Remark 2.1. Jin & Zhou (2008) consider a problem which has a related and seemingly more general objective function than (2.22). In their paper, they include probability weighting functions on both the positive and negative parts, and when both weighting functions are the identity function, the objective function reduces to the one given in Eq. (2.22). Although adding probability weighting functions generalizes the model, problem (2.22) differs from theirs in at least two aspects. First, every feasible decision variable Z in problem (2.22) is non-negative, whereas there is no pre-specified lower bound on the terminal portfolio value in the model of Jin & Zhou (2008). In their paper, they do require the terminal portfolio value to be bounded from below, but the lower bound depends on the trading strategy under consideration. Consequently, their solution does not work for the problem with a lower bound specified as a constraint. Second, because there is no pre-specified lower bound on the terminal portfolio value in their model, their problem is unbounded if the probability weighting function on the negative part is an identity function, which means that their model does not encompass problem (2.22) as a special case.

[b]The ratio has been linearized. The optimization problem is still non-linear in Z.

The following proposition provides the justification for considering the linearized problem (2.22) in solving problem (2.14).

即 $v(\lambda, x_0) = E(U(Z_\lambda^* - L)) - \lambda E(D)$

Proposition 2.3. *Assume $x_0 < e^{-rT}L$. For each $\lambda \geq 0$, let Z_λ^* be a solution to problem (2.22), and suppose there exists a constant $\lambda^* \geq 0$ such that*

$$\lambda^* = \frac{\mathbb{E}\{U[(Z_{\lambda^*}^* - L)_+]\}}{\mathbb{E}\{D[(L - Z_{\lambda^*}^*)_+]\}}. \iff v(\lambda^*, x_0) = 0 \tag{2.23}$$

① → λ^* 存在性

② → 线性问题

Then $Z^* := Z_{\lambda^*}^*$ solves problem (2.14), and λ^* is the optimal value.

Proof. The proof is similar to the proof of the analogous result for nonlinear fractional programs in Dinkelbach (1967). By the optimality of $Z_{\lambda^*}^*$ for problem (2.22), for $\forall\, Z \in \mathcal{M}_+$ satisfying $\mathbb{E}[\xi_T Z] \leq x_0$, we have:

$$\begin{aligned} 0 &= \mathbb{E}\{U[(Z_{\lambda^*}^* - L)_+]\} - \lambda^* \mathbb{E}\{D[(L - Z_{\lambda^*}^*)_+]\} \\ &\geq \mathbb{E}\{U[(Z - L)_+]\} - \lambda^* \mathbb{E}\{D[(L - Z)_+]\} \\ &= \mathbb{E}\{U[(Z - L)_+]\} - \frac{\mathbb{E}\{U[(Z_{\lambda^*}^* - L)_+]\}}{\mathbb{E}\{D[(L - Z_{\lambda^*}^*)_+]\}} \mathbb{E}\{D[(L - Z)_+]\}. \end{aligned} \tag{2.24}$$

$E(U(Z_{\lambda^*}^* - L)_+) - \lambda^* E(D(L - Z_{\lambda^*}^*)) = 0$

Furthermore, $\mathbb{E}[\xi_T Z] \leq x_0$ implies that $Z < L$ holds with some positive probability; otherwise, $x_0 \geq \mathbb{E}[\xi_T Z] \geq \mathbb{E}[\xi_T] L = e^{-rT} L$, contradicting the assumption that $x_0 < e^{-rT} L$. Thus $\mathbb{E} \geq \{D[(L - Z)_+]\} > 0$ and

$$\lambda^* = \frac{\mathbb{E}\{U[(Z_{\lambda^*}^* - L)_+]\}}{\mathbb{E}\{D[(L - Z_{\lambda^*}^*)_+]\}} \geq \frac{\mathbb{E}\{U[(Z - L)_+]\}}{\mathbb{E}\{D[(L - Z)_+]\}} \tag{2.25}$$

for any $Z \in \mathcal{M}_+$ satisfying $\mathbb{E}[\xi_T Z] \leq x_0$. □

Remark 2.2. Note that at optimality the budget constraint must be binding, i.e. we must have $\mathbb{E}[\xi_T Z_\lambda^*] = x_0$, for if $\mathbb{E}[\xi_T Z_\lambda^*] < x_0$, then

$$\tilde{Z} := Z_\lambda^* + e^{rT}(x_0 - \mathbb{E}[\xi_T Z_\lambda^*]) \tag{2.26}$$

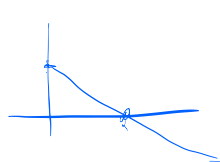
would still be feasible and yields a larger objective value. We can then obtain an optimal trading strategy π^* with initial value x_0.

In the rest of the section, we study the existence of λ^* satisfying the conditions in Proposition 2.3. This study requires the following two propositions. Proposition 2.4 summarizes some relevant properties of the value function v defined in (2.22). Its proof requires the following additional assumption on the asymptotic behavior of the "Arrow–Pratt relative risk aversion" of U:

(H4) $\liminf_{x \to \infty} \left(-\frac{xU''(x)}{U'(x)}\right) > 0$.

Proposition 2.4. *Suppose that $x_0 < e^{-rT}L$ and assumptions* (H1)–(H4) *hold.*

(a) $0 < v(0; x_0) < \infty$.
(b) *v is non-increasing in λ.*



(c) $v(\lambda; x_0)$ *is convex in* λ *for each fixed* $x_0 > 0$.
(d) $v(\cdot; x_0)$ *is Lipschitz continuous.*

Proof. (a) We begin by explicitly showing that v is finite. By Lemmas A.1 and A.2 in Appendix A, the problem $\sup_{Z\in\mathcal{C}(x_0)} \mathbb{E}[U(Z)]$ has a finite optimal value. Since $\mathbb{E}[U((Z-L)_+] \leq \mathbb{E}[U(Z)]$, we obtain $v(0; x_0)$. It is easy to find Z for which $\mathbb{P}(Z > L) > 0$ and $\mathbb{E}[\xi_T Z] = x_0$ (e.g. the payoff generated by putting all the money in one of the stocks), and therefore $v(0; x_0) > 0$.

(b) The proof is similar to the proof of the analogous result in Dinkelbach (1967).[c]

(c) As in part (b), the proof is similar to the proof of the analogous result in Dinkelbach (1967).

(d) Since $v(\cdot; x_0)$ is convex, it is locally Lipschitz on the interior of its domain. Global Lipschitz continuity can be proved directly as follows. For $Z \in \mathcal{C}(x_0)$, denote

$$G_\lambda(Z) = \mathbb{E}[U((Z-L)_+) - \lambda D((L-Z)_+)]. \tag{2.27}$$

Then

$$|G_{\lambda_1}(Z) - G_{\lambda_2}(Z)| = |\lambda_1 - \lambda_2|\mathbb{E}[D((L-Z)_+)] \leq D(L)|\lambda_1 - \lambda_2|. \tag{2.28}$$

Let $\varepsilon > 0$ and Z_i be such that $G_{\lambda_i}(Z_i) \geq v(\lambda_i; x_0) - \varepsilon$, $i = 1, 2$. Then

$$\begin{aligned} v(\lambda_2; x_0) &\geq G_{\lambda_2}(Z_1) \\ &\geq G_{\lambda_1}(Z_1) - D(L)|\lambda_1 - \lambda_2| \\ &\geq v(\lambda_1; x_0) - \varepsilon - D(L)|\lambda_1 - \lambda_2|. \end{aligned} \tag{2.29}$$

Since $\varepsilon > 0$ was arbitrary,

$$v(\lambda_1; x_0) - v(\lambda_2; x_0) \leq D(L)|\lambda_1 - \lambda_2|. \tag{2.30}$$

Symmetry yields

$$v(\lambda_2; x_0) - v(\lambda_1; x_0) \leq D(L)|\lambda_1 - \lambda_2| \tag{2.31}$$

and thus

$$|v(\lambda_1; x_0) - v(\lambda_2; x_0)| \leq D(L)|\lambda_1 - \lambda_2|. \tag{2.32}$$

□

We are seeking a λ^* such that $v(\lambda^*; x_0) = 0$ in order to apply Proposition 2.3 and obtain a solution of problem (2.14). To do so, we show that $\lim_{\lambda\to\infty} v(\lambda; x_0) = -\infty$ and invoke the intermediate value theorem. We define the set

$$C^{\mathrm{eq}}(x_0) := \{Z \in \mathcal{M}_+ \,|\, \mathbb{E}[\xi_T Z] = x_0\} = \{Z \in \mathcal{M}_+ \,|\, \mathbb{E}^{\mathbb{Q}}[Z] = e^{rT} x_0\}. \tag{2.33}$$

[c]In Dinkelbach (1967), existence of an optimal solution follows from a compactness assumption, which we do not make here. Existence of an optimal solution for our problem is proved in the next section (the properties of the value function asserted here can also be derived using ε-optimal solutions).

$\lambda \to +\infty$ $\quad V = \sup E(U) - \lambda E(D))$

$V(\lambda, x_0) \to -\infty$

Proposition 2.5. *Let* $M = \sup_{Z \in C^{\mathrm{eq}}(x_0)} \mathbb{E}[U((Z-L)_+)]$ *and* $m = \inf_{Z \in C^{\mathrm{eq}}(x_0)} \times \mathbb{E}[D((L-Z)_+)]$. *Then* $M < \infty$, *and* $m > 0$.

Proof. The fact that $M < \infty$ has already been shown in the proof of Proposition 2.4. Suppose $m = 0$. Let $\{Z_n\}$ be a sequence in $C^{\mathrm{eq}}(x_0)$ such that $\lim_{n\to\infty} \mathbb{E}[D((L-Z_n)_+)] = 0$. Then $D((L-Z_n)_+)$ converges to 0 in probability with respect to the measure $\mathbb{P}$, and consequently so does $(L-Z_n)_+$. Thus, $(L-Z_n)_+$ also converges to 0 in probability with respect to $\mathbb{Q}$.[d] So $(L-Z_n)_+$ is a bounded sequence that converges to 0 in probability, and consequently also converges to zero in L^1, contradicting the fact that

$$\mathbb{E}^{\mathbb{Q}}[(L-Z_n)_+] \geq \mathbb{E}^{\mathbb{Q}}[L-Z_n] = L - x_0 e^{rT} > 0. \tag{2.34}$$

□

Corollary 2.1. $\lim_{\lambda\to\infty} v(\lambda; x_0) = -\infty$.

Proof. It was noted in Remark 2.2 that the budget constraint is binding at optimality. Thus,

$$v(\lambda; x_0) = \sup_{Z \in C^{\mathrm{eq}}(x_0)} \mathbb{E}[U((Z-L)_+) - \lambda D((L-Z)_+)] \leq M - \lambda m, \tag{2.35}$$

which implies the desired result. □

Combining Proposition 2.4 and Corollary 2.1 yields the existence of the multiplier λ^* to satisfy (2.23) as shown in the proposition below.

Proposition 2.6. *Under assumptions* (H1)–(H4), *there exists a* $\lambda^* \geq 0$ *such that* (2.23) *holds.*

3. Optimal Solutions to Problems (2.14) and (2.22)

Henceforth, we assume that (H1)–(H4) hold and $x_0 < e^{-rT} L$. We will first analyze problem (2.22), and then summarize the optimal solution to problem (2.14) at the end of this section. Our analysis will focus on the cases of either a concave penalty function D or a strictly convex D.

3.1. *Lagrangian duality and pointwise optimization*

The analysis in the last section motivates us to focus on the linearized optimal payoff problem (2.22), which we solve by a Lagrangian duality method and a pointwise optimization procedure. This entails introducing the following optimization problems with multipliers β, for each $\lambda \geq 0$:

$E \leq x_0$

$$\sup_{Z \in \mathcal{M}_+} \mathbb{E}\{h_\lambda(Z) - \beta \xi_T Z\}, \quad \beta > 0, \tag{3.1}$$

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[d]Every subsequence has a further subsequence converging to zero a.s. $\mathbb{P}$, and therefore a.s. $\mathbb{Q}$.

where

$$h_\lambda(x) := U[(x-L)_+] - \lambda D[(L-x)_+], \quad x \in \mathbb{R}. \tag{3.2}$$

We solve the above problem by resorting to a pointwise optimization procedure and consider the following problem indexed by $\lambda \geq 0$ and $y > 0$:

$$\sup_{x\in\mathbb{R}_+} \{h_\lambda(x) - yx\}, \tag{3.3}$$

where $\mathbb{R}_+$ denotes the set of nonnegative real numbers.

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Lemma 3.1. (a) *Let $x^*_\lambda(y)$ be a Borel measurable function such that $x^*_\lambda(y)$ is an optimal solution to problem* (3.3) *for each $y > 0$ and $\lambda \geq 0$. Then, $Z^*_{\lambda,\beta} := x^*_\lambda(\beta\xi_T)$ solves problem* (3.1).

