5: FILTRATIONS AND CONDITIONING

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Outline

We will examine the following mathematical concepts:

1. Partitions and σ-fields.
2. Filtrations and adapted stochastic processes.
3. Conditional expectation with respect to a partition or a σ-field.
4. Change of a probability measure and the Radon-Nikodym density.

Our goals are to:

1. Introduce the concept of a dynamic information flow (filtration).
2. Model the stock price as a discrete time stochastic process.
3. Find the conditional distributions for the stock price.
4. Compute the conditional expectations for the stock price.

Definition of a σ-field

- Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of all natural numbers.

Definition (σ-field)

A collection $\mathcal{F}$ of subsets of Ω is called the σ-**field** whenever:

1. $\Omega \in \mathcal{F}$,
2. if $A \in \mathcal{F}$ then $A^c := \Omega \setminus A \in \mathcal{F}$,
3. if $A_i \in \mathcal{F}$ for all $i \in \mathbb{N}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

- The notion of a σ-field (also known as the σ-**algebra**) of sets is a mathematical formalism introduced in **Measure Theory** (that is, the general theory of integration) at the beginning of the 20th century.
- In applications of Probability Theory, the concept of a σ-field $\mathcal{F}_t$ is used to describe the amount of information (e.g., the history of market data) available at any given moment t.

Interpretation of a σ-field

- The set of information has to contain all possible states, so that we postulate that Ω belongs to any σ-field.
- Any set $A \in \mathcal{F}$ is interpreted as an observed **event**.
- If an event $A \in \mathcal{F}$ is given, that is, some collection of states is given, then the remaining states can also be identified and thus the complement A^c is also an event.
- The idea of a σ-field is to model a certain level of information.
- In particular, as the σ-field becomes larger, more and more events can be identified.
- We introduce the concept of an **information flow** indexed by the time parameter, which is formally represented by an increasing family of σ-fields. It will be called a **filtration**.

Partition of Ω

We take $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and we define three σ-fields:

$$\mathcal{F}_0 = \{\emptyset, \Omega\}$$
$$\mathcal{F}_1 = \Big\{\emptyset, \Omega, \{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_1, \omega_2, \omega_3\}, \{\omega_1, \omega_2, \omega_4\}, \{\omega_3, \omega_4\}\Big\}$$
$$\mathcal{F}_2 = 2^{\Omega} \quad \text{(the power set of } \Omega\text{, i.e., the class of all subsets of } \Omega\text{).}$$

Note that $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2$, that is, the information increases:

- $\mathcal{F}_0$: no specific information, except for the set Ω (all possible outcomes).
- $\mathcal{F}_1$: a partial information, since we cannot distinguish between ω_1 or ω_2,
- $\mathcal{F}_2$: the full information, since $\{\omega_1\}, \{\omega_2\}, \{\omega_3\}$ and $\{\omega_4\}$ can be observed.

Definition (Partition)

A **partition** of Ω is any collection $\mathcal{P} = (A_i)_{i \in I}$ of non-empty subsets of Ω such that the sets A_i are pairwise disjoint ($A_i \cap A_j = \emptyset$ when $i \neq j$) and $\bigcup_{i \in I} A_i = \Omega$.

Partition Associated with a σ-Field

Lemma

For any σ-field $\mathcal{F}$ of subsets of a finite (or countable) state space Ω, a partition $\mathcal{P}$ associated with $\mathcal{F}$ exists and is unique. We then say that σ-field $\mathcal{F}$ is **generated** *by the partition $\mathcal{P}$ and we write $\mathcal{F} = \sigma(\mathcal{P})$.*

- We have $\mathcal{P}_0 = \mathcal{F}_0$ and $\mathcal{P}_2 = \Big\{ \{\omega_1\}, \{\omega_2\} \{\omega_3\}, \{\omega_4\} \Big\}$.
- The unique partition associated with $\mathcal{F}_1$ is given by

$$\mathcal{P}_1 = \Big\{ \{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4\} \Big\}.$$

- Define the probabilities

$$\mathbb{P}\left(\{\omega_1, \omega_2\}\right) = 0.6, \quad \mathbb{P}\left(\{\omega_3\}\right) = 0.2, \quad \mathbb{P}\left(\{\omega_4\}\right) = 0.2.$$

- Then for any event $A \in \mathcal{F}_1$ the probability of A can be computed, e.g.,

$$\mathbb{P}\left(\{\omega_1, \omega_2, \omega_4\}\right) = \mathbb{P}\left(\{\omega_1, \omega_2\}\right) + \mathbb{P}\left(\{\omega_4\}\right) = 0.8.$$

Filtration

- To model the information flow, we use the concept of a **filtration**.
- The following definition covers the cases of both discrete time and continuous time market models.

