4A: MARTINGALE MEASURES

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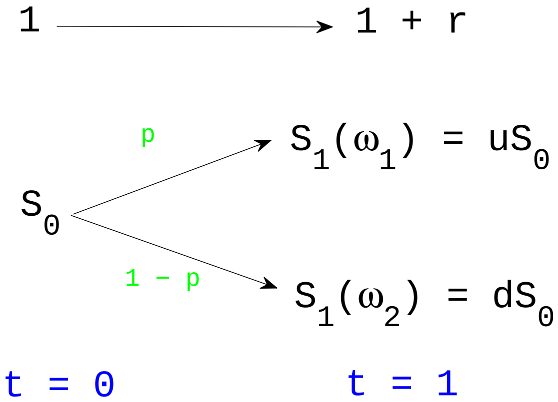
PART A

ELEMENTARY MARKET MODEL WITH A SINGLE STOCK AND $\Omega = \{\omega_1, \omega_2\}$

Elementary Market Model $\mathcal{M} = (B, S)$

- An investor has an initial wealth x at $t = 0$ and is allowed to invest in the risk-free asset B (bank account) and the risky asset S (stock).
- She purchases ϕ shares of the stock and invests the remaining funds in her bank account (or borrows cash from the bank).
- Notation:
  - Sample space is $\Omega = \{\omega_1, \omega_2\}$ and thus $n = 1$ and $k = 2$.
  - Probability measure: $\mathbb{P}(\omega_1) = p > 0$ and $\mathbb{P}(\omega_2) = 1 - p > 0$.
  - A deterministic interest rate $r > -1$. We set $B_0 = 1$ and $B_1 = 1 + r$.
  - Let S_t be the price of a risky asset at time t for $t = 0, 1$.
  - We assume that $S_0 > 0$ and we denote $u = \frac{S_1(\omega_1)}{S_0}$ and $d = \frac{S_1(\omega_2)}{S_0}$.
- We suppose that $0 < d < u$, so that there are two distinct values of the future stock price: $S_1(\omega_1) > S_1(\omega_2)$.
- It is not assumed that $d < 1$ and $u > 1$ (although d stands for 'down' and u stands for 'up' the actual meaning is 'low' and 'high').


1
1 + r
p
S1(ω1) = uS0
S0
1 − p
S1(ω2) = dS0
t = 0
t = 1

Properties of the Elementary Market Model

The properties of the elementary market model:

1. The two-state single-period market model $\mathcal{M} = (B, S)$ is arbitrage-free if and only if $d < 1 + r < u$, that is, whenever $S_1(\omega_2) < S_0(1 + r) < S_1(\omega_1)$.
2. The martingale measure $\widetilde{\mathbb{P}}$ exists and is unique if and only if $d < 1 + r < u$ (that is, whenever $\mathcal{M}$ is arbitrage-free). It satisfies $\mathbb{E}_{\widetilde{\mathbb{P}}}(S_1) = S_0(1 + r)$.
3. The arbitrage-free property of the model $\mathcal{M} = (B, S)$ does not depend on the real-world probability measure $\mathbb{P}$.
4. An arbitrary contingent claim X can be replicated by means of a (unique) trading strategy. Hence we say that the market model is complete.
5. The initial value of a replicating strategy for X is called the arbitrage price for X and we denote it by $\pi_0(X)$.
6. The arbitrage price $\pi_0(X)$ of any contingent claim X can be computed using the risk-neutral valuation formula.

Theorem A

Theorem (A. Elementary Market Model)

- *The model $\mathcal{M} = (B, S)$ is arbitrage-free whenever $d < 1 + r < u$.*
- *Any contingent claim $X \in \mathbb{R}^2$ can be replicated and thus the model $\mathcal{M} = (B, S)$ is complete. Formally, $X = V_1(x, \phi)$ for some $(x, \phi) \in \mathbb{R}^2$.*
- *If $d < 1 + r < u$ then any contingent claim X has the unique arbitrage price $\pi_0(X) := x = V_0(x, \phi)$ where $X = V_1(x, \phi)$ where $V_0(x, \phi) = x$.*
- *If $d < 1 + r < u$ then $\widetilde{\mathbb{P}} = (\widetilde{\mathbb{P}}(\omega_1), \widetilde{\mathbb{P}}(\omega_2)) = (\widetilde{p}, 1 - \widetilde{p})$ where $\widetilde{p} = \frac{1+r-d}{u-d}$ is a unique martingale measure and thus $\mathbb{M} = \{\widetilde{\mathbb{P}}\}$.*
- *If either $1 + r \leq d$ or $u \leq 1 + r$ then no martingale measure exists and thus $\mathbb{M} = \emptyset$.*
- *If $d < 1 + r < u$ then the arbitrage price $\pi_0(X)$ of X satisfies*

$$\pi_0(X) = \mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{X}{1+r}\right) = \mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{V_1(x, \phi)}{1+r}\right) = V_0(x, \phi) = x.$$

