2: SECURITIES MARKETS

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Semester 2, 2020


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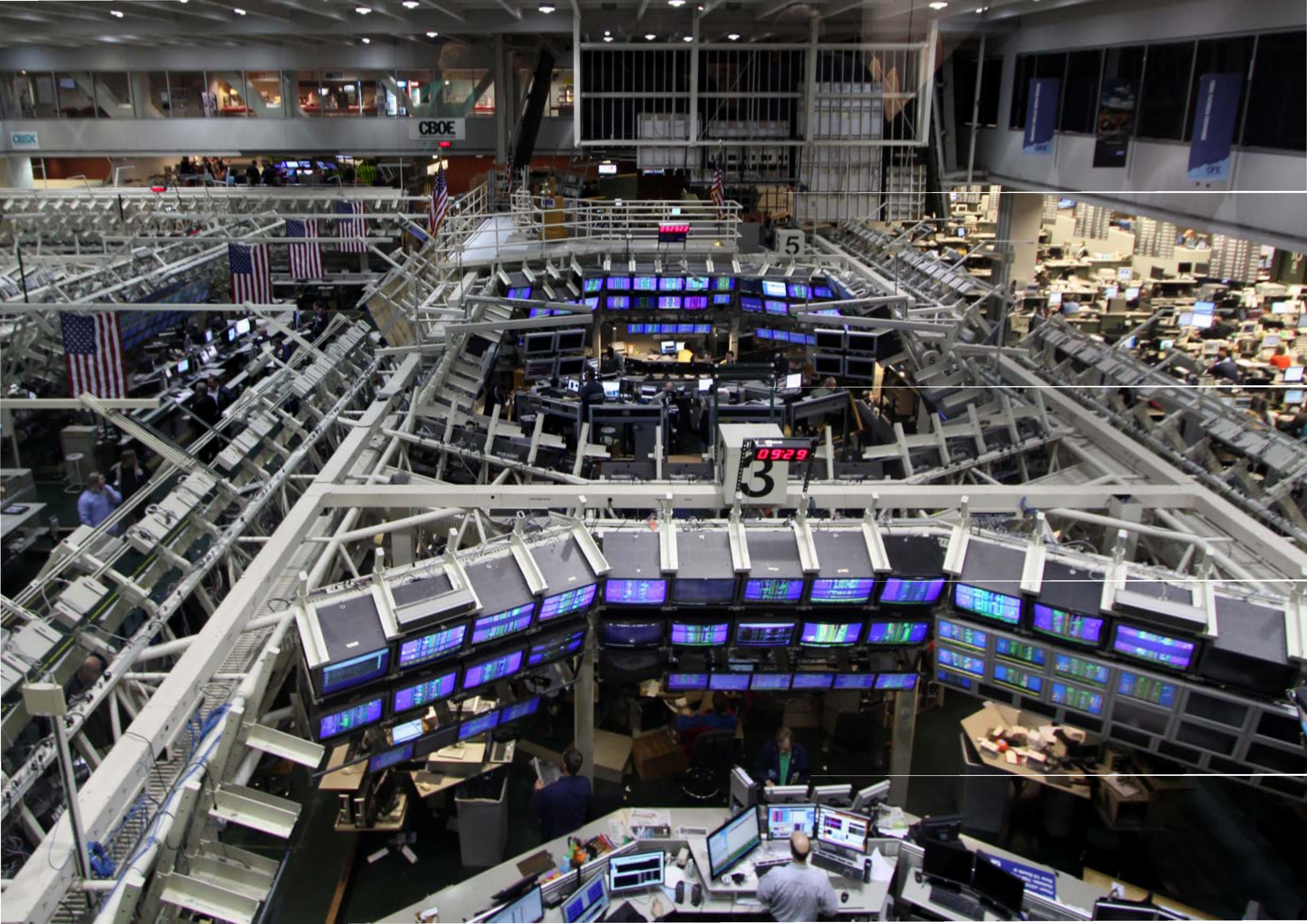
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Books

Recommended readings:

- Hull, J. C. (2014). *Options, Futures, and Other Derivatives.* 9th Edition. Prentice Hall.
- Van der Hoek, J. and Elliott, R. E. (2006). *Binomial Models in Finance.* Springer.
- Pliska, S. R. (1997). *Introduction to Mathematical Finance: Discrete Time Models*. Blackwell Publishing.
- Shreve, S. E. (2004). *Stochastic Calculus for Finance. Volume 1: The Binomial Asset Pricing Model*. Springer.
- Seydel, R. (2009). *Tools for Computational Finance.* 4th Edition. Springer.
- Musiela, M. and Rutkowski, M. (2005). *Martingale Methods in Financial Modelling.* 2nd Edition. Springer.

Classic Black-Scholes Model

- In the paper *The Pricing of Options and Corporate Liabilities* by Fischer Black and Myron Scholes published in 1973, the authors developed ideas, which are nowadays widely used by practitioners to price derivative securities in continuous-time market models.
- The advanced theory of securities markets requires some knowledge of Stochastic Calculus and numerical methods for partial differential equations (PDEs).
- By restricting attention to discrete-time financial models, one can acquire a good grasp of fundamental ideas of arbitrage pricing.
- Furthermore, In practice, some discrete time models are also used by practitioners to price derivatives. In particular, the Black-Scholes model can be approximated by some sequences of discrete-time models.

Goals of the Course

- Our concerns in this course are:
  - How to describe the randomness inherent in securities prices?
  - What is a hedging strategy?
  - Why martingales are useful in financial modelling?
  - How to compute the “fair price” of a derivative security?
- In this course, some useful models will be presented and the above questions will be given detailed and rigorous answers.
- Students enrolled in MATH3075 will get familiar with discrete-time market models. In particular, they will learn about the classical Cox-Ross-Rubinstein binomial model of the stock price, as well as the Black-Scholes options pricing formula.
- Students enrolled in MATH3975 will also learn about theoretical underpinning for benchmark Black-Scholes continuous-time model.

Long and Short Positions

- A financial security (or an asset) is a contract which records some entitlement or obligation and which can be traded at financial markets (either organised exchanges or over-the-counter (OTC)).
- Some most typical examples of financial securities: stocks (equities), sovereign and corporate bonds, options, swaps, and futures.
- In principle, we say that an agent takes a **long (short) position** in an asset if the agent owns (owes) it.
- In modern financial markets, it may not be necessary to own an asset to “sell” it. The transaction of selling a borrowed asset is referred to as **short-selling** of an asset.
- For instance, an agent can borrow a number of stocks and sell them to undertake other investments. Then the agent needs to buy them back and return to the original owner at a predetermined time.

Exchanges and OTC Markets

- If a large number of traders short sell a certain stock, it is very likely that the stock price will fall. This has drawn some criticism in the last couple of years and restrictions on short-selling were introduced in some countries.
- Two types of financial markets:
  - **Exchanges** (stock exchange, options exchange, futures exchange, etc.)
  - **Over-the-counter (OTC)** markets.
- A securities exchange provides an organised forum for trading securities whose contracts are standarised. Buyers and sellers do not directly contact each other and all trades are done by the market makers. Prices of securities traded on exchanges are widely disseminated.

Exchanges

Some examples of exchanges and securities traded on them:

- Equity market = stock market: stocks/shares and stock/index exchange-traded funds (ETF).
- Bond market: bills (zero-coupon bonds), notes, government bonds, corporate bonds, convertible bonds, municipal bonds.
- Foreign exchange (FX = Forex) market: foreign currencies, options on foreign currencies.
- Futures market: futures contracts on stock indexes, currencies, bonds and interest rates.
- Options market: equity options and interest rate options.
- Commodities market: oil, wheat, corn, metals futures with either a physical delivery or cash settlement.

OTC Markets

- OTC markets are less organised and two institutions such as a bank and an investment company may be simply involved.
- Some examples of OTC markets and most typical securities traded on them:
  - Interest rate market (mostly OTC): forward rate agreements (FRAs), caps, floors, interest rate swaps and swaptions.
  - Exotic options market: barrier options, lookback options, compound options.
  - Credit market (mostly OTC): credit derivatives such as CDSs (Credit Default Swaps) and CDOs (Collateralized Debt Obligations).
- Prices (quotes) of OTC traded securities are generally hard to obtain.

Reasons of Trading in Securities

- **Profit**
  - A trader believing that the price of a security will rise may follow the idea of buying low and selling high. If he believes that the price of a security will decline then short-selling is the right strategy.
- **Protection**
  - For example, a company dependent on imports may prefer to fix the exchange rate in advance and apply it to a trade later. To this end, one can purchase a futures contract on the exchange rate. It is also possible to enter an OTC forward contract on the exchange rate.
- **Hedging**
  - To reduce the risk that a trader's portfolio is exposed to, by holding a certain position in related securities.
- **Diversification**
  - Risks affecting specific securities behave in different manner and thus they may cancel out.

Bear and Bull Markets

- Market prices of traded securities are driven by the forces of **supply** and **demand** (that is, the willingness to sell and buy, respectively).
- In over-supply or under-demand, security prices will generally fall. This is referred to as the **bear market.**
- In under-supply or over-demand, security prices will generally rise. This is called the **bull market.**
- Some factors, such as: market information, rumour mill and human psychology, cause **random fluctuations** in supply and demand and hence the market price as well.
- Prices that satisfy supply and demand are called to be in **equilibrium** (at least, in the academic literature).

Perfect Markets

- **Assumption 1.** Markets are frictionless, i.e., no transaction costs, no taxes or other costs and there are no penalties for short selling.
- **Assumption 2.** Infinitely divisible security prices. It is possible to buy or sell any (also non-integer) quantity of any security.
- **Assumption 3.** All traders have access to the same information.
- According to the economic theory, the Efficient Market Hypothesis (EMH) has three forms:
    - **Weak-form:** Current security prices reflect all past market prices and data. Technical analysis is of no benefit.
    - **Semi-strong form:** Current security prices reflect all publicly available information. Fundamental analysis is of no benefit.
    - **Strong form:** Current security prices reflect all available information. Inside information is of no benefit.

Arbitrage-free Markets

- **Assumption 4.** Markets are assumed to be arbitrage-free.
- **Arbitrage** is the guarantee of a positive cash flow in the future without any current investment. Modern financial theory holds under the assumption that efficient financial markets should forbid such an arbitrage opportunity.
- **Law of one price**. It follows from the **no-arbitrage principle** that if two financial securities have the same pattern of future cash-flows, they should have the same price today.
- Standing assumptions 1. to 4. provide a framework in which we can develop mathematical models for **primary securities** and propose pricing methods for **derivative securities**.
- Assumptions 1. to 3. will be implicitly taken into account when describing **trading**. We will thus mainly focus on the no-arbitrage principle when dealing with **valuation**.

Arbitrage Principle

	Cash-Flow at t_0	Cash-Flow at t_1	$\cdots$	Cash-Flow at t_n
Buy S_1	$-P_1$	X_1	$\cdots$	X_n
Sell S_2	P_2	$-X_1$	$\cdots$	$-X_n$
Totals	$-P_1 + P2$	0	$\cdots$	0

Table: Arbitrage Table

- The law of one price is a trivial consequence of the no-arbitrage principle. It may thus be surprising that the no-arbitrage principle, when combined with random models of securities market and the concept of dynamic trading, furnishes a powerful method for **fair valuation** of derivative assets, such as equity options.

European Call Option

Definition (European Call Option)

A **European call option** is a financial security, which gives its buyer the right (but not the obligation) to buy an asset at a future time $T > 0$ for a price $K > 0$, which is known as the strike price.

- The underlying asset, the **expiration** (or maturity) date T and the **strike** (or exercise) price K are specified in the contract.
- We assume that the **underlying asset** is one share of the stock with price S_t at time t.

At expiry T, a rational holder of a European call option should proceed as follows:

- If the stock price $S_T > K$, he should buy the stock at time T for the price K from the seller of the option (an option is **exercised**) and immediately sell it on the market for the market price S_T, leading to a positive payoff of $S_T - K$.

Call Option Payoff

- If $S_T \leq K$, then it does not make sense to buy the stock for the price K from the seller, since it can be bought for a lower price on the market. In that case, the holder should waive his right to buy (an option is **abandoned**) and this leads to a payoff of 0.

These arguments show that a European call option is formally equivalent to the following **random payoff** C_T at time T

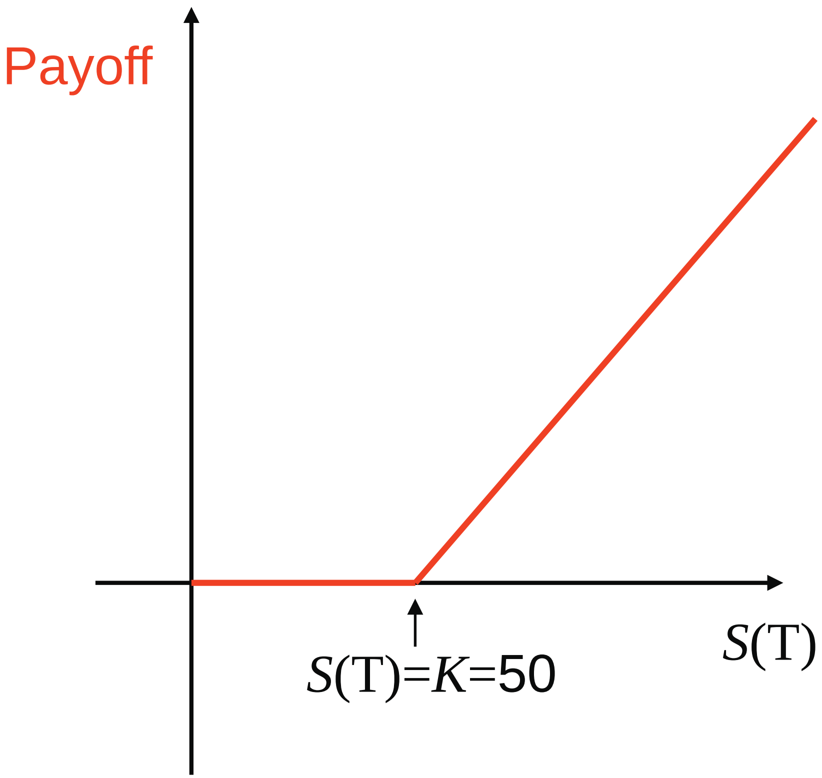
$$C_T := \max\left(S_T - K, 0\right) = \left(S_T - K\right)^+$$

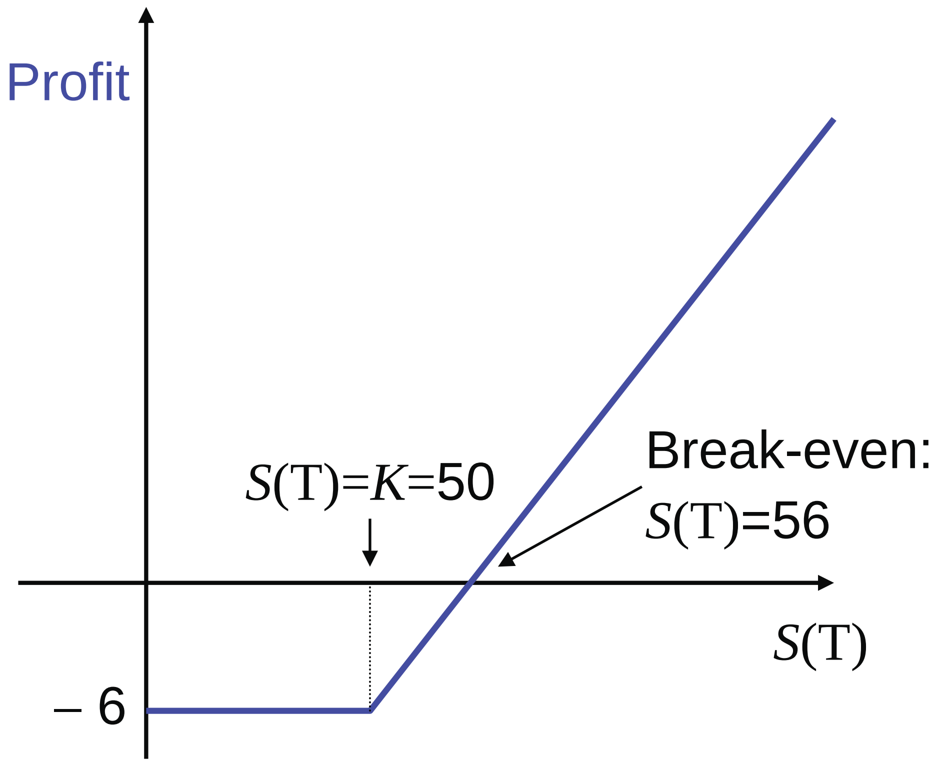
where we denote $x^+ = \max\left(x, 0\right)$ for any real number x.

