7: THE CRR MODEL

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Outline

We will examine the following issues:

1. The Cox-Ross-Rubinstein Market Model
2. The CRR Call Option Pricing Formula
3. The CRR Put Option Pricing Formula
4. Call and Put Options of American Style
5. Dynamic Programming Approach to American Claims
6. Examples: American Call and Put Options
7. Implementations of the CRR Model

PART 1

THE COX-ROSS-RUBINSTEIN MARKET MODEL

Introduction

- The **Cox-Ross-Rubinstein (1979) model** (aka CRR model) is an example of a multi-period market model of the stock price.
- At each point in time, the stock price is assumed to either go 'up' by a fixed factor u or go 'down' by a fixed factor d.
- Only three parameters are needed to specify the binomial asset pricing model: $u > d > 0$ and $r > -1$. Recall that we set

$$B_t = (1 + r)^t, \quad t = 0, 1, \ldots, T.$$

- Note that we do not postulate that $d < 1 < u$.
- The real-world probability of an 'up' movement is assumed to be the same $0 < p < 1$ for each period and is assumed to be independent of all previous stock price movements.

Bernoulli Processes

Definition (Bernoulli Process)

A process $X = (X_t)_{1 \leq t \leq T}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called the **Bernoulli process** with parameter $0 < p < 1$ if the random variables $X_1, X_2, \dots, X_T$ are independent and have the following common probability distribution

$$\mathbb{P}(X_t = 1) = 1 - \mathbb{P}(X_t = 0) = p.$$

Definition (Bernoulli Counting Process)

The **Bernoulli counting process** $N = (N_t)_{0 \leq t \leq T}$ is defined by setting $N_0 = 0$ and, for every $t = 1, 2, \dots, T$ and $\omega \in \Omega$,

$$N_t(\omega) := X_1(\omega) + X_2(\omega) + \cdots + X_t(\omega).$$

The process N is a special case of an **additive random walk**.

Stock Price

Definition (Stock Price)

The stock price process in the CRR model is defined via an initial value $S_0 > 0$ and, for $t = 1, 2, \ldots, T$ and all $\omega \in \Omega$,

$$S_t(\omega) := S_0 u^{N_t(\omega)} d^{\,t - N_t(\omega)}.$$

- The underlying Bernoulli process X governs the movements of the stock.
- The stock price moves 'up' at time t if $X_t(\omega) = 1$ and 'down' if $X_t(\omega) = 0$.
- The dynamics of the stock price in the CRR model can be seen as an example of a **multiplicative random walk**.
- The Bernoulli counting process N counts the up movements. Before and including time t, the stock price moves up N_t times and down $t - N_t$ times.

Distribution of the Stock Price under $\mathbb{P}$

- For every fixed $t = 1, 2, \ldots, T$, the random variable N_t has the **binomial distribution** with parameters p and t.
- For every fixed $t = 1, 2, \ldots, T$ and every $k = 0, 1, \ldots, t$ we have that

$$\mathbb{P}(N_t = k) = \binom{t}{k} p^k (1-p)^{t-k}$$

where

$$\binom{t}{k} = \frac{t\,!}{k\,!(t-k)\,!}.$$

- The equality of events $\{N_t = k\} = \{S_t = S_0 u^k d^{\,t-k}\}$ is satisfied for every $k = 0, 1, \ldots, t$.
- Hence the probability distribution of the stock price S_t at time t is given by

$$\mathbb{P}(S_t = S_0 u^k d^{\,t-k}) = \binom{t}{k} p^k (1-p)^{t-k}$$

for every fixed $t = 1, 2, \ldots, T$ and every $k = 0, 1, \ldots, t$.

Stock Price Lattice

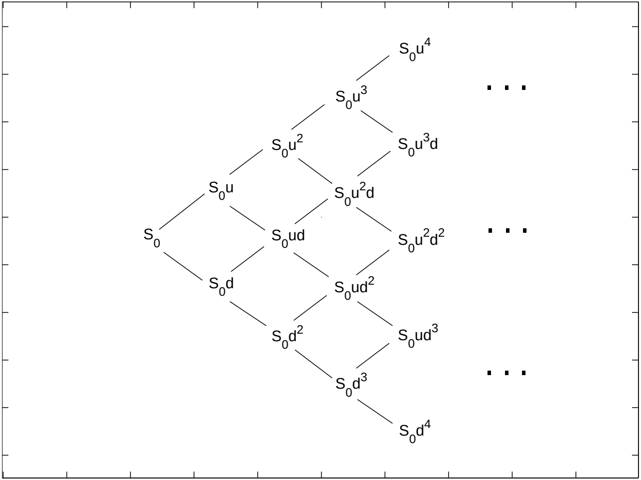


Figure: Stock Price Lattice in the CRR Model

Martingale Measure $\widetilde{\mathbb{P}}$

Proposition (7.1)

Assume that $d < 1 + r < u$. *Then a probability measure* $\widetilde{\mathbb{P}}$ *on* $(\Omega, \mathcal{F}_T)$ *is a martingale measure for the CRR market model* $\mathcal{M} = (B, S)$ *with parameters* p, u, d, r *and time horizon* T *if and only if:*

1. $X_1, X_2, X_3, \dots, X_T$ *are independent under the probability measure* $\widetilde{\mathbb{P}}$,
2. $0 < \widetilde{p} := \widetilde{\mathbb{P}}\,(X_t = 1) < 1$ *for all* $t = 1, 2, \dots, T$,
3. $\widetilde{p}u + (1 - \widetilde{p})d = (1 + r)$,

where X *is the Bernoulli process governing the stock price* S. *Hence the probability distribution under* $\widetilde{\mathbb{P}}$ *of the stock price* S_t *at time* t *is given by*

$$\widetilde{\mathbb{P}}(S_t = S_0 u^k d^{t-k}) = \binom{t}{k} \widetilde{p}^k (1 - \widetilde{p})^{t-k}$$

for every fixed $t = 1, 2, \dots, T$ *and every* $k = 0, 1, \dots, t$.

Risk-Neutral Probability Measure

Proposition (7.2)

If $d < 1 + r < u$ then the CRR market model $\mathcal{M} = (B, S)$ is arbitrage-free and complete.

