MATH3075/3975 Financial Derivatives

Tutorial 4: Solutions

Exercise 1 We consider the elementary market model $\mathcal{M} = (B, S)$ with $S_0 > 0$ and $0 < d < 1 + r < u$.

(a) Our aim is to find the probability measure $\widehat{\mathbb{P}}$ such that $\mathbb{E}_{\widehat{\mathbb{P}}}(\widehat{B}_T) = \widehat{B}_0$ where $\widehat{B}_t = B_t/S_t$ for $t = 0, 1$. We will also compute the Radon-Nikodym density L of $\widehat{\mathbb{P}}$ with respect to $\widetilde{\mathbb{P}}$ and we will show that $\mathbb{E}_{\widetilde{\mathbb{P}}}(L) = 1$.

- We denote $\widehat{\mathbb{P}}(\omega_1) = \widehat{p}$ and $\widehat{\mathbb{P}}(\omega_2) = \widehat{q} = 1 - \widehat{p}$. The postulated equality $\mathbb{E}_{\widehat{\mathbb{P}}}(\widehat{B}_T) = \widehat{B}_0$ means that

$$\mathbb{E}_{\widehat{\mathbb{P}}}\left(\frac{B_T}{S_T}\right) = \frac{B_0}{S_0},$$

which can be expanded to the following equation for $\widehat{p}$

$$\widehat{p}\,\frac{1+r}{S^u} + \widehat{q}\,\frac{1+r}{S^d} = \frac{1}{S_0}.$$

We obtain

$$\begin{aligned}\widehat{p} &= \left(\frac{1}{S^d} - \frac{1}{(1+r)S_0}\right)\frac{S^uS^d}{S^u - S^d} = \frac{(1+r)S_0 - S^d}{(1+r)S_0S^d}\,\frac{S^uS^d}{S^u - S^d} \\ &= \frac{1+r-d}{u-d}\,\frac{u}{1+r} = \widetilde{p}\,\frac{u}{1+r}\end{aligned}$$

and

$$\begin{aligned}\widehat{q} &= \left(\frac{1}{(1+r)S_0} - \frac{1}{S^u}\right)\frac{S^uS^d}{S^u - S^d} = \frac{S^u - (1+r)S_0}{(1+r)S_0S^u}\,\frac{S^uS^d}{S^u - S^d} \\ &= \frac{u-(1+r)}{u-d}\,\frac{d}{1+r} = \widetilde{q}\,\frac{d}{1+r}\end{aligned}$$

where we denote $\widetilde{q} = 1 - \widetilde{p}$. It is easy to see that $\widehat{p} > 0, \widehat{q} > 0$ and $\widehat{p} + \widehat{q} = 1$. Hence $\widehat{\mathbb{P}} = (\widehat{\mathbb{P}}(\omega_1), \widehat{\mathbb{P}}(\omega_2)) = (\widehat{p}, \widehat{q})$ is a probability on $\Omega = (\omega_1, \omega_2)$ and it is equivalent to the risk-neutral probability measure $\widetilde{\mathbb{P}}$ (hence also equivalent to the probability measure $\mathbb{P}$).

The Radon-Nikodym density L of $\widehat{\mathbb{P}}$ with respect to $\widetilde{\mathbb{P}}$ equals

$$L(\omega_1) = \frac{d\widehat{\mathbb{P}}}{d\widetilde{\mathbb{P}}}(\omega_1) = \frac{\widehat{\mathbb{P}}(\omega_1)}{\widetilde{\mathbb{P}}(\omega_1)} = \frac{\widehat{p}}{\widetilde{p}} = \frac{u}{1+r}, \quad L(\omega_2) = \frac{d\widehat{\mathbb{P}}}{d\widetilde{\mathbb{P}}}(\omega_2) = \frac{\widehat{\mathbb{P}}(\omega_2)}{\widetilde{\mathbb{P}}(\omega_2)} = \frac{\widehat{q}}{\widetilde{q}} = \frac{d}{1+r}.$$