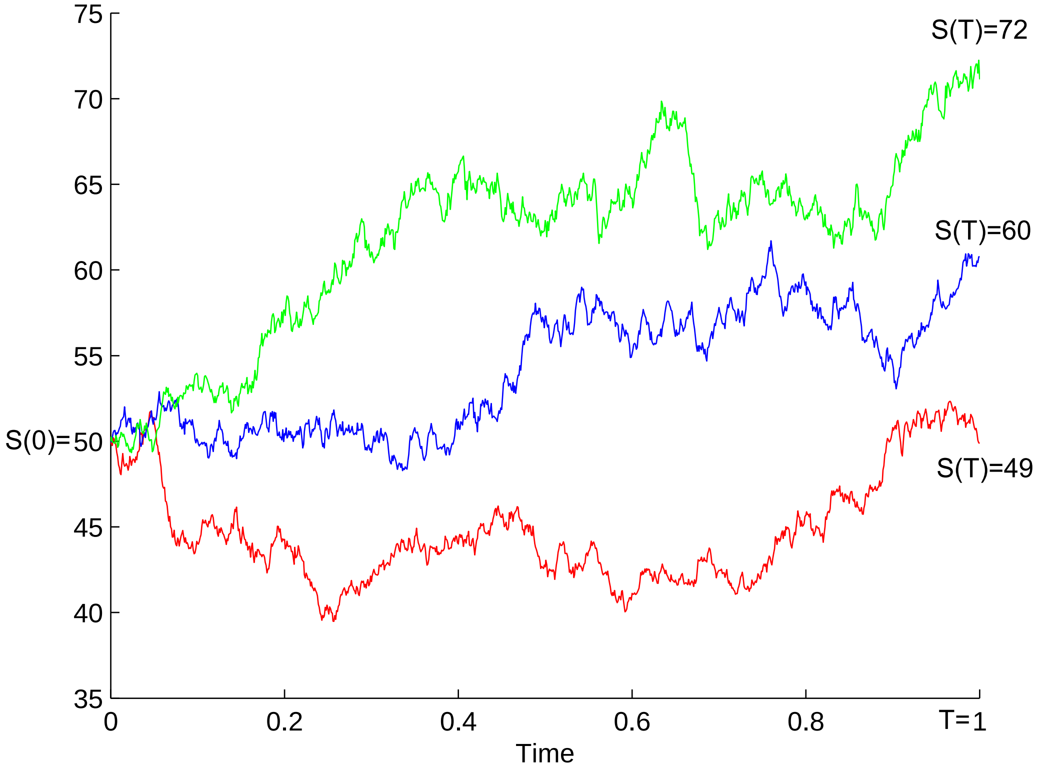
- Note that the payoff C_T is **path-independent**, meaning that it only depends on the stock price at expiry date T, and not on the whole **sample path** $S_t,\ t \in [0, T]$ of the stock price.

Long Call Position

- Inputs: $K = \$50,\ C(t,K,T) = \$6$
- Payoff: $max\ [S(T) - 50,\ 0]$
- Profit: $max\ [S(T) - 50,\ 0] - 6$






75
70
65
60
55
S(0)=50
45
40
35
0
0.2
0.4
0.6
0.8
T=1
Time
S(T)=72
S(T)=60
S(T)=49

European Put Option

Definition (European Put Option)

A **European put option** is a financial security, which gives its buyer the right (but not the obligation) to sell an asset at a future date $T > 0$ for the strike price $K > 0$.

One can show that a European put option on the stock S is formally equivalent to the following random payoff P_T at time T

$$P_T := \max\left(K - S_T, 0\right) = (K - S_T)^+.$$

It is also easy to see that $C_T - P_T = S_T - K$.

- European call and put options are actively traded on organised exchanges.
- A pertinent theoretical question thus arises:
  - What should be the 'fair' prices C_t and P_t of European call and put options prior to the expiry date T?

Interest Rates and Zero-Coupon Bonds

- All participants of financial markets have access to **riskless cash** through borrowing and lending from the money market.
- An investor who borrows cash must pay back money borrowed from the lender with **interest** on a predetermined date in the future.
- An investor who lends cash will receive from the borrower the nominal value and interest payment on a predetermined date in the future.
- We assume throughout that the **borrowing rate** is equal to the **lending rate**.

Definition (Zero-Coupon Bond)

The unit **zero-coupon bond** maturing at T is a financial security returning to its holder one unit of cash at time T. We denote by $B(t,T)$ the bond price at time $t \in [0,T]$. Hence $B(T,T) = 1$.

Discretely Compounded Interest

- The set of dates is $\{0, 1, 2, \dots\}$.
- Let a real number $r > -1$ represent the **simple** interest rate over each period $[t, t+1]$ for $t = 0, 1, 2, \dots$.
- One unit of cash invested at time 0 in the **money market account** yields the following amount at time $t = 0, 1, 2, \dots$

$$B_t = (1 + r)^t.$$

- From the law of one price, it follows that

$$B(t, T) = \frac{B_t}{B_T} = (1 + r)^{-(T-t)}.$$

- In general, if $r(t) > -1$ is the deterministic simple interest rate over the time period $[t, t+1]$ then we obtain, for every $t = 0, 1, 2, \dots$,

$$B_t = \prod_{u=0}^{t-1} (1 + r(u)), \quad B(t, T) = \prod_{u=t}^{T-1} (1 + r(u))^{-1}.$$

Continuously Compounded Interest

- In the continuous-time setup, the **instantaneous** interest rate is either as a real number r or a deterministic function $r(t)$.
- Hence the **money market account** satisfies the ODE $dB_t = r(t)B_t\,dt$.
- If r is constant then one unit of cash invested at time 0 in the money market account yields the following amount at time t

$$B_t = e^{rt}.$$

- The price of a unit zero-coupon bond maturing at T equals for $0 \le t \le T$,

$$B(t,T) = e^{-r(T-t)}.$$

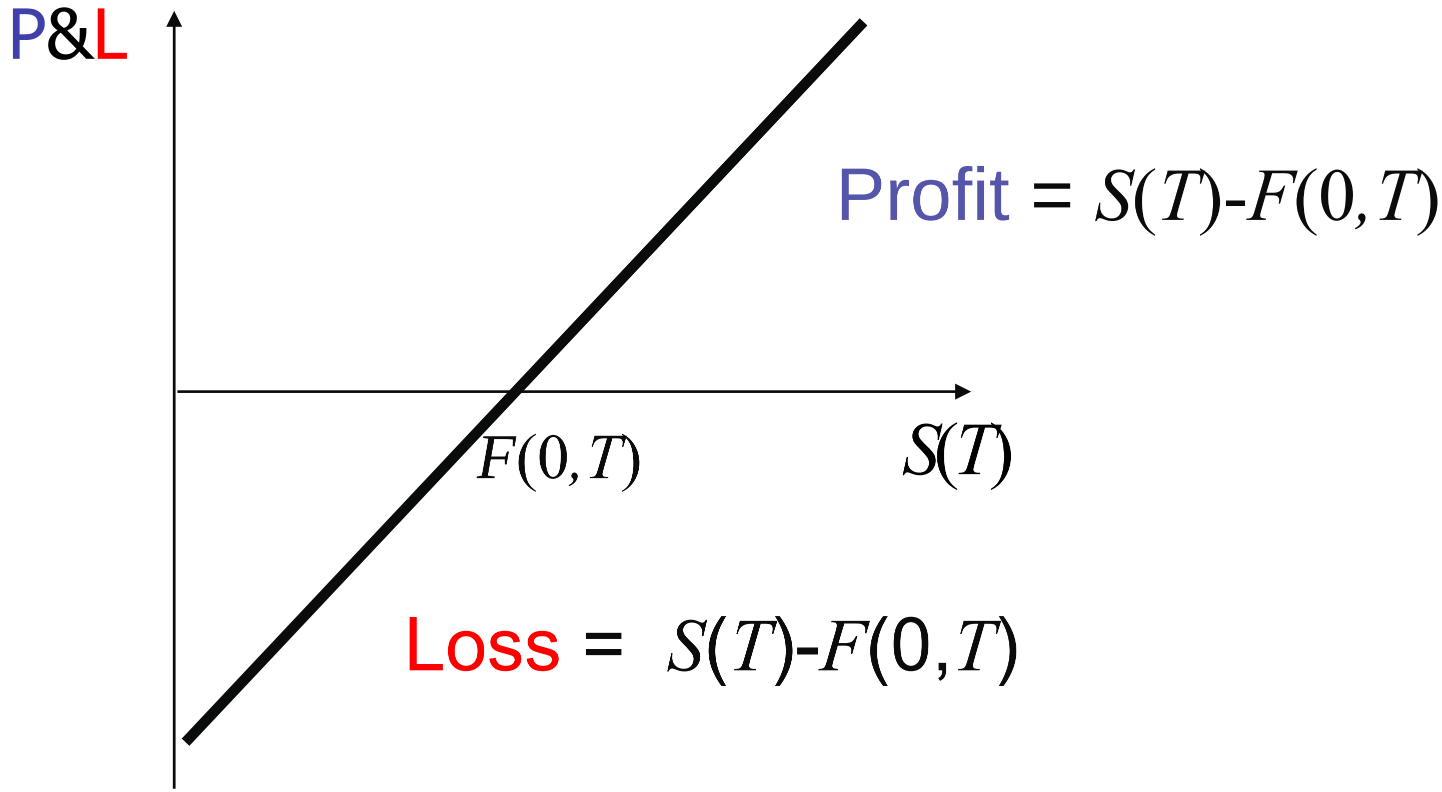
- When $r : \mathbb{R}_+ \to \mathbb{R}$ is a function, then

$$B_t = \exp\left(\int_0^t r(u)\,du\right), \quad B(t,T) = \frac{B_t}{B_T} = \exp\left(-\int_t^T r(u)\,du\right).$$

Summary: Equities and Equity Derivatives

- Spot market (Stock exchanges (e.g., NYSE, NASDAQ, ASX) and OTC)
- Forward contracts (OTC)
- Futures contracts and futures options (Futures exchanges)
- Equity options (e.g., Apple (AAPL) or J.P. Morgan Chase (JPM) options)
  - Call and put equity options of European or American style (Options exchanges and OTC)
  - Exotic options (OTC)
- Index options (e.g., DJX on DJIA or SPX on S&P500)
  - Call and put index options of European or American style (Options exchanges and OTC)
  - Exotic index options (OTC)
- Implied volatility futures and options (e.g., CBOE VIX Option on Volatility Index).

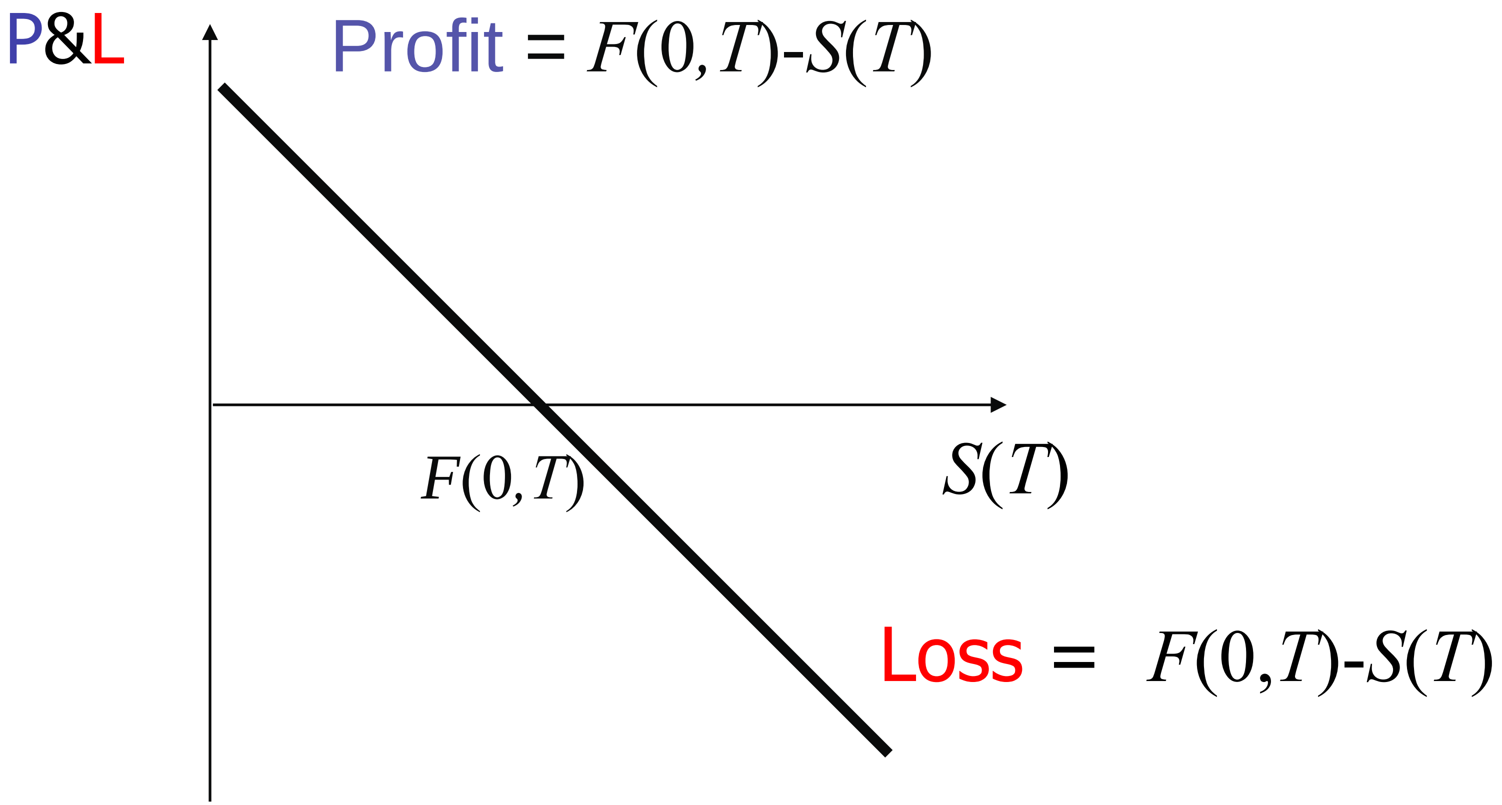
Profit/loss from a long forward contract



$S(T)$ is the spot price of the underlying asset S at maturity T

$F(0,T)$ is the forward price of S for maturity T

Profit/loss from a short forward contract



$S(T)$ is the spot price of the underlying asset S at maturity T

$F(0,T)$ is the forward price of S for maturity T

Long Call

	Outcome at maturity	
	OTM: $S(T) \le K$	ITM: $S(T) > K$
Payoff:	0	$S(T) - K > 0$
Profit:	$-C(t, K, T)$	$S(T) - K - C(t, K, T)$

