# 8: THE BLACK-SCHOLES MODEL

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# Outline

We will examine the following issues:

1. The Wiener Process and its Properties
2. The Black-Scholes Market Model
3. The Black-Scholes Call Option Pricing Formula
4. The Black-Scholes Partial Differential Equation
5. Implied Volatility Surface
6. Random Walk Approximations

## PART 1

## THE WIENER PROCESS AND ITS PROPERTIES

# The Origin of the Wiener Process

- The **Brownian motion** is a mathematical model used to describe the random movements of particles. It was named after Scottish botanist Robert Brown (1773-1858) who published in 1827 a paper in which he examined the chaotic movements of pollen suspended in water.
- The Brownian motion was used by Louis Bachelier in his PhD thesis, which was completed in 1900 and devoted to pricing of options.
- The Brownian motion was also used by physicists to describe the diffusion movements of particles, in particular, by Albert Einstein (1879-1955) in his celebrated paper published in 1905.
- The Brownian motion is also known as the **Wiener process** in honour of the famous American mathematician Norbert Wiener (1894-1964).
- The Brownian motion is nowadays widely used to model uncertainty in engineering, economics and finance.

# Wiener Process: Definition

## Definition (Wiener Process)

A stochastic process $W = (W_t,\, t \in \mathbb{R}_+)$ is called the **Wiener process** (or the **standard Brownian motion**) if the following conditions hold:

1. $W_0 = 0$,
2. sample paths of the process $W$, that is, the maps $t \to W_t(\omega)$ are continuous functions,
3. the process $W$ has the Gaussian (i.e., normal) distribution with the expected value $\mathbb{E}_{\mathbb{P}}(W_t) = 0$ for all $t \geq 0$ and the covariance

$$\mathit{Cov}\,(W_s, W_t) = \min\,(s,t), \quad s,t \geq 0.$$

# Wiener Process: Equivalent Definition

## Definition (Wiener Process: Equivalent Definition)

A stochastic process $W = (W_t,\, t \in \mathbb{R}_+)$ on $\Omega$ is called the **Wiener process** if the following conditions hold:

1. $W_0 = 0$.
2. sample paths of $W$ are continuous functions,
3. for any $0 \leq s < t$, $W_t - W_s$ is normally distributed with the mean 0 and the variance $t - s$,
4. for any $0 \leq t_1 < t_2 < \cdots < t_n$,

$$W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$$

are mutually independent.

# Existence of the Wiener Process

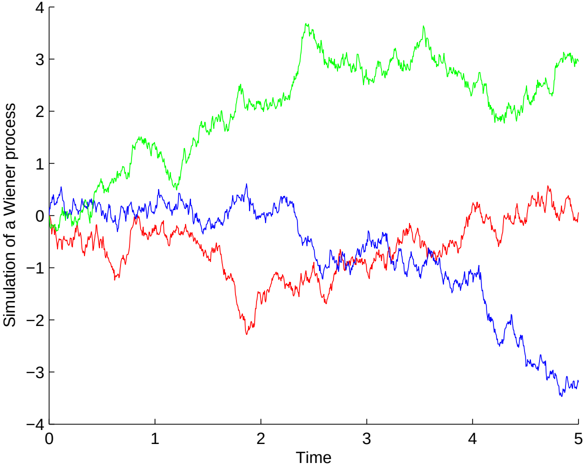
- The existence of a stochastic process satisfying the definition of a Wiener process is not obvious.
- The following theorem was first rigorously established by Norbert Wiener in his paper published in 1923.

**Theorem (Wiener (1923))**

*There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a process $W$ defined on this space, such that conditions 1)-3) of the definition of the Wiener process are met.*

- It is known that almost all sample paths of the Wiener process are continuous functions of the time parameter, but they are non-differentiable everywhere. This striking feature makes the Wiener process rather difficult to analyse.

# Sample Paths of the Wiener Process



# Gaussian Distribution

## Remark (Gaussian Distribution)

- *We say that $X$ has the* **Gaussian (normal) distribution** *with the mean $\mu \in \mathbb{R}$ and the variance $\sigma^2 > 0$ if its pdf equals*

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for } x \in \mathbb{R}.$$

  *We write $X \sim N(\mu, \sigma^2)$.*

- *One can show that*

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = 1.$$

- *We have*

$$\mathbb{E}_{\mathbb{P}}(X) = \mu \quad \text{and} \quad Var(X) = \sigma^2.$$

# Standard Normal Distribution

## Remark (Standard Normal Distribution)

- *If we set* $\mu = 0$ *and* $\sigma^2 = 1$ *then we obtain the* **standard normal distribution** $N(0,1)$ *with the following pdf*

$$n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{for } x \in \mathbb{R}.$$

- *The cdf of the probability distribution* $N(0,1)$ *equals*

$$N(x) = \int_{-\infty}^{x} n(u)\, du = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du \quad \text{for } x \in \mathbb{R}.$$

- *The values of* $N(x)$ *can be found in the* **cumulative standard normal table** *(also known as the* **Z table***).*
- *If* $X \sim N(\mu, \sigma^2)$ *then* $Z := \frac{X-\mu}{\sigma} \sim N(0,1)$.

# Marginal Distributions of the Wiener Process

- Let $N(\mu, \sigma^2)$ denote the Gaussian (normal) distribution with mean $\mu$ and variance $\sigma^2$.
- For any $t > 0$, $W_t \sim N(0, t)$ and thus $(\sqrt{t})^{-1} W_t \sim N(0, 1)$.
- The random variable $W_t$ has the pdf $p(t, x)$ given by

$$p(t, x) = \frac{1}{\sqrt{2\pi t}} \, e^{-x^2/2t}, \quad \text{for } x \in \mathbb{R}.$$

- Hence for any real numbers $a \leq b$

$$\begin{aligned} \mathbb{P}(W_t \in [a, b]) &= \int_a^b \frac{1}{\sqrt{2\pi t}} \, e^{-x^2/2t} \, dx = \int_{\frac{a}{\sqrt{t}}}^{\frac{b}{\sqrt{t}}} \frac{1}{\sqrt{2\pi}} \, e^{-x^2/2} \, dx \\ &= \int_{\frac{a}{\sqrt{t}}}^{\frac{b}{\sqrt{t}}} n(x) \, dx = N\left(\frac{b}{\sqrt{t}}\right) - N\left(\frac{a}{\sqrt{t}}\right). \end{aligned}$$

# Markov Property (MATH3975)

## Proposition (8.1)

*The Wiener process* $W$ *is a Markov process in the following sense: for every* $n \geq 1$*, any sequence of times* $0 < t_1 < \ldots < t_n < t$ *and any real numbers* $x_1, \ldots, x_n$*, the following holds for all* $x \in \mathbb{R}$

$$\mathbb{P}\left(W_t \leq x \middle| W_{t_1} = x_1, \ldots, W_{t_n} = x_n\right) = \mathbb{P}\left(W_t \leq x \middle| W_{t_n} = x_n\right).$$

*Moreover, for all* $s < t$ *and* $x, y \in \mathbb{R}$ *we have*

$$\mathbb{P}\left(W_t \leq y \middle| W_s = x\right) = \int_{-\infty}^{y} p(t-s, z-x)\, dz$$

*where*

$$p(t-s, z-x) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(z-x)^2}{2(t-s)}\right)$$

*is the transition probability density function of the Wiener process.*

# Martingale Property (MATH3975)

## Proposition (8.2)

*Let $W$ be the Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the process $W$ is a martingale with respect to its natural filtration $\mathcal{F}_t = \mathcal{F}_t^W$, that is, the filtration generated by $W$.*

## Proof of Proposition 8.2.

For all $0 \leq s < t$, using the independence of increments of the Wiener process $W$, we obtain

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}(W_t \,|\, \mathcal{F}_s) &= \mathbb{E}_{\mathbb{P}}\big((W_t - W_s) + W_s \,|\, \mathcal{F}_s\big) \\ &= \mathbb{E}_{\mathbb{P}}(W_t - W_s) + W_s \\ &= W_s.\end{aligned}$$

Hence $W$ is a martingale with respect to the filtration $\mathbb{F}$. $\square$

# PART 2

# THE BLACK-SCHOLES MARKET MODEL

# The Black-Scholes Market

Assumptions of the Black-Scholes market:

- There are no arbitrage opportunities in the class of trading strategies.
- It is possible to either borrow or lend any amount of cash at a constant continuously compounded interest rate $r$.
- The stock price dynamics are governed by a geometric Brownian motion (aka continuous-time random walk).
- It is possible to purchase any amount of a stock and short-selling of shares is allowed without restrictions.
- The market is frictionless: there are no transaction costs or any other costs, such as, e.g., capital gains tax (CGT).
- The underlying stock does not pay any dividends.

# Practice: shareholding as investment

A **shareholder** is a person who holds shares for the purpose of earning income from dividends and similar receipts. For a shareholder:

- the cost of purchase of shares is not an allowable deduction against current year income, but is a capital cost,
- receipts from the sale of shares are not assessable income  but any capital gain on the shares is subject to capital gains tax,
- a net capital loss from the sale of shares can't be offset against income from other sources, but can be offset against another capital gain or carried forward to offset against future capital gains,
- the transaction costs is not an allowable deduction against income, but are taken into account in determining the capital gain,
- dividends and other similar receipts from the shares are included in assessable income,
- costs such as interest on borrowed money are an allowable deduction against current year income.

# Practice: share trading as business

A **share trader** is a person who carries out business activities for the purpose of earning income from buying and selling shares.

For a share trader:

- receipts from the sale of shares constitute assessable income,
- purchased shares are regarded as trading stock,
- costs incurred in buying or selling shares – including the cost of the shares – are an allowable deduction in the year in which they are incurred,
- dividends and other similar receipts are included in assessable income.

The question of whether a person is a share trader or a shareholder is determined by considering: the nature of the activities, the volume and regularity of the activities, a registered business name and an Australian business number (ABN), the amount of capital invested.

# Stock Price Process

- We note that the values of the Wiener process $W$ can be negative and thus it cannot be used to directly model the movements of the stock price.
- As in papers by Samuelson (1965) and Black and Scholes (1973), we postulate that the stock price process $S$ is governed under the risk-neutral probability measure $\widetilde{\mathbb{P}}$ by the **stochastic differential equation (SDE)**
$$dS_t = r\, S_t\, dt + \sigma S_t\, dW_t \tag{1}$$
with a constant initial value $S_0 > 0$.
- The term $\sigma S_t\, dW_t$ gives a plausible description of the uncertainty of the stock price.
- The **volatility** parameter $\sigma > 0$ is used to control the size of random fluctuations of the stock price.

# Stochastic Differential Equation

- Sample path of the Wiener process $W$ are not differentiable so that equation (1) cannot be represented as

$$dS_t = r\, S_t\, dt + \sigma S_t W_t'\, dt.$$

- It should be understood as the **stochastic integral equation**

$$S_t = S_0 + \int_0^t r S_u\, du + \int_0^t \sigma S_u\, dW_u$$

  where the second integral is the Itō stochastic integral.

- The Itō stochastic integration theory, which extends the classic integrals and underpins financial modelling in continuous time, is beyond the scope of this course.

# The Black-Scholes Model $\mathcal{M} = (B, S)$

- It turns out that stochastic differential equation (1) can be solved explicitly yielding the unique solution

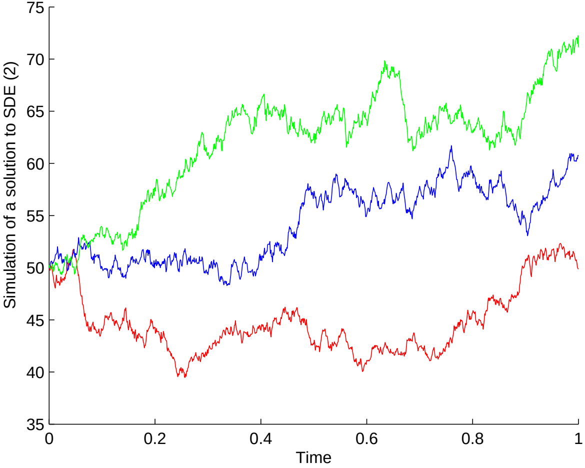
$$S_t = S_0 \exp\left(\sigma W_t + \left(r - \frac{1}{2}\sigma^2\right)t\right). \tag{2}$$

  The process $S$ is called the **geometric Brownian motion**.
- Observe that $S_t$ has the **lognormal** distribution for every $t > 0$.
- It can be shown that $S$ is a Markov process. Note, however, that $S$ is not a process of independent increments.
- We assume that the continuously compounded interest rate $r$ is constant. Hence the **savings account** $B$ satisfies

$$B_t = B_0 e^{rt}, \quad t \geq 0,$$

  where $B_0 = 1$ and thus $dB_t = rB_t \, dt$ for $t \geq 0$.

# Sample Paths of the Stock Price



# Discounted Stock Price (MATH3975)

As in a multi-period market, the discounted stock price $\hat{S}$ is a martingale.

