MATH3075/3975 Financial Derivatives

Tutorial 3: Solutions

Exercise 1 Let (x, φ) be the replicating strategy for the contingent claim $X = g(S_1) = S_1$, that is,

$$V_1(x, \varphi) := (x - \varphi S_0)(1 + r) + \varphi S_1 = S_1.$$

More explicitly, for every $\omega_i \in \Omega = \{\omega_1, \omega_2\}$,

$$V_1(x, \varphi)(\omega_i) = (x - \varphi S_0)(1 + r) + \varphi S_1(\omega_i) = S_1(\omega_i).$$

The unique solution reads: $\varphi = 1$ and

$$x = \frac{1 - \varphi}{1 + r} S_1(\omega) + \varphi S_0 = S_0.$$

Alternatively, we may argue that if a contingent claim X is offered at a price different from S_0 then an arbitrage opportunity arises. Hence the unique fair value of the claim $X = S_1$ at time 0 equals S_0.

Exercise 2 We fix $K > 0$ and we write $C = C(K)$ and $P = P(K)$. We start by observing that

$$C_T - P_T = (S_T - K)^+ - (K - S_T)^+ = S_T - K. \tag{1}$$

The arbitrage price π_0 is an additive map on the space of contingent claims, so that

$$\begin{aligned} C_0 - P_0 = \pi_0(C_T) - \pi_0(P_T) &\overset{\text{additive}}{=} \pi_0(C_T - P_T) \overset{(1)}{=} \pi_0(S_T - K) \\ &\overset{\text{additive}}{=} \pi_0(S_T) - \pi_0(K) = S_0 - (1 + r)^{-1} K, \end{aligned}$$

since we know that $\pi_0(S_T) = S_0$ and $\pi_0(K) = (1+r)^{-1}K$. Alternatively, we may use the risk-neutral valuation formula

$$\begin{aligned} C_0 - P_0 = \pi_0(C_T - P_T) &= \mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{C_T - P_T}{1 + r}\right) = \mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{S_T - K}{1 + r}\right) \\ &= \mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{S_T}{1 + r}\right) - \mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{K}{1 + r}\right) = S_0 - (1 + r)^{-1} K. \end{aligned}$$

Exercise 3 We consider the options defined in Examples 2.1.1 and 2.1.2. Recall that $T = 1$ (say, one month) and $S_0 = K = 1$ (say, one AUD).

- For the call option, we need to solve

  $$V_1(x, \varphi) := (x - \varphi S_0)(1 + r) + \varphi S_1 = C_1 = (S_1 - K)^+,$$

  or, more explicitly,

  $$\begin{cases} \frac{4}{3}(x - \varphi) + 2\varphi = 1, \\ \frac{4}{3}(x - \varphi) + \frac{1}{2}\varphi = 0. \end{cases}$$

  We obtain $(x, \varphi) = (5/12, 2/3)$ so that

  $$x - \varphi S_0 = 5/12 - 8/12 = -3/12,$$

  meaning that after selling (aka "writing" or "issuing") the call option at the price $5/12$, in order to buy $2/3$ shares of the stock S at time 0, we need to borrow $3/12$ units of cash. Notice that here we buy shares and we borrow cash at time 0.

- For the put option, we solve

$$V_1(x, \varphi) := (x - \varphi S_0)(1 + r) + \varphi S_1 = P_1 = (K - S_1)^+,$$

that is,

$$\begin{cases} \frac{4}{3}(x - \varphi) + 2\varphi = 0, \\ \frac{4}{3}(x - \varphi) + \frac{1}{2}\varphi = \frac{1}{2}. \end{cases}$$

We now obtain $(x, \varphi) = (1/6, -1/3)$. Note that $x - \varphi S_0 = 1/6 + 1/3 = 1/2$ meaning that after selling the put option at the price $1/6$ and short-selling of $1/3$ share of the stock at time 0, we invest $1/2$ units of cash in the savings account. Notice that here we short sell shares and we lend cash at time 0.

