3: ELEMENTARY MARKET MODEL

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Outline

We will examine the following issues:

1. Single-Period Two-State Market Model
2. Trading Strategies and Arbitrage-Free Models
3. Replication of Contingent Claims
4. Risk-Neutral Probability Measure
5. Put-Call Parity Relationship
6. Summary of the Elementary Market Model
7. Generalisation of the Elementary Market Model

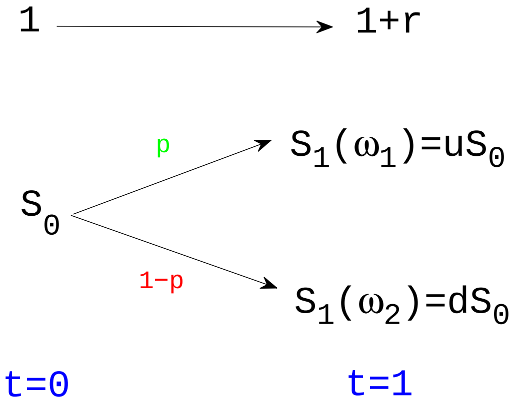
Single-Period Two-State Market Model

- Only one period is considered.
  - The beginning of the period is usually set as $t = 0$.
  - The end of the period is usually set as $t = 1$ (or $T = 1$).
- At $t = 0$, the prices of all assets are known and an investor can choose the investment.
- At $t = 1$, the prices of all assets are observed and an investor obtains a payoff corresponding to the current portfolio value.
- A single-period market model is not realistic, but it allows us to illustrate important economic principles without suffering from (or enjoying) sophisticated mathematics.
- A full understanding of advantages and shortcomings of a single-period market model is essential for further developments.

Elementary Market Model $\mathcal{M} = (B, S)$

- An investor has an initial wealth x at $t = 0$ and is allowed to invest in the risk-free asset B (bank account) and the risky asset S (stock).
- She purchases ϕ shares of the stock and invests the remaining funds in her bank account (or borrows cash from the bank).
- Notation:
  - Sample space: $\Omega = \{\omega_1, \omega_2\}$.
  - Probability measure: $\mathbb{P}(\omega_1) = p > 0$ and $\mathbb{P}(\omega_2) = 1 - p > 0$.
  - A deterministic interest rate $r > -1$. We set $B_0 = 1$ and $B_1 = 1 + r$.
  - The price of a risky asset at time t is denoted by S_t.
  - We assume that $S_0 > 0$ and we set $u = \frac{S_1(\omega_1)}{S_0}$ and $d = \frac{S_1(\omega_2)}{S_0}$.
- We suppose that $0 < d < u$, so that there are two distinct values of the future stock price: $S_1(\omega_1) > S_1(\omega_2)$.
- We do not assume that $d < 1$ and $u > 1$ (although d stands for 'down' and u stands for 'up' the actual meaning is 'low' and 'high').

Elementary Market Model



Why the elementary market model?

- In reality, the stock price movements are obviously more complicated than those given by the elementary market model.
- Hence we do not pretend that the elementary model furnishes a realistic picture of the stock price fluctuations.
- Nevertheless, even in this simplistic framework, the random character of the stock price is already visible and thus the options pricing problem is non-trivial.
- There are two important reasons why we consider this model:
  - First, the concept of arbitrage-free pricing of derivative securities can be clearly explained.
  - Second, the binomial asset pricing model, which is used in practice, can be seen as a concatenation of elementary market models.

PART 1

TRADING STRATEGIES AND ARBITRAGE-FREE MODELS

Trading Strategy and Wealth Process

- For arbitrary real numbers x and ϕ, where x is the initial endowment and ϕ is the number of shares of stock purchased or sold short at time 0, the pair (x, ϕ) is called the **trading strategy**.
- The **initial wealth** (or the **initial value**) equals (recall that $B_0 = 1$)

$$V_0(x, \phi) := x = (x - \phi S_0) \, B_0 + \phi S_0 = \phi_0^0 B_0 + \phi_0^1 S_0.$$

- The investor liquidates her portfolio (x, ϕ) at time 1 and thus obtains the random value $V_1(x, \phi)$, which equals for $i = 1, 2$

$$V_1(x, \phi)(\omega_i) := (x - \phi S_0) \, (1 + r) + \phi S_1(\omega_i).$$

Definition (Value Process)

The **wealth process** (or the **value process**) of the trading strategy (x, ϕ) is given by $(V_0(x, \phi), V_1(x, \phi))$ where $V_0(x, \phi) = x$ and $V_1(x, \phi)$ is the random variable $V_1(x, \phi) := (x - \phi S_0) \, (1 + r) + \phi S_1$.

