Optimal management of pension fund under relative performance ratio and VaR constraint

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Abstract

This article studies a stochastic control problem of non-self-financing, non-linear and with Value-at-Risk constraints, which belongs to the problem of defined contribution pension risk management. Different from the traditional expected utility theory, we use the relative performance ratio (the ratio of high performance to low performance) of wealth at the time of retirement to evaluate investment performance at the time of retirement, which leads to the non-linearity of this problem. Due to the consideration of defined contribution items, this problem becomes non-self-financing.

We first convert the problem into an equivalent self-financing model by introducing an auxiliary process, and then solve the original planning problem based on the martingale method, fractional programming method, Lagrange multiplier method, and concave (dual) method, thus obtaining the optimal decision process, optimal terminal wealth and optimal value of pension. At the end of the article, we conduct a sensitivity analysis and discussion on the optimal value.

Keywords: Performance ratio, Value-at-Risk constraint, Martingale method, Lagrange multiplier method, Defined contribution pension plan

1 The investment problem for a DC pension fund

This part is mainly related to Dong and Zheng (2020).

2 Wealth process

Consider a complete probability space $(\Omega, \mathscr{F}, P)$, above which we consider the augment filtration $\mathbb{F} := \{\mathscr{F}_t | 0 \leqslant t \leqslant T\}$ of the natural σ-filtration generated by the standard Brownian motion $W := \{W(t) | 0 \leqslant t \leqslant T\}$. The pension fund starts at the initial time 0 and the retirement time is T. The pension fund manager can adjust the strategy within time horizon $[0, T]$. All the processes introduced below are well-defined and adapted to $\{\mathcal{F}_t, t \in [0, T]\}$.

2.1 Financial market

Because the investment horizon of a DC pension fund is relatively long, often 20-40 years, the wealth of the fund is faced with many risks from the financial market. Especially, inflation risk will decrease the real purchasing power of the fund and plays a prominent part in the risk management. In our work, we consider the effect of inflation risk on the DC pension fund. We introduce the financial market with inflation risk and three different assets. To simplify the financial model, we assume that the nominal interest rate and the real interest rate are deterministic. We are mainly concerned with the impact of inflation on the behavior of the pension fund. The financial market in our model consists of cash, bond and stock.

The price of the risk-free (i.e., cash) asset $S_0(t)$ is characterized by

$$\frac{\mathrm{d}S_0(t)}{S_0(t)} = r_n(t)\mathrm{d}t, \qquad S_0(0) = S_0, \tag{2.1}$$

where S_0 is a constant. $r_n(t)$ represents nominal interest rate in the market.

Next, we derive the financial model of the inflation index by the Fisher equation. The Fisher equation describes relationships among the nominal interest rate $r_n(t)$, the real interest rate $r_r(t)$ and the inflation index $I(t)$. The inflation index $I(t)$ reflects a reduction in purchasing power per unit of money.

We present the following extended continuous-time Fisher equation given by Zhang(cf.Zhang et al. (2007)(2007), **?**(2014)):

$$\begin{cases} r_n(t) - r_r(t) = \lim\limits_{\Delta t \to 0} \dfrac{1}{\Delta t} \widetilde{\mathbf{E}}[i(t, t+\Delta t) | \mathcal{F}_t], \\ i(t, t+\Delta t) = \dfrac{I(t+\Delta t) - I(t)}{I(t)}, \end{cases} \tag{2.2}$$

where $\widetilde{\mathbf{E}}$ is the expectation under risk neutral measure $\widetilde{\mathbf{P}}$ and $i(t, t+\Delta t)$ is the inflation rate from time t to $t + \Delta t$. Thus, we characterize the risk of the inflation by the Brownian motion $\widetilde{W}_I(t)$, and the following model is an efficient model of $I(t)$ to satisfy the extended Fisher equation:

$$\frac{\mathrm{d}I(t)}{I(t)} = (r_n(t) - r_r(t))\mathrm{d}t + \sigma_I \mathrm{d}\widetilde{W}_I(t), \tag{2.3}$$

where $\widetilde{W}_I(t)$ is a standard Brownian motion under the risk-neutral measure $\widetilde{\mathbf{P}}$. Besides, $\widetilde{W}_I(t)$ is independent of $W_0(t)$.

Denote market price of risk of $W_I(t)$ by λ_I. Then, $W_I(t)$ is also independent of $W_0(t)$.

Therefore, by Girsanov's theorem we can derive the stochastic inflation index $I(t)$ w.r.t. the original measure $\mathbf{P}$ as follows:

$$\frac{\mathrm{d}I(t)}{I(t)}=(r_n(t)-r_r(t))\mathrm{d}t+\sigma_I[\lambda_I\mathrm{d}t+\mathrm{d}W_I(t)], I(0)=I_0. \tag{2.4}$$

In order to hedge risk of the inflation, we introduce here an inflation-indexed zero coupon bond. An inflation-indexed zero coupon bond $P(t,T)$ is a contract at time t with final payment of real money \$1 at maturity T. Different from the general zero-coupon bond, $P(t,T)$ delivers $I(T)$ at maturity T. Therefore, based on pricing formula of derivatives, the price of $P(t,T)$ is $P(t,T) = \widetilde{\mathbf{E}}[\exp(-\int_t^T r_n(s)\mathrm{d}s)I(T)|\mathcal{F}_t]$. Since the nominal interest rate in our model is deterministic, a simple calculation can show that explicit form of $P(t,T)$ is

$$P(t,T) = I(t)\exp[-\int_t^T r_r(s)\mathrm{d}s].$$

We can see that $P(t,T)$ is in fact the nominal wealth of the discounted wealth of \$1 in the real market. Thus $P(t,T)$ also satisfies the following backward stochastic differential equation:

$$\begin{cases} \frac{\mathrm{d}P(t,T)}{P(t,T)} = r_n(t)\mathrm{d}t + \sigma_I[\lambda_I\mathrm{d}t + \mathrm{d}W_I(t)], \\ P(T,T) = I(T). \end{cases} \tag{2.5}$$