(b) *Assume that, given $\lambda \geq 0$, there exists a constant $\beta^* > 0$ such that $Z^*_{\lambda,\beta^*} \in \mathcal{M}_+$ solves problem* (3.1) *for $\beta = \beta^*$ and satisfies $\mathbb{E}[\xi_T Z^*_{\lambda,\beta^*}] = x_0$. Then, $Z^*_\lambda := Z^*_{\lambda,\beta^*}$ solves problem* (2.22).

Proof. The proof is in parallel with those of Lemmas 3.1 and 3.2 in Lin *et al.* (2017), and thus omitted. □

3.2. *Solutions to the pointwise optimization problem*

Figure 1 presents the curve of the function h_λ for $\lambda = 0.5$, $L = 40$ and some special forms (power or linear) for the functions U and D. As can be seen from the figure, h_λ is not globally concave, but concave for sufficiently large input values. In order to maximize $h_\lambda(x) - yx$ with respect to x, it is convenient to employ the

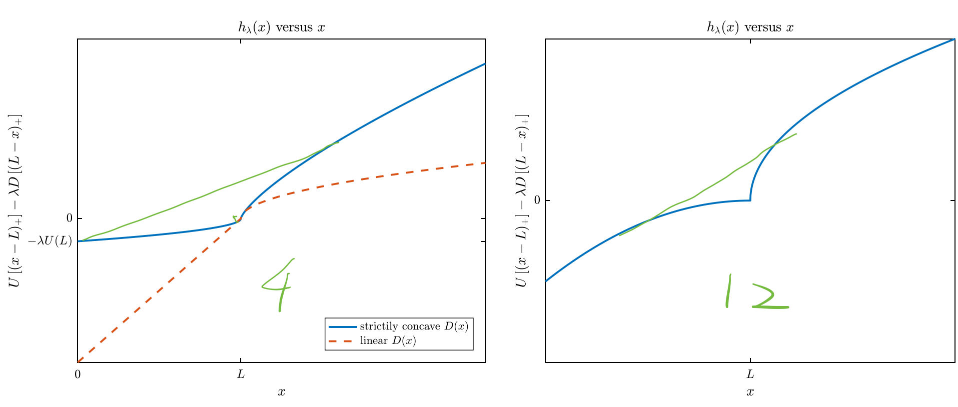

Fig. 1. Shapes of $h_\lambda(x) := U[(x-40)_+] - \lambda D[(40-x)_+]$. (a) The solid curve is for $U(x) = x^{0.75}$ and $D(x) = x^{0.5}$. The dashed curve is for $U(x) = x^{0.5}$ and $D(x) = x$. (b) The curve is for $U(x) = x^{0.5}$ and $D(x) = x^2$.

concavification method (e.g., Carpenter (2000), and He & Kou (2018)). We denote the concave envelope of a given function f with a domain G by f^c.

$$\begin{aligned} f^c(x) := \inf\{g(x) \,|\, g : G \to \mathbb{R} \ \text{ is a concave function,} \\ g(t) \geq f(t),\ \forall\, t \in G\}, \quad x \in G. \end{aligned} \tag{3.4}$$

Note that for $a, b \in \mathbb{R}$, the concave envelope of $f(x) + ax + b$ is $f^c(x) + ax + b$. Thus, the concavified version of (3.3) is:

$$\sup_{x \in \mathbb{R}_+} [h_\lambda^c(x) - yx], \quad \lambda \geq 0 \ \text{ and } \ y > 0. \tag{3.5}$$

The following result provides a connection between the solutions to problems (3.5) and (3.3).

Lemma 3.2. *Given $\lambda \geq 0$ and $y > 0$, if $x_\lambda^*(y)$ is a solution to problem* (3.5) *and $h_\lambda^c(x_\lambda^*(y)) = h_\lambda(x_\lambda^*(y))$, then $x_\lambda^*(y)$ solves problem* (3.3).

Proof. The proof is straightforward; see Lin *et al.* (2017) for details. □

Based on the shape of h_λ, the following two lemmas may be employed to calculate h_λ^c.

Lemma 3.3. *Suppose $f : [0, \infty) \to [0, \infty)$ is continuous and satisfies $f(0) = 0$; f is concave on $[\tilde{z}, \infty)$ with $\tilde{z} > 0$; $f(x) \leq kx$ on $[0, \tilde{z}]$ with $f'_+(\tilde{z}) \leq k := \frac{f(\tilde{z})}{\tilde{z}} > 0$. Then the concave envelope of f is given by*

$$f^c(x) = \begin{cases} kx, & x \in [0, \tilde{z}), \\ f(x), & x \in [\tilde{z}, \infty). \end{cases} \tag{3.6}$$

Proof. See Lemma A.1 of Lin *et al.* (2017). □

Lemma 3.4. *Suppose $f : [0, \infty) \to [0, \infty)$ is continuous and satisfies*:

(1) $f(0) = 0$;
(2) *f is concave on $[0, \tilde{z}_1]$ and $[\tilde{z}_2, \infty)$, with $\tilde{z}_2 > \tilde{z}_1 > 0$;*
(3) *$f(x) \leq kx + c$ on $[\tilde{z}_1, \tilde{z}_2]$, with*

$$k = \frac{f(\tilde{z}_2) - f(\tilde{z}_1)}{\tilde{z}_2 - \tilde{z}_1} > 0 \tag{3.7}$$

and

$$c = \frac{\tilde{z}_2 f(\tilde{z}_1) - \tilde{z}_1 f(\tilde{z}_2)}{\tilde{z}_2 - \tilde{z}_1}; \tag{3.8}$$

(4) $f'_+(\tilde{z}_2) \leq k \leq f'_-(\tilde{z}_1)$.

Then the concave envelope of f is given by

$$f^c(x) = \begin{cases} f(x), & x \in [0, \tilde{z}_1] \cup [\tilde{z}_2, \infty), \\ kx + c, & x \in (\tilde{z}_1, \tilde{z}_2). \end{cases} \tag{3.9}$$

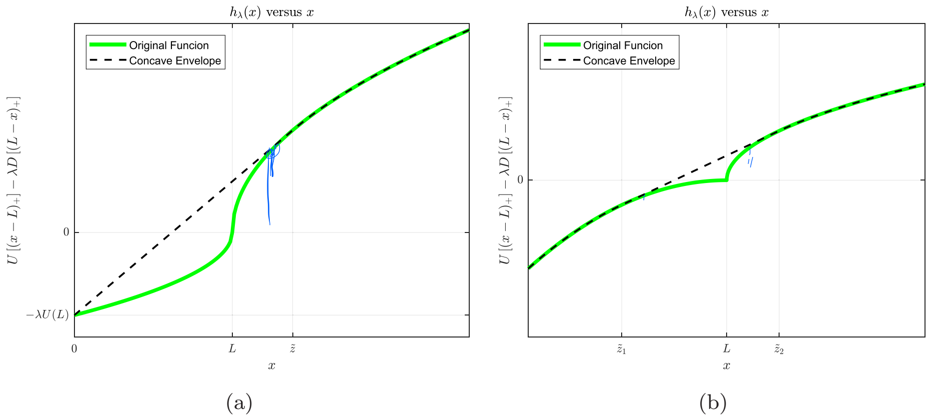

Fig. 2. Concave envelope of $h_\lambda(x) := U[(x-40)_+] - \lambda D[(40-x)_+]$. (a) The solid curve is for $h_\lambda(x)$ with $U(x) = x^{0.5}$ and $D(x) = x^{0.5}$. (b) The solid curve is for $h_\lambda(x)$ with $U(x) = x^{0.5}$ and $D(x) = x^2$.

Proof. The proof is given in Appendix B. $\square$

Example 3.1. The two graphs in Fig. 2 illustrate the concave envelopes of two functions with shapes described in Lemmas 3.3 and 3.4, respectively. In Lemma 3.5 below, we will apply the concavification results of the two lemmas to obtain the concave envelope of the objective function in the pointwise optimization problem (3.3).

An application of Lemmas 3.3 and 3.4 yields the following result.

Lemma 3.5. *Let $f(x) := h_\lambda(x) + \lambda D(L)$, and $\widehat{z} > L$ be the unique solution to*

$$p_1(\widehat{z}) := U(\widehat{z} - L) + \lambda D(L) - U'(\widehat{z} - L) \cdot \widehat{z} = 0. \tag{3.10}$$

on (L, ∞). Then, the concave envelope of f is given as follows:

(a) *If D is an increasing concave function, then f^c is given by Eq. (3.6) with $\tilde{z} = \widehat{z}$ and $k = f'(\tilde{z})$.*
(b) *If D is an increasing strictly convex function with $f'(\widehat{z}) \geq f'_+(0) = \lambda D'(L)$, then $f^c(x)$ is given by Eq. (3.6) with $\tilde{z} = \widehat{z}$ and $k = f'(\tilde{z})$.*
(c) *If D is an increasing strictly convex function with $\lim_{x \searrow 0} D'(x) = 0$, and $f'(\widehat{z}) < f'_+(0) = \lambda D'(L)$. Then, $f^c(x)$ is given by Eq. (3.9) with*

$$k = f'(\tilde{z}_1) = f'(\tilde{z}_2) = \lambda D'(L - \tilde{z}_1) = U'(\tilde{z}_2 - L) \tag{3.11}$$

and $c = f(\tilde{z}_2) - k\tilde{z}_2$, where the pair $(\tilde{z}_1, \tilde{z}_2)$ is the unique solution on $[0, L) \times (L, \infty)$ to the system of equations:

$$\begin{cases} p_2(\tilde{z}_1, \tilde{z}_2) := U'(\tilde{z}_2 - L) - \lambda D'(L - \tilde{z}_1) = 0, \\ p_3(\tilde{z}_1, \tilde{z}_2) := U(\tilde{z}_2 - L) + \lambda D(L - \tilde{z}_1) - U'(\tilde{z}_2 - L) \cdot (\tilde{z}_2 - \tilde{z}_1) = 0. \end{cases} \tag{3.12}$$

Proof. The definition of p_1 and the Inada condition for reward function U imply

$$\lim_{z \searrow L} p_1(z) = 0 + \lambda D(L) - \lim_{y \searrow 0} U'(y) = -\infty. \tag{3.13}$$

The strict concavity of U and $U(0) = 0$ together imply

$$U(z - L) - zU'(z - L) > -LU'(z - L) \quad \text{for } z > L \tag{3.14}$$

and thus

$$p_1(z) > \lambda D(L) - LU'(z - L) \to \lambda D(L) > 0 \quad \text{as } z \to \infty \tag{3.15}$$

by the Inada condition. So a root $\hat{z}$ of (3.10) exists on (L, ∞) and indeed, it is unique, since

$$p_1'(z) = -U''(z - L)z > 0 \quad \text{for } z > L. \tag{3.16}$$

Note that f is concave on $[L, \infty)$ with

$$f(x) = U(x - L) + \lambda D(L) > 0 \quad \text{for } x \geq L. \tag{3.17}$$

Also,

$$f(x) = -D(L - x) + \lambda D(L) \quad \text{for } x \leq L. \tag{3.18}$$

Accordingly, $f'(x) = U'(x - L)$ for $x > L$ and $f'_+(0) = \lambda D'_+(L)$.

(a) Since $\widehat{z} > L$, f is concave on $[\tilde{z}, \infty)$ with $\tilde{z} = \widehat{z}$. By Lemma 3.3, it remains to show $f(x) \leq kx$ on $[0, \tilde{z}]$ with $k = \frac{f(\tilde{z})}{\tilde{z}} > 0$ and $f'_+(\tilde{z}) \leq k$. We have $\tilde{z}f'(\tilde{z}) = f(\tilde{z})$ (this is how Eq. (3.10) is defined) and $k = f'(\tilde{z}) = \frac{f(\tilde{z})}{\tilde{z}} > 0$. Concavity of f on $[L, \tilde{z}]$ implies

$$f(x) \leq f(\tilde{z}) + (x - \tilde{z})f'(\tilde{z}) = kx \quad \text{for } x \in [L, \tilde{z}], \tag{3.19}$$

so that $f(L) \leq k \cdot L$ as well. Further, noticing that D is concave and thus f is convex on $[0, L]$, we obtain $f(x) \leq f'(x)x$ and

$$\begin{aligned} f(x) &\leq f(L) - (L - x)f'(x) \\ &\leq f(L) - (L - x)\frac{f(x)}{x} \\ &= f(L) + \left(1 - \frac{L}{x}\right) f(x). \end{aligned} \tag{3.20}$$

Rearranging this inequality yields

$$f(x) \leq \frac{x}{L} f(L) \leq \frac{x}{L} L \cdot k = kx, \tag{3.21}$$

which implies $f(x) \leq kx$ for $x \in [0, L]$.

(b) The proof is similar to part (a) and thus omitted.

