7: GAME CONTINGENT CLAIMS
MATH3975

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Outline

We will examine the following concepts:

1. Convertible Bonds
2. Game Contingent Claims (Israeli options)
3. Dynkin Stopping Games
4. Nash Equilibrium for a Dynkin Game
5. Valuation of Game Contingent Claims
6. Rational Exercise Times
7. Dynamic Programming Approach to Game Options

Convertible Bonds

- We start by describing a particular class of assets which have the features of a game contingent claim.
- Convertible bonds fall into the class of hybrid financial securities. They are a type of corporate bond with characteristics resembling those of straight bonds as well as equity instruments.
- Like a straight bond, the issuer of a convertible bond pays the holder a regular interest payment, or coupon, with the issuer repaying the face value of the bond on maturity.
- However, unlike a straight bond, the holder is able to sell the convertible bond back to the issuer (the so-call **put** feature) for a predetermined put price.
- Alternatively, the holder may choose to **convert** the convertible bond for a predetermined number of shares of issuer's stock.

Convertible Bonds

- Furthermore, contingent on the issuer's **call**, the holder of a convertible is able to redeem the convertible prior to maturity.
- This feature is restricted to a **call notice period** during which the holder may either convert the convertible or redeem the convertible at the end of the call notice period for a predetermined call price.
- The delivery of these payments or exercise of these features are subject to the issuer's credit risk. The issuer may not be able to completely fulfil their contractual obligations to the holder.
- The issuer may not be able to deliver its own stock, as it may be worthless, the issuer may not be able to deliver the coupons or face value as it may be required or it may not be able to buy back the convertible bond.
- However, upon default the holder may be entitled to receive a recoverable amount.

Game Contingent Claim: Definition

Definition

A **game contingent claim** $X^g = (L, H, \mathcal{T}^e, \mathcal{T}^c)$ with expiry T consists of a pair of $\mathbb{F}$-adapted payoff processes L and H, a set $\mathcal{T}^e$ of admissible **exercise times**, and a set $\mathcal{T}^c$ of admissible **cancellation times**.

- Random variable L_t is the payoff received by the holder of X^g when he exercises a game claim at time t.
- Random variable H_t is the payoff received by the holder of X^g when the issuer cancels a game claim at time t.
- We henceforth assume that $L_t \leq H_t$ for all $t = 0, 1, \dots, T$.
- Note that this is a **zero-sum game.** The amount received by the holder has to be paid by the issuer (and vice versa).
- General concept of a **game option** (aka **Israeli option**) was introduced and studied by Yuri Kifer (2000).

Game Contingent Claim: Specification

- By convention, we say that the game claim can be **exercised** by its holder at $t \leq T$ or **cancelled** by its issuer at $t < T$ (provided that it has not been exercised before or at t).
- The game claim is always **exercised** by its holder at time T.
- Hence the **payoff process** of the game contingent claim equals, for $t = 0, 1, \ldots, T$,
$$X_t = \mathbb{1}_{\{\tau = t \leq \sigma\}} L_t + \mathbb{1}_{\{\sigma = t < \tau\}} H_t.$$
- τ and σ are called the **exercise** and **cancellation** times, respectively.
- Typically, the set of admissible exercise policies τ and σ is restricted to the class $\mathcal{T}_{[0,T]}$ of all $\mathbb{F}$-**stopping times** with values in $\{0, 1, \ldots, T\}$.
- In other words, we will set $\mathcal{T}^e = \mathcal{T}^c = \mathcal{T}_{[0,T]}$.

Game Contingent Claim: Comments

- If the class of all admissible cancellation times is assumed to be given as $\mathcal{T}^c = \{T\}$ then a game claim X^g becomes an American claim X^a with the running payoff $X = L$.
- Hence the definition of a game claim covers as a special case the notion of an American claim.
- If we postulate, in addition, that $\mathcal{T}^e = \{T\}$ then a game claim reduces to a European claim $X = L_T$ maturing at time T.
- We know that the valuation and hedging of an American claim is related to the **optimal stopping problem**.
- Game contingent claims are related to **Dynkin games**, which are natural extensions of optimal stopping problems.
- Stochastic stopping games were first studied by Eugene Borisovich Dynkin in 1969.