The third asset is a stock in the market. The price $S_1(t)$ of the stock is as follows:

$$\begin{cases} \frac{\mathrm{d}S_1(t)}{S_1(t)} = r_n(t)\mathrm{d}t + \sigma_{S_1}(\lambda_I\mathrm{d}t + \mathrm{d}W_I(t)) + \sigma_{S_2}(\lambda_S\mathrm{d}t + \mathrm{d}W_S(t)), \\ S_1(t) = S_1, \end{cases} \tag{2.6}$$

where σ_{S_1} and σ_{S_2} are positive constants and represent the volatilities of the stock. $W_S(t)$ is a standard Brownian motion on space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbf{P})$ and independent of $W_0(t)$ and $W_I(t)$. Moreover, λ_S is the price of market risk of $W_S(t)$.

2.2 DC pension fund

Before retirement, the pension particpants put part of the salary in the DC pension fund. As the salary is often not determinate while depends on the marcoeconomics, it is realist to assume that the contribution rate is a stochastic process. Denote the contribution rate at time t by $c(t)$. We suppose that $c(t)$ satisfies the following equation,

$$\frac{dc(t)}{c(t)} = \mu dt + \sigma_{C_1}\mathrm{d}W_I(t) + \sigma_{C_2}\mathrm{d}W_S(t), c(0) = c_0. \tag{2.7}$$

In order to hedge the risks from the financial market, the pension manager invests in the market continuously within time horizon $[0,T]$. Assume that there are no transaction costs or taxes in the market and short buying is also allowed. Denote money invested in the cash, inflation-indexed zero coupon bond and stock by $\pi_0(t)$, $\pi_P(t)$ and $\pi_S(t)$, respectively. Then the wealth of the DC fund is as follows:

$$\begin{cases} \mathrm{d}X(t) = c(t)dt + \pi_0(t)\frac{\mathrm{d}S_0(t)}{S_0(t)} + \pi_P(t)\frac{\mathrm{d}P(t,T)}{P(t,T)} + \pi_S(t)\frac{\mathrm{d}S_1(t)}{S_1(t)}, \\ X(0) = X_0. \end{cases} \tag{2.8}$$

X_0 is the initial wealth of the pension fund. Substituting (2.1), (2.5) and (2.6) into the last equation, we can obtain the compact form of the wealth process:

$$\begin{cases} \mathrm{d}X(t) = & c(t)dt + r_n(t)X(t)\mathrm{d}t + \pi_P(t)\sigma_I[\lambda_I\mathrm{d}t + \mathrm{d}W_I(t)] \\ & +\pi_S(t)\sigma_{S_1}[\lambda_I\mathrm{d}t + \mathrm{d}W_I(t)] + \pi_S(t)\sigma_{S_2}[\lambda_S\mathrm{d}t + \mathrm{d}W_S(t)], \\ X(0) = & X_0 \end{cases} \tag{2.9}$$

where we have just used $X(t) = \pi_0(t) + \pi_P(t) + \pi_S(t)$. Denote $\pi(t) = (\pi_P(t), \pi_S(t))$, which represents investment strategies. We call $\pi(t)$ an admissible strategy if it satisfies the following conditions:

(i) $\pi_P(t), \pi_S(t)$ are progressively measurable w.r.t. $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbf{P})$;

(ii) $\mathbf{E}\{\int_0^T [\pi_P(t)^2\sigma_I^2 + \pi_S(t)^2\sigma_{S_1}^2 + \pi_S(t)^2\sigma_{S_2}^2]\mathrm{d}t\} < +\infty$;

(iii) Eq.(2.9) has a unique strong solution for the initial data

$(t_0, I(0), X(0)) \in [0,T] \times (0,+\infty)^2$.

Denote set of all admissible reinsurance and investment strategies of $\pi(t)$ by Π. We are only cared about the admissible strategies.

3 Optimal management with performance ratio

The traditional investment optimization problem considers maximizing the expected utility of the Sharpe ratio, which does not distinguish investors' preferences on gains and losses. Therefore, we consider the relative performance ratio to characterize the performance of the invested pension funds.

Keating and Shadwick (2002) considered an "Ω-measure": given a reference level value θ, the "Ω-measure" of the random variable R is defined as

$$\Omega_\theta(R) = \frac{\mathbb{E}[(R-\theta)_+]}{\mathbb{E}[(\theta-R)_+]}, \tag{3.1}$$

where $x_+ := \max\{x, 0\}$ represents the positive part of x. Because inflation risk exists in the financial market, the penrison manager is concerned with the real terminal wealth at retirement. Therefore, if we consider the optimization problem of the real wealth of the portfolio at the time of retirement determined by the equation (**??**) regarding the "Ω-measure", the optimaization problem is formulated as:

$$\max_{\pi\in\mathcal{A}(x_0)}\left\{\Omega_\theta(\frac{X^\pi(T)}{I(T)}):=\frac{\mathbb{E}[(\frac{X^\pi(T)}{I(T)}-\theta)_+]}{\mathbb{E}[(\theta-\frac{X^\pi(T)}{I(T)})_+]}\right\}. \tag{3.2}$$

