MATH3075/3975 Financial Derivatives

Tutorial 6: Solutions

Exercise 1 (a) To show that the model is arbitrage-free, we need to show that there is no strategy (x, φ) satisfying conditions of Definition 2.2.3 of an arbitrage opportunity:

(i) $x = V_0(x, \varphi) = 0$, (ii) $V_1(x, \varphi) \geq 0$, (iii) $\mathbb{E}_{\mathbb{P}}(V_1(x, \varphi)) > 0$.

For $x = 0$, the wealth at time 1 of a strategy $(0, \varphi)$ equals

$$V_1(0, \varphi) = \varphi(S_1 - S_0(1 + r)).$$

Since $S_0 = 4$ and $r = 0.1$, we have $S_0(1 + r) = 4.4$ and thus

$$\begin{aligned}
V_1(x, \varphi)(\omega_1) &= (8 - 4.4)\varphi = 3.6\varphi, \\
V_1(x, \varphi)(\omega_2) &= (5 - 4.4)\varphi = 0.6\varphi, \\
V_1(x, \varphi)(\omega_3) &= (3 - 4.4)\varphi = -1.4\varphi.
\end{aligned}$$

There are now three cases to consider, namely,

$$\begin{cases}
\varphi > 0 & \Longrightarrow \quad V_1(x, \varphi)(\omega_3) = -1.4\varphi < 0, \\
\varphi < 0 & \Longrightarrow \quad V_1(x, \varphi)(\omega_1) = 3.6\varphi < 0, \\
\varphi = 0 & \Longrightarrow \quad \mathbb{E}_{\mathbb{P}}(V_1(x, \varphi)) = 0.
\end{cases}$$

In all three cases we have a contradiction, since either condition (ii) or condition (iii) in Definition 2.2.3 is not satisfied. We thus conclude that the model $\mathcal{M} = (B, S)$ is arbitrage-free.

(b) We will show that the call option with strike $K = 4$ is not attainable. To this end, we try to compute the replicating strategy directly, that is, by solving the equation

$$\begin{bmatrix} 1.1 & 8 \\ 1.1 & 5 \\ 1.1 & 3 \end{bmatrix} \begin{bmatrix} \varphi^0 \\ \varphi^1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$$

where we denote $\varphi^0 = x - \varphi^1 S_0$ and $\varphi^1 = \varphi$. If we now augment the right-hand side and row reduce the matrix then we find

$$\begin{bmatrix} 1.1 & 8 & 4 \\ 1.1 & 5 & 1 \\ 1.1 & 3 & 0 \end{bmatrix} \Longrightarrow \text{ row operations } \Longrightarrow \begin{bmatrix} 1.1 & 8 & 4 \\ 0 & 3 & 3 \\ 0 & 0 & -1 \end{bmatrix}.$$

We conclude that the above linear system has no solution and thus the call option with strike $K = 4$ is not attainable. Of course, the non-existence of a solution can be verified using other arguments, for instance, by finding the unique solution to the first two equations and checking that it fails to satisfy the last equation.

(c) Since $\Omega = \{\omega_1, \omega_2, \omega_3\}$ we may identify the class of all contingent claims with the vector space $\mathbb{R}^3$. Let $Y_1 = (1,1,1)$ and $Y_2 = (8,5,3)$. Then the space of all attainable claims is given by

$$\left\{ X \in \mathbb{R}^3 \,|\, X = \lambda_1 Y_1 + \lambda_2 Y_2,\ \lambda_1, \lambda_2 \in \mathbb{R} \right\},$$

that is, it is the subspace of $\mathbb{R}^3$ spanned by the vectors Y_1 and Y_2. More explicitly, the space of all attainable claims is the plane given by

$$\left\{ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, \lambda_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \lambda_1 \begin{bmatrix} 8 \\ 5 \\ 3 \end{bmatrix},\ \lambda_1, \lambda_2 \in \mathbb{R} \right\}.$$

(d) We note that a martingale measure in this model is not unique. Indeed, we already know from part (b) that the call option with strike $K = 4$ is not an attainable claim. Hence the model is incomplete and thus the non-uniqueness of a martingale measure follows from Proposition 2.2.7 (or Theorem 2.2.2). Our goal is now to compute explicitly the set of all martingale measures

$$\mathbb{Q} = (\mathbb{Q}(\omega_1), \mathbb{Q}(\omega_2), \mathbb{Q}(\omega_3)) = (q_1, q_2, q_3) \in \mathbb{M}$$

for the model $\mathcal{M}$.