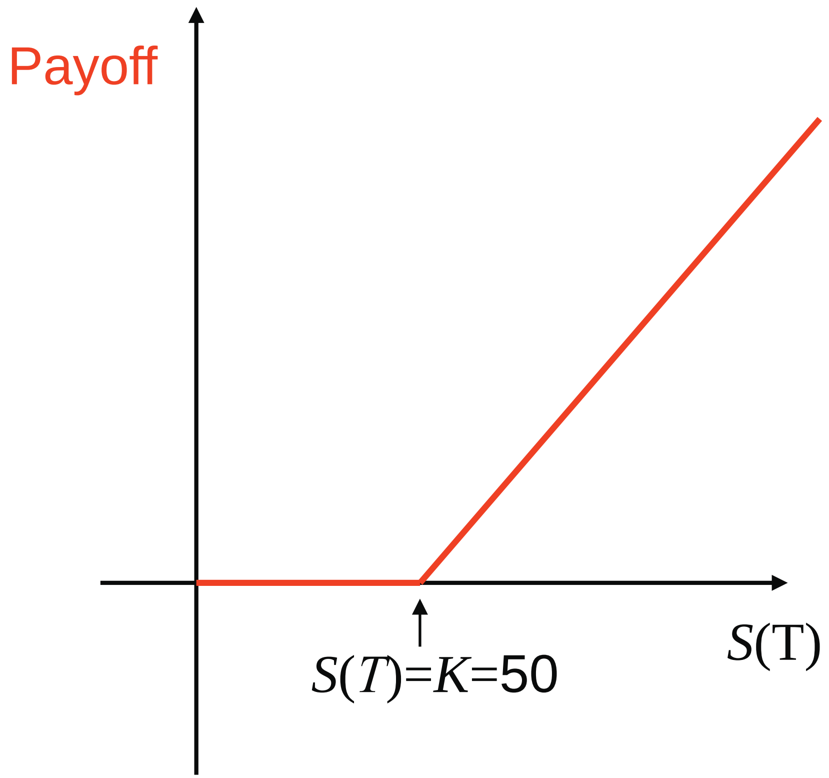
A more compact notation:

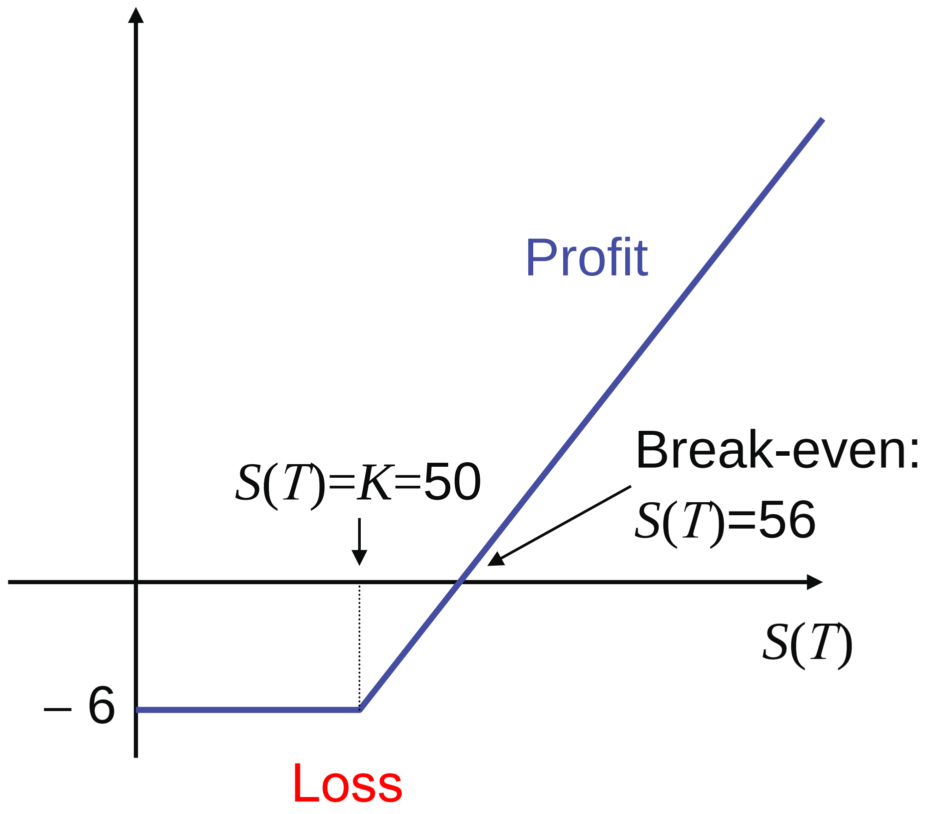
Payoff: $max\,[S(T) - K, 0] = (S(T)\text{-}K)^{+}$

Profit: $max\,[S(T) - K, 0] - C(t,K,T)$

Long Call

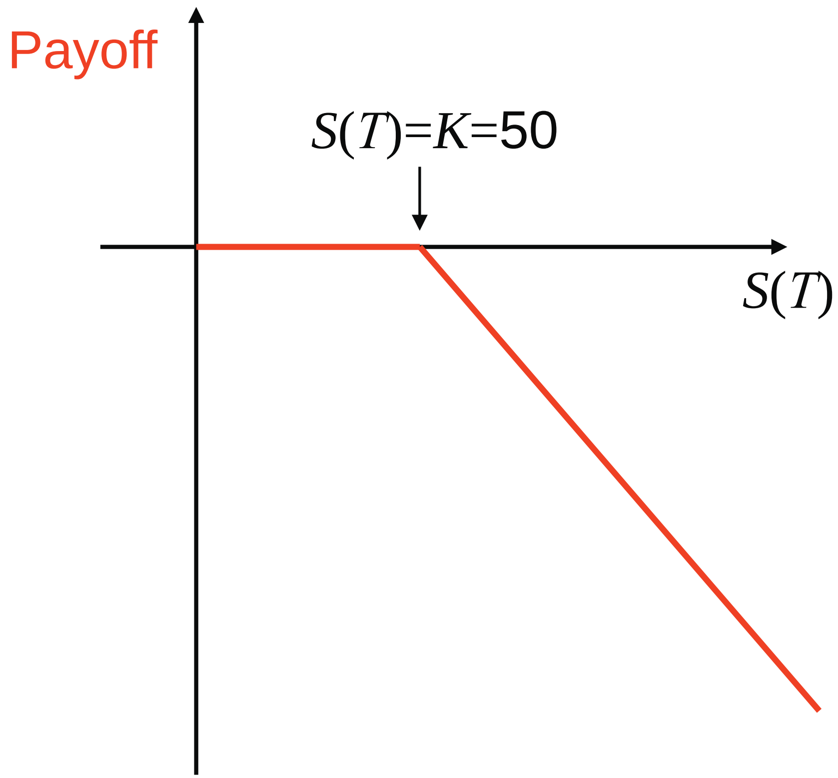
- Inputs: $K = \$50,\ C(t,K,T) = \$6$
- Payoff: $max\ [S(T) - 50,\ 0]$
- Profit: $max\ [S(T) - 50,\ 0] - 6$

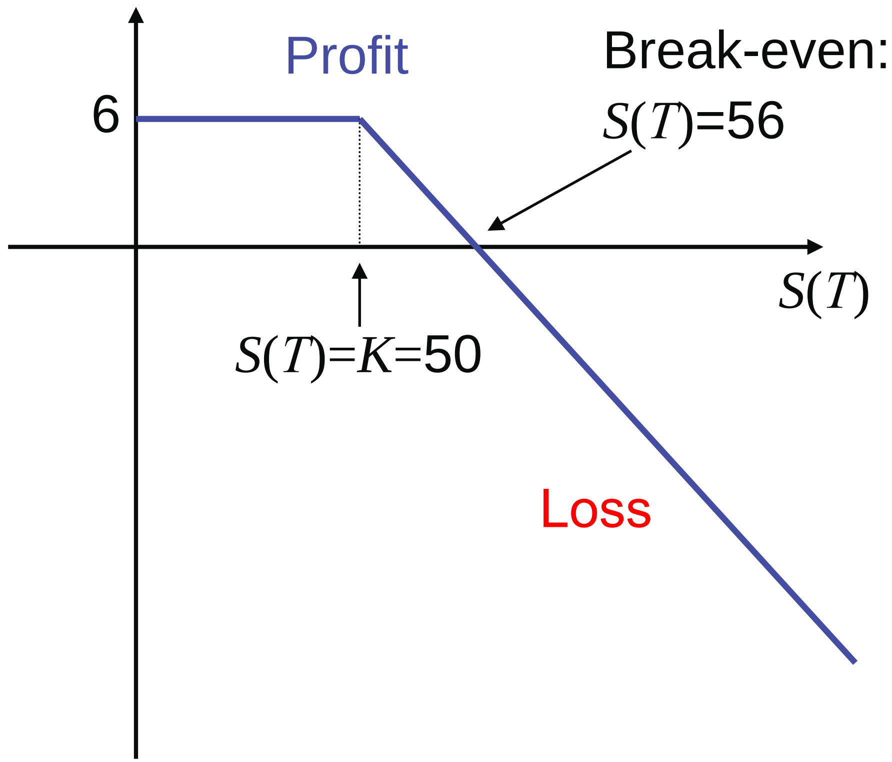




Short Call

- Inputs: $K = \$50,\ C(t,K,T) = \$6$
- Payoff: $-\ max\ [S(T) - 50,\ 0]$
- Profit: $6 - max\ [S(T) - 50,\ 0]$





Long Put

	Outcome at maturity	
	ITM: $S(T) < K$	OTM: $S(T) > K$
Payoff:	$K - S(T) > 0$	0
Profit:	$K - S(T) - P(t, K, T)$	$-P(t, K, T)$

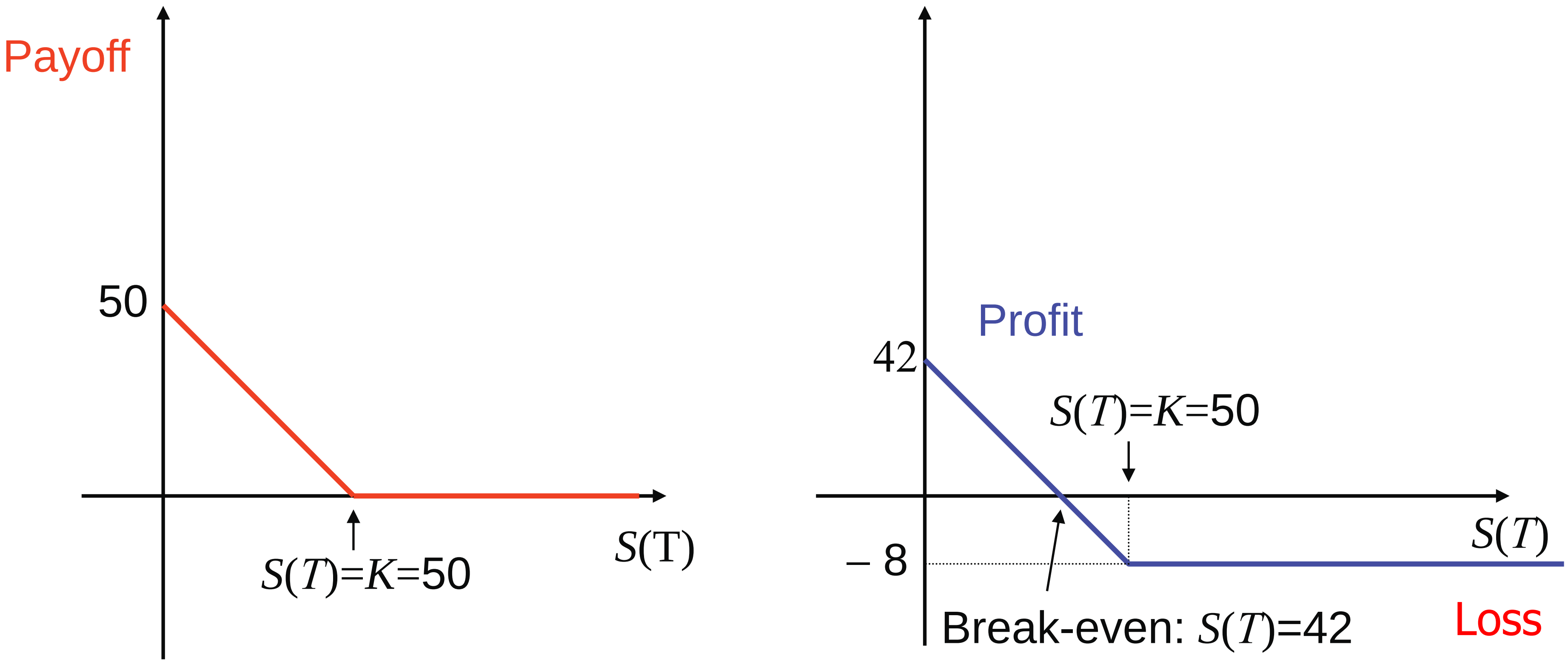
A more compact notation:

Payoff: $max\ [K - S(T), 0] = (K\text{-}S(T))^+$

Profit: $max\ [K - S(T), 0] - P(t, K, T)$

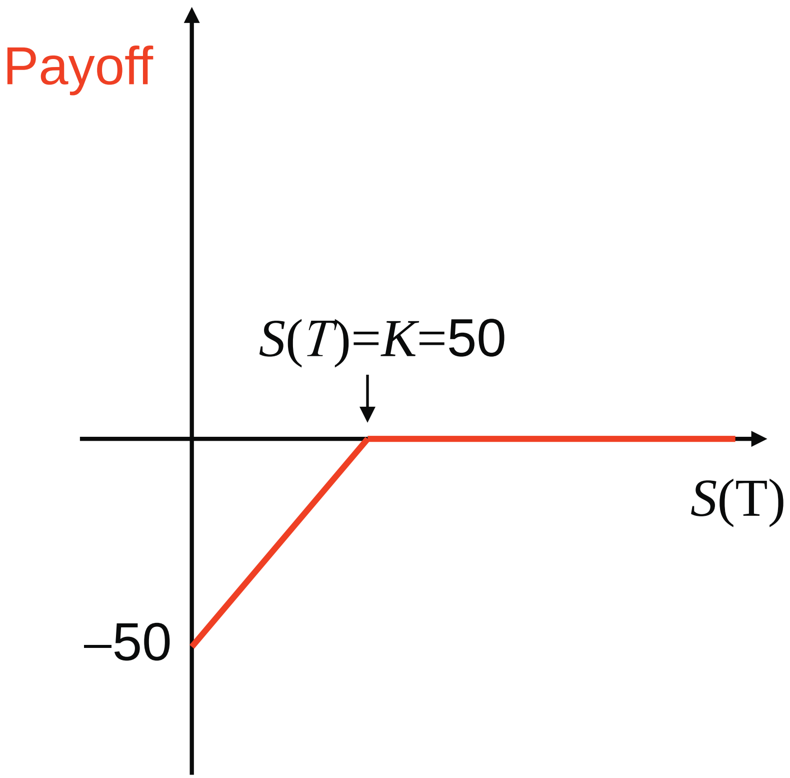
Long Put

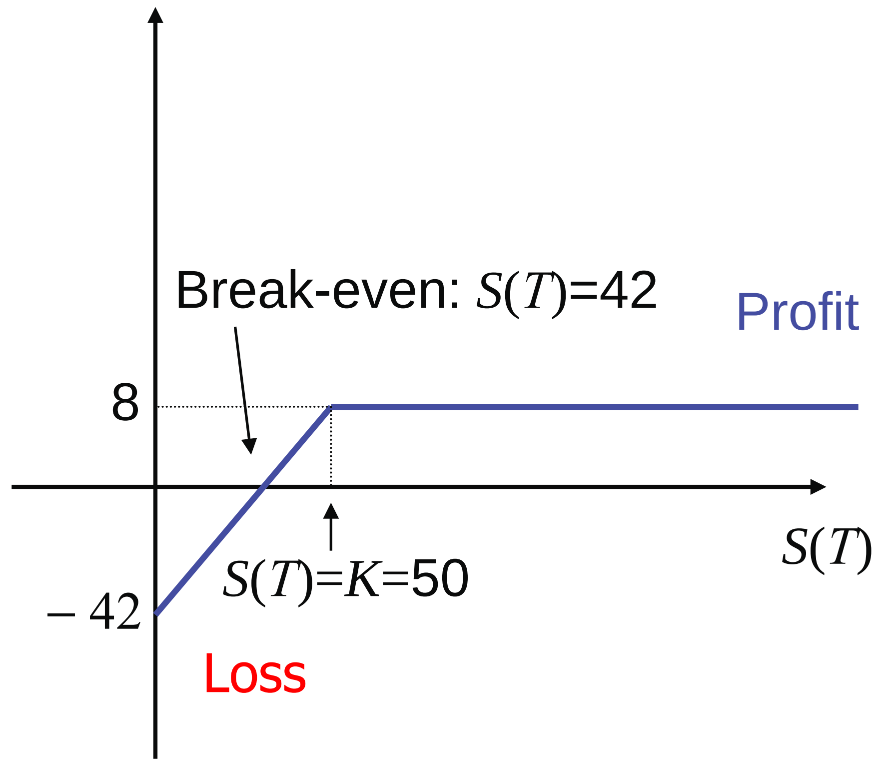
- Inputs: $K = \$50,\ P(t,K,T) = \$8$
- Payoff: $max\ [50 - S(T), 0]$
- Profit: $max\ [50 - S(T), 0] - 8$



