4: SINGLE-PERIOD MARKET MODELS
MATH3975

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General Single-Period Market Models

- The main differences between the elementary and general single period market models are:
  - The investor is allowed to invest in several risky securities instead of only one.
  - The sample set Ω has more $k \geq 2$ elements, that is, there are more possible states of the world at time $t = 1$ than only two.
- The sample space is $\Omega = \{\omega_1, \omega_2, \dots, \omega_k\}$ with $\mathcal{F} = 2^\Omega$.
- An investor's personal beliefs about the future behaviour of stock prices are given by the probabilities $\mathbb{P}(\omega_i) = p_i > 0$ for $i = 1, 2, \dots, k$.
- The price of the jth stock is denoted by S_t^j for $t = 0, 1$ and $j = 1, \dots, n$. Then $S_0^j > 0$ and S_1^j is a random variable on Ω.
- The savings account B equals $B_0 = 1$ and $B_1 = 1 + r$ for some constant $r > -1$.
- A contingent claim $X = (X(\omega_1), \dots, X(\omega_k))$ is a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Questions

1. Under which conditions on $B, S^1, \dots, S^n$ a general single-period market model $\mathcal{M} = (B, S^1, \dots, S^n)$ is arbitrage-free?
2. How to define the concept of a risk-neutral probability measure for a model?
3. How to use risk-neutral probability measures to analyse a general single-period market model?
4. Under which conditions a general single-period market model is complete?
5. Is completeness of a market model related to risk-neutral probability measures?
6. How to define an arbitrage price of an attainable claim?
7. Can we still apply the risk-neutral valuation formula to compute the 'fair' price of an attainable claim?
8. How to deal with contingent claims that are not attainable?
9. How to use the class of risk-neutral probability measures to value non-attainable claims?

Outline

We will examine the following issues:

1. Trading Strategies and Arbitrage-Free Models
2. Fundamental Theorem of Asset Pricing (FTAP)
3. Examples of Market Models
4. Risk-Neutral Valuation of Contingent Claims
5. Stochastic Volatility Model
6. Completeness of a Market Model

PART 1

TRADING STRATEGIES AND ARBITRAGE-FREE MODELS

Trading Strategy

Definition (Trading Strategy)

A **trading strategy** (or a **portfolio**) in a general single-period market model is defined as the vector

$$(x, \phi^1, \dots, \phi^n) \in \mathbb{R}^{n+1}$$

where x is the initial wealth of an investor and ϕ^j stands for the number of shares of the jth stock purchased/sold at time $t = 0$.

- If an investor adopts the trading strategy $(x, \phi^1, \dots, \phi^n)$ at time $t = 0$ then the cash value of his portfolio at time $t = 1$ equals

$$V_1(x, \phi^1, \dots, \phi^n) := \left(x - \sum_{j=1}^{n} \phi^j S_0^j \right) (1 + r) + \sum_{j=1}^{n} \phi^j S_1^j.$$

Wealth Process of a Trading Strategy

Definition (Wealth Process)

The **wealth process** (or the **value process**) of a trading strategy $(x, \phi^1, \dots, \phi^n)$ is the pair

$$(V_0(x, \phi^1, \dots, \phi^n), V_1(x, \phi^1, \dots, \phi^n)).$$

The real number $V_0(x, \phi^1, \dots, \phi^n)$ is simply the initial wealth x so that

$$V_0(x, \phi^1, \dots, \phi^n) := x$$

and the real-valued random variable $V_1(x, \phi^1, \dots, \phi^n)$ represents the cash value of the portfolio at time $t = 1$

$$V_1(x, \phi^1, \dots, \phi^n) := \left(x - \sum_{j=1}^{n} \phi^j S_0^j \right) (1 + r) + \sum_{j=1}^{n} \phi^j S_1^j.$$

Undiscounted Gains Process

- Nominal profits or losses an investor obtains from the investment can be calculated by subtracting $V_0(\cdot)$ from $V_1(\cdot)$. That quantity defines the undiscounted gains process but it is not very useful.
- A 'gain' can be negative; hence it may also represent a 'loss'.

Definition (Gains Process)

The (undiscounted) **gains process** is defined as $G_0(x, \phi^1, \dots, \phi^n) = 0$ and

$$\begin{aligned} G_1(x, \phi^1, \dots, \phi^n) &:= V_1(x, \phi^1, \dots, \phi^n) - V_0(x, \phi^1, \dots, \phi^n) \\ &= \left(x - \sum_{j=1}^{n} \phi^j S_0^j \right) r + \sum_{j=1}^{n} \phi^j \Delta S_1^j \end{aligned}$$

where the random variable $\Delta S_1^j = S_1^j - S_0^j$ represents the nominal change in the price of the jth stock.

Discounted Stock Price and Value Process

- To understand whether the jth stock appreciates in real terms, we consider the **discounted stock prices** of the jth stock

$$\widehat{S}_0^j := S_0^j = \frac{S_0^j}{B_0}, \quad \widehat{S}_1^j := \frac{S_1^j}{1+r} = \frac{S_1^j}{B_1}.$$

- Similarly, we define the **discounted wealth process** as

$$\widehat{V}_0(x, \phi^1, \dots, \phi^n) := x, \quad \widehat{V}_1(x, \phi^1, \dots, \phi^n) := \frac{V_1(x, \phi^1, \dots, \phi^n)}{B_1}.$$

- It is easy to see that

$$\begin{aligned} \widehat{V}_1(x, \phi^1, \dots, \phi^n) &= \left(x - \sum_{j=1}^n \phi^j S_0^j \right) + \sum_{j=1}^n \phi^j \widehat{S}_1^j \\ &= x + \sum_{j=1}^n \phi^j (\widehat{S}_1^j - \widehat{S}_0^j). \end{aligned}$$

Discounted Gains Process

Definition (Discounted Gains Process)

The **discounted gains process** for the investor is defined as

$$\widehat{G}_0(x, \phi^1, \ldots, \phi^n) = 0$$

and

$$\begin{aligned}\widehat{G}_1(x, \phi^1, \ldots, \phi^n) &:= \widehat{V}_1(x, \phi^1, \ldots, \phi^n) - \widehat{V}_0(x, \phi^1, \ldots, \phi^n) \\ &= \sum_{j=1}^{n} \phi^j \Delta \widehat{S}_1^j\end{aligned}$$

where $\Delta \widehat{S}_1^j = \widehat{S}_1^j - \widehat{S}_0^j$ is the change in the discounted price of the jth stock.

- Notice that the discounted gains process does not depend on the initial wealth x so that $\widehat{G}_1(x, \phi^1, \ldots, \phi^n) = \widehat{G}_1(0, \phi^1, \ldots, \phi^n)$ for every $x \in \mathbb{R}$.

Arbitrage: Definition

- The concept of an **arbitrage opportunity** in a general single-period market model is essentially the same as in the elementary market model.
- It is worth noting that in the definition below the real-world probability $\mathbb{P}$ can be replaced by any equivalent probability measure $\mathbb{Q}$.

Definition (Arbitrage)

A trading strategy $(x, \phi^1, \dots, \phi^n)$ in a general single-period market model is called an **arbitrage opportunity** if

- A.1. $V_0(x, \phi^1, \dots, \phi^n) = 0$,
- A.2. $V_1(x, \phi^1, \dots, \phi^n)(\omega_i) \geq 0$ for $i = 1, 2, \dots, k$,
- A.3. There exists $\omega \in \Omega$ such that $V_1(x, \phi^1, \dots, \phi^n)(\omega) > 0$.

Arbitrage: Equivalent Conditions

Under A.2, the following condition A.3′ is equivalent to A.3:

- A.3′. $\mathbb{E}_{\mathbb{P}}\left\{V_1(x, \phi^1, \ldots, \phi^n)\right\} > 0$, that is,

$$\sum_{i=1}^{k} V_1(x, \phi^1, \ldots, \phi^n)(\omega_i)\,\mathbb{P}(\omega_i) > 0.$$

It is important to observe that the definition of arbitrage can be formulated using the discounted wealth and gains processes.

Proposition (4.1)

A trading strategy $(x, \phi^1, \ldots, \phi^n)$ in a general single-period market model is an arbitrage opportunity if and only if one of the following conditions holds:

1. *Assumptions A.1-A.3 in the definition of an arbitrage opportunity hold with $\widehat{V}(x, \phi^1, \ldots, \phi^n)$ instead of $V(x, \phi^1, \ldots, \phi^n)$.*
2. *$x = 0$ and A.2-A.3 in the definition of an arbitrage opportunity are satisfied with $\widehat{G}_1(x, \phi^1, \ldots, \phi^n)$ instead of $V_1(x, \phi^1, \ldots, \phi^n)$.*

Proof of Proposition 4.1

Proof of Proposition 4.1: First step.

We will show that the following two statements are true:

- The definition of an arbitrage opportunity and condition 1 in Proposition 4.1 are equivalent.
- In Proposition 4.1, condition 1 is equivalent to condition 2.

To prove the first statement, we use the relationships between $V(x, \phi^1, \dots, \phi^n)$ and $\widehat{V}(x, \phi^1, \dots, \phi^n)$:

$$\widehat{V}_0(x, \phi^1, \dots, \phi^n) = V_0(x, \phi^1, \dots, \phi^n) = x,$$

$$\widehat{V}_1(x, \phi^1, \dots, \phi^n) = \frac{1}{1+r} V_1(x, \phi^1, \dots, \phi^n).$$

This shows that the first statement holds. $\square$

Proof of Proposition 4.1

Proof of Proposition 4.1: Second step.