First method. Recall that $\mathbb{Q} \in \mathbb{M}$ if $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and $\mathbb{E}_{\mathbb{Q}}(\widehat{S}_1) = S_0$. More explicitly, (q_1, q_2, q_3) satisfies

$$\begin{aligned} S_0 &= \frac{1}{1+r}\big(q_1 S_1(\omega_1) + q_2 S_1(\omega_2) + q_3 S_1(\omega_3)\big), \\ q_1 &+ q_2 + q_3 = 1, \quad q_i > 0,\ i = 1,2,3. \end{aligned}$$

Equivalently, we need to solve the following system

$$\begin{bmatrix} 8 & 5 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 4.4 \\ 1 \end{bmatrix}$$

subject to the constraints $0 < q_i < 1$ for $i = 1,2,3$. By solving these equations, we conclude that the set of all martingale measures $\mathbb{M}$ can be represented as follows (note that $\lambda = q_3$ was chosen as a parameter)

$$\mathbb{M} = \left\{ \mathbb{Q} = (q_1, q_2, q_3) = \left(\tfrac{2\lambda}{3} - \tfrac{1}{5}, -\tfrac{5\lambda}{3} + \tfrac{6}{5}, \lambda\right),\ \lambda \in \left(\tfrac{3}{10}, \tfrac{18}{25}\right) \right\},$$

that is,

$$\mathbb{M} = \left\{ \mathbb{Q} = (q_1, q_2, q_3) = \left(-\tfrac{1}{5}, \tfrac{6}{5}, 0\right) + \lambda\left(\tfrac{2}{3}, -\tfrac{5}{3}, 1\right),\ \lambda \in \left(\tfrac{3}{10}, \tfrac{18}{25}\right) \right\}.$$

This confirms our claim that several martingale measures for the model $\mathcal{M}$ exist.

Second method. It is also possible to use two-state sub-models. Let $\mathbb{Q}_1$ (respectively, $\mathbb{Q}_2$) be the unique martingale measure for the two-state sub-model with the elementary event ω_2 (respectively, ω_1) discarded. Simple computations for two-state sub-models show that $\mathbb{Q}_1 = (0.28, 0, 0.72)$ and $\mathbb{Q}_2 = (0, 0.7, 0.3)$. Hence this method gives the following representation for the class $\mathbb{M}$

$$\mathbb{M} = \left\{ \mathbb{Q} = (q_1, q_2, q_3) \in \mathbb{R}^3 \,|\, \mathbb{Q} = \alpha \mathbb{Q}_1 + (1-\alpha)\mathbb{Q}_2,\ \alpha \in (0,1) \right\}.$$

Let us stress that the probability measures $\mathbb{Q}_1$ and $\mathbb{Q}_2$ do not belong to the class $\mathbb{M}$ since they are not equivalent to $\mathbb{P}$ (since $q_2 = 0$ for $\mathbb{Q}_1$ and $q_1 = 0$ for $\mathbb{Q}_2$).

(e) We search for the range of expected values

$$\mathbb{E}_{\mathbb{Q}}\left(\frac{(S_1-4)^+}{1+r}\right) = \tfrac{4}{1.1}\,q_1 + \tfrac{1}{1.1}\,q_2 + \tfrac{0}{1.1}\,q_3$$

where $\mathbb{Q} = (q_1, q_2, q_3) \in \mathbb{M}$ is any martingale measure.