Short Put

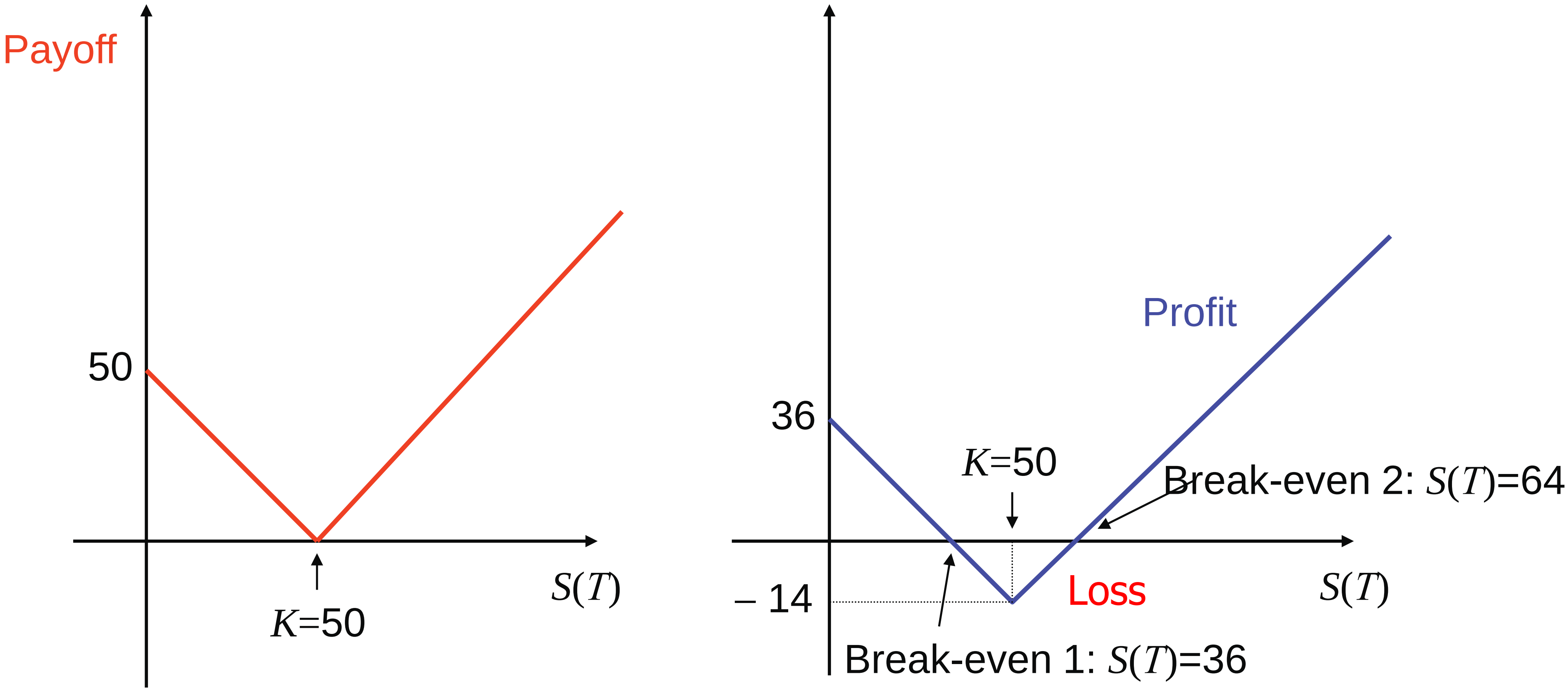
- Inputs: K = \$50, $P(t,K,T)$ = \$8
- Payoff: $-\ max\ [50 - S(T), 0]$
- Profit: $8 - max\ [50 - S(T), 0]$





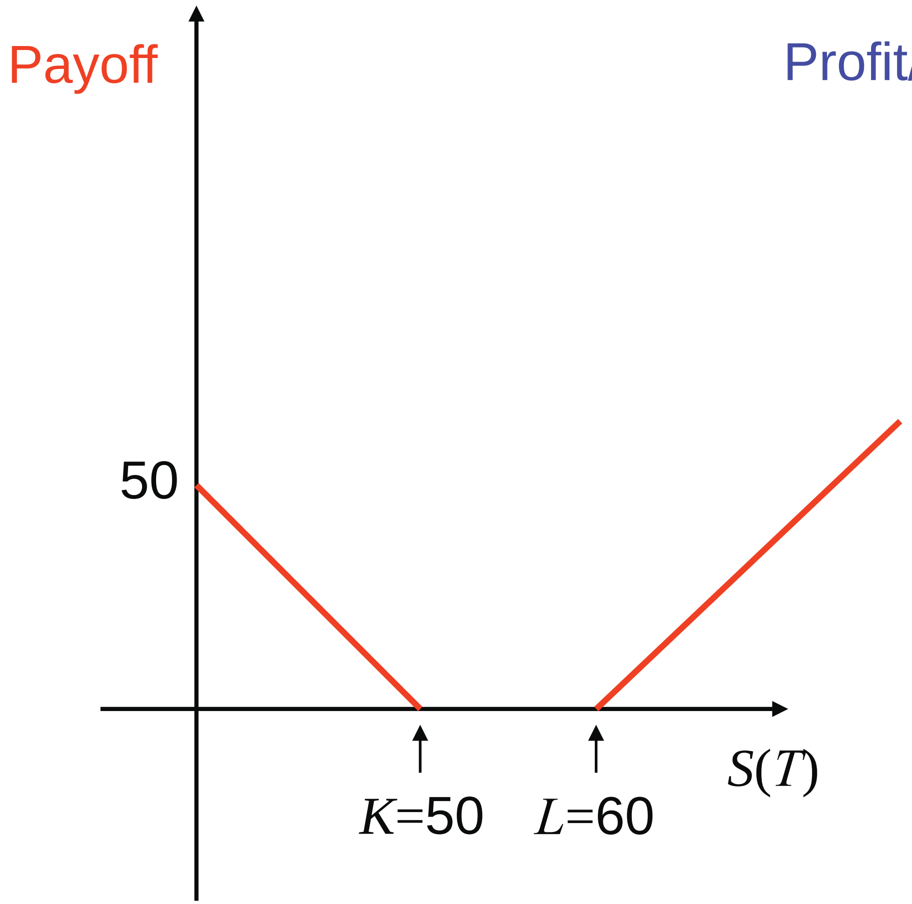
Bottom Straddle

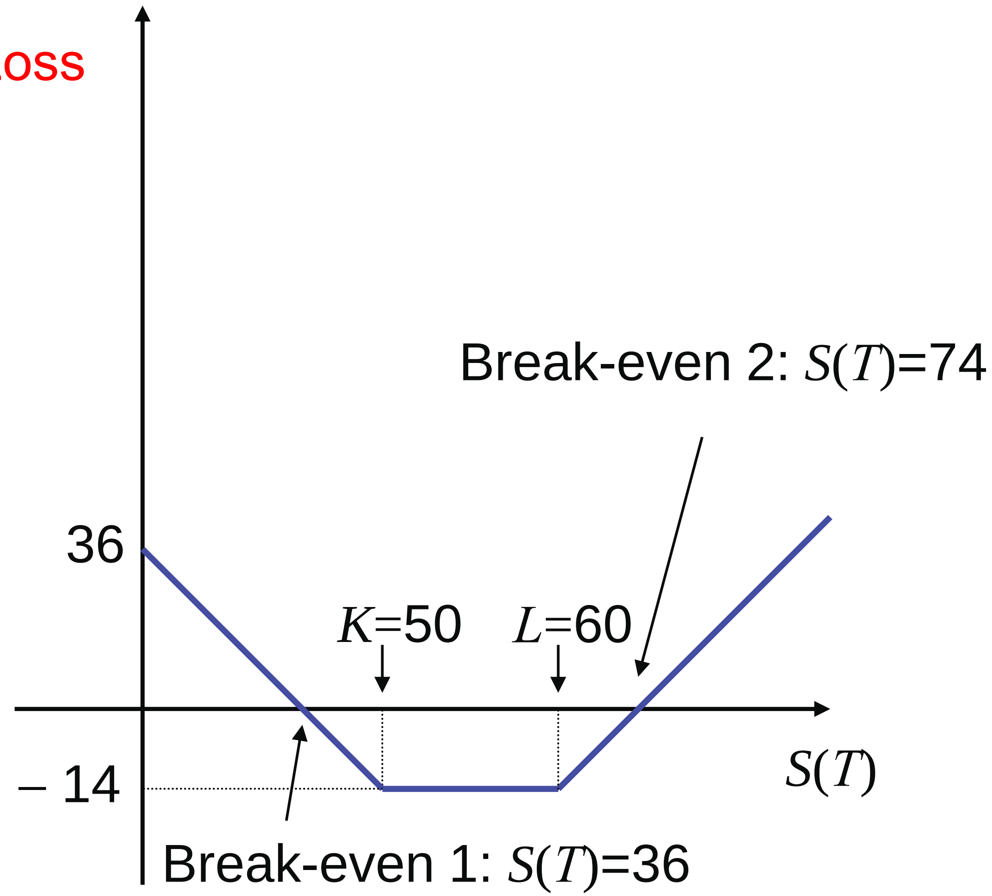
Inputs: K = \$50, $P(K)$ = \$8, $C(K)$ = \$6



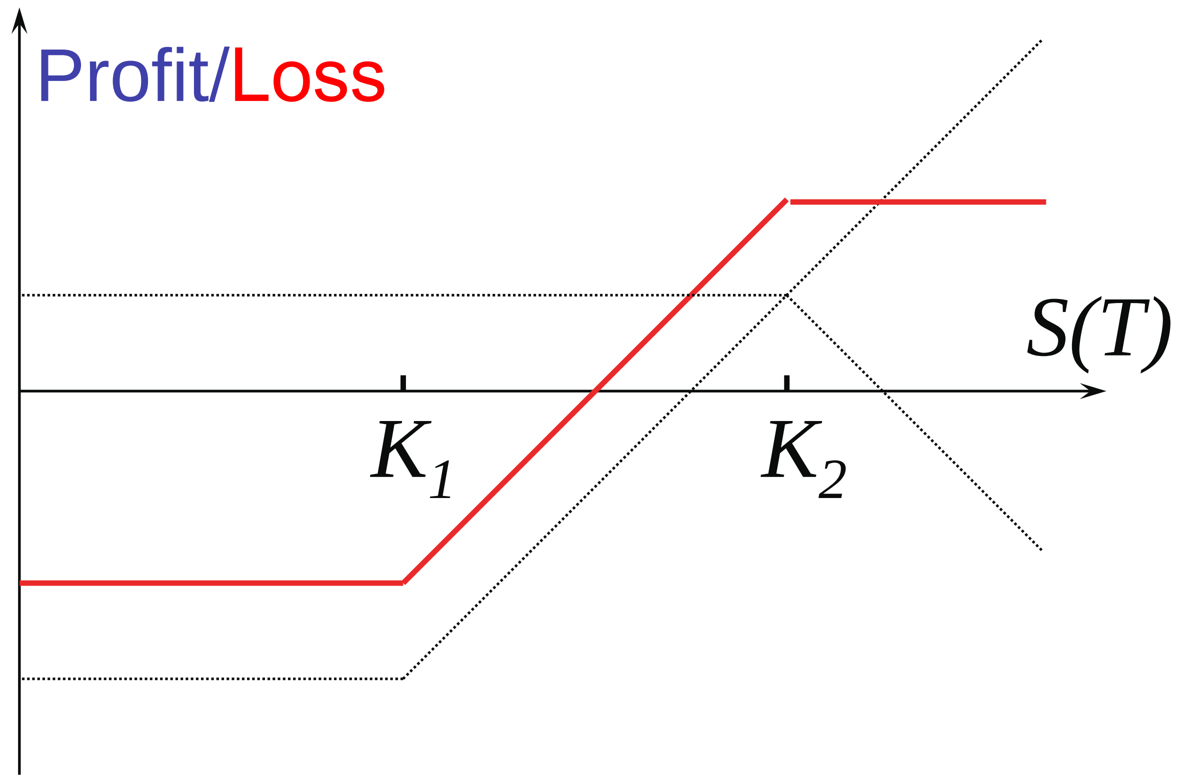
Bottom Strangle

Inputs: K = \$50, L= \$60, $P(K)$ = \$8, $C(L)$ = \$6





Bull Spread Using Calls



Bull Spread (Calls)

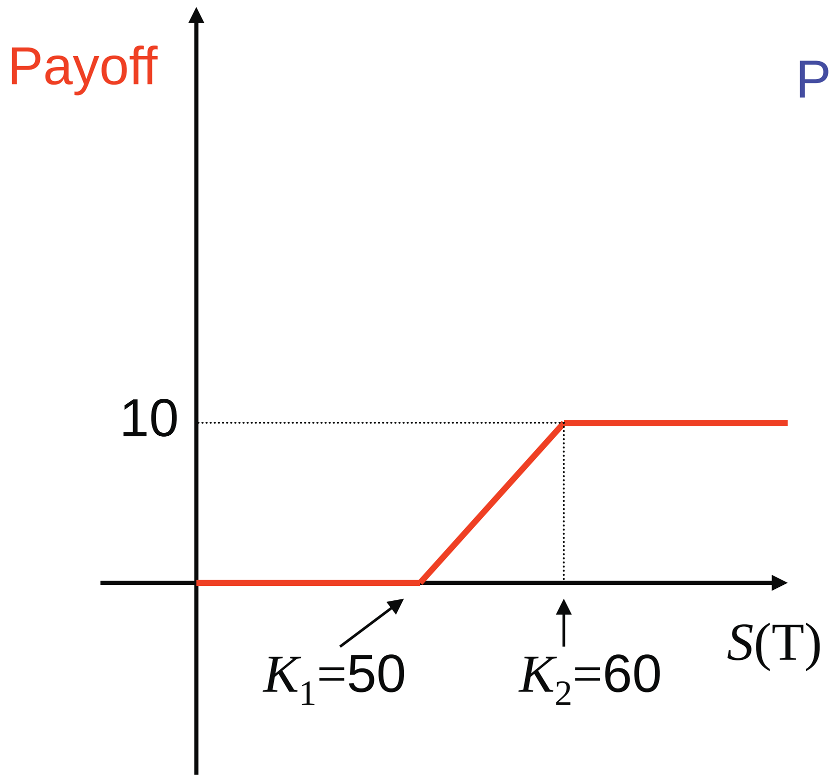
- Two strike prices: K_1, K_2 with $K_1 < K_2$
- Shorthand notation: $C(K_1), C(K_2)$