(c) For each $z_1 \in [0, L)$, equation $p_2(z_1, z_2) = 0$ is equivalent to

$$z_2 = L + (U')^{-1}[\lambda D'(L - z_1)]. \tag{3.22}$$

This means that there is a unique solution $z_2 > L$ to the equation $p_2(z_1, z_2) = 0$ for any $z_1 \in [0, L)$. Write

$$z_2(z_1) := L + (U')^{-1}[\lambda D'(L - z_1)] \tag{3.23}$$

to get $\frac{dz_2}{dz_1} > 0$ and

$$\begin{aligned}\frac{dp_3(z_1, z_2(z_1))}{dz_1} &= U'(z_2 - L)\frac{dz_2}{dz_1} - \lambda D'(L - z_1) \\ &\quad - U''(z_2 - L)(z_2 - z_1)\frac{dz_2}{dz_1} - U'(z_2 - L)\left(\frac{dz_2}{dz_1} - 1\right) \\ &= -U''(z_2 - L)(z_2 - z_1)\frac{dz_2}{dz_1} > 0 \end{aligned} \tag{3.24}$$

which implies that $p_4(z_1) := p_3(z_1, z_2(z_1))$ is increasing in z_1 on $[0, L)$. Furthermore, by concavity and (3.10):

$$U(z_2 - L) \le U(\widehat{z} - L) + U'(\widehat{z} - L)(z_2 - \widehat{z}) = U'(\widehat{z} - L)z_2 - \lambda D(L). \tag{3.25}$$

So:

$$\begin{aligned} p_3(z_1, z_2) &\le z_2(U'(\widehat{z} - L) - U'(z_2 - L)) \\ &\quad + \lambda(D(L - z_1) - D(L)) + U'(z_2 - L)z_1. \end{aligned} \tag{3.26}$$

Using $U'(z_2(z_1) - L) = \lambda D'(L - z_1)$ to get

$$\begin{aligned} p_4(z_1) &\le z_2(z_1)(U'(\widehat{z} - L) - U'(z_2(z_1) - L)) \\ &\quad + \lambda(D(L - z_1) - D(L)) + \lambda D'(L - z_1)z_1. \end{aligned} \tag{3.27}$$

As $z_1 \searrow 0$, the last two terms in the above expression tend to zero. The first term is strictly negative for small enough z_1 by assumption since $z_2(z_1) > L$, and

$$\begin{aligned} U'(z_2(z_1) - L) &= \lambda D'(L - z_1) \\ &\to D'(L) > f'(\widehat{z}) \\ &= U'(\widehat{z} - L), \quad \text{as } z_1 \searrow 0. \end{aligned} \tag{3.28}$$

Moreover, by the concavity of U, we have $U(x) \ge U(y) - U'(x)(y - x)$ for $x, y \ge 0$. Therefore,

$$U(z_2 - L) \ge U(z_1) - U'(z_2 - L)[z_1 - (z_2 - L)] \tag{3.29}$$

and

$$\begin{aligned} p_4(z_1) &\ge U(z_1) + \lambda D(L - z_1) \\ &\quad - \lambda D'(L - z_1) \cdot L \to U(L) > 0, \quad \text{as } z_1 \nearrow L. \end{aligned} \tag{3.30}$$

Combining the above analysis, we conclude that there exists a unique solution $(\tilde{z}_1, \tilde{z}_2)$ on $(L, \infty) \times [0, L)$ to the system (3.12). For this solution $(\tilde{z}_1, \tilde{z}_2)$,

$$f'(\tilde{z}_1) = f'(\tilde{z}_2) = \frac{f(\tilde{z}_2) - f(\tilde{z}_1)}{\tilde{z}_2 - \tilde{z}_1} \quad \text{(this is how (3.12) is defined)}, \tag{3.31}$$

and $k = f'(\tilde{z}_1) > 0$. By definition f is concave on $[0, \tilde{z}_1]$ and $[\tilde{z}_2, \infty)$ and moreover, $f(x) \leq kx + c$ for $x \in (\tilde{z}_1, \tilde{z}_2)$ by the concavity on $(\tilde{z}_1, L]$ and on $[L, \tilde{z}_2]$. □

The concave envelope of h_λ can be obtained from Lemma 3.5 as

$$h_\lambda^c = f^c - \lambda D(L). \tag{3.32}$$

The solution to problem (3.3) can be obtained based on solving problem (3.5) as shown in the following proposition.

Proposition 3.1. *For fixed $\lambda \geq 0$ and $y > 0$, $x_\lambda^*(y)$ defined below solves both problems* (3.3) *and* (3.5) *in each of the following cases, where $\widehat{z} > L$ is the unique root of the function p_1 defined in* (3.10).

(a) *If D is an increasing concave function satisfying the Inada condition, i.e. $\lim_{x\to 0} D'(x) = \infty$, then*

$$x_\lambda^*(y) = \begin{cases} (U')^{-1}(y) + L, & 0 < y \leq k, \\ 0, & y > k, \end{cases} \tag{3.33}$$

where $k = f'(\hat{z})$.

(b) *Assume that D is an increasing strictly convex function satisfying*

$$\lim_{x\to 0} D'(x) = 0. \tag{3.34}$$

(b1) *For $U'(\widehat{z} - L) \geq \lambda D'(L)$, $x_\lambda^*(y)$ is given as in equation* (3.33) *where $k = f'(\widehat{z})$.*

(b2) *For $U'(\widehat{z} - L) < \lambda D'(L)$,*

$$x_\lambda^*(y) = \begin{cases} (U')^{-1}(y) + L, & 0 < y \leq k, \\ L - (D')^{-1}\left(\dfrac{y}{\lambda}\right), & k < y < \lambda D'(L), \\ 0, & y \geq \lambda D'(L), \end{cases} \tag{3.35}$$

where

$$k = f'(\tilde{z}_1) = f'(\tilde{z}_2) = U'(\tilde{z}_2 - L) = \lambda D'(L - \tilde{z}_1) \tag{3.36}$$

and the pair $(\tilde{z}_1, \tilde{z}_2)$ is the unique solution to (3.12) *satisfying*

$$0 \leq \tilde{z}_1 < L < \tilde{z}_2. \tag{3.37}$$

Proof. The concave envelope of h_λ is given by $h_\lambda^c(x) = f^c(x) - \lambda D(L)$, where f^c is defined in Lemma 3.5. To find a maximizer of $h_\lambda^c(x) - yx$, for a given $y > 0$ and $\lambda \geq 0$, we simply need to find the points $x_\lambda^*(y)$ for which 0 is in the superdifferential of $h_\lambda^c(x) - yx$, which can be determined by straightforward but tedious calculation. Further, observing that $x_\lambda^*(y) \in \{x \geq 0 : h_\lambda(x) = h_\lambda^c(x)\}$ yields the result. □

3.3. *Solutions to the linearized optimal payoff problem* (2.22)

The derivation of solutions to problem (2.22) relies on the function x_λ^* given in Proposition 3.1. To proceed, for each fixed $\lambda \geq 0$, we define $Z_{\lambda,\beta}^* := x_\lambda^*(\beta\xi_T)$ for $\beta > 0$. Then, part (a) of Lemma 3.1 together with Proposition 3.1 implies that $Z_{\lambda,\beta}^*$ solves problem (3.1). Consequently, by part (b) of Lemma 3.1, if there exists a positive constant β^* satisfying $\mathbb{E}[\xi_T x_\lambda^*(\beta^*\xi_T)] = x_0$ or equivalently $\mathbb{E}[\xi_T Z_{\lambda,\beta}^*] = x_0$, then $Z_\lambda^* := Z_{\lambda,\beta^*}^*$ solves the auxiliary problem (2.22).

Proposition 3.2. *For each $\lambda \geq 0$, there exists a unique constant $\beta^* > 0$ such that $Z_\lambda^* := Z_{\lambda,\beta^*}^* \equiv x_\lambda^*(\beta^*\xi_T)$ satisfies $\mathbb{E}[\xi_T Z_{\lambda,\beta^*}^*] = x_0$, where the function x_λ^* is given in Proposition* 3.1.

Proof. Define

$$H_\lambda(\beta) := \mathbb{E}[\xi_T Z_{\lambda,\beta}^*] \equiv \mathbb{E}[\xi_T x_\lambda^*(\beta\xi_T)]. \tag{3.38}$$

First, we observe that $\xi_T x_\lambda^*(\beta\xi_T)$ is nonnegative, decreasing in β and tends to 0 and ∞ respectively with probability one as β goes to ∞ and 0. Furthermore, for a fixed β', we note that

$$\begin{aligned} H_\lambda(\beta') &= \mathbb{E}[\xi_T x_\lambda^*(\beta'\xi_T)] \\ &\leq \mathbb{E}[\xi_T (U')^{-1}(\beta'\xi_T)] + \mathbb{E}[\xi_T L] \\ &= \mathbb{E}[\xi_T (U')^{-1}(\beta'\xi_T)] + Le^{-rT} < \infty, \end{aligned} \tag{3.39}$$

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where the last inequality follows from Lemma A.2 in Appendix A under assumption (H4). The Monotone Convergence Theorem then implies that $\lim_{\beta\to\infty} H_\lambda(\beta) = 0$ and $\lim_{\beta\to 0^+} H_\lambda(\beta) = \infty$.

Next, we show the continuity of $H_\lambda(\beta)$ with respect to β on $(0,\infty)$. Fix $\beta \in (0,\infty)$ and take a sequence $\beta_n \in (0,\infty)$ with $\beta_n \to \beta$ as $n \to \infty$. Given $\varepsilon > 0$, there exists N such that

$$0 \leq \xi_T x_\lambda^*(\beta_n\xi_T) \leq \xi_T[(U')^{-1}((\beta - \varepsilon)\xi_T) + L] \tag{3.40}$$

for all $n \geq N$, and the upper bound is integrable. Thus, it follows from the Dominated Convergence Theorem that

$$\begin{aligned} \lim_{\beta_n\to\beta} H_\lambda(\beta_n) &= \lim_{\beta_n\to\beta} \mathbb{E}[\xi_T x_\lambda^*(\beta_n\xi_T)] = \mathbb{E}\left[\lim_{\beta_n\to\beta} \xi_T x_\lambda^*(\beta_n\xi_T)\right] \\ &= \mathbb{E}[\xi_T x_\lambda^*(\beta\xi_T)] = H_\lambda(\beta), \end{aligned} \tag{3.41}$$

where the third equality follows from the continuity of $x^*_\lambda(y)$ with respect to y almost everywhere. Thus, H_λ is continuous on $(0,\infty)$, and the existence of β^* is proved.

To prove the uniqueness of β^*, it is sufficient to show the strict monotonicity of H_λ. For $\beta_1 > \beta_2 > 0$, we define sets

$$E_i = \{\omega \in \Omega \,|\, \xi_T(\omega)x^*_\lambda(\beta_i\xi_T(\omega)) > 0\}, \quad i = 1,2. \tag{3.42}$$

Then,

$$\mathbb{P}(E_i) \geq \mathbb{P}(\{\omega \,|\, \beta_i\xi_T(\omega) \leq k\}) > 0, \quad i = 1,2. \tag{3.43}$$

The strict monotonicity of x^*_λ implies

$$\xi_T x^*_\lambda(\beta_1\xi_T) < \xi_T x^*_\lambda(\beta_2\xi_T) \quad \text{for } \omega \in E_1 \tag{3.44}$$

and also that $E_1 \subseteq E_2$. As a consequence, we obtain

$$\begin{aligned} H_\lambda(\beta_1) &= \mathbb{E}[\xi_T x^*_\lambda(\beta_1\xi_T)] \\ &= \int_{E_1} \xi_T(\omega)x^*_\lambda(\beta_1\xi_T(\omega))d\mathbb{P}(\omega) \\ &< \int_{E_1} \xi_T(\omega)x^*_\lambda(\beta_2\xi_T(\omega))d\mathbb{P}(\omega) \\ &\leq \int_{E_2} \xi_T(\omega)x^*_\lambda(\beta_2\xi_T(\omega))d\mathbb{P}(\omega) \\ &= \mathbb{E}[\xi_T x^*_\lambda(\beta_2\xi_T)] \\ &= H_\lambda(\beta_2), \end{aligned} \tag{3.45}$$

which means that $H_\lambda(\beta)$ is strictly decreasing in β. □

Remark 3.1. The proof of Proposition 3.2 also implies that $H_\lambda(\beta) = \mathbb{E}[\xi_T Z^*_{\lambda,\beta}]$ is strictly decreasing as a function of β over the interval $(0,\infty)$, for each fixed $\lambda \geq 0$. In numerical implementations, in which we solve for β^* numerically, the monotonicity of $H_\lambda(\beta)$ is a useful property.