First method. From part (d), we obtain

$$\mathbb{E}_{\mathbb{Q}}\left(\frac{(S_1-4)^+}{1+r}\right) = \tfrac{4}{1.1}\left(\tfrac{2\lambda}{3} - \tfrac{1}{5}\right) + \tfrac{1}{1.1}\left(-\tfrac{5\lambda}{3} + \tfrac{6}{5}\right) + \tfrac{0}{1.1}\,\lambda.$$

We thus see that the range of arbitrage prices for the call option coincides with the range of the linear function $f(\lambda) = 4/11 + (10/11)\lambda$ where $\lambda \in (3/10, 18/25)$. Since the range of values of f equals $(7/11, 56/55)$, we conclude that the arbitrage price of the option may take any value from the open interval $(7/11, 56/55)$.

Second method. This result can also be confirmed by noting that

$$\mathbb{E}_{\mathbb{Q}}\left(\frac{(S_1-4)^+}{1+r}\right) = \alpha\,\mathbb{E}_{\mathbb{Q}_1}\left(\frac{(S_1-4)^+}{1+r}\right) + (1-\alpha)\,\mathbb{E}_{\mathbb{Q}_2}\left(\frac{(S_1-4)^+}{1+r}\right)$$

where $\alpha \in (0,1)$ and $\mathbb{Q}_1 = (0.28, 0, 0.72)$ and $\mathbb{Q}_2 = (0, 0.7, 0.3)$. Since

$$\mathbb{E}_{\mathbb{Q}_1}\left(\frac{(S_1-4)^+}{1+r}\right) = \tfrac{4}{1.1}\,0.28 = \tfrac{56}{55}$$

and

$$\mathbb{E}_{\mathbb{Q}_2}\left(\frac{(S_1-4)^+}{1+r}\right) = \tfrac{1}{1.1}\,0.7 = \tfrac{7}{11}$$

we conclude once again that the range of arbitrage prices for the call option is the open interval $(7/11, 56/55)$.

(f) (MATH3975) Generally speaking, when a contingent claim is not attainable, then by the *superhedging price* for X we mean the minimal value of $x \in \mathbb{R}$ for which there exists $\varphi \in \mathbb{R}$ such that $V_1(x,\varphi)(\omega) \geq X(\omega)$ for every $\omega \in \Omega$. One can check that when a contingent claim X is attainable, then the superhedging price for X coincides with the arbitrage price for X, which is obtained through replication.

In our case, we already know that the claim $X = C_T = (S_1 - 4)^+$ is not attainable and the superhedging conditions for X read:

$$1.1\,x + 3.6\,\varphi_1 \geq 4, \tag{1}$$
$$1.1\,x + 0.6\,\varphi_1 \geq 1, \tag{2}$$
$$1.1\,x - 1.4\,\varphi_1 \geq 0. \tag{3}$$

It is now sufficient to sketch solutions (half-planes) for the above inequalities and find the corner point in the feasible region with the lowest x. It appears to be the intersect of lines corresponding to equations (1) and (3). Observe that the second inequality is strict in that case. This means that an arbitrage opportunity will arise for the seller if she is able to sell the option at the superhedging price computed below and ω_2 occurs at time $T = 1$.

We now proceed to explicit computations. By solving equations corresponding to inequalities (1) and (3), that is,

$$\begin{aligned} 1.1\,x + 3.6\,\varphi_1 &= 4, \\ 1.1\,x - 1.4\,\varphi_1 &= 0, \end{aligned}$$

we obtain $x = 56/55$ and $\varphi = 4/5$. Then inequality (2) is strict, namely,

$$1.1\,x + 0.6\,\varphi_1 = 1.1\,\tfrac{56}{55} + 0.6\,\tfrac{4}{5} > 1.$$

The minimal superhedging price and the corresponding hedge ratio are thus given by $(x, \varphi) = (56/55, 4/5)$. It is worth noting that the minimal superhedging price is equal to the upper bound for the arbitrage price of the option computed in part (e), that is,

$$\min \left\{ x \in \mathbb{R} \,|\, V_1(x, \varphi) \geq C_T \text{ for some } \varphi \in \mathbb{R} \right\} = \sup_{\mathbb{Q} \in \mathbb{M}} \mathbb{E}_{\mathbb{Q}} \left(\frac{C_T}{1+r} \right). \tag{4}$$

In fact, it is possible to show that equality (4) is valid for an arbitrary contingent claim X and not only for the claim $X = C_T = (S_1 - 4)^+$.