A. Comments on replication. We deal with two independent questions:
a) how to replicate the payoff C_1 of the call option?
b) how to replicate the payoff P_1 of the put option?
This means that we need to find the *initial wealth* of each replicating strategy as well as the number of shares we need to buy or short sell to replicate a given option. The *initial wealth* should not be confused with our "initial endowment", which is assumed to be null. The respective answers for the call and put are:
a) $x = 5/12$ (initial wealth) and $\varphi = 2/3$ (hence we buy shares),
b) $x = 1/6$ (initial wealth) and $\varphi = -1/3$ (hence we short sell shares).
Notice that the initial wealth of replicating portfolio gives the *fair price* (also called the *arbitrage price*) for the option we sell, that is, $C_0 = 5/12$ and $P_0 = 1/6$.

B. Comments on the put-call parity. Recall that the put-call parity in a single-period case states that $C_0 - P_0 = S_0 - K(1 + r)^{-1}$. Since we have found that $C_0 = 5/12$ (from part (a)) and $P_0 = 1/6$ (from part (b)) so that

$$\text{LHS} = C_0 - P_0 = 5/12 - 1/6 = 3/12 = 1/4$$

and the right-hand side equals

$$\text{RHS} = S_0 - K(1 + r)^{-1} = 1 - 1(1 + 1/3)^{-1} = 1 - 3/4 = 1/4.$$

This confirms that our approach to the valuation and hedging problems (a) and (b) is consistent with the put-call parity. However, we cannot deduce solutions to (a) or (b) above from the put-call parity.

C. Comments on the concept of hedging. A *hedging strategy* (or a *hedging portfolio*) is a pair (x, φ) and our goal was to "hedge" the risk of our short position in either a call and put option in case (a) and (b), respectively. If we sell either a call or a put option, then our transactions can be summarised as follows:

1. at time 0, we sell an option at some initial price,
2. then, also at time 0, we hedge the risk exposure by establishing a replicating portfolio composed of our (either positive or negative) positions in shares and cash in the bank account (or, equivalently, bonds),
3. at time T, we first liquidate (unwind) our hedging portfolio and,
4. then, again at time T, we settle the contract by delivering the payoff of the option to the holder. This corresponds to the case of cash settlement; the case of physical delivery is somewhat different in practice, but mathematically equivalent.

Liquidate means to convert assets into cash or cash equivalents by selling them on the open market. Since the wealth at time T is equal to our liability (the amount of cash we have to pay to the option holder), we see that we will be always left with nothing. In particular, there will be no loss from the option contract, meaning that we are perfectly "hedged" (you may wish to say "protected," "sheltered," "covered" or "insured") against an eventual loss at time T, but also we will never make any profit. This means that arbitrage pricing hinges on a complete elimination of profits and losses, as opposed to *speculation* where agents hope to make profits, but also run the risk of a loss.

It is important to observe that analogous arguments can be applied to the buyer (holder) of the option and thus fair valuation is bilateral: the unique amount of cash the seller is prepared to accept for the option is equal to the amount of cash the buyer is ready to pay. This property justifies why we can use the name: *fair pricing.*

Exercise 4 We assume that $r = \frac{1}{4}$, $S_0 = 1$, $u = 3$, $d = \frac{1}{3}$, $p = \frac{4}{5}$ and we consider the digital call option with the following payoff

$$g(S_1) = \begin{cases} 1, & \text{if } S_1 \geq K, \\ 0, & \text{otherwise.} \end{cases}$$

The unique risk-neutral probability measure $\widetilde{\mathbb{P}} = (\widetilde{\mathbb{P}}(\omega_1), \widetilde{\mathbb{P}}(\omega_2)) = (\widetilde{p}, 1 - \widetilde{p})$ where

$$\widetilde{p} = \frac{1 + r - d}{u - d} = \frac{1 + \frac{1}{4} - \frac{1}{3}}{3 - \frac{1}{3}} = \frac{11}{32}.$$

