MATH3075/3975
Financial Derivatives

School of Mathematics and Statistics
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Tutorial sheet 7

Background: Chapter 3 – Multi-Period Market Models.

Exercise 1 We consider the conditional expectation $\mathbb{E}_{\mathbb{P}}(X \mid \mathcal{G})$ where $\mathcal{G}$ is generated by a finite partition $(A_i)_{i \in I}$ of the sample space $\Omega = \{\omega_1, \dots, \omega_k\}$. Specifically, let $k = 5$ and

$$A_1 = \{\omega_1, \omega_2\},\ A_2 = \{\omega_3\},\ A_3 = \{\omega_4, \omega_5\}.$$

Let the probability measure $\mathbb{P}$ be given by

$$\mathbb{P}(\omega_1) = \mathbb{P}(\omega_2) = 0.1,\ \mathbb{P}(\omega_3) = 0.3,\ \mathbb{P}(\omega_4) = 0.2,\ \mathbb{P}(\omega_5) = 0.3.$$

Consider the random variable $X : \Omega \to \mathbb{R}$ given by $X(\omega_i) = i$ for $i = 1, \dots, 5$.

(a) Find the probability distribution of the random variable X.

(b) Compute the conditional expectation $\mathbb{E}_{\mathbb{P}}(X \mid \mathcal{G})$.

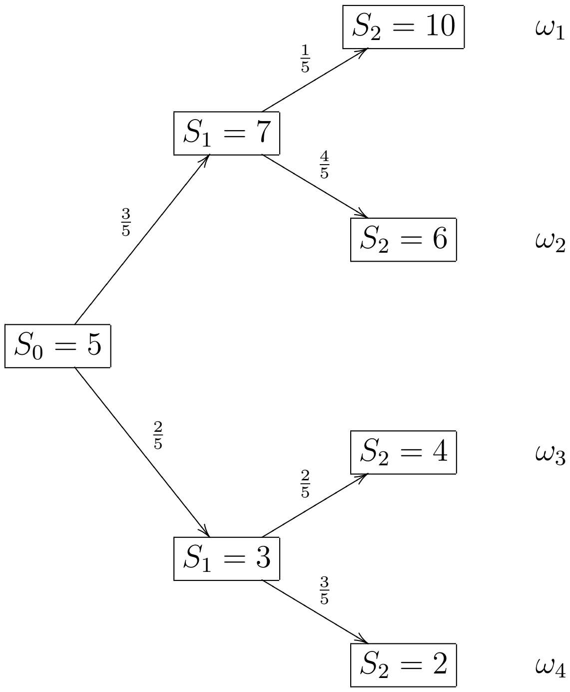
(c) Find the probability distribution of the random variable $Y := \mathbb{E}_{\mathbb{P}}(X \mid \mathcal{G})$.

(d) Show that $\mathbb{E}_{\mathbb{P}}(X) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(X \mid \mathcal{G}))$.

Exercise 2 Consider the two-period market model $\mathcal{M} = (B, S)$ with the savings account B given by

$$B_0 = 1, \quad B_1 = 1 + r, \quad B_2 = (1 + r)^2$$

with $r = 0.25$ and the stock price S evolving according to the following diagram



(a) Compute the probabilities of the states $\omega_1, \omega_2, \omega_3, \omega_4$.

(b) Compute the conditional expectation $\mathbb{E}_{\mathbb{P}}(S_2 \,|\, \mathcal{F}_1)$:

(b1) using the formula

$$\mathbb{E}_{\mathbb{P}}(S_2|\, \mathcal{F}_1) = \sum_{i=1}^{m} \frac{\mathbb{1}_{A_i}}{\mathbb{P}(A_i)} \sum_{\omega \in A_i} S_2(\omega)\mathbb{P}(\omega),$$

(b2) using directly the conditional probabilities.

(c) Compute $\mathbb{E}_{\mathbb{P}}(S_2)$ directly and using the equality

$$\mathbb{E}_{\mathbb{P}}(S_2) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(S_2|\, \mathcal{F}_1)).$$

Exercise 3 **(MATH3975)** Consider a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an arbitrary σ-field $\mathcal{G} \subset \mathcal{F}$. Let X be any $\mathcal{F}$-measurable random variable.

(a) Show that the conditional expectation $\mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{G})$ satisfies

$$\sum_{\omega \in G} X(\omega)\mathbb{P}(\omega) = \sum_{\omega \in G} \mathbb{E}_{\mathbb{P}}(X|\, \mathcal{G})(\omega)\mathbb{P}(\omega), \quad \forall\, G \in \mathcal{G}.$$

Deduce from this equality that $\mathbb{E}_{\mathbb{P}}(X) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(X|\, \mathcal{G}))$.