Exercise 2 (a) A martingale measure $\mathbb{Q}$ satisfies: $q_1 + q_2 + q_3 = 1$, $0 < q_i < 1$ and

$$\mathbb{E}_{\mathbb{Q}}(\widehat{S}_1) = \mathbb{E}_{\mathbb{Q}} \left(\frac{S_1}{B_1} \right) = \mathbb{E}_{\mathbb{Q}} \left(\frac{S_1}{1.1} \right) = 7q_1 + 5q_2 + 4q_3 = S_0 = 5$$

or, equivalently, $\mathbb{E}_{\mathbb{Q}}(\widehat{S}_1 - S_0) = 2q_1 - q_3 = 0$. Let $q_3 = \alpha$. Then

$$q_1 = \frac{\alpha}{2} = \frac{1}{3}\,(1-\gamma), \quad q_2 = 1 - \frac{3}{2}\alpha = \gamma, \quad q_3 = \alpha = \frac{2}{3}\,(1-\gamma)$$

where $0 < \alpha < 2/3$ (or, equivalently, $\gamma \in (0,1)$). We obtain

$$\mathbb{M} = \left\{ (q_1, q_2, q_3) \,\middle|\, q_1 = \frac{\alpha}{2},\ q_2 = 1 - \frac{3}{2}\alpha,\ q_3 = \alpha,\ 0 < \alpha < 2/3 \right\}$$

and thus the market model $\mathcal{M}$ is not complete.

(b) It suffices to solve the following equations

$$\begin{aligned} 1.1\varphi^0 + 7.7\,\varphi^1 &= 5.5, \\ 1.1\varphi^0 + 5.5\,\varphi^1 &= 3.3, \\ 1.1\varphi^0 + 4.4\,\varphi^1 &= 2.2. \end{aligned}$$

It is clear that the strategy $(\varphi^0, \varphi^1) = (-2, 1)$ replicates X and thus X is attainable. The price of X at time $t = 0$ thus equals $\pi_0(X) = V_0(\varphi) = \varphi^0 + \varphi^1 S_0 = -2 + 1 \cdot 5 = 3$.

(c) For an arbitrary $0 < \alpha < 2/3$, we obtain (recall that $B_1 = 1.1$)

$$\mathbb{E}_{\mathbb{Q}} \left(\frac{Y}{B_1} \right) = (1.1)^{-1} \big(3\,\frac{\alpha}{2} + 1 \left(1 - \frac{3}{2}\alpha \right) + 0\alpha \big) = (1.1)^{-1}.$$

Hence the claim Y is attainable.

(d) For $0 < \alpha < 2/3$, we obtain (recall that $B_1 = 1.1$)

$$\mathbb{E}_{\mathbb{Q}}\left(\frac{Z}{B_1}\right) = 4\,\frac{\alpha}{2} - 3\alpha = -\alpha.$$

Hence the range of prices is the interval $(-2/3, 0)$ and thus the claim Z is not attainable.

(e) If $\pi_0(Y) = -0.5$ then $\alpha = 0.5$ and thus the unique martingale measure for the extended model $\widetilde{\mathcal{M}} = (B, S^1, S^2)$ equals.

$$\widetilde{\mathbb{Q}} = \big(\widetilde{\mathbb{Q}}(\omega_1), \widetilde{\mathbb{Q}}(\omega_2), \widetilde{\mathbb{Q}}(\omega_3)\big) = (1/4, 1/4, 1/2).$$

The martingale measure for $\widetilde{\mathcal{M}}$ is unique and thus the model $\widetilde{\mathcal{M}} = (B, S^1, S^2)$ is complete.