However, for this problem, we will find that it is unbounded in the following note (4.3). Therefore, we consider transforming "Ω-measure": introducing two weighting function $U:\mathbb{R}_+\mapsto\mathbb{R}$ and $D:\mathbb{R}_+\mapsto\mathbb{R}$, which are both monotonically increasing measurable function, at which time the objective function becomes:

$$R(X^\pi(T)):=\frac{\mathbb{E}\{U[(\frac{X^\pi(T)}{I(T)}-\theta)_+]\}}{\mathbb{E}\{D[(\theta-\frac{X^\pi(T)}{I(T)})_+]\}}. \tag{3.3}$$

At this time, the numerator $\mathbb{E}\{U[(\frac{X^\pi(T)}{I(T)}-\theta)_+]\}$ represents the income when the wealth value at the time of retirement exceeds the reference level, and the denominator $\mathbb{E}\{D[(\frac{X^\pi(T)}{I(T)}-\theta)_+]\}$ represents the punishment when the wealth value at the time of retirement is lower than the reference level. Therefore, the function U is also called the reward function, and the function D is called the penalty function. Then, the problem becomes

$$\max_{\pi\in\mathcal{A}(x_0)}\left\{\frac{\mathbb{E}\{U[(\frac{X^\pi(T)}{I(T)}-\theta)_+]\}}{\mathbb{E}\{D[(\theta-\frac{X^\pi(T)}{I(T)})_+]\}}\right\}, \tag{3.4}$$

which is the model in Lin et al. (2019).

Furthermore, in insurance risk management, it is necessary to ensure the solvency of insurance companies. We impose VaR constraints on pension funds at the time of retirement to ensure their solvency here: the probability of the real wealth higher than another reference value L at retirement time is required to be at least $1-\varepsilon$. That is to say, we need

$$P(\frac{X^\pi(T)}{I(T)}\geqslant L)\geqslant 1-\varepsilon, \tag{3.5}$$

where $0\leqslant\varepsilon\leqslant 1$ is a given constant. Then the optimization problem becomes

$$\begin{cases}\max\limits_{\pi\in\mathcal{A}(x_0)} & \dfrac{\mathbb{E}\{U[(\frac{X^\pi(T)}{I(T)}-\theta)_+]\}}{\mathbb{E}\{D[(\theta-\frac{X^\pi(T)}{I(T)})_+]\}},\\ s.t. & \dfrac{X^\pi(T)}{I(T)} \text{ satisfies } (\textbf{??}),\\ & P(\dfrac{X^\pi(T)}{I(T)}\geqslant L)\geqslant 1-\varepsilon,\end{cases} \tag{3.6}$$

which is the optimization problem considered in this article.

It is easy to see: when $\varepsilon=1$, the constraint is equivalent to no constraint, and when $\varepsilon=0$, the wealth is required to be higher than a certain level at retirement time, which is the optimization problem considered by Basak (1995).

4 Optimal Strategy

4.1 Auxillary process

Because of the existence of fixed contributions, the wealth process X^{π} is not a self-financing process. For the convenience of processing, we introduce the auxiliary wealth process $\tilde{X}^{\pi} := \{\tilde{X}^{\pi}(t)|0 \leqslant t \leqslant T\}$ to convert the model into an equivalent self-financing problem. First, we introduce the value $D(t,s)$ of a derivative at time t with final payment $c(s)$ at maturity s. Based on the theory of derivative pricing,

$$D(t,s) = \widetilde{\mathbf{E}}[\exp(-\int_t^s r_n(u)du)c(s)|\mathcal{F}_t].$$

By the Feyman-Kac's formula, $D(t,s)$ satisfies the following partial differential equation,

The above equation can be solved explicitly. We can obtain the explicit form of $D(t,s)$ as follows,

Moreover, we can derive the differential equation of $D(t,s)$,

Next, we integrate $D(t,s)$ w.r.t. s from time t to time T and construct a new process $F(t)$.

$$F(t) = \int_t^T D(t,s)ds.$$

$F(t)$ represents the expected value of the accumulated contribution rate from time t to T at time t. Based on the differential form of $D(t,s)$, the dynamics of $F(t)$ are

Next, we construct an auxillary process as

$$\tilde{X}^{\pi}(t) = \frac{X^{\pi}(t) + F(t)}{I(t)}. \tag{4.1}$$

Because $F(T) = 0$, $\tilde{X}^{\pi}(T) = \frac{X^{\pi}(T)}{I(T)}$. So the optimization problem over $\frac{X^{\pi}(T)}{I(T)}$ is equilivalent to the optimization problem over $\tilde{X}^{\pi}(T)$. Then from the equation (**??**) we have

$$d\tilde{X}^{\pi}(t) = \tag{4.2}$$

and $\tilde{X}^{\pi}(0) = \frac{X^{\pi}(0)+F(0)}{I(0)} := \tilde{x}_0$. Then the problem becomes

$$\begin{cases} \max\limits_{\pi \in \tilde{\mathcal{A}}(\tilde{x}_0)} & \dfrac{\mathbb{E}\{U[(\tilde{X}^{\pi}(T) - \theta)_+]\}}{\mathbb{E}\{D[(\theta - \tilde{X}^{\pi}(T))_+]\}}, \\ s.t. & \tilde{X}^{\pi}(t) \text{ satisfies (4.2)}, \tilde{X}^{\pi}(0) = \tilde{x}_0, \\ & P(\tilde{X}^{\pi}(T) \geqslant L) \geqslant 1 - \varepsilon. \end{cases} \tag{4.3}$$

That is, we have transformed the original non-self-financing problem(3.6) into the self-financing problem(4.3).