**Proposition (8.3)**

*The discounted stock price, that is, the process $\hat{S}$ given by the formula*

$$\hat{S}_t = \frac{S_t}{B_t} = e^{-rt} S_t$$

*is a martingale with respect to its natural filtration under $\widetilde{\mathbb{P}}$. This means that for every $0 \leq s \leq t$*

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\big(\hat{S}_t \,\big|\, \hat{S}_u,\, u \leq s\big) = \hat{S}_s.$$

# Proof of Proposition 8.3 (MATH3975)

## Proof of Proposition 8.3.

- We observe that equality (2) yields

$$\hat{S}_t = S_0\, e^{\sigma W_t - \frac{1}{2}\sigma^2 t} = \hat{S}_s\, e^{\sigma (W_t - W_s) - \frac{1}{2}\sigma^2 (t-s)}. \tag{3}$$

- Hence if we know $\hat{S}_t$ then we also know the value of $W_t$ and vice versa. This immediately implies that $\mathbb{F}^{\hat{S}} = \mathbb{F}^W$.
- Therefore, the following conditional expectations coincide

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\big(X \,\big|\, \hat{S}_u,\, u \le s\big) = \mathbb{E}_{\widetilde{\mathbb{P}}}\big(\, X \,|\, W_u,\, u \le s\big) \tag{4}$$

for any integrable random variable $X$

□

# Proof of Proposition 8.3 (MATH3975)

## Proof of Proposition 8.3 (Continued).

- We obtain the following chain of equalities

$$\begin{aligned}
&\mathbb{E}_{\widetilde{\mathbb{P}}}\big(\hat{S}_t \,\big|\, \hat{S}_u,\, u \le s\big) \\
&= \mathbb{E}_{\widetilde{\mathbb{P}}}\left(\hat{S}_s\, e^{\sigma\left(W_t - W_s - \frac{1}{2}\sigma^2(t-s)\right)} \,\Big|\, \hat{S}_u,\, u \le s\right) && \text{(from (3))} \\
&= \hat{S}_s\, e^{-\frac{1}{2}\sigma^2(t-s)}\, \mathbb{E}_{\widetilde{\mathbb{P}}}\left(e^{\sigma(W_t - W_s)} \,\Big|\, \hat{S}_u,\, u \le s\right) && \text{(conditioning)} \\
&= \hat{S}_s\, e^{-\frac{1}{2}\sigma^2(t-s)}\, \mathbb{E}_{\widetilde{\mathbb{P}}}\left(e^{\sigma(W_t - W_s)} \,\Big|\, W_u,\, u \le s\right) && \text{(from (4))} \\
&= \hat{S}_s\, e^{-\frac{1}{2}\sigma^2(t-s)}\, \mathbb{E}_{\widetilde{\mathbb{P}}}\left(e^{\sigma(W_t - W_s)}\right). && \text{(independence)}
\end{aligned}$$

- It remains to compute the expected value above.

□

# Proof of Proposition 8.3 (MATH3975)

## Proof of Proposition 8.3 (Continued).

- Recall also that $W_t - W_s = \sqrt{t-s}\, Z$ where $Z \sim N(0,1)$, and thus

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\big(\hat{S}_t \,\big|\, \hat{S}_u,\, u \le s\big) = \hat{S}_s\, e^{-\frac{1}{2}\sigma^2(t-s)}\, \mathbb{E}_{\widetilde{\mathbb{P}}}\big(e^{\sigma\sqrt{t-s}Z}\big).$$

- Let us finally observe that if $Z \sim N(0,1)$ then for any real $a$

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\big(e^{aZ}\big) = e^{a^2/2}.$$

- By setting $a = \sigma\sqrt{t-s}$, we finally obtain

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\big(\hat{S}_t \,\big|\, \hat{S}_u,\, u \le s\big) = \hat{S}_s\, e^{-\frac{1}{2}\sigma^2(t-s)}\, e^{\frac{1}{2}\sigma^2(t-s)} = \hat{S}_s$$

which shows that $\hat{S}$ is indeed a martingale.

$\square$

## PART 3

## THE BLACK-SCHOLES CALL OPTION PRICING FORMULA

# Call Option

- Recall that the **European call option** written on the stock $S$ pays at its maturity $T$ the random amount

$$C_T = (S_T - K)^+$$

where $K > 0$ is a fixed strike price.

- We take for granted that for $t \leq T$ the price $C_t(s)$ of the call option when $S_t = s$ is given by the **risk-neutral pricing formula**

$$C_t(s) = e^{-r(T-t)}\, \mathbb{E}_{\widetilde{\mathbb{P}}} \left( (S_T - K)^+ \big|\, S_t = s \right).$$

- The RNV formula can be supported by the replication principle. However, this requires the knowledge of the Itō **stochastic integration** theory with respect to the Wiener process, which was developed in 1944 by Japanese mathematician Kiyosi Itō (1915-2008).

# The Black-Scholes Call Pricing Formula

- The following call option pricing result was established by Fischer Black (1938-1995) and Myron Scholes (1941–) in their seminal paper published in 1973 in *Journal of Political Economy*.

**Theorem (8.1)**

*The arbitrage price* $C_t = C_t(S_t)$ *of the call option at time* $t \leq T$ *equals*

$$C_t = S_t\, N\big(d_+(S_t, T-t)\big) - K e^{-r(T-t)} N\big(d_-(S_t, T-t)\big)$$

*where*

$$d_\pm(S_t, T-t) = \frac{\ln \frac{S_t}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

*and* $N$ *is the standard normal cumulative distribution function.*

# Proof of Theorem 8.1 (MATH3975)

## Proof of Theorem 8.1.

- Our goal is to compute the conditional expectation

$$C_t(s) = e^{-r(T-t)}\,\mathbb{E}_{\widetilde{\mathbb{P}}}\left( (S_T - K)^+ \,\middle|\, S_t = s \right).$$

- We can represent the stock price $S_T$ as follows

$$S_T = S_t\, e^{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(W_T - W_t)}.$$

- As in the proof of Proposition 8.3, we write

$$W_T - W_t = \sqrt{T-t}Z$$

where $Z$ has the standard Gaussian probability distribution, that is, $Z \sim N(0,1)$.

□

# Proof of Theorem 8.1 (MATH3975)

## Proof of Theorem 8.1 (Continued).

- Using the independence of increments of the Wiener process $W$, for a generic value $s > 0$ of the stock price $S_t$ at time $t$, we get

$$
\begin{aligned}
C_t(s) &= e^{-r(T-t)}\,\mathbb{E}_{\widetilde{\mathbb{P}}}\left(\left(S_t\, e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma(W_T-W_t)} - K\right)^+ \,\Big|\, S_t = s\right) \\
&= e^{-r(T-t)}\,\mathbb{E}_{\widetilde{\mathbb{P}}}\left(s\, e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}Z} - K\right)^+ \\
&= e^{-r(T-t)}\int_{-\infty}^{\infty}\left(s\, e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}z} - K\right)^+ n(z)\,dz
\end{aligned}
$$

where $n$ is the pdf of $Z$, that is, the standard normal pdf. $\square$

# Proof of Theorem 8.1 (MATH3975)

## Proof of Theorem 8.1 (Continued).

- It is clear that the function under the integral sign is non-zero if and only if the following inequality holds

$$s\,e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}z} - K \geq 0.$$

- This in turn is equivalent to the following inequality

$$z \geq \frac{\ln\frac{K}{s} - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} = -d_-(s, T-t).$$

- Note that

$$d_-(s, T-t) = d_+(s, T-t) - \sigma\sqrt{T-t}.$$

- We shall write $d_+ = d_+(s, T-t)$ and $d_- = d_-(s, T-t)$.

□

## Proof of Theorem 8.1 (Continued).

$$
\begin{aligned}
C_t(s) &= e^{-r(T-t)} \int_{-d_-}^{\infty} \left( s\, e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}z} - K \right) n(z)\, dz \\
&= s\, e^{-\frac{1}{2}\sigma^2(T-t)} \int_{-d_-}^{\infty} e^{\sigma\sqrt{T-t}z} n(z)\, dz - K e^{-r(T-t)} \int_{-d_-}^{\infty} n(z)\, dz \\
&= s\, e^{-\frac{1}{2}\sigma^2(T-t)} \int_{-d_-}^{\infty} e^{\sigma\sqrt{T-t}z} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2}\, dz - K e^{-r(T-t)} N\left(d_-\right) \\
&= s \int_{-d_-}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(z-\sigma\sqrt{T-t}\right)^2}\, dz - K e^{-r(T-t)} N\left(d_-\right) \\
&= s \int_{-d_- -\sigma\sqrt{T-t}}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-u^2/2}\, du - K e^{-r(T-t)} N\left(d_-\right) \\
&= s \int_{-d_+}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-u^2/2}\, du - K e^{-r(T-t)} N\left(d_-\right) \\
&= sN\left(d_+\right) - K e^{-r(T-t)} N\left(d_-\right).
\end{aligned}
$$

# Put-Call Parity

- The price of the put option can be computed from the put-call parity

$$C_t - P_t = S_t - Ke^{-r(T-t)} = S_t - KB(t,T).$$

- It is easy to check that the put option price equals

$$P_t = Ke^{-r(T-t)} N\big(-d_-(S_t, T-t)\big) - S_t\, N\big(-d_+(S_t, T-t)\big).$$

- It is worth noting that $C_t > 0$ and $P_t > 0$.
- One can check that the prices of call and put options are increasing functions of the volatility $\sigma$ (all other quantities being fixed). Hence options become more expensive when the underlying stock becomes more risky.
- The price of a call (put) option is an increasing (decreasing) function of the interest rate $r$.

# Forward Prices

Let $F_S(t,T)$ and $F_C(t,T)$ be the forward prices of the stock and the call option

$$F_S(t,T) = \frac{S_t}{B(t,T)} = S_t e^{r(T-t)}, \quad F_C(t,T) = \frac{C_t}{B(t,T)} = C_t e^{r(T-t)}.$$

## Corollary

*The forward price $F_C(t,T)$ of the call option at time $t \le T$ equals*

$$F_C(t,T) = F_S(t,T)\, N\big(\widetilde{d}_+(F_S(t,T), T-t)\big) - KN\big(\widetilde{d}_-(F_S(t,T), T-t)\big)$$

*where*

$$\widetilde{d}_\pm(F, T-t) = \frac{\ln \frac{F}{K} \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

# Example A: Call Option

## Example (Price of the call option)

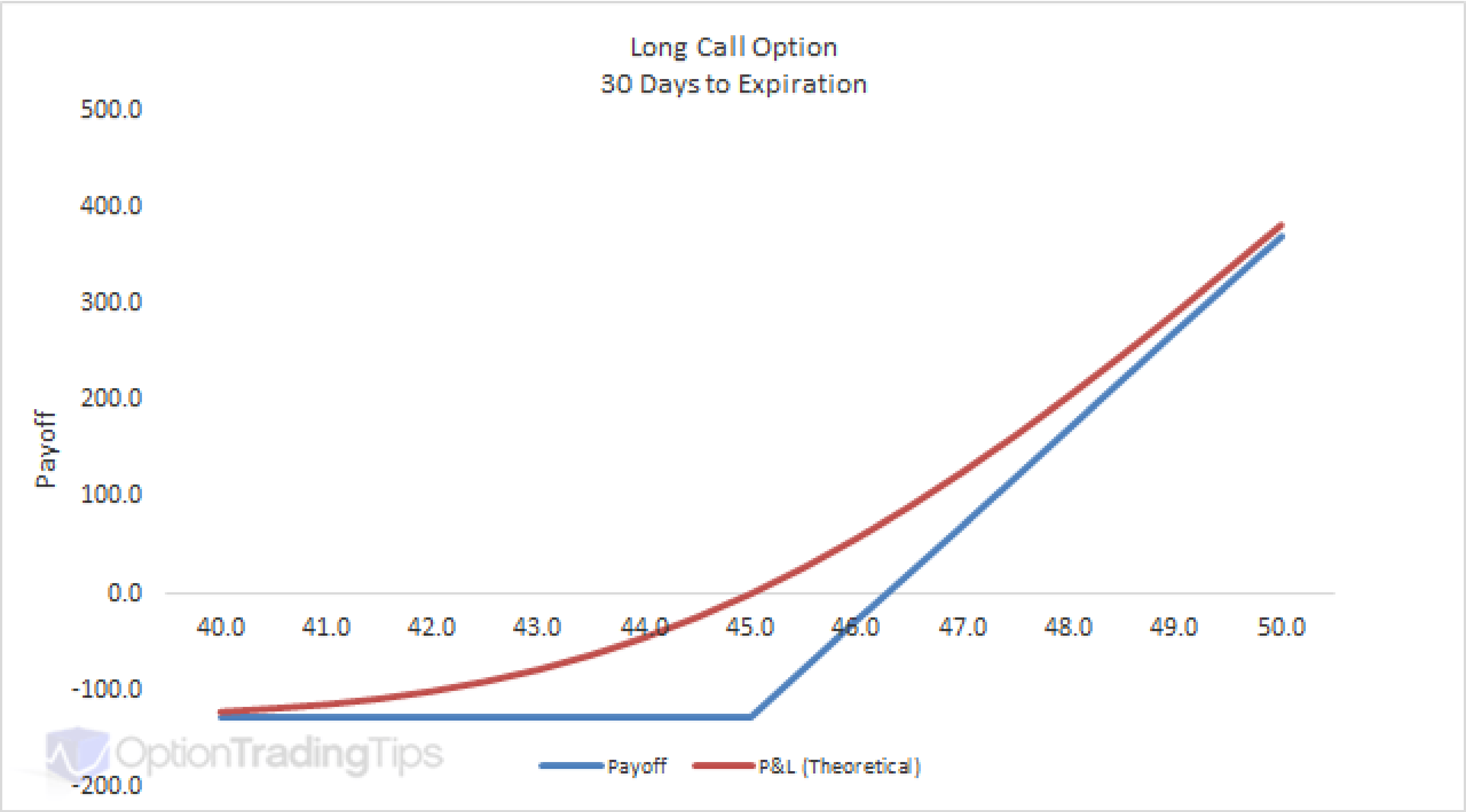
- Suppose that the current stock price equals \$31, the stock price volatility is $\sigma = 10\%$ per annum, and the risk-free rate is $r = 5\%$ per annum with continuous compounding.
- Consider a call option $S$ with strike \$30 and 3 months to expiry. We may assume that $t = 0$ and $T = 0.25$. We get $d_+(S_0, T) = 0.93$ and thus

$$d_-(S_0, T) = d_+(S_0, T) - \sigma\sqrt{T} = 0.88.$$

- The Black-Scholes call option pricing formula yields (approximately)

$$C_0 = 31N(0.93) - 30e^{-0.05/4}N(0.88) = 25.42 - 23.9 = 1.52$$

since $N(0.93) \approx 0.82$ and $N(0.88) \approx 0.81$.