Arbitrage Opportunity with Zero Initial Wealth

- An essential feature of an efficient market is that for any trading strategy can turn nothing into something, an investor who adopts it must also face the risk of a loss.
- The following definition is thus a crucial step in the arbitrage pricing methodology.

Definition (Arbitrage opportunity)

A trading strategy (x, ϕ) in the single-period market model is called an **arbitrage opportunity** if the following conditions are satisfied:

- A.1. $x = 0$, that is, no initial investment is required,
- A.2. $V_1(x, \phi) \geq 0$, that is, no risk of losing money,
- A.3. there exists an ω_i such that $V_1(x, \phi)(\omega_i) > 0$.

Arbitrage-Free Model

- If A.2 holds, then condition A.3 is equivalent to

  A.3′. $\mathbb{E}_{\mathbb{P}}\big(V_1(x,\phi)\big) > 0$, that is, a strictly positive expected payoff.

Definition (Arbitrage-Free Model)

A single-period market model is said to be **arbitrage-free** if no arbitrage opportunity exists in the model.

- Real markets may exhibit arbitrages, but the forces of supply and demand take actions to remove it as soon as someone discovers it.
- Market model that admit arbitrage are not suitable for pricing of financial derivatives or portfolio optimisation.

Proposition (3.1)

The model $\mathcal{M} = (B, S)$ is arbitrage-free if and only if $d < 1 + r < u$.

Proof of Proposition (3.1)

Proof of Proposition 3.1: ($\Rightarrow$).

To prove the 'only if' part, we argue by contradiction:

- Assume first that $d \geq 1 + r$:
  - At $t = 0$, the investor borrows the amount S_0 of cash on the money market and buys one share of the stock.
  - At $t = 1$, the investor sells the stock and thus receives either uS_0 or dS_0. She also pays back $(1 + r)S_0$.
- Let us now assume that $u \leq 1 + r$:
  - At $t = 0$, the investor borrows one share of the stock from the stock market and sells it immediately. She then invests S_0 in the money market.
  - At $t = 1$, the investor obtains $(1 + r)S_0$ from the bank account. She spends either uS_0 or dS_0 to buy one share of the stock and returns it to the original owner.
- This completes the proof of the 'only if' part.

Proof of Proposition (3.1)

Proof of Proposition 3.1: ($\Leftarrow$) and ($\Rightarrow$).

The ‘if’ part is also easy to establish:

- We start by noting that for $x = 0$ the equality

$$V_1(x, \phi) = (x - \phi S_0)\,(1 + r) + \phi S_1$$

  becomes $V_1(0, \phi) = \phi(S_1 - (1 + r)S_0)$.

- More explicitly,

$$\begin{aligned} V_1(0, \phi)(\omega_1) &= \phi(S_1(\omega_1) - (1 + r)S_0) = \phi S_0(u - (1 + r)) \\ V_1(0, \phi)(\omega_2) &= \phi(S_1(\omega_2) - (1 + r)S_0) = \phi S_0(d - (1 + r)) \end{aligned}$$

- Hence if $d < 1 + r < u$ then an arbitrage opportunity does not exist.
- In fact, we can also prove the ‘only if’ part using the equalities above.

$\square$

Arbitrage Opportunity with Non-Zero Initial Wealth

- There is no need to postulate that $x = 0$ in the definition of an arbitrage opportunity since if $x \neq 0$, then it can be extended as follows.

Definition (Extended arbitrage opportunity)

A trading strategy (x, ϕ) in the single-period market model is called an **extended arbitrage opportunity** if:

- B.1. $V_1(x, \phi) \geq (1 + r)x = B_1 x$,
- B.2. there exists an ω_i such that $V_1(x, \phi)(\omega_i) > (1 + r)x = B_1 x$.

- All (linear) arbitrage-free market models according to the first definition of an arbitrage opportunity are also arbitrage-free with respect to extended arbitrage opportunities.

PART 2

REPLICATION OF CONTINGENT CLAIMS

European Options

Definition (European Call and Put Options)

A **European call (put) option** is a contract which gives the buyer the right to buy (sell) an asset at a future time T for a predetermined price K.