Dynkin Game: Definition

We consider the (zero-sum) **Dynkin game** associated with the payoff

$$Z(\sigma,\tau) = \mathbb{1}_{\{\tau \le \sigma\}} L_\tau + \mathbb{1}_{\{\sigma < \tau\}} H_\sigma$$

where $L \le H$ are $\mathbb{F}$-adapted stochastic processes defined on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F}$.

Definition

For a fixed date $t = 0, 1, \dots, T$, by the **Dynkin game** started at time t we mean a stochastic game in which the **min-player**, who controls a stopping time $\sigma \in \mathcal{T}_{[t,T]}$, aims to minimize the conditional expectation

$$\mathbb{E}_{\mathbb{P}}(Z(\sigma,\tau) \,|\, \mathcal{F}_t)$$

while the goal of **max-player**, who controls a stopping time $\tau \in \mathcal{T}_{[t,T]}$, is to maximize the conditional expectation.

Solution: Nash Equilibrium

Definition

We say that the **Nash equilibrium** holds for a Dynkin game if for any t there exist stopping times $\sigma_t^*, \tau_t^* \in \mathcal{T}_{[t,T]}$ such that the inequalities

$$\mathbb{E}_{\mathbb{P}}\big(Z(\sigma_t^*, \tau) \,|\, \mathcal{F}_t\big) \leq \mathbb{E}_{\mathbb{P}}(Z(\sigma_t^*, \tau_t^*) \,|\, \mathcal{F}_t) \leq \mathbb{E}_{\mathbb{P}}\big(Z(\sigma, \tau_t^*) \,|\, \mathcal{F}_t\big)$$

are satisfied for all stopping times $\tau, \sigma \in \mathcal{T}_{[t,T]}$. In other words, the pair (σ_t^*, τ_t^*) is a **saddle point** of a Dynkin game.

- This concept was first used in 1838 by Antoine Cournot and later extended by John Forbes Nash in his 1951 paper **Non-Cooperative Games** where mixed strategies were introduced and studied.
- In the proof, Nash used the Kakutani fixed-point theorem for set-valued functions proven by Shizuo Kakutani in 1941.

Candidate for the Value Process

We first introduces a plausible candidate for the **value process** of a Dynkin game.

Definition

The process Y is defined by setting $Y_T = L_T$ and for every $t = 0, 1, \dots, T-1$

$$Y_t = \min \Big\{ H_t, \, \max \left\{ L_t, \mathbb{E}_{\mathbb{P}}(Y_{t+1} \,|\, \mathcal{F}_t) \right\} \Big\}.$$

- Since $L \leq H$, we also have that for every $t = 0, 1, \dots, T-1$

$$Y_t = \max \Big\{ L_t, \, \min \left\{ H_t, \mathbb{E}_{\mathbb{P}}(Y_{t+1} \,|\, \mathcal{F}_t) \right\} \Big\}.$$

- It is thus clear that $L_t \leq Y_t \leq H_t$ for $t = 0, 1, \dots, T$.
- In particular, if the equality $L_t = H_t$ holds then $Y_t = L_t = H_t$.

Dynkin Game: Nash Equilibrium

Proposition

Let the stopping times σ_t^*, τ_t^* *be given by*

$$\sigma_t^* = \min \left\{ u \in \{t, t+1, \dots, T\} \,|\, Y_u = H_u \right\}$$

and

$$\tau_t^* = \min \left\{ u \in \{t, t+1, \dots, T\} \,|\, Y_u = L_u \right\} \wedge T.$$

Then for arbitrary stopping times $\tau, \sigma \in \mathcal{T}_{[t,T]}$

$$\mathbb{E}_{\mathbb{P}}\big(Z(\sigma_t^*, \tau) \,|\, \mathcal{F}_t \big) \leq Y_t \leq \mathbb{E}_{\mathbb{P}}\big(Z(\sigma, \tau_t^*) \,|\, \mathcal{F}_t \big)$$

and thus also

$$\mathbb{E}_{\mathbb{P}}\big(Z(\sigma_t^*, \tau) \,|\, \mathcal{F}_t \big) \leq \mathbb{E}_{\mathbb{P}}\big(Z(\sigma_t^*, \tau_t^*) \,|\, \mathcal{F}_t \big) \leq \mathbb{E}_{\mathbb{P}}\big(Z(\sigma, \tau_t^*) \,|\, \mathcal{F}_t \big)$$

so that the Nash equilibrium holds.