- We henceforth assume that the inequalities $d < 1 + r < u$ are satisfied and thus $\widetilde{p} = \frac{1+r-d}{u-d} \in (0, 1)$ a martingale measure $\widetilde{\mathbb{P}}$ is well defined.
- Since the CRR market model is complete, the unique arbitrage price of any European contingent claim can be computed using the risk-neutral valuation formula
$$\pi_t(X) = B_t \, \mathbb{E}_{\widetilde{\mathbb{P}}} \left(\frac{X}{B_T} \,\middle|\, \mathcal{F}_t \right).$$
- We will use this formula to compute the price of the call option on the stock, that is, the claim X with maturity T given by
$$C_T = (S_T - K)^+ = \max\big(S_T - K, 0\big).$$

PART 2

THE CRR CALL OPTION PRICING FORMULA

CRR Call Option Pricing Formula

Proposition (7.3)

The arbitrage price at time $t = 0$ of the European call option with the payoff $C_T = (S_T - K)^+$ in the binomial market model $\mathcal{M} = (B, S)$ is given by the CRR call pricing formula

$$C_0 = S_0 \sum_{k=\hat{k}}^{T} \binom{T}{k} \hat{p}^k (1-\hat{p})^{T-k} - \frac{K}{(1+r)^T} \sum_{k=\hat{k}}^{T} \binom{T}{k} \widetilde{p}^k (1-\widetilde{p})^{T-k}$$

where

$$\widetilde{p} = \frac{1+r-d}{u-d}, \quad \hat{p} = \frac{\widetilde{p}u}{1+r}$$

and $\hat{k}$ is the smallest integer k such that

$$k \log\left(\frac{u}{d}\right) > \log\left(\frac{K}{S_0 d^T}\right).$$

Proof of Proposition 7.3

Proof of Proposition 7.3.

- The price at time $t = 0$ of the claim $X = C_T = (S_T - K)^+$ can be computed using the risk-neutral valuation under $\widetilde{\mathbb{P}}$

$$C_0 = \frac{1}{(1+r)^T} \, \mathbb{E}_{\widetilde{\mathbb{P}}} \left(C_T \right).$$

In view of Proposition 7.1, the probability distribution of the stock price S_T at time T is given by

$$\widetilde{\mathbb{P}}(S_T = S_0 u^k d^{T-k}) = \binom{T}{k} \widetilde{p}^k (1 - \widetilde{p})^{T-k}$$

for $k = 0, 1, \ldots, t$ and thus

$$C_0 = \frac{1}{(1+r)^T} \sum_{k=0}^{T} \binom{T}{k} \widetilde{p}^k (1 - \widetilde{p})^{T-k} \max\left(0, S_0 u^k d^{T-k} - K\right).$$

Proof of Proposition 7.3

Proof of Proposition 7.3 (Continued).

- We note that $S_0 u^k d^{T-k} - K > 0$ whenever

$$\left(\frac{u}{d}\right)^k > \frac{K}{S_0 d^T} \ \Leftrightarrow\ k \log\left(\frac{u}{d}\right) > \log\left(\frac{K}{S_0 d^T}\right).$$

- Let $\hat{k} = \hat{k}(S_0, T)$ be the smallest integer k such that the last inequality is satisfied.
- Then we obtain

$$\begin{aligned} C_0 &= \frac{1}{(1+r)^T} \sum_{k=\hat{k}}^{T} \binom{T}{k} \widetilde{p}^k (1-\widetilde{p})^{T-k} \left(S_0 u^k d^{T-k} - K\right) \\ &= \frac{S_0}{(1+r)^T} \sum_{k=\hat{k}}^{T} \binom{T}{k} \widetilde{p}^k (1-\widetilde{p})^{T-k} u^k d^{T-k} - \frac{K}{(1+r)^T} \sum_{k=\hat{k}}^{T} \binom{T}{k} \widetilde{p}^k (1-\widetilde{p})^{T-k}. \end{aligned}$$

□

Proof of Proposition 7.3 (Continued).

- Consequently,

$$
\begin{aligned}
C_0 \quad &= \quad S_0 \sum_{k=\hat{k}}^{T} \binom{T}{k} \left(\frac{\widetilde{p}u}{1+r} \right)^{k} \left(\frac{(1-\widetilde{p})d}{1+r} \right)^{T-k} \\
&\qquad - \frac{K}{(1+r)^T} \sum_{k=\hat{k}}^{T} \binom{T}{k} \widetilde{p}^{k} (1-\widetilde{p})^{T-k}
\end{aligned}
$$

and thus

$$
C_0 = S_0 \sum_{k=\hat{k}}^{T} \binom{T}{k} \hat{p}^{k} (1-\hat{p})^{T-k} - \frac{K}{(1+r)^T} \sum_{k=\hat{k}}^{T} \binom{T}{k} \widetilde{p}^{k} (1-\widetilde{p})^{T-k}
$$

where we denote $\hat{p} = \frac{\widetilde{p}u}{1+r}$ so that $1 - \hat{p} = \frac{(1-\widetilde{p})d}{1+r}$.

Arbitrage Pricing Through Backward Induction

- The following recursion holds, for every $t = 0, 1, \ldots, T-1$,

$$C_t = B_t \, \mathbb{E}_{\widetilde{\mathbb{P}}}\big(B_T^{-1}(S_T - K)^+ \,|\, \mathcal{F}_t\big) = (1+r)^{-1} \mathbb{E}_{\widetilde{\mathbb{P}}}\big(C_{t+1} \,|\, \mathcal{F}_t\big)$$

  with the terminal condition $C_T = (S_T - K)^+$.

- Notice that C_t is the price of the call option at time t and not the payoff of the call option with expiration date t.

- In general, the unique arbitrage price in the CRR model for any contingent claim $X : \Omega \to \mathbb{R}$ satisfies for every $t = 0, 1, \ldots, T-1$

$$\pi_t(X) = B_t \, \mathbb{E}_{\widetilde{\mathbb{P}}}\big(B_T^{-1} X \,|\, \mathcal{F}_t\big) = (1+r)^{-1} \mathbb{E}_{\widetilde{\mathbb{P}}}\big(\pi_{t+1}(X) \,|\, \mathcal{F}_t\big)$$

  with the terminal condition $\pi_T(X) = X$.