4.2 Martingale method

Problem(4.3) is a non-linear optimization problem with constrained objectives, which cannot be solved by traditional dynamic programming methods. Since the market is complete, we refer to the Martingale method adopted by Cox and Huang (1989) for solution.

First consider the pricing kernel process $H(t)$

$$H(t) := \exp\left\{-\left(r+\frac{\xi^2}{2}\right)t - \xi W(t)\right\}. \tag{4.4}$$

That is

$$\begin{cases} dH(t) = H(t)[-rdt - \xi dW(t)], \\ H(0) = 1. \end{cases} \tag{4.5}$$

By the following theorem, we can transform the initial dynamic programming problem of optimizing over the portfolio process into a static optimization problem of optimizing over measurable random variables.

Theorem 4.1. *The optimization problem (4.3) and the following problem (4.6) have the same optimal value, where $\mathcal{M}_+$ denotes the set of non-negative $\mathcal{F}_T$-measurable random variables.*

$$\begin{cases} \max\limits_{Z\in\mathcal{M}_+} & \dfrac{\mathbb{E}\{U[(Z-\theta)_+]\}}{\mathbb{E}\{D[(\theta-Z)_+]\}}, \\ s.t. & \mathbb{E}[H(T)Z] \leqslant \tilde{x}_0, \\ & P(Z \geqslant L) \geqslant 1-\varepsilon. \end{cases} \tag{4.6}$$

Proof. We first proof that the optimization problem(4.3) is equivalent to the problem (4.7)

$$\begin{cases} \max\limits_{\pi\in\tilde{\mathcal{A}}(\tilde{x}_0)} & \dfrac{\mathbb{E}\{U[(\tilde{X}^\pi(T)-\theta)_+]\}}{\mathbb{E}\{D[(\theta-\tilde{X}^\pi(T))_+]\}}, \\ s.t. & \tilde{X}^\pi(t) \text{ satisfies (4.2)}, \tilde{X}^\pi(0) = \tilde{x}_0, \\ & \mathbb{E}[H(T)\tilde{X}^\pi(T)] \leqslant \tilde{x}_0, \\ & P(\tilde{X}^\pi(T) \geqslant L) \geqslant 1-\varepsilon. \end{cases} \tag{4.7}$$

Comparing with problem(4.3), it is easy to see that the problem (4.7) has just one more restraint:

$$\mathbb{E}[H(T)\tilde{X}^\pi(T)] \leqslant \tilde{x}_0. \tag{4.8}$$

Using ItÔ's formula and (4.5), (4.2), we can easily get

$$\begin{aligned} d[H(t)\tilde{X}^\pi(t)] =& H(t)[r\tilde{X}^\pi(t) + \pi(t)\sigma\xi]dt + H(t)\pi(t)\sigma dW(t) \\ &+ \tilde{X}^\pi(t)H(t)[-rdt - \xi dW(t)] \\ &+ H(t)(-\xi)\times\pi(t)\sigma dt \\ =& H(t)[\pi(t)\sigma - \xi\tilde{X}^\pi(t)]dW(t). \end{aligned} \tag{4.9}$$

Then we have

$$H(t)\tilde{X}^{\pi}(t)=\tilde{X}^{\pi}(0)+\int_0^t H(s)[\pi(s)\sigma-\xi\tilde{X}^{\pi}(s)]dW(s),\quad t\in[0,T]. \tag{4.10}$$

While the right part of the integration about t is a local martingale, then we have $H\tilde{X}^{\pi}:=\{H(t)\tilde{X}^{\pi}(t),t\geqslant 0\}$ is a non-negative local martingale about $\{\mathcal{F}_t,t\geqslant 0\}$, so is a super-martingale (see Karatzas and Shreve (1991)). So we have

$$\mathbb{E}[H(T)\tilde{X}^{\pi}(T)]\leqslant\mathbb{E}[H(0)\tilde{X}^{\pi}(0)]=\tilde{x}_0. \tag{4.11}$$

That is, the inequality constraint (4.8) is contained in the condition (4.2), so it's easy to see the problem (4.7) is equivalent to the problem (4.3).

It is easy to see that compared with the problem (4.7), the problem (4.6) has less restrictions on the condition (4.2), so the optimal target value of the problem (4.6) is not less than the optimal target value of the problem (4.7).

On the other hand, from the following proposition (4.2), it can be seen that for any optimal solution to the problem (4.6), a portfolio process can be found so that the wealth at the time of retirement is just the optimal solution of problem (4.6). So it is known that the optimal target value of the optimization problem (4.7) is not less than the optimal target value of the optimization problem (4.6). Finally, we see the optimal value of the two is equal. □

Proposition 4.2. *Assume Z^* is the optimal solution to problem (4.6), then there is a portfolio process $\pi^*\in\tilde{\mathcal{A}}(\tilde{x}_0)$ which satisfies $\tilde{X}^{\pi^*}(T)=Z^*$.*