PART B

GENERALISED MARKET MODEL WITH A SINGLE STOCK AND $\Omega = \{\omega_1, \omega_2, \dots, \omega_k\}$

Market Model with a Single Stock

1. We still deal with two primary traded assets, the bond B and the stock S.
2. The sample space is $\Omega = \{\omega_1, \omega_2, \ldots, \omega_k\}$ where $k \geq 3$.
3. Hence $S_1 = (S_1(\omega_1), \ldots, S_1(\omega_k))$ where we may assume that
$$S_1(\omega_k) < S_1(\omega_{k-1}) < \cdots < S_1(\omega_2) < S_1(\omega_1).$$
4. The model is arbitrage-free if and only if
$$S_1(\omega_k) < S_0(1+r) < S_1(\omega_1).$$
5. A martingale measure $\mathbb{Q}$ exists if and only if $S_1(\omega_k) < S_0(1+r) < S_1(\omega_1)$ but it is never unique if $n = 1$ (single stock) and $k \geq 3$.
6. The model is incomplete if $n = 1$ and $k \geq 3$, that is, no replicating strategy exists for some contingent claims $X = (X(\omega_1), \ldots, X(\omega_k))$.

Theorem B

Theorem (B. Market Model with a Single Stock)

- *The market model $\mathcal{M} = (B, S)$ with $n = 1$ and $k \geq 3$ is arbitrage-free if and only if $S_1(\omega_k) < S_0(1 + r) < S_1(\omega_1)$.*
- *The martingale measure $\mathbb{Q}$ for the market model $\mathcal{M} = (B, S)$ exists if and only if $S_1(\omega_k) < S_0(1 + r) < S_1(\omega_1)$ but it is not unique. The class $\mathbb{M}$ is a convex subset of $\mathcal{P}^+$.*
- *Only some (but not all) contingent claims X can be replicated and thus the model $\mathcal{M} = (B, S)$ is incomplete.*
- *If $S_1(\omega_k) < S_0(1 + r) < S_1(\omega_1)$ then any attainable claim X admits a unique arbitrage price $\pi_0(X)$ given by a (unique) replicating strategy.*
- *If $S_1(\omega_k) < S_0(1 + r) < S_1(\omega_1)$ then the arbitrage price $\pi_0(X)$ of any attainable claim X satisfies for any martingale measure $\mathbb{Q} \in \mathbb{M}$*

$$\pi_0(X) = \mathbb{E}_{\mathbb{Q}} \left(\frac{X}{1 + r} \right).$$

How to find the class $\mathbb{M}$?

Direct method

- We consider the equality $\mathbb{E}_{\mathbb{Q}}(\Delta \widehat{S}_1) = 0$ or, equivalently, $\mathbb{E}_{\mathbb{Q}}(S_1) = S_0(1+r)$. By assumption
$$S_1(\omega_k) < S_0(1+r) < S_1(\omega_1)$$
and thus it suffices to solve the following problem:
$$\begin{aligned} &q_1 S_1(\omega_1) + q_2 S_1(\omega_2) + \cdots + q_k S_1(\omega_k) = S_0(1+r), \\ &q_1 + q_2 + \cdots + q_k = 1, \quad 0 < q_i < 1 \text{ for all } i = 1, 2, \ldots, k. \end{aligned}$$
- We take $q_3, q_4, \ldots, q_k$ as parameters and solve the equations above for q_1 and q_2.
- Then we obtain two linear functions
$$q_1 = f(q_3, q_4, \ldots, q_k), \quad q_2 = g(q_3, q_4, \ldots, q_k)$$
and we identify all values of $q_3, q_4, \ldots, q_k \in (0,1)$ for which $q_1, q_2 \in (0,1)$.
- This is easy when $k = 3$ and feasible when $k = 4$ but quite hard if $k \geq 5$.

How to find the class $\mathbb{M}$?