Long Call Option
30 Days to Expiration
500.0
400.0
300.0
200.0
100.0
0.0
-100.0
-200.0
Payoff
40.0
41.0
42.0
43.0
44.0
45.0
46.0
47.0
48.0
49.0
50.0
Payoff
P&L (Theoretical)
OptionTradingTips

# Example A: Replicating Strategy

## Example (Replicating portfolio)

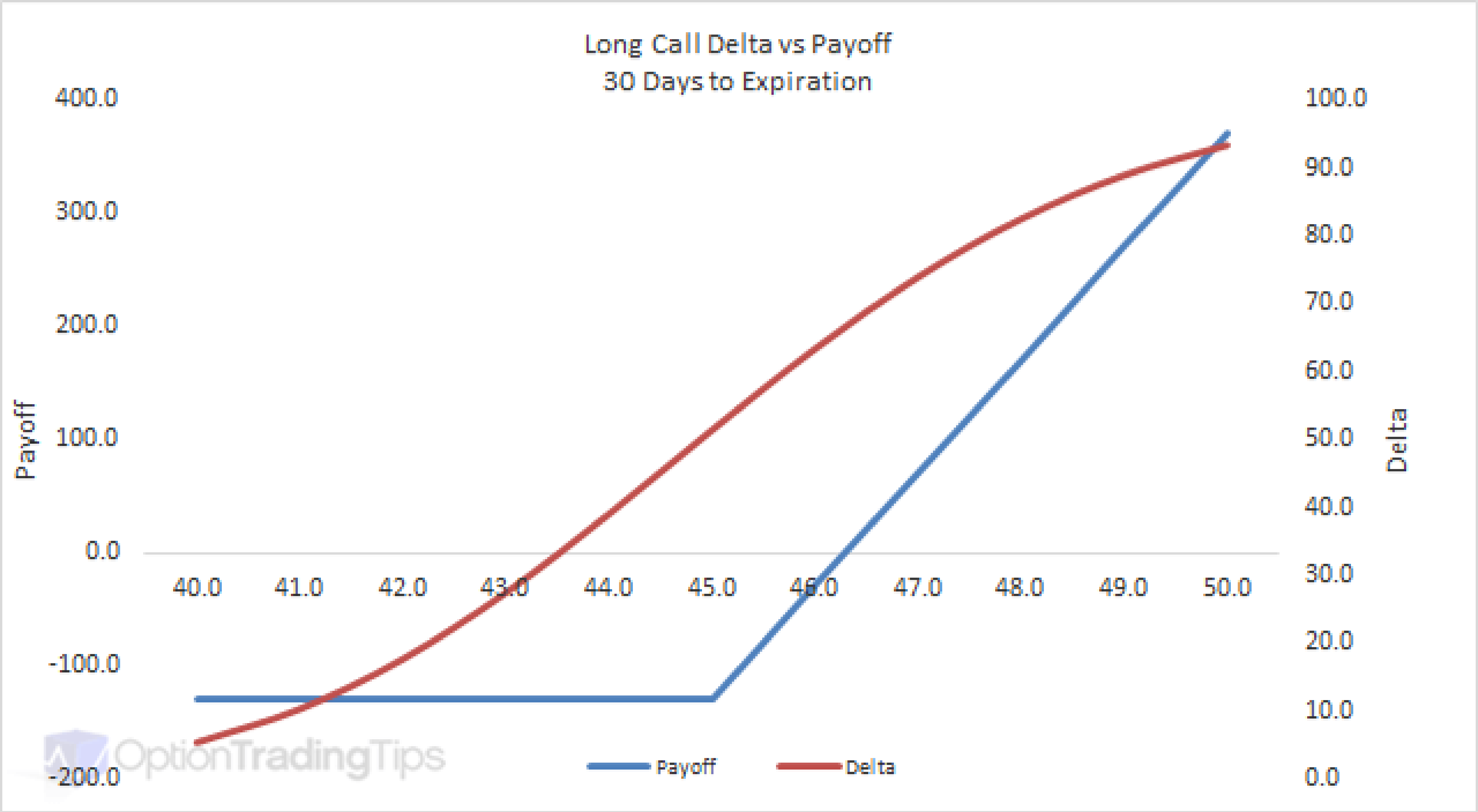
- Let $C_t = \phi_t^0 B_t + \phi_t^1 S_t$. The hedge ratio for the call option is known to be given by the formula

$$\phi_t^1 = N(d_+(S_t, T-t)).$$

- Hence the replicating portfolio at time $t = 0$ is given by

$$\phi_0^0 = -23.9, \quad \phi_0^1 = N(d_+(S_0, T)) = 0.82.$$

- This means that to hedge a short position in the call option, which was sold at the arbitrage price $C_0 = \$1.52$, the writer needs to buy at time $0$ the number $\delta = 0.82$ shares of stock.
- The purchase of shares requires an additional borrowing of \$23.9.


Long Call Delta vs Payoff
30 Days to Expiration
Payoff
Delta
400.0
300.0
200.0
100.0
0.0
-100.0
-200.0
100.0
90.0
80.0
70.0
60.0
50.0
40.0
30.0
20.0
10.0
0.0
40.0
41.0
42.0
43.0
44.0
45.0
46.0
47.0
48.0
49.0
50.0
Payoff
Delta
OptionTradingTips

# Example A: Riskiness of Options

## Example (Price elasticity of the call)

- The **elasticity** at time 0 of the call option price with respect to the stock price equals

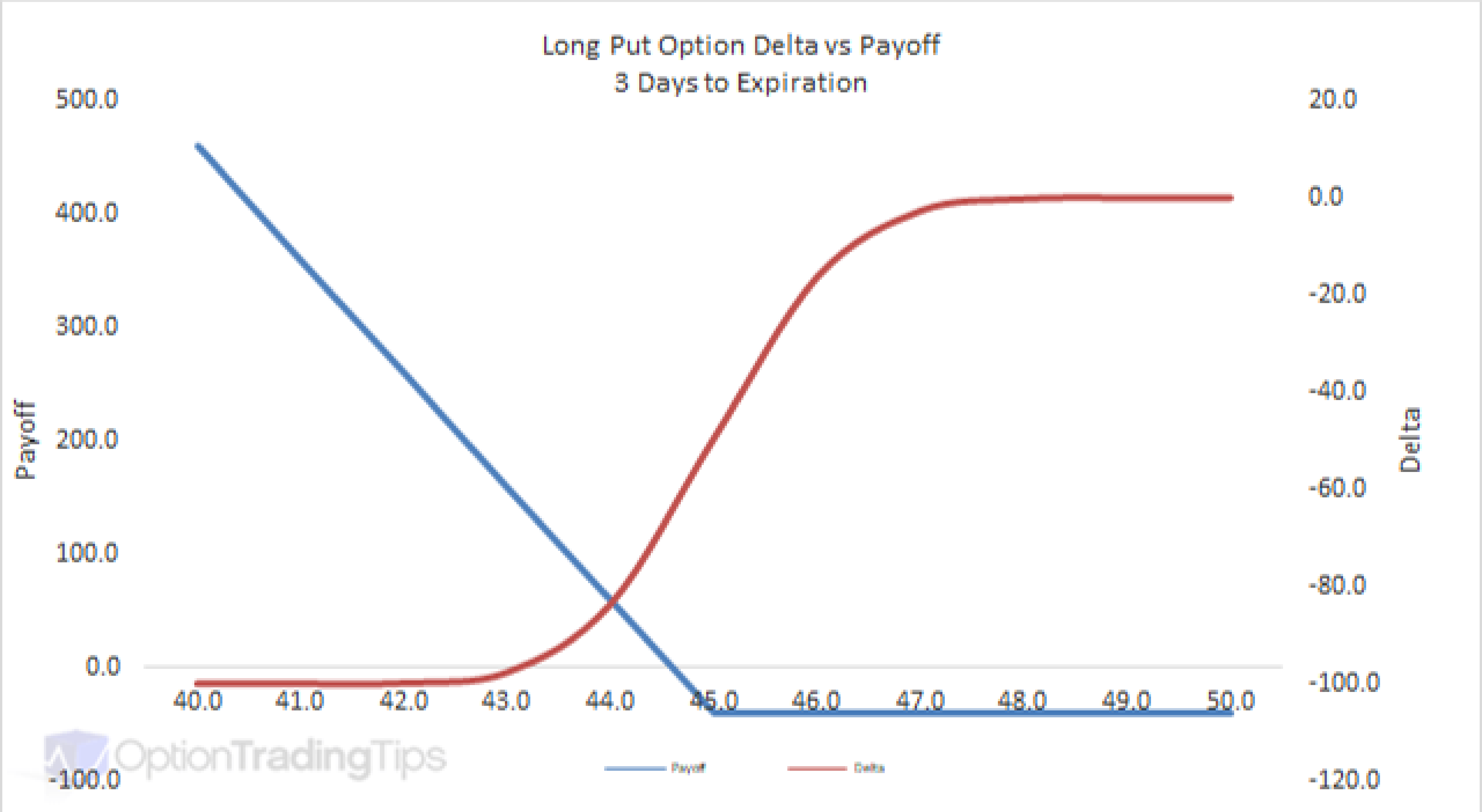
$$\eta_0^c := \frac{\partial C}{\partial S}\left(\frac{C_0}{S_0}\right)^{-1} = \frac{N\big(d_+(S_0,T)\big)S_0}{C_0} = 16.72.$$

- Suppose that the stock price rises immediately from \$31 to \$31.2 yielding the instantaneous return rate of 0.65%.
- Then the call price will move by approximately 16.5 cents from \$1.52 to \$1.685, giving the instantaneous return rate of 10.86%.
- The call option has nearly 17 times the return rate of the stock.
- This also means that it will drop 17 times as fast if the price of the stock decreases from \$31 to \$30.8,

# Example A: Riskiness of Options

## Example (Price elasticity of the put)

- The price of the put option at time 0 equals
$$P_0 = 30e^{-0.05/4}N(-0.88) - 31N(-0.93) = 5.73 - 5.58 = 0.15.$$
- The hedge ratio for the short position in the put option equals approximately $-0.18$ (since $N(-0.93) \approx 0.18$).
- To hedge the exposure the issuer needs to short $0.18$ shares of stock for one put option. The proceeds from the option and share-selling transactions (altogether \$5.73) are invested in risk-free bonds.
- Note that the elasticity of the put is here larger than the elasticity of the call. For instance, if the stock price rises immediately from \$31 to \$31.2, then the price of the put option will drop to less than 12 cents yielding the negative instantaneous return rate of $-20\%$.


Long Put Option Delta vs Payoff
3 Days to Expiration
500.0
400.0
300.0
200.0
100.0
0.0
-100.0
Payoff
20.0
0.0
-20.0
-40.0
-60.0
-80.0
-100.0
-120.0
Delta
40.0
41.0
42.0
43.0
44.0
45.0
46.0
47.0
48.0
49.0
50.0
OptionTradingTips


Option delta
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
Delta
$5
$6
$7
$8
$9
$10
$11
$12
$13
$14
$15
Δ = 0.637
Exercise $10
Δ = -0.363
Call
Put

## PART 5

## THE BLACK-SCHOLES PDE

# Black-Scholes Call Pricing Function

- We denote by $c(s,\tau)$ the function $c:\mathbb{R}_+\times[0,T]\to\mathbb{R}$ such that $C_t=c(S_t,T-t)$ and we assume that $r\geq 0$.
- Notice that $\tau$ stands here for the deterministic time parameter, which represents the length of the time period between the current date $t$ and the option's maturity date $T$.
- From Theorem 8.1, we know that $c(s,\tau)=c(s,\tau,K,\sigma,r)$ equals

$$c(s,\tau)=sN\big(d_+(s,\tau)\big)-Ke^{-r\tau}N\big(d_-(s,\tau)\big)$$

where

$$d_\pm(s,\tau)=\frac{\ln\frac{s}{K}+\left(r\pm\frac{1}{2}\sigma^2\right)(\tau)}{\sigma\sqrt{\tau}}$$

and $N$ is the standard Gaussian (i.e., normal) cumulative distribution function.

# Sensitivities of the Call Price

- Claim: the prices of call (put) options satisfy the Black-Scholes PDE.
- Let $c(s,\tau)$ stand for the function $c : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that $C_t = c(S_t, T-t)$. Then

$$
\begin{aligned}
c_s &= N(d_+) = \delta_c > 0, && (delta)\\
c_{ss} &= \frac{n(d_+)}{s\sigma\sqrt{\tau}} = \gamma_c > 0, && (gamma)\\
c_\tau &= \frac{s\sigma}{2\sqrt{\tau}}\, n(d_+) + Kre^{-r\tau}N(d_-) = \theta_c > 0, && (theta)\\
c_\sigma &= s\sqrt{\tau}n(d_+) = v_c > 0, && (vega)\\
c_r &= \tau K e^{-r\tau}N(d_-) = \rho_c > 0, && (rho)\\
c_K &= -e^{-r\tau}N(d_-) < 0,
\end{aligned}
$$

where $d_+ = d_+(s,\tau)$, $d_- = d_-(s,\tau)$ and $n$ is the standard Gaussian probability density function.

# Sensitivities of the Put Price

- We denote by $p(s,\tau)$ the function $p : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that $P_t = p(S_t, T-t)$. Then

$$\begin{aligned}
p_s &= N(d_+) - 1 = -N(-d_+) = \delta_p < 0, && (delta)\\
p_{ss} &= \frac{n(d_+)}{s\sigma\sqrt{\tau}} = \gamma_p > 0, && (gamma)\\
p_\tau &= \frac{s\sigma}{2\sqrt{\tau}}\, n(d_+) + Kre^{-r\tau}(N(d_-) - 1) = \theta_p > 0, && (theta)\\
p_\sigma &= s\sqrt{\tau}n(d_+) = \upsilon_p > 0, && (vega)\\
p_r &= \tau K e^{-r\tau}(N(d_-) - 1) = \rho_p < 0, && (rho)\\
p_K &= e^{-r\tau}(1 - N(d_-)) > 0,
\end{aligned}$$

where $d_+ = d_+(s,\tau)$, $d_- = d_-(s,\tau)$ and $n$ is the standard Gaussian probability density function.