- To prove the second statement, we recall the relationship between $\widehat{V}(x, \phi^1, \dots, \phi^n)$ and $\widehat{G}_1(x, \phi^1, \dots, \phi^n)$

$$\begin{aligned}\widehat{G}_1(x, \phi^1, \dots, \phi^n) &= \widehat{V}_1(x, \phi^1, \dots, \phi^n) - \widehat{V}_0(x, \phi^1, \dots, \phi^n) \\ &= \widehat{V}_1(x, \phi^1, \dots, \phi^n) - x.\end{aligned}$$

  It is now clear that for $x = 0$ we have

$$\widehat{G}_1(x, \phi^1, \dots, \phi^n) = \widehat{V}_1(x, \phi^1, \dots, \phi^n).$$

  Hence the second statement is true as well.

- We have already observed that $\widehat{G}_1(x, \phi^1, \dots, \phi^n)$ does not depend on x since $\widehat{G}_1(x, \phi^1, \dots, \phi^n) = \sum_{j=1}^n \phi^j \Delta \widehat{S}_1^j$.

$\square$

Verification of the Arbitrage-Free Property

- It can be sometimes hard to check directly whether arbitrage opportunities exist in a given market model, especially when dealing with several risky assets or in the multi-period setup.
- We have introduced the risk-neutral probability measure in the elementary market model and we noticed that it can be used to compute the arbitrage price of any contingent claim.
- We will show that the concept of a risk-neutral probability measure is also a convenient tool for checking whether a general single-period market model is arbitrage-free or not.
- In addition, we will argue that a risk-neutral probability measure can also be used for the purpose of valuation of a contingent claim (either attainable or not attainable).

Risk-Neutral Probability

Definition (Risk-Neutral Probability)

A probability measure $\mathbb{Q}$ on Ω is called a **risk-neutral probability measure** for a general single-period market model $\mathcal{M} = (B, S^1, S^2, \ldots, S^n)$ if:

- R.1 $\mathbb{Q}(\omega_i) > 0$ for all $\omega_i \in \Omega$,
- R.2 $\mathbb{E}_{\mathbb{Q}}\big(\Delta \widehat{S}_1^j\big) = 0$ for $j = 1, 2, \ldots, n$.

We denote by $\mathbb{M}$ the class of all risk-neutral probability measures for $\mathcal{M}$.

- R.1 means that $\mathbb{Q}$ and $\mathbb{P}$ are equivalent probability measures on Ω.
- R.2 is equivalent to $\mathbb{E}_{\mathbb{Q}}\big(\widehat{S}_1^j\big) = \widehat{S}_0^j$ or, more explicitly,

$$\mathbb{E}_{\mathbb{Q}}\big(S_1^j\big) = (1+r)S_0^j.$$

- A risk-neutral probability measure is also known as a **martingale measure**.

Example A: Stock Prices

Example (Stock prices)

- We consider the following model featuring two stocks S^1 and S^2 given on the sample space $\Omega = \{\omega_1, \omega_2, \omega_3\}$.
- The interest rate $r = \frac{1}{10}$ so that $B_0 = 1$ and $B_1 = 1 + \frac{1}{10} = 1.1$.
- We deal here with the market model $\mathcal{M} = (B, S^1, S^2)$.
- The stock prices at $t = 0$ are given by $S_0^1 = 2$ and $S_0^2 = 3$.
- The stock prices at $t = 1$ are represented in the table:

	ω_1	ω_2	ω_3
S_1^1	1	5	3
S_1^2	3	1	6

Example A: Wealth Process

Example (Wealth process)

- For any trading strategy $(x, \phi^1, \phi^2) \in \mathbb{R}^3$, we have

$$V_1(x, \phi^1, \phi^2) = \left(x - 2\phi^1 - 3\phi^2\right)\left(1 + \frac{1}{10}\right) + \phi^1 S_1^1 + \phi^2 S_1^2.$$

- We denote $\phi^0 := x - 2\phi^1 - 3\phi^2$ so that ϕ^0 is the amount of cash invested in the savings account B at time 0.

- Then $V_1(x, \phi^1, \phi^2)$ equals

$$\begin{aligned}
V_1(x, \phi^1, \phi^2)(\omega_1) &= 1.1\phi^0 + \phi^1 + 3\phi^2, \\
V_1(x, \phi^1, \phi^2)(\omega_2) &= 1.1\phi^0 + 5\phi^1 + \phi^2, \\
V_1(x, \phi^1, \phi^2)(\omega_3) &= 1.1\phi^0 + 3\phi^1 + 6\phi^2.
\end{aligned}$$

Example A: Undiscounted Gains Process

Example (Undiscounted gains process)

- The increments ΔS_1^j are represented by the following table

	ω_1	ω_2	ω_3
ΔS_1^1	-1	3	1
ΔS_1^2	0	-2	3

- The undiscounted gains process is thus given by $G_0(x, \phi^1, \phi^2) = 0$ and $G_1(x, \phi^1, \phi^2) = V_1(x, \phi^1, \phi^2) - x$ that is

$$G_1(x, \phi^1, \phi^2)(\omega_1) = 0.1\phi^0 - \phi^1 + 0\phi^2,$$
$$G_1(x, \phi^1, \phi^2)(\omega_2) = 0.1\phi^0 + 3\phi^1 - 2\phi^2,$$
$$G_1(x, \phi^1, \phi^2)(\omega_3) = 0.1\phi^0 + \phi^1 + 3\phi^2.$$

Example A: Discounted Stock Prices

Example (Discounted stock prices)

- Out next goal is to compute the discounted wealth process $\widehat{V}(x, \phi^1, \phi^2)$ and the discounted gains process $\widehat{G}_1(x, \phi^1, \phi^2)$.
- To this end, we first compute the discounted stock prices.
- Of course, $\widehat{S}_0^j = S_0^j$ for $j = 1, 2$.
- The following table represents the discounted stock prices $\widehat{S}_1^j$ for $j = 1, 2$

	ω_1	ω_2	ω_3
$\widehat{S}_1^1$	$\frac{10}{11}$	$\frac{50}{11}$	$\frac{30}{11}$
$\widehat{S}_1^2$	$\frac{30}{11}$	$\frac{10}{11}$	$\frac{60}{11}$

Example A: Discounted Wealth Process

Example (Discounted wealth process)

- The discounted wealth process $\widehat{V}(x, \phi^1, \phi^2)$ is thus given by

$$\widehat{V}_0(x, \phi^1, \phi^2) = V_0(x, \phi^1, \phi^2) = x$$

and

$$\begin{aligned}
\widehat{V}_1(x, \phi^1, \phi^2)(\omega_1) &= \phi^0 + \frac{10}{11}\phi^1 + \frac{30}{11}\phi^2, \\
\widehat{V}_1(x, \phi^1, \phi^2)(\omega_2) &= \phi^0 + \frac{50}{11}\phi^1 + \frac{10}{11}\phi^2, \\
\widehat{V}_1(x, \phi^1, \phi^2)(\omega_3) &= \phi^0 + \frac{30}{11}\phi^1 + \frac{60}{11}\phi^2,
\end{aligned}$$

where $\phi^0 = x - 2\phi^1 - 3\phi^2$ is the amount of cash invested in B at time 0 (as opposed to the initial wealth given by x).

Example (Discounted gains)

- The increments of the discounted stock prices equal

	ω_1	ω_2	ω_3
$\Delta \widehat{S}^1_1$	$-\frac{12}{11}$	$\frac{28}{11}$	$\frac{8}{11}$
$\Delta \widehat{S}^2_1$	$-\frac{3}{11}$	$-\frac{23}{11}$	$\frac{27}{11}$

- Hence the discounted gains $\widehat{G}_1(x, \phi^1, \phi^2)$ are given by

$$\widehat{G}_1(x, \phi^1, \phi^2)(\omega_1) = -\frac{12}{11}\phi^1 - \frac{3}{11}\phi^2,$$
$$\widehat{G}_1(x, \phi^1, \phi^2)(\omega_2) = \frac{28}{11}\phi^1 - \frac{23}{11}\phi^2,$$
$$\widehat{G}_1(x, \phi^1, \phi^2)(\omega_3) = \frac{8}{11}\phi^1 + \frac{27}{11}\phi^2.$$

Example A: Arbitrage-Free Property

Example (No-arbitrage)

- The condition $\widehat{G}_1(x, \phi^1, \phi^2) \geq 0$ is equivalent to

$$\begin{aligned} -12\phi^1 - 3\phi^2 &\geq 0 \\ 28\phi^1 - 23\phi^2 &\geq 0 \\ 8\phi^1 + 27\phi^2 &\geq 0 \end{aligned}$$

- Can we find $(\phi^1, \phi^2) \in \mathbb{R}^2$ such that all inequalities are valid and at least one of them is strict?
- It appears that the answer is negative, since it is easy to check that the unique vector satisfying all inequalities above is $(\phi^1, \phi^2) = (0, 0)$.
- Hence the single-period market model $\mathcal{M} = (B, S^1, S^2)$ is arbitrage-free.