Let β^* be the unique constant that satisfies $\mathbb{E}[\xi_T x^*_\lambda(\beta^*\xi_T)] = x_0$. We characterize the optimal value $v(\lambda; x_0)$ of problem (2.22) in the following proposition, where we use notation that makes explicit the dependence of $k, \beta^*, \hat{z}, \tilde{z}_1$ and $\tilde{z}_2$ on λ (this dependence has been heretofore suppressed for ease of notation). From the above analysis,

$$v(\lambda; x_0) = f_1(\lambda) - \lambda f_2(\lambda), \tag{3.46}$$

where

$$f_1(\lambda) := \mathbb{E}\{U[(Z^*_{\lambda,\beta^*} - L)_+]\} \tag{3.47}$$

and

$$f_2(\lambda) := \mathbb{E}\{D[(L - Z^*_{\lambda,\beta^*})_+]\}. \tag{3.48}$$

Proposition 3.3. *For any $\lambda \geq 0$, let $\widehat{z}(\lambda) > L$ be the unique root of the function p_1 defined in* (3.10).

(a) *If D is an increasing concave function satisfying the Inada condition, i.e. $\lim_{x\to 0} D'(x) = \infty$, then*

$$\begin{cases} f_1(\lambda) = \mathbb{E}\{U[(U')^{-1}(\beta^*(\lambda)\xi_T)]\mathbf{1}_{\{\beta^*(\lambda)\xi_T \leq k(\lambda)\}}\}, \\ f_2(\lambda) = D(L)\mathbb{P}[\beta^*(\lambda)\xi_T > k(\lambda)], \end{cases} \tag{3.49}$$

where

$$k(\lambda) = f'(\hat{z}(\lambda)) = U'[(\hat{z}(\lambda) - L)_+]. \tag{3.50}$$

(b) *Assume that D is an increasing strictly convex function satisfying*

$$\lim_{x\to 0} D'(x) = 0. \tag{3.51}$$

(b1) *If $f'(\widehat{z}(\lambda)) \geq \lambda D'(L)$, then $f_1(\lambda)$ and $f_2(\lambda)$ are given as in* (3.49) *with $k(\lambda) = f'(\hat{z}(\lambda))$.*

(b2) *If $f'(\widehat{z}(\lambda)) < \lambda D'(L)$, then*

$$\begin{cases} f_1(\lambda) = \mathbb{E}\{U[(U')^{-1}(\beta^*(\lambda)\xi_T)]\mathbf{1}_{\{\beta^*(\lambda)\xi_T \leq k(\lambda)\}}\}, \\ f_2(\lambda) = \mathbb{E}\left\{D\left[(D')^{-1}\left(\dfrac{\beta^*(\lambda)\xi_T}{\lambda}\right)\right]\mathbf{1}_{\{k(\lambda) < \beta^*(\lambda)\xi_T < \lambda D'(L)\}}\right\} \\ \qquad\qquad + D(L)\mathbb{P}[\beta^*(\lambda)\xi_T \geq \lambda D'(L)], \end{cases} \tag{3.52}$$

where

$$\begin{aligned} k(\lambda) &= f'(\tilde{z}_1(\lambda)) = f'(\tilde{z}_2(\lambda)) = U'[(\tilde{z}_2(\lambda) - L)_+] \\ &= \lambda D'[(L - \tilde{z}_1(\lambda))_+] \end{aligned} \tag{3.53}$$

and the pair $(\tilde{z}_1(\lambda), \tilde{z}_2(\lambda))$ is the unique solution to (3.12).

Proof. The claims follow immediately from Proposition 3.1. □

3.4. *Optimal solution to optimal payoff problem* (2.14)

Based on the previous analysis, we can summarize the solution to problem (2.14) as follows. Let

$$Z^*_\lambda := x^*_\lambda(\beta^*\xi_T) \tag{3.54}$$

with a unique β^* satisfying $\mathbb{E}[\xi_T Z^*_\lambda] = x_0$ where the function x^*_λ is given in Proposition 3.1, and the existence of a β^* for each λ is insured by Proposition 3.2. Further, by Proposition 2.6, there exists a constant $\lambda^* \geq 0$ satisfying (2.23), and therefore, it follows from Proposition 2.3 that

$$Z^* := x^*_{\lambda^*}(\beta^* \xi_T) \tag{3.55}$$

is a solution to problem (2.14).

As such, we derive a solution Z^* to problem (2.14) by the following algorithm:

Algorithm 1 (Numerical algorithm to derive Z^* for problem (2.14)).

Step 1. Derive the optimal function x^*_λ for the pointwise optimization problem (3.3) using Eqs. (3.33) and (3.35) from Proposition 3.1;
Step 2. For each $\lambda \geq 0$, search for the unique solution to equation $\mathbb{E}[\xi_T x^*_\lambda(\beta^*(\lambda)\xi_T)] = x_0$ and set $Z^*_\lambda = x^*_\lambda(\beta^*(\lambda)\xi_T)$;
Step 3. Invoke Proposition 2.3 to get $Z^* := Z^*_{\lambda^*}$ by solving for λ^* from equation (2.23) or equivalently $v(\lambda^*; x_0) \equiv f_1(\lambda^*) - \lambda^* f_2(\lambda^*) = 0$.

4. Optimal Trading Strategies under Power Functions

In the preceding section, we have studied how to derive the optimal solution Z^* for the optimal payoff problem (2.14). Given this solution Z^*, in principle, we can invoke Proposition 2.1 to obtain the optimal trading strategies. The implementation involves the computation of relevant quantities from (2.17)–(2.19). In this section, we study the optimal trading strategy by assuming both U and D are power functions, and obtain a more explicit solution. As we already showed in Proposition 2.2, the portfolio selection problem (2.12) is ill-posed for a convex reward function U. So, throughout this section, we assume $U(x) = x^{\gamma_1}$ for $0 < \gamma_1 < 1$ which is strictly concave, and study the optimal trading strategies with $D(x) = x^{\gamma_2}$ for $0 < \gamma_2 \leq 1$ and $\gamma_2 > 1$ in two separate subsections. It is easy to verify that assumptions (H1)–(H4) are all satisfied in this setting. We follow the steps outlined in Algorithm 1 for the determination of optimal solutions.

4.1. *Optimal trading strategies when D is a concave power function*

In this section, we consider

$$U(x) = x^{\gamma_1} \quad \text{for } 0 < \gamma_1 < 1 \tag{4.1}$$

and

$$D(x) = x^{\gamma_2} \quad \text{for } 0 < \gamma_2 \leq 1. \tag{4.2}$$

In this case, part (a) of Proposition 3.1 is applicable and for each $\lambda \geq 0$, the solution to problem (3.3) is given by

$$x^*_\lambda(y) = \begin{cases} \left(\dfrac{y}{\gamma_1}\right)^{\frac{1}{\gamma_1 - 1}} + L, & 0 < y \leq k(\lambda), \\ 0, & y > k(\lambda), \end{cases} \tag{4.3}$$

where

$$k(\lambda) = \gamma_1(\widetilde{z}_1(\lambda) - L)^{\gamma_1 - 1} \tag{4.4}$$

and $\tilde{z}_1(\lambda)$ is the unique solution to

$$[(1-\gamma_1)\tilde{z}_1(\lambda) - L](\tilde{z}_1(\lambda) - L)^{\gamma_1 - 1} + \lambda L^{\gamma_2} = 0. \tag{4.5}$$

Therefore, we set

$$Z^*_\lambda := Z^*_{\lambda,\beta^*(\lambda)} \equiv x^*_\lambda(\beta^*(\lambda)\xi_T) = \left[\left(\frac{\beta^*(\lambda)\xi_T}{\gamma_1}\right)^{\frac{1}{\gamma_1-1}} + L\right]\mathbf{1}_{\{\beta^*(\lambda)\xi_T \le k(\lambda)\}}, \tag{4.6}$$

where $\beta^*(\lambda)$ is determined by the equation $\mathbb{E}[\xi_T Z^*_{\lambda,\beta^*(\lambda)}] = x_0$ for each $\lambda \ge 0$.

To proceed, we use Φ and ϕ to denote the standard normal distribution function and its density function, and define

$$\begin{cases} d_{1,t}(\beta) := \dfrac{\ln\beta - \ln\xi_t + \left(r - \dfrac{1}{2}\zeta^2\right)(T-t)}{\zeta\sqrt{T-t}}, \\ d_{2,t}(\beta;\gamma) := d_{1,t}(\beta) + \dfrac{\zeta\sqrt{T-t}}{1-\gamma}, \\ K(\beta;\gamma) := \phi[d_{1,t}(\beta)]\left(1 + \dfrac{\zeta\sqrt{T-t}}{1-\gamma}\dfrac{\Phi[d_{2,t}(\beta;\gamma)]}{\phi[d_{2,t}(\beta;\gamma)]}\right). \end{cases} \tag{4.7}$$

Noticing

$$\left(\frac{\beta^*(\lambda)\xi_T}{\gamma_1}\right)^{\frac{1}{\gamma_1-1}} + L \ge L, \tag{4.8}$$

we use Eq. (3.49) from Proposition 3.3 to obtain

$$\begin{cases} f_1(\lambda) := \mathbb{E}\{U[(Z^*_\lambda - L)_+]\} \\ \qquad = \mathbb{E}\left\{\left[\left(\dfrac{\beta^*(\lambda)\xi_T}{\gamma_1}\right)^{\frac{\gamma_1}{\gamma_1-1}}\right]\mathbf{1}_{\{\beta^*(\lambda)\xi_T \le k(\lambda)\}}\right\} \\ \qquad = e^{-rT}\cdot\beta^*(\lambda)\cdot\gamma_1^{\frac{\gamma_1}{1-\gamma_1}}\dfrac{\phi\left[d_{1,0}\left(\dfrac{1}{\beta^*(\lambda)}\right)\right]}{\phi\left[d_{2,0}\left(\dfrac{1}{\beta^*(\lambda)};\gamma_1\right)\right]}\Phi\left[d_{2,0}\left(\dfrac{k(\lambda)}{\beta^*(\lambda)};\gamma_1\right)\right], \\ f_2(\lambda) := \mathbb{E}\{D[(L - Z^*_\lambda)_+]\} \\ \qquad = \mathbb{E}[(L)^{\gamma_2}\mathbf{1}_{\{\beta^*(\lambda)\xi_T > k(\lambda)\}}] \\ \qquad = L^{\gamma_2}\left\{1 - \Phi\left[d_{1,0}\left(\dfrac{k(\lambda)}{\beta^*(\lambda)}\right) + \zeta\sqrt{T}\right]\right\}. \end{cases} \tag{4.9}$$

With the above expressions for f_1 and f_2, we determine a $\lambda^* > 0$ to satisfy

$$f_1(\lambda^*) - \lambda^* f_2(\lambda^*) = 0. \tag{4.10}$$

The existence of such a λ^* is guaranteed by Proposition 2.6.

Given $\lambda^* > 0$, we can derive the optimal solution and portfolio value for the portfolio optimization problem (2.12) as shown in the proposition below.

Proposition 4.1. *Let $\lambda^* > 0$ be a constant satisfying equation* (2.23) *or equivalently $v(\lambda^*; x_0) = f_1(\lambda^*) - \lambda^* f_2(\lambda^*) = 0$. Let*

$$k(\lambda^*) = \gamma_1(\widetilde{z}_1(\lambda^*) - L)^{\gamma_1 - 1} \tag{4.11}$$

and $\tilde{z}_1(\lambda^*)$ be the solution to Eq. (4.5) *with $\lambda = \lambda^*$, the optimal portfolio value, the optimal trading strategy and the corresponding terminal portfolio value are given as follows.*

(1) *The optimal portfolio value at time t, $0 \le t < T$, is given by*

$$\begin{cases} X_t^* = \mathrm{e}^{-r(T-t)}(A_1 + A_2), \\ A_1 = \left(\dfrac{k(\lambda^*)}{\gamma_1}\right)^{\frac{1}{\gamma_1 - 1}} \dfrac{\phi\left[d_{1,t}\left(\dfrac{k(\lambda^*)}{\beta^*(\lambda^*)}\right)\right]}{\phi\left[d_{2,t}\left(\dfrac{k(\lambda^*)}{\beta^*(\lambda^*)}; \gamma_1\right)\right]} \\ \qquad \cdot \Phi\left[d_{2,t}\left(\dfrac{k(\lambda^*)}{\beta^*(\lambda^*)}; \gamma_1\right)\right], \\ A_2 = L\Phi\left[d_{1,t}\left(\dfrac{k(\lambda^*)}{\beta^*(\lambda^*)}\right)\right]. \end{cases} \tag{4.12}$$

(2) *For $0 \le t < T$, an optimal amount to invest in the risky asset at time t is given by π_t^* as follows*:

$$\begin{cases} \pi_t^* = \dfrac{e^{-r(T-t)}}{\sigma\sqrt{T-t}}(a_1 + a_2), \\ a_1 = \left(\dfrac{k(\lambda^*)}{\gamma_1}\right)^{\frac{1}{\gamma_1 - 1}} K\left(\dfrac{k(\lambda^*)}{\beta^*(\lambda^*)}; \gamma_1\right), \\ a_2 = L\phi\left[d_{1,t}\left(\dfrac{k(\lambda^*)}{\beta^*(\lambda^*)}\right)\right]. \end{cases} \tag{4.13}$$

(3) *The optimal terminal portfolio value is*

$$X_T^* = \left[\left(\frac{\beta^*(\lambda^*)\xi_T}{\gamma_1}\right)^{\frac{1}{\gamma_1 - 1}} + L\right] \mathbf{1}_{\{\beta^*(\lambda^*)\xi_T \le k(\lambda^*)\}}. \tag{4.14}$$

Table 1. Parameter setting for numerical illustration.

x_0	T	r	L	μ	σ
100	5	0.03	150	0.07	0.3

Proof. With the optimal $\lambda^* > 0$, the proposition depends on the propositions and lemmas in Secs. 2 and 3. By Proposition 2.1 and standard arguments, $\pi^* = \{\pi_t^*, 0 \leq t \leq T\}$, valued in $\mathbb{R}$, solves problem (2.12) with an optimal portfolio value at time t given by

$$X_t^{\pi^*} = Y_t^*, \quad t \in [0, T], \tag{4.15}$$

where

$$\pi_t^* = \sigma^{-1}\xi_t^{-1}\theta_t^* + \sigma^{-1}\zeta Y_t^*, \tag{4.16}$$

θ_t^* valued in $\mathbb{R}$ comes from the martingale representation theorem,

$$Y_t^* := \xi_t^{-1}\mathbb{E}[\xi_T Z^* \mid \mathcal{F}_t] \tag{4.17}$$

and $Z^* := Z_{\lambda^*}^* = Z_{\lambda^*, \beta^*(\lambda^*)}^*$.[e] The calculation of the solution is straightforward, but tedious, and follows from a similar procedure to that in the proof of Proposition 4.1 in Lin *et al.* (2017). □

Example 4.1. We consider the parameter values given in Table 1.