Alternative method

- We assume here that $S_0(1+r) \neq S_1(\omega_i)$ for every $i = 1, 2, \dots, k$.
- We consider sub-models by taking all combinations of stock prices at time 1 such that $S_1(\omega_l) < S_0(1+r) < S_1(\omega_m)$. For every pair (m, l) there exists a unique solution to the problem

$$\begin{aligned} & p_{l,m} S_1(\omega_l) + (1 - p_{l,m}) S_1(\omega_m) = S_0(1+r), \\ & 0 < p_{l,m} < 1. \end{aligned}$$

- Since this is an elementary sub-model, it is clear that

$$p_{l,m} = \frac{S_0(1+r) - S_1(\omega_l)}{S_1(\omega_m) - S_1(\omega_l)}, \quad q_{l,m} = 1 - p_{l,m}.$$

- Let $\mathbb{Q}_{l,m} \in \mathbb{R}^k$ be given by

$$\mathbb{Q}_{l,m} = (0, 0, \dots, p_{l,m}, 0, \dots, 0, 1 - p_{l,m}, 0, \dots, 0) = p_{l,m} e_l + (1 - p_{l,m}) e_m.$$

  Then the class $\mathbb{M}$ can be represented as follows:

$$\mathbb{M} = \Big\{ \mathbb{Q} = \sum_{l,m} \alpha_{l,m} \mathbb{Q}_{l,m} \;\Big|\; \sum_{l,m} \alpha_{l,m} = 1,\ \alpha_{l,m} > 0 \Big\}.$$

Example

- Suppose that $k = 3$ and $r = 0.1$. Let $S_0 = 10$ and $S_1 = (14, 12, 8)$.
- Then we need to consider two sub-models and we obtain

$$p_{1,3} = \frac{11 - 8}{14 - 8} = 1/2, \quad q_{1,3} = 1/2$$

  and

$$p_{2,3} = \frac{11 - 8}{12 - 8} = 3/4, \quad q_{2,3} = 1/4.$$

  Hence $\mathbb{Q}_{1,3} = (1/2, 0, 1/2)$ and $\mathbb{Q}_{2,3} = (0, 3/4, 1/4)$.
- The class $\mathbb{M}$ can be represented as follows:

$$\mathbb{M} = \left\{ \mathbb{Q} = \alpha \mathbb{Q}_{1,3} + (1 - \alpha)\mathbb{Q}_{2,3} \,\middle|\, \alpha \in (0, 1) \right\}.$$

- It is clear the for every $\mathbb{M} \in \mathbb{Q}$ we have that

$$\mathbb{E}_{\mathbb{Q}}(S_1) = \mathbb{E}_{\alpha\mathbb{Q}_{1,3}+(1-\alpha)\mathbb{Q}_{2,3}}(S_1) = \alpha\, \mathbb{E}_{\mathbb{Q}_{1,3}}(S_1) + (1-\alpha)\, \mathbb{E}_{\mathbb{Q}_{2,3}}(S_1) = S_0(1+r).$$

PART C

MARKET MODEL WITH SEVERAL STOCKS

The Fundamental Theorem of Asset Pricing

1. A single-period market model $\mathcal{M} = (B, S^1, S^2, \dots, S^n)$ is arbitrage-free if and only if it admits at least one martingale measure, that is, $\mathbb{M} \neq \emptyset$.
2. An arbitrage-free single-period market model $\mathcal{M}$ is complete if and only if a martingale measure $\mathbb{Q}$ is unique, that is, $\mathbb{M} = \{\mathbb{Q}\}$.
3. If a single-period model $\mathcal{M}$ is arbitrage-free, then:
   - Any attainable claim X (that is, any claim for which a replicating strategy exists) has the unique arbitrage price $\pi_0(X)$.
   - The arbitrage price $\pi_0(X)$ of any attainable claim X can be computed using the risk-neutral valuation formula with any martingale measure $\mathbb{Q} \in \mathbb{M}$.
   - If X is not attainable then we may define a price $p_0(X)$ for X, which is consistent with the no-arbitrage principle. It can be computed using the risk-neutral valuation formula, but it always depends on a choice of a martingale measure $\mathbb{Q} \in \mathbb{M}$.

How to find the class $\mathbb{M}$?