Hence the price at time 0 of the digital call option equals

$$\pi_0(g(S_1)) = \mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{g(S_1)}{1+r}\right) = \tfrac{4}{5}\,\mathbb{E}_{\widetilde{\mathbb{P}}}\big(g(S_1)\big) = \begin{cases} \frac{4}{5}, & \text{if } \ K \leq dS_0 = \frac{1}{3}, \\ \frac{4}{5} \cdot \frac{11}{32} = \frac{11}{40}, & \text{if } \frac{1}{3} < K \leq 3, \\ 0, & \text{if } \ K > uS_0 = 3, \end{cases}$$

since

$$\begin{aligned} K \leq dS_0 \quad &\Rightarrow \quad g(S_1) = 1, \\ dS_0 < K \leq uS_0 \quad &\Rightarrow \quad g(S_1) = \mathbb{1}_{\{S_1 = 3\}}, \\ K > uS_0 \quad &\Rightarrow \quad g(S_1) = 0. \end{aligned}$$

Note that the value of $p = \frac{4}{5}$ was not used in the computation of the price.

Exercise 5 Assume that $d < 1 + r < u$ and consider any trading strategy of the form $(0, \varphi)$. Then

$$V_1(0, \varphi) := -\varphi S_0(1 + r) + \varphi S_1,$$

so that, for every $\omega_i \in \Omega = \{\omega_1, \omega_2\}$,

$$V_1(0, \varphi)(\omega_i) = \varphi\big(S_1(\omega_i) - S_0(1 + r)\big).$$

Since $S_1(\omega_1) = uS_0$ and $S_1(\omega_2) = dS_0$, we obtain

$$\begin{cases} V_1(0, \varphi)(\omega_1) = \varphi S_0\big(u - (1 + r)\big), \\ V_1(0, \varphi)(\omega_2) = \varphi S_0\big(d - (1 + r)\big), \end{cases}$$

where $u - (1 + r) > 0$ and $d - (1 + r) < 0$.

- If we take $\varphi = 0$ then, obviously, $V_1(0,\varphi)(\omega_i) = 0$ for $i = 1, 2$.
- If $\varphi \neq 0$ then $V_1(0,\varphi)(\omega_1)$ and $V_1(0,\varphi)(\omega_2)$ have the opposite signs, namely,

$$V_1(0,\varphi)(\omega_1)\, V_1(0,\varphi)(\omega_2) = \varphi^2 S_0^2 \big(u - (1+r)\big)\big(d - (1+r)\big) < 0.$$

We conclude that there is no strategy $(0,\varphi)$ satisfying Definition 2.1.3 of an arbitrage opportunity and thus the elementary market model is arbitrage free if $d < 1 + r < u$.

Furthermore, if either $1 + r \leq d$ or $1 + r \geq u$ then is is easy to see that arbitrage opportunities exist – it suffices to consider either buying (when $1 + r \leq d$) or short-selling (when $1 + r \geq u$) of the stock S.

Exercise 6 Note that $\varphi^0 = x - \varphi^1 S_0$ is the amount invested in the savings account B. The initial wealth of our portfolio is thus given by $x = V_0(x,\varphi) = \varphi^0 + \varphi^1 S_0$ and the portfolio's wealth at time $t = 1$ equals

$$V_1(x,\varphi) = (x - \varphi^1 S_0)(1+r) + \varphi^1 S_1 = \varphi^0(1+r) + \varphi^1 S_1.$$

(a) For the call option with the payoff $C_1 = (S_1 - 28.5)^+$, the unique replicating strategy (φ^0, φ^1) can be found by solving the following equations:

$$\begin{bmatrix} 1.1 & 31 \\ 1.1 & 28 \end{bmatrix} \begin{bmatrix} \varphi^0 \\ \varphi^1 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \end{bmatrix} \implies (\varphi^0, \varphi^1) = (-700/33, 5/6).$$