The behavior of $v(\lambda; x_0) = f_1(\lambda) - \lambda f_2(\lambda)$, is shown in Fig. 3. As shown earlier, $v(\lambda; x_0)$ is convex and decreasing. Meanwhile, $f_1(\lambda)$ and $f_2(\lambda)$ are decreasing as well and $v(\lambda; x_0)$ crosses zero for λ around 1.3. We can pick two different λ's that lead to a positive value and a negative value for $v(\lambda; x_0)$ and then use the bisection method to approach λ^* such that $v(\lambda^*; x_0) = 0$, where we select the tolerance for root finding to be 1.0×10^{-10}. Using the bisection method, we obtain $\lambda^* = 1.3664$ with $v(\lambda^*; x_0) = -8.6066 \times 10^{-11}$, $f_1(\lambda^*) = 4.2426$ and $f_2(\lambda^*) = 3.1048$. The ratio, i.e. $\frac{f_1(\lambda^*)}{f_2(\lambda^*)}$, agrees with λ^*. The optimal λ^* is the optimal objective value of the original problem (2.12) for the given parameter set.

With the obtained λ^*, we are able to find the optimal portfolio value on $[0, T]$, and the optimal amount of investment in the risky asset by using Proposition 4.1. Figure 4 shows the relationship between π_t^* and X_t^* for $t = 4$, one year before maturity (in the left panel), and how X_T^* varies with ξ_T (in the right panel).

The figure in the left panel exhibits a "peak-and-valley" pattern with two turning points. When the optimal portfolio value X_t^* is close to zero, the optimal amount of investment in the risky asset approaches zero as well. When X_t^* is large enough, π_t^* increases with X_t^*. On the other hand, the figure in the right panel reveals that

[e]Note we denote $X_t^* := X_t^{\pi^*}, t \in [0, T]$, by dropping π from the superscript.

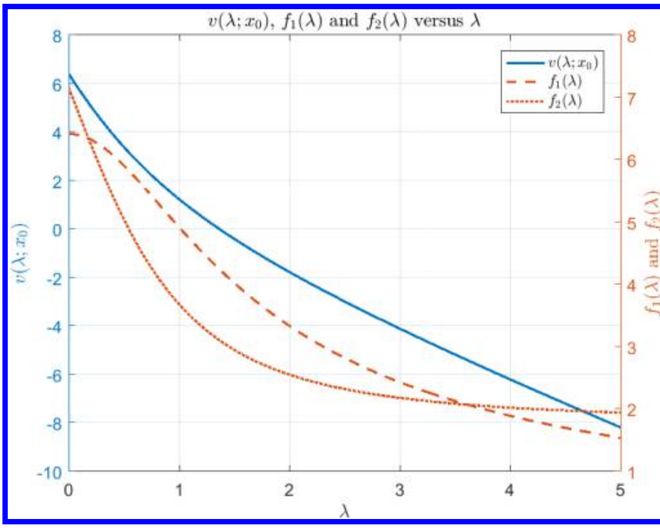
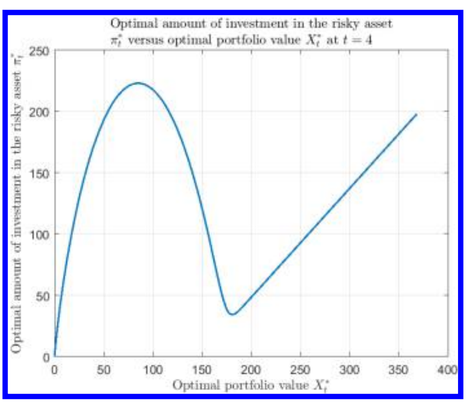
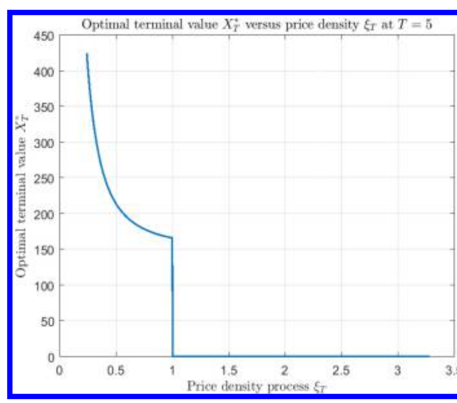


Fig. 3. This figure shows how the optimal objective value $v(\lambda; x_0)$ of problem (2.22) and the two quantities $f_1(\lambda)$ and $f_2(\lambda)$ respond to change in λ, where $f_1(\lambda)$ and $f_2(\lambda)$ are respectively given by Eqs. (3.47) and (3.48) so that $v(\lambda; x_0) = f_1(\lambda) - \lambda f_2(\lambda)$. In the figure, both the reward and penalty functions are given by $U(x) = D(x) = x^{0.5}$, and the threshold $L = 150$.



(a)



(b)

Fig. 4. This figure shows: (a) how the optimal amount of investment π_t^* versus with the optimal portfolio value X_t^* for $t = 4$; and (b) how the optimal terminal portfolio value X_T^* versus with the pricing kernel ξ_T, where $T = 5$ is the terminal time. The model parameters are given by $\gamma_1 = 0.5$, $\gamma_2 = 0.5$ and $L = 150$, respectively.

the optimal terminal portfolio value X_T^* decreases with ξ_T and drops to zero when ξ_T is around 1. Recall from (4.14) that when $\xi > \frac{k(\lambda^*)}{\beta^*(\lambda^*)}$, $X_T^* = 0$. The numerical results tell us that $\frac{k(\lambda^*)}{\beta^*(\lambda^*)} = 1.0034$ and also that $\tilde{z}_1(\lambda) = 166.0221$ which is the vertical distance of the drop of X_T^* at $\xi_T = \frac{k(\lambda^*)}{\beta^*(\lambda^*)}$, as shown in the figure. The mathematical intuition is that we start from the origin and draw a tangent line to touch the original objective function where the tangent point is $\tilde{z}_1(\lambda^*)$ and thus we obtain that the optimal terminal portfolio value $X_T^* \in \{0\} \cup [\tilde{z}_1(\lambda^*), \infty)$.

4.2. *Optimal trading strategies when D is a convex power function*

In this section, we consider

$$U(x) = x^{\gamma_1}, \quad 0 < \gamma_1 < 1 \tag{4.18}$$

and

$$D(x) = x^{\gamma_2}, \quad \gamma_2 > 1. \tag{4.19}$$

In this case, part (b) of Proposition 3.1 is applicable and for each $\lambda \geq 0$, the solution to problem (3.3) is given as follows.

(1) If $\gamma_1(\widehat{z} - L)^{\gamma_1 - 1} \geq \lambda\gamma_2 \cdot (L)^{\gamma_2 - 1}$ with $\widehat{z} > L$ being the unique solution to (4.5), then $x_\lambda^*(y)$ is given by (4.3) where

$$k = \gamma_1(\widehat{z} - L)^{\gamma_1 - 1} \quad \text{and} \quad \tilde{z}_1 = \widehat{z}. \tag{4.20}$$

(2) If $\gamma_1(\widehat{z} - L)^{\gamma_1 - 1} < \lambda\gamma_2 \cdot (L)^{\gamma_2 - 1}$ with $\widehat{z} > L$ being the unique solution to (4.5), then $x_\lambda^*(y)$ is given by

$$x_\lambda^*(y) = \begin{cases} \left(\dfrac{y}{\gamma_1}\right)^{\frac{1}{\gamma_1 - 1}} + L, & 0 < y \leq k, \\ L - \left(\dfrac{y}{\lambda\gamma_2}\right)^{\frac{1}{\gamma_2 - 1}}, & k < y \leq \lambda\gamma_2 \cdot (L)^{\gamma_2 - 1}, \\ 0, & y > k, \end{cases} \tag{4.21}$$

where

$$k = \gamma_1(\tilde{z}_2 - L)^{\gamma_1 - 1} = \lambda\gamma_2(L - \tilde{z}_1)^{\gamma_2 - 1} \tag{4.22}$$

and the pair $(\tilde{z}_1, \tilde{z}_2)$ is the unique solution to

$$\begin{cases} \gamma_1(\tilde{z}_2 - L)^{\gamma_1 - 1} - \lambda\gamma_2(L - \tilde{z}_1)^{\gamma_2 - 1} = 0, \\ (\tilde{z}_2 - L)^{\gamma_1} + \lambda(L - \tilde{z}_1)^{\gamma_2} - \gamma_1(\tilde{z}_2 - L)^{\gamma_1 - 1} \cdot (\tilde{z}_2 - \tilde{z}_1) = 0. \end{cases} \tag{4.23}$$

The optimal solution Z_λ^* for the linearized problem 2.22 can be obtained for each of the above two cases separately. For the first case, it can be obtained through (4.6) and both $f_1(\lambda)$ and $f_2(\lambda)$ are specified in (4.9). The optimal solution to the portfolio selection problem (2.12) is as given in Proposition 4.1.

In the second case, we are able to write down the optimal solution to problem (2.22), with the notation $\tilde{z}_1(\lambda)$, $\tilde{z}_2(\lambda)$, $k(\lambda)$ and $\beta^*(\lambda)$ to be consistent with the previous section, as follows:

$$
\begin{aligned}
Z_\lambda^* := Z_{\lambda,\beta^*(\lambda)}^* &\equiv x^*(\beta^*(\lambda)\xi_T) \\
&= \left[\left(\frac{\beta^*(\lambda)\xi_T}{\gamma_1}\right)^{\frac{1}{\gamma_1-1}} + L\right]\mathbf{1}_{\{\beta^*(\lambda)\xi_T \le k(\lambda)\}} \\
&\quad + \left[L - \left(\frac{\beta^*(\lambda)\xi_T}{\lambda\gamma_2}\right)^{\frac{1}{\gamma_2-1}}\right]\mathbf{1}_{\{k(\lambda)<\beta^*(\lambda)\xi_T\le\lambda\gamma_2\cdot(L)^{\gamma_2-1}\}},
\end{aligned}
\tag{4.24}
$$

where $\beta^*(\lambda)$ is determined by the equation $\mathbb{E}[\xi_T Z_{\lambda,\beta^*(\lambda)}^*] = x_0$.