Definition (Filtration)

A family $(\mathcal{F}_t)_{0 \leq t \leq T}$ of σ-fields on Ω is called a **filtration** if $\mathcal{F}_s \subset \mathcal{F}_t$ for every $s \leq t$. For brevity, we denote $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$.

- We interpret the σ-field $\mathcal{F}_t$ as the information available to an agent at time t. In particular, $\mathcal{F}_0$ represents the information available at time 0, that is, the initial information.
- We assume that the information accumulated over time can only grow, so no event from the past is forgotten.

Probability Measure

Definition (Probability Measure)

A map $\mathbb{P} : \mathcal{F} \to [0,1]$ is called a **probability measure** if:

1. $\mathbb{P}(\Omega) = 1$,
2. for any sequence $A_i, i \in \mathbb{N}$ of pairwise disjoint events we have

$$\mathbb{P}\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) = \sum_{i\in\mathbb{N}} \mathbb{P}(A_i).$$

The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **probability space**.

- By convention, the probability of all possibilities is 1 (see 1).
- By definition, a probability measure is σ-additive (see 2).
- $\mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$ for any $A \in \mathcal{F}$.

Conditional Expectation: Framework

- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite (or countable) probability space.
- Let X be an arbitrary $\mathcal{F}$-**measurable** random variable. This means that for every $x \in \mathbb{R}$, the set $\{\omega \in \Omega \,|\, X(\omega) = x\}$ belongs to $\mathcal{F}$.
- Assume that $\mathcal{G}$ is a **sub-σ-field** of $\mathcal{F}$, meaning that $\mathcal{G} \subset \mathcal{F}$. More explicitly, all events belonging to $\mathcal{G}$ belong to $\mathcal{F}$ as well.
- Let $A \subset \Omega$. We define the **indicator function** $\mathbb{1}_A : \Omega \to \{0, 1\}$ by setting $\mathbb{1}_A(\omega) = 1$ if $\omega \in A$ and $\mathbb{1}_A(\omega) = 0$ if $\omega \notin A$.
- Let $\mathcal{P} = (A_i)_{i \in I}$ be the unique partition associated with $\mathcal{G}$ where I is the index set.
- Then a random variable X is $\mathcal{G}$-measurable if it is constant on each event $A_i \in \mathcal{P}$. Hence $X = \sum_{i \in I} c_i \mathbb{1}_{A_i}$ for some real numbers c_i, $i \in I$.
- Our goal is to define the **conditional expectation** $\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})$, that is, the conditional expectation of a random variable X with respect to a σ-field $\mathcal{G}$.
- The expected value $\mathbb{E}_{\mathbb{P}}(X)$ can be obtained from $\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})$ by setting $\mathcal{G} = \mathcal{F}_0$, that is, $\mathbb{E}_{\mathbb{P}}(X) = \mathbb{E}_{\mathbb{P}}(X|\,\mathcal{F}_0)$.
- Alternatively, the expected value $\mathbb{E}_{\mathbb{P}}(X)$ can be computed using the equality $\mathbb{E}_{\mathbb{P}}(X) = \mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})\big)$ for any sub-σ-field $\mathcal{G}$ of $\mathcal{F}$.