Example A: Risk-Neutral Probability

Example (Risk-neutral probability)

- We will now show that this market model admits a unique risk-neutral probability measure on $\Omega = \{\omega_1, \omega_2, \omega_3\}$.
- Let us denote $q_i = \mathbb{Q}(\omega_i)$ for $i = 1, 2, 3$.
- By the definition of a risk-neutral probability measure $\mathbb{Q}$, we have that $\mathbb{E}_{\mathbb{Q}}\big(\Delta \widehat{S}_1^j\big) = 0$ for $j = 1, 2$.
- We obtain the following conditions: $0 < q_i < 1$ and

$$
\begin{aligned}
-\frac{12}{11} q_1 + \frac{28}{11} q_2 + \frac{8}{11} q_3 &= 0, \\
-\frac{3}{11} q_1 - \frac{23}{11} q_2 + \frac{27}{11} q_3 &= 0, \\
q_1 + q_2 + q_3 &= 1.
\end{aligned}
$$

- The unique solution equals $\mathbb{Q} = (q_1, q_2, q_3) = \big(\frac{47}{80}, \frac{15}{80}, \frac{18}{80}\big)$.

PART 2

FUNDAMENTAL THEOREM OF ASSET PRICING

Fundamental Theorem of Asset Pricing (FTAP)

- In Example A, we have shown that $\mathcal{M} = (B, S^1, S^2)$ is arbitrage-free and a unique risk-neutral probability measure for $\mathcal{M}$ exists.
- Is there any relation between no arbitrage property of a market model $\mathcal{M}$ and the existence of a risk-neutral probability measure for $\mathcal{M}$?
- The following important result, which is due to Harrison and Pliska (1981), gives a complete answer to this question within the present setup.

Theorem (FTAP)

A general single-period model $\mathcal{M} = (B, S^1, \dots, S^n)$ is arbitrage-free if and only if there exists a risk-neutral probability measure for $\mathcal{M}$, that is, the class $\mathbb{M} \neq \emptyset$.

- J. M. Harrison and S. R. Pliska: Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and their Applications* 11 (1981), 215–260.

Proof of ($\Leftarrow$) in FTAP

Proof of ($\Leftarrow$) in FTAP.

($\Leftarrow$) We first prove the 'if' part.

- We assume that the class $\mathbb{M}$ is nonempty so that a risk-neutral probability measure $\mathbb{Q}$ for $\mathcal{M}$ exists.
- Let $(0, \phi) = (0, \phi^1, \dots, \phi^n)$ be any trading strategy with null initial wealth. Then for any $\mathbb{Q} \in \mathbb{M}$

$$\mathbb{E}_{\mathbb{Q}}\big(\widehat{V}_1(0,\phi)\big) = \mathbb{E}_{\mathbb{Q}}\big(\widehat{G}_1(0,\phi)\big) = \mathbb{E}_{\mathbb{Q}}\Big(\sum_{j=1}^{n} \phi^j \Delta \widehat{S}_1^j\Big) = \sum_{j=1}^{n} \phi^j \underbrace{\mathbb{E}_{\mathbb{Q}}\big(\Delta \widehat{S}_1^j\big)}_{=0} = 0.$$

- If we assume that $\widehat{V}_1(0,\phi) \geq 0$ then the last equation implies that the equality $\widehat{V}_1(0,\phi)(\omega) = 0$ must hold for all $\omega \in \Omega$.
- Hence no trading strategy satisfying all conditions of an arbitrage opportunity may exist.

Geometric Interpretation of X and $\mathbb{Q}$

- The proof of the implication ($\Rightarrow$) in the FTAP is harder and thus the full proof is for MATH3975 only. We will now find an equivalent geometric representation of the arbitrage-free property.
- Any random variable on Ω can be identified with a vector in $\mathbb{R}^k$ since

$$X = (X(\omega_1), \dots, X(\omega_k))^T = (x_1, \dots, x_k)^T \in \mathbb{R}^k.$$

- An arbitrary probability measure $\mathbb{Q}$ on Ω can also be interpreted as a vector in $\mathbb{R}^k$

$$\mathbb{Q} = (\mathbb{Q}(\omega_1), \dots, \mathbb{Q}(\omega_k)) = (q_1, \dots, q_k) \in \mathbb{R}^k.$$

- We note that

$$\mathbb{E}_{\mathbb{Q}}(X) = \sum_{i=1}^{k} X(\omega_i)\mathbb{Q}(\omega_i) = \sum_{i=1}^{k} x_i q_i = \langle X, \mathbb{Q} \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product of two vectors in $\mathbb{R}^k$.

Auxiliary Subsets of $\mathbb{R}^k$

- We define the following classes:

$$\mathbb{W} = \left\{ X \in \mathbb{R}^k \,\middle|\, X = \widehat{V}_1(0, \phi^1, \dots, \phi^n) \text{ for some } \phi^1, \dots, \phi^n \right\}$$

$$\mathbb{W}^{\perp} = \left\{ Z \in \mathbb{R}^k \mid \langle X, Z \rangle = 0 \text{ for all } X \in \mathbb{W} \right\}$$

- The set $\mathbb{W}$ is the image of the map $\widehat{V}_1(0, \cdot, \dots, \cdot) : \mathbb{R}^n \to \mathbb{R}^k$.
- We note that $\mathbb{W}$ represents all discounted values at $t = 1$ of trading strategies with null initial wealth.
- The set $\mathbb{W}^{\perp}$ is the set of all vectors in $\mathbb{R}^k$ orthogonal to $\mathbb{W}$.
- We introduce the following sets of k-dimensional vectors:

$$\mathbb{A} = \left\{ X \in \mathbb{R}^k \,\middle|\, X \neq 0,\ x_i \geq 0 \text{ for } i = 1, \dots, k \right\}$$

$$\mathcal{P}^+ = \left\{ \mathbb{Q} \in \mathbb{R}^k \,\middle|\, \sum_{i=1}^{k} q_i = 1,\ \ q_i > 0 \right\}$$

$\mathbb{W}$ and $\mathbb{W}^\perp$ as Vector Spaces

Corollary

The sets $\mathbb{W}$ and $\mathbb{W}^\perp$ are vector (linear) subspaces of $\mathbb{R}^k$.

Proof.

- It suffices to observe that the mapping $\widehat{V}_1(0, \cdot, \dots, \cdot) : \mathbb{R}^n \to \mathbb{R}^k$ is linear.
- In other words, for any trading strategies $(0, \eta^1, \dots, \eta^n)$ and $(0, \kappa^1, \dots, \kappa^n)$ and arbitrary real numbers α, β

$$(0, \phi^1, \dots, \phi^n) = \alpha(0, \eta^1, \dots, \eta^n) + \beta(0, \kappa^1, \dots, \kappa^n)$$

  is also a trading strategy. Hence the set $\mathbb{W}$ is a vector subspace of $\mathbb{R}^k$. In particular, the zero vector $(0, 0, \dots, 0)$ belongs to $\mathbb{W}$.
- It is easy to check that $\mathbb{W}^\perp$, that is, the orthogonal complement of $\mathbb{W}$ is a vector subspace of $\mathbb{R}^k$ as well.

$\square$

Risk-Neutral Probability Measures

Lemma (4.1)

A single-period market model $\mathcal{M} = (B, S^1, \dots, S^n)$ *is arbitrage-free if and only if* $\mathbb{W} \cap \mathbb{A} = \emptyset$.

Proof.

The proof is obvious since it suffices to apply Proposition 4.1. □

Lemma (4.2)

- *A probability* $\mathbb{Q}$ *is a risk-neutral probability measure for a single-period market model* $\mathcal{M} = (B, S^1, \dots, S^n)$ *if and only if* $\mathbb{Q} \in \mathbb{W}^\perp \cap \mathcal{P}^+$.

- Hence the set $\mathbb{M}$ of all risk-neutral probability measures for the model $\mathcal{M}$ satisfies $\mathbb{M} = \mathbb{W}^\perp \cap \mathcal{P}^+$ and thus

$$\mathbb{M} \neq \emptyset \quad \Leftrightarrow \quad \mathbb{W}^\perp \cap \mathcal{P}^+ \neq \emptyset.$$

Proof of Lemma 4.2

Proof of (⇒) in Lemma 4.2.

(⇒) Assume that $\mathbb{Q}$ is any risk-neutral probability measure, that is, $\mathbb{Q} \in \mathbb{M}$.

- By the property R.1, it is obvious that $\mathbb{Q}$ belongs to $\mathcal{P}^+$.
- Using the property R.2, we obtain for any vector $X = \widehat{V}_1(0,\phi) \in \mathbb{W}$

$$\langle X, \mathbb{Q} \rangle = \mathbb{E}_{\mathbb{Q}}\big(\widehat{V}_1(0,\phi)\big) = \mathbb{E}_{\mathbb{Q}}\big(\widehat{G}_1(0,\phi)\big) = \mathbb{E}_{\mathbb{Q}}\Big(\sum_{j=1}^n \phi^j \Delta \widehat{S}_1^j\Big)$$

$$= \sum_{j=1}^n \phi^j \underbrace{\mathbb{E}_{\mathbb{Q}}\big(\Delta \widehat{S}_1^j\big)}_{=0} = 0$$

  and thus $\mathbb{Q}$ belongs to the set $\mathbb{W}^\perp$.

- We conclude that $\mathbb{Q} \in \mathbb{W}^\perp \cap \mathcal{P}^+$ and thus $\mathbb{M} \subset \mathbb{W}^\perp \cap \mathcal{P}^+$.

$\square$

Proof of Lemma 4.2

Proof of ($\Leftarrow$) in Lemma 4.2.