It is important to notice that

$$L = \frac{d\widehat{\mathbb{P}}}{d\widetilde{\mathbb{P}}} = \frac{S_TB_0}{S_0B_T} = \frac{\widehat{S}_T}{\widehat{S}_0}.$$

Furthermore,

$$\mathbb{E}_{\widetilde{\mathbb{P}}}(L) = \widetilde{p}\,\frac{u}{1+r} + (1-\widetilde{p})\,\frac{u}{1+r} = \widehat{p} + \widehat{q} = 1.$$

(b) We wish to show that the price $\pi_0(X)$ of any contingent claim $X = g(S_T)$ satisfies

$$\pi_0(X) = S_0\,\mathbb{E}_{\widehat{\mathbb{P}}}\left(\frac{X}{S_T}\right) = S_0\,\mathbb{E}_{\widehat{\mathbb{P}}}\left(\frac{g(S_T)}{S_T}\right).$$

• We already know from lectures that any contingent claim X can be replicated in the elementary market model $\mathcal{M} = (B, S)$ and its arbitrage price at time 0 can be computed using the risk-neutral valuation formula

$$\pi_0(X) = B_0\,\mathbb{E}_{\widetilde{\mathbb{P}}}\big((1+r)^{-1}X\big). \tag{1}$$

First method. To show that $\pi_0(X)$ satisfies also the equality

$$\pi_0(X) = S_0\,\mathbb{E}_{\widehat{\mathbb{P}}}\big(S_T^{-1}X\big), \tag{2}$$

we may use Radon-Nikodym density L of $\widehat{\mathbb{P}}$ with respect to $\widetilde{\mathbb{P}}$. It suffices to observe that

$$S_0\,\mathbb{E}_{\widehat{\mathbb{P}}}\big(S_T^{-1}X\big) = S_0\,\mathbb{E}_{\widetilde{\mathbb{P}}}\big(LS_T^{-1}X\big) = S_0\,\mathbb{E}_{\widetilde{\mathbb{P}}}\big(\widehat{S}_T\widehat{S}_0^{-1}S_T^{-1}X\big) = B_0\,\mathbb{E}_{\widetilde{\mathbb{P}}}\big(B_T^{-1}X\big) = \mathbb{E}_{\widetilde{\mathbb{P}}}\big((1+r)^{-1}X\big).$$

Second method. Alternatively, we consider any portfolio $\varphi = (\varphi^0, \varphi^1)$ where $\varphi^0 = x - \varphi^1 S_0 = V_0(\varphi) - \varphi^1 S_0$ and

$$V_T(\varphi) = \varphi^0(1+r) + \varphi^1 S_T = \varphi^0 B_T + S_0^{-1}(V_0(\varphi) - \varphi^0)S_T.$$

Then (recall that $B_0 = 1$)

$$\frac{V_T(\varphi)}{S_T} = \frac{V_0(\varphi)}{S_0} + \varphi^0\left(\frac{B_T}{S_T} - \frac{B_0}{S_0}\right),$$

so that, using the definition of the probability $\widehat{\mathbb{P}}$, we obtain

$$\mathbb{E}_{\widehat{\mathbb{P}}}\big(S_T^{-1}V_T(\varphi)\big) = \frac{V_0(\varphi)}{S_0} + \mathbb{E}_{\widehat{\mathbb{P}}}\left[\varphi^0\left(\frac{B_T}{S_T} - \frac{B_0}{S_0}\right)\right] = \frac{V_0(\varphi)}{S_0} + \varphi^0\,\mathbb{E}_{\widehat{\mathbb{P}}}(\widehat{B}_T - \widehat{B_0}) = \frac{V_0(\varphi)}{S_0}$$

since $\mathbb{E}_{\widehat{\mathbb{P}}}(\widehat{B}_T - \widehat{B_0}) = 0$. Hence if a portfolio φ replicates X so that $V_T(\varphi) = X$, then

$$\mathbb{E}_{\widehat{\mathbb{P}}}\big(S_T^{-1}X\big) = \frac{V_0(\varphi)}{S_0} = \frac{\pi_0(X)}{S_0}.$$