Conditional Expectation: Definition

Definition (Conditional Expectation)

The **conditional expectation** $\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})$ of X with respect to $\mathcal{G}$ is defined as a $\mathcal{G}$-measurable random variable, which is given by, for every i and all $\omega \in A_i$

$$\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})(\omega) = \frac{1}{\mathbb{P}(A_i)} \sum_{\omega_l \in A_i} X(\omega_l)\mathbb{P}(\omega_l) = \sum_{x_m} x_m \, \mathbb{P}(X = x_m \,|A_i)$$

where the summation is over all possible values of X and the conditional probability of the event $\{\omega \in \Omega \,|\, X(\omega) = x_m\}$ given the event A_i, denoted as $\mathbb{P}(X = x_m \,|A_i)$, satisfies

$$\mathbb{P}(X = x_m \,|\, A_i) = \frac{\mathbb{P}(\{X = x_m\} \cap A_i)}{\mathbb{P}(A_i)}.$$

Hence the conditional expectation $\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G}) : \Omega \to \mathbb{R}$ equals for every $\omega \in \Omega$

$$\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})(\omega) = \sum_{i \in I} \frac{1}{\mathbb{P}(A_i)} \, \mathbb{E}_{\mathbb{P}}(X \mathbb{1}_{A_i}) \, \mathbb{1}_{A_i}(\omega).$$

Conditional Expectation: Computation

- The conditional expectation $\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})$ can be represented as follows

$$\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G}) = \sum_{i \in I} c_i \mathbb{1}_{A_i}$$

where $c_i \in \mathbb{R}$ can be computed from either the equality

$$c_i = \frac{1}{\mathbb{P}(A_i)} \sum_{\omega_l \in A_i} X(\omega_l)\mathbb{P}(\omega_l)$$

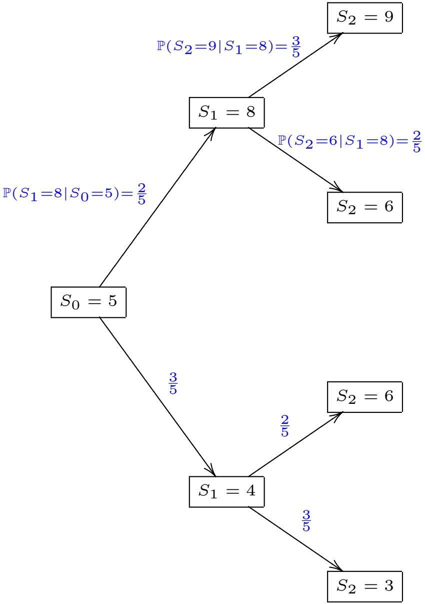
or the equality

$$c_i = \sum_{x_m} x_m \, \mathbb{P}(X = x_m \,|A_i).$$

- It is clear that the random variable $\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})$ is constant on every set A_i from the partition $\mathcal{P}$ associated with $\mathcal{G}$ and thus it is indeed a $\mathcal{G}$-measurable random variable.
- It is also easy to check that

$$\mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})) = \sum_{i \in I} c_i \mathbb{P}(A_i) = \sum_{i \in I} \sum_{\omega_l \in A_i} X(\omega_l)\mathbb{P}(\omega_l) = \mathbb{E}_{\mathbb{P}}(X).$$

Example: Conditional Probabilities for S



$\omega_1 = (u, u) \qquad \mathbb{P}(\omega_1) = 6/25 = 0.24$

$\omega_2 = (u, d) \qquad \mathbb{P}(\omega_2) = 4/25 = 0.16$

$\omega_3 = (d, u) \qquad \mathbb{P}(\omega_3) = 6/25 = 0.24$

$\omega_4 = (d, d) \qquad \mathbb{P}(\omega_4) = 9/25 = 0.36$

Example: Conditional Probabilities for S

- The probability space is given by $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $\mathbb{P} = (0.24, 0.16, 0.24, 0.36)$.
- At time $t = 0$, the σ-field $\mathcal{F}_0^S = \{\emptyset, \Omega\}$ is trivial (as always).
- At time $t = 1$, the stock can take two possible values, $x_1 = 8$ and $x_2 = 4$, and

$$S_1(\omega) = \begin{cases} x_1 = 8 & \text{if } \omega \in \{\omega_1, \omega_2\} = A_1 \\ x_2 = 4 & \text{if } \omega \in \{\omega_3, \omega_4\} = A_2 \end{cases}$$

  so that $\mathcal{P}_1^S = \{A_1, A_2\} = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$ and $\mathcal{F}_1^S = \{\emptyset, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \Omega\}$.
- At time $t = 2$, we have $\mathcal{F}_1^S = 2^\Omega$ (all subsets of Ω).
- Note that all conditional probabilities can be computed from $\mathbb{P}$, for instance,