($\Leftarrow$) We now assume that $\mathbb{Q}$ is an arbitrary vector in $\mathbb{W}^\perp \cap \mathcal{P}^+$.

- Since $\mathbb{Q} \in \mathcal{P}^+$, we see that $\mathbb{Q}$ defines a probability measure satisfying condition R.1, that is, $\mathbb{Q}$ is equivalent to $\mathbb{P}$.
- It remains to show that $\mathbb{Q}$ satisfies condition R.2 as well. To this end, for a fixed (but arbitrary) $j = 1, 2, \dots, n$, we consider the trading strategy $(0, \phi^1, \dots, \phi^n)$ with

$$(\phi^1, \dots, \phi^n) = (0, \dots, 0, 1, 0, \dots, 0) =: e_j.$$

  This trading strategy only invests in the savings account and the jth asset.
- The discounted wealth at time $t = 1$ of this strategy equals $\widehat{V}_1(0, e_j) = \Delta \widehat{S}_1^j$.

$\square$

Proof of Lemma 4.2

Proof of ($\Leftarrow$) in Lemma 4.2 (Continued).

- Since $\widehat{V}_1(0, e_j) \in \mathbb{W}$ and $\mathbb{Q} \in \mathbb{W}^\perp$, we obtain

$$0 = \langle \widehat{V}_1(0, e_j), \mathbb{Q} \rangle = \langle \Delta \widehat{S}_1^j, \mathbb{Q} \rangle = \mathbb{E}_{\mathbb{Q}}\big(\Delta \widehat{S}_1^j\big).$$

- Since j is here arbitrary, we see that $\mathbb{Q}$ satisfies condition R.2.
- We conclude that any $\mathbb{Q} \in \mathbb{W}^\perp \cap \mathcal{P}^+$ is a martingale measure and thus $\mathbb{W}^\perp \cap \mathcal{P}^+ \subset \mathbb{M}$.

$\square$

In view of Lemmas 4.1 and 4.2, the FTAP can be restated the follows.

Proposition (Geometric FTAP)

The following equivalence holds: $\quad \mathbb{W} \cap \mathbb{A} = \emptyset \quad \Leftrightarrow \quad \mathbb{W}^\perp \cap \mathcal{P}^+ \neq \emptyset.$

Separating Hyperplane Theorem: Statement

Theorem (Separating Hyperplane Theorem)

Let $B, C \subset \mathbb{R}^k$ be nonempty, closed, convex sets such that $B \cap C = \emptyset$. Assume, in addition, that at least one of these sets is compact (that is, bounded and closed). Then there exist vectors $a, y \in \mathbb{R}^k$ such that

$$\langle b - a, y \rangle < 0 \quad \text{for all} \quad b \in B$$

and

$$\langle c - a, y \rangle > 0 \quad \text{for all} \quad c \in C.$$

Proof of the Separating Hyperplane Theorem.

The proof can be found in any textbook of convex analysis or functional analysis. It is sketched in the course notes. □

Separating Hyperplane Theorem: Interpretation

- Let the vectors $a, y \in \mathbb{R}^k$ be as in the statement of the Separating Hyperplane Theorem
- It is clear that $y \in \mathbb{R}^k$ is never a zero vector.
- We define the $(k-1)$-dimensional **hyperplane** $H \subset \mathbb{R}^k$ by setting

$$H = a + \left\{ x \in \mathbb{R}^k \,|\, \langle x, y \rangle = 0 \right\} = a + \{y\}^{\perp}.$$

- Then we say that the hyperplane H **strictly separates** the convex sets B and C.
- Intuitively, the sets B and C lie on different sides of the hyperplane H and thus they can be seen as geometrically separated by H.
- Note that the compactness of at least one of the sets is a necessary condition for the **strict** separation of B and C.

Separating Hyperplane Theorem: Corollary

- The following corollary is a consequence of the separating hyperplane theorem.
- It is more suitable for our purposes: it will be later applied to $B = \mathbb{W}$ and $C = \mathbb{A}^+ := \{X \in \mathbb{A} \,|\, \langle X, \mathbb{P} \rangle = 1\} \subset \mathbb{A}$.

Corollary (4.1)

Assume that $B \subset \mathbb{R}^k$ is a vector subspace and set C is a compact convex set such that $B \cap C = \emptyset$. Then there exists a vector $y \in \mathbb{R}^k$ such that

$$\langle b, y \rangle = 0 \quad \textit{for all} \quad b \in B$$

that is, $y \in B^\perp$, and

$$\langle c, y \rangle > 0 \quad \textit{for all} \quad c \in C.$$

Proof of Corollary 4.1

Proof of Corollary 4.1: First step.

- We note that any vector subspace of $\mathbb{R}^k$ is a closed and convex set.
- From the separating hyperplane theorem, there exist $a, y \in \mathbb{R}^k$ such that the inequality

$$\langle b, y \rangle < \langle a, y \rangle$$

  is satisfied for all vectors $b \in B$.
- Since B is a vector subspace, the vector λb belongs to B for any $\lambda \in \mathbb{R}$. Hence for any $b \in B$ and $\lambda \in \mathbb{R}$ we have

$$\langle \lambda b, y \rangle = \lambda \langle b, y \rangle < \langle a, y \rangle.$$

- This in turn implies that $\langle b, y \rangle = 0$ for any vector $b \in B$, meaning that $y \in B^{\perp}$. Also, we have that $\langle a, y \rangle > 0$.

$\square$

Proof of Corollary 4.1

Proof of Corollary 4.1: Second step.

To establish the second inequality, we observe that from the separating hyperplane theorem, we obtain

$$\langle c, y \rangle > \langle a, y \rangle \quad \text{for all} \quad c \in C.$$

Consequently, for any $c \in C$

$$\langle c, y \rangle > \langle a, y \rangle > 0.$$

We conclude that $\langle c, y \rangle > 0$ for all $c \in C$. $\square$

- We now are ready to establish the implication $(\Rightarrow)$ in the FTAP:

$$\mathbb{W} \cap \mathbb{A} = \emptyset \quad \Rightarrow \quad \mathbb{W}^{\perp} \cap \mathcal{P}^{+} \neq \emptyset.$$

Proof of ($\Rightarrow$) in FTAP: 1

Proof of ($\Rightarrow$) in FTAP: First step.

- We assume that the model is arbitrage-free. From Lemma 4.1, this is equivalent to the condition $\mathbb{W} \cap \mathbb{A} = \emptyset$.
- Our goal is to show that the class $\mathbb{M}$ is non-empty.
- In view of Lemma 4.2, it thus suffices to show that

$$\mathbb{W} \cap \mathbb{A} = \emptyset \quad \Rightarrow \quad \mathbb{W}^{\perp} \cap \mathcal{P}^{+} \neq \emptyset.$$

- We define an auxiliary set $\mathbb{A}^{+} = \{X \in \mathbb{A} \,|\, \langle X, \mathbb{P} \rangle = 1\}$.
- Observe that $\mathbb{A}^{+}$ is a closed, bounded (hence compact) and convex subset of $\mathbb{R}^{k}$. Since $\mathbb{A}^{+} \subset \mathbb{A}$, it is clear that

$$\mathbb{W} \cap \mathbb{A} = \emptyset \;\Rightarrow\; \mathbb{W} \cap \mathbb{A}^{+} = \emptyset.$$

- Hence in the next step we may assume that $\mathbb{W} \cap \mathbb{A}^{+} = \emptyset$.

Proof of ($\Rightarrow$) in FTAP: 2

Proof of ($\Rightarrow$) in FTAP: Second step.

- By applying Corollary 4.1 to $B = \mathbb{W}$ and $C = \mathbb{A}^+$, we see that there exists a vector $Y \in \mathbb{W}^\perp$ such that

$$\langle X, Y \rangle > 0 \quad \text{for all} \quad X \in \mathbb{A}^+. \tag{1}$$

- Our goal is to show that Y can be used to define a risk-neutral probability $\mathbb{Q}$. We need first to show that $y_i > 0$ for every i.
- For this purpose, for any fixed $i = 1, 2, \dots, k$, we define

$$X_i = (\mathbb{P}(\omega_i))^{-1}\,(0, \dots, 0, 1, 0 \dots, 0) = (\mathbb{P}(\omega_i))^{-1}\,e_i$$

so that $X_i \in \mathbb{A}^+$ since

$$\mathbb{E}_{\mathbb{P}}(X_i) = \langle X_i, \mathbb{P} \rangle = 1.$$

□

Proof of ($\Rightarrow$) in FTAP: 3

Proof of ($\Rightarrow$) in FTAP: Third step.

- Let y_i be the ith component of Y. It follows from (1) that

$$0 < \langle X_i, Y \rangle = (\mathbb{P}(\omega_i))^{-1}\, y_i$$

and thus $y_i > 0$ for all $i = 1, 2, \ldots, k$. We set $\mathbb{Q}(\omega_i) = q_i$ where

$$q_i := \frac{y_i}{y_1 + \cdots + y_k} = c y_i > 0$$

It is clear that $\mathbb{Q}$ is a probability measure and $\mathbb{Q} \in \mathcal{P}^+$.