Proof. As Z^* is the optimal solution to the problem (4.6), it is easy to see that there must be inequality constraints $\mathbb{E}[H(T)Z]\leqslant\tilde{x}_0$ just happens to be equal, that is, there must be $\mathbb{E}[H(T)Z^*]=\tilde{x}_0$. Otherwise, consider

$$\tilde{Z}:=Z^*+e^{rT}(\tilde{x}_0-\mathbb{E}[H(T)Z^*]). \tag{4.12}$$

In the process of investing excess assets in risk-free returns, it is easy to find that $\tilde{Z}$ is still a feasible solution to the problem (4.6), but corresponds to a larger target value. Define the process

$$Y^*(t):=H(t)^{-1}\mathbb{E}[H(T)Z^*|\mathcal{F}_t],\quad 0\leqslant t\leqslant T. \tag{4.13}$$

It's easy to verify that $\{H(t)Y^*(t),t\geqslant 0\}$ is a martingale about $\{\mathcal{F}_t,t\geqslant 0\}$. By the representation of martingale, there exists a $\mathbb{F}$-progressively measurable process $\{\psi(t),0\leqslant t\leqslant T\}$ such that $\int_0^T\psi(t)^2dt<\infty,a.s.$ and

$$H(t)Y^*(t)=\tilde{x}_0+\int_0^t\psi(s)dW(s),\quad 0\leqslant t\leqslant T,\ a.s.. \tag{4.14}$$

Write

$$\pi^*(t)=H(t)^{-1}\psi(t)\sigma^{-1}+Y^*(t)\xi\sigma^{-1},\quad 0\leqslant t\leqslant T. \tag{4.15}$$

Then we have $\pi^* \in \tilde{\mathcal{A}}(\tilde{x}_0)$ and $\tilde{X}^{\pi^*}(t) = Y^*(t), 0 \leqslant t \leqslant T, a.s.$. Particularly, we have $\tilde{X}^{\pi^*}(T) = Y^*(T) = Z^*$. □

Remark 4.1. It can be seen from the above proof that Theorem 4.1 and Proposition 4.2 not only show that the optimal values of problem (4.3) and problem (4.6) are equal, but also give the relationship between the solutions of the two problems. Thus we have turned solving the dynamic optimization problem (4.3) into solving optimization problems (4.6). So in the following solution, we only need to solve the optimization problem (4.6).

4.3 Linearization

The objective problem here is different from the traditional maximum expected utility problem. The relative performance ratio we adopt is a non-linear objective. In order to make the problem solvable, we first make some assumptions about the reward function U and the penalty function D:

(H1) Assume that U and D are both twice differentiable and strictly increaseable functions with an initial value of 0.

Remark 4.2. The monotony of U and D is easily explained by its meaning, and the condition of the initial value of 0 makes the problem close to the "Ω-measure" problem considered by the predecessors when the wealth at retirement time is approximately equal to the reference level.

Remark 4.3. According to the results of Jin and Zhou (2008), a list of random variables Z_n can be constructed satisfying $\mathbb{E}[H(T)Z_n] = \tilde{x}_0$ but $\mathbb{E}[Z_n] \to \infty$. So it is easy to see that if the reward function U is convex, the optimization problem (4.6) is unbounded without VaR constraints. Therefore, in order to ensure the convergence of the problem and at the same time for the convenience of solving, we assume that the reward function U is a concave function (this is also more common in the usual maximization problem, and the objective function is a concave function, indicating that investment is a risk avoidance type).

More specifically, we assume that the reward function U meets the following conditions:

(H2) Assuming that the reward function U satisfies the Inada condition:

$$\lim_{x \searrow 0} U'(x) = \infty, \quad \lim_{x \to \infty} U'(x) = 0. \tag{4.16}$$

(H3) Assuming that the reward function U is strictly concave, that is, assuming $U''(z) < 0, \forall z \in (0, \infty)$.

For the penalty function D, let us assume that it is a concave function or a strictly convex function.

At this time, the objective function is the ratio of two expectation. For this, we adopt the linearization method: for the parameter $\nu \geqslant 0$, consider solving a family of optimization problems:

$$v(\nu, \tilde{x}_0) := \sup_{Z \in \mathcal{C}(\tilde{x}_0), P(Z \geqslant L) \geqslant 1-\varepsilon} \mathbb{E}\{U[(Z-\theta)_+]\} - \nu \mathbb{E}\{D[(\theta - Z)_+]\}, \tag{4.17}$$

where

$$\mathcal{C}(\tilde{x}_0) = \{Z \in \mathcal{M}_+ : \mathbb{E}[H(T)Z] \leqslant \tilde{x}_0\}. \tag{4.18}$$

First, the problem (4.17) has the same preference as the problem (4.6): both tend to take the reward function U larger and the penalty function D smaller. Secondly, the following theorem shows that it is only necessary to solve such a family of linearized problems (4.17) to solve the problem (4.6).

Remark 4.4. Similar to the proof of the proposition (4.2), it is easy to see that when the problem (4.17) takes the optimal value, there must be inequality constraints, that is, there must be

$$v(\nu, \tilde{x}_0) = \sup_{Z \in \mathcal{M}, \mathbb{E}[H(T)Z] = \tilde{x}_0, P(Z \geqslant L) \geqslant 1-\varepsilon} \mathbb{E}\{U[(Z-\theta)_+]\} - \nu \mathbb{E}\{D[(\theta - Z)_+]\}. \tag{4.19}$$

Theorem 4.3. *Asuume $\tilde{x}_0 < e^{-rT}\theta$, $\forall \nu \geqslant 0$, let Z_ν be the solution to problem (4.17), that is*

$$v(\nu, \tilde{x}_0) = \mathbb{E}\{U[(Z_\nu - \theta)_+]\} - \nu \mathbb{E}\{D[(\theta - Z_\nu)_+]\}. \tag{4.20}$$

If there exists $\nu^* \geqslant 0$ such that

$$\nu^* = \frac{\mathbb{E}\{U[(Z_{\nu^*} - \theta)_+]\}}{\mathbb{E}\{D[(\theta - Z_{\nu^*})_+]\}}, \tag{4.21}$$

that is, there exists $\nu^* \geqslant 0$ such that $v(\nu^*, \tilde{x}_0) = 0$, then we have $Z^* := Z_{\nu^*}$ the solution to problem (4.6), and ν^* is the optimal valve.