From the expression (4.24), it is easy to verify that

$$
\left(\frac{\beta^*(\lambda)\xi_T}{\gamma_1}\right)^{\frac{1}{\gamma_1-1}} + L \ge L, \quad \text{a.s.,} \tag{4.25}
$$

and also that

$$
L - \left(\frac{\beta^*(\lambda)\xi_T}{\lambda\gamma_2}\right)^{\frac{1}{\gamma_2-1}} \le L, \quad \text{a.s.,} \tag{4.26}
$$

Therefore, from Proposition 3.3 we obtain:

$$
\begin{cases}
\begin{aligned}
f_1(\lambda) &:= \mathbb{E}\{U[(Z_\lambda^* - L)_+]\} \\
&= \mathbb{E}\left\{\left[\left(\frac{\beta^*(\lambda)\xi_T}{\gamma_1}\right)^{\frac{\gamma_1}{\gamma_1-1}}\right]\mathbf{1}_{\{\beta^*(\lambda)\xi_T\le k(\lambda)\}}\right\} \\
&= e^{-rT}\cdot\beta^*(\lambda)\cdot\gamma_1^{\frac{\gamma_1}{1-\gamma_1}}\frac{\phi\left[d_{1,0}\left(\frac{1}{\beta^*(\lambda)}\right)\right]}{\phi\left[d_{2,0}\left(\frac{1}{\beta^*(\lambda)};\gamma_1\right)\right]}\Phi\left[d_{2,0}\left(\frac{k(\lambda)}{\beta^*(\lambda)};\gamma_1\right)\right], \\
f_2(\lambda) &:= \mathbb{E}\{D[(L - Z_\lambda^*)_+]\} \\
&= \mathbb{E}\left\{\left[\left(\frac{\beta^*(\lambda)\xi_T}{\lambda\gamma_2}\right)^{\frac{\gamma_2}{\gamma_2-1}}\right]\mathbf{1}_{\{k(\lambda)<\beta^*(\lambda)\xi_T\le\lambda\gamma_2\cdot(L)^{\gamma_2-1}\}}\right\} \\
&\quad + \mathbb{E}[(L)^{\gamma_2}\mathbf{1}_{\{\beta^*(\lambda)\xi_T>\lambda\gamma_2\cdot(L)^{\gamma_2-1}\}}] \\
&= e^{-rT}\cdot\beta^*(\lambda)\cdot(\lambda\gamma_2)^{\frac{\gamma_2}{1-\gamma_2}}\frac{\phi\left[d_{1,0}\left(\frac{1}{\beta^*(\lambda)}\right)\right]}{\phi\left[d_{2,0}\left(\frac{1}{\beta^*(\lambda)};\gamma_2\right)\right]} \\
&\quad \times\left\{\Phi\left[d_{2,0}\left(\frac{\lambda\gamma_2\cdot(L)^{\gamma_2-1}}{\beta^*(\lambda)};\gamma_2\right)\right] - \Phi\left[d_{2,0}\left(\frac{k(\lambda)}{\beta^*(\lambda)};\gamma_2\right)\right]\right\} \\
&\quad + L^{\gamma_2}\left\{1 - \Phi\left[d_{1,0}\left(\frac{\lambda\gamma_2\cdot(L)^{\gamma_2-1}}{\beta^*(\lambda)}\right) + \zeta\sqrt{T}\right]\right\}.
\end{aligned}
\end{cases}
\tag{4.27}
$$

Similarly, with the above expressions for f_1 and f_2, we determine a $\lambda^* > 0$ to satisfy

$$f_1(\lambda^*) - \lambda^* f_2(\lambda^*) = 0. \tag{4.28}$$

The existence of such a λ^* is guaranteed by Proposition 2.6.

Given $\lambda^* > 0$, we can derive the optimal solution and portfolio value for the optimization problem (2.12) as shown in the proposition below.

Proposition 4.2. *Given $\lambda^* > 0$ such that* (2.23) *holds, the optimal portfolio value, the optimal trading strategy and the corresponding terminal portfolio value are given as follows*:

(1) *If $\gamma_1(\widehat{z} - L)^{\gamma_1 - 1} \geq \lambda^* \gamma_2 \cdot (L)^{\gamma_2 - 1}$ where $\widehat{z} > L$ is the unique solution to* (4.5), *then the optimal portfolio value at time t, $0 \leq t < T$, is given by* (4.12), *the optimal amount to invest in the risky asset at time t, $0 \leq t < T$, is given by* (4.13) *and the optimal terminal portfolio value is given by* (4.14) *with*

$$k(\lambda^*) = \gamma_1(\widetilde{z}(\lambda^*) - L)^{\gamma_1 - 1}, \quad \text{and} \quad \tilde{z}_1(\lambda^*) = \widehat{z}. \tag{4.29}$$

(2) *If $\gamma_1(\widehat{z} - L)^{\gamma_1 - 1} < \lambda^* \gamma_2 \cdot (L)^{\gamma_2 - 1}$ where $\widehat{z} > L$ is the unique solution to* (4.5), *then*

$$k(\lambda^*) = \gamma_1(\widetilde{z}_2(\lambda^*) - L)^{\gamma_1 - 1} = \lambda^* \gamma_2 (L - \tilde{z}_1(\lambda^*))^{\gamma_2 - 1} \tag{4.30}$$

and the pair $(\tilde{z}_1(\lambda^*), \tilde{z}_2(\lambda^*))$ is the unique solution to (4.23). *Furthermore*,

(2.1) *the optimal portfolio value at time t, $0 \leq t < T$, is given by*

$$\begin{cases} X_t^* = \mathrm{e}^{-r(T-t)}(B_1 + B_2 - B_3), \\[2ex] B_1 = \left(\dfrac{k(\lambda^*)}{\gamma_1}\right)^{\frac{1}{\gamma_1 - 1}} \dfrac{\phi\left[d_{1,t}\left(\dfrac{k(\lambda^*)}{\beta^*(\lambda^*)}\right)\right]}{\phi\left[d_{2,t}\left(\dfrac{k(\lambda^*)}{\beta^*(\lambda^*)}; \gamma_1\right)\right]} \cdot \Phi\left[d_{2,t}\left(\dfrac{k(\lambda^*)}{\beta^*(\lambda^*)}; \gamma_1\right)\right], \\[2ex] B_2 = L\Phi\left[d_{1,t}\left(\dfrac{\lambda^* \gamma_2 \cdot (L)^{\gamma_2 - 1}}{\beta^*(\lambda^*)}\right)\right], \\[2ex] B_3 = (\lambda^* \gamma_2)^{\frac{1}{1-\gamma_2}} \dfrac{\phi\left[d_{1,t}\left(\dfrac{1}{\beta^*(\lambda)}\right)\right]}{\phi\left[d_{2,t}\left(\dfrac{1}{\beta^*(\lambda)}; \gamma_2\right)\right]} \\[2ex] \qquad \times \left\{\Phi\left[d_{2,t}\left(\dfrac{\lambda \gamma_2 \cdot (L)^{\gamma_2 - 1}}{\beta^*(\lambda)}; \gamma_2\right)\right] - \Phi\left[d_{2,t}\left(\dfrac{k(\lambda)}{\beta^*(\lambda)}; \gamma_2\right)\right]\right\}. \end{cases} \tag{4.31}$$

(2.2) *For $0 \leq t < T$, an optimal amount to invest in the risky asset at time t is given by π_t^* as follows*:

$$\begin{cases} \pi_t^* = \dfrac{e^{-r(T-t)}}{\sigma\sqrt{T-t}}(b_1 + b_2 + b_3), \\ b_1 = \left(\dfrac{k(\lambda^*)}{\gamma_1}\right)^{\frac{1}{\gamma_1-1}} K\left(\dfrac{k(\lambda^*)}{\beta^*(\lambda^*)}; \gamma_1\right), \\ b_2 = L\phi\left[d_{1,t}\left(\dfrac{\lambda^*\gamma_2 \cdot (L)^{\gamma_2-1}}{\beta^*(\lambda^*)}\right)\right], \\ b_3 = L \times K\left(\dfrac{\lambda^*\gamma_2 \cdot (L)^{\gamma_2-1}}{\beta^*(\lambda^*)}; \gamma_2\right) - \left(\dfrac{k(\lambda^*)}{\lambda^*\gamma_2}\right)^{\frac{1}{\gamma_2-1}} K\left(\dfrac{k(\lambda^*)}{\beta^*(\lambda^*)}; \gamma_2\right). \end{cases} \tag{4.32}$$

(2.3) *The optimal terminal portfolio value when $t = T$ is*

$$X_T^* = \left[\left(\frac{\beta^*(\lambda^*)\xi_T}{\gamma_1}\right)^{\frac{1}{\gamma_1-1}} + L\right] \mathbf{1}_{\{\beta^*(\lambda^*)\xi_T \leq k(\lambda^*)\}} + \left[L - \left(\frac{\beta^*(\lambda^*)\xi_T}{\lambda^*\gamma_2}\right)^{\frac{1}{\gamma_2-1}}\right] \mathbf{1}_{\{k(\lambda^*) < \beta^*(\lambda^*)\xi_T \leq \lambda^*\gamma_2 \cdot (L)^{\gamma_2-1}\}}. \tag{4.33}$$

Proof. The results are derived in a way similar to those in Proposition 4.1. □

Example 4.2. We also numerically implement the results for illustration based on the set of parameters specified in Table 1. The behavior of $v(\lambda; x_0)$ is shown in Fig. 5. The range of λ is chosen to be $[0, 0.1]$, within which $v(\lambda; x_0)$ is decreasing in λ, and crosses zero when $\lambda \in (0.02, 0.04)$. Compared with Fig. 3 where λ is taken in the range $[0, 5]$, we can see that a larger γ_2 leads to a smaller λ^*. The reason is due to the fact that the negative part $f_2(\lambda)$ dominates the positive part $f_1(\lambda)$ when γ_2 is large. Using the bisection method, we are able to obtain $\lambda^* = 0.0251$ and $v(\lambda^*; x_0) = -4.5648 \times 10^{-11}$, which is within the selected tolerance 1.0×10^{-10}. Also $f_1(\lambda^*) = 4.0125$ and $f_2(\lambda^*) = 159.7092$. Their ratio $\frac{f_1(\lambda^*)}{f_2(\lambda^*)}$ coincides with λ^* as well.

With the obtained λ^*, we are also able to obtain the optimal portfolio value in $[0, T]$ and the optimal amount of investment in the risky asset. Figure 6 demonstrates the relationship between π_t^* and X_t^* for $t = 4$, one year before maturity (in the left panel), and how X_T^* varies with ξ_T (in the right panel).

The "peak-and-valley" pattern revealed in the left panel is similar to the previous case demonstrated in Fig. 4(a). In the right panel of Fig. 6, X_T^* is a decreasing function of ξ_T, with a drop at a certain point and decreasing until it hits zero. With $\lambda^* = 0.0251$ and $\gamma_2 = 1.3$, we can verify from the numerical results that X_t^*, π_t^* and

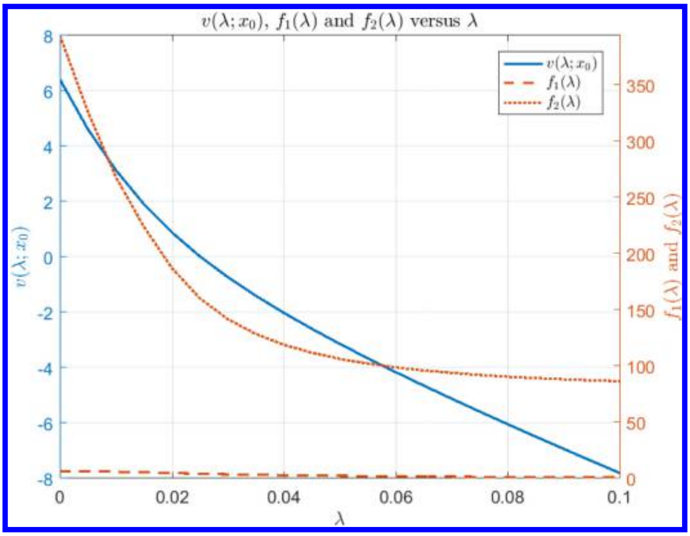
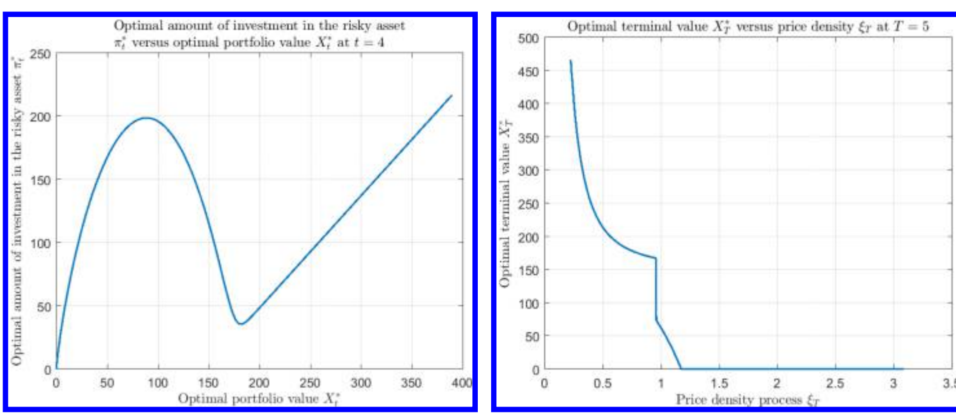


Fig. 5. This figure shows how the optimal objective value $v(\lambda; x_0)$ of problem (2.22) and the two quantities $f_1(\lambda)$ and $f_2(\lambda)$ respond to change in λ, where $f_1(\lambda)$ and $f_2(\lambda)$ are respectively given by Eqs. (3.47) and (3.48) so that $v(\lambda; x_0) = f_1(\lambda) - \lambda f_2(\lambda)$. In the figure, the reward and penalty functions are given by $U(x) = x^{0.5}$ and $D(x) = x^{1.3}$, respectively, and the threshold $L = 150$.