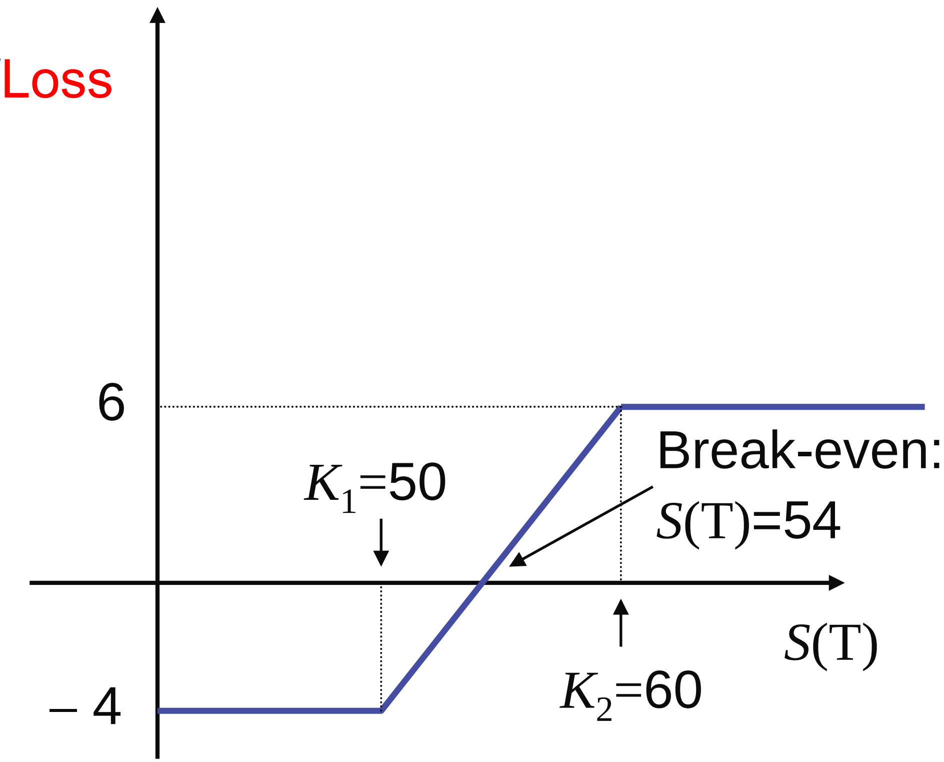
Outcome at expiration

	$S(T) \leq K_1$	$K_1 < S(T) \leq K_2$	$S(T) > K_2$
Payoff:	0	$S(T) - K_1$	$S(T) - K_1 - (S(T) - K_2) =$ $= K_2 - K_1$
Profit:	$C(K_2) - C(K_1)$	$C(K_2) - C(K_1)$ $+ S(\mathrm{T}) - K_1$	$C(K_2) - C(K_1) + K_2 - K_1$

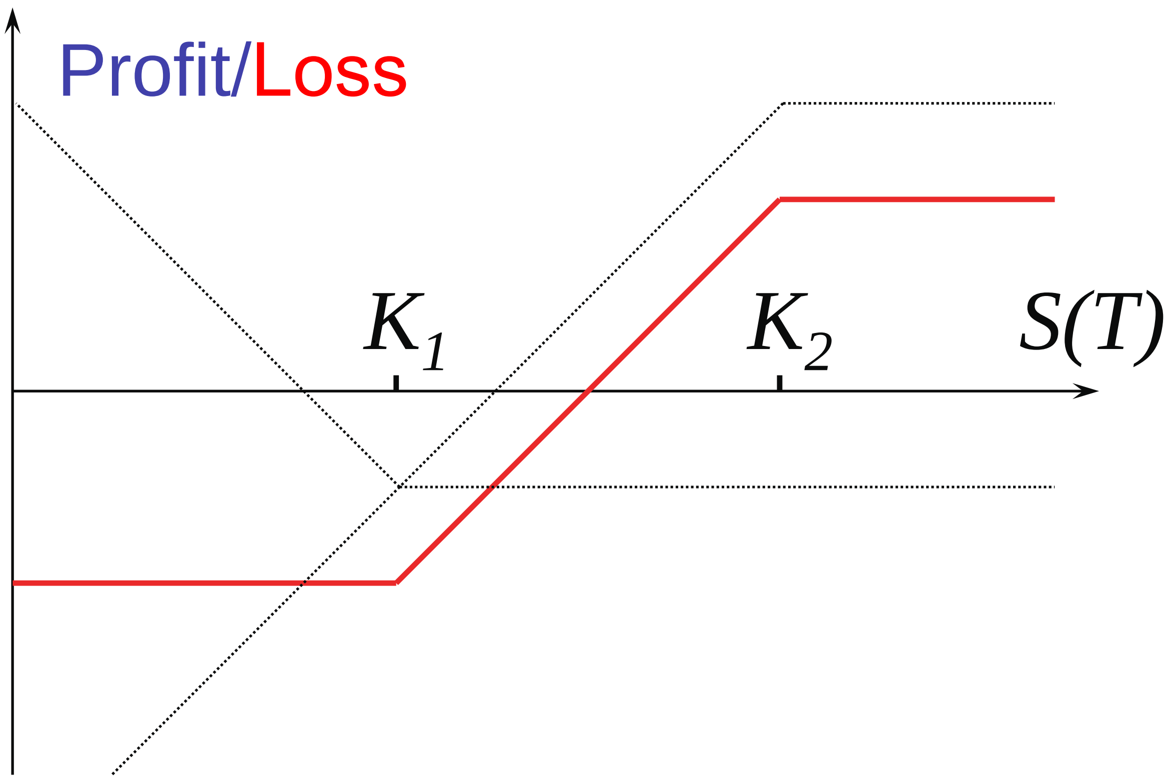
Bull Spread (Calls)

- Inputs: $K_1 = \$50,\ K_2 = \$60,\ C(K_1) = \$10,\ C(K_2) = \$6$
- Payoff: $max\ [S(T) - 50, 0] - max\ [S(T) - 60, 0]$
- Profit: $(6 - 10) + max\ [S(T) - 50,0] - max\ [S(T) - 60,0]$

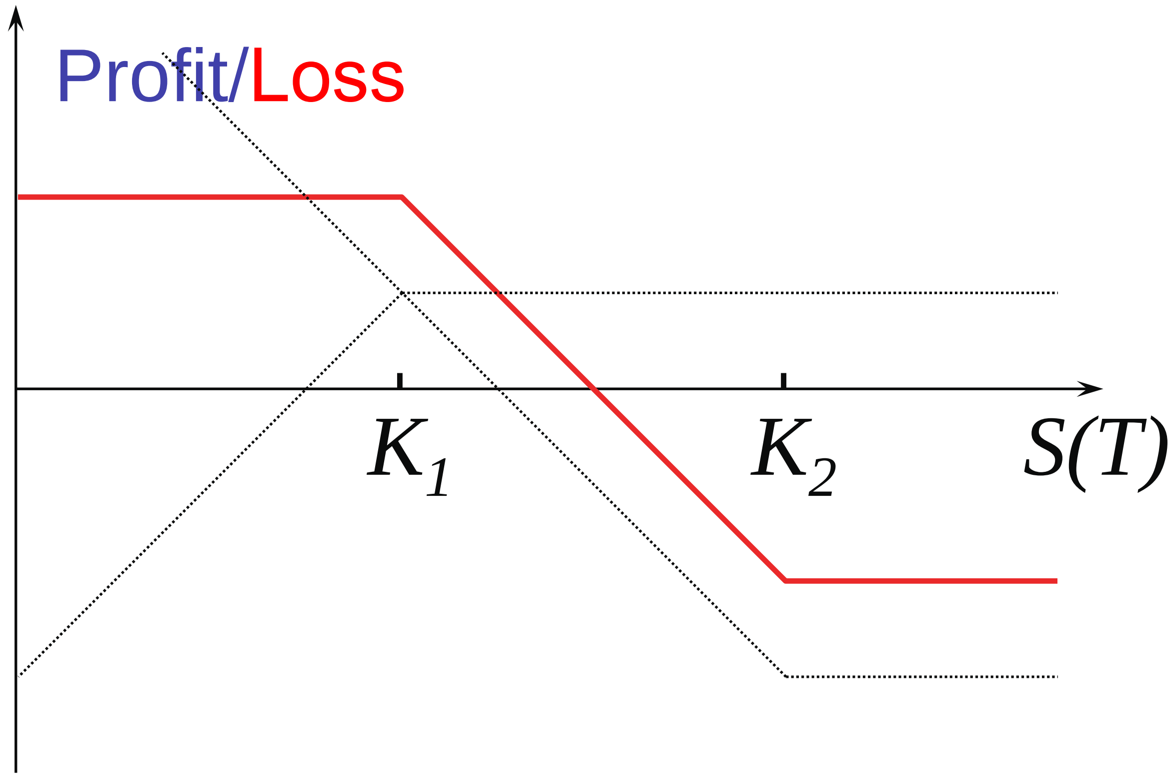




Bull Spread Using Puts



Bear Spread Using Puts



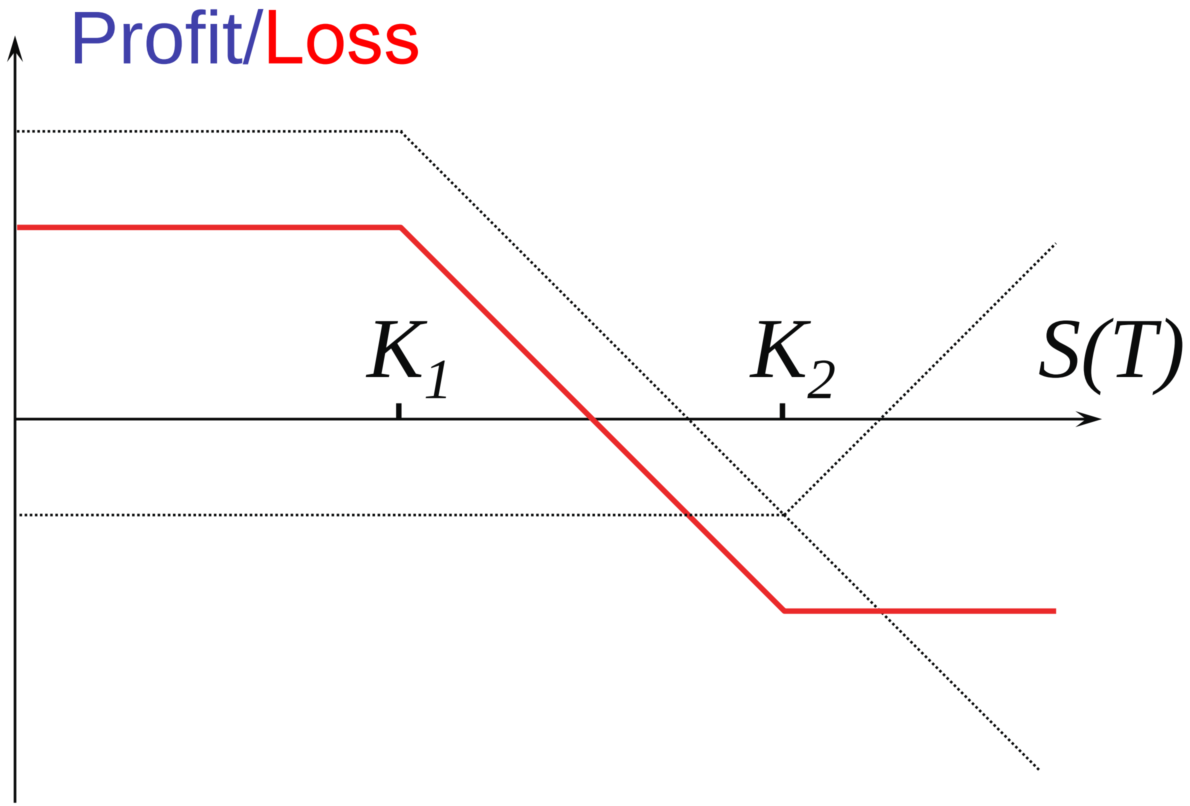
Bear Spread (Puts)

- Again two strikes: K_1, K_2 with $K_1 < K_2$
- Shorthand notation: $P(K_1), P(K_2)$

Outcome at expiration

	$S(T) \le K_1$	$K_1 < S(T) \le K_2$	$S(T) > K_2$
Payoff:	$K_2 - S(\mathrm{T}) - (K_1 - S(\mathrm{T})) =$ $= K_2 - K_1$	$K_2 - S(T)$	0
Profit:	$P(K_1) - P(K_2) + K_2 - K_1$	$P(K_1) - P(K_2) +$ $+ K_2 - S(T)$	$P(K_1) - P(K_2)$

Bear Spread Using Calls



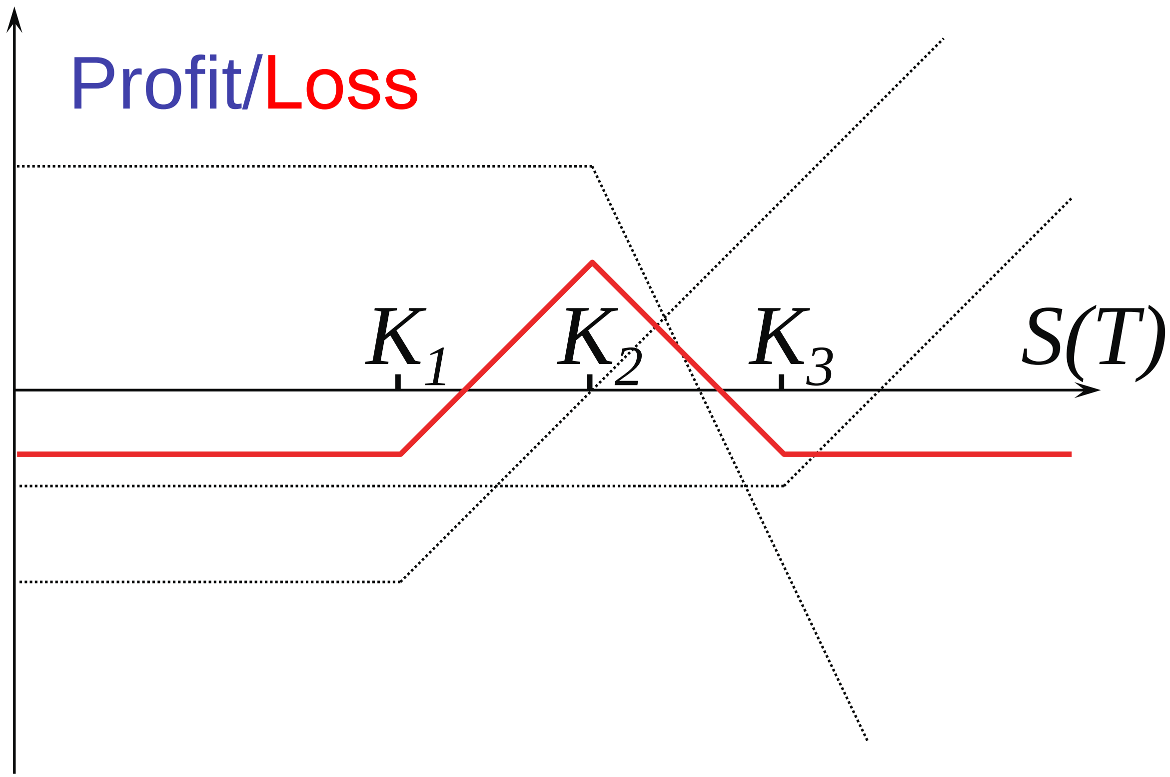
Butterfly Spread

- Positions in **three** options of the same class with the same maturity but different strikes $K_1 < K_2 < K_3$