## The Black-Scholes PDE

### Proposition (8.4)

*Consider a path-independent contingent claim* $X = g(S_T)$. *Denote by* $v(s,t)$ *the price of the claim* $X$ *at time* $t$ *given that* $S_t = s$. *Then* $v(s,t)$ *is a unique solution of the* **Black-Scholes partial differential equation**

$$\frac{\partial v}{\partial t}(s,t) + \frac{\sigma^2 s^2}{2}\frac{\partial^2 v}{\partial s^2}(s,t) + rs\,\frac{\partial v}{\partial s}(s,t) - rv(s,t) = 0$$

*with the terminal condition* $v(s,T) = g(s)$.

### Proof of Proposition 8.4.

The statement is an immediate consequence of the risk-neutral valuation formula and the Feynman-Kac theorem. □

# Feynman-Kac Theorem (MATH3975)

- The following result underpins the Monte Carlo method for solving parabolic PDEs.
- It was established by Richard Feynman and Mark Kac in 1949.

**Theorem**

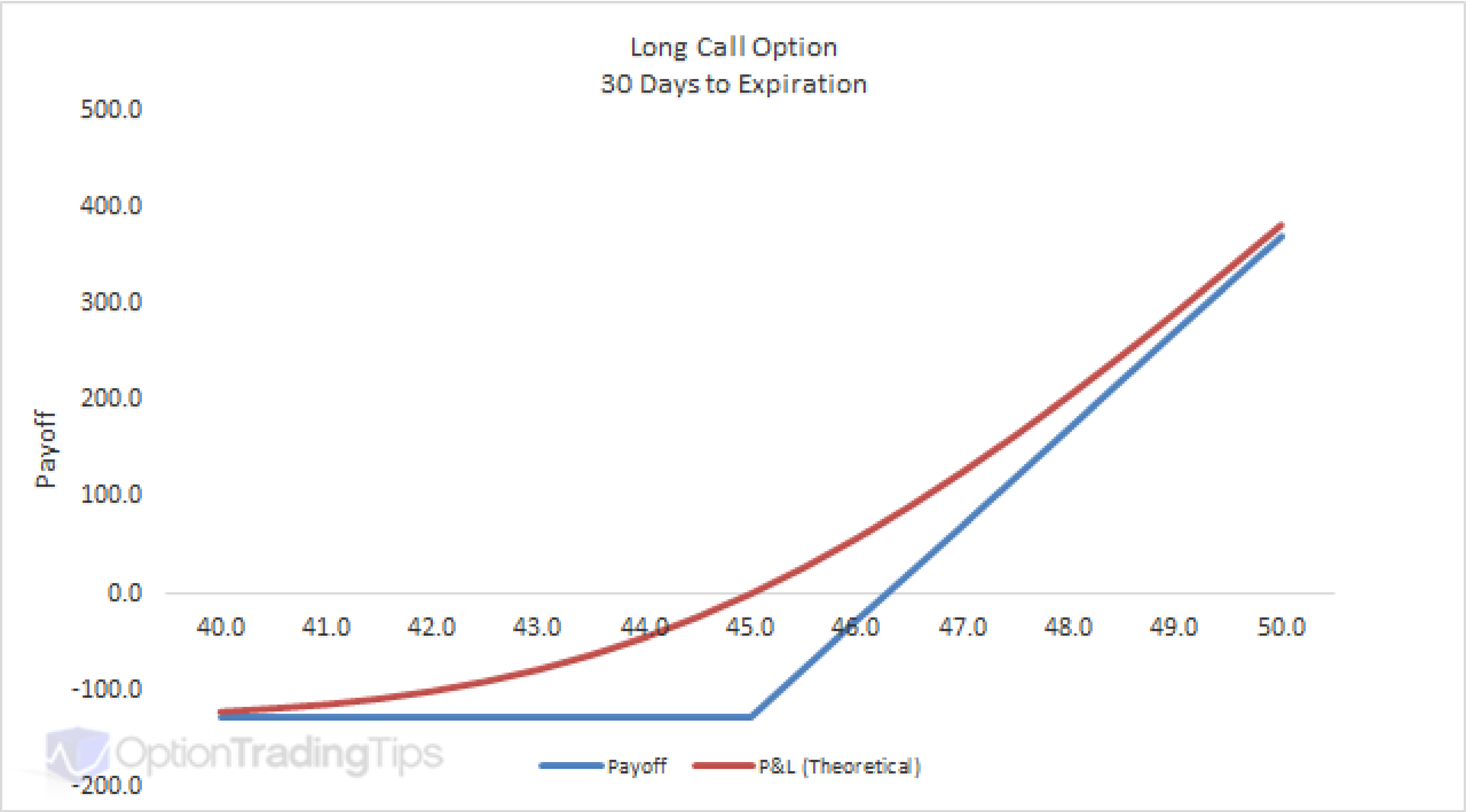
*Let* $g : \mathbb{R} \to \mathbb{R}$ *be a given function. For a Brownian motion* $W$ *we define*

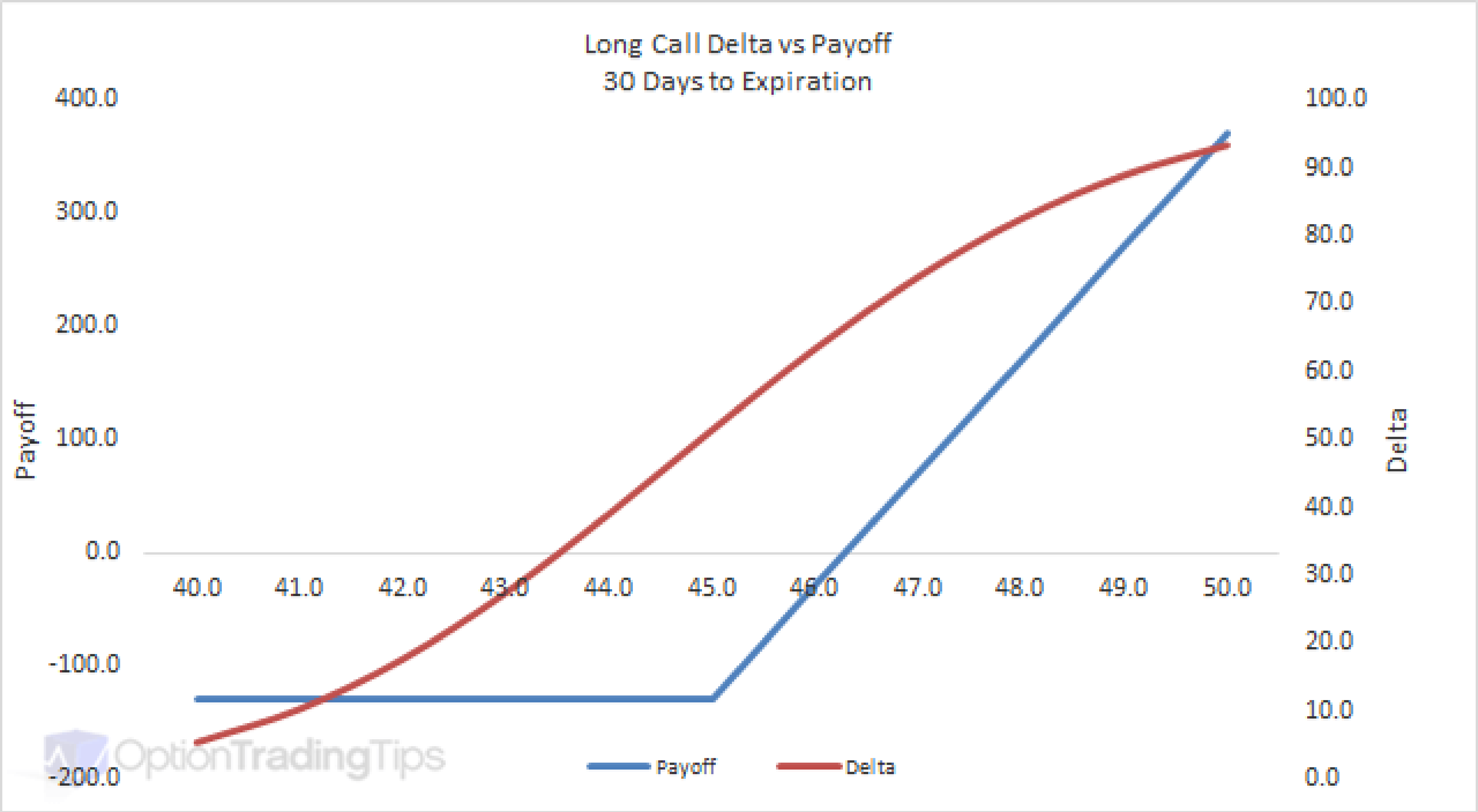
$$u(z,t) = e^{-r(T-t)} \mathbb{E}_{\mathbb{P}} \big( g(W_T) \,|\, W_t = z \big).$$

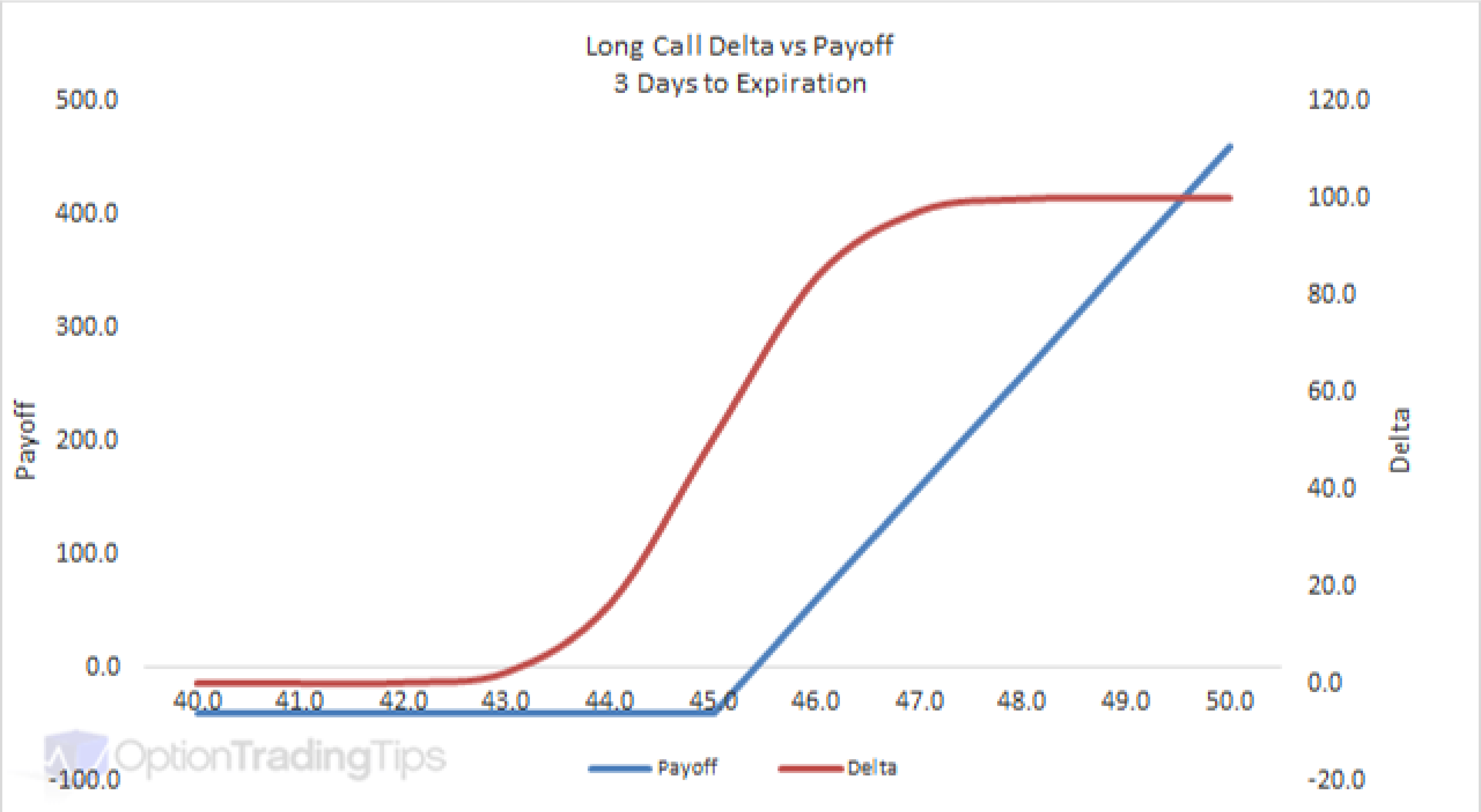
*Then* $u(z,t) : \mathbb{R} \times [0,T] \to \mathbb{R}$ *solves the parabolic partial differential equation*