(a) (b)

Fig. 6. This figure shows: (a) how the optimal amount of investment π_t^* versus with the optimal portfolio value X_t^* for $t = 4$; and (b) how the optimal terminal portfolio value X_T^* versus with the pricing kernel ξ_T, where $T = 5$ is the terminal time. The model parameters are given by $\gamma_1 = 0.5$, $\gamma_2 = 1.3$ and $L = 150$, respectively.

X_T^* are as given in (4.31)–(4.33). We obtain $\frac{k(\lambda^*)}{\beta^*(\lambda^*)} = 0.9575$ where the drop in value of X_T^* occurs and the vertical distance of the drop is equal to $\tilde{z}_2(\lambda^*) - \tilde{z}_1(\lambda^*)$ with $\tilde{z}_1(\lambda^*) = 74.2832$ and $\tilde{z}_2(\lambda^*) = 167.4731$. The mathematical intuition is similar to that for example 4.1. The concave envelope is constructed through a tangent line where the two tangent points are $\tilde{z}_1(\lambda^*)$ and $\tilde{z}_2(\lambda^*)$, and thus the optimal terminal portfolio value $X_T^* \in [0, \tilde{z}_1(\lambda^*)] \cup [\tilde{z}_2(\lambda^*), \infty)$.

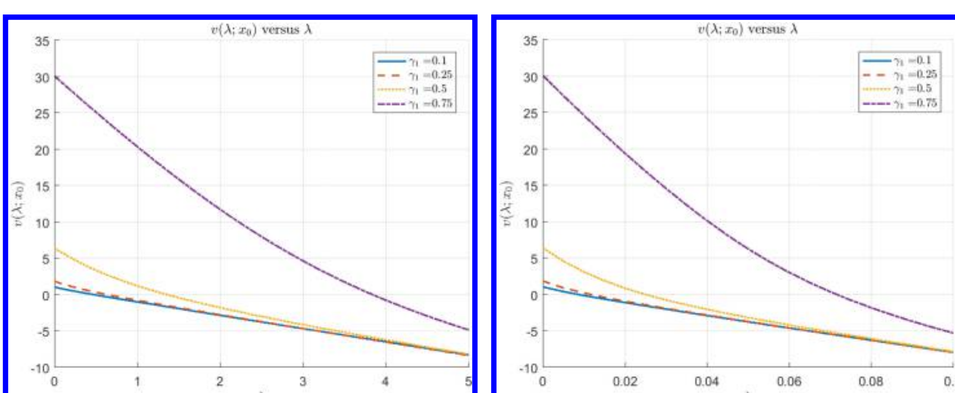
5. Sensitivity Analysis

In the previous section, with U and D specified as power functions, we obtained closed-form solutions to the performance ratio maximization problem. In this section, we conduct a sensitivity analysis with respect to the model parameters, obtaining further insights into the behavior of both $v(\lambda; x_0)$ and λ^* such that $v(\lambda^*; x_0) = 0$.

5.1. *Sensitivity with respect to γ_1*

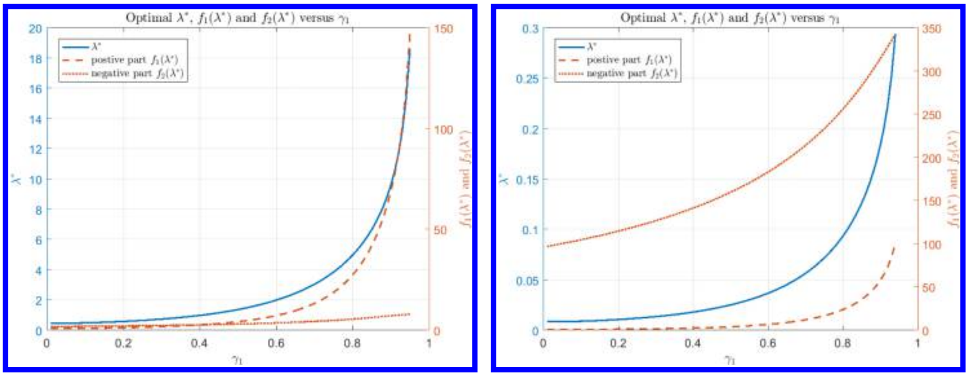
In Proposition 2.2, we have ruled out the case in which problem (2.12) is unbounded when U is a convex function. Thus, in the previous section, the parameter γ_1 of the power function U is constrained to be strictly between 0 and 1. It is interesting to investigate the behavior of both $v(\lambda; x_0)$ and λ^* with respect to γ_1, especially when γ_1 is approaching 1. We use the same parameters as specified in Table 1, unless stated otherwise.

We fix γ_2 to be 0.5 and 1.3 for analysis in two distinct cases. For each γ_2 we vary the choice of $\gamma_1 \in \{0.1, 0.25, 0.5, 0.75\}$ to illustrate the behavior of $v(\lambda; x_0)$ with respect to λ, as shown in Fig. 7. Firstly, $v(\lambda; x_0)$ is always decreasing in λ, as expected. Secondly, fixing a λ, $v(\lambda; x_0)$ is increasing in γ_1, as revealed by both the left and right panels in the figure. Thirdly, while the shapes of the graphs in both panels are similar, the scale of λ is different. With $\gamma_2 = 0.5$, the range of λ presented



(a) $\gamma_2 = 0.5$ (b) $\gamma_2 = 1.3$

Fig. 7. This figure shows how the optimal value $v(\lambda; x_0)$ of problem (2.22) responds to different values of γ_1 while γ_2 is fixed at either 0.5 or 1.3, where the reward and penalty functions are given by $U(x) = x^{\gamma_1}$ and $D(x) = x^{\gamma_2}$, respectively, and the threshold $L = 150$.



(a) $\gamma_2 = 0.5$ (b) $\gamma_2 = 1.3$

Fig. 8. This figure shows how three quantities (i.e. λ^*, $f_1(\lambda^*)$ and $f_2(\lambda^*)$) respond to change in γ_1 while γ_2 is fixed at either 0.5 or 1.3. Here λ^* is given by equation (2.23), $f_1(\lambda^*)$ and $f_2(\lambda^*)$ are defined in Eqs. (3.47) and (3.48), respectively. The threshold $L = 150$.

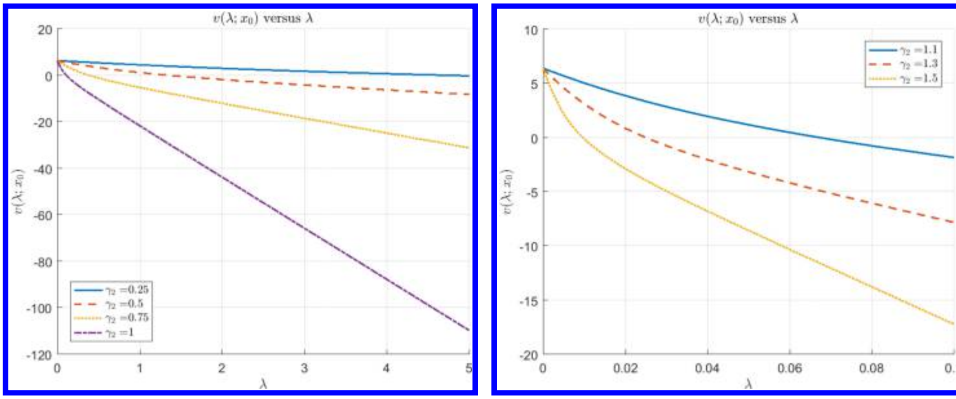
is $[0, 5]$, while in the case with $\gamma_2 = 1.3$, the range is reduced to $[0, 0.1]$. The change in slope is due to the choice of γ_2; a convex penalty function penalizes losses more and makes the optimal λ^* much smaller than a concave penalty function.

Since people hold different attitudes toward reward and risk, their utility and penalty functions on reward and risk are different. It is of interest to see how the optimal objective value λ^* behaves. Figure 8 shows the relationship between λ^* and γ_1 for γ_2 to be 0.5 and 1.3, respectively, corresponding to a concave and a convex penalties. First of all, as expected, the slope of v is very different in the two cases, which explains the different scales in the two panels of the figure. Secondly, the optimal λ^* is increasing with respect to γ_1. In the figure, we set the range to be $[0.01, 0.95]$ with step-size 0.01 since $\gamma_1 = 1$ corresponds to a linear utility function U, for which problem (2.12) is unbounded. As γ_1 approaches 1, λ^* keeps increasing and shows a trend to increase to infinity, which is also the reason for the numerical difficulty that arises if we choose γ_1 to be greater than 0.95. Figure 8 also shows the behavior of $f_1(\lambda)$ and $f_2(\lambda)$ as functions of λ.

5.2. *Sensitivity with respect to γ_2*

We now consider the behavior of $v(\lambda; x_0)$ and λ^* for a fixed γ_1 but varying γ_2. We set γ_1 to be 0.5 and use the values in Table 1 for the other parameters. Fig. 9(a) presents the relationship between $v(\lambda; x_0)$ and λ for γ_2 set to be less than or equal to 1, corresponding to a concave penalty function, while in Fig. 9(b) we select convex penalty functions for illustration. With a fixed λ, $v(\lambda; x_0)$ is decreasing in γ_2.

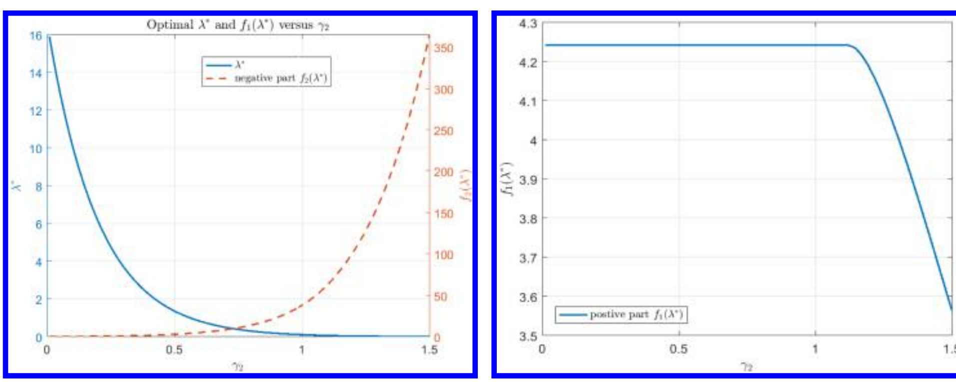
We also plot the optimal λ^* with respect to γ_2 in Fig. 10. The range of γ_2 is set to be $(0, 1.5]$ where we start from 0.01 with step-size 0.01. The optimal λ^* decreases



(a) $\gamma_1 = 0.5$

(b) $\gamma_1 = 0.5$

Fig. 9. This figure shows how the optimal objective value $v(\lambda; x_0)$ of problem (2.22) responds to the change in γ_2 while γ_1 is fixed at 0.5, where the reward and penalty functions are given by $U(x) = x^{\gamma_1}$ and $D(x) = x^{\gamma_2}$, respectively, and the threshold $L = 150$.



(a) $\gamma_1 = 0.5$

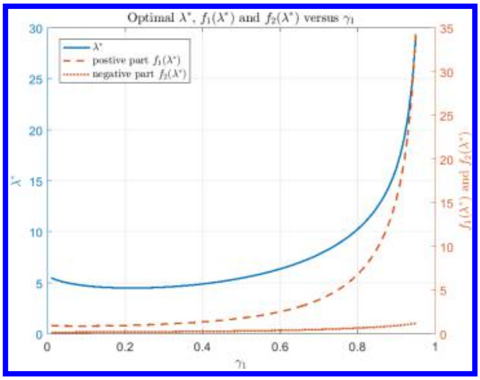
(b) $\gamma_1 = 0.5$

Fig. 10. This figure shows how three quantities (i.e. λ^*, $f_1(\lambda^*)$ and $f_2(\lambda^*)$) respond to the change in γ_2 while γ_1 is fixed at either 0.5. Here λ^* is given by Eq. (2.23), $f_1(\lambda^*)$ and $f_2(\lambda^*)$ are defined in Eqs. (3.47) and (3.48), respectively, and the threshold $L = 150$.