- The underlying asset, the maturity time T and the strike price K are specified in the contract.
- Payoff of a **European call option**:
  - If the stock price S_1 at $T = 1$ is above K then the holder obtains the payoff $S_1 - K > 0$ from exercising the contract.
  - If the stock price S_1 at $T = 1$ is below K then the holder does not exercise the contract and this leads to the null payoff.
  - Hence the payoff of a European call option at time $T = 1$ is

$$C_T = h_1(S_1) = \max\{S_1 - K, 0\} = (S_1 - K)^+.$$

European Options

- Payoff of a **European put option**:
  - If the stock price S_1 at $T = 1$ is above K then the holder does not exercise the contract; hence the payoff equals 0.
  - If the stock price S_1 at $T = 1$ is below K then the holder exercises the option and obtains the payoff $K - S_1 > 0$.
  - Hence the payoff of a European put option at time $T = 1$ equals

$$P_T = h_2(S_1) = \max\{K - S_1, 0\} = (K - S_1)^+.$$

- European calls and puts are examples of **contingent claims**. Their payoffs C_T and P_T at the expiration date T are random, but they only depend on the stock price S_1 and strike K.

We will now address the following general question:

- How to select an initial investment x and a trading strategy (x, ϕ) in order to obtain the same wealth $V_1(x, \phi)$ at time 1 as the payoff of a given **contingent claim** $X = h(S_1)$?

Replication of a Contingent Claim

Definition (Replicating Strategy)

A **replicating strategy** (or a **hedge**) (x, ϕ) for the contingent claim $X = h(S_1)$ in the elementary market model $\mathcal{M} = (B, S)$ is a trading strategy which satisfies $V_1(x, \phi) = h(S_1)$, that is,

$$(x - \phi S_0)\,(1 + r) + \phi S_1(\omega_1) = h(S_1(\omega_1)), \tag{1}$$

$$(x - \phi S_0)\,(1 + r) + \phi S_1(\omega_2) = h(S_1(\omega_2)). \tag{2}$$

The following definition is consistent with the law of one price.

Definition (Arbitrage Price)

Assume that the elementary market model $\mathcal{M} = (B, S)$ is arbitrage-free. If (x, ϕ) is a replicating strategy of a contingent claim then x is called the **arbitrage price** (or **price**) for the claim at $t = 0$. We write $x = \pi_0(X)$.

Replication of a Contingent Claim: Example 1

- A replicating strategy (x, ϕ) for $X = h(S_1)$ can be computed from

$$x(1 + r) + \phi(S_1(\omega_1) - S_0(1 + r)) = h(S_1(\omega_1)),$$
$$x(1 + r) + \phi(S_1(\omega_2) - S_0(1 + r)) = h(S_1(\omega_2)).$$

- **Example 1.** Let $S_0 = 280, S_1(\omega_1) = 320, S_1(\omega_2) = 220$ and $K = 280$.
- If $X = C_1 = h_1(S_1) = (S_1 - K)^+$ then $C_1(\omega_1) = 40$ and $C_1(\omega_2) = 0$.
- If $r = 0.01$ then $S_0(1 + r) = 282.8$ and we need to solve

$$1.01x + 37.2\phi = 40,$$
$$1.01x - 62.8\phi = 0.$$

  Hence $\phi = 0.4$ and $x = (1.01)^{-1} \times 0.4 \times 62.8 = \$24.871 = C_0$.
- If $r = 0$ then the price of the call equals $\$24$.
- **Exercise 1.** Compute the call price when $r = 0.02$.

Replication of a Contingent Claim: Example 2

- We now denote $\beta = x - \phi S_0$ and $\alpha = \phi$. Then a replicating strategy (x, ϕ) for $X = h(S_1)$ can be found by solving

$$\beta(1 + r) + \alpha S_1(\omega_1) = h(S_1(\omega_1)),$$
$$\beta(1 + r) + \alpha S_1(\omega_2) = h(S_1(\omega_2)).$$

- **Example 2.** Let $S_0 = 280, S_1(\omega_1) = 320, S_1(\omega_2) = 220$ and $K = 280$.
- If $X = C_1 = h_1(S_1) = (S_1 - K)^+$ then $C_1(\omega_1) = 40$ and $C_1(\omega_2) = 0$.
- If $r = 0.01$ then we need to solve

$$1.01\beta + 320\alpha = 40,$$
$$1.01\beta + 220\alpha = 0.$$

  Hence $\alpha = \phi = 0.4$ and $\beta = -(1.01)^{-1} \times 0.4 \times 220 = -87.129$.
- The price $C_0 = \beta + \alpha S_0 = -87.129 + 0.4 \times 280 = \$24.871$.
- Check that if $r = 0$ then the price of the call equals $\$24$.
- **Exercise 2.** Compute the call price when $r = 0.02$.