- For instance, for $t = T-1$ we have $t+1 = T$ and thus

$$\pi_{T-1}(X) = (1+r)^{-1} \mathbb{E}_{\widetilde{\mathbb{P}}}\big(\pi_T(X) \,|\, \mathcal{F}_{T-1}\big) = (1+r)^{-1} \mathbb{E}_{\widetilde{\mathbb{P}}}\big(X \,|\, \mathcal{F}_{T-1}\big).$$

Pricing Formula at Time t for the Call Option

- For any date $t = 0, 1, \ldots, T-1$ we have

$$
\begin{aligned}
C_t = & S_t \sum_{k=\hat{k}(S_t, T-t)}^{T-t} \binom{T-t}{k} \hat{p}^k (1-\hat{p})^{T-t-k} \\
& - \frac{K}{(1+r)^{T-t}} \sum_{k=\hat{k}(S_t, T-t)}^{T-t} \binom{T-t}{k} \widetilde{p}^k (1-\widetilde{p})^{T-t-k}
\end{aligned}
$$

where $\hat{k}(S_t, T-t)$ is the smallest integer k such that

$$
k \log\left(\frac{u}{d}\right) > \log\left(\frac{K}{S_t d^{T-t}}\right).
$$

- Note that $C_t = C(S_t, T-t)$ meaning that the call option price depends on the time to maturity $T-t$ and the level S_t of the stock price observed at time t, but it is independent of the history of the stock price prior to t.

Replicating Strategy for the Call Option

- The CRR model can be seen as a concatenation of single-period models and thus to compute the replicating strategy, we use the same idea as in the single-period case. For instance, let us find the hedging portfolio at time 0.
- At time $t = 0$, the replicating portfolio (ϕ_0^0, ϕ_0^1) for the call option satisfies $\phi_0^0 + \phi_0^1 S_0 = V_0(\phi) = C_0$ and

$$\phi_0^0(1+r) + \phi_0^1 S_1^u = C_1^u,$$
$$\phi_0^0(1+r) + \phi_0^1 S_1^d = C_1^d,$$

  where $C_1^u = C(uS_0, T-1)$ and $C_1^d = C(dS_0, T-1)$.
- Hence

$$\phi_0^0 = C_0 - \phi_0^1 S_0,$$
$$\phi_0^1 = \frac{C_1^u - C_1^d}{S_1^u - S_1^d} = \frac{C(uS_0, T-1) - C(dS_0, T-1)}{S_0(u-d)}.$$

Put-Call Parity

- Since $C_T - P_T = S_T - K$, we see that the following **put-call parity** holds at any date $t = 0, 1, \dots, T$

$$C_t - P_t = S_t - K(1+r)^{-(T-t)} = S_t - KB(t,T)$$

  where

$$B(t,T) = (1+r)^{-(T-t)}$$

  is the price at time t of zero-coupon bond maturing at T.

- Using Proposition 7.3 and the put-call parity relationship, one can derive an explicit pricing formula for the European put option with the payoff $P_T = (K - S_T)^+$. This is left as an exercise.

- Moreover, one can derive explicit pricing formulae for the call and put options at any date $t = 0, 1, \dots, T$.

Martingale Approach (MATH3975)

- Check that $0 < \hat{p} = \frac{\widetilde{p}u}{1+r} < 1$ whenever $0 < \widetilde{p} = \frac{1+r-d}{u-d} < 1$.
- Let $\widehat{\mathbb{P}}$ be the probability on $(\Omega, \mathcal{F}_T)$ obtained by setting $\widetilde{p} = \widehat{p}$ in Proposition 7.1. Show that the process $\frac{B}{S}$ is a martingale under $\widehat{\mathbb{P}}$ so that

$$\widehat{\mathbb{P}}(S_t = S_0 u^k d^{t-k}) = \binom{t}{k} \widehat{p}^k (1 - \widehat{p})^{t-k}$$

  for every fixed $t = 1, 2, \ldots, T$ and every $k = 0, 1, \ldots, t$.
- For $t = 0$, the price of the call satisfies

$$C_0 = S_0 \, \widehat{\mathbb{P}}(D) - KB(0,T) \, \widetilde{\mathbb{P}}(D)$$

  where $D = \{S_T > K\} = \{\omega \in \Omega : S_T(\omega) > K\}$. Using the abstract Bayes formula, show that

$$C_t = B_t \, \mathbb{E}_{\widetilde{\mathbb{P}}}\big(B_T^{-1}(S_T - K)^+ \,|\, \mathcal{F}_t\big) = S_t \, \widehat{\mathbb{P}}(D \,|\, \mathcal{F}_t) - KB(t,T) \, \widetilde{\mathbb{P}}(D \,|\, \mathcal{F}_t).$$

PART 3

CALL AND PUT OPTIONS OF AMERICAN STYLE

American Options

- In contrast to a contingent claim of a European style, a claim of an American style can by exercised by its holder at any date before its expiration date T.

Definition (American Call and Put Options)

An American call (put) option is a contract which gives the holder the right to buy (sell) an asset at any time $t \leq T$ at strike price K.

- It the study of an American claim, we are concerned with the price process and the 'optimal' exercise policy by its holder.
- If the holder of an American option exercises it at some date $\tau \in [0, T]$, then the moment τ is called the **exercise time**.

Stopping Times

- An admissible exercise time belongs to the class of **stopping times**.

Definition (Stopping Time)

A **stopping time** with respect to $\mathbb{F}$ is a map $\tau : \Omega \to \{0, 1, \dots, T\}$ such that for any $t = 0, 1, \dots, T$ the event $\{\omega \in \Omega \,|\, \tau(\omega) = t\}$ belongs to the σ-field $\mathcal{F}_t$.

- Intuitively, this means that the decision whether to stop a given process at time t (for instance, whether to exercise an option at time t or not) depends on the stock price fluctuations up to time t only.

Definition

Let $\mathcal{T}_{[t,T]}$ be the subclass of stopping times τ with respect to $\mathbb{F}$ satisfying the inequalities $t \leq \tau \leq T$.

American Call Option

Definition

By an **arbitrage price** of the American call we mean a price process C_t^a, $t \leq T$, such that the extended financial market model – that is, a market with trading in riskless bonds, stocks and an American call option – remains arbitrage-free.

Proposition (7.4)

The price of an American call option in the CRR arbitrage-free market model with $r \geq 0$ coincides with the arbitrage price of a European call option with the same expiry date and strike price.

Proof.

- It suffices to show that the American call option should never be exercised before maturity, since otherwise the issuer of the option would be able to make riskless profit.

$\square$

Proof of Proposition 7.4

Proof of Proposition 7.4 – Step 1.

- The argument hinges on the inequality, for $t = 0, 1, \ldots, T$,

$$C_t \geq (S_t - K)^+. \tag{1}$$

- An intuitive way of deriving (1) is based on no-arbitrage arguments.
- Notice that since the price C_t is always non-negative, it suffices to consider the case when the current stock price is greater than the exercise price: $S_t - K > 0$.
- Suppose, on the contrary, that $C_t < S_t - K$ for some t.
- Then it would be possible, with zero net initial investment, to buy at time t a call option, short a stock, and invest the sum $S_t - C_t > K$ in the savings account.