(c) We consider the put option with the payoff $P_T(K) = (K - S_T)^+$ for some $K > 0$. We will show that the arbitrage price $P_0(K)$ admits the following representation

$$P_0(K) = K(1+r)^{-1}\,\widetilde{\mathbb{P}}(S_T < K) - S_0\,\widehat{\mathbb{P}}(S_T < K).$$

• We denote $A = \{S_T < K\}$ so that

$$P_T(K) = (K - S_T)^+ = (K - S_T)\mathbb{1}_A = \mathbb{1}_A K - \mathbb{1}_A S_T = X_1 - X_2.$$

where, by definition, $\mathbb{1}_A = 1$ on the event A and it equals 0 on the complement of A. Therefore, by applying (1) to X_1 and (2) to X_2, we obtain

$$\begin{aligned}
P_0(K) &= \pi_0\big(X_1\big) - \pi_0\big(X_2\big) = \pi_0\big(\mathbb{1}_A K\big) - \pi_0\big(\mathbb{1}_A S_T\big) \\
&= B_0\,\mathbb{E}_{\widetilde{\mathbb{P}}}\big((1+r)^{-1}\mathbb{1}_A K\big) - S_0\,\mathbb{E}_{\widehat{\mathbb{P}}}\big(S_T^{-1}\mathbb{1}_A S_T\big) \\
&= K(1+r)^{-1}\,\widetilde{\mathbb{P}}(S_T < K) - S_0\,\widehat{\mathbb{P}}(S_T < K).
\end{aligned}$$

Let $C_T(K) = (S_T - K)^+$ for some $K > 0$. It is easy to show that the price $C_0(K)$ satisfies

$$C_0(K) = S_0\, \widehat{\mathbb{P}}(S_T > K) - K(1+r)^{-1}\, \widetilde{\mathbb{P}}(S_T > K).$$

(d) Our goal is to show that the extended model $\mathcal{M}^e = (B, S, P(K))$ is arbitrage-free, in the sense of Definition 2.2.3 from the course notes.

- For any trading strategy $(x, \varphi^1, \varphi^2) \in \mathbb{R}^3$, the wealth satisfies $V_0(x, \varphi^1, \varphi^2) = x$ and

$$V_1(x, \varphi^1, \varphi^2) = \big(x - \varphi^1 S_0 - \varphi^2 P_0(K)\big)(1+r) + \varphi^1 S_1 + \varphi^2 P_1(K).$$

If $x = 0$, then for every $(\varphi^1, \varphi^2) \in \mathbb{R}^2$

$$\begin{aligned} V_1(0, \varphi^1, \varphi^2) &= \varphi^1\big(S_T - S_0(1+r)\big) + \varphi^2\big(P_T(K) - P_0(K)(1+r)\big) \\ &= \varphi^1\big(S_T - S_0(1+r)\big) + \varphi^2\big(X - (1+r)\pi_0(X)\big) \end{aligned}$$

where we denote $X = P_T(K)$ and $\pi_0(X) = P_0(K)$. Therefore,

$$\mathbb{E}_{\widetilde{\mathbb{P}}}(V_T(0, \varphi^1, \varphi^2)) = \varphi^1\, \mathbb{E}_{\widetilde{\mathbb{P}}}\big(S_T - S_0(1+r)\big) + \varphi^2\, \mathbb{E}_{\widetilde{\mathbb{P}}}\big(X - (1+r)\pi_0(X)\big) = 0.$$

If the wealth $V_T(0, \varphi^1, \varphi^2)$ is non-negative and has the expected value equal to zero, then necessarily $V_T(0, \varphi^1, \varphi^2)(\omega_i) = 0$ for $i = 1, 2$. We conclude that arbitrage opportunities do not exist in the extended model $\mathcal{M}^e = (B, S, P(K))$.

(e) We take a fixed K such that $S_0 d < K < S_0 u$ and we consider the modified market model $\mathcal{N} = (B, P(K))$. We will show that the price of an arbitrary claim X computed in $\mathcal{N} = (B, P(K))$ coincides with its arbitrage price computed in $\mathcal{M} = (B, S)$. We will also find the arbitrage price at time 0 for the claim $X = S_T$.