Exercise 3 (MATH3975) (a) Consider the relative wealth $\widehat{V}(\varphi)$, which is given by

$$\widehat{V}_t(\varphi) := \frac{V_t(\varphi)}{S_t^2}, \quad t = 0, 1.$$

For any trading strategy $\varphi = (\varphi_0^1, \varphi_0^2) \in \mathbb{R}^2$ with $V_0(\varphi) = 0$, we obtain

$$\begin{aligned} \widehat{V}_1(\varphi) &= \widehat{V}_1(\varphi) - \widehat{V}_0(\varphi) = \frac{V_1(\varphi)}{S_1^2} - \frac{V_0(\varphi)}{S_0^2} \\ &= \frac{\varphi_0^1 S_1^1 + \varphi_0^2 S_1^2}{S_1^2} - \frac{\varphi_0^1 S_0^1 + \varphi_0^2 S_0^2}{S_0^2} = \varphi_0^1\left(\frac{S_1^1}{S_1^2} - \frac{S_0^1}{S_0^2}\right). \end{aligned}$$

More explicitly

$$\widehat{V}_1(\varphi)(\omega_1) = \varphi_0^1\left(\frac{s_1}{z_1} - \frac{s_0}{z_0}\right),$$
$$\widehat{V}_1(\varphi)(\omega_2) = \varphi_0^1\left(\frac{s_2}{z_2} - \frac{s_0}{z_0}\right),$$

where φ_0^1 is an arbitrary real number. By standard arguments, we obtain the following necessary and sufficient conditions for the arbitrage-free property of the model $\mathcal{M} = (S^1, S^2)$

$$(A) \quad \frac{s_1}{z_1} < \frac{s_0}{z_0} < \frac{s_2}{z_2} \quad \text{or} \quad (B) \quad \frac{s_2}{z_2} < \frac{s_0}{z_0} < \frac{s_1}{z_1} \quad \text{or} \quad (C) \quad \frac{s_1}{z_1} = \frac{s_0}{z_0} = \frac{s_2}{z_2}.$$

(b) Let us now examine the completeness of the model. To this end, we ask whether the system of equations

$$V_1(\varphi)(\omega_1) = \varphi_0^1 s_1 + \varphi_0^2 z_1 = X_1,$$
$$V_1(\varphi)(\omega_2) = \varphi_0^1 s_2 + \varphi_0^2 z_2 = X_2,$$

has a solution $\varphi = (\varphi_0^1, \varphi_0^2) \in \mathbb{R}^2$ for any claim $X = (X_1, X_2) \in \mathbb{R}^2$.

This holds if and only if the determinant $\Delta := s_1 z_2 - z_1 s_2 \neq 0$, that is, whenever the vectors (s_1, s_2) and (z_1, z_2) are not collinear. Observe that conditions (A) and (B) imply that $\Delta \neq 0$. Hence under either (A) or (B) the model is arbitrage-free and complete.

It is clear that if (C) holds then $\Delta = 0$ and thus the model is still arbitrage-free, but it is incomplete. We also observe that if $\Delta = 0$ then the model is incomplete and it is arbitrage-free whenever

$$\frac{s_1}{z_1} = \frac{s_0}{z_0} = \frac{s_2}{z_2}.$$

(c) We need to find a vector $\varphi = (\varphi_0^1, \varphi_0^2) \in \mathbb{R}^2$ such that

$$\begin{aligned}\varphi_0^1 s_1 + \varphi_0^2 z_1 &= (s_1 - z_1)^+,\\ \varphi_0^1 s_2 + \varphi_0^2 z_2 &= (s_2 - z_2)^+.\end{aligned}$$