□

Proof of Proposition 7.4

Proof of Proposition 7.4 – Step 1.

- By holding this portfolio unchanged up to the maturity date T, we would be able to lock in a riskless profit.
- Indeed, the value of our portfolio at time T would satisfy (recall that $r \geq 0$)

$$\begin{aligned} &C_T - S_T + (1+r)^{T-t}(S_t - C_t) \\ &> (S_T - K)^+ - S_T + (1+r)^{T-t}K \geq 0. \end{aligned}$$

- We conclude that inequality (1) is necessary for the absence of arbitrage opportunities.
- In the next step, we assume that (1) holds.

□

Proof of Proposition 7.4

Proof of Proposition 7.4 – Step 2.

- Taking (1) for granted, we may now deduce the property $C_t^a = C_t$ using simple no-arbitrage arguments.
- Suppose, on the contrary, that the issuer of the American call is able to sell the option at time 0 at the price $C_0^a > C_0$.
- In order to profit from this transaction, the issuer of the American call establishes a dynamic portfolio ϕ that replicates the value process of the European call and invests the remaining funds in the savings account.
- Suppose that the holder of the option decides to exercise it at time t before the expiry date T.

□

Proof of Proposition 7.4

Proof of Proposition 7.4 – Step 2.

- Then the issuer of the option locks in a riskless profit, since the value of his portfolio at time t satisfies

$$C_t - (S_t - K)^+ + (1+r)^t (C_0^a - C_0) > 0.$$

- The above reasoning shows that the European and American call options are equivalent from the point of view of arbitrage pricing theory.
- Both options have the same price and an American call should never be exercised by its holder before expiry.
- Note that the assumption $r \geq 0$ was necessary to obtain (1).

$\square$

American Put Option

- Recall that the American put is an option to sell a specified number of shares, which may be exercised at any time before or at the expiry date T.
- For the American put on stock with strike K and expiry date T, we have the following valuation result.

Proposition (7.5)

The arbitrage price P_t^a of an American put option equals

$$P_t^a = \max_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}_{\widetilde{\mathbb{P}}}\big((1+r)^{-(\tau-t)}(K - S_\tau)^+ \,|\, \mathcal{F}_t\big), \quad \forall\, t \leq T.$$

For any $t \leq T$, the stopping time τ_t^* which realizes the maximum is given by the expression

$$\tau_t^* = \min\{u \geq t \,|\, P_u^a = (K - S_u)^+\}.$$

PART 4

DYNAMIC PROGRAMMING APPROACH TO AMERICAN CLAIMS

Dynamic Programming Recursion

- The stopping time τ_t^* is called the **rational exercise time** of an American put option that is assumed to be still alive at time t.
- By an application of the classic **Bellman principle** (1952), it is possible to reduce the optimal stopping problem in Proposition 7.5 to an explicit recursive procedure for the arbitrage price process.
- The following corollary to Proposition 7.5 gives the **dynamic programming recursion** for the arbitrage price of an American put option.
- It is clear that this is an extension of the backward induction approach to arbitrage pricing of European contingent claims.
- We will later see that it can be applied to any contingent claim of American style, for instance, an American call option with variable strike.

Dynamic Programming Recursion

Corollary (Bellman Principle)

Let the process U be defined recursively by the dynamic programming principle: $U_T = (K - S_T)^+$ and, for every $t \leq T - 1$,

$$U_t = \max\Big\{(K - S_t)^+,\ (1+r)^{-1}\,\mathbb{E}_{\widetilde{\mathbb{P}}}(U_{t+1}\,|\,\mathcal{F}_t)\Big\}.$$

Then the arbitrage price P_t^a of the American put option at time t equals U_t and the rational exercise time on $[t, T]$ has the following representation

$$\tau_t^* = \min\{u \geq t\,|\,U_u = (K - S_u)^+\}.$$

Therefore, $\tau_T^* = T$ and for every $t = 0, 1, \dots, T-1$ the Bellman principle holds, that is,

$$\tau_t^* = t\mathbb{1}_{\{U_t = (K - S_t)^+\}} + \tau_{t+1}^*\mathbb{1}_{\{U_t > (K - S_t)^+\}}.$$

Dynamic Programming Recursion

- It is also possible to show directly that the price P_t^a satisfies the recursive relationship, for $t \leq T-1$,

$$P_t^a = \max \Big\{ (K - S_t)^+ ,\, (1+r)^{-1} \, \mathbb{E}_{\widetilde{\mathbb{P}}} \big(P_{t+1}^a \,|\, \mathcal{F}_t \big) \Big\}$$

  subject to the terminal condition $P_T^a = (K - S_T)^+$.

- In the case of the CRR model, this formula reduces the pricing problem to the simple single-period case.

- To show this we shall argue by contradiction. Assume first that (2) fails to hold for $t = T-1$. If this is the case, one may easily construct at time $T-1$ a portfolio which produces riskless profit at time T. Hence, we conclude that necessarily

$$P_{T-1}^a = \max \Big\{ (K - S_{T-1})^+ ,\, (1+r)^{-1} \, \mathbb{E}_{\widetilde{\mathbb{P}}} \big((K - S_T)^+ \,|\, \mathcal{F}_T \big) \Big\}.$$

- This procedure may be repeated as many times as needed.

American Put Option: Summary

To summarise:

- In the CRR model, the arbitrage pricing of the American put option reduces to the following recursive recipe, for $t \leq T-1$,

$$P_t^a = \max\left\{(K - S_t)^+,\, (1+r)^{-1}\big(\widetilde{p}P_{t+1}^{au} + (1-\widetilde{p})P_{t+1}^{ad}\big)\right\}$$

  with the terminal condition

$$P_T^a = (K - S_T)^+.$$

- The quantities P_{t+1}^{au} and P_{t+1}^{ad} represent the values of the American put in the next step corresponding to the upward and downward movements of the stock price starting from a given node on the CRR lattice.

General American Claim

Definition

An **American contingent claim** $X^a = (X, \mathcal{T}_{[0,T]})$ expiring at T consists of a sequence of payoffs $(X_t)_{0 \le t \le T}$ where the random variable X_t is $\mathcal{F}_t$-measurable for $t = 0, 1, \ldots, T$ and the set $\mathcal{T}_{[0,T]}$ of admissible exercise policies.

- We interpret X_t as the **payoff** received by the holder of the claim X^a upon exercising it at time t.
- The set of admissible exercise policies is restricted to the class $\mathcal{T}_{[0,T]}$ of all **stopping times** with values in $\{0, 1, \ldots, T\}$.
- Let $g : \mathbb{R} \times \{0, 1, \ldots, T\} \to \mathbb{R}$ be an arbitrary function. We say that X^a is a **path-independent** American claim with the payoff function g if the equality $X_t = g(S_t, t)$ holds for every $t = 0, 1, \ldots, T$.