$$\mathbb{P}(S_2 = 9 \,|\, S_1 = 8) = \mathbb{P}(S_2 = 9 \,|\, A_1) = \frac{\mathbb{P}(\{X = 9\} \cap A_1)}{\mathbb{P}(A_1)} = \frac{6/25}{6/25 + 4/25} = 3/5,$$

$$\mathbb{P}(S_2 = 6 \,|\, S_1 = 8) = \mathbb{P}(S_2 = 6 \,|\, A_1) = \frac{\mathbb{P}(\{X = 6\} \cap A_1)}{\mathbb{P}(A_1)} = \frac{4/25}{6/25 + 4/25} = 2/5.$$

- As expected, the conditional probabilities satisfy

$$\mathbb{P}(S_2 = 9 \,|\, S_1 = 8) + \mathbb{P}(S_2 = 6 \,|\, S_1 = 8) = 1,$$
$$\mathbb{P}(S_2 = 6 \,|\, S_1 = 4) + \mathbb{P}(S_2 = 3 \,|\, S_1 = 4) = 1.$$

Example: Conditional Probabilities for S

- Recall that $A_1 = \{\omega_1, \omega_2\} = \{S_1 = 8\}$ and $A_2 = \{\omega_3, \omega_4\} = \{S_1 = 4\}$.
- Then $\mathbb{P}(A_1) = 2/5$ and $\mathbb{P}(A_2) = 3/5$. Therefore, the conditional probabilities are

$$\mathbb{P}(S_2 = 9 \,|\, A_1) = \frac{\mathbb{P}(\{\omega \in A_1\} \cap \{S_2(\omega) = 9\})}{\mathbb{P}(A_1)} = \frac{\mathbb{P}(\omega_1)}{\mathbb{P}(\{\omega_1, \omega_2\})} = \frac{\frac{6}{25}}{\frac{2}{5}} = \frac{3}{5}$$

$$\mathbb{P}(S_2 = 6 \,|\, A_1) = \frac{\mathbb{P}(\{\omega \in A_1\} \cap \{S_2(\omega) = 6\})}{\mathbb{P}(A_1)} = \frac{\mathbb{P}(\omega_2)}{\mathbb{P}(\{\omega_1, \omega_2\})} = \frac{\frac{4}{25}}{\frac{2}{5}} = \frac{2}{5}$$

$$\mathbb{P}(S_2 = 3 \,|\, A_1) = \frac{\mathbb{P}(\{\omega \in A_1\} \cap \{S_2(\omega) = 3\})}{\mathbb{P}(A_1)} = \frac{\mathbb{P}(\emptyset)}{\mathbb{P}(\{\omega_1, \omega_2\})} = 0$$

$$\mathbb{P}(S_2 = 9 \,|\, A_2) = \frac{\mathbb{P}(\{\omega \in A_2\} \cap \{S_2(\omega) = 9\})}{\mathbb{P}(A_2)} = \frac{\mathbb{P}(\emptyset)}{\mathbb{P}(\{\omega_3, \omega_4\})} = 0$$

$$\mathbb{P}(S_2 = 6 \,|\, A_2) = \frac{\mathbb{P}(\{\omega \in A_2\} \cap \{S_2(\omega) = 6\})}{\mathbb{P}(A_2)} = \frac{\mathbb{P}(\omega_3)}{\mathbb{P}(\{\omega_3, \omega_4\})} = \frac{\frac{6}{25}}{\frac{3}{5}} = \frac{2}{5}$$

$$\mathbb{P}(S_2 = 3 \,|\, A_2) = \frac{\mathbb{P}(\{\omega \in A_2\} \cap \{S_2(\omega) = 3\})}{\mathbb{P}(A_2)} = \frac{\mathbb{P}(\omega_4)}{\mathbb{P}(\{\omega_3, \omega_4\})} = \frac{\frac{9}{25}}{\frac{3}{5}} = \frac{3}{5}$$