Proof. By $\tilde{x}_0 < e^{-rT}\theta$ and the defination of $\mathcal{C}(\tilde{x}_0)$ we confirm that $P(Z < \theta) > 0$, or we have $\tilde{x}_0 \geqslant \mathbb{E}[H(T)Z] \geqslant \mathbb{E}[H(T)]\theta = e^{-rT}\theta$ which contradict with $\tilde{x}_0 < e^{-rT}\theta$. While by the optimality of Z^*, we konw $\forall Z \in \mathcal{M}_+, \mathbb{E}[H(T)Z] \leqslant \tilde{x}_0, P(Z \geqslant L) \geqslant 1 - \varepsilon$, thus we have

$$\begin{aligned}
0 =& v(\nu^*, \tilde{x}_0) \\
=& \mathbb{E}\{U[(Z_{\nu^*} - \theta)_+]\} - \nu^* \mathbb{E}\{D[(\theta - Z_{\nu^*})_+]\} \\
\geqslant& \mathbb{E}\{U[(Z_\nu - \theta)_+]\} - \nu^* \mathbb{E}\{D[(\theta - Z_\nu)_+]\} \\
=& \mathbb{E}\{U[(Z - \theta)_+]\} - \frac{\mathbb{E}\{U[(Z_{\nu^*} - \theta)_+]\}}{\mathbb{E}\{D[(\theta - Z_{\nu^*})_+]\}} \mathbb{E}\{D[(\theta - Z)_+]\}.
\end{aligned} \tag{4.22}$$

That is

$$\frac{\mathbb{E}\{U[(Z_{\nu^*} - \theta)_+]\}}{\mathbb{E}\{D[(\theta - Z_{\nu^*})_+]\}} \geqslant \frac{\mathbb{E}\{U[(Z - \theta)_+]\}}{\mathbb{E}\{D[(\theta - Z)_+]\}}. \tag{4.23}$$

So we have $Z^* = Z_{\nu^*}$ is the solution to problem (4.6) and ν^* is the optimal value, which we call the optimal parameter. □

So here we only need to do two things

- To prove the existence of the optimal parameter, that is, the existence of the root of equation $v(\cdot,\tilde{x}_0)=0$.
- To solve the optimization problem (4.17) after linearization.

4.4 The existence of optimal parameters

The existence of the optimal parameter ν^* requires additional conditions:

(H4) Assume the reward function U has the asymptotic behavior of the "Arrow-Pratt relative risk aversion" and satisfies the asymptotic elasticity condition:

$$\liminf_{x\to\infty}\left(-\frac{xU''(x)}{U'(x)}\right)>0 \quad \lim_{x\to\infty}\frac{xU''(x)}{U'(x)}<1. \tag{4.24}$$

The following will explore the property of the function $v(\cdot,\tilde{x}_0)$ to give the existence of the optimal parameter, that is, the existence of the root of equation $v(\cdot,\tilde{x}_0)=0$, for which we need the following lemma.

Lemma 4.4. *Write*

$$\begin{aligned} M &= \sup_{Z\in\mathcal{M},\mathbb{E}[H(T)Z]=\tilde{x}_0}\mathbb{E}\{U[(Z-\theta)_+]\},\\ m &= \inf_{Z\in\mathcal{M},\mathbb{E}[H(T)Z]=\tilde{x}_0}\mathbb{E}\{D[(\theta-Z)_+]\}. \end{aligned} \tag{4.25}$$

When $\tilde{x}_0<e^{-rT}\theta$, we have $M<\infty, m>0$.

Proof. For the proof of $M<\infty$, see Jin et al. (2008). Assume that $m=0$ and $Z_n\in\mathcal{M},\mathbb{E}[H(T)Z_n]=\tilde{x}_0$ such that $\lim_{n\to\infty}\mathbb{E}\{D[(\theta-Z_n)_+]\}=0$. Then we have $D[(\theta-Z_n)_+]$ converge to 0 in probability, thus $\theta-Z_n$ converge to 0 in probability, so do $H(T)(\theta-Z_n)$. Then we have $H(T)(\theta-Z_n)$ converge to 0 in L^1 norm and so do $\mathbb{E}\{H(T)(\theta-Z_n)\}\to 0$, which contradicts with the assumption $\mathbb{E}\{H(T)(\theta-Z_n)\}=\theta-\tilde{x}_0e^{rT}>0$. So we have $m>0$. $\square$

Proposition 4.5. *When the reward function U and the penalty function D satisfy the above assumptions (H1)-(H4), the initial value $\tilde{x}_0<e^{-rT}\theta$ and the problem (4.6) has a nontrivial feasible solution, that is, when $Z\in\mathcal{M}_+$ satisfies*

$$\begin{cases} \mathbb{E}[H(T)Z]\leqslant\tilde{x}_0,\\ P(Z\geqslant L)\geqslant 1-\varepsilon, ,\\ P(Z\geqslant\theta)\geqslant 0. \end{cases} \tag{4.26}$$

the function $v(\cdot,\tilde{x}_0)$ has the following properties

(i) $0<v(0,\tilde{x}_0)<\infty$.