- It suffices to observe that the probability measure $\mathbb{Q} = \widetilde{\mathbb{P}}$ is the unique martingale measure for the model $\mathcal{N} = (B, P(K))$ since

$$\mathbb{E}_{\widetilde{\mathbb{P}}}(P_1(K)) = (1+r)P_0(K).$$

Hence the arbitrage price of any contingent claim X computed in $\mathcal{N} = (B, P(K))$ and $\mathcal{M} = (B, S)$ are identical. In particular, the arbitrage price of the claim $X = S_T$ in the model $\mathcal{N}$ can be computed from the risk-neutral valuation

$$\pi_0(X) = B_0\, \mathbb{E}_{\mathbb{Q}}\big(B_T^{-1} X\big) = B_0\, \mathbb{E}_{\widetilde{\mathbb{P}}}\big(B_T^{-1} X\big) = B_0\, \mathbb{E}_{\widetilde{\mathbb{P}}}\big(B_T^{-1} S_T\big) = S_0$$

where the last equality is a consequence of the definition of $\widetilde{\mathbb{P}}$. One may also show directly that the claim $X = S_T$ can be replicated by a portfolio composed of B and $P(K)$.

Exercise 2 Consider a trading strategy $(x, \varphi) = (x, \varphi^1, \dots, \varphi^n)$. Its wealth satisfies $V_0(x, \varphi) = x$ and

$$V_1(x, \varphi) = \left(x - \sum_{j=1}^n \varphi^j S_0^j\right)(1+r) + \sum_{j=1}^n \varphi^j S_1^j$$

and thus

$$\begin{aligned} \widehat{V}_1(x, \varphi) := \frac{V_1(x, \varphi)}{B_1} = \frac{V_1(x, \varphi)}{1+r} &= \left(x - \sum_{j=1}^n \varphi^j S_0^j\right) + \sum_{j=1}^n \varphi^j (1+r)^{-1} S_1^j \\ &= \left(x - \sum_{j=1}^n \varphi^j S_0^j\right) + \sum_{j=1}^n \varphi^j \widehat{S}_1^j \end{aligned}$$

where $\widehat{S}_1^j := (1+r)^{-1} S_1^j$.

Consequently,

$$\begin{aligned}
\widehat{G}_1(x,\varphi) &:= \widehat{V}_1(x,\varphi) - \widehat{V}_0(x,\varphi) = \widehat{V}_1(x,\varphi) - V_0(x,\varphi) = \widehat{V}_1(x,\varphi) - x \\
&= \left(x - \sum_{j=1}^{n} \varphi^j S_0^j\right) + \sum_{j=1}^{n} \varphi^j \widehat{S}_1^j - x = \sum_{j=1}^{n} \varphi^j (\widehat{S}_1^j - S_0^j) \\
&= \sum_{j=1}^{n} \varphi^j (\widehat{S}_1^j - \widehat{S}_0^j) = \sum_{j=1}^{n} \varphi^j \Delta \widehat{S}_1^j.
\end{aligned}$$

Exercise 3 In view of Definition 2.2.4, we need to find all probability measures $\mathbb{Q}$ on the space $\Omega = \{\omega_1, \omega_2, \omega_3\}$ such that $\mathbb{Q}$ is equivalent to $\mathbb{P}$ (that is, $\mathbb{Q}(\omega_i) > 0$ for $i = 1, 2, 3$) and

$$\mathbb{E}_{\mathbb{Q}}(\Delta \widehat{S}_1) = 0$$

where

$$\Delta \widehat{S}_1 = \widehat{S}_1 - \widehat{S}_0 = \tfrac{9}{10}\left(\tfrac{60}{9}, \tfrac{40}{9}, \tfrac{30}{9}\right) - (5, 5, 5) = (1, -1, -2).$$

Let us denote $\mathbb{Q} = (q_1, q_2, q_2)$. Then we search for all solutions (q_1, q_2, q_2) to the system

$$\begin{cases} 0 < q_i < 1 \text{ for } i = 1, 2, 3, \\ q_1 - q_2 - 2q_3 = 0, \\ q_1 + q_2 + q_3 = 1. \end{cases}$$