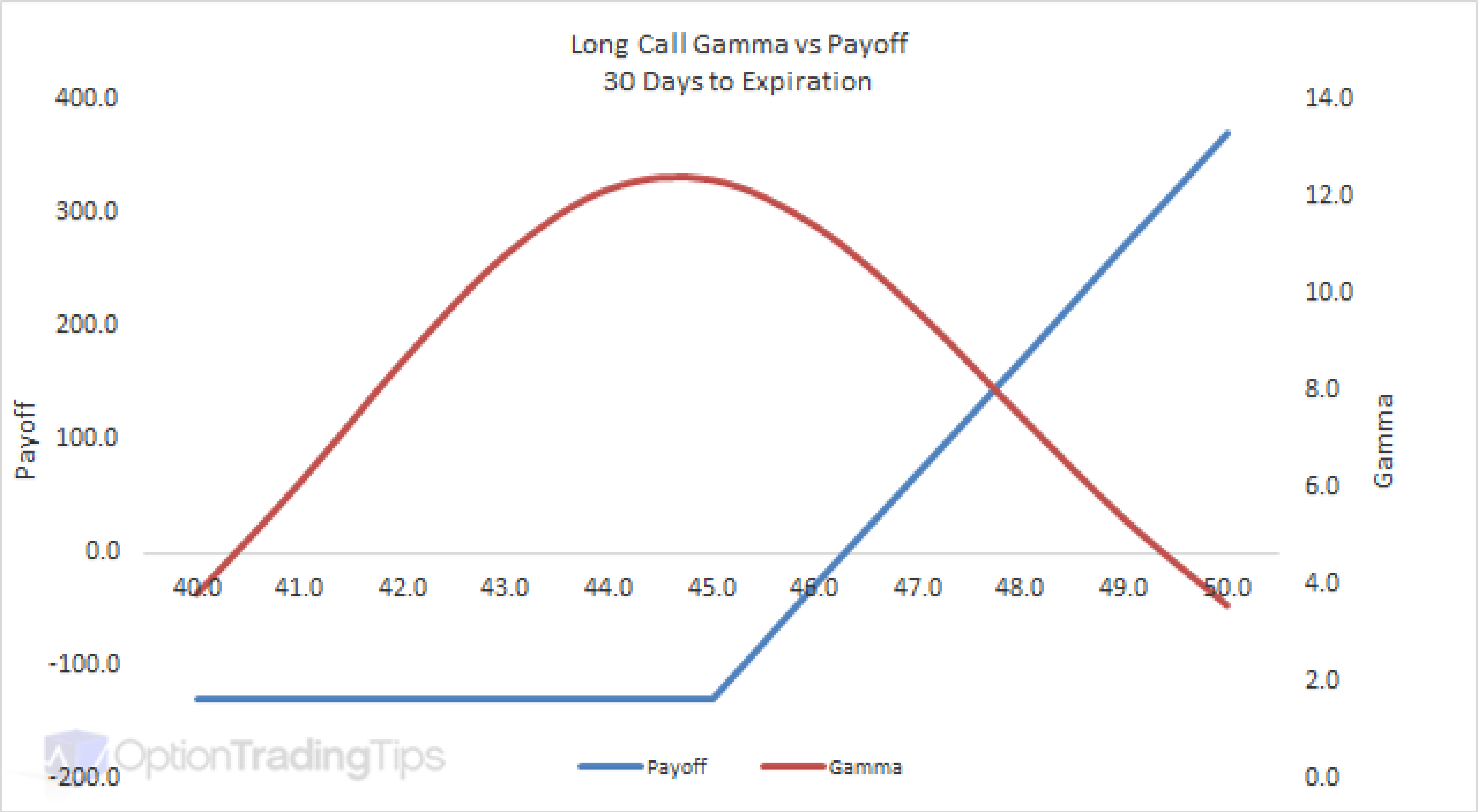
$$\frac{\partial u}{\partial t}(z,t) + \frac{1}{2} \frac{\partial^2 u}{\partial z^2}(z,t) - ru(z,t) = 0$$

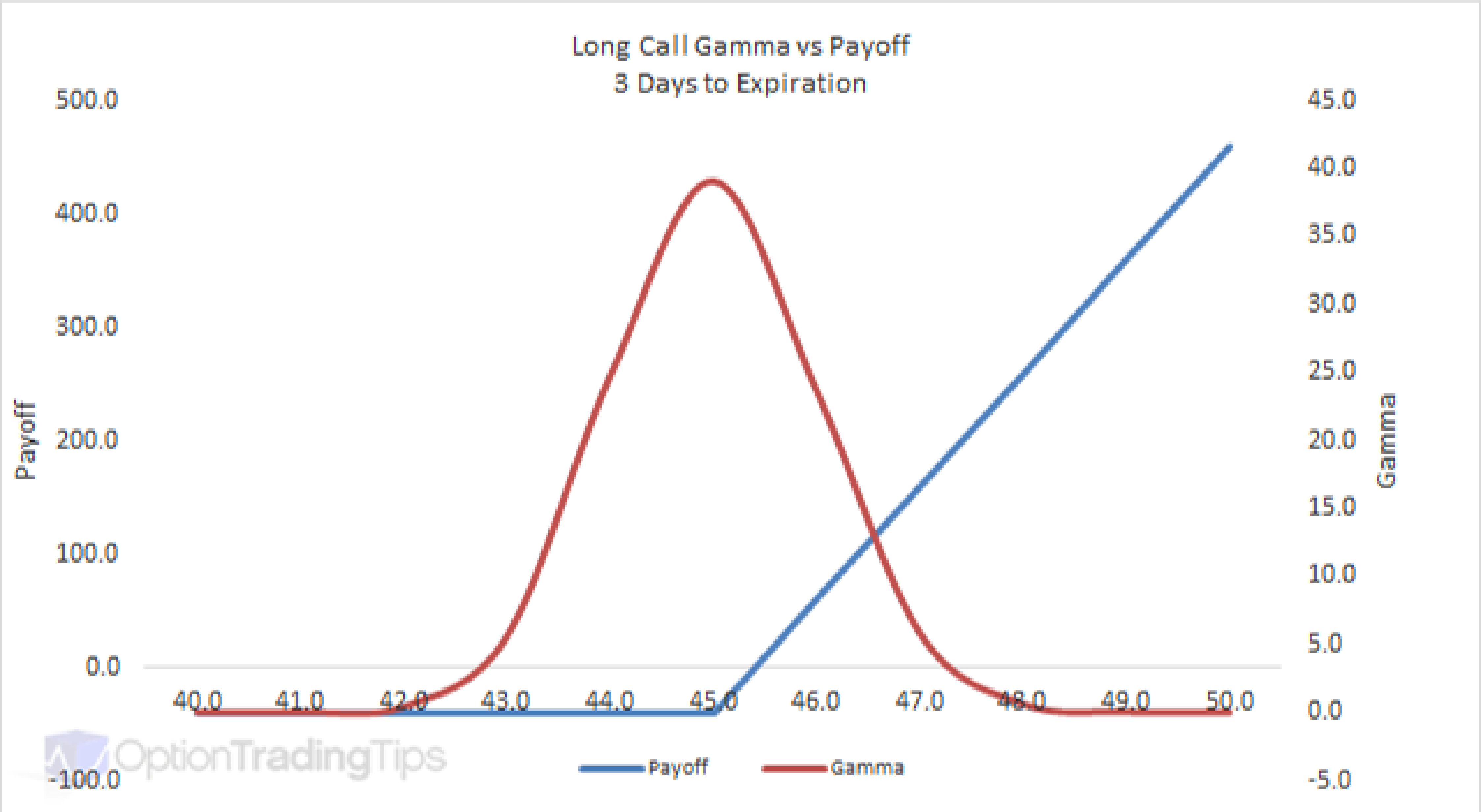
*with the terminal condition* $u(z,T) = g(z)$.


Long Call Option
30 Days to Expiration
Payoff
500.0
400.0
300.0
200.0
100.0
0.0
-100.0
-200.0
40.0
41.0
42.0
43.0
44.0
45.0
46.0
47.0
48.0
49.0
50.0
Payoff
P&L (Theoretical)
OptionTradingTips

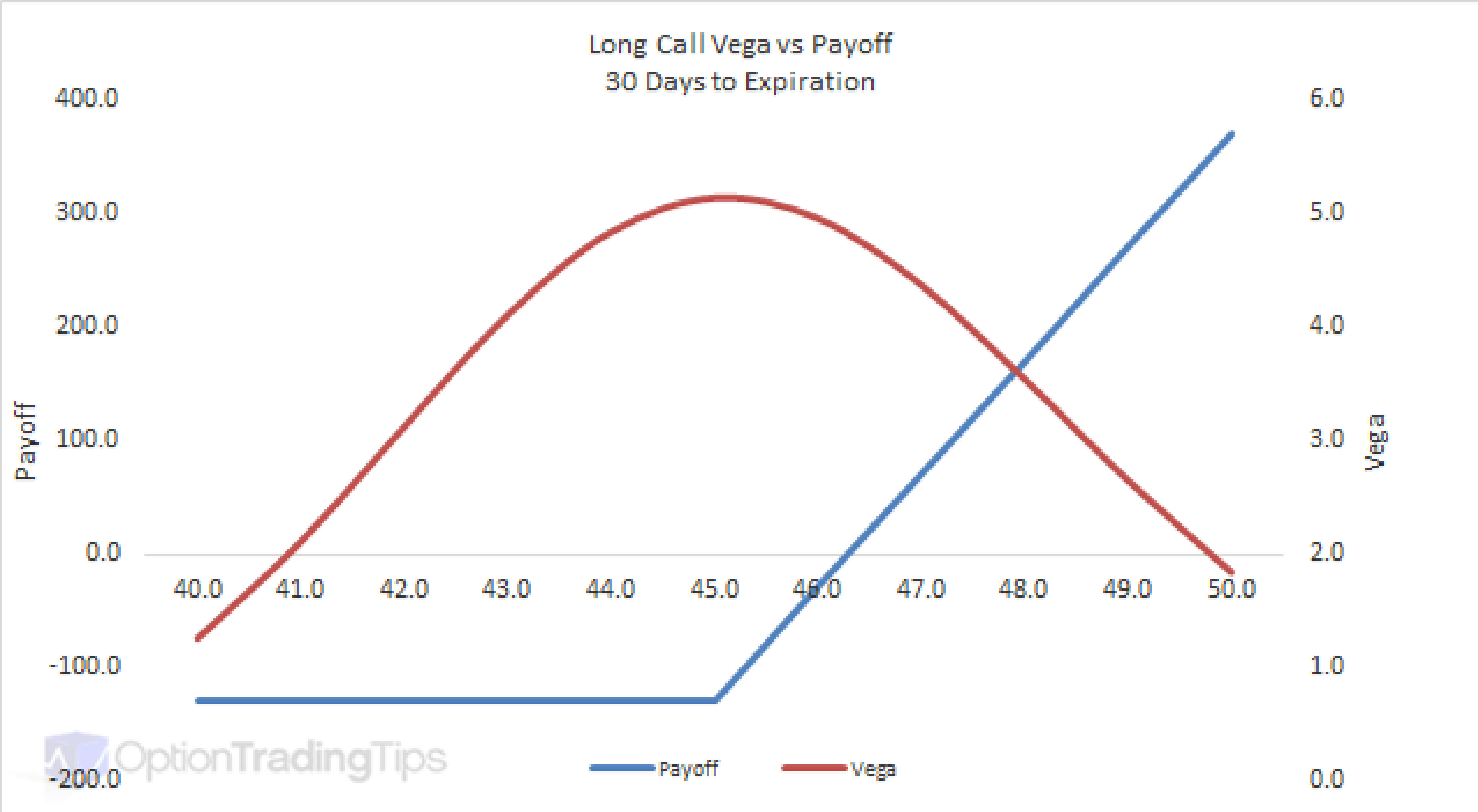

Long Call Delta vs Payoff
30 Days to Expiration
Payoff
400.0
300.0
200.0
100.0
0.0
-100.0
-200.0
Delta
100.0
90.0
80.0
70.0
60.0
50.0
40.0
30.0
20.0
10.0
0.0
40.0
41.0
42.0
43.0
44.0
45.0
46.0
47.0
48.0
49.0
50.0
Payoff
Delta
OptionTradingTips


Long Call Delta vs Payoff
3 Days to Expiration
500.0
400.0
300.0
200.0
100.0
0.0
-100.0
Payoff
120.0
100.0
80.0
60.0
40.0
20.0
0.0
-20.0
Delta
40.0
41.0
42.0
43.0
44.0
45.0
46.0
47.0
48.0
49.0
50.0
Payoff
Delta
OptionTradingTips

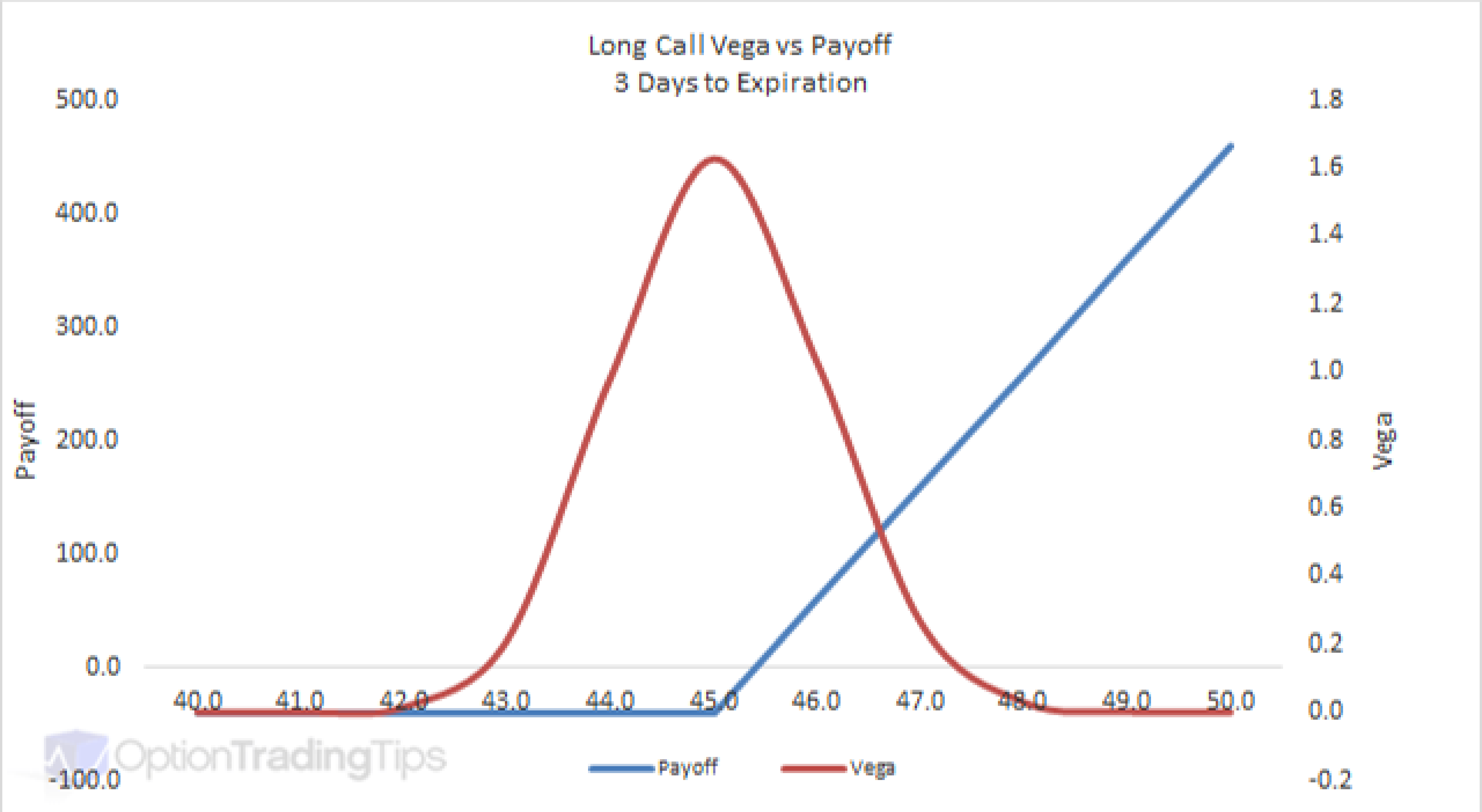

Long Call Gamma vs Payoff
30 Days to Expiration
Payoff
Gamma
400.0
300.0
200.0
100.0
0.0
-100.0
-200.0
14.0
12.0
10.0
8.0
6.0
4.0
2.0
0.0
40.0
41.0
42.0
43.0
44.0
45.0
46.0
47.0
48.0
49.0
50.0
Payoff
Gamma
OptionTradingTips

