MATH3075/3975
Financial Derivatives

School of Mathematics and Statistics
University of Sydney

Semester 2, 2020

Tutorial sheet 9

Background: Chapter 4 – European Options in the CRR Model.

Exercise 1 Consider the CRR model $\mathcal{M} = (B, S)$ with the horizon date $T = 2$, the risk-free rate $r = 0.1$, and $S_0 = 10$, $S_1^u = 13.2$, $S_1^d = 9.9$. Let X be a European contingent claim with the maturity date $T = 2$ and the payoff at maturity given by the formula

$$X = \big(\min(S_1, S_2) - 10\big)^+.$$

(a) Find the martingale measure $\widetilde{\mathbb{P}}$ for the market model $\mathcal{M} = (B, S)$.

(b) Show explicitly that X is a path-dependent contingent claim.

(c) Let $\mathcal{F}_t = \mathcal{F}_t^S = \sigma(S_0, \dots, S_t)$ for $t = 0, 1, 2$. Compute the arbitrage price $(\pi_t(X),\, t = 0, 1, 2)$ using the risk-neutral valuation formula, for $t = 0, 1, 2$,

$$\pi_t(X) = B_t\, \mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{X}{B_T}\,\Big|\, \mathcal{F}_t\right).$$

(d) Find the replicating strategy $(\varphi_t,\, t = 0, 1)$ for the claim X and check that the wealth process $V(\varphi)$ of the unique replicating strategy for X coincides with the price process $\pi(X)$ computed in part (c).

Exercise 2 We take for granted the CRR call option pricing formula

$$C_0 = S_0 \sum_{k=\widehat{k}}^{T} \binom{T}{k} \widehat{p}^k (1 - \widehat{p})^{T-k} - \frac{K}{(1+r)^T} \sum_{k=\widehat{k}}^{T} \binom{T}{k} \widetilde{p}^k (1 - \widetilde{p})^{T-k}$$

where $\widehat{k}$ is the smallest integer k such that

$$k \log\left(\frac{u}{d}\right) > \log\left(\frac{K}{S_0 d^T}\right).$$

Assume that the initial stock price equals $S_0 = 9$, the risk-free interest rate is $r = 0.01$ and the stock price volatility equals $\sigma = 0.1$ per annum. Use the CRR parametrization for the parameters u and d, that is, set

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = \frac{1}{u},$$

with the time increment $\Delta t = 1$ (year).

(a) Compute the arbitrage price C_0 of the European call option with strike price $K = 10$ and maturity date $T = 5$ years.

(b) Compute the prices C_1^u and C_1^d at time $t = 1$ for the same option using a suitable version of the CRR call option pricing formula.

(c) Find the hedge ratio for the option at time 0.

Exercise 3 **(MATH3975)** Consider any arbitrage-free multi-period model $\mathcal{M} = (B, S)$ where B is deterministic and S is an $\mathbb{F}$-adapted process defined on the finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F}$. We assume that B and S are strictly positive. Let $\widetilde{\mathbb{Q}}$ be any martingale measure for the process S/B and let $\widehat{\mathbb{Q}}$ be a probability measure equivalent to $\widetilde{\mathbb{Q}}$ such that the Radon-Nikodym density of $\widehat{\mathbb{Q}}$ with respect to $\widetilde{\mathbb{Q}}$ on $\mathbb{F}$ equals L. Assume that the process L is given by the following expression

$$L_t = \frac{d\widehat{\mathbb{Q}}}{d\widetilde{\mathbb{Q}}}|\mathcal{F}_t := \frac{B_0}{S_0}\frac{S_t}{B_t}, \quad t = 0, 1, \ldots, T.$$

In your answers, you may use results from Exercises 4 and 5 in week 7.

(a) Show that $L_0 = 1$ and L is a strictly positive martingale with respect to the filtration $\mathbb{F}$ under $\widetilde{\mathbb{Q}}$ so that the probability measure $\widehat{\mathbb{Q}}$ is well defined.

(b) Check that the process B/S is a martingale with respect to the filtration $\mathbb{F}$ under $\widehat{\mathbb{Q}}$.

(c) Using the abstract Bayes formula and the expression for the Radon-Nikodym density L show that for any $s > t$ and an arbitrary $\mathcal{F}_s$-measurable random variable Y the following equality holds

$$S_t\, \mathbb{E}_{\widehat{\mathbb{Q}}}\left(\frac{Y}{S_s}\,\Big|\, \mathcal{F}_t\right) = B_t\, \mathbb{E}_{\widetilde{\mathbb{Q}}}\left(\frac{Y}{B_s}\,\Big|\, \mathcal{F}_t\right)$$

(d) Assume that the call option with strike K can be replicated in $\mathcal{M}$. Using part (c), show that the arbitrage price C_t at time $t \leq T$ can be represented as follows

$$C_t = S_t \, \widehat{\mathbb{Q}}(D \,|\, \mathcal{F}_t) - K B(t,T) \, \widetilde{\mathbb{Q}}(D \,|\, \mathcal{F}_t)$$

where $D = \{\omega \in \Omega : S_T(\omega) > K\}$ and $B(t,T) := B_t/B_T$.

(e) Show that $B(t,T)$ is the arbitrage price at time t of the zero-coupon bond, which pays one unit of cash at time T.