We obtain

$$\begin{cases} q_1 = \frac{1}{2} q_3 + \frac{1}{2}, \\ q_2 = -\frac{3}{2} q_3 + \frac{1}{2}, \end{cases}$$

and we check that the condition $q_i \in (0, 1)$ is satisfied for every $i = 1, 2, 3$ whenever $q_3 \in (0, 1/3)$ since it is clear that q_3 should satisfy the following inequalities

$$\begin{cases} 0 < \frac{1}{2} q_3 + \frac{1}{2} < 1, \\ 0 < -\frac{3}{2} q_3 + \frac{1}{2} < 1, \end{cases}$$

and this holds whenever $q_3 \in (0, 1/3)$. Hence the set of all risk-neutral probability measures for $\mathcal{M}$ can be represented as follows

$$\mathbb{M} = \left\{ \mathbb{Q} = \left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right) + q_3 \left(\tfrac{1}{2}, -\tfrac{3}{2}, 1\right),\ q_3 \in \left(0, \tfrac{1}{3}\right) \right\}.$$

Exercise 4 (a) For any trading strategy $(x, \varphi) = (x, \varphi^1, \varphi^2)$, we have

$$V_1(x,\varphi) = \left(x - \sum_{j=1}^{2} \varphi^j S_0^j\right)(1 + r) + \sum_{j=1}^{2} \varphi^j S_1^j$$

or, more explicitly,

$$V_1(x,\varphi)(\omega_i) = \begin{cases} \frac{10}{9} x + \frac{10}{9} \varphi^1 + \frac{20}{9} \varphi^2, & i = 1, \\ \frac{10}{9} x + \frac{10}{9} \varphi^1 - \frac{20}{9} \varphi^2, & i = 2, \\ \frac{10}{9} x - \frac{10}{9} \varphi^1 - \frac{20}{9} \varphi^2, & i = 3, \\ \frac{10}{9} x - \frac{30}{9} \varphi^1 + \frac{20}{9} \varphi^2, & i = 4. \end{cases}$$

Since

$$G_1(x,\varphi) := V_1(x,\varphi) - V_0(x,\varphi) = V_1(x,\varphi) - x,$$

we obtain

$$G_1(x,\varphi)(\omega_i) = \begin{cases} \frac{1}{9}\,x + \frac{10}{9}\,\varphi^1 + \frac{20}{9}\,\varphi^2, & i=1, \\ \frac{1}{9}\,x + \frac{10}{9}\,\varphi^1 - \frac{20}{9}\,\varphi^2, & i=2, \\ \frac{1}{9}\,x - \frac{10}{9}\,\varphi^1 - \frac{20}{9}\,\varphi^2, & i=3, \\ \frac{1}{9}\,x - \frac{30}{9}\,\varphi^1 + \frac{20}{9}\,\varphi^2, & i=4. \end{cases}$$

Next

$$\widehat{V}_1(x,\varphi) := (1+r)^{-1}\,V_1(x,\varphi) = \tfrac{9}{10}\,V_1(x,\varphi)$$

so that

$$\widehat{V}_1(x,\varphi)(\omega_i) = \begin{cases} x + \varphi^1 + 2\varphi^2, & i=1, \\ x + \varphi^1 - 2\varphi^2, & i=2, \\ x - \varphi^1 - 2\varphi^2, & i=3, \\ x - 3\varphi^1 + 2\varphi^2, & i=4. \end{cases}$$

Finally,

$$\widehat{G}_1(x,\varphi) := \widehat{V}_1(x,\varphi) - \widehat{V}_0(x,\varphi) = \widehat{V}_1(x,\varphi) - x$$

and thus

$$\widehat{G}_1(x,\varphi)(\omega_i) = \begin{cases} \varphi^1 + 2\varphi^2, & i=1, \\ \varphi^1 - 2\varphi^2, & i=2, \\ -\varphi^1 - 2\varphi^2, & i=3, \\ -3\varphi^1 + 2\varphi^2, & i=4. \end{cases}$$

