MATH3075/3975 Financial Derivatives

Tutorial 5: Solutions

Exercise 1 For any trading strategy (x, φ), the wealth at time 1 equals

$$V_1(x, \varphi) = (x - \varphi S_0)(1 + r) + \varphi S_1.$$

Hence the class of all attainable contingent claims is the two-dimensional subspace of the linear space $\mathbb{R}^3$ spanned by the vectors $(1, 1, 1)$ and $(6, 4, 3)$. This means that the considered model is incomplete since the space $\mathbb{R}^3$ of all contingent claims is three-dimensional.

We wish to find out for which values of the strike K the call option with the payoff $C_T = (S_1 - K)^+$ is an attainable claim. To this end, we need to examine four subcases:

- We first assume that $K \leq \frac{30}{9}$. Then $C_T = S_1 - K$ and thus it is an attainable claim with the unique arbitrage price at time 0 given by $C_0 = S_0 - \frac{9}{10} K$.
- Next, we assume that $\frac{30}{9} < K < \frac{40}{9}$. Then
$$C_T = (S_1 - K)^+ = \Big(\frac{60}{9} - K, \frac{40}{9} - K, 0\Big),$$
so that we now search for $\alpha, \beta \in \mathbb{R}$ satisfying
$$\begin{cases} \alpha + 6\beta = \frac{60}{9} - K, \\ \alpha + 4\beta = \frac{40}{9} - K, \\ \alpha + 3\beta = 0. \end{cases}$$
From the last two equations, we obtain $\beta = \frac{40}{9} - K$ and $\alpha = -3\beta$. Then the first equation becomes
$$3\beta = \tfrac{120}{9} - 3K = \tfrac{60}{9} - K,$$
which yields $K = \frac{30}{9}$. Hence the option is not attainable when $\frac{30}{9} < K < \frac{40}{9}$.
- We now assume that $\frac{40}{9} \leq K < \frac{60}{9}$. Then
$$C_T = (S_1 - K)^+ = \Big(\frac{60}{9} - K, 0, 0\Big)$$
and thus we search for $\alpha, \beta \in \mathbb{R}$ such that
$$\begin{cases} \alpha + 6\beta = \frac{60}{9} - K, \\ \alpha + 4\beta = 0, \\ \alpha + 3\beta = 0. \end{cases}$$
The last two equations give $\alpha = \beta = 0$ and thus the first equation is not satisfied. Hence the option is not attainable when $\frac{40}{9} \leq K < \frac{60}{9}$.
- Finally, we assume that $K \geq \frac{60}{9}$. Then $C_T = 0$ and thus it is an attainable claim with the unique arbitrage price at time 0 given by $C_0 = 0$.

We conclude that the call option is attainable only when either $K \leq \frac{30}{9}$ or $K \geq \frac{60}{9}$. However, in the former case $C_T = S_1 - K$ and thus we deal with the forward contract, and in the latter case $C_T = 0$ so that the contract is trivial. In contrast, if we take any $K \in (\frac{30}{9}, \frac{60}{9})$, then the call option cannot be replicated in our model since the claim $C_T = (S_T - K)^+$ is not attainable. This confirms that the model is incomplete, as was observed before.

Exercise 2 The model $\mathcal{M} = (B, S)$ introduced in Example 2.2.3 postulates that the stock price S_1 satisfies

$$S_1(\omega) = \begin{cases} (1+h)S_0 & \text{if } \omega = \omega_1, \\ (1+l)S_0 & \text{if } \omega = \omega_2, \\ (1-l)S_0 & \text{if } \omega = \omega_3, \\ (1-h)S_0 & \text{if } \omega = \omega_4, \end{cases}$$

where $0 < l < h < 1$. The savings account B satisfies $B_0 = 1$, $B_1 = 1 + r$ where, by assumption, $0 \le r < h$.

(a) For a trading strategy (x, φ) we have

$$V_1(x, \varphi) = (x - \varphi S_0)(1 + r) + \varphi S_1.$$

Hence the class of all attainable contingent claims is the two-dimensional subspace of $\mathbb{R}^4$ spanned by the vectors $(1, 1, 1, 1)$ and $(1+h, 1+l, 1-l, 1-h)$. It is thus clear that the model is not complete.

(b) By Definition 2.2.4 of the martingale measure, a probability measure $\mathbb{Q} = (q_1, q_2, q_3, q_4)$ belongs to $\mathbb{M}$ when $0 < q_i < 1$ for $i = 1, 2, 3, 4$ and $\mathbb{E}_{\mathbb{Q}}(\widehat{S}_1) = S_0$. More explicitly,

$$S_0 = \frac{1}{1+r} \sum_{i=1}^{4} q_i S_1(\omega_i),$$

that is,

$$(1 + r)S_0 = q_1(1 + h)S_0 + q_2(1 + l)S_0 + q_3(1 - l)S_0 + q_4(1 - h)S_0.$$

After simplifications, we obtain the following system:

$$\begin{cases} q_1 + q_2 + q_3 + q_4 = 1, \\ q_1 h + q_2 l - q_3 l - q_4 h = r, \end{cases}$$

with the constraints $0 < q_i < 1$ for $i = 1, 2, 3, 4$. By multiplying the first equation by h, we obtain

$$\begin{cases} q_1 h + q_2 h + q_3 h + q_4 h = h, \\ q_1 h + q_2 l - q_3 l - q_4 h = r. \end{cases}$$