Pricing of a Contingent Claim: Example 3

- Let $S_0 = 280$, $S_1(\omega_1) = 320$, $S_1(\omega_2) = 220$ and $r = 0.01$.
- We define

$$\widetilde{p} = \frac{(1+r)S_0 - S_1(\omega_2)}{S_1(\omega_1) - S_1(\omega_2)} = \frac{282.8 - 220}{320 - 220} = 0.628.$$

  Then $\widetilde{\mathbb{P}} = (\widetilde{p}, 1 - \widetilde{p}) = (0.628, 0.372)$ is a probability on $\Omega = \{\omega_1, \omega_2\}$.
- **Example 3.** If $X = C_1 = (S_1 - K)^+$ where $K = 280$ then $C_1(\omega_1) = 40$ and $C_1(\omega_2) = 0$.
- We compute

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\big((1+r)^{-1}C_1\big) = (1+r)^{-1}\widetilde{p}C_1(\omega_1) = (1.01)^{-1}\times 0.628\times 40 = \$24.871 = C_0.$$

- If $r = 0$ then $\widetilde{p} = 0.6$ and the price of the call equals $\$24$.
- If $r = 0.02$, then $\widetilde{p} = 0.656$ and $C_0 = (1.02)^{-1} \times 0.656 \times 40 = \$25.725$.
- This is by no means a coincidence and in fact the notion of a **risk-neutral probability** $\widetilde{\mathbb{P}}$ is the main tool in the study of arbitrage-free models.

Pricing of a Contingent Claim: Example 4

- Let $S_0 = 280$, $S_1(\omega_1) = 320$, $S_1(\omega_2) = 220$ and $r = 0.01$.
- Recall that

$$\widetilde{p} = \frac{(1+r)S_0 - S_1(\omega_2)}{S_1(\omega_1) - S_1(\omega_2)} = \frac{282.8 - 220}{320 - 220} = 0.628.$$

- **Example 4.** Suppose that we can apply the same method to the put option with strike $K = 280$.
- Then $X = P_1 = h_2(S_1) = (K - S_1)^+$ so that $P_1(\omega_1) = 0$ and $P_1(\omega_2) = 60$.
- Since $\widetilde{q} = 1 - \widetilde{p} = 0.372$ we obtain $\mathbb{E}_{\widetilde{\mathbb{P}}}\big((1+r)^{-1}P_1\big) = (1+r)^{-1}\widetilde{q}P_1(\omega_2) = (1.01)^{-1} \times 0.372 \times 60 = \$22.099 = P_0$.
- If $r = 0$ then $\widetilde{q} = 0.4$ and the price of the put equals $\$24$.
- If $r = 0.02$, then $\widetilde{q} = 0.344$ and $P_0 = (1.02)^{-1} \times 0.344 \times 60 = \$20.235$.
- Notice that $C_0 = P_0$ when $r = 0$ and $S_0 = K$. Moreover, the price of the put decreases when r increases.

Pricing and Hedging of a Contingent Claim

Question. How to compute the hedge ratio and the (unique) arbitrage price x for an arbitrary contingent claim $X = h(S_1)$?

- By subtracting (2) from (1), we find the **hedge ratio**

$$\phi = \frac{h(S_1(\omega_1)) - h(S_1(\omega_2))}{S_1(\omega_1) - S_1(\omega_2)} = \frac{h(uS_0) - h(dS_0)}{(u-d)S_0} =: \delta. \tag{3}$$

- Equality (3) is known as the **delta hedging formula**.
- One can substitute (3) into (1) or (2) in order to compute x.
- To derive a convenient general representation for x, we compute

$$\widetilde{p} := \frac{1 + r - d}{u - d} \in (0, 1). \tag{4}$$

- Then we define the probability $\widetilde{\mathbb{P}}$ by: $\widetilde{\mathbb{P}}(\omega_1) = \widetilde{p}$ and $\widetilde{\mathbb{P}}(\omega_2) = 1 - \widetilde{p}$.