We observe that

$$\widehat{G}_1(x,\varphi) = \varphi^1 \Delta \widehat{S}_1^1 + \varphi^2 \Delta \widehat{S}_1^2 = \varphi^1(1,1,-1,-3) + \varphi^2(2,-2,-2,2).$$

(b) It is clear that $G_1(x,\varphi)$ depends on the initial wealth x, but $\widehat{G}_1(x,\varphi)$ does not, so that for any $x,y \in \mathbb{R}$ and arbitrary $\varphi \in \mathbb{R}^2$ we have $\widehat{G}_1(x,\varphi) = \widehat{G}_1(y,\varphi)$. In particular, $\widehat{G}_1(x,\varphi) = \widehat{G}_1(0,\varphi)$ for every $x \in \mathbb{R}$ and $\varphi \in \mathbb{R}^2$.

Exercise 5 (MATH3975) (a) We have $k=4$ and $n=2$. Recall that

$$\mathbb{W} := \{X \in \mathbb{R}^4 \,|\, X = \widehat{G}_1(x,\varphi) \text{ for some } (x,\varphi) \in \mathbb{R}^3\}$$

where in fact $\widehat{G}_1(x,\varphi)$ does not depend on x. More explicitly, $X \in \mathbb{W}$ if and only if $X \in \mathbb{R}^4$ and $X \in \widehat{G}_1(x,\varphi)$ that is (from the previous exercise)

$$X = \begin{cases} \varphi^1 + 2\varphi^2, & i=1, \\ \varphi^1 - 2\varphi^2, & i=2, \\ -\varphi^1 - 2\varphi^2, & i=3, \\ -3\varphi^1 + 2\varphi^2, & i=4, \end{cases}$$

for some real numbers φ^1 and φ^2. This means that $\mathbb{W}$ is a plane given by

$$\mathbb{W} = \{X \in \mathbb{R}^4 \,|\, X = \varphi^1(1,1,-1,-3) + \varphi^2(2,-2,-2,2),\ \ \varphi^1,\varphi^2 \in \mathbb{R}\}$$

or, more precisely, the two-dimensional linear subspace of $\mathbb{R}^4$ spanned by the vectors $(1,1,-1,-3)$ and $(2,-2,-2,2)$.

(b) Recall that the space $\mathbb{W}^\perp$ is the orthogonal complement of $\mathbb{W}$, that is,

$$\mathbb{W}^\perp := \{Z \in \mathbb{R}^4 \,|\, \langle X, Z\rangle = 0 \text{ for all } X \in \mathbb{W}\}.$$

Hence a vector $Z \in \mathbb{R}^4$ belongs to $\mathbb{W}^\perp$ whenever it satisfies

$$\begin{cases} z_1 + z_2 - z_3 - 3z_4 = 0, \\ z_1 - z_2 - z_3 + z_4 = 0. \end{cases}$$

Therefore, $z_1 = z_3 + z_4$ and $z_2 = 2z_4$ and thus

$$\mathbb{W}^\perp = \{Z \in \mathbb{R}^4 \,|\, Z = z_3(1,0,1,0) + z_4(1,2,0,1),\ z_3, z_4 \in \mathbb{R}\}$$

which is the two-dimensional linear subspace of $\mathbb{R}^4$ spanned by the vectors $(1,0,1,0)$ and $(1,2,0,1)$. We conclude that $\mathbb{W}^\perp$ is the plane orthogonal to the plane $\mathbb{W}$.

(c) We will show that the model $\mathcal{M} = (B, S^1, S^2)$ is arbitrage-free using two methods.