For the put option with the payoff $P_1 = (28.5 - S_1)^+$, the unique replicating strategy $(\varphi^0, \varphi^1)$is computed through the following system of linear equations:

$$\begin{bmatrix} 1.1 & 31 \\ 1.1 & 28 \end{bmatrix} \begin{bmatrix} \varphi^0 \\ \varphi^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} \implies (\varphi^0, \varphi^1) = (155/33, -1/6).$$

(b) Using part (a), we can compute the price of the put and the call. We know that the initial endowment x of the replicating portfolio is given by $x = \varphi^0 + \varphi^1 S_0$. Hence the prices of the put and call options are equal to

$$C_0 = -700/33 + (5/6)27 = 85/66, \quad P_0 = 155/33 - (1/6)27 = 13/66.$$

(c) On the one hand, we have $C_0 - P_0 = 85/66 - 13/66 = 72/66 = 12/11$. On the other hand, we obtain

$$S_0 - (1+r)^{-1}K = 27 - 28.5/1.1 = (297 - 285)/11 = 12/11.$$

Hence the put call-parity relationship holds.

(d) Recall that

$$\widetilde{\mathbb{P}}(\omega_1) = \frac{1 + r - d}{u - d}, \qquad \widetilde{\mathbb{P}}(\omega_2) = 1 - \widetilde{\mathbb{P}}(\omega_1).$$

We have $u = 31/27$ and $d = 28/17$. Hence the unique risk-neutral probability measure $\widetilde{\mathbb{P}}$ (also called the *martingale measure*) is given by

$$\widetilde{\mathbb{P}}(\omega_1) = \frac{1.1 - \frac{28}{27}}{\frac{31}{27} - \frac{28}{27}} = \frac{17}{30}, \quad \widetilde{\mathbb{P}}(\omega_2) = 1 - \widetilde{\mathbb{P}}(\omega_1) = \frac{13}{30}.$$

The arbitrage prices of options can be computed using the risk-neutral valuation formula. The price of the call option satisfies

$$C_0 = \mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{C_1}{1+r}\right) = \frac{1}{1+r}\,\mathbb{E}_{\widetilde{\mathbb{P}}}\big((S_1-K)^+\big) = \frac{1}{1.1}\,\frac{17}{30}2.5 = \frac{85}{66}.$$

and price of the put option equals

$$P_0 = \mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{P_1}{1+r}\right) = \frac{1}{1+r}\,\mathbb{E}_{\widetilde{\mathbb{P}}}\big((K-S_1)^+\big) = \frac{1}{1.1}\,\frac{13}{30}0.5 = \frac{13}{66}.$$

(e) Let us write $\widehat{r} = 0.05$. The new replicating strategy (φ^0, φ^1) for the call option is given by

$$\begin{bmatrix} 1.05 & 31 \\ 1.05 & 28 \end{bmatrix} \begin{bmatrix} \varphi^0 \\ \varphi^1 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \end{bmatrix} \implies (\varphi^0, \varphi^1) = (-200/9, 5/6).$$

For the put option, we obtain

$$\begin{bmatrix} 1.05 & 31 \\ 1.05 & 28 \end{bmatrix} \begin{bmatrix} \varphi^0 \\ \varphi^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} \implies (\varphi^0, \varphi^1) = (310/63, -1/6).$$

The new price of the call is

$$\widehat{C}_0 = -200/9 + (5/6)27 = 5/18 = 35/126 < 85/66 = C_0$$

and the new price of the put equals

$$\widehat{P}_0 = 310/63 - (1/6)27 = 53/126 > 13/66 = P_0.$$

Hence the price of the call for the interest rate $\widehat{r} = 5\%$ is lower than for the interest rate $r = 10\%$, but the price of the put option is higher when the interest rate is lower. This is due to the fact that we borrow cash when we replicate the call option, but we put cash into money market account when we replicate the put option. As expected, the put-call parity still holds:

$$\widehat{C}_0 - \widehat{P}_0 = -18/126 = -1/7$$

and

$$S_0 - K(1+\widehat{r})^{-1} = 27 - 28.5/1.05 = (2835 - 2850)/105 = -15/105 = -1/7.$$