Pricing of a Contingent Claim

- It is easy to check that $\widetilde{\mathbb{P}}$ satisfies

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{S_1}{1+r}\right) = \frac{1}{1+r}\left[\widetilde{p}S_1(\omega_1) + (1-\widetilde{p})\, S_1(\omega_2)\right] = S_0. \qquad (5)$$

- We rewrite (1) and (2) in the following way:

$$x + \left[\frac{S_1(\omega_1)}{1+r} - S_0\right]\phi = \frac{1}{1+r}h(S_1(\omega_1))$$

$$x + \left[\frac{S_1(\omega_2)}{1+r} - S_0\right]\phi = \frac{1}{1+r}h(S_1(\omega_2))$$

- We multiply (6) and (7) by $\widetilde{p}$ and $1-\widetilde{p}$, respectively, and add them.
- Then we obtain

$$\begin{aligned} & x + \left[\frac{1}{1+r}\left[\widetilde{p}S_1(\omega_1) + (1-\widetilde{p})\, S_1(\omega_2)\right] - S_0\right]\phi \\ & = \frac{1}{1+r}\left[\widetilde{p}h(S_1(\omega_1)) + (1-\widetilde{p})\, h(S_1(\omega_2))\right]. \end{aligned} \qquad (6)$$

Model Completeness

- In view of (5), the term with ϕ vanishes. Therefore, equation (8) yields the following convenient representation for the price x

$$x = \frac{1}{1+r}\left[\widetilde{p}h(S_1(\omega_1)) + (1-\widetilde{p})\, h(S_1(\omega_2))\right]. \qquad (7)$$

- It can be seen from (9) that the price x depends on $\widetilde{p}$ and $1-\widetilde{p}$, but it is independent of the real-world probabilities p and $1-p$.
- Note that equalities (3) and (9) hold for an arbitrary payoff function $h(S_1)$. Hence for any contingent claim X we have found a unique replicating strategy and arbitrage price.

Definition (Completeness)

Since every contingent claim (that is, every derivative security) in the elementary market model has a replicating strategy, the model is said to be **complete**.

PART 3

RISK-NEUTRAL PROBABILITY MEASURE

Probability Measure on a Finite Sample Space

- A **probability** $\mathbb{P} : \Omega \mapsto [0,1]$ on a sample space $\Omega = \{\omega_1, \omega_2, \dots, \omega_k\}$ uniquely specifies probabilities of all events $A_k := \{\omega_k\}$.
- It is common to write $\mathbb{P}(\{\omega_k\}) = \mathbb{P}(\omega_k) = p_k$.
- One may show that any probability on a discrete sample space Ω generates a unique probability measure on $(\Omega, \mathcal{F})$ where $\mathcal{F}$ is the class of all **events** (all subsets of Ω).

Proposition

Let $\mathbb{P} : \Omega \mapsto [0,1]$ be a probability on a discrete sample space Ω. Then the unique probability measure on $(\Omega, \mathcal{F})$ generated by $\mathbb{P}$ satisfies for all $A \in \mathcal{F}$

$$\mathbb{P}(A) = \sum_{\omega_k \in A} \mathbb{P}(\omega_k) = \sum_{\omega_k \in A} p_k.$$

Radon-Nikodym Density

Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures on a discrete sample space Ω.

Definition (Equivalence of Probability Measures)

We say that the probability measures $\mathbb{P}$ and $\mathbb{Q}$ are **equivalent** and we write $\mathbb{P} \sim \mathbb{Q}$ if for all $\omega \in \Omega$ we have that: $\mathbb{P}(\omega) > 0 \iff \mathbb{Q}(\omega) > 0$.

Definition (Radon-Nikodym Density)

The **Radon-Nikodym density** of $\mathbb{Q}$ with respect to $\mathbb{P}$ is the random variable $L : \Omega \to \mathbb{R}_+$ given by

$$L(\omega) = \frac{\mathbb{Q}(\omega)}{\mathbb{P}(\omega)}.$$

Note that

$$\mathbb{E}_{\mathbb{Q}}(X) = \sum_{\omega \in \Omega} X(\omega)\,\mathbb{Q}(\omega) = \sum_{\omega \in \Omega} X(\omega) L(\omega)\,\mathbb{P}(\omega) = \mathbb{E}_{\mathbb{P}}(LX).$$

Johann Radon (1887-1956) and Otto Nikodym (1887-1974)

Example: Radon-Nikodym Density

- The sample space Ω is defined as $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.
- Consider the probability measures $\mathbb{P}$ and $\mathbb{Q}$ on Ω given by

$$(\mathbb{P}(\omega_1), \mathbb{P}(\omega_2), \mathbb{P}(\omega_3), \mathbb{P}(\omega_4)) = \left(\frac{1}{8}, \frac{3}{8}, \frac{2}{8}, \frac{2}{8}\right)$$

$$(\mathbb{Q}(\omega_1), \mathbb{Q}(\omega_2), \mathbb{Q}(\omega_3), \mathbb{Q}(\omega_4)) = \left(\frac{4}{8}, \frac{1}{8}, \frac{2}{8}, \frac{1}{8}\right).$$