Example: Conditional Expectations for S

- We have

$$\mathbb{E}_{\mathbb{P}}(S_2 \,|\, \mathcal{F}_1^S)(\omega) = 9 \cdot \frac{3}{5} + 6 \cdot \frac{2}{5} = \frac{39}{5} = 7.8 \quad \text{for } \omega \in A_1$$

$$\mathbb{E}_{\mathbb{P}}(S_2 \,|\, \mathcal{F}_1^S)(\omega) = 6 \cdot \frac{2}{5} + 3 \cdot \frac{3}{5} = \frac{21}{5} = 4.2 \quad \text{for } \omega \in A_2$$

and thus

$$\mathbb{E}_{\mathbb{P}}(S_2 |\, \mathcal{F}_1^S) = 7.8\, \mathbb{1}_{A_1} + 4.2\, \mathbb{1}_{A_2} = \begin{cases} 7.8 & \text{if } \ \omega \in \{\omega_1, \omega_2\} \\ 4.2 & \text{if } \ \omega \in \{\omega_3, \omega_4\} \end{cases}$$

- To check that $\mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}(S_2 |\, \mathcal{F}_1^S)\big) = \mathbb{E}_{\mathbb{P}}(S_2 |\, \mathcal{F}_0^S) = \mathbb{E}_{\mathbb{P}}(S_2)$, we compute

$$\mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}(S_2 |\, \mathcal{F}_1^S)\big) = \frac{39}{5} \cdot \frac{2}{5} + \frac{21}{5} \cdot \frac{3}{5} = \frac{141}{25} = 5.64,$$

$$\mathbb{E}_{\mathbb{P}}(S_2) = 9 \cdot \frac{6}{25} + 6 \cdot \frac{4}{25} + 6 \cdot \frac{6}{25} + 3 \cdot \frac{9}{25} = \frac{141}{25} = 5.64.$$

- We have $\mathbb{E}_{\mathbb{P}}(S_1) = 8 \cdot \frac{10}{25} + 4 \cdot \frac{15}{25} = \frac{140}{25} = 5.6$ and thus $\mathbb{E}_{\mathbb{P}}(S_0) < \mathbb{E}_{\mathbb{P}}(S_1) < \mathbb{E}_{\mathbb{P}}(S_2)$.

Properties of Conditional Expectation

Proposition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be endowed with sub-σ-fields $\mathcal{G}$ and $\mathcal{G}_1 \subset \mathcal{G}_2$ of $\mathcal{F}$. Then

1. **Tower property:** *If $X : \Omega \to \mathbb{R}$ is an $\mathcal{F}$-measurable r.v. then*
$$\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G}_1) = \mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G}_2)\,|\,\mathcal{G}_1\big) = \mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G}_1)\,|\,\mathcal{G}_2\big).$$
2. **Taking out what is known:** *If $X : \Omega \to \mathbb{R}$ is a $\mathcal{G}$-measurable r.v. and $Y : \Omega \to \mathbb{R}$ is an $\mathcal{F}$-measurable r.v. then*
$$\mathbb{E}_{\mathbb{P}}(XY|\,\mathcal{G}) = X\,\mathbb{E}_{\mathbb{P}}(Y|\,\mathcal{G}).$$
3. **Trivial conditioning:** *If $X : \Omega \to \mathbb{R}$ is an $\mathcal{F}$-measurable r.v., which is independent of $\mathcal{G}$, then*
$$\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G}) = \mathbb{E}_{\mathbb{P}}(X).$$

- We say that X is **independent** of $\mathcal{G}$ if for every $x \in \mathbb{R}$ and every $A \in \mathcal{G}$
$$\mathbb{P}\big(\{X = x\} \cap A\big) = \mathbb{P}\big(\{X = x\}\big)\mathbb{P}\big(A\big).$$

  Recall that $\{X = x\} = \{\omega \in \Omega \,|\, X(\omega) = x\}$.

Equivalence of Probability Measures

Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures on a finite (or countable) sample space Ω.

Definition

We say that the probability measures $\mathbb{P}$ and $\mathbb{Q}$ are **equivalent** and we write $\mathbb{P} \sim \mathbb{Q}$ if for all $\omega \in \Omega$ we have that: $\mathbb{P}(\omega) > 0 \;\Leftrightarrow\; \mathbb{Q}(\omega) > 0$.