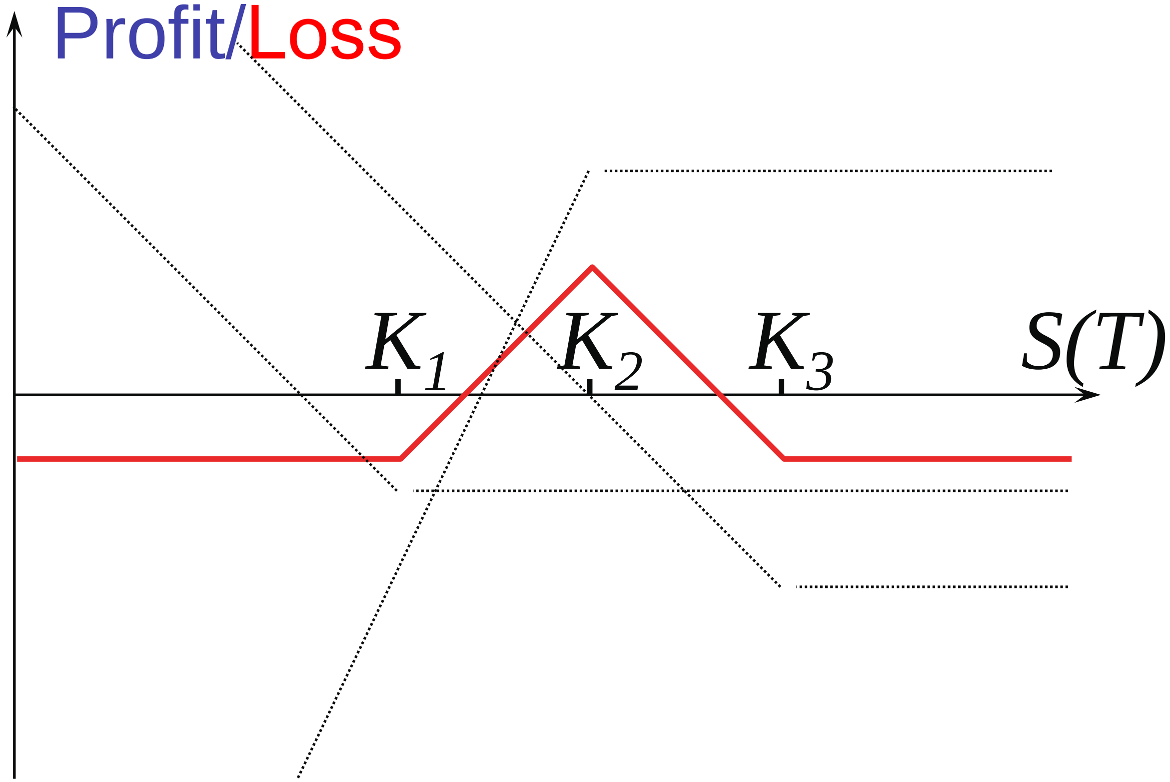
Long butterfly spreads: buy one option each with strikes K_1, K_3, sell two with strike K_2

- $K_2 = (K_1 + K_3) / 2$

Butterfly Spread Using Calls

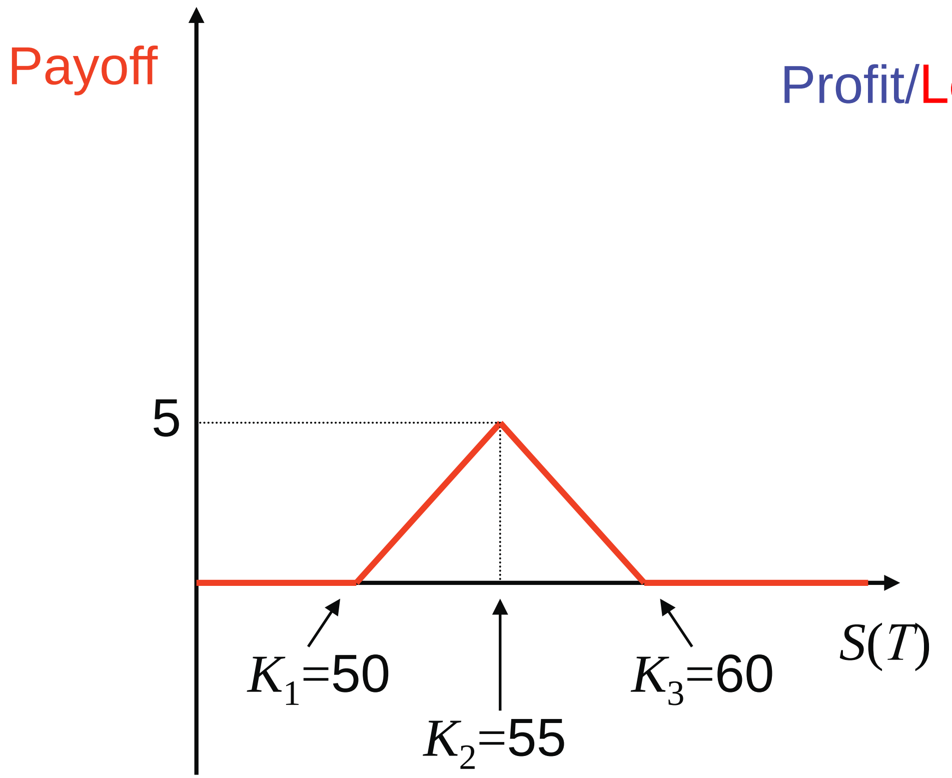


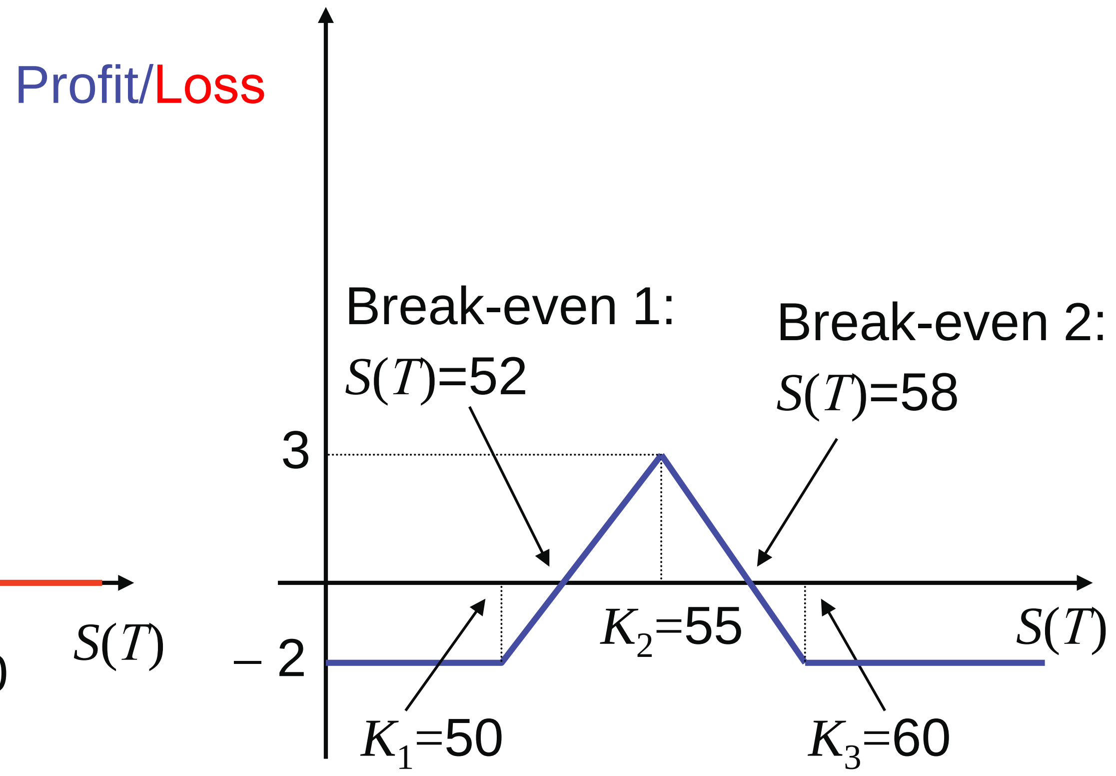
Butterfly Spread Using Puts



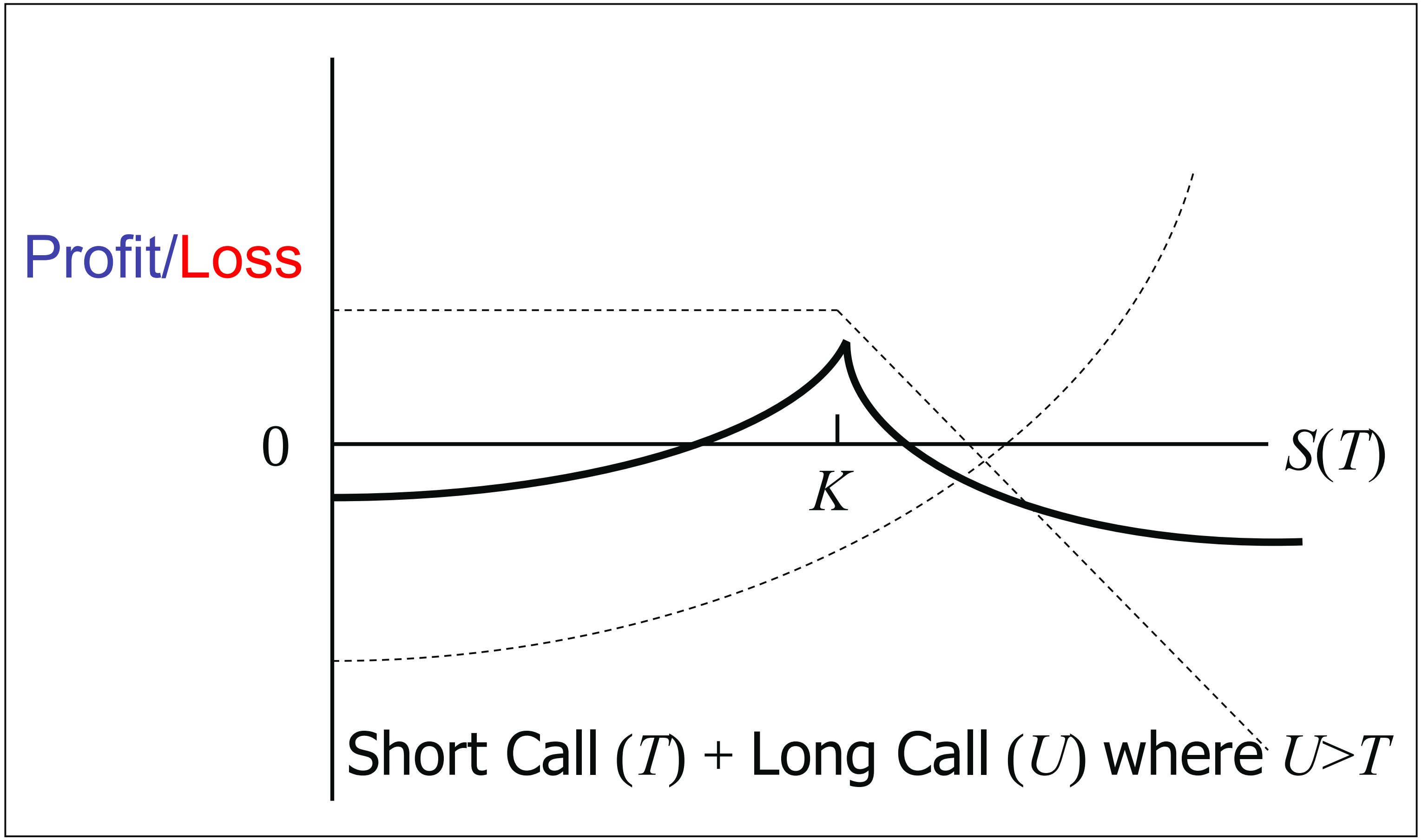
Long Butterfly Spread (Puts)