Dynkin Game: Nash Equilibrium

Proposition

The process Y is the **value process** *of a Dynkin game, that is, for every $t = 0, 1, \dots, T$,*

$$Y_t = \min_{\sigma \in \mathcal{T}_{[t,T]}} \max_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}_{\mathbb{P}}\big(Z(\sigma, \tau) \,|\, \mathcal{F}_t\big) = \mathbb{E}_{\mathbb{P}}\big(Z(\sigma^*_t, \tau^*_t) \,|\, \mathcal{F}_t\big)$$

and thus the stopping times σ^*_t and τ^*_t are optimal as of time t.

- We are now going to apply this result to the case of a game option. Note that both parties can be seen as 'holders' of a game contingent claim since they can 'exercise.'
- To solve the valuation problem, we consider discounted payoffs under the unique martingale measure $\widetilde{\mathbb{P}}$ in the CRR model.
- The proof of the next proposition is omitted.

General Game Options: Arbitrage Pricing

Proposition

The arbitrage price $\pi(X^g)$ of a game contingent claim is unique and it is given by the value process of the associated Dynkin game with discounted processes $\widehat{L}$ and $\widehat{H}$ under the martingale measure $\widetilde{\mathbb{P}}$. In terms of discounted values, we have that

$$\widehat{\pi}_t(X^g) = \min \Big\{ \widehat{H}_t, \, \max \big\{ \widehat{L}_t, \, \mathbb{E}_{\widetilde{\mathbb{P}}} \big(\widehat{\pi}_{t+1}(X^g) \,|\, \mathcal{F}_t \big) \big\} \Big\}$$

with the terminal value $\widehat{\pi}_T(X^g) = \widehat{L}_T$.

- More explicitly, the price $\pi(X^g)$ is given by the recursive formula

$$\pi_t(X^g) = \min \Big\{ H_t, \, \max \big\{ L_t, \, B_t \, \mathbb{E}_{\widetilde{\mathbb{P}}} \big(B_{t+1}^{-1} \pi_{t+1}(X^g) \,|\, \mathcal{F}_t \big) \big\} \Big\}$$

with the terminal value $\pi_T(X^g) = L_T$.

General Game Options: Rational Exercise Times

- As in the case of an American option, we also need to know how to exercise (that is, to stop) the contract in a rational way.
- For a fixed $t = 0, 1, \dots, T$, we define the following stopping times taking values in $\{t, t+1, \dots, T\}$

$$\tau_t^* = \min \left\{ u \geq t \,|\, \pi_t(X^g) = L_u \right\} \wedge T$$

  and

$$\sigma_t^* = \min \left\{ u \geq t \,|\, \pi_t(X^g) = H_u \right\}.$$

- One can show that τ_t^* and σ_t^* are **rational exercise times** for the holder and for the issuer, respectively, in the game claim starting at time t.
- If the claim is sold at time 0, the **replicating strategy** should be computed up to the random time $\tau_0^* \wedge \sigma_0^*$.

Path-Independent Game Options: Summary

- Assume that the payoffs $H_t = h(S_t, t)$ and $L_t = \ell(S_t, t)$ are given in terms of the current value of the stock price at time t.
- We use the shorthand notation $X_t^g = \pi_t(X^g)$ for all t.
- Then the recursive pricing formula for the game option can be represented as follows

$$X_t^g = \min\Big\{h(S_t,t), \max\big\{\ell(S_t,t), (1+r)^{-1}\big(\widetilde{p}X_{t+1}^{gu} + (1-\widetilde{p})X_{t+1}^{gd}\big)\big\}\Big\}$$

  with the terminal condition $X_T^g = \ell(S_T, T)$.
- The holder should exercise at time t whenever $X_t^g = \ell(S_t, t)$.
- The issuer should cancel at time t whenever $X_t^g = h(S_t, t)$.
- This exercise/cancel rule is a direct extension of the rational exercise rule for the holder of an American claim.