(ii) $v(\cdot,\tilde{x}_0)$ *is nonincreasing in* $\mathbb{R}$.

(iii) For a given $0 < \tilde{x}_0$, $v(\cdot, \tilde{x}_0)$ *is convex in* $\mathbb{R}$.

(iv) $\lim_{\nu\to\infty} v(\nu, \tilde{x}_0) = -\infty$.

Proof. $0 < v(0, \tilde{x}_0)$ can easily be seen from the assumption (4.26), while $v(\nu, \tilde{x}_0) \leqslant M - \nu m$ can be get from lemma (4.4) and remark (4.4). Then we have $v(0, \tilde{x}_0) < \infty$ and $\lim_{\nu\to\infty} v(\nu, \tilde{x}_0) = -\infty$.

$\forall \nu_1 < \nu_2, t \in [0, 1]$, assume that Z_{ν_2} maximize $v(\nu_2, \tilde{x}_0)$ and Z_t maximize $v(t\nu_1 + (1-t)\nu_2, \tilde{x}_0)$, then we have

$$\begin{aligned} v(\nu_2, \tilde{x}_0) &= \mathbb{E}\{U[(Z_{\nu_2} - \theta)_+] - \nu_2 D[(\theta - Z_{\nu_2})_+]\} \\ &\leqslant \mathbb{E}\{U[(Z_{\nu_2} - \theta)_+] - \nu_1 D[(\theta - Z_{\nu_2})_+]\} \\ &\leqslant v(\nu_1, \tilde{x}_0), \end{aligned} \tag{4.27}$$

that is, $v(\cdot, \tilde{x}_0)$ is a non-increasing function. Furthermore

$$\begin{aligned} v(t\nu_1 + (1-t)\nu_2, \tilde{x}_0) =& \mathbb{E}\{U[(Z_t - \theta)_+] - (t\nu_1 + (1-t)\nu_2) D[(\theta - Z_t)_+]\} \\ =& t\mathbb{E}\{U[(Z_t - \theta)_+] - \nu_1 D[(\theta - Z_t)_+]\} \\ &+ (1-t)\mathbb{E}\{U[(Z_t - \theta)_+] - \nu_2 D[(\theta - Z_t)_+]\} \\ \leqslant& t v(\nu_1, \tilde{x}_0) + (1-t) v(\nu_2, \tilde{x}_0). \end{aligned} \tag{4.28}$$

So $v(\cdot, \tilde{x}_0)$ is convex in $\mathbb{R}$. □

Corollary 4.1. *There is a unique* $\nu^* > 0$ *such that* $v(\nu^*, \tilde{x}_0) = 0$.

Proof. Because $v(\cdot, \tilde{x}_0)$ is convex in $\mathbb{R}$, it is also continuous. Because $0 < v(0, \tilde{x}_0) < \infty$, $\lim_{\nu\to\infty} v(\nu, \tilde{x}_0) = -\infty$, we can easily know that there is ν^* such that $v(\nu^*, \tilde{x}_0) = 0$ by the intermediate value theorem for continuous function. Assume $\nu_1 < \nu_2$ are both the root of $v(\cdot, \tilde{x}_0) = 0$. As $\lim_{\nu\to\infty} v(\nu, \tilde{x}_0) = -\infty$, we can find $\nu_3 > \nu_2$ such that $v(\nu_3, \tilde{x}_0) < v(\nu_2, \tilde{x}_0) = 0$, then we have

$$\nu_2 = \frac{\nu_2 - \nu_1}{\nu_3 - \nu_1}\nu_3 + \frac{\nu_3 - \nu_2}{\nu_3 - \nu_1}\nu_1, \tag{4.29}$$

then

$$v(\nu_2, \tilde{x}_0) = 0 > \frac{\nu_2 - \nu_1}{\nu_3 - \nu_1} v(\nu_3, \tilde{x}_0) + \frac{\nu_3 - \nu_2}{\nu_3 - \nu_1} v(\nu_1, \tilde{x}_0), \tag{4.30}$$

which contradicts with the assumption $v(\cdot, \tilde{x}_0)$ is convex in $\mathbb{R}$. So we confirm that ν^* is unique. □

Next, we only need to solve the optimization problem after linearization (4.17). For this optimization problem with VaR constraints, we first consider transforming the VaR constraints

4.5 Transformation of VaR constraints

For convenience, write

$$f_\nu(Z) = U[(Z-\theta)_+] - \nu D[(\theta - Z)_+], \tag{4.31}$$

$$f_{\nu,\lambda}(Z) = U[(Z-\theta)_+] - \nu D[(\theta - Z)_+] + \lambda \mathbf{1}_{Z\geqslant L}. \tag{4.32}$$

For the optimization problem after linearization (4.17), that is

$$\begin{aligned} \max_{Z\in\mathcal{M}_+} \quad & \mathbb{E}\{U[(Z-\theta)_+] - \nu D[(\theta-Z)_+]\} = \mathbb{E}\{f_\nu(Z)\}, \\ s.t. \quad & \mathbb{E}[H(T)Z] \leqslant \tilde{x}_0, \\ & P(Z\geqslant L) \geqslant 1-\varepsilon. \end{aligned} \tag{4.33}$$