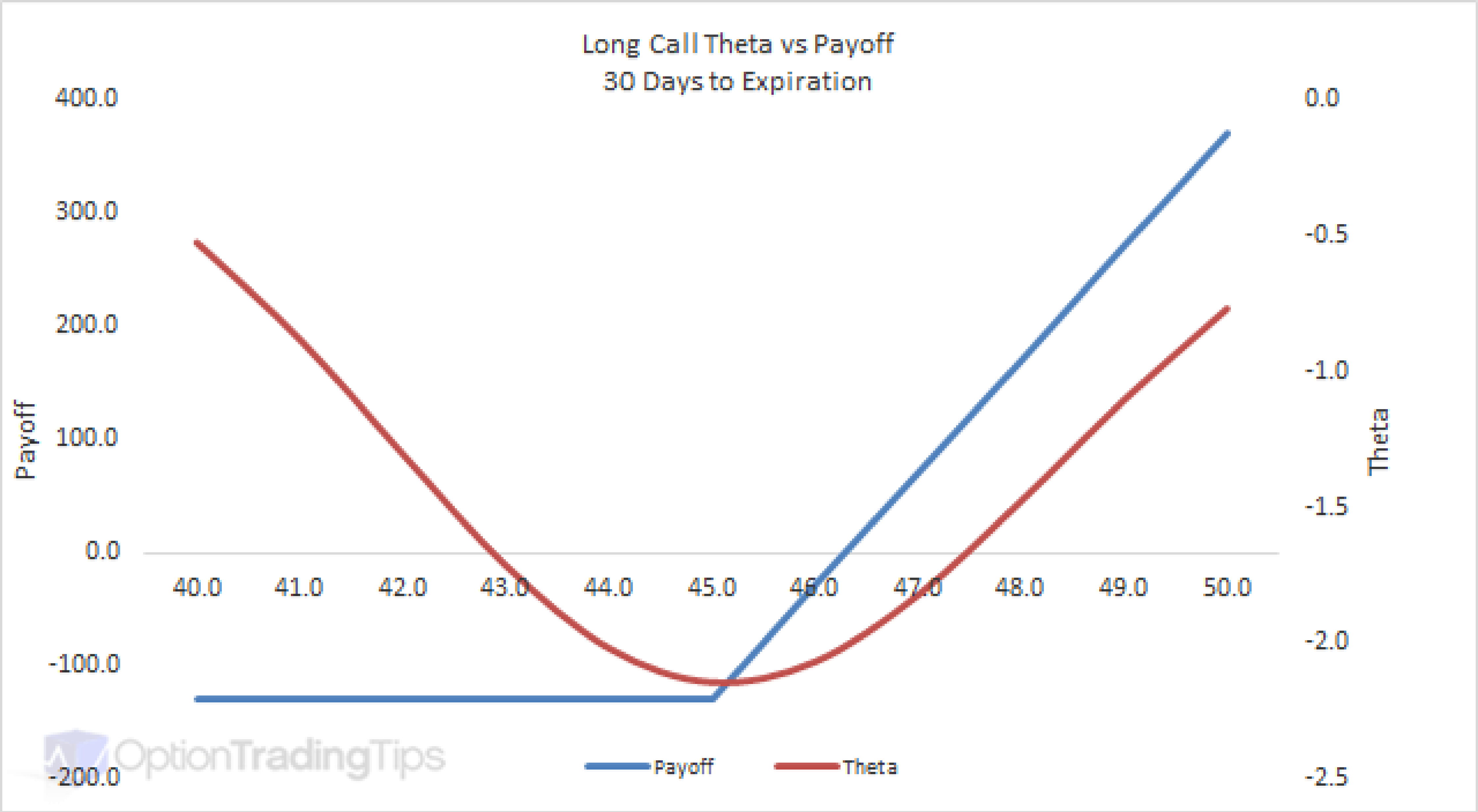


Long Call Gamma vs Payoff
3 Days to Expiration
Payoff
Gamma
500.0
400.0
300.0
200.0
100.0
0.0
-100.0
45.0
40.0
35.0
30.0
25.0
20.0
15.0
10.0
5.0
0.0
-5.0
40.0
41.0
42.0
43.0
44.0
45.0
46.0
47.0
48.0
49.0
50.0
Payoff
Gamma
OptionTradingTips

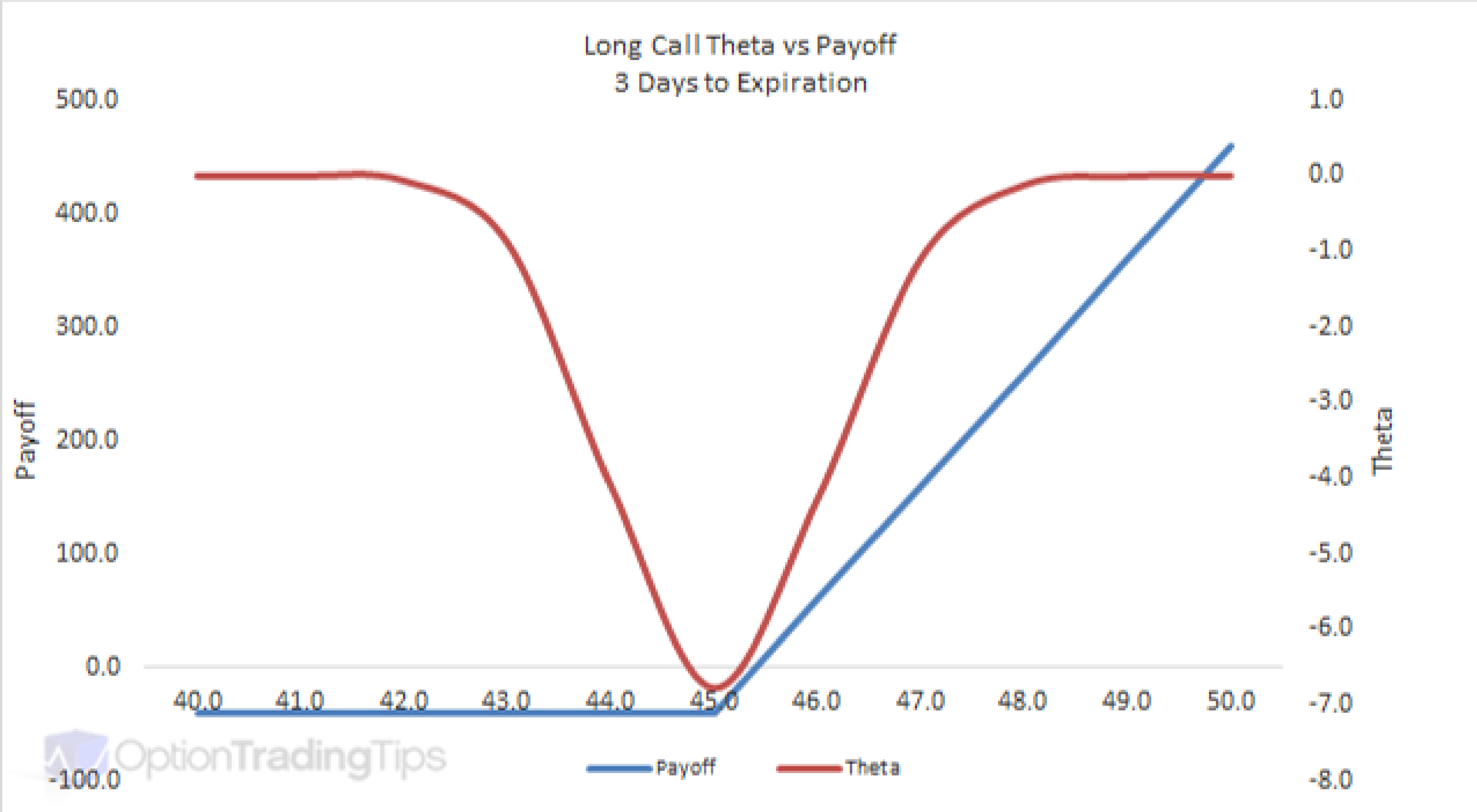


Long Call Vega vs Payoff
30 Days to Expiration
Payoff
Vega
400.0
300.0
200.0
100.0
0.0
-100.0
-200.0
6.0
5.0
4.0
3.0
2.0
1.0
0.0
40.0
41.0
42.0
43.0
44.0
45.0
46.0
47.0
48.0
49.0
50.0
Payoff
Vega
OptionTradingTips


Long Call Vega vs Payoff
3 Days to Expiration
Payoff
Vega
500.0
400.0
300.0
200.0
100.0
0.0
-100.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
40.0
41.0
42.0
43.0
44.0
45.0
46.0
47.0
48.0
49.0
50.0
Payoff
Vega
OptionTradingTips



Long Call Theta vs Payoff
30 Days to Expiration
400.0
300.0
200.0
100.0
0.0
-100.0
-200.0
Payoff
0.0
-0.5
-1.0
-1.5
-2.0
-2.5
Theta
40.0
41.0
42.0
43.0
44.0
45.0
46.0
47.0
48.0
49.0
50.0
Payoff
Theta
OptionTradingTips



Long Call Theta vs Payoff
3 Days to Expiration
Payoff
Theta
500.0
400.0
300.0
200.0
100.0
0.0
-100.0
1.0
0.0
-1.0
-2.0
-3.0
-4.0
-5.0
-6.0
-7.0
-8.0
40.0
41.0
42.0
43.0
44.0
45.0
46.0
47.0
48.0
49.0
50.0
Payoff
Theta
OptionTradingTips

## PART 6

## IMPLIED VOLATILITY SURFACE

# Black-Scholes Model: Volatility Parameter

- Suppose that the interest rate is non-negative.
- We already know that in any arbitrage-free model the price of a call option satisfies, for every maturity $T$ and strike $K$

$$(S_0 - Ke^{-rT})^+ \leq C_0(S_0, K, T) \leq S_0.$$

- It is easy to show that if $C_0(S_0, K, T, \sigma, r)$ is computed using the Black-Scholes call pricing formula, then

$$\lim_{\sigma \to 0} C_0(S_0, K, T, \sigma, r) = (S_0 - Ke^{-rT})^+,$$
$$\lim_{\sigma \to \infty} C_0(S_0, K, T, \sigma, r) = S_0.$$

- Recall also that the vega of a call is strictly positive and thus the price of a call option is a strictly increasing function of the volatility parameter $\sigma > 0$. Hence $(S_0 - Ke^{-rT})^+ < C_0(S_0, K, T) < S_0$ in the Black-Scholes model.