General American Claim (MATH3975)

Proposition (7.6)

For every $t \le T$, the arbitrage price $\pi(X^a)$ of an American claim X^a in the CRR model equals

$$\pi_t(X^a) = \max_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}_{\widetilde{\mathbb{P}}}\Big((1+r)^{-(\tau-t)} X_\tau \,|\, \mathcal{F}_t\Big).$$

The price process $\pi(X^a)$ satisfies the following recurrence relation, for $t \le T-1$,

$$\pi_t(X^a) = \max\Big\{X_t,\, \mathbb{E}_{\widetilde{\mathbb{P}}}\Big((1+r)^{-1}\pi_{t+1}(X^a) \,|\, \mathcal{F}_t\Big)\Big\}$$

with $\pi_T(X^a) = X_T$ and the rational exercise time τ_t^* equals

$$\tau_t^* = \min\Big\{u \ge t \,|\, X_u = \mathbb{E}_{\widetilde{\mathbb{P}}}\Big((1+r)^{-1}\pi_{u+1}(X^a) \,|\, \mathcal{F}_u\Big)\Big\}$$

or, equivalently,

$$\tau_t^* = \min\Big\{u \ge t \,|\, X_u \ge \mathbb{E}_{\widetilde{\mathbb{P}}}\Big((1+r)^{-1}\pi_{u+1}(X^a) \,|\, \mathcal{F}_u\Big)\Big\}.$$

Path-Independent American Claim: Valuation

- For a generic value of the stock price S_t at time t, we denote by $\pi^u_{t+1}(X^a)$ and $\pi^d_{t+1}(X^a)$ the values of the price $\pi_{t+1}(X^a)$ at the nodes corresponding to the upward and downward movements of the stock price during the period $[t, t+1]$, that is, for the values uS_t and dS_t of the stock price at time $t+1$, respectively.

Proposition (7.7)

For a path-independent American claim X^a with the payoff process $X_t = g(S_t, t)$ we obtain, for every $t \leq T-1$,

$$\pi_t(X^a) = \max\Big\{ g(S_t, t),\ (1+r)^{-1}\big(\widetilde{p}\,\pi^u_{t+1}(X^a) + (1-\widetilde{p})\,\pi^d_{t+1}(X^a)\big)\Big\}.$$

Path-Independent American Claim: Summary

- We consider a path-independent American claim X^a with the running payoff $g(S_t, t)$ for $t = 0, 1, \dots, T$.
- Let $X_t^a = \pi_t(X^a)$ be the arbitrage price at time t of X^a.
- Then the pricing formula becomes

$$X_t^a = \max \Big\{ g(S_t, t),\ (1+r)^{-1} \big(\widetilde{p}\, X_{t+1}^{au} + (1-\widetilde{p})\, X_{t+1}^{ad} \big) \Big\}$$

  with the terminal condition $X_T^a = g(S_T, T)$. Moreover

$$\tau_t^* = \min \,\{ u \geq t \,|\, g(S_u, u) = X_u^a \}$$

  or, equivalently,

$$\tau_t^* = \min \{ u \geq t \,|\, g(S_u, u) \geq X_u^a \}.$$

- The risk-neutral valuation formula given above is valid for an arbitrary path-independent American claim with a payoff function g in the CRR market model.

Example A: American Call Option

Example (American call)

- We consider here the CRR binomial model with the horizon date $T = 2$ and the risk-free rate $r = 0.2$.
- The stock price S for $t = 0$ and $t = 1$ equals

$$S_0 = 10, \quad S_1^u = 13.2, \quad S_1^d = 10.8.$$

- Let X^a be the American call option with maturity date $T = 2$ and the following payoff process

$$g(S_t, t) = (S_t - K_t)^+.$$

- The strike K_t is **variable** and satisfies

$$K_0 = 9, \quad K_1 = 9.9, \quad K_2 = 12.$$

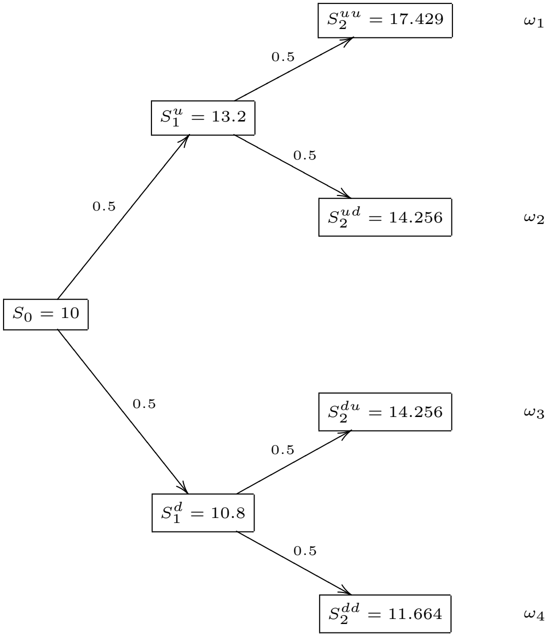
Example A: Risk-Neutral Probability

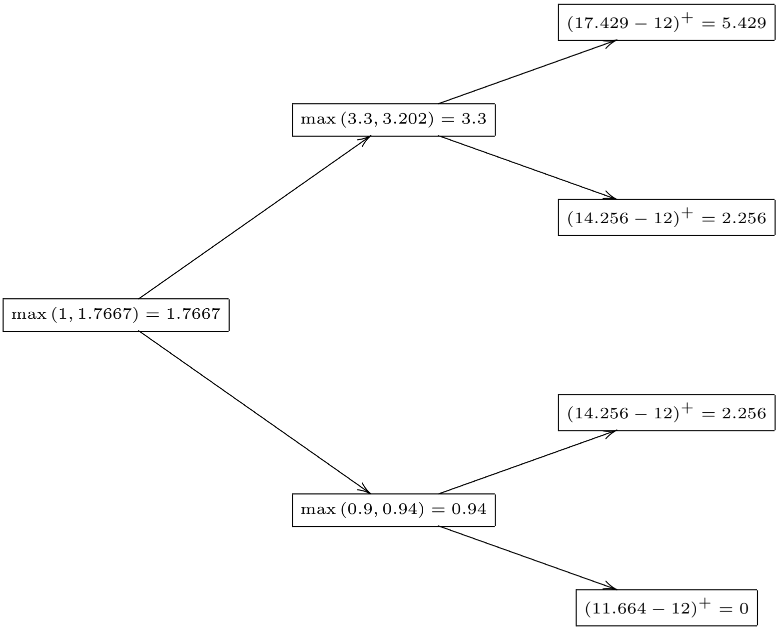
Example (Risk-neutral probability)