We consider transforming VaR constraints into Lagrange penalty terms into the objective function: Consider the following optimization problem

$$\begin{aligned} \max_{Z\in\mathcal{M}_+} \quad & \mathbb{E}\{U[(Z-\theta)_+] - \nu D[(\theta-Z)_+] + \lambda \mathbf{1}_{Z\geqslant L}\} = \mathbb{E}\{f_{\nu,\lambda}(Z)\}, \\ s.t. \quad & \mathbb{E}[H(T)Z] \leqslant \tilde{x}_0. \end{aligned} \tag{4.34}$$

The following theorem will give the relationship between the new optimization problem (4.34) and the optimization problem (4.33)

Theorem 4.6. *For all $\lambda \geqslant 0$, let $Z_{\nu,\lambda}$ be the solution to problem (4.34). If there is a $\lambda^* \geqslant 0$such that*

$$P(Z_{\nu,\lambda^*} \geqslant L) \geqslant 1-\varepsilon, \tag{4.35}$$

$$\lambda^*[P(Z_{\nu,\lambda^*} \geqslant L) - (1-\varepsilon)] = 0. \tag{4.36}$$

Then we have $Z_\nu := Z_{\nu,\lambda^*}$ is the solution to problem (4.33).

Proof. First, we know from the condition (4.35) that Z_{ν,λ^*} is a feasible solution to the problem (4.33), so $\mathbb{E}\{f_\nu(Z_{\nu,\lambda^*})\}$ does not exceed the optimal target value of the optimization problem (4.33). On the other hand, for all feasible solution Z to the optimization problem (4.33), Z is also a feasible solution for the optimization problem (4.34), so$\mathbb{E}\{f_{\nu,\lambda^*}(Z)\} \leqslant \mathbb{E}\{f_{\nu,\lambda^*}(Z_{\nu,\lambda^*})\}$, that is

$$\mathbb{E}\{f_\nu(Z)\} + \lambda^* P(Z\geqslant L) \leqslant \mathbb{E}\{f_\nu(Z_{\nu,\lambda^*})\} + \lambda^* P(Z_{\nu,\lambda^*} \geqslant L), \tag{4.37}$$

then we have

$$\begin{aligned} \mathbb{E}\{f_\nu(Z)\} \leqslant & \mathbb{E}\{f_\nu(Z_{\nu,\lambda^*})\} + \lambda^* \left[P(Z_{\nu,\lambda^*} \geqslant L) - P(Z\geqslant L)\right] \\ = & \mathbb{E}\{f_\nu(Z_{\nu,\lambda^*})\} + \lambda^* \left[1-\varepsilon - P(Z\geqslant L)\right] \\ \leqslant & \mathbb{E}\{f_\nu(Z_{\nu,\lambda^*})\}. \end{aligned} \tag{4.38}$$

Therefore, the optimal value of the optimization problem (4.33) does not exceed $\mathbb{E}\{f_\nu(Z_{\nu,\lambda^*})\}$. In summary, Z_{ν,λ^*} is the optimal solution to the problem (4.33). □

Remark 4.5. The $\lambda^* \geqslant 0$ that satisfies the theorem (4.6) exists. In fact, under certain conditions, such $\lambda^* \geqslant 0$ is unique. For proof, see Dong and Zheng (2020).

4.6 Non-randomization and transformation of inequality constraints

Similar to the transformation of the VaR constraint above, we consider transforming the inequality constraint $\mathbb{E}[H(T)Z] \leqslant \tilde{x}_0$ of the optimization problem (4.34) into a Lagrange penalty term into the objective function. For each $\beta > 0$, consider the following optimization problem

$$\max_{Z \in \mathcal{M}_+} \quad \mathbb{E}\{f_{\nu,\lambda}(Z) - \beta H(T)Z\}, \tag{4.39}$$

and the non-randomized version of the optimization problem (4.39): consider for each $y > 0$

$$\max_{x \in \mathbb{R}_+} \quad f_{\nu,\lambda}(x) - yx. \tag{4.40}$$

We have the following theorem to give the relationship between the solution of the optimization problem (4.34), the optimization problem (4.39) and the optimization problem (4.40):

Theorem 4.7. *For each $\nu \geqslant 0, \lambda \geqslant 0$, we have the following properties*

(a) Let Borel measurable function $x^*_{\nu,\lambda}(y)$ is the solution to problem (4.40) for each $\nu \geqslant 0, \lambda \geqslant 0$ and $y > 0$. Then $Z_{\nu,\lambda,\beta} := x^*_{\nu,\lambda}(\beta H(T))$ is the solution to problem (4.39).

(b) If there is $\beta^* > 0$, makes Z_{ν,λ,β^*} the optimal solution for the optimization problem (4.39) and meets the inequality constraints to get the equal sign, that is $\mathbb{E}[H(T)Z_{\nu,\lambda,\beta^*}] = \tilde{x}_0$. Then $Z_{\nu,\lambda} := Z_{\nu,\lambda,\beta^*}$ is the solution to problem (4.34), at this time, β^* is called the optimal multiplier.

Proof. See the article Lin et al. (2017) for proof. □

So the following only needs to do two things:

1. Prove the existence of the optimal multiplier β^*, that is, the existence of $\beta^* > 0$ makes Z_{ν,λ,β^*} is an optimization problem (4.39) and satisfy the inequality constraints to get the equal sign.
2. Solve non-stochastic optimization problems (4.40).

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