- Since $Y \in \mathbb{W}^\perp$, $\mathbb{Q} = cY$ for some scalar c and $\mathbb{W}^\perp$ is a vector space, we have that $\mathbb{Q} \in \mathbb{W}^\perp$. We conclude that $\mathbb{Q} \in \mathbb{W}^\perp \cap \mathcal{P}^+$ so that $\mathbb{W}^\perp \cap \mathcal{P}^+ \neq \emptyset$.
- From Lemma 4.2, $\mathbb{Q}$ is a risk-neutral probability and thus $\mathbb{M} \neq \emptyset$.

$\square$

PART 3

EXAMPLES OF MARKET MODELS

Example A: Arbitrage-Free Market Model

Example (Revisited)

- We consider the market model $\mathcal{M} = (B, S^1, S^2)$ introduced in Example A.
- The interest rate $r = 0.1$ so that $B_0 = 1$ and $B_1 = 1.1$.
- The stock prices at $t = 0$ are given by $S_0^1 = 2$ and $S_0^2 = 3$.
- We have shown that the increments of the discounted stock prices $\widehat{S}^1$ and $\widehat{S}^2$ are given by:

	ω_1	ω_2	ω_3
$\Delta \widehat{S}_1^1$	$-\frac{12}{11}$	$\frac{28}{11}$	$\frac{8}{11}$
$\Delta \widehat{S}_1^2$	$-\frac{3}{11}$	$-\frac{23}{11}$	$\frac{27}{11}$

Example A: Arbitrage-Free Market Model

Example (Spaces $\mathbb{W}$ and $\mathbb{W}^{\perp}$)

- The vector spaces $\mathbb{W}$ and $\mathbb{W}^{\perp}$ are given by

$$\mathbb{W} = \left\{ \alpha \begin{pmatrix} -12 \\ 28 \\ 8 \end{pmatrix} + \beta \begin{pmatrix} -3 \\ -23 \\ 27 \end{pmatrix} \,\middle|\, \alpha, \beta \in \mathbb{R} \right\}$$

and

$$\mathbb{W}^{\perp} = \left\{ \gamma \begin{pmatrix} 47 \\ 15 \\ 18 \end{pmatrix} \,\middle|\, \gamma \in \mathbb{R} \right\}.$$

- We first show the model is arbitrage-free using Lemma 4.1.
- It thus suffices to check that $\mathbb{W} \cap \mathbb{A} = \emptyset$.

Example A: Arbitrage-Free Market Model

Example ($\mathbb{W} \cap \mathbb{A} = \emptyset$)

- If there exists a vector $X \in \mathbb{W} \cap \mathbb{A}$ then the following three inequalities are satisfied by $X = (x_1, x_2, x_3) \in \mathbb{R}^3$

$$
\begin{aligned}
x_1 &= x_1(\alpha, \beta) = -12\alpha - 3\beta \geq 0, \\
x_2 &= x_2(\alpha, \beta) = 28\alpha - 23\beta \geq 0, \\
x_3 &= x_3(\alpha, \beta) = 8\alpha + 27\beta \geq 0,
\end{aligned}
$$

with at least one strict inequality where $\alpha, \beta \in \mathbb{R}$ are arbitrary.

- It can be verified that such a vector $X \in \mathbb{R}^3$ does not exist and thus $\mathbb{W} \cap \mathbb{A} = \emptyset$. This is left as an easy exercise.
- In view of Lemma 4.1, we conclude that the market model is arbitrage-free.
- In the next step, our goal is to show that the class $\mathbb{M}$ is non-empty.

Example A: Arbitrage-Free Market Model

Example (Risk-neutral probability)

- Lemma 4.2 tells us that $\mathbb{M} = \mathbb{W}^{\perp} \cap \mathcal{P}^{+}$. If $\mathbb{Q} \in \mathbb{W}^{\perp}$ then

$$\mathbb{Q} = \gamma \begin{pmatrix} 47 \\ 15 \\ 18 \end{pmatrix} \quad \text{for some } \gamma \in \mathbb{R}.$$

  If $\mathbb{Q} \in \mathcal{P}^{+}$ then $47\gamma + 15\gamma + 18\gamma = 1$ so that $\gamma = \frac{1}{80} > 0$.

- We conclude that the unique martingale measure $\mathbb{Q}$ equals

$$\mathbb{Q} = \frac{1}{80} \begin{pmatrix} 47 \\ 15 \\ 18 \end{pmatrix}.$$

- Hence the FTAP confirms that the market model $\mathcal{M} = (B, S^1, S^2)$ is arbitrage-free.

Example B: Market Model with Arbitrage

Example (Stock prices)

- We consider the following model featuring two stocks S^1 and S^2 on the sample space $\Omega = \{\omega_1, \omega_2, \omega_3\}$.
- The interest rate $r = \frac{1}{10}$ so that $B_0 = 1$ and $B_1 = 1 + \frac{1}{10}$.
- The stock prices at $t = 0$ are given by $S_0^1 = 1$ and $S_0^2 = 2$ and the stock prices at $t = 1$ are represented in the table:

	ω_1	ω_2	ω_3
S_1^1	1	$\frac{1}{2}$	3
S_1^2	$\frac{5}{2}$	4	$\frac{1}{10}$

- Does this market model admit an arbitrage opportunity?

Example B: Market Model with Arbitrage

Example (Discounted stocks)

- Once again, we will analyse this problem using Lemma 4.1, Lemma 4.2 and the FTAP.
- To tell whether a model is arbitrage-free it suffices to know the increments of discounted stock prices.
- The increments of discounted stock prices are represented in the following table:

	ω_1	ω_2	ω_3
$\Delta \widehat{S}_1^1$	$-\frac{1}{11}$	$-\frac{6}{11}$	$\frac{20}{11}$
$\Delta \widehat{S}_1^2$	$\frac{3}{11}$	$\frac{18}{11}$	$-\frac{21}{11}$

Example B: Market Model with Arbitrage

Example ($\mathbb{W} \cap \mathbb{A} \neq \emptyset$)

- Recall that
$$\widehat{G}_1(x, \phi^1, \phi^2) = \phi^1 \Delta \widehat{S}_1^1 + \phi^2 \Delta \widehat{S}_1^2$$
Hence, by the definition of $\mathbb{W}$, we have
$$\mathbb{W} = \left\{ \alpha \begin{pmatrix} -1 \\ -6 \\ 20 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 18 \\ -21 \end{pmatrix} \,\middle|\, \alpha, \beta \in \mathbb{R} \right\}.$$
- Let us take $\alpha = 3$ and $\beta = 1$. Then we obtain the vector $(0, 0, 39)^T$, which manifestly belongs to $\mathbb{A}$.
- We conclude that $\mathbb{W} \cap \mathbb{A} \neq \emptyset$ and thus, by Lemma 4.1, the market model is not arbitrage-free.

Example B: Market Model with Arbitrage

Example (No risk-neutral probability)

- We note that
$$\mathbb{W}^{\perp} = \left\{ \gamma \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix} \,\middle|\, \gamma \in \mathbb{R} \right\}.$$
- If there exists a risk-neutral probability measure $\mathbb{Q}$ then $\mathbb{Q} \in \mathbb{W}^{\perp} \cap \mathcal{P}^{+}$.
- Since $\mathbb{Q} \in \mathbb{W}^{\perp}$, we obtain $\mathbb{Q}(\omega_1) = -6\mathbb{Q}(\omega_2)$.
- However, $\mathbb{Q} \in \mathcal{P}^{+}$ implies that $\mathbb{Q}(\omega) > 0$ for all $\omega \in \Omega$.
- We conclude that $\mathbb{W}^{\perp} \cap \mathcal{P}^{+} = \emptyset$ and thus, by Lemma 4.2, no martingale measure exists, that is, $\mathbb{M} = \emptyset$.
- Hence the FTAP confirms that the model is not arbitrage-free.

PART 4

RISK-NEUTRAL VALUATION OF CONTINGENT CLAIMS IN ARBITRAGE-FREE MARKET MODELS

Contingent Claims

- We now know how to check whether a given model is arbitrage-free and thus we henceforth assume that $\mathcal{M}$ is arbitrage-free.
- Our next question reads: What should be the 'fair' price of a call or put option in a general **arbitrage-free** single-period market model?
- The idea of pricing European options can be extended to any contingent claim, for instance, $X = g(S_1^1, S_1^2, \ldots, S_1^n)$.

Definition (Contingent Claim)

A **contingent claim** is a real-valued random variable X defined on Ω and representing the payoff at the maturity date.

- Derivatives nowadays are usually quite complicated and thus it makes sense to analyse valuation and hedging of a general contingent claim and not only European call and put options.

Extended Market Model and No-Arbitrage Principle

Let X be an arbitrary contingent claim. We denote by $p_0(X)$ a real number representing a putative price for X at time 0.

Definition

We say that a price $p_0(X)$ of a contingent claim X is **consistent with the no-arbitrage principle** if the extended model, which consists of B, the original stocks $S^1, \dots, S^n$, as well as an additional asset S^{n+1} satisfying $S_0^{n+1} = p_0(X)$ and $S_1^{n+1} = X$, is arbitrage-free.

The interpretation of that definition is as follows:

- We assume that the model $\mathcal{M} = (B, S^1, \dots, S^n)$ is arbitrage-free.
- We consider X as an additional tradable risky asset in the extended market model $\widetilde{\mathcal{M}} = (B, S^1, \dots, S^{n+1})$.
- Then its price at time 0 should be selected in such a way that the extended market model $\widetilde{\mathcal{M}}$ is still arbitrage-free.