- We will first compute the arbitrage price $\pi_t(X^a)$ of this option at times $t = 0, 1, 2$ and the rational exercise time τ_0^*.
- Subsequently, we will compute the replicating strategy for X^a up to the rational exercise time τ_0^*.
- We start by noting that the unique risk-neutral probability measure $\widetilde{\mathbb{P}}$ satisfies

$$\widetilde{p} = \frac{1 + r - d}{u - d} = \frac{(1+r)S_0 - S_1^d}{S_1^u - S_1^d} = \frac{12 - 10.8}{13.2 - 10.8} = 0.5$$

- The dynamics of the stock price under $\widetilde{\mathbb{P}}$ are given by the first exhibit (note that $S_2^{ud} = S_2^{du}$).
- The second exhibit represents the price of the American call option with variable strike K_t.


$S_2^{uu} = 17.429$
ω_1
0.5
$S_1^u = 13.2$
0.5
$S_2^{ud} = 14.256$
ω_2
0.5
$S_0 = 10$
0.5
$S_2^{du} = 14.256$
ω_3
0.5
$S_1^d = 10.8$
0.5
$S_2^{dd} = 11.664$
ω_4


$(17.429 - 12)^+ = 5.429$
$\max(3.3, 3.202) = 3.3$
$(14.256 - 12)^+ = 2.256$
$\max(1, 1.7667) = 1.7667$
$(14.256 - 12)^+ = 2.256$
$\max(0.9, 0.94) = 0.94$
$(11.664 - 12)^+ = 0$

Example A: American Call Option

Example (Computation of the price process $\pi_t(X^a)$)

- **At time 1.** On the event $\{S_1 = uS_0\} = \{\omega_1, \omega_2\}$

$$\pi_1(X^a) = \max\big(3.3, (1.2)^{-1}0.5(5.429 + 2.256)\big) = \max(3.3, 3.202) = 3.3.$$

On the event $\{S_1 = dS_0\} = \{\omega_3, \omega_4\}$

$$\pi_1(X^a) = \max\big(0.9, (1.2)^{-1}0.5(2.256 + 0)\big) = \max(0.9, 0.94) = 0.94.$$

- **At time 0.** At time $t = 0$, we have for every ω

$$\pi_0(X^a) = \max\big(1, (1.2)^{-1}0.5(3.3 + 0.94)\big) = \max(1, 1.7667) = 1.7667.$$

Example A: American Call Option

Example (Holder's rational exercise time)

- **Holder.** The rational holder should exercise the American option at time $t = 1$ if the stock price rises during the first period. Otherwise, the option should be held till time $T = 2$. Hence $\tau_0^* : \Omega \to \{0, 1, 2\}$ equals

$$\tau_0^*(\omega) = 1 \ \text{ for } \ \omega \in \{\omega_1, \omega_2\}$$
$$\tau_0^*(\omega) = 2 \ \text{ for } \ \omega \in \{\omega_3, \omega_4\}$$

- **Issuer.** We now examine the situation of the issuer of the option. At time $t = 0$, we need to solve

$$1.2\,\phi_0^0 + 13.2\,\phi_0^1 = 3.3$$
$$1.2\,\phi_0^0 + 10.8\,\phi_0^1 = 0.94$$

Hence $(\phi_0^0, \phi_0^1) = (-8.067, 0.983)$ for all ω.

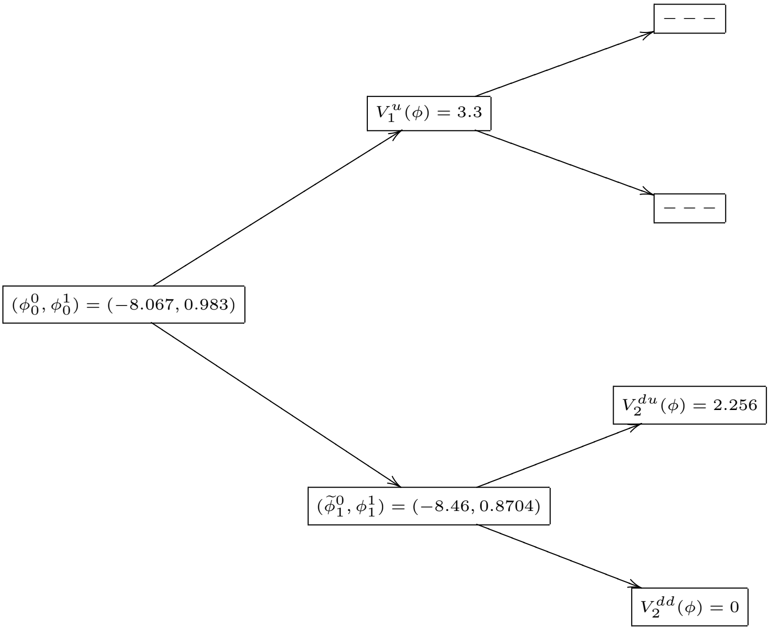
Example A: Replicating Strategy

Example (Issuer's replicating strategy)

- If the stock price rises during the first period, the option is exercised and thus we do not need to compute the strategy at time 1 for $\omega \in \{\omega_1, \omega_2\}$.
- If the stock price falls during the first period, we solve

$$1.2\,\widetilde{\phi}_1^0 + 14.256\,\phi_1^1 = 2.256$$
$$1.2\,\widetilde{\phi}_1^0 + 11.664\,\phi_1^1 = 0$$

- Hence $(\widetilde{\phi}_1^0, \phi_1^1) = (-8.46, 0.8704)$ for $\omega \in \{\omega_3, \omega_4\}$.
- Note that $\widetilde{\phi}_1^0 = -8.46$ is the amount of cash borrowed at time 1, rather than the number of units of the savings account B.
- The replicating strategy $\phi = (\phi^0, \phi^1)$ is defined at time 0 for all ω and it is defined at time 1 on the event $\{\omega_3, \omega_4\}$ only.