- It is clear that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, that is, $\mathbb{P} \sim \mathbb{Q}$.
- The Radon-Nikodym density L of $\mathbb{Q}$ with respect to $\mathbb{P}$ equals

$$L = (L(\omega_1), L(\omega_2), L(\omega_3), L(\omega_4)) = \left(4, \frac{1}{3}, 1, \frac{1}{2}\right).$$

- Check that for any random variable X: $\mathbb{E}_{\mathbb{Q}}(X) = \mathbb{E}_{\mathbb{P}}(LX)$.

Risk-Neutral Probability Measure

Definition (Risk-Neutral Probability Measure)

A probability measure $\mathbb{Q}$ on the sample space $\Omega = \{\omega_1, \omega_2\}$ is called a **risk-neutral probability measure (equivalent martingale measure)** for the market model $\mathcal{M} = (B, S)$ if $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and the following equality holds

$$\mathbb{E}_{\mathbb{Q}}\left(\frac{S_1}{1+r}\right) = S_0.$$

Proposition (3.2)

- *The risk-neutral probability measure for the model* $\mathcal{M} = (B, S)$ *is unique and it satisfies* $\mathbb{Q} = \widetilde{\mathbb{P}}$ *if and only if* $d < 1 + r < u$.
- *If* $1 + r \leq d$ *or* $u \leq 1 + r$ *then no risk-neutral probability exists.*

Proof of Proposition 3.2

Proof of Proposition 3.2.

- It suffices to observe that the equality

$$\mathbb{E}_{\mathbb{Q}}\left(\frac{S_1}{1+r}\right) = S_0$$

means that $\mathbb{Q}(\omega_1) \geq 0$, $\mathbb{Q}(\omega_2) \geq 0$, $\mathbb{Q}(\omega_1) + \mathbb{Q}(\omega_2) = 1$ and

$$\mathbb{Q}(\omega_1)S_1(\omega_1) + \mathbb{Q}(\omega_2)S_1(\omega_2) = (1+r)S_0.$$

- The last equality above yields

$$\mathbb{Q}(\omega_1) = \frac{1+r-d}{u-d} = \widetilde{\mathbb{P}}(\omega_1). \tag{8}$$

- If $d = 1+r$ or $u = 1+r$ then $\mathbb{Q}$ given by (10) is well defined, but it is not equivalent to $\mathbb{P}$ since either $\mathbb{Q} = (1,0)$ or $\mathbb{Q} = (0,1)$.

Expected Rates of Return on Traded Assets

- Assume that $d < 1 + r < u$. Then the risk-neutral probability measure $\mathbb{Q} = \widetilde{\mathbb{P}}$ exists and is unique.
- The expected rate of return on the savings account B equals

$$\mathbb{E}_{\widetilde{\mathbb{P}}}(r_B) = \mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{B_1 - B_0}{B_0}\right) = \frac{B_1 - B_0}{B_0} = r.$$

- The expected rate of return on the stock under $\widetilde{\mathbb{P}}$ equals r. This follows from the equality

$$\mathbb{E}_{\widetilde{\mathbb{P}}}(r_S) = \mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{S_1 - S_0}{S_0}\right) = \frac{(1+r)S_0 - S_0}{S_0} = r.$$

- The probability measure $\widetilde{\mathbb{P}}$ is the unique probability measure $\mathbb{Q}$ under which the equality $\mathbb{E}_{\mathbb{Q}}(r_B) = \mathbb{E}_{\mathbb{Q}}(r_S)$ holds.

Risk-Neutral Valuation Formula

Proposition (3.3)

For any claim $X = h(S_1)$, the arbitrage price of X at time 0 in the arbitrage-free elementary market model $\mathcal{M} = (B, S)$ satisfies

$$\pi_0(X) = \frac{1}{1+r}\,\mathbb{E}_{\widetilde{\mathbb{P}}}\left(h(S_1)\right) = \mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{X}{1+r}\right). \tag{9}$$

Proof of Proposition 3.3.

- We know (see (9)) that the price x satisfies

$$x = \frac{1}{1+r}\left[\widetilde{p}h(S_1(\omega_1)) + (1-\widetilde{p})\,h(S_1(\omega_2))\right].$$

  Equality (11) now follows immediately.
- We will now give a shorter proof for equality (11).

Proof of Proposition 3.3

Proof of Proposition 3.3.