Definition (Radon-Nikodym Density)

If $\mathbb{P} \sim \mathbb{Q}$, then the **Radon-Nikodym density** of $\mathbb{Q}$ with respect to $\mathbb{P}$ is the random variable $L : \Omega \to \mathbb{R}_+$ given by, for every $\omega \in \Omega$,

$$L(\omega) = \frac{\mathbb{Q}(\omega)}{\mathbb{P}(\omega)}.$$

Note that

$$\mathbb{E}_{\mathbb{Q}}(X) = \sum_{\omega \in \Omega} X(\omega)\, \mathbb{Q}(\omega) = \sum_{\omega \in \Omega} X(\omega) L(\omega)\, \mathbb{P}(\omega) = \mathbb{E}_{\mathbb{P}}(XL).$$

The following question arises: is it true that if $\mathcal{G}$ is a sub-σ-field then

$$\mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{G}) = \mathbb{E}_{\mathbb{P}}(XL \,|\, \mathcal{G})?$$

We will see that the answer to that question is negative, in general.

Example: Radon-Nikodym Density

- The sample space Ω is defined as $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$.
- Consider the probability measures $\mathbb{P}$ and $\mathbb{Q}$ on Ω given by

$$\big(\mathbb{P}(\omega_1), \mathbb{P}(\omega_2), \mathbb{P}(\omega_3), \mathbb{P}(\omega_4), \mathbb{P}(\omega_5)\big) = \left(\frac{1}{10}, \frac{3}{10}, \frac{2}{10}, \frac{3}{10}, \frac{1}{10}\right)$$
$$\big(\mathbb{Q}(\omega_1), \mathbb{Q}(\omega_2), \mathbb{Q}(\omega_3), \mathbb{Q}(\omega_4), \mathbb{Q}(\omega_5)\big) = \left(\frac{3}{10}, \frac{2}{10}, \frac{1}{10}, \frac{2}{10}, \frac{2}{10}\right).$$

- It is clear that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, that is, $\mathbb{P} \sim \mathbb{Q}$.
- The Radon-Nikodym density L of $\mathbb{Q}$ with respect to $\mathbb{P}$ equals

$$L = \big(L(\omega_1), L(\omega_2), L(\omega_3), L(\omega_4), L(\omega_5)\big) = \left(3, \frac{2}{3}, \frac{1}{2}, \frac{2}{3}, 2\right).$$

- Check that for any random variable X: $\mathbb{E}_{\mathbb{Q}}(X) = \mathbb{E}_{\mathbb{P}}(XL)$.
- Show be means of an example that $\mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{G}) \neq \mathbb{E}_{\mathbb{P}}(XL \,|\, \mathcal{G})$, in general.

Change of a Probability Measure

- Let $\mathbb{P}$ and $\mathbb{Q}$ be two equivalent probability measures on $(\Omega, \mathcal{F})$.
- Let the Radon-Nikodym density of $\mathbb{Q}$ with respect to $\mathbb{P}$ be

$$L(\omega) = \frac{\mathbb{Q}(\omega)}{\mathbb{P}(\omega)} \quad \text{or} \quad L(\omega) = \frac{d\mathbb{Q}}{d\mathbb{P}}(\omega).$$

meaning that L is $\mathcal{F}$-measurable and for every $A \in \mathcal{F}$

$$\mathbb{Q}(A) = \sum_{\omega \in A} L(\omega)\, \mathbb{P}(\omega) \quad \text{or} \quad \mathbb{Q}(A) = \int_A L\, d\mathbb{P} = \int_A L(\omega)\, d\mathbb{P}(\omega)$$

- If Ω is finite then for every $A \in \mathcal{F}$

$$\sum_{\omega \in A} X(\omega)\, \mathbb{Q}(\omega) = \sum_{\omega \in A} X(\omega) L(\omega)\, \mathbb{P}(\omega).$$

- If X is an arbitrary $\mathcal{F}$-measurable and $\mathbb{Q}$-integrable r.v., then

$$\int_A X\, d\mathbb{Q} = \int_A X L\, d\mathbb{P}.$$

- Equality $\mathbb{E}_{\mathbb{Q}}(X) = \mathbb{E}_{\mathbb{P}}(XL)$ holds for any X (it suffices to take $A = \Omega$).
- L is strictly positive $\mathbb{P}$-a.s. and $\mathbb{E}_{\mathbb{P}}(L) = 1$.