$- - -$
$V_1^u(\phi) = 3.3$
$- - -$
$(\phi_0^0, \phi_0^1) = (-8.067, 0.983)$
$V_2^{du}(\phi) = 2.256$
$(\widetilde{\phi}_1^0, \phi_1^1) = (-8.46, 0.8704)$
$V_2^{dd}(\phi) = 0$

PART 5

IMPLEMENTATION OF THE CRR MODEL

Derivation of u and d from r and σ

- We fix T and we assume that the **continuously compounded** interest rate r is constant so that $B(0,T) = e^{-rT}$.
- From the market data for stock prices, one can estimate the stock price **volatility** σ per one time unit (typically, one year).
- Note that until now we assumed that $t = 0, 1, 2, \dots, T$, which means that $\Delta t = 1$. In general, the length of each period can be any positive number smaller than 1. We set $n = T/\Delta t$.
- Two widely used conventions for obtaining u and d from σ and r are:
  - **The Cox-Ross-Rubinstein (CRR) parametrisation**:
$$u = e^{\sigma\sqrt{\Delta t}} \quad \text{and} \quad d = \frac{1}{u} = e^{-\sigma\sqrt{\Delta t}}.$$
  - **The Jarrow-Rudd (JR) parameterisation**:
$$u = e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}} \quad \text{and} \quad d = e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}}.$$

The Cox-Ross-Rubinstein parameterisation

Proposition (7.8)

Assume that $B_{k\Delta t} = (1 + r\Delta t)^k$ *for* $k = 0, 1, \ldots, n$ *and* $u = d^{-1} = e^{\sigma\sqrt{\Delta t}}$ *in the CRR model. Then the risk-neutral probability measure* $\widetilde{\mathbb{P}}$ *satisfies*

$$\widetilde{\mathbb{P}}\left(S_{t+\Delta t} = S_t u \,|\, S_t\right) = \frac{1}{2} + \frac{r - \frac{\sigma^2}{2}}{2\sigma}\sqrt{\Delta t} + o(\sqrt{\Delta t})$$

provided that Δt *is sufficiently small.*

Proof of Proposition 7.8.

The risk-neutral probability measure for the CRR model is given by

$$\widetilde{p} = \widetilde{\mathbb{P}}\left(S_{t+\Delta t} = S_t u \,|\, S_t\right) = \frac{1 + r\Delta t - d}{u - d}$$

$\square$

The CRR parameterisation

Proof of Proposition 7.8.

Under the CRR parametrisation, we obtain

$$\widetilde{p} = \frac{1 + r\Delta t - d}{u - d} = \frac{1 + r\Delta t - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}.$$

The Taylor expansions up to the second order term are

$$\begin{aligned} e^{\sigma\sqrt{\Delta t}} &= 1 + \sigma\sqrt{\Delta t} + \frac{\sigma^2}{2}\Delta t + o(\Delta t), \\ e^{-\sigma\sqrt{\Delta t}} &= 1 - \sigma\sqrt{\Delta t} + \frac{\sigma^2}{2}\Delta t + o(\Delta t). \end{aligned}$$

$\square$

The CRR parameterisation

Proof of Proposition 7.8.

By substituting the Taylor expansions into the risk-neutral probability measure, we obtain

$$
\begin{aligned}
\widetilde{p} &= \frac{1 + r\Delta t - \left(1 - \sigma\sqrt{\Delta t} + \frac{\sigma^2}{2}\Delta t\right) + o(\Delta t)}{\left(1 + \sigma\sqrt{\Delta t} + \frac{\sigma^2}{2}\Delta t\right) - \left(1 - \sigma\sqrt{\Delta t} + \frac{\sigma^2}{2}\Delta t\right) + o(\Delta t)} \\
&= \frac{\sigma\sqrt{\Delta t} + \left(r - \frac{\sigma^2}{2}\right)\Delta t + o(\Delta t)}{2\sigma\sqrt{\Delta t} + o(\Delta t)} \\
&= \frac{1}{2} + \frac{r - \frac{\sigma^2}{2}}{2\sigma}\sqrt{\Delta t} + o(\sqrt{\Delta t})
\end{aligned}
$$

as was required to show. $\square$

The CRR parameterisation

- To summarise, for Δt sufficiently small, we get

$$\widetilde{p} = \frac{1 + r\Delta t - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \approx \frac{1}{2} + \frac{r - \frac{\sigma^2}{2}}{2\sigma}\sqrt{\Delta t}.$$

- Note that $1 + r\Delta t \approx e^{r\Delta t}$ when Δt is sufficiently small.
- Hence the risk-neutral probability measure can also be represented as follows

$$\widetilde{p} \approx \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}.$$

- More formally, if we define $\hat{r}$ such that $(1 + \hat{r})^n = e^{rT}$ for a fixed T and $n = T/\Delta t$ then $\hat{r} \approx r\Delta t$ since $\ln(1 + \hat{r}) = r\Delta t$ and $\ln(1 + \hat{r}) \approx \hat{r}$ when $\hat{r}$ is close to zero.

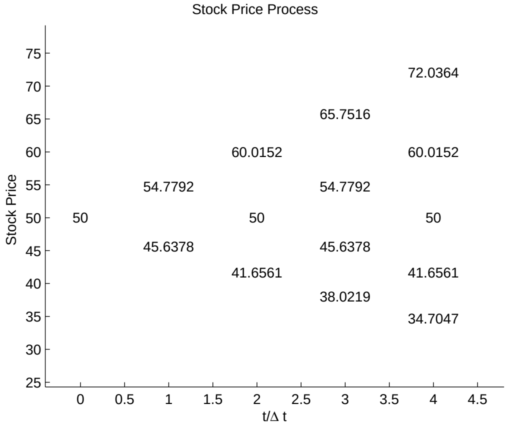
Example B: CRR Parameterisation

Example (CRR parameterisation)

- Let the annualized variance of logarithmic returns be $\sigma^2 = 0.1$.
- The interest rate is set to $r = 0.1$ per annum.
- Suppose that the current stock price is $S_0 = 50$.
- We examine European and American put options with strike price $K = 53$ and maturity $T = 4$ months (i.e. $T = \frac{1}{3}$).
- The length of each period is $\Delta t = \frac{1}{12}$, that is, one month.
- Hence $n = \frac{T}{\Delta t} = 4$ steps.
- We adopt the CRR parameterisation to derive the stock price.
- Then $u = 1.0956$ and $d = 1/u = 0.9128$.
- We compute $1 + r\Delta t = 1.00833 \approx e^{r\Delta t}$ and $\widetilde{p} = 0.5228$.