- It is assumed that $d < 1 + r < u$. Let (x, ϕ) be any trading strategy. From the equality

$$\frac{V_1(x,\phi)}{1+r} = x + \phi\left(\frac{S_1}{1+r} - S_0\right)$$

and equality (5), we deduce that investing is a 'fair game' under $\widetilde{\mathbb{P}}$ meaning that: for any strategy (x, ϕ) we have

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{V_1(x,\phi)}{1+r}\right) = x = V_0(x,\phi).$$

- In particular, if (x, ϕ) replicates X then $V_1(x, \phi) = X$ and thus we obtain the **risk-neutral valuation formula** (11) since $x = \pi_0(X)$.

$\square$

PART 4

PUT-CALL PARITY RELATIONSHIP

Example: Call and Put Options

Example (3.1)

- Consider the elementary market model $\mathcal{M} = (B, S)$ with parameters $r = \frac{1}{3}$, $S_0 = 1$, $u = 2$, $d = \frac{1}{2}, p = \frac{3}{5}$ and $T = 1$.
- Recall that the risk-neutral probability measure $\widetilde{\mathbb{P}}$ is given as $\widetilde{\mathbb{P}}(\omega_1) = \widetilde{p}$ and $\widetilde{\mathbb{P}}(\omega_2) = 1 - \widetilde{p}$ where
$$\widetilde{p} = \frac{1 + r - d}{u - d}.$$
- Hence the risk-neutral probability measure $\widetilde{\mathbb{P}}$ equals
$$\widetilde{\mathbb{P}}(\omega_1) = \widetilde{p} = \frac{1 + \frac{1}{3} - \frac{1}{2}}{2 - \frac{1}{2}} = \frac{5}{9}, \quad \widetilde{\mathbb{P}}(\omega_2) = 1 - \widetilde{p} = \frac{4}{9}.$$

Example (3.1 Continued)

$$\boxed{S_1(\omega_1) = 2} \qquad C_1(\omega_1) = 1 \qquad P_1(\omega_1) = 0$$

$$\widetilde{p}=5/9$$

$$\boxed{S_0 = 1} \qquad K = 1 \qquad C_1 = (S_1 - K)^+ \qquad P_1 = (K - S_1)^+$$

$$1-\widetilde{p}=4/9$$

$$\boxed{S_1(\omega_2) = 0.5} \qquad C_1(\omega_2) = 0 \qquad P_1(\omega_2) = 0.5$$

Example: Call and Put Options

Example (3.1 Continued)

- The price of the European call with strike price $K = 1$ equals

$$\begin{aligned} C_0 &= \frac{1}{1+r}\,\mathbb{E}_{\widetilde{\mathbb{P}}}\left(C_T\right) = \frac{1}{1+r}\left(\widetilde{p}C_T(\omega_1) + (1-\widetilde{p})C_T(\omega_2)\right) \\ &= \frac{1}{1+r}\,\widetilde{p}(uS_0 - K) = \frac{3}{4} \times \frac{5}{9} \times 1 = \frac{5}{12}. \end{aligned}$$

- The price of the European put with strike price $K = 1$ equals

$$\begin{aligned} P_0 &= \frac{1}{1+r}\,\mathbb{E}_{\widetilde{\mathbb{P}}}\left(P_T\right) = \frac{1}{1+r}\left(\widetilde{p}P_T(\omega_1) + (1-\widetilde{p})P_T(\omega_2)\right) \\ &= \frac{1}{1+r}\,(1-\widetilde{p})(K - dS_0) = \frac{3}{4} \times \frac{4}{9} \times \frac{1}{2} = \frac{1}{6}. \end{aligned}$$

Put-Call Parity

- The arbitrage prices at time 0 computed in Example (3.1) satisfy

$$C_0 - P_0 = \frac{1}{4} = 1 - \frac{3}{4} = S_0 - \frac{1}{1+r}K. \tag{10}$$

- Equality (12) is a special case of the **put-call parity**.

Proposition (3.4)

The put-call parity can be represented as follows, for $t = 0, 1$,

$$C_t - P_t = S_t - B(t,T)K \tag{11}$$

where $B(t,T) = (1+r)^{-(T-t)}$ so that $B(0,1) = (1+r)^{-1}$ and $B(1,1) = 1$.

- Recall that we have already checked that $C_T - P_T = S_T - K$ where T is the expiration date.
- Equality (13) is an easy consequence of Proposition 3.3.