Exercise 7 (MATH3975) From Proposition 2.1.4, we know that the arbitrage price x of any contingent claim $X = g(S_1)$ in the elementary market model satisfies

$$\pi_0(g(S_1)) = \mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{g(S_1)}{1+r}\right)$$

where $\widetilde{\mathbb{P}}$ is the unique risk-neutral probability measure. We note that the probability measures $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent and thus the Radon-Nikodym density L of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$ is well defined, specifically,

$$L(\omega_i) = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}(\omega_i) = \frac{\widetilde{\mathbb{P}}(\omega_i)}{\mathbb{P}(\omega_i)} = \begin{cases} \frac{\widetilde{p}}{p}, & i = 1, \\ \frac{1-\widetilde{p}}{1-p}, & i = 2. \end{cases}$$

Consequently, if a trading strategy (x, φ) replicates the claim $X = g(S_1)$ then

$$x = \pi_0\big(g(S_1)\big) = \mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{g(S_1)}{1+r}\right) = \mathbb{E}_{\mathbb{P}}\left(L\,\frac{g(S_1)}{1+r}\right) = \mathbb{E}_{\mathbb{P}}\big(Zg(S_1)\big)$$

where we set $Z := (1+r)^{-1}L$. The *pricing kernel* Z can thus be used to price any contingent claim using the real-world probability $\mathbb{P}$ in the elementary market model.

Exercise 8 (MATH3975) For $S_0 = 100.23$, we observed the following mid-prices of European call and put options on JPMorgan stock on 1 September (for the most current market data, enter JPM as a stock symbol on http://www.cboe.com/delayedquote/quote-table)

Call $C_0(K)$	**Strike** K	**Put** $P_0(K)$
\$3.95	\$98	\$2.19
\$3.65	\$99	\$2.45
\$3.12	\$100	\$2.91
\$2.65	\$101	\$3.42
\$2.23	\$102	\$4.02

If $r = 0, d > 1$ and $0 < d = u^{-1} < 1$, then $d < 1 + r < u$ and thus the martingale measure $\widetilde{\mathbb{P}}$ is well defined and $\widetilde{p}$ satisfies

$$\widetilde{p} = \frac{1 + r - d}{u - d} = \frac{1 - d}{u - d} = \frac{1 - u^{-1}}{u - u^{-1}} = \frac{u - 1}{u^2 - 1} = \frac{1}{u + 1}.$$

Hence, if the strike K satisfies $dS_0 < K < uS_0$, then

$$C_0(K) = \widetilde{p}(uS_0 - K) = \frac{uS_0 - K}{u + 1}. \tag{2}$$

If $S_0 = 100.23$ and $K = 100$, then we obtain the following equalities

$$C_0(K) = \frac{uS_0 - K}{u + 1} = \frac{100.23u - 100}{u + 1} = 3.12.$$

It now suffices to compute u and thus also $\widetilde{p}$ and apply (2) in order to find call prices for other strikes. We have $u = 1.0619$ and $\widetilde{p} = 0.485$. Therefore, the model price of the call with strike $K = 99$ equals

$$C_0(99) = \widetilde{p}(uS_0 - K) = 0.485(106.43 - 99) = 3.60$$

Similarly, for $K = 101$ we obtain

$$C_0(101) = \widetilde{p}(uS_0 - K) = 0.485(106.43 - 101) = 2.63.$$

For the put option, we use the equality

$$P_0(K) = \widetilde{q}(K - dS_0). \tag{3}$$

where $d = u^{-1} = 0.9417$ and $\widetilde{q} = 1 - \widetilde{p} = 0.515$. For instance, for $K = 100$ we obtain

$$P_0(100) = \widetilde{q}(K - dS_0) = 0.515(100 - 94.3866) = 2.89.$$