(b) Let η be a random variable such that η is $\mathcal{G}$-measurable and

$$\sum_{\omega \in G} X(\omega)\mathbb{P}(\omega) = \sum_{\omega \in G} \eta(\omega)\mathbb{P}(\omega), \quad \forall\, G \in \mathcal{G}.$$

Show that $\eta = \mathbb{E}_{\mathbb{P}}(X|\, \mathcal{G})$.

Exercise 4 **(MATH3975)** Let $\mathbb{P}$ and $\mathbb{Q}$ be two equivalent probability measures on a (finite) probability space $(\Omega, \mathcal{F})$. Let $\mathbb{F}$ be an arbitrary filtration. For any fixed $t = 0, 1, \ldots, T$, we denote by L_t the Radon-Nikodym density of $\mathbb{Q}$ with respect to $\mathbb{P}$ when $\mathbb{Q}$ and $\mathbb{P}$ are restricted to the σ-field $\mathcal{F}_t$.

(a) Show that $\mathbb{E}_{\mathbb{P}}(L_s \,|\, \mathcal{F}_t) = L_t$ for every $0 \leq t \leq s \leq T$. You may use part (b) in Exercise 3.

(b) Using the abstract Bayes formula, establish the following equality, for an arbitrary $\mathcal{F}_s$-measurable random variable Y and for every $0 \leq t \leq s$

$$\mathbb{E}_{\mathbb{Q}}(Y \,|\, \mathcal{F}_t) = (L_t)^{-1}\, \mathbb{E}_{\mathbb{P}}(Y L_s \,|\, \mathcal{F}_t).$$

(c) Let M be a process such that M_t is $\mathcal{F}_t$-measurable for every t. Show that the following conditions are equivalent:
(i) $\mathbb{E}_{\mathbb{Q}}(M_s \,|\, \mathcal{F}_t) = M_t$ for every $0 \leq t \leq s \leq T$,
(ii) $\mathbb{E}_{\mathbb{P}}(L_s M_s \,|\, \mathcal{F}_t) = L_t M_t$ for every $0 \leq t \leq s \leq T$.

If a process M is such that M_t is $\mathcal{F}_t$-measurable for every t, then we say that M is $\mathbb{F}$-*adapted*. If for an $\mathbb{F}$-adapted process M the equality $\mathbb{E}_{\mathbb{Q}}(M_s \,|\, \mathcal{F}_t) = M_t$ is satisfied for every $0 \leq t \leq s \leq T$, then we say that M is an $\mathbb{F}$-*martingale under* $\mathbb{Q}$. Hence it was shown in part (c) that the following conditions are equivalent for an $\mathbb{F}$-adapted process M:
(i) the process M is an $\mathbb{F}$-martingale under $\mathbb{Q}$,
(ii) the process LM is an $\mathbb{F}$-martingale under $\mathbb{P}$.
Note also that it was shown in (a) that the Radon-Nikodym density process L of $\mathbb{Q}$ with respect to $\mathbb{P}$ is an $\mathbb{F}$-martingale under $\mathbb{P}$.

Exercise 5 **(MATH3975)** Using the tower property of conditional expectation, show that if M is an $\mathbb{F}$-adapted process, then the following conditions are equivalent:
(i) the process M is a martingale under $\mathbb{P}$,
(ii) $\mathbb{E}_{\mathbb{P}}(M_{t+1} \,|\, \mathcal{F}_t) = M_t$ for every $0 \leq t \leq T-1$,
(iii) $\mathbb{E}_{\mathbb{P}}(M_T \,|\, \mathcal{F}_t) = M_t$ for every $0 \leq t \leq T$.
Deduce that if X an $\mathcal{F}_T$-measurable random variable, then the process $M_t := \mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{F}_t)$ is the unique martingale under $\mathbb{P}$ with the terminal value $M_T = X$.

Exercise 6 **(MATH3975)** Consider the process S from Exercise 2.

(a) Show that S is not a martingale under $\mathbb{P}$.

(b) Find the unique probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_2)$ such that S is a martingale under $\mathbb{Q}$.

(c) Find the Radon-Nikodym density process L of $\mathbb{Q}$ with respect to $\mathbb{P}$ and show that L is a martingale under $\mathbb{P}$.