Complete case. We first assume that either (A) or (B) holds, so that the determinant $\Delta := s_1 z_2 - z_1 s_2 \neq 0$. Hence the model is arbitrage-free and complete. Then the unique replicating strategy $(\varphi_0^1, \varphi_0^2)$ can be easily computed for any contingent claim $X = (X_1, X_2)$. Specifically, the unique solution to equations

$$\begin{aligned}\varphi_0^1 s_1 + \varphi_0^2 z_1 &= X_1,\\ \varphi_0^1 s_2 + \varphi_0^2 z_2 &= X_2,\end{aligned}$$

reads

$$\varphi_0^1 = \frac{X_1 z_2 - X_2 z_1}{s_1 z_2 - z_1 s_2}, \quad \varphi_0^2 = \frac{X_1 s_2 - X_2 s_1}{s_1 z_2 - z_1 s_2}.$$

The arbitrage price of X at time 0 equals

$$\pi_0(X) = \varphi_0^1 s_0 + \varphi_0^2 z_0$$

or, more explicitly,

$$\pi_0(X) = \frac{X_1 z_2 - X_2 z_1}{s_1 z_2 - z_1 s_2}\, s_0 + \frac{X_1 s_2 - X_2 s_1}{s_1 z_2 - z_1 s_2}\, z_0$$

Of course, this formula can be applied to the claim

$$X = (S_1^1 - S_1^2)^+ = \big((s_1 - z_1)^+,\, (s_2 - z_2)^+\big).$$

Incomplete case. Assume now that condition (C) is satisfied so that the determinant $\Delta := s_1 z_2 - z_1 s_2 = 0$. Hence the model is arbitrage-free and incomplete. It is now clear that $(s_1, s_2) = \lambda(z_1, z_2)$ for some strictly positive real number λ. Consequently, the system of equations for the replicating strategy $\varphi = (\varphi_0^1, \varphi_0^2)$ becomes

$$\begin{aligned}\varphi_0^1 \lambda z_1 + \varphi_0^2 z_1 &= (\lambda z_1 - z_1)^+ = z_1(\lambda - 1)^+,\\ \varphi_0^1 \lambda z_2 + \varphi_0^2 z_2 &= (\lambda z_2 - z_2)^+ = z_2(\lambda - 1)^+.\end{aligned}$$

- Let us first assume that $0 < \lambda \leq 1$. Then $\varphi = (\varphi_0^1, \varphi_0^2)$ solves

  $$\varphi_0^1 \lambda + \varphi_0^2 = 0.$$

  Hence $\varphi = (\varphi_0^1, -\lambda\varphi_0^1)$ for any real number φ_0^1. Note that, in view of condition (C), we have that $s_0 = \lambda z_0$ as well. We conclude that for $0 < \lambda \leq 1$ the price equals

  $$\pi_0(X) = \varphi_0^1 s_0 - \lambda \varphi_0^1 z_0 = 0.$$

- Assume now that $\lambda > 1$. Then $\varphi = (\varphi_0^1, \varphi_0^2)$ solves

$$\varphi_0^1 \lambda + \varphi_0^2 = \lambda - 1 > 0.$$

Hence $\varphi = (\varphi_0^1, \lambda - 1 - \lambda\varphi_0^1)$ for any real number φ_0^1. We conclude that for $\lambda > 1$ the price satisfies

$$\pi_0(X) = \varphi_0^1 s_0 - (\lambda - 1 - \lambda\varphi_0^1) z_0 = (\lambda - 1) z_0.$$

To summarise, the unique price of X in an arbitrage-free and incomplete model equals $\pi_0(X) = (\lambda - 1)^+ z_0$ where $\lambda = s_0/z_0$ or, simply, $\pi_0(X) = (s_0 - z_0)^+$.

(d) We note that

$$X - Y = (S_1^1 - S_1^2)^+ - (S_1^2 - S_1^1)^+ = S_1^1 - S_1^2$$

and thus, by the linearity of the arbitrage price map, we obtain

$$\pi_0(X) - \pi_0(Y) = \pi_0(X - Y) = \pi_0(S_1^1 - S_1^2) = \pi_0(S_1^1) - \pi_0(S_1^2) = S_0^1 - S_0^2 = s_0 - z_0.$$

Hence the put-call parity relationship reads

$$\pi_0(X) - \pi_0(Y) = s_0 - z_0.$$