Arbitrage Pricing via Replication

Definition (Replication and Arbitrage Price)

A trading strategy $(x, \phi^1, \dots, \phi^n)$ is called a **replicating strategy** (or a **hedging strategy**) for a claim X when $V_1(x, \phi^1, \dots, \phi^n) = X$. Then the initial wealth is denoted as $\pi_0(X)$ and it is called the **arbitrage price** of X.

Notice that the initial wealth x is the same for **all** replicating strategies for X.

Proposition (Arbitrage Price)

Assume that a contingent claim X can be replicated by means of a trading strategy $(x, \phi^1, \dots, \phi^n)$. Then the unique price $p_0(X)$ for X at 0 consistent with the no-arbitrage principle equals $V_0(x, \phi^1, \dots, \phi^n) = x$, that is, $p_0(X) = \pi_0(X)$.

Proof.

If the price $p_0(X)$ for X is higher (lower) than x, one can sell (buy) X for $p_0(X)$ and buy (sell) the replicating portfolio for x. This is an arbitrage opportunity in the extended market in which X is traded at time $t = 0$ at $p_0(X)$. $\square$

Valuation of Attainable Contingent Claims

Definition (Attainable Contingent Claim)

A contingent claim X is called to be **attainable** if there exists at least one replicating strategy for X.

Let us summarise the known properties of attainable claims:

- It is clear how to price attainable contingent claims by the replicating principle.
- There might be more than one replicating strategy, but since $\mathcal{M}$ is arbitrage-free the initial wealth x is unique.
- In the two-state single-period market model, one can use the risk-neutral probability measure to price contingent claims.
- Our next objective is to extend the **risk-neutral valuation formula** to any attainable contingent claim within the framework of a general single-period market model. We assume that $\mathbb{M} \neq \emptyset$.

Arbitrage Pricing via Risk-Neutral Valuation

Recall that if X is an attainable contingent claim, then its unique arbitrage price $\pi_0(X)$ at $t = 0$ is defined by replication. The next result shows that $\pi_0(X)$ can also be computed using the **risk-neutral valuation formula.**

Proposition (4.2)

If X is an attainable contingent claim then $\pi_0(X)$ satisfies

$$\pi_0(X) = \frac{1}{1+r}\,\mathbb{E}_{\mathbb{Q}}\left(X\right)$$

where $\mathbb{Q} \in \mathbb{M}$ is an arbitrary martingale measure. In particular, the expected value $\mathbb{E}_{\mathbb{Q}}\left(X\right)$ does not depend on $\mathbb{Q} \in \mathbb{M}$.

Proof of Proposition 4.2.

Recall that a trading strategy $(x, \phi^1, \dots, \phi^n)$ is a replicating strategy for X whenever $V_1(x, \phi^1, \dots, \phi^n) = X$. We wish to compute the initial wealth x. □

Proof of the Risk-Neutral Valuation Formula

Proof of Proposition 4.2.

We divide both sides by $1+r$, to obtain

$$\frac{X}{1+r} = \frac{V_1(x,\phi^1,\dots,\phi^n)}{1+r} = \widehat{V}_1(x,\phi^1,\dots,\phi^n).$$

Hence

$$\begin{aligned}\frac{1}{1+r}\,\mathbb{E}_{\mathbb{Q}}\left(X\right) &= \mathbb{E}_{\mathbb{Q}}\left\{\widehat{V}_1(x,\phi^1,\dots,\phi^n)\right\} = \mathbb{E}_{\mathbb{Q}}\left\{x+\widehat{G}_1(x,\phi^1,\dots,\phi^n)\right\}\\ &= x+\mathbb{E}_{\mathbb{Q}}\left\{\sum_{j=1}^n \phi^j\Delta\widehat{S}_1^j\right\} = x+\sum_{j=1}^n \phi^j\,\underbrace{\mathbb{E}_{\mathbb{Q}}\big(\Delta\widehat{S}_1^j\big)}_{=0} = x.\end{aligned}$$

Recall that the uniqueness of x (hence its independence of $\mathbb{Q}$) is an easy consequence of the assumption that the model $\mathcal{M}$ is arbitrage-free. $\square$

Valuation of Non-Attainable Claims

- We already know that the risk-neutral valuation formula returns the arbitrage price for any attainable contingent claim.
- The next result shows that it also yields a price consistent with the no-arbitrage principle when it is applied to any non-attainable claim.
- We will later see that the arbitrage price obtained in this way is not unique, however, unless a claim X is attainable so that $p_0(X) = \pi_0(X)$.

Proposition (4.3)

Let X be a possibly non-attainable contingent claim and $\mathbb{Q}$ is an arbitrary risk-neutral probability measure. Then $p_0(X)$ given by

$$p_0(X) := \frac{1}{1+r}\,\mathbb{E}_{\mathbb{Q}}\left(X\right) \tag{2}$$

defines a price for X at $t=0$, which is consistent with the no-arbitrage principle, that is, the extended model $\widetilde{\mathcal{M}}$ is arbitrage-free.

Proof of Proposition 4.3

Proof of Proposition 4.3.

- Let $\mathbb{Q} \in \mathbb{M}$ be an arbitrary martingale measure for $\mathcal{M}$.
- We will show that $\mathbb{Q}$ is also a martingale measure for the extended model $\widetilde{\mathcal{M}} = (B, S^1, \ldots, S^{n+1})$ in which $S_0^{n+1} = p_0(X)$ and $S_1^{n+1} = X$.
- For this purpose, we check that

$$\mathbb{E}_{\mathbb{Q}}\left(\Delta \widehat{S}_1^{n+1}\right) = \mathbb{E}_{\mathbb{Q}}\left\{\frac{X}{1+r} - p_0(X)\right\} = 0$$

  and thus $\mathbb{Q} \in \widetilde{\mathbb{M}}$ is indeed a martingale measure for the extended market model.
- By the FTAP, the extended model $\widetilde{\mathcal{M}}$ is arbitrage-free and thus the price $p_0(X)$ given by (1) complies with the no-arbitrage principle.

□

PART 5

STOCHASTIC VOLATILITY MODEL

Example C: Stochastic Volatility Model

Example (Stochastic volatility)

- In the elementary market model, a replicating strategy for any contingent claim always exists. However, in a general single-period market model, a replicating strategy may fail to exist for some contingent claims.
- For instance, when there are more sources of randomness than there are stocks available for investment then replicating strategies do not exist for some contingent claims.
- Consider a market model consisting of bond B, stock S, and a random variable v called the **volatility**.
- The volatility is used to specify the size of the stock price movement over one period.
- This is a simple example of a **stochastic volatility model**.

Example C: Stock Price

Example (Stock price)

- The sample space is given by

$$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$$

and the volatility is defined as

$$v(\omega_i) = \begin{cases} h & \text{for} \quad i = 1, 4, \\ l & \text{for} \quad i = 2, 3. \end{cases}$$

- We furthermore assume that $0 < l < h < 1$. The stock price S_1 is given by

$$S_1(\omega_i) = \begin{cases} (1 + v(\omega_i))S_0 & \text{for} \quad i = 1, 2, \\ (1 - v(\omega_i))S_0 & \text{for} \quad i = 3, 4. \end{cases}$$

Example C: Incompleteness

Example (Incompleteness)

- It is easy to check that the model is arbitrage-free whenever

$$1 - h < 1 + r < 1 + h.$$

- We will check that for some contingent claims a replicating strategy does not exist, meaning that they are not **attainable**.
- To this end, we consider the **digital call option** X with the following payoff

$$X = \begin{cases} 1 & \text{if } S_1 > K, \\ 0 & \text{otherwise,} \end{cases}$$

where $K > 0$ is the strike price.

Example C: Digital Call Option

Example (Digital call)

- We assume that $(1+l)S_0 < K < (1+h)S_0$, so that

$$(1-h)S_0 < (1-l)S_0 < (1+l)S_0 < K < (1+h)S_0$$

and thus

$$X(\omega_i) = \begin{cases} 1 & \text{for } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

- Suppose that (x, ϕ) is a replicating strategy for X. Equality $V_1(x, \phi) = X$ becomes

$$(x - \phi S_0) \begin{pmatrix} 1+r \\ 1+r \\ 1+r \\ 1+r \end{pmatrix} + \phi \begin{pmatrix} (1+h)S_0 \\ (1+l)S_0 \\ (1-l)S_0 \\ (1-h)S_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Example C: Digital Call Option

Example (Digital call)

- Upon setting $\beta = \phi S_0$ and $\alpha = (1+r)x - \phi S_0 r$, we see that the existence of a solution (x, ϕ) to this system is equivalent to the existence of a solution (α, β) to the system

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} h \\ l \\ -l \\ -h \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- It is obvious that the above system of equations has no solution and thus the digital call is not an attainable contingent claim within the framework of the stochastic volatility model.
- Intuitively, the randomness generated by the volatility cannot be hedged, since the volatility is not a traded asset.