- *First method.* In view of Remark 2.2.3, to check whether the model is arbitrage-free, it suffices to show that $\mathbb{W} \cap \mathbb{A} = \emptyset$ where

$$\mathbb{A} = \{X \in \mathbb{R}^4 \,|\, X \neq 0,\ x_i \geq 0,\ i = 1, \dots, 4\}.$$

  Since for all $X \in \mathbb{W}$ we have $x_3 = -x_1$ and for all $X \in \mathbb{A}$ we have $x_i \geq 0$, it is clear that if $X \in \mathbb{W} \cap \mathbb{A}$ then $x_1 = x_3 = 0$. This in turn implies that $\varphi^1 = -2\varphi^2$ and thus $x_2 = -4\varphi^2$ and $x_4 = 8\varphi^2$. Condition $x_i \geq 0$ implies that $\varphi^2 = 0$ and thus also $x_1 = x_2 = x_3 = x_4 = 0$. We conclude that $\mathbb{W} \cap \mathbb{A} = \emptyset$ and thus the market model is arbitrage-free.

- *Second method.* In view of Remark 2.2.4, in order to confirm that the model is arbitrage-free, one may check that $\mathbb{W}^\perp \cap \mathcal{P}^+ \neq \emptyset$ where the set $\mathcal{P}^+$ is given by

$$\mathcal{P}^+ := \Big\{\mathbb{Q} \in \mathbb{R}^4 \,\Big|\, \sum_{i=1}^4 q_i = 1,\ q_i > 0\Big\}.$$

  Assume that $Z \in \mathbb{W}^\perp \cap \mathcal{P}^+$. We observe that the condition $q_i > 0$ for $i = 1, \dots, 4$ implies that $z_3 > 0$ and $z_4 > 0$. Then $z_1 = z_3 + z_4 > 0$ and $z_2 = 2z_4 > 0$ and thus it suffices to impose the condition that Z is a probability measure so that

$$\sum_{i=1}^4 z_i = 1 \ \Rightarrow\ 2z_3 + 4z_4 = 1.$$

  It is now clear that $\mathbb{W}^\perp \cap \mathcal{P}^+ \neq \emptyset$ and thus the market model is arbitrage-free.

  For instance, we may take $z_3 = 1/4$ and $z_4 = 1/8$. Then we obtain the following vector

$$Z = \tfrac{1}{4}(1,0,1,0) + \tfrac{1}{8}(1,2,0,1) = \big(\tfrac{3}{8}, \tfrac{2}{8}, \tfrac{2}{8}, \tfrac{1}{8}\big) \in \mathbb{W}^\perp \cap \mathcal{P}^+.$$

  It is easy to check that for $\mathbb{Q} = Z$ we have that $\mathbb{E}_{\mathbb{Q}}\big(\Delta \widehat{S}_1^i\big) = 0$ for $i = 1, 2$ since $\Delta \widehat{S}_1^1 = (1, 1, -1, -3)$ and $\Delta \widehat{S}_1^2 = (2, -2, -2, 2)$.

(d) From Lemma 2.2.1, we know that $\mathbb{M} = \mathbb{W}^\perp \cap \mathcal{P}^+$. The class $\mathbb{M}$ of all risk-neutral probabilities for this model is thus non-empty and non-uniqueness of a risk-neutral probability holds. We may represent $\mathbb{M}$ as follows

$$\mathbb{M} = \Big\{\mathbb{Q} \in \mathbb{R}^4 \,\Big|\, \mathbb{Q} = q_3(1,0,1,0) + q_4(1,2,0,1),\ q_3 > 0,\ q_4 > 0,\ 2q_3 + 4q_4 = 1\Big\}$$

or, equivalently,

$$\mathbb{M} = \Big\{\mathbb{Q} \in \mathbb{R}^4 \,\Big|\, \mathbb{Q} = q_3(1,0,1,0) + \tfrac{1-2q_3}{4}(1,2,0,1),\ q_3 \in (0, \tfrac{1}{2})\Big\}.$$

Exercise 6 (MATH3975) The proof of Proposition 2.2.1 is straightforward. It suffices to observe that $V_1(x,\varphi)$ and $\widehat{V}_1(x,\varphi)$ have the same properties, since $\widehat{V}_1(x,\varphi) = cV_1(x,\varphi)$ for a strictly positive constant c (specifically, $c = (1+r)^{-1}$). This argument shows that the first statement in Proposition 2.2.1 is valid.

Furthermore, we observe that in Definition 2.2.3 we only consider trading strategies with $x = 0$. Therefore, we may use the equality $\widehat