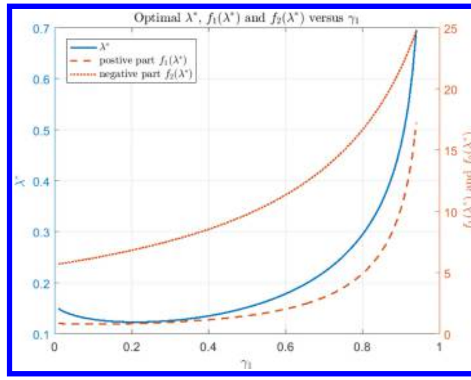
with γ_2, which means that if the penalty for underperformance is increased, the optimal objective becomes smaller. As shown in the figure, the penalty $f_2(\lambda^*)$ increases along with γ_2 while the positive part $f_1(\lambda^*)$ stays at the level of roughly 4.24, and eventually decreases as γ_2 becomes greater than the turning point in Fig. 10(b). This turning point occurs at the transition between the two cases in Proposition 4.2.

5.3. *On the choice of L*

In the previous section, we selected the initial wealth x_0 to be 100 and the threshold L to be 150. The present value of the threshold is $Le^{-rT} = 129.1062$, which is



(a) $\gamma_2 = 0.5$



(b) $\gamma_2 = 1.3$

Fig. 11. This figure shows how three quantities (i.e. λ^*, $f_1(\lambda^*)$ and $f_2(\lambda^*)$) respond to the change in γ_1 while γ_2 is fixed at either 0.5 or 1.3. Here λ^* is given by Eq. (2.23), $f_1(\lambda^*)$ and $f_2(\lambda^*)$ are defined in Eqs. (3.47) and (3.48), respectively. Optimal λ^* for the original problem (2.12) versus γ_1 with distinct γ_2 and $L = 120$.

29.1062% larger than the initial wealth. If $x_0 \geq Le^{-rT}$, the strategy of simply investing in the risk-free asset will make the objective function undefined. We wish to study the behavior of the optimal λ^* if we select Le^{-rT} close to x_0, so we set $L = 120$, with $Le^{-rT} = 103.285$, which is only 3.285% larger than the initial wealth. We start by varying γ_1 and keeping γ_2 to be 0.5 or 1.3. The results are reported in Fig. 11. Compared to the patten displayed in Fig. 8 which is for a benchmark $L = 150$, it is interesting to see in Fig. 11 that when γ_1 is small, roughly in the range $(0, 0.22)$, the optimal λ^* decreases with respect to γ_1.

We now fix $\gamma_1 = 0.5$ and vary γ_2 to investigate the behavior of λ^* in response to the change of γ_2. The results are displayed in Fig. 12. As expected, λ^* decreases with γ_2. Meanwhile, when γ_2 is less than 1.03, $f_1(\lambda^*)$ remains constant due to the form of the solution, as noted above. When γ_2 exceeds 1.03, then $f_1(\lambda)$ starts to decrease. The turning point corresponds to the threshold where the transition occurs from one case to the other as described in Proposition 4.2. However, in the entire interval $(0, 1.5]$, $f_2(\lambda)$ keeps increasing in γ_2, resulting in a decrease in the optimal value λ^*. This pattern is the same as observed in Fig. 10.

In addition, we also carry out sensitivity analysis with respect to L. The results are shown in Fig. 13. As we can see, the optimal performance ratio, i.e. λ^*, decreases with L. In fact, the monotonicity of λ^* with respect to L can be proved by noting that $\frac{U[(x-L)_+]}{D[(L-x)_+]}$ is decreasing with respect to L since both U and D satisfy the assumptions (H1)–(H3). Intuitively speaking, when the benchmark L is larger, it is more difficult to construct a portfolio to outperform the benchmark, making the performance ratio smaller. Furthermore, it is obvious that both $f_1(\lambda^*)$ and $f_2(\lambda^*)$ are at a similar magnitude in Fig, 13(a), while the negative part $f_2(\lambda^*)$ is much larger than $f_1(\lambda^*)$ in Fig. 13(b). This observation agrees with our

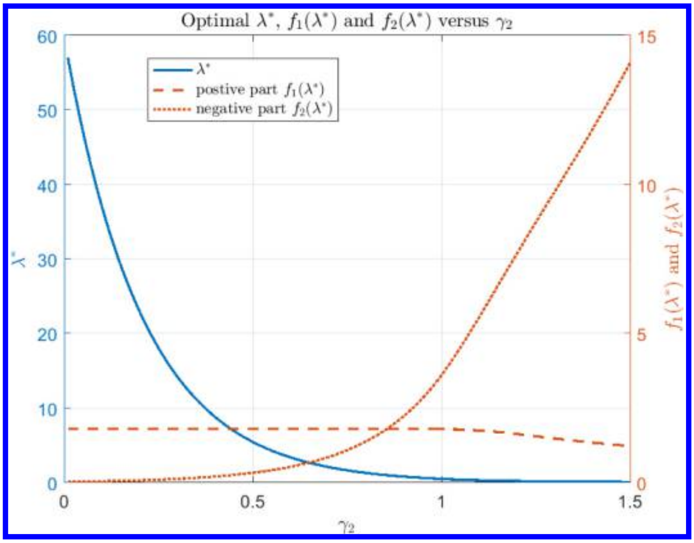
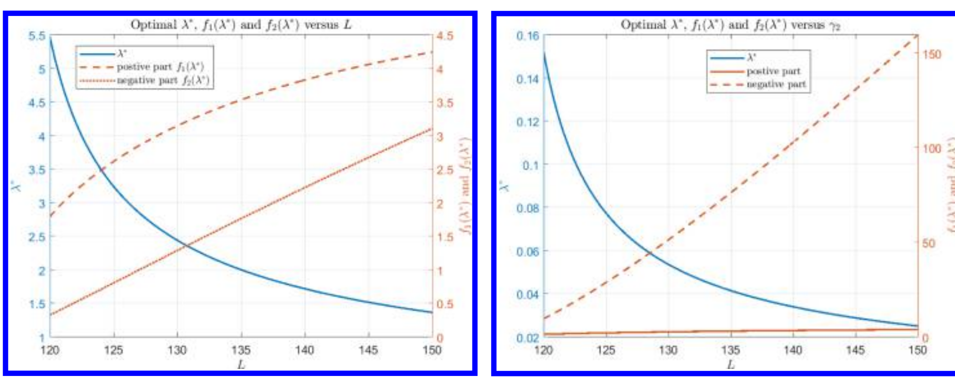


Fig. 12. This figure shows how three quantities (i.e. λ^*, $f_1(\lambda^*)$ and $f_2(\lambda^*)$) respond to the change in γ_2 while γ_1 is fixed at 0.5 and the threshold $L = 120$. Here λ^* is given by Eq. (2.23), $f_1(\lambda^*)$ and $f_2(\lambda^*)$ are defined in Eqs. (3.47) and (3.48), respectively.



(a) $\gamma_1 = 0.5$ and $\gamma_2 = 0.5$

(b) $\gamma_1 = 0.5$ and $\gamma_2 = 1.3$

Fig. 13. This figure shows how three quantities (i.e. λ^*, $f_1(\lambda^*)$ and $f_2(\lambda^*)$) respond to the change in the threshold L with fixed γ_1 and γ_2. Here λ^* is given by Eq. (2.23), $f_1(\lambda^*)$ and $f_2(\lambda^*)$ are defined in Eqs. (3.47) and (3.48), respectively.

intuition that the convex penalty function D imposes more substantial penalties on underperformance.

6. Conclusion

In this paper, we consider a portfolio selection problem for a performance ratio maximizing agent. Employing a strategy from fractional programming, we relate the problem to a family of solvable ones. Relying on the martingale approach and the pointwise optimization technique, we obtain a closed-form solution. In the pointwise optimization procedure we adopt a concavification technique. In the end, we recover the optimal solution to the original portfolio selection problem. With the optimal solution in hand, we present numerical examples for power functions and a sensitivity analysis with respect to several model parameters.

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Appendix A. Results from Jin *et al.* (2008)

This section summarizes some results from Jin *et al.* (2008). Interested readers may refer to the paper for detailed proofs. Consider the following optimization problem:

$$\begin{cases} \sup\limits_{Z \in \mathcal{M}_+} & \mathbb{E}[U(Z)], \\ \text{subject to} & \mathbb{E}[\xi_T Z] \le x_0, \end{cases} \tag{A.1}$$

where $x_0 > 0$, ξ_T is a given scalar-valued random variable, $U : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a twice differentiable, strictly increasing, strictly concave function with $U(0) = 0$, $\lim_{x\to 0} U(x) = \infty$ and $\lim_{x\to\infty} U(x) = 0$.

Lemma A.1. *If there exists a constant $\beta^* > 0$ such that $\mathbb{E}[\xi_T \cdot (U')^{-1}(\beta^* \xi_T)] = x_0 < \infty$ and $\mathbb{E}[U((U')^{-1}(\beta^* \xi_T))] < \infty$, then $Z^* = (U')^{-1}(\beta^* \xi_T)$ is optimal for problem* (A.1).

Lemma A.2. *Suppose $\liminf_{x\to\infty}(-\frac{xU''(x)}{U'(x)}) > 0$ and $\mathbb{E}[\xi_T^{-\alpha}] < \infty$, $\forall\, \alpha \ge 1$, then $\mathbb{E}[\xi_T \cdot (U')^{-1}(\beta \xi_T)] < 0$ for all $\beta > 0$ and problem* (A.1) *admits a unique optimal solution $Z^* = (U')^{-1}(\beta^* \xi_T)$ for any $x_0 > 0$.*

Lemma (A.2) is actually Corollary 5.1 from Jin *et al.* (2008). The condition $\liminf_{x\to\infty} -\frac{xU''(x)}{U'(x)} > 0$ involves the behavior of the *Arrow-Pratt index of risk aversion* of the utility function when x is large enough. It ensures the existence of an optimal Lagrange multiplier such that the budget constraint is binding. Most commonly used utility functions, e.g. the power utility function $U(x) = x^\gamma, 0 < \gamma < 1$, satisfy this condition. The condition $\mathbb{E}[\xi_T^{-\alpha}] < \infty$, $\forall \alpha \ge 1$, guarantees that the obtained solution with the optimal Lagrange multiplier will result in a finite objective value. In the literature, ξ_T usually has a log-normal distribution, and the condition holds automatically.

Appendix B. Proof of Lemma 3.4

Proof. By definition $f^c \geq f$. Let g be any concave function with $g \geq f$. Then $g(x) \geq f(x) = f^c(x)$ for $x \in [0, \tilde{z}_1] \cup [\tilde{z}_2, \infty)$. Further, any $x \in (\tilde{z}_1, \tilde{z}_2)$ can be written as $x = \alpha\tilde{z}_1 + (1-\alpha)\tilde{z}_2$ for some $\alpha \in (0,1)$, and the concavity of g implies:

$$\begin{aligned} g(x) &= g(\alpha\tilde{z}_1 + (1-\alpha)\tilde{z}_2) \geq \alpha g(\tilde{z}_1) + (1-\alpha)g(\tilde{z}_2) \\ &\geq \alpha(k\tilde{z}_1 + c) + (1-\alpha)(k\tilde{z}_2 + c) = kx + c = f^c(x). \end{aligned} \tag{B.1}$$

To complete the proof, we need to show that f^c defined in Eq. (3.9) is concave on $[0, \infty)$. Recall that h is concave if:

$$h(x_\alpha) \geq \alpha h(x_1) + (1-\alpha)h(x_0) \tag{B.2}$$

for any $x_0, x_1 \in [0, \infty)$ with $x_0 < x_1$, where $x_\alpha = (1-\alpha)x_0 + \alpha x_1$, $\alpha(0,1)$. Define

$$f_1(x) = \begin{cases} f^c(x), & x \leq \tilde{z}_1, \\ kx + c, & x > \tilde{z}_1 \end{cases} \tag{B.3}$$

and

$$f_2(x) = \begin{cases} kx + c, & x < \tilde{z}_2, \\ f^c(x), & x \geq z_2. \end{cases} \tag{B.4}$$

Note that the hypotheses of the lemma imply that $f_i(x) \leq kx + c$ for all x and $i = 1, 2$. Consider f_1. If $x_0 \geq \tilde{z}_1$, or $x_1 \leq \tilde{z}_1$, then (B.2) is immediate. If $x_\alpha \geq \tilde{z}_1$, (B.2) follows from $f_1(x_0) \leq kx_0 + c$. Finally, if $x_0 < x_\alpha < \tilde{z}_1 < x_1$, we note that

$$\frac{f_1(x_1) - f_1(x_0)}{x_1 - x_0} \leq \frac{f_1(\tilde{z}_1) - f_1(x_0)}{\tilde{z}_1 - x_0} \leq \frac{f_1(x_\alpha) - f_1(x_0)}{x_\alpha - x_0}, \tag{B.5}$$

where the first inequality follows from $f_1(x_0) \leq kx_0 + c$, and the second follows from the supposed concavity of f on $[0, \tilde{z}_1]$. (B.2) follows immediately from the outer two terms of the above inequality.

The proof of the concavity of f_2 is similar. The concavity of $f^c = f_1 \wedge f_2$ follows immediately. □

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