Direct method

- We now need to consider the equality $\mathbb{E}_{\mathbb{Q}}(\Delta \widehat{S}_1^j) = 0$ for every $j = 1, 2, \ldots, n$, that is, $\mathbb{E}_{\mathbb{Q}}(S_1^j) = S_0^j(1+r)$. We assume that for every j

$$\min_{l=1,2,\ldots,k} S_1^j(\omega_l) < S_0^j(1+r) < \max_{l=1,2,\ldots,k} S_1^j(\omega_l)$$

and we obtain the following problem with k unknowns and $n+1$ equations:

$$\begin{aligned}
&q_1 S_1^1(\omega_1) + q_2 S_1^1(\omega_2) + \cdots + q_k S_1^1(\omega_k) = S_0^1(1+r), \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&q_1 S_1^n(\omega_1) + q_2 S_1^n(\omega_2) + \cdots + q_k S_1^n(\omega_k) = S_0^n(1+r), \\
&q_1 + q_2 + \cdots + q_k = 1, \ 0 < q_i < 1 \text{ for all } i = 1, 2, \ldots, k.
\end{aligned}$$

- If $k > n+1$ then we take $q_{n+1}, \ldots, q_k$ as parameters and solve the equations above for $q_1, \ldots, q_n$. Then we check conditions $0 < q_i < 1$ for all $i = 1, 2, \ldots, k$.
- If $k = n+1$ and the matrix is non-singular then we obtain a complete model with a unique martingale measure $\mathbb{Q}$ provided that a unique solution $q_1, q_2, \ldots, q_k$ satisfies the inequalities $0 < q_i < 1$ for all $i = 1, 2, \ldots, k$.

How to find the class $\mathbb{M}$?

Alternative method

- We first find the class $\mathbb{M}_j$ of martingale measures for each 'partial' model $\mathcal{M}_j = (B, S^j)$ for $j = 1, 2, \dots, n$.
- Next, we set $\mathbb{M} = \mathbb{M}_1 \cap \mathbb{M}_2 \cap \dots \cap \mathbb{M}_n$ and we check if $\mathbb{M}$ is nonempty.
- Then any $\mathbb{Q} \in \mathbb{M}$ is a martingale measure for all stocks and thus it is a martingale measure for the 'full' model $\mathcal{M} = (B, S^1, S^2, \dots, S^n)$.
- The class $\mathbb{Q} \in \mathbb{M}$ is a convex subset of $\mathcal{P}^+$ in the sense that:
  - if $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathbb{M}$ then for every real number $\alpha \in [0, 1]$ the probability measure $\alpha\mathbb{Q}_1 + (1 - \alpha)\mathbb{Q}_2$ belongs to $\mathbb{M}$.
  - the convex combination $\alpha\mathbb{Q}_1 + (1 - \alpha)\mathbb{Q}_2$ should be understood as the convex combination of vectors $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathbb{R}^k$.
- Hence the problem of finding the class $\mathbb{M}$ reduces to finding the class $\mathbb{M}_i$, which can be addressed as in the case of $n = 1$ and $k \geq 3$.
- In general, it may happen that
  - (a) $\mathbb{M}$ is empty,
  - (b) $\mathbb{M}$ is a singleton, or
  - (c) several martingale measures exist for $\mathcal{M}$.

Can we find a good single-period model?

- The elementary market model postulates only two values of a stock price at time 1. It is complete and arbitrage-free when $S_1(\omega_2) < S_0(1+r) < S_1(\omega_1)$ but it does not yield realistic prices for options and other derivatives.
- It we enlarge the number of stock values at time 1, then the model is arbitrage free if $S_1(\omega_k) < S_0(1+r) < S_1(\omega_1)$ but it is never complete when $k \geq 3$ and thus we may only find unique prices for some options.
- We may also add more stocks to complete the model with k states but then the prices of, say, a call option on the stock S^1 will depend on other stocks since all of them will be used in replication. This is not realistic.
- We need a flexible model with a large number of states and several stocks and thus the only viable option is to consider dynamic multi-period models.
- Then we will be able to get reliable pricing and hedging of single-asset and multi-asset derivatives, for instance, *lookback options* or *basket options*, as well as various kinds of options of American style.

Examples of exotic options

- **Lookback options**: A lookback option is an option on a maximum or minimum of a stock price during the option's lifetime, for instance, a *call option on a maximum*:

$$X_T^1 = \Big(\max_{0 \leq t \leq T} S_t - K \Big)^+.$$

- **Basket options**: A basket option is an option on a portfolio of stocks, for instance, a *call option on the stock index*:

$$X_T^2 = \Big(\sum_{j=1}^{n} S_T^j - K \Big)^+.$$

- A large variety of other single-asset and multi-asset options of either a European or American style are traded by investment banks.