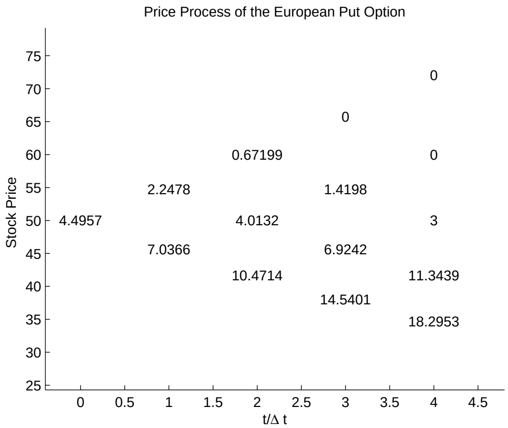
Example B: American Put Option

Example (Stock price process)



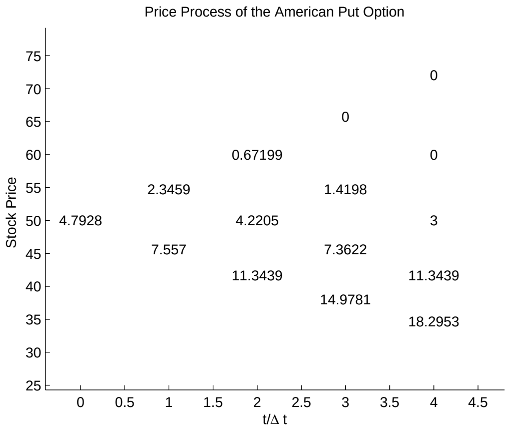
Example B: European Put Option

Example (European put option price)



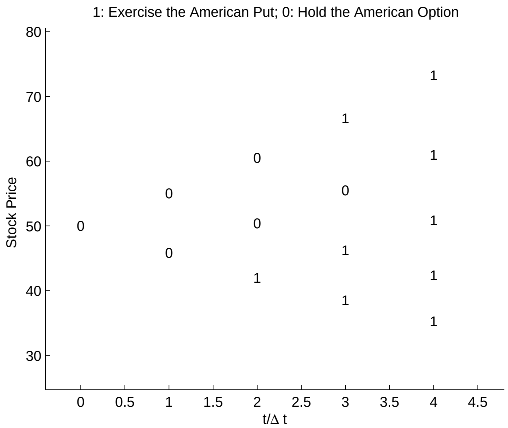
Example B: American Put Option

Example (American put option price)



Example B: American Put Option

Example (Rational exercise time)



The Jarrow-Rudd parameterisation

The next result deals with the Jarrow-Rudd parametrisation.

Proposition (7.9)

Let $B_{k\Delta t} = (1 + r\Delta t)^k$ *for* $k = 0, 1, \dots, n$. *We assume that*

$$u = e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}}$$

and

$$d = e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}}.$$

Then the risk-neutral probability measure $\widetilde{\mathbb{P}}$ *satisfies*

$$\widetilde{\mathbb{P}}\left(S_{t+\Delta t} = S_t u \,|\, S_t\right) = \frac{1}{2} + o(\Delta t)$$

provided that Δt *is sufficiently small.*

The JR parameterisation

Proof of Proposition 7.9.

Under the JR parametrisation, we have

$$\widetilde{p} = \frac{1 + r\Delta t - d}{u - d} = \frac{1 + r\Delta t - e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}}}{e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}} - e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}}}.$$

The Taylor expansions up to the second order term are

$$e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}} = 1 + r\Delta t + \sigma\sqrt{\Delta t} + o(\Delta t)$$
$$e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}} = 1 + r\Delta t - \sigma\sqrt{\Delta t} + o(\Delta t)$$

and thus

$$\widetilde{p} = \frac{1}{2} + o(\Delta t).$$

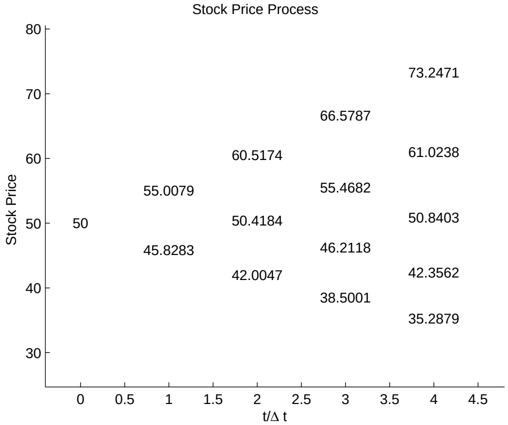
$\square$

Example C: JR Parameterisation

Example (JR parameterisation)

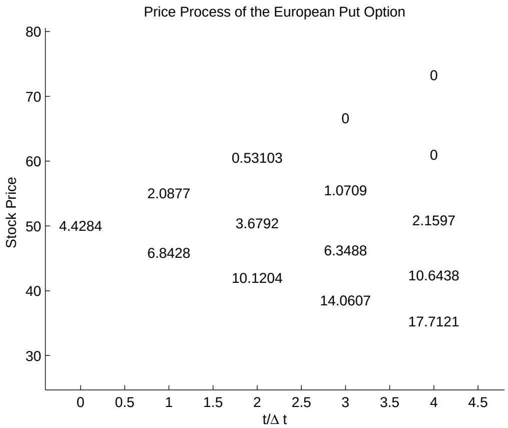
- We consider Example B, but with parameters u and d computed using the JR parameterisation. We obtain $u = 1.1002$ and $d = 0.9166$.
- As before, $1 + r\Delta t = 1.00833 \approx e^{r\Delta t}$, but $\widetilde{p} = 0.5$.
- We compute the price processes for the stock, the European put option and the American put option.
- If we compare with Example B, then we see that the prices are slightly different, although the rational exercise time is identical.
- The CRR and JR parameterisations were both devised to give a good approximation of the continuous-time Black-Scholes model.
- If Δt is sufficiently small, then the prices computed under the two parametrisations will be very close to each other.

Example (Stock price)



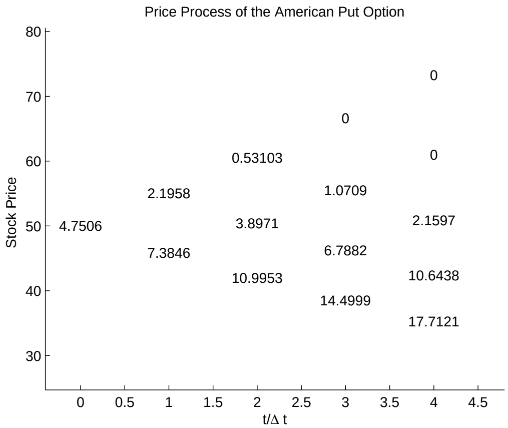
Example C: European Put Option

Example (European put option price)



Example C: American Put Option

Example (American put option price)



Example C: Rational Exercise Time

Example (Rational exercise time)