## Volatility Parameter

- Recall that

$$d_{\pm}(S,T) = \frac{\ln\frac{S}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)(T)}{\sigma\sqrt{T}}.$$

- We have

$$\begin{aligned}
&\lim_{\sigma\to 0} d_{\pm}(S,T) = -\infty \quad \text{if } S < Ke^{-rT}\\
&\lim_{\sigma\to 0} d_{\pm}(S,T) = \infty \quad \text{if } S > Ke^{-rT}\\
&\lim_{\sigma\to \infty} d_{+}(S,T) = \infty, \quad \lim_{\sigma\to \infty} d_{-}(S,T) = -\infty
\end{aligned}$$

- and thus

$$\begin{aligned}
&\lim_{\sigma\to 0} N(d_{\pm}(S,T)) = 0 \quad \text{if } S < Ke^{-rT}\\
&\lim_{\sigma\to 0} N(d_{\pm}(S,T)) = 1 \quad \text{if } S > Ke^{-rT}\\
&\lim_{\sigma\to \infty} N(d_{+}(S,T)) = 1, \quad \lim_{\sigma\to \infty} N(d_{-}(S,T)) = 0.
\end{aligned}$$

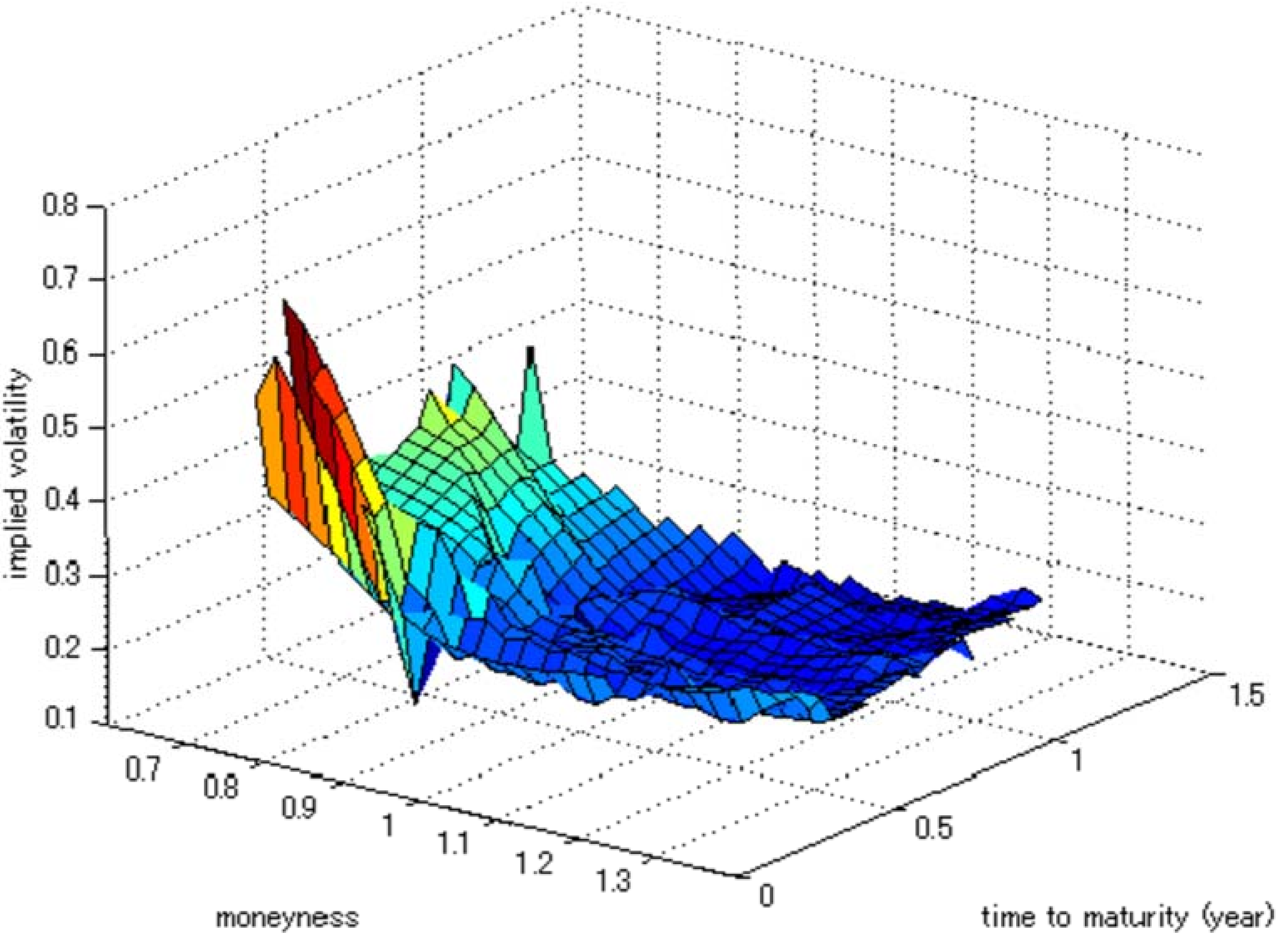
# Market Data: Implied Volatility Surface

- Let $t = 0$ and let the market quote for the call be $\bar{C}_0(S_0, K, T)$.
- If $(S_0 - Ke^{-rT})^+ < \bar{C}_0(S_0, K, T) < S_0$, then there exists a unique value of the volatility parameter $\bar{\sigma}$ in the Black-Scholes call pricing formula such that $C_0(S_0, K, T, \bar{\sigma}) = \bar{C}_0(S_0, K, T)$.

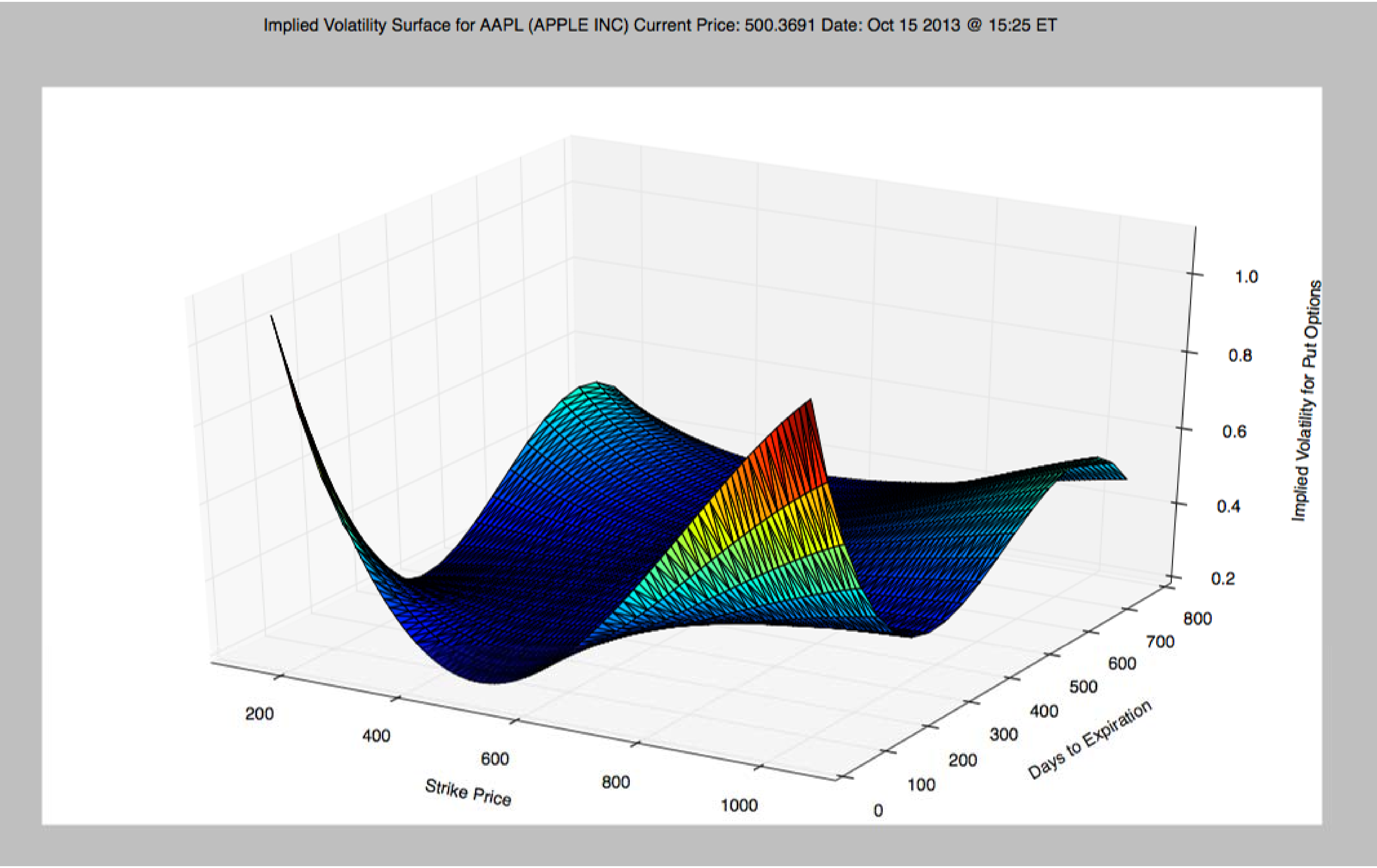
**Definition**

If $\bar{C}_0(S_0, K, T) = C_0(S_0, K, T, \bar{\sigma})$ for some $\bar{\sigma} > 0$, then $\bar{\sigma}$ is called the **implied volatility** and denoted as $\bar{\sigma}(K, T)$ or $\sigma_{\text{imp}}(K, T)$.

- The mapping $(K, T) \mapsto \sigma_{\text{imp}}(K, T)$ is called the **implied volatility surface** (IVS). The shape of IVS represents the current market data.
- If prices of all options are computed using the Black-Scholes formula for a given volatility $\sigma$, then the implied volatility surface is flat since $\sigma_{\text{imp}}(K, T) = \sigma$ for all $(K, T)$.



0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
implied volatility
0.7
0.8
0.9
1
1.1
1.2
1.3
moneyness
0
0.5
1
1.5
time to maturity (year)


Implied Volatility Surface for AAPL (APPLE INC) Current Price: 500.3691 Date: Oct 15 2013 @ 15:25 ET
Implied Volatility for Put Options
1.0
0.8
0.6
0.4
0.2
Strike Price
200
400
600
800
1000
Days to Expiration
0
100
200
300
400
500
600
700
800

# Heston's (1993) Stochastic Volatility Model

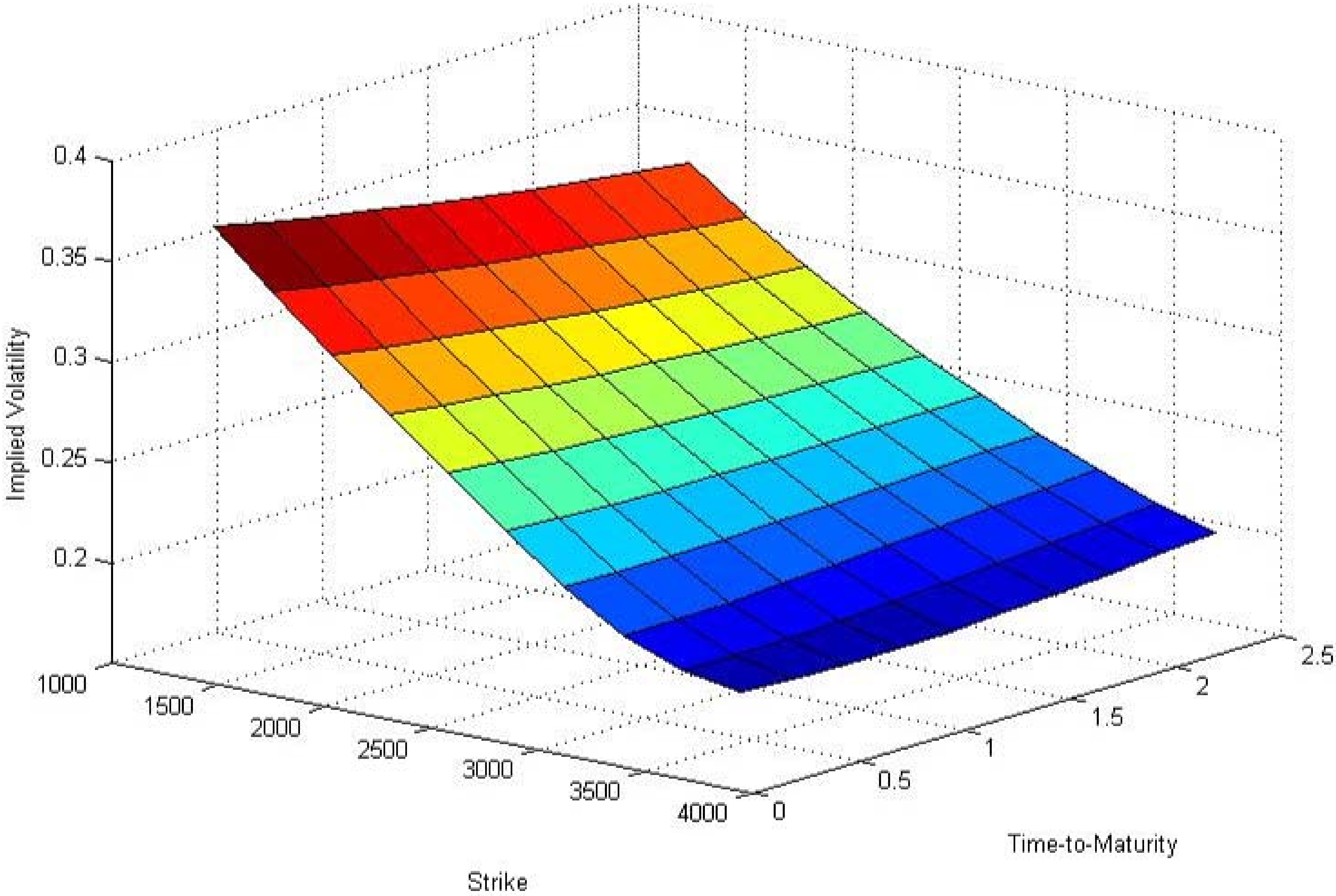
- If prices are computed using some other stochastic model, then the shape of the IVS can be chosen to be close to the IVS obtained from market quotes: **model IVS** $\approx$ **market IVS**.
- For instance, the **stochastic volatility model** proposed by Heston (1993) assumes that the asset price $S$ satisfies

$$dS_t = S_t\big(\mu_t\, dt + \sqrt{\nu_t}\, dW_t\big)$$

with the instantaneous variance $\nu$ governed by the SDE

$$d\nu_t = \widehat{\kappa}(\widehat{\nu} - \nu_t)\, dt + \eta\sqrt{\nu_t}\, d\widehat{W}_t.$$

- The processes $W$ and $\widehat{W}$ are one-dimensional Brownian motions defined on $(\Omega, \mathbb{F}, \mathbb{P})$ with the cross-variation $\langle W, \widehat{W}\rangle_t = \rho t$ for some correlation coefficient $\rho \in [-1, 1]$.