- $K_1 = \$50$, $K_2 = \$55$, $K_3 = \$60$
- $P(K_1) = \$4$, $P(K_2) = \$6$, $P(K_3) = \$10$





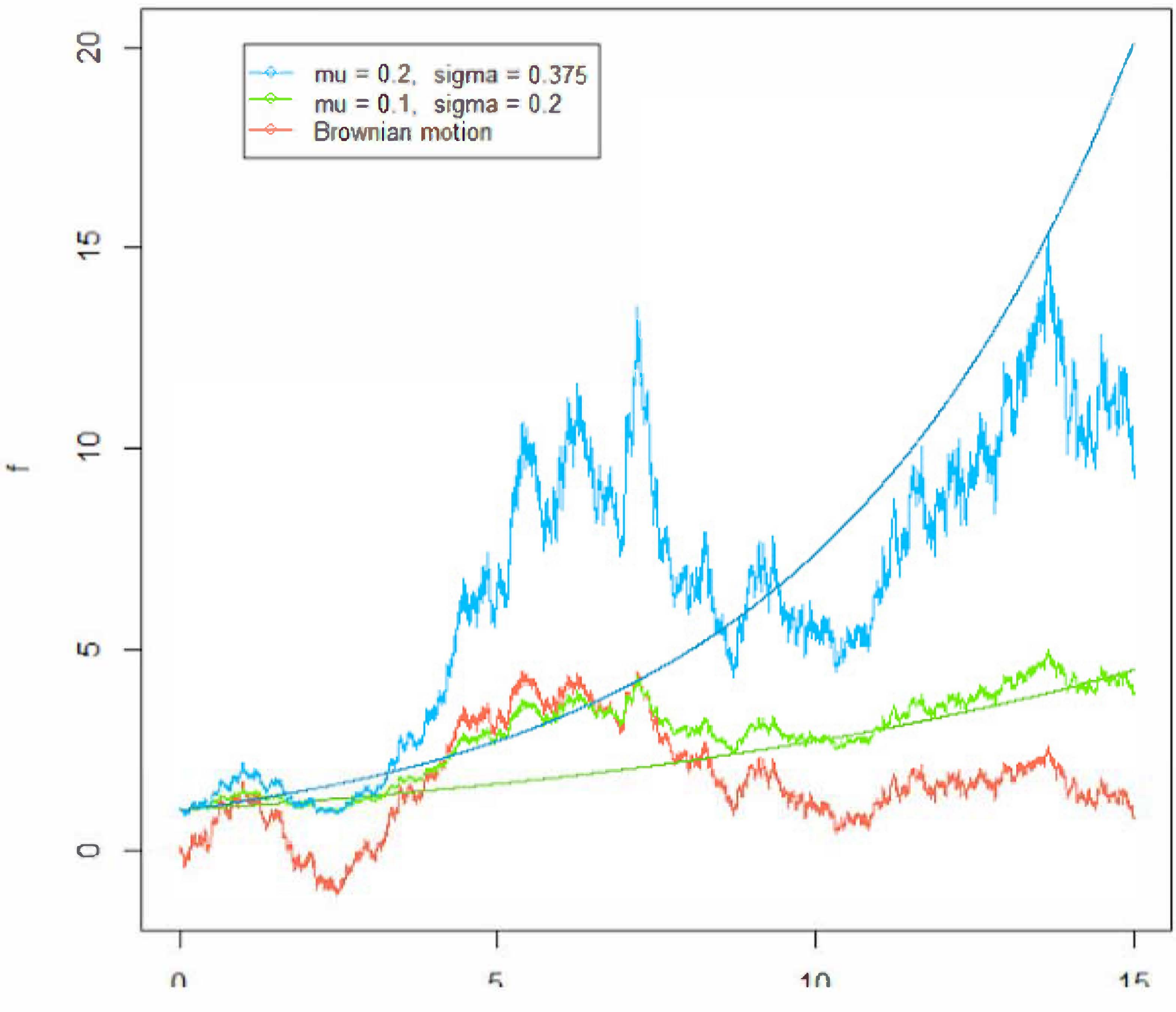
Calendar Spread



Myron Scholes

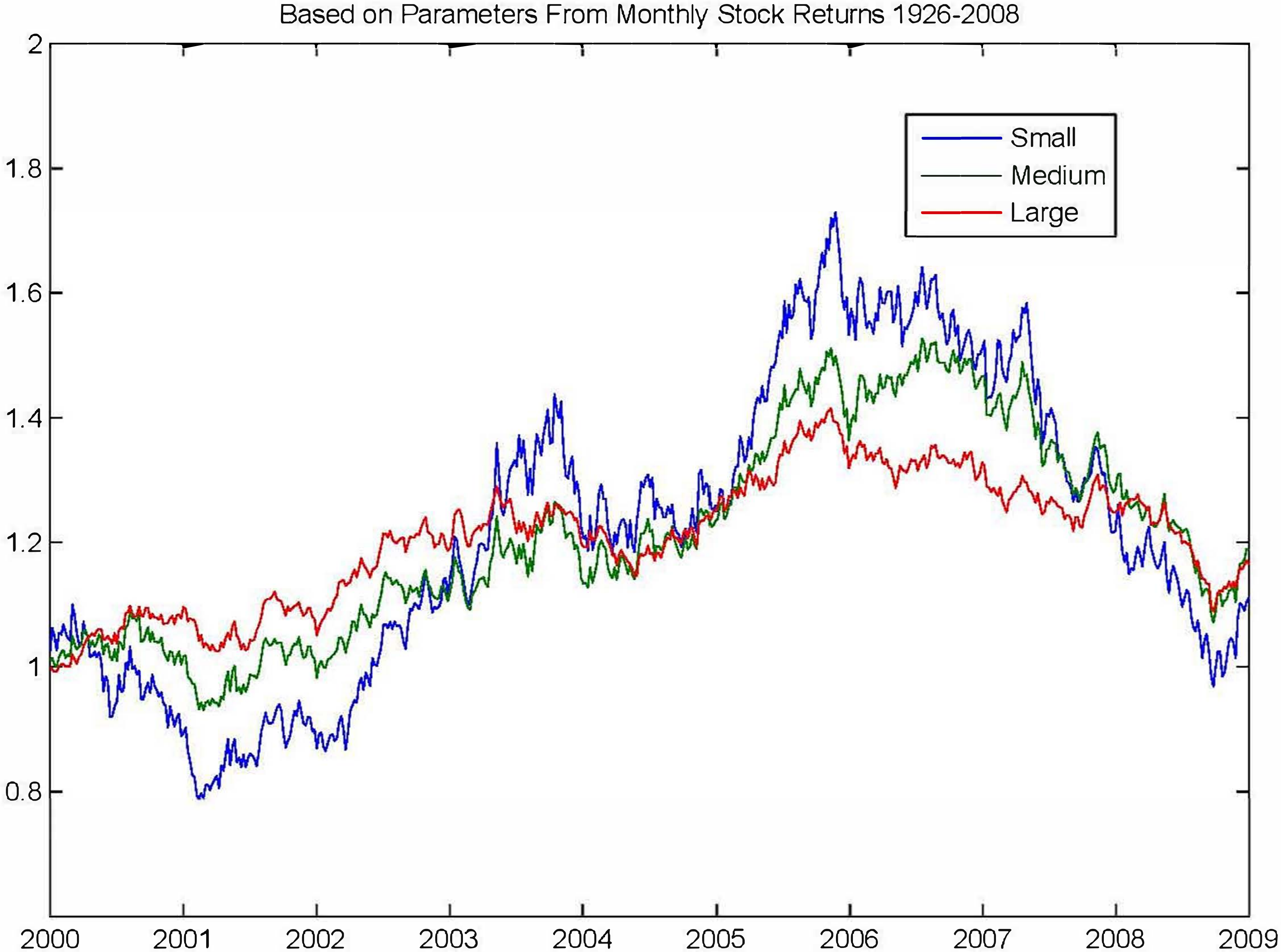
Fischer Black

Robert C. Merton



mu = 0.2, sigma = 0.375
mu = 0.1, sigma = 0.2
Brownian motion
f
0
5
10
15
20
0
5
10
15

Small, Medium and Large-Cap Stock Prices

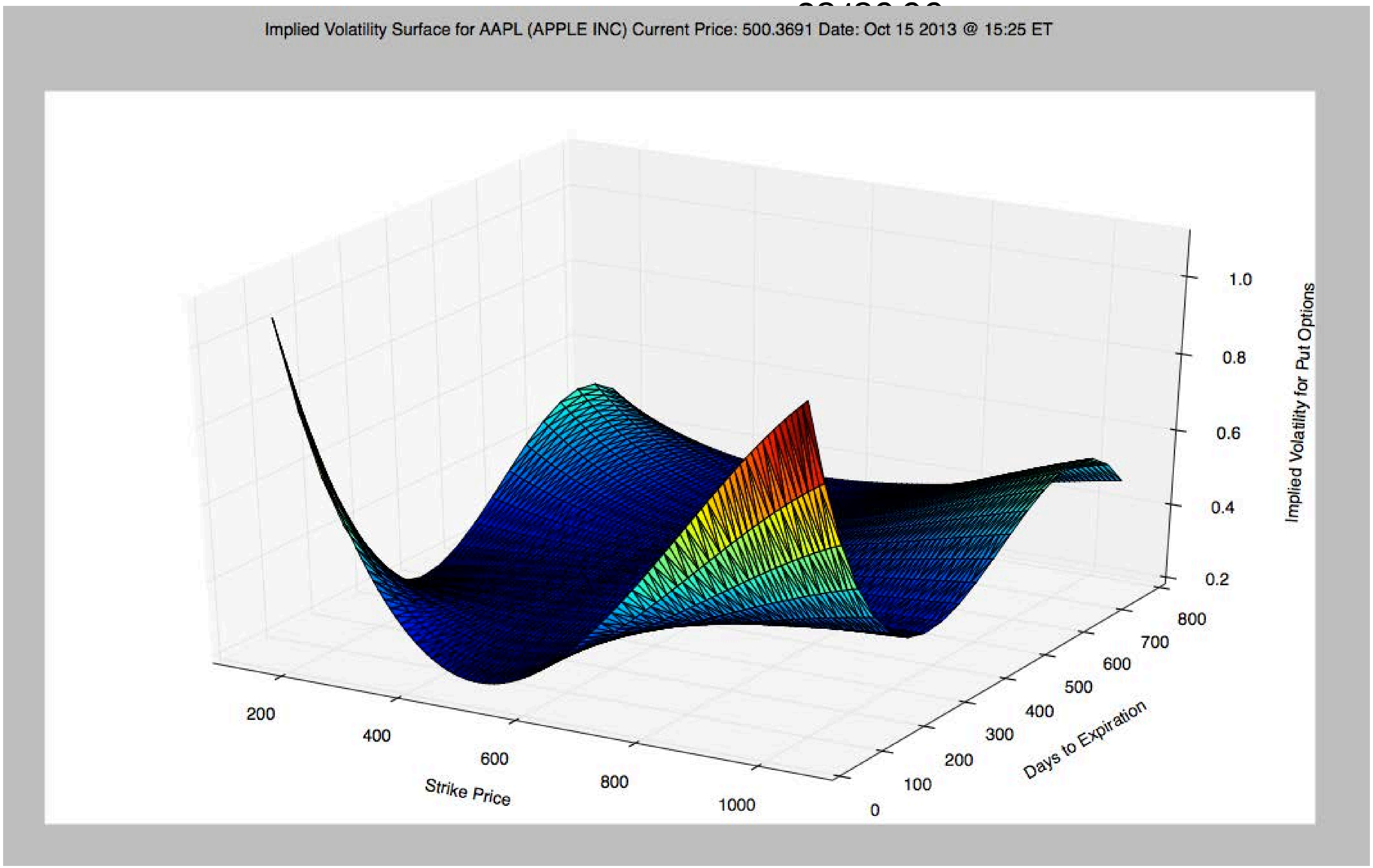


CONVICTION OF BUYERS AND SELLERS NEAR THE S&P 500'S 200-DAY MOVING AVERAGE (2017-2019)

www.ccmmarketmodel.com

For illustrative purposes only.

DJIA	S&P 500	NASDAQ	FTSE 100	NIKKEI 225 Future	EUR-USD	WTI Crude
27,778.07	3,389.78	11,210.84	6,070.37		1.1933	42.64
▼-66.84		▲+81.12			▲+0.0002	



SPX:IND

S&P 500 Index

3,389.78 USD

+7.79 +0.23% ▲

MARKET CLOSED

AS OF 08/18/2020 EDT

1D 1M 6M **YTD** 1Y 5Y

S&P 500 Index in 2020

Kiyoshi Ito

Kiyoshi Itō (伊藤 清, *Itō Kiyoshi*[?]) (September 7, 1915 – 10 November 2008) was a Japanese mathematician whose work is now called Itō calculus. The basic concept of this calculus is the Itō integral, and among the most important results is Itō's lemma. The Itō calculus facilitates mathematical understanding of random events. His theory is widely applied in various fields, and is perhaps best known for its use in financial mathematics.

Black-Scholes formula

- For an European call option $g(s) = (s - K)^+$, the solution of the PDE is given by the Black-Scholes formula:

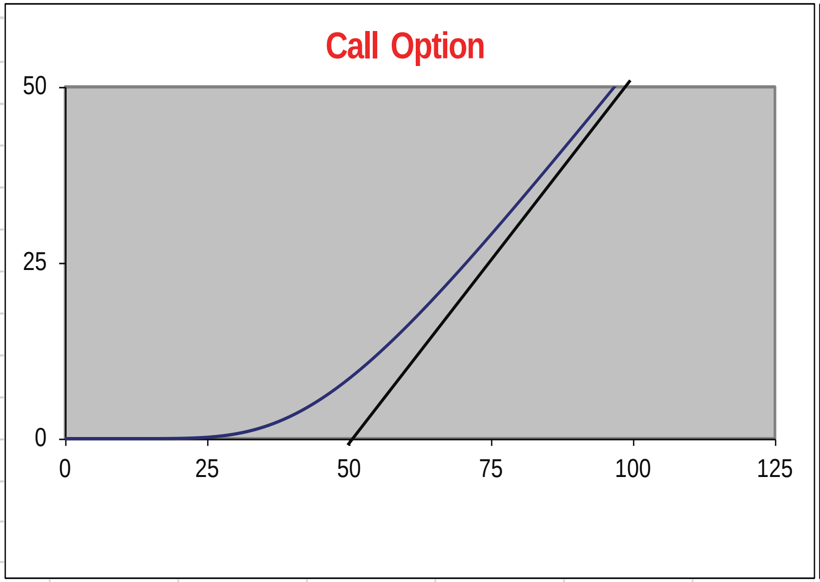
$$C(t, S(t)) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2)$$

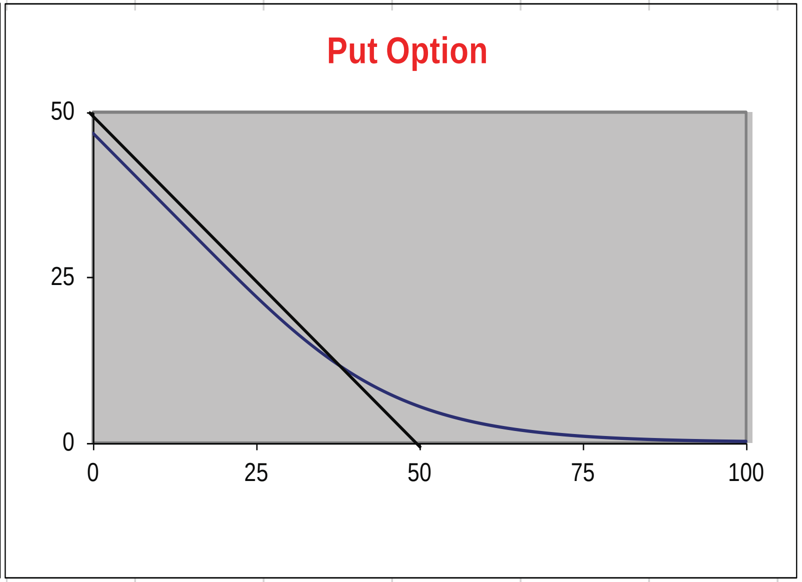
where

$$N(x) := P[Z \leq x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$$

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}}[\log(S(t)/K) + (r + \sigma^2/2)(T-t)] \\ d_2 &= \frac{1}{\sigma\sqrt{T-t}}[\log(S(t)/K) + (r - \sigma^2/2)(T-t)] \\ &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