Hence q_1 and q_4 can be expressed in terms of q_2 and q_3, specifically,

$$q_1 = \frac{h+r}{2h} - \frac{h+l}{2h} q_2 - \frac{h-l}{2h} q_3,$$

and

$$q_4 = \frac{h-r}{2h} - \frac{h-l}{2h} q_2 - \frac{h+l}{2h} q_3.$$

Let us write $q_1 = f(q_2, q_3)$ and $q_4 = g(q_2, q_3)$. We denote by D the following domain in $\mathbb{R}^2$

$$D := \left\{ (q_2, q_3) \in (0, 1)^2 \,\middle|\, 0 < f(q_2, q_3) < 1,\ 0 < g(q_2, q_3) < 1 \right\}.$$

After sketching this domain, we realise that it is non-empty. We conclude that the class of all martingale measures for $\mathcal{M}$ is a non-empty set, which can be represented as follows

$$\mathbb{M} = \left\{ (q_2, q_3) \in D \,\middle|\, \left(\tfrac{h+r}{2h}, 0, 0, \tfrac{h-r}{2h}\right) + q_2\left(-\tfrac{h+l}{2h}, 1, 0, -\tfrac{h-l}{2h}\right) + q_3\left(-\tfrac{h-l}{2h}, 0, 1, -\tfrac{h+l}{2h}\right) \right\}.$$

(c) (MATH3975) We assume that $S_0(1+l) < K < S_0(1+h)$ and thus the call option can be identified in our model with the following payoff

$$C_T = ((1+h)S_0 - K, 0, 0, 0).$$

According to Proposition 2.2.5, an arbitrage price of any contingent claim X is given by the equality

$$\pi_0(X) = \mathbb{E}_{\mathbb{Q}}\big((1+r)^{-1}X\big)$$

where $\mathbb{Q}$ is an arbitrary martingale measure for $\mathcal{M}$. Recall that we denote $q_1 = f(q_2, q_3)$. Therefore, the set of arbitrage prices at time 0 for the call option is given by

$$\Big\{ f(q_2, q_3)(1+r)^{-1}\big((1+h)S_0 - K\big),\ (q_2, q_3) \in D \Big\}$$

or, more explicitly,

$$\Big\{ \Big(\tfrac{h+r}{2h} - \tfrac{h+l}{2h}\, q_2 - \tfrac{h-l}{2h}\, q_3 \Big)(1+r)^{-1}\big((1+h)S_0 - K\big),\ (q_2, q_3) \in D \Big\}.$$

(d) (MATH3975) We now assume that $r = 0$. As before, we have that

$$C_T = ((1+h)S_0 - K, 0, 0, 0)$$

and thus we search for $\alpha, \beta \in \mathbb{R}$ such that

$$\begin{cases} \alpha + \beta(1+h) = (1+h)S_0 - K, \\ \alpha + \beta(1+l) = 0, \\ \alpha + \beta(1-l) = 0, \\ \alpha + \beta(1-h) = 0. \end{cases}$$

It is obvious that no solution (α, β) exists since $(1+h)S_0 - K > 0$ and thus the option is not attainable. Because any arbitrage price for C_T at time 0 is equal to $((1+h)S_0 - K)q_1$ for some value of q_1, it suffices to find the lower and upper bounds for q_1 when $\mathbb{Q}$ ranges over the class $\mathbb{M}$.

- The lower bound for q_1 equals 0, since for an arbitrarily small value of q_1 there exists a risk-neutral probability $\mathbb{Q} \in \mathbb{M}$.
- The upper bound for q_1 can be found by considering the situation when q_2 and q_3 are arbitrarily small. It is then easy to verify that the upper bound equals 0.5. Finally, one may check directly that there is no martingale measure $\mathbb{Q}$ such that $q_1 \geq 0.5$. Indeed, for $q_1 = 0.5$, we obtain $q_2 = q_3 = 0$ and $q_4 = 0.5$, and thus $\mathbb{Q}$ is not equivalent to $\mathbb{P}$. If $q_1 > 0.5$ then we get $q_4 < 0$ and thus $\mathbb{Q}$ is not a probability measure.

We conclude that $q_1 \in (0, 0.5)$ when $\mathbb{Q}$ ranges over the class $\mathbb{M}$ of all martingale measures. Hence the set of all possible arbitrage prices for the call option in an extended arbitrage-free market model is the open interval $(0, c)$ where $c = 0.5((1+h)S_0 - K)$.

Remark. For a slightly different approach, you may consult Example 2.2.4 from the course notes.