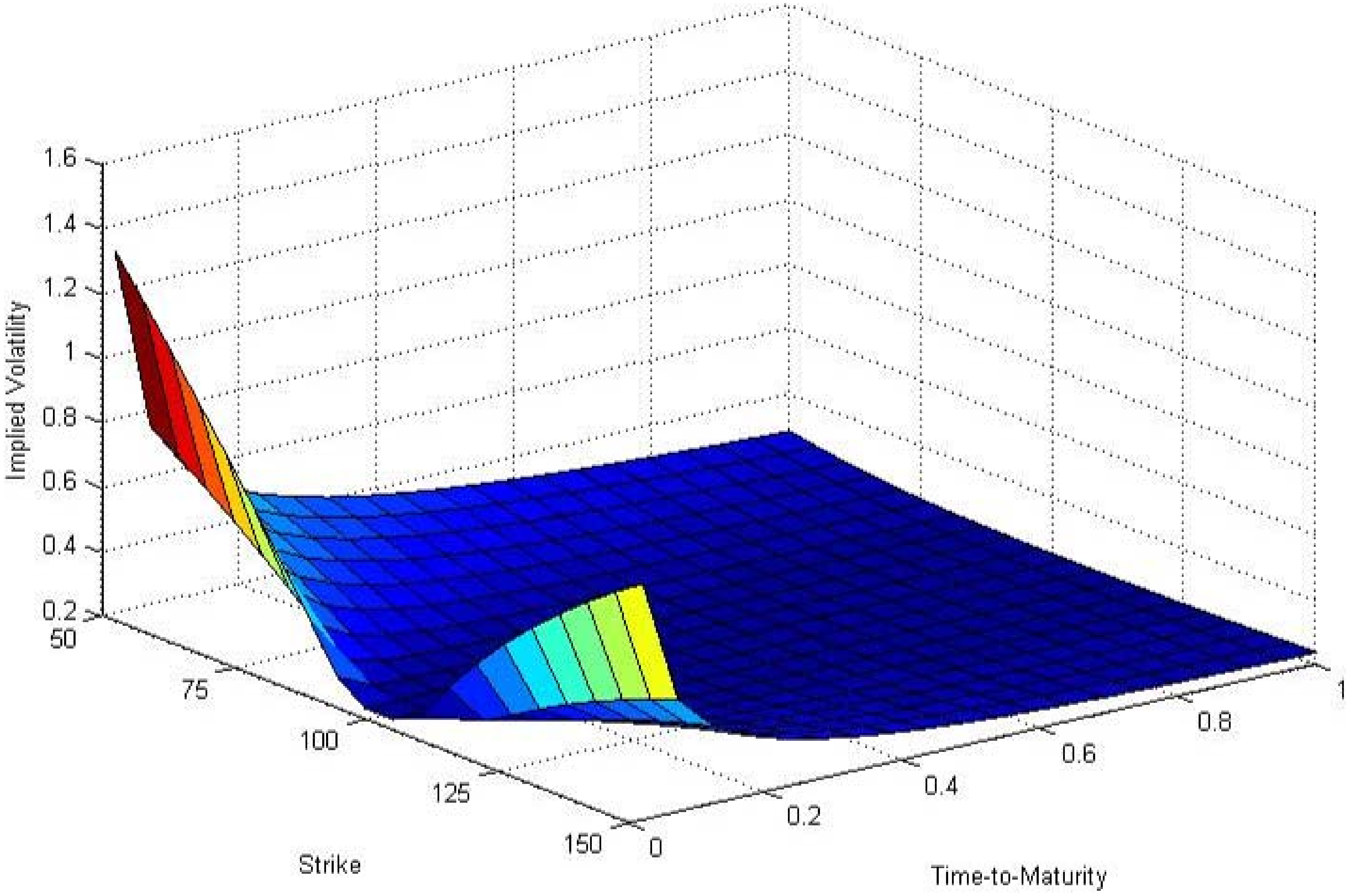
Implied Volatility
0.4
0.35
0.3
0.25
0.2
1000
1500
2000
2500
3000
3500
4000
Strike
0
0.5
1
1.5
2
2.5
Time-to-Maturity

# Merton's (1976) Jump-Diffusion Model

- One may also propose a model of the stock price with jumps.
- A simple model of this kind was proposed by Merton (1976) who postulated that the asset price $S$ is given by

$$S_t = S_0 \exp\left(\sigma W_t + \left(r - \frac{1}{2}\sigma^2\right)t\right) \prod_{i=1}^{N_t} Y_i$$

  where $W$ is the Brownian motion and $N$ is the Poisson process with intensity $\lambda$.
- Random variables $Y_1, Y_2, \ldots$ are independent, identically distributed and have the lognormal distribution with parameters $\mu_Y$ and $\sigma_Y$.
- Random variables $Y_1, Y_2, \ldots$ are also and independent of the Brownian motion $W$ and the Poisson process $N$.
- Merton's jump-diffusion model is capable of generating/fitting IVS with steep volatility skew for options with short time-to-maturity.


1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
Implied Volatility
50
75
100
125
150
Strike
0
0.2
0.4
0.6
0.8
1
Time-to-Maturity

# Dupire's (1993) Model of Local Volatility

- Let the stock price $S$ be given by the stochastic differential equation
$$dS_t = S_t\big(r\,dt + \sigma(S_t,t)\,dW_t\big).$$
- We denote by $c(K,T)$ be the corresponding call pricing function
$$c(K,T) = e^{-rT}\,\mathbb{E}_{\widetilde{\mathbb{P}}}\big((S_T-K)^+\big).$$

**Proposition**

*If $c$ is of class $C^{1,2}(\mathbb{R}\times[0,T],\mathbb{R})$, then it satisfies Dupire's PDE*
$$-c_T(K,T) + \frac{1}{2}\,\sigma^2(K,T)K^2 c_{KK}(K,T) - rKc_K(K,T) = 0$$
*with the initial condition $c(K,0) = (S_0-K)^+$.*

- Hence $\sigma_{\text{loc}}(K,T)$ can be computed if $\bar{C}_0(K,T)$ or, equivalently, $\sigma_{\text{imp}}(K,T)$ is given (hence we get a perfect fit to market data).

# PART 7

# RANDOM WALK APPROXIMATIONS

# Random Walk Approximations

- We wish to examine an approximation of the Black-Scholes model by a sequence of CRR models.
- We will first examine an approximation of the Wiener process by a sequence of symmetric random walks.
- Next, we will use this result in order to show how to approximate the Black-Scholes stock price process by a sequence of the CRR stock price models.
- We will also recognise that the proposed approximation of the stock price leads to the Jarrow-Rudd parametrisation of the CRR model in terms of the short term rate $r$ and the stock price volatility $\sigma$.

# Symmetric Random Walk

## Definition (Symmetric Random Walk)

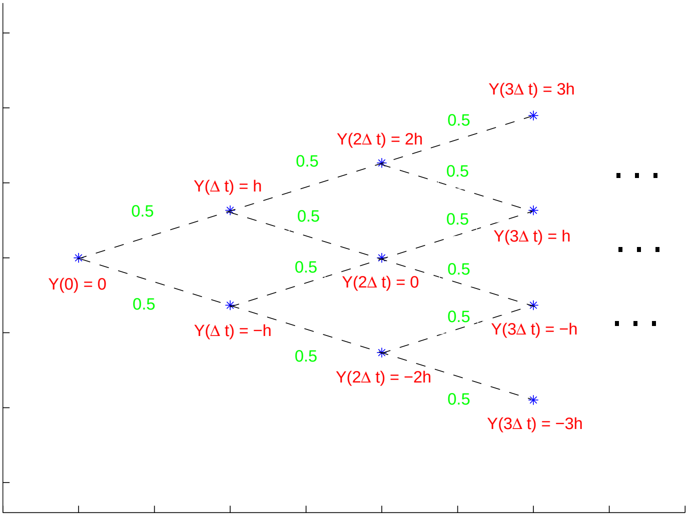
A process $Y = (Y_k,\, k = 0, 1, \dots)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called the **symmetric random walk** started at zero if $Y_0 = 0$ and $Y_k = \sum_{i=1}^{k} X_i$ where the random variables $X_1, X_2, \dots$ are independent with the following common probability distribution $\mathbb{P}(X_i = 1) = 0.5 = \mathbb{P}(X_i = -1)$.

The **scaled random walk** $Y^h$ is obtained from $Y$ as follows: we fix $h = \sqrt{\Delta t}$ and for every $k = 0, 1, \dots$ we set

$$Y^h_{k\Delta t} = \sqrt{\Delta t} Y_k = \sum_{i=1}^{k} \sqrt{\Delta t} X_i$$

Of course, for $h = \sqrt{\Delta t} = 1$ we obtain $Y^h_{k\Delta t} = Y^1_k = Y_k$.

# Scaled Random Walk



# Central Limit Theorem (CLT)

Let us recall the classic version of the Central Limit Theorem.

**Theorem (Central Limit Theorem)**

*Assume that $X_1, X_2, \dots$ are independent and identically distributed random variables with mean $\mu$ and variance $\sigma^2 > 0$. Then for all real $x$*

$$\lim_{n\to\infty} \mathbb{P}\left\{ \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \leq x \right\} = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du = N(x).$$

Note that if we denote $Y_n = \sum_{i=1}^{n} X_i$ then

$$\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} = \frac{Y_n - \mathbb{E}_{\mathbb{P}}(Y_n)}{\sqrt{\mathit{Var}(Y_n)}}.$$

# Approximation of the Wiener Process

- The following result is an easy consequence of the classic Central Limit Theorem (CLT) for sequences of independent and identically distributed (i.i.d.) random variables.

**Theorem (8.2)**

*Let $Y_t^h$ for $t = 0, \Delta t, \dots,$ be a random walk starting at 0 with increments $\pm h = \pm\sqrt{\Delta t}$. If*

$$\mathbb{P}\left(Y^h_{t+\Delta t} = y + h \,\middle|\, Y^h_t = y\right) = \mathbb{P}\left(Y^h_{t+\Delta t} = y - h \,\middle|\, Y^h_t = y\right) = 0.5$$

*then, for any fixed $t \geq 0$, the limit $\lim_{h\to 0} Y_t^h$ exists in the sense of probability distribution. Specifically, $\lim_{h\to 0} Y_t^h \sim W_t$ where $W$ is the Wiener process and $\sim$ denotes the equality of probability distributions. In other words, $\lim_{h\to 0} Y_t^h \sim N(0,t)$.*

# Proof of Theorem 8.2

## Proof of Theorem 8.2.

- We fix $t > 0$ and we set $k = t/\Delta t$. Hence if $\Delta t \to 0$ then $k \to \infty$. We recall that $h = \sqrt{\Delta t}$ and
$$Y^h_{k\Delta t} = \sum_{i=1}^{k} \sqrt{\Delta t} X_i.$$
- Since $\mathbb{E}_{\mathbb{P}}(X_i) = 0$ and $\mathit{Var}(X_i) = \mathbb{E}_{\mathbb{P}}(X_i^2) = 1$, we obtain
$$\mathbb{E}_{\mathbb{P}}(Y^h_{k\Delta t}) = \sqrt{\Delta t} \sum_{i=1}^{k} \mathbb{E}_{\mathbb{P}}(X_i) = 0$$
$$\mathit{Var}(Y^h_{k\Delta t}) = \sum_{i=1}^{k} \Delta t \, \mathit{Var}(X_i) = \sum_{i=1}^{k} \Delta t = k\Delta t = t.$$
- Hence the statement follows from the (slightly extended) CLT.

# Approximation of the Wiener Process

The sequence of random walks $Y^h$ approximates the Wiener process $W$ when $h = \sqrt{\Delta t} \to 0$ meaning that:

- For a fixed $t \geq 0$, the convergence $\lim_{h\to 0} Y_t^h \sim W_t$ holds, where $\sim$ denotes the equality of probability distributions on $\mathbb{R}$. This follows from Theorem 8.2.
- For any fixed $n$ and any dates $0 \leq t_1 < t_2 < \cdots < t_n$, we have

$$\lim_{h\to 0}(Y_{t_1}^h, \dots, Y_{t_n}^h) \sim (W_{t_1}, \dots, W_{t_n})$$

  where $\sim$ denotes the equality of probability distributions on the space $\mathbb{R}^n$. This can be proven by extending Theorem 8.2.
- In view of the theorem established by Monroe Donsker in 1951, the sequence of piecewise linear versions of the random walks $Y^h$ converges to a continuous time process $W$ in the sense of the weak convergence of stochastic processes.

# Approximation of the Stock Price

- Recall that the JR parameterisation for the CRR binomial model postulates that

$$u = e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}} \text{ and } d = e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}},$$

  whereas under the CRR convention we set $u = e^{\sigma\sqrt{\Delta t}} = 1/d$.
- We will show that it corresponds to a particular approximation of the stock price process $S$
- Recall that for all $0 \le s \le t$

$$\begin{aligned} S_t &= S_0 e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t} \\ &= S_0 e^{\left(r - \frac{\sigma^2}{2}\right)s + \sigma W_s} \, e^{\left(r - \frac{\sigma^2}{2}\right)(t-s) + \sigma(W_t - W_s)} \qquad (5) \\ &= S_s e^{\left(r - \frac{\sigma^2}{2}\right)(t-s) + \sigma(W_t - W_s)}. \end{aligned}$$

# Approximation of the Stock Price

- Let us set $t - s = \Delta t$ and let us replace the Wiener process $W$ by the random walk $Y^h$ in equation (5). Then

$$W_{t+\Delta t} - W_t \approx Y^h_{t+\Delta t} - Y^h_t = \pm h = \pm\sqrt{\Delta t}.$$

- Consequently, we obtain the following approximation

$$S_{t+\Delta t} \approx \begin{cases} S_t e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}} & \text{if the price increases,} \\ S_t e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}} & \text{if the price decreases.} \end{cases}$$

- More explicitly, for $k = 0, 1, \dots$

$$S^h_{k\Delta t} = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)k\Delta t + \sigma Y^h_{k\Delta t}}.$$

If we denote $t = k\Delta t$ then

$$S^h_t = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma Y^h_t}.$$

# Jarrow-Rudd Parametrisation

- We observe that this approximation of the stock price process leads to the Jarrow-Rudd parameterisation

$$u = e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}} \quad \text{and} \quad d = e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}}.$$

- The convergence of the sequence of random walks $Y^h$ to the Wiener process $W$ (Donsker's Theorem) implies that the sequence $S^h$ of CRR stock price models converges to the Black-Scholes stock price $S$.
- The convergence of $S^h$ to the stock price process $S$ justifies the claim that the JR parametrisation is more suitable than the CRR method.
- This is especially important when dealing with valuation and hedging of path-dependent and American contingent claims.

# THE END