PART 5

SUMMARY OF THE ELEMENTARY MARKET MODEL

Summary: Properties

Let us summarise the properties of the elementary market model:

1. The two-state single-period market model $\mathcal{M} = (B, S)$ is arbitrage-free if and only if $d < 1 + r < u$.
2. The arbitrage-free property of the model $\mathcal{M} = (B, S)$ does not depend on the real-world probability measure $\mathbb{P}$.
3. An arbitrary contingent claim X can be replicated by means of a unique trading strategy (hence the model is complete).
4. The initial endowment of a replicating strategy for X is called the arbitrage price for X and is denoted as $\pi_0(X)$.
5. The risk-neutral probability measure $\widetilde{\mathbb{P}}$ exists and is unique if and only if $d < 1 + r < u$ (that is, whenever the model $\mathcal{M}$ is arbitrage-free). Recall that, by definition, $\widetilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$.
6. The arbitrage price $\pi_0(X)$ of any contingent claim X can be computed from the risk-neutral valuation formula.

Summary: Theorem

Theorem (3.1 Elementary Market Model)

- *The elementary market model* $\mathcal{M} = (B, S)$ *is arbitrage-free if and only if* $d < 1 + r < u$.
- *Any contingent claim* X *can be replicated so the model is complete. Formally,* $X = V_1(x, \phi)$ *for some* $(x, \phi) \in \mathbb{R}^2$.
- *If* $d < 1 + r < u$ *then any contingent claim* X *admits the unique arbitrage price* $\pi_0(X) := x = V_0(x, \phi)$ *where* $X = V_1(x, \phi)$.
- *The risk-neutral probability measure* $\widetilde{\mathbb{P}}$ *for the model* $\mathcal{M}$ *exists and is unique if and only if* $d < 1 + r < u$.
- *If* $1 + r \leq d$ *or* $u \leq 1 + r$ *then no risk-neutral probability* $\mathbb{Q}$ *exists.*
- *If* $d < 1 + r < u$ *then the arbitrage price* $\pi_0(X)$ *of* X *satisfies*

$$\pi_0(X) = \mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{X}{1+r}\right) = \mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{V_1(x,\phi)}{1+r}\right) = V_0(x, \phi).$$

PART 6

GENERALISATION OF THE ELEMENTARY MARKET MODEL

Generalisation of the Elementary Market Model

We generalise the elementary market model by postulating that:

1. We still deal with two primary traded assets, the bond B and the stock S.
2. The sample space $\Omega = \{\omega_1, \omega_2, \dots, \omega_k\}$ where $k \geq 3$.
3. Hence $S_1 = (S_1(\omega_1), \dots, S_1(\omega_k))$ where we assume that
$$S_1(\omega_k) < S_1(\omega_{k-1}) < \dots < S_1(\omega_2) < S_1(\omega_1).$$
4. The model is arbitrage-free if and only if
$$S_1(\omega_k) < S_0(1+r) < S_1(\omega_1).$$
5. The risk-neutral probability measure $\mathbb{Q}$ exists if and only if $S_1(\omega_k) < S_0(1+r) < S_1(\omega_1)$, but it is not unique.
6. The model is incomplete: no replicating strategy exists for some contingent claims $X = (X(\omega_1), \dots, X(\omega_k))$.
7. The model can be seen as a special case of a general single-period market model.

Generalisation of the Elementary Market Model

Theorem (3.2 Generalised Elementary Market Model)

- *The generalised elementary market model* $\mathcal{M} = (B, S)$ *with* $k \geq 3$ *is arbitrage-free if and only if* $S_1(\omega_k) < S_0(1 + r) < S_1(\omega_1)$.
- *Only some (but not all) contingent claims* X *can be replicated, that is, are attainable. Hence* $\mathcal{M} = (B, S)$ *is incomplete.*
- *If* $S_1(\omega_k) < S_0(1 + r) < S_1(\omega_1)$ *then any attainable claim* X *has the unique arbitrage price* $\pi_0(X)$.
- *The risk-neutral probability* $\mathbb{Q}$ *for the model* $\mathcal{M}$ *exists if and only if* $S_1(\omega_k) < S_0(1 + r) < S_1(\omega_1)$. *It is not unique.*
- *If* $S_1(\omega_k) < S_0(1 + r) < S_1(\omega_1)$ *then the arbitrage price* $\pi_0(X)$ *of any attainable claim* X *satisfies for any risk-neutral probability* $\mathbb{Q}$

$$\pi_0(X) = \mathbb{E}_{\mathbb{Q}}\left(\frac{X}{1 + r}\right).$$