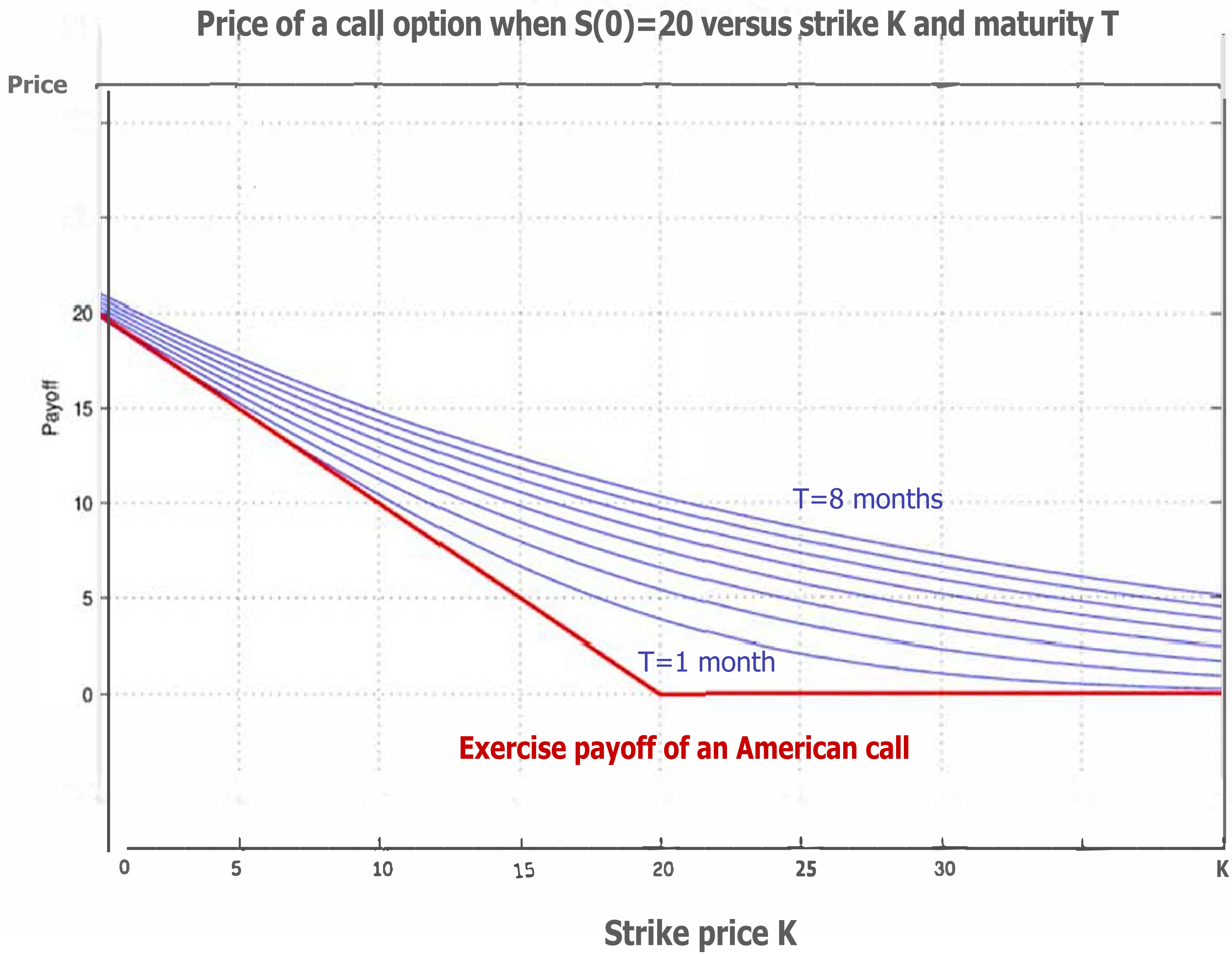
European Call and Put Prices

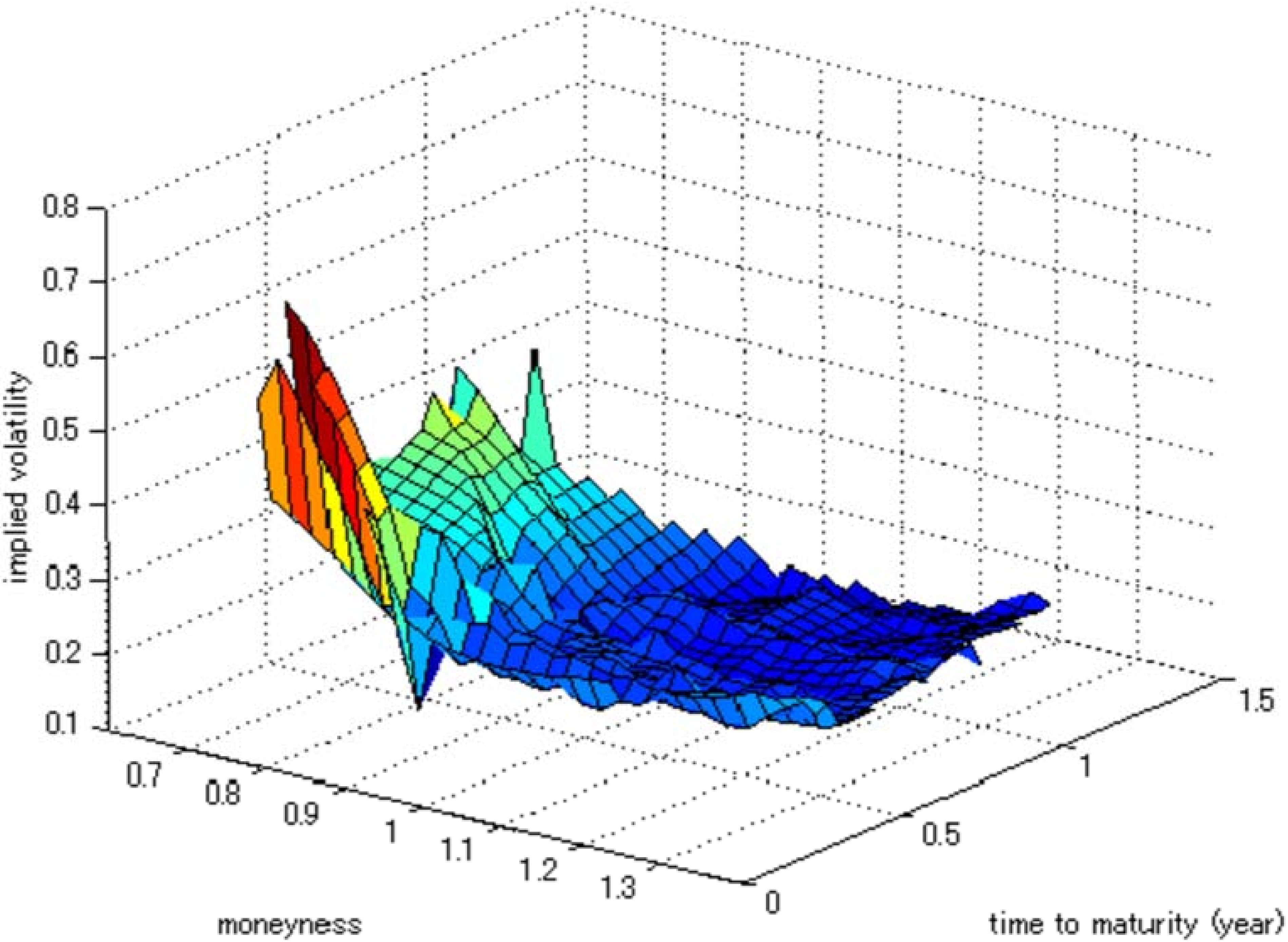




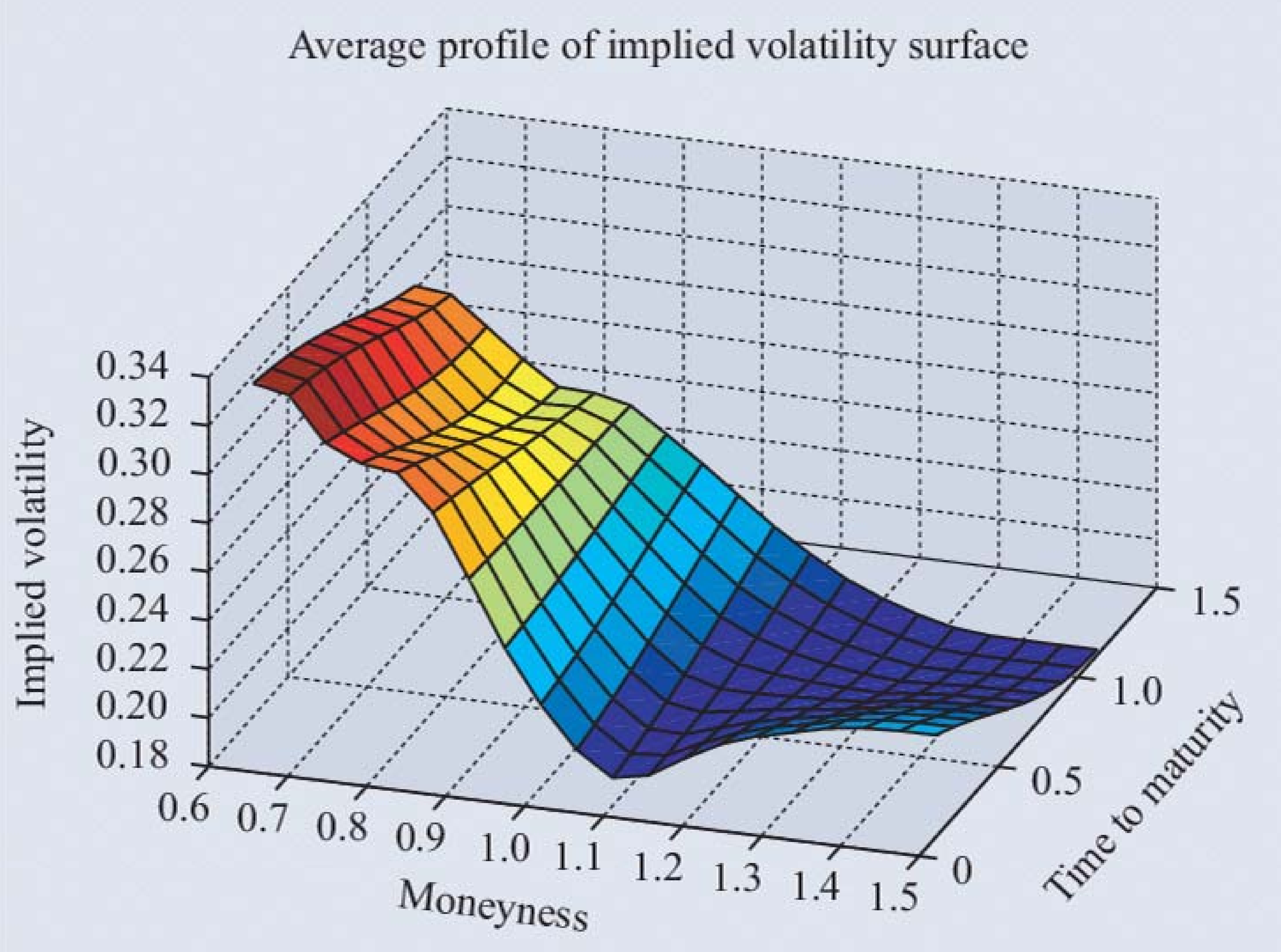
CBOE DJX and SPX Options Quotes

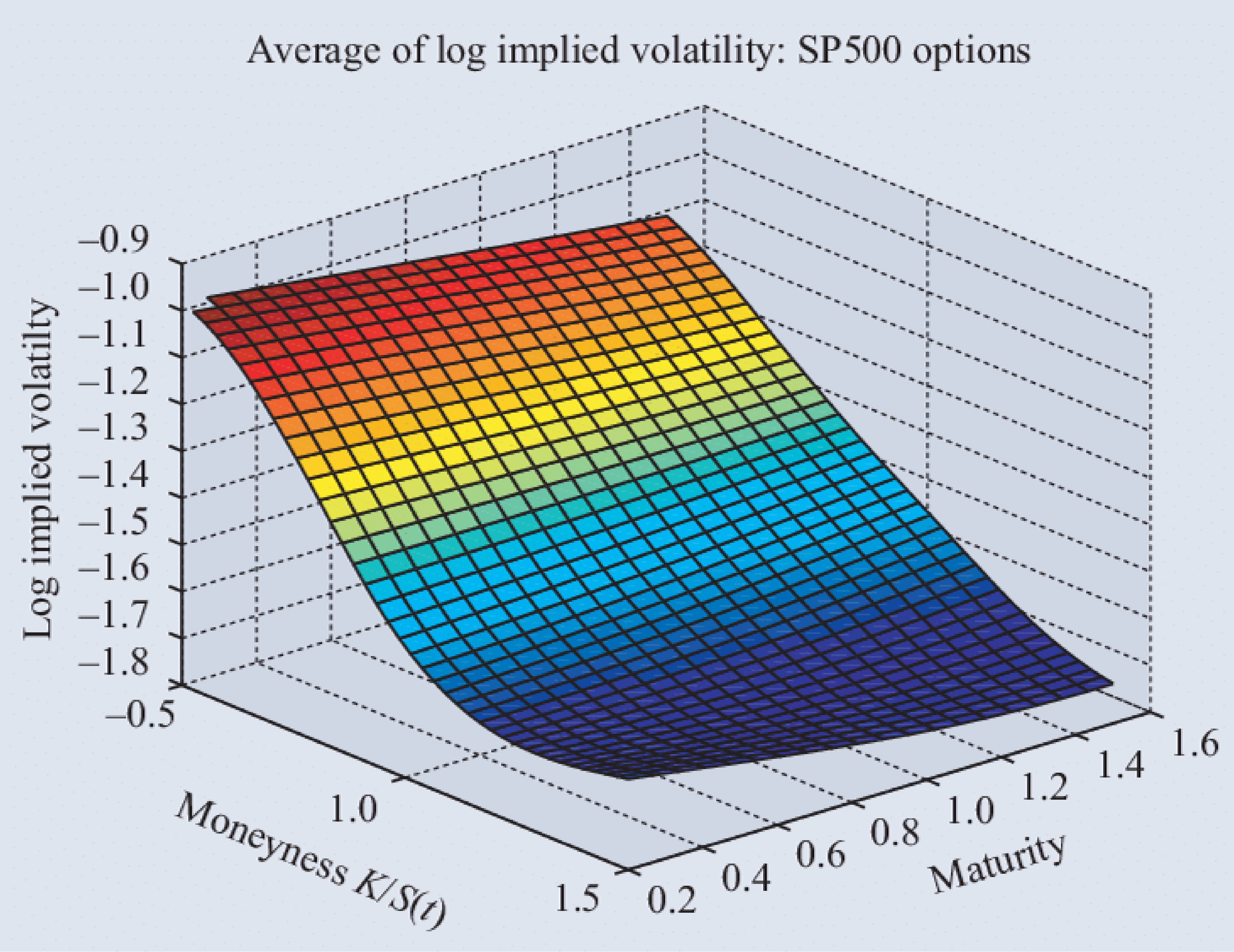
http://www.cboe.com/delayedquote/quote-table


Price of a call option when S(0)=20 versus strike K and maturity T
Price
Payoff
20
15
10
5
0
T=8 months
T=1 month
Exercise payoff of an American call
0
5
10
15
20
25
30
K
Strike price K

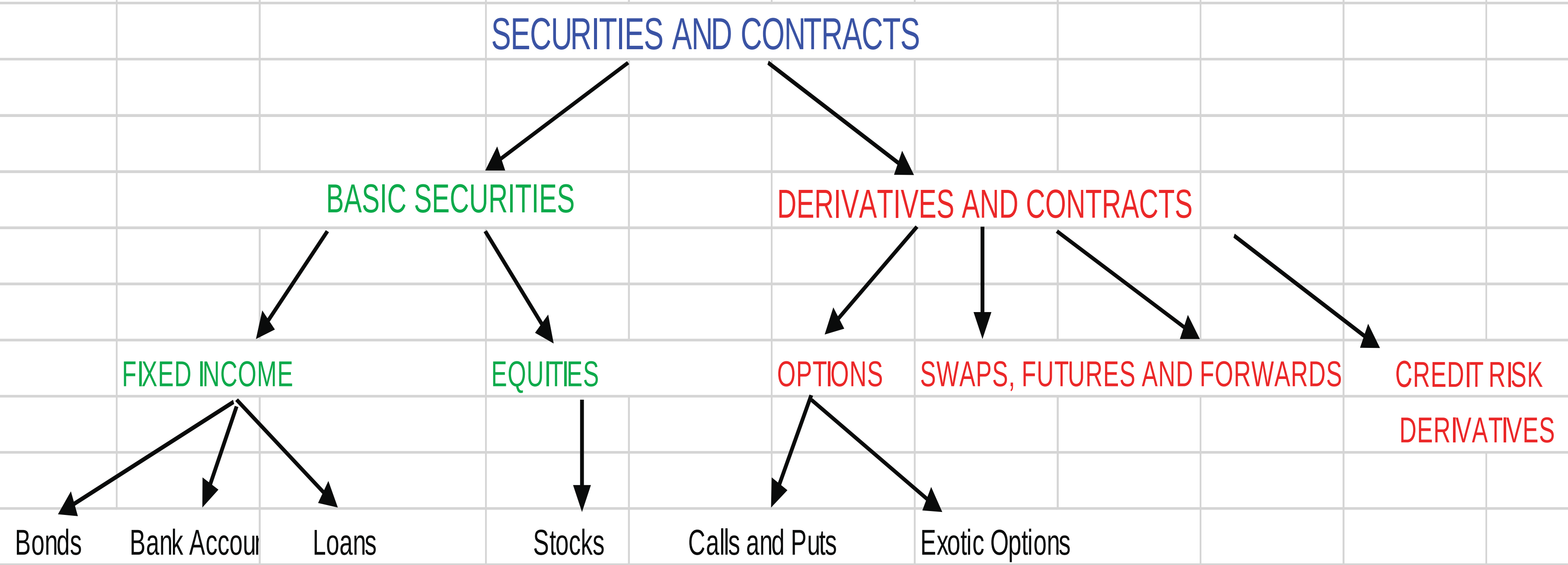


0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
implied volatility
0.7
0.8
0.9
1
1.1
1.2
1.3
moneyness
0
0.5
1
1.5
time to maturity (year)


Average profile of implied volatility surface
Implied volatility
0.34
0.32
0.30
0.28
0.26
0.24
0.22
0.20
0.18
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.3
1.4
1.5
Moneyness
0
0.5
1.0
1.5
Time to maturity


Average of log implied volatility: SP500 options
Log implied volatilty
−0.9
−1.0
−1.1
−1.2
−1.3
−1.4
−1.5
−1.6
−1.7
−1.8
−0.5
1.0
1.5
Moneyness K/S(t)
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
Maturity

Classification of Financial Instruments



MATH3075/3975 FINANCIAL DERIVATIVES

NO-ARBITRAGE PRICING THEORY