Abstract Bayes Formula

Lemma (Bayes Formula)

Let X be an arbitrary $\mathcal{F}$-measurable and $\mathbb{Q}$-integrable random variable. If $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$, then the **Bayes formula** *holds*

$$\mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{G}) = \frac{\mathbb{E}_{\mathbb{P}}(XL \,|\, \mathcal{G})}{\mathbb{E}_{\mathbb{P}}(L \,|\, \mathcal{G})}. \tag{1}$$

Proof.

- The random variable $\mathbb{E}_{\mathbb{P}}(L \,|\, \mathcal{G})$ is strictly positive so that the right-hand side in (1) is well defined.
- Therefore, it suffices to show that the equality

$$\mathbb{E}_{\mathbb{P}}(XL \,|\, \mathcal{G}) = \mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{G}) \, \mathbb{E}_{\mathbb{P}}(L \,|\, \mathcal{G}) \tag{2}$$

  holds for every random variable X.
- We first give the proof for the special case when Ω is finite and then we consider the general case.

$\square$

MATH3075: Proof for Partitions (Finite Ω)

Proof.

Let $\mathcal{G} = \sigma(A_1, A_2, \dots, A_p)$ and $\mathbb{Q}(\omega_l) = L(\omega_l)\mathbb{P}(\omega_l)$. For any fixed $i = 1, 2, \dots, p$, we obtain on the event A_i

$$\mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{G}) = \frac{1}{\mathbb{Q}(A_i)} \sum_{\omega_l \in A_i} X(\omega_l)\mathbb{Q}(\omega_l)$$

$$\mathbb{E}_{\mathbb{P}}(L \,|\, \mathcal{G}) = \frac{1}{\mathbb{P}(A_i)} \sum_{\omega_l \in A_i} L(\omega_l)\mathbb{P}(\omega_l) = \frac{\mathbb{Q}(A_i)}{\mathbb{P}(A_i)}$$

$$\mathbb{E}_{\mathbb{P}}(XL \,|\, \mathcal{G}) = \frac{1}{\mathbb{P}(A_i)} \sum_{\omega_l \in A_i} X(\omega_l)L(\omega_l)\mathbb{P}(\omega_l) = \frac{1}{\mathbb{P}(A_i)} \sum_{\omega_l \in A_i} X(\omega_l)\mathbb{Q}(\omega_l)$$

Hence on every event A_i from the partition $\mathcal{P} = \{A_1, \dots, A_p\}$ we obtain the equality

$$\mathbb{E}_{\mathbb{P}}(XL \,|\, \mathcal{G}) = \mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{G})\, \mathbb{E}_{\mathbb{P}}(L \,|\, \mathcal{G}).$$

This shows that (2) and thus also (1) hold when the σ-field $\mathcal{G}$ is generated by a partition $\mathcal{P} = \{A_1, A_2, \dots, A_p\}$. $\square$

MATH3975: Proof for Arbitrary σ-fields (General Ω)

Proof.

Since the right-hand side in (2) defines a $\mathbb{G}$-measurable random variable, it suffices to verify that we have, for every event $G \in \mathcal{G}$,

$$\int_G XL\,d\mathbb{P} = \int_G \mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{G})\,\mathbb{E}_{\mathbb{P}}(L \,|\, \mathcal{G})\,d\mathbb{P}.$$

For any $G \in \mathcal{G}$, we obtain

$$\begin{aligned}
\int_G XL\,d\mathbb{P} &\overset{L-density}{=} \int_G X\,d\mathbb{Q} \overset{\mathbb{Q}-cond.}{=} \int_G \mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{G})\,d\mathbb{Q} \\
&\overset{L-density}{=} \int_G \mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{G})L\,d\mathbb{P} \overset{\mathbb{P}-cond}{=} \int_G \mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{G})L \,\big|\, \mathcal{G}\big)\,d\mathbb{P} \\
&\overset{Proposition}{=} \int_G \mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{G})\,\mathbb{E}_{\mathbb{P}}(L \,|\, \mathcal{G})\,d\mathbb{P}.
\end{aligned}$$

We have used here the following property of the conditional expectation

$$\int_G X\,d\mathbb{P} = \int_G \mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})\,d\mathbb{P}, \quad \forall\, G \in \mathcal{G}.$$

as well as the 'taking out what is known' property of the conditional expectation. $\square$