Example C: Discounted Stock Price

Example (Discounted stock)

- Propositions 4.2 and 4.3 show that martingale measures can be used to price contingent claims.
- We henceforth assume that the interest rate $r = 0$.
- Then the model is arbitrage-free since we assume that $0 < h < 1$ and thus $1 - h < 1 + r = 1 < 1 + h$. We will check that the class $\mathbb{M}$ is non-empty.
- The increments of the discounted stock price $\widehat{S}$ are represented in the following table

	ω_1	ω_2	ω_3	ω_4
$\Delta\widehat{S}_1$	hS_0	lS_0	$-lS_0$	$-hS_0$

Example C: Spaces $\mathbb{W}$ and $\mathbb{W}^\perp$

Example (Spaces $\mathbb{W}$ and $\mathbb{W}^\perp$)

- By the definition of the linear subspace $\mathbb{W} \subset \mathbb{R}^4$, we have

$$\mathbb{W} = \left\{ \gamma \begin{pmatrix} h \\ l \\ -l \\ -h \end{pmatrix} \,\middle|\, \gamma \in \mathbb{R} \right\}.$$

- The orthogonal complement of $\mathbb{W}$ is thus the three-dimensional subspace of $\mathbb{R}^4$ given by

$$\mathbb{W}^\perp = \left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \in \mathbb{R}^4 \,\middle|\, \left\langle \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}, \begin{pmatrix} h \\ l \\ -l \\ -h \end{pmatrix} \right\rangle = 0 \right\}.$$

Example C: Martingale Measures

Example (Martingale measures)

- Recall that a vector $(q_1, q_2, q_3, q_4)^\top \in \mathbb{R}^4$ belongs to $\mathcal{P}^+$ if and only if the equality $\sum_{i=1}^4 q_i = 1$ holds and $q_i > 0$ for $i = 1, 2, 3, 4$.
- Since the set of martingale measures is given by $\mathbb{M} = \mathbb{W}^\perp \cap \mathcal{P}^+$, we have

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} \in \mathbb{M} \quad \Leftrightarrow \quad \Big\{ (q_1, q_2, q_3, q_4)^\top \in \mathbb{R}^4 \,\Big|\, q_i > 0, \ \sum_{i=1}^4 q_i = 1$$

$$\text{and} \quad h(q_1 - q_4) + l(q_2 - q_3) = 0 \Big\}.$$

Example C: Martingale Measures

Example (Martingale measures)

- The class $\mathbb{M}$ of all martingale measures in our stochastic volatility model is therefore given by

$$\mathbb{M} = \left\{ \left(\begin{array}{c} q_1 \\ q_2 \\ q_3 \\ 1 - q_1 - q_2 - q_3 \end{array} \right) \;\middle|\; \begin{array}{c} q_1 > 0,\, q_2 > 0,\, q_3 > 0, \\ q_1 + q_2 + q_3 < 1, \\ l(q_2 - q_3) = h(1 - 2q_1 - q_2 - q_3) \end{array} \right\}.$$

- This set appears to be non-empty and thus we conclude that our stochastic volatility model is arbitrage-free.
- Recall that we have already shown that the digital call option is not attainable when K satisfies

$$(1+l)S_0 < K < (1+h)S_0.$$

Example C: Pricing of the Digital Call

Example (Pricing of the digital call)

- It is not difficult to check that for every $0 < q_1 < \frac{1}{2}$ there exists a probability measure $\mathbb{Q} \in \mathbb{M}$ such that $\mathbb{Q}(\omega_1) = q_1$.
- Indeed, it suffices to take any $q_1 \in (0, \frac{1}{2})$ and to set

$$q_4 = q_1, \quad q_2 = q_3 = \frac{1}{2} - q_1.$$

- Let us apply the risk-neutral valuation formula to the digital call $X = (1, 0, 0, 0)^\top$. For any $\mathbb{Q} = (q_1, q_2, q_3, q_4)^\top \in \mathbb{M}$, we obtain

$$\mathbb{E}_{\mathbb{Q}}(X) = q_1 \cdot 1 + q_2 \cdot 0 + q_3 \cdot 0 + q_4 \cdot 0 = q_1.$$

- Since q_1 is any number from $(0, \frac{1}{2})$, we see that every value from the open interval $(0, \frac{1}{2})$ can be achieved. Hence an arbitrage price for X is not unique.

PART 6

COMPLETENESS OF A MARKET MODEL

Complete and Incomplete Models

- The non-uniqueness of arbitrage prices is a serious theoretical problem, which is still not completely resolved.
- We categorise market models into two classes: **complete** and **incomplete** models.

Definition (Completeness)

A financial market model is called **complete** if for any contingent claim X there exists a replicating strategy $(x, \phi) \in \mathbb{R}^{n+1}$. A model is **incomplete** when there exists a claim X for which a replicating strategy does not exist.

- Given an arbitrage-free and complete model, the issue of pricing all contingent claims by replication is completely solved.
- How can we tell whether a given model is complete?

Algebraic Criterion for Market Completeness

Proposition (4.4)

Assume that a single-period market model $\mathcal{M} = (B, S^1, \dots, S^n)$ *defined on the sample space* $\Omega = \{\omega_1, \dots, \omega_k\}$ *is arbitrage-free. Then* $\mathcal{M}$ *is complete if and only if the* $k \times (n+1)$ *matrix* A

$$A = \begin{pmatrix} 1+r & S_1^1(\omega_1) & \cdot & \cdot & \cdot & S_1^n(\omega_1) \\ 1+r & S_1^1(\omega_2) & \cdot & \cdot & \cdot & S_1^n(\omega_2) \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ 1+r & S_1^1(\omega_k) & \cdot & \cdot & \cdot & S_1^n(\omega_k) \end{pmatrix} = (A_0, A_1, \dots, A_n)$$

has a full row rank, that is, rank $(A) = k$. *Equivalently,* $\mathcal{M}$ *is complete whenever the linear subspace spanned by the vectors* $A_0, A_1, \dots, A_n$ *coincides with the full space* $\mathbb{R}^k$.

Proof of Proposition 4.4

Proof of Proposition 4.4.

- By the linear algebra, A has a full row rank if and only if for every $X \in \mathbb{R}^k$ the equation $AZ = X$ has a solution $Z \in \mathbb{R}^{n+1}$.
- If we set $\phi^0 = x - \sum_{j=1}^n \phi^j S_0^j$ then we have

$$\begin{pmatrix} 1+r & S_1^1(\omega_1) & \cdot & \cdot & \cdot & S_1^n(\omega_1) \\ 1+r & S_1^1(\omega_2) & \cdot & \cdot & \cdot & S_1^n(\omega_2) \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ 1+r & S_1^1(\omega_k) & \cdot & \cdot & \cdot & S_1^n(\omega_k) \end{pmatrix} \begin{pmatrix} \phi^0 \\ \phi^1 \\ \cdot \\ \cdot \\ \cdot \\ \phi^n \end{pmatrix} = \begin{pmatrix} V_1(\omega_1) \\ V_1(\omega_2) \\ \cdot \\ \cdot \\ \cdot \\ V_1(\omega_k) \end{pmatrix}$$

  where $V_1(\omega_i) = V_1(x, \phi)(\omega_i)$.

- This shows that computing a replicating strategy for X is equivalent to solving the equation $AZ = X$.

Example C: Incomplete Model

Example (Matrix A)

- Consider the stochastic volatility model from Example C.
- We already know that this model is incomplete, since the digital call is not an attainable claim.
- The matrix A is given by

$$A = \begin{pmatrix} 1+r & S_1^1(\omega_1) \\ 1+r & S_1^1(\omega_2) \\ 1+r & S_1^1(\omega_3) \\ 1+r & S_1^1(\omega_4) \end{pmatrix}$$

- The rank of A is 2, and thus it is not equal to $k = 4$.
- In view of Proposition 4.4, this confirms that the model is incomplete.

Probabilistic Criterion for Attainability

- Proposition 4.4 yields a method for determining whether a market model is complete.
- Given an incomplete model, how to recognize an attainable claim?
- Recall that if a model $\mathcal{M}$ is arbitrage-free then the class $\mathbb{M}$ is non-empty.

Proposition (4.5)

Assume that a single-period model $\mathcal{M} = (B, S^1, \dots, S^n)$ is arbitrage-free. Then a contingent claim X is attainable if and only if the expected value

$$\mathbb{E}_{\mathbb{Q}}\left((1+r)^{-1}X\right)$$

has the same value for all martingale measures $\mathbb{Q} \in \mathbb{M}$.

- The proof of Proposition 4.5 is for MATH3975 only.

Proof of Proposition 4.5: 1

Proof of Proposition 4.5.

($\Rightarrow$) It is immediate from Proposition 4.2 that if a contingent claim X is attainable then the expected value

$$\mathbb{E}_{\mathbb{Q}}\left((1+r)^{-1}X\right)$$

has the same value for all $\mathbb{Q} \in \mathbb{M}$.

($\Leftarrow$) **(MATH3975)** We prove this implication by contrapositive. Let us thus assume that the contingent claim X is not attainable. Our goal is to find two risk-neutral probabilities, say $\mathbb{Q}$ and $\widehat{\mathbb{Q}}$, for which

$$\mathbb{E}_{\mathbb{Q}}\left((1+r)^{-1}X\right) \neq \mathbb{E}_{\widehat{\mathbb{Q}}}\left((1+r)^{-1}X\right). \tag{3}$$

□

Proof of Proposition 4.5: 2

Proof of Proposition 4.5.

- Consider the matrix A introduced in Proposition 4.4.
- Since the claim X is not attainable, there is no solution $Z \in \mathbb{R}^{n+1}$ to the linear system
$$AZ = X.$$
- We define the following subsets of $\mathbb{R}^k$
$$B = \text{image}\,(A) = \left\{ AZ \,|\, Z \in \mathbb{R}^{n+1} \right\} \subset \mathbb{R}^k$$
and $C = \{X\}$.
- Then B is a proper subspace of $\mathbb{R}^k$ and, obviously, the set C is convex and compact. Moreover, $B \cap C = \emptyset$.

$\square$

Proof of Proposition 4.5: 3

Proof of Proposition 4.5.

- In view of Corollary 4.1, there exists a non-zero vector $Y = (y_1, \ldots, y_k) \in \mathbb{R}^k$ such that

$$\begin{aligned} \langle b, Y \rangle &= 0 \quad \text{for all} \quad b \in B, \\ \langle c, Y \rangle &> 0 \quad \text{for all} \quad c \in C. \end{aligned}$$

- In view of the definition of B and C, this means that for every $j = 0, 1, \ldots, n$

$$\langle A_j, Y \rangle = 0 \quad \text{and} \quad \langle X, Y \rangle > 0 \tag{4}$$

  where A_j is the jth column of the matrix A.

- It is worth noting that the vector Y depends on X.

□

Proof of Proposition 4.5: 4

Proof of Proposition 4.5.

- We assumed that the market model is arbitrage-free and thus, by the FTAP, the class $\mathbb{M}$ is non-empty.
- Let $\mathbb{Q} \in \mathbb{M}$ be an arbitrary martingale measures.
- We may choose a real number $\lambda > 0$ to be small enough in order to ensure that for every $i = 1, 2, \ldots, k$

$$\widehat{\mathbb{Q}}(\omega_i) := \mathbb{Q}(\omega_i) + \lambda(1 + r)y_i > 0. \tag{5}$$

- In the next step, our next goal is to show that $\widehat{\mathbb{Q}}$ is also martingale measures and it is different from $\mathbb{Q}$.
- In the last step, we will show that inequality (3) is valid.

$\square$

Proof of Proposition 4.5: 5

Proof of Proposition 4.5.

- From the definition of A in Proposition 4.4 and the first equality in (4) with $j = 0$, we obtain

$$\sum_{i=1}^{k} \lambda(1+r)y_i = \lambda\langle A_0, Y\rangle = 0.$$

- It then follows from (5) that

$$\sum_{i=1}^{k} \widehat{\mathbb{Q}}(\omega_i) = \sum_{i=1}^{k} \mathbb{Q}(\omega_i) + \sum_{i=1}^{k} \lambda(1+r)y_i = 1$$

  and thus $\widehat{\mathbb{Q}}$ is a probability measure on the space Ω.

- In view of (5), it is clear that $\widehat{\mathbb{Q}}$ satisfies condition R.1.

$\square$

Proof of Proposition 4.5: 6

Proof of Proposition 4.5.

- It remains to check that $\widehat{\mathbb{Q}}$ satisfies also condition R.2.
- We examine the behaviour under $\widehat{\mathbb{Q}}$ of the discounted stock price $\widehat{S}_1^j$.
- For every $j = 1, 2, \ldots, n$, we have

$$
\begin{aligned}
\mathbb{E}_{\widehat{\mathbb{Q}}}\big(\widehat{S}_1^j\big) &= \sum_{i=1}^{k} \widehat{\mathbb{Q}}(\omega_i)\widehat{S}_1^j(\omega_i) \\
&= \sum_{i=1}^{k} \mathbb{Q}(\omega_i)\widehat{S}_1^j(\omega_i) + \lambda \sum_{i=1}^{k} \widehat{S}_1^j(\omega_i)(1+r)y_i \\
&= \mathbb{E}_{\mathbb{Q}}\big(\widehat{S}_1^j\big) + \lambda \underbrace{\langle A_j, Y \rangle}_{=0} && \text{(in view of (4))} \\
&= \widehat{S}_0^j && \text{(since } \mathbb{Q} \in \mathbb{M}\text{)}
\end{aligned}
$$

Proof of Proposition 4.5: 7

Proof of Proposition 4.5.

- We conclude that $\mathbb{E}_{\widehat{\mathbb{Q}}}\big(\Delta\widehat{S}_1^j\big) = 0$ and thus $\widehat{\mathbb{Q}} \in \mathbb{M}$, that is, $\widehat{\mathbb{Q}}$ is a risk-neutral probability measure for the market model $\mathcal{M}$.
- From (5), it is clear that $\mathbb{Q} \neq \widehat{\mathbb{Q}}$. We have thus proven that if $\mathcal{M}$ is arbitrage-free and incomplete then there exists more than one risk-neutral probability measure.
- To complete the proof, it remains to show that inequality (3) is satisfied for a contingent claim X.
- Recall that X was a fixed non-attainable contingent claim and we constructed a risk-neutral probability measure $\widehat{\mathbb{Q}}$ corresponding to X.

$\square$

Proof of Proposition 4.5: 8

Proof of Proposition 4.5.

- We observe that

$$\begin{aligned}
\mathbb{E}_{\widehat{\mathbb{Q}}}\left(\frac{X}{1+r}\right) &= \sum_{i=1}^{k} \widehat{\mathbb{Q}}(\omega_i)\frac{X(\omega_i)}{1+r} \\
&= \sum_{i=1}^{k} \mathbb{Q}(\omega_i)\frac{X(\omega_i)}{1+r} + \lambda \underbrace{\sum_{i=1}^{k} y_i X(\omega_i)}_{>0} \\
&> \sum_{i=1}^{k} \mathbb{Q}(\omega_i)\frac{X(\omega_i)}{1+r} = \mathbb{E}_{\mathbb{Q}}\left(\frac{X}{1+r}\right)
\end{aligned}$$

since the inequalities $\langle X, Y\rangle > 0$ and $\lambda > 0$ imply that the braced expression is strictly positive.

$\square$

Probabilistic Criterion for Market Completeness

Theorem (4.1)

Assume that a single-period model $\mathcal{M} = (B, S^1, \ldots, S^n)$ is arbitrage-free. Then $\mathcal{M}$ is complete if and only if the class $\mathbb{M}$ consists of a single element, that is, there exists a unique martingale measure for $\mathcal{M}$.

Proof of ($\Leftarrow$) in Theorem 4.1.

Since $\mathcal{M}$ is assumed to be arbitrage-free, it follows from the FTAP that there exists at least one risk-neutral probability measure, that is, the class $\mathbb{M}$ is non-empty.

($\Leftarrow$) Assume first that a martingale measure for $\mathcal{M}$ is unique. Then the condition of Proposition 4.5 is trivially satisfied for any claim X. Hence any claim X is attainable and thus the model $\mathcal{M}$ is complete. □

Proof of (⇒) in Theorem 4.1

Proof of (⇒) Theorem 4.1.

(⇒) Assume that $\mathcal{M}$ is complete and consider any two martingale measures $\mathbb{Q}$ and $\widehat{\mathbb{Q}}$ from $\mathbb{M}$. For a fixed, but arbitrary, $i = 1, 2, \ldots, k$, let the contingent claim X^i be given by

$$X^i(\omega) = \begin{cases} 1 + r & \text{if } \omega = \omega_i, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathcal{M}$ is now assumed to be complete, the contingent claim X^i is attainable and thus, from Proposition 4.2, it follows that

$$\mathbb{Q}(\omega_i) = \mathbb{E}_{\mathbb{Q}}\left(\frac{X^i}{1+r}\right) = \pi_0(X^i) = \mathbb{E}_{\widehat{\mathbb{Q}}}\left(\frac{X^i}{1+r}\right) = \widehat{\mathbb{Q}}(\omega_i).$$

Since i was arbitrary, we see that the equality $\mathbb{Q} = \widehat{\mathbb{Q}}$ holds. □

Arrow-Debreu Prices

- In financial economics, by a canonical **Arrow-Debreu security** we mean a security that pays one unit of cash if a particular state of the world is reached and zero otherwise (a binary claim).
- The price of such a security is called a **state price**.
- Any European claim X whose payoff is a function of the price S_T of the underlying asset can be decomposed as linear combination of Arrow-Debreu securities.
- Since the work of Breeden and Litzenberger (1978), researchers have used traded options to extract Arrow-Debreu prices for a variety of applications in financial economics.
- Kenneth J. Arrow and Gérard Debreu: Existence of an equilibrium for a competitive economy. *Econometrica* 22(3) (1954), 265–290.
- Douglas T. Breeden and Robert H. Litzenberger: Prices of state-contingent claims implicit in option prices. *Journal of Business* 51(4) (1978), 621–651.

Summary

Let us summarise the properties of single-period market models:

1. A single-period market model $\mathcal{M}$ is arbitrage-free if and only if it admits at least one martingale measure, that is, $\mathbb{M} \neq \emptyset$.
2. An arbitrage-free single-period market model $\mathcal{M}$ is complete if and only if a martingale measure $\mathbb{Q}$ is unique, that is, $\mathbb{M} = \{\mathbb{Q}\}$.
3. If a single-period model $\mathcal{M}$ is arbitrage-free, then:
    - Any attainable claim X (that is, any claim for which a replicating strategy exists) has the unique arbitrage price $\pi_0(X)$.
    - The arbitrage price $\pi_0(X)$ of any attainable claim X can be computed from the risk-neutral valuation formula using any martingale measure $\mathbb{Q} \in \mathbb{M}$.
    - If X is not attainable then we may define a price $p_0(X)$ for X, which is consistent with the no-arbitrage principle. It can be computed using the risk-neutral valuation formula, but it always depends on a choice of a martingale measure $\mathbb{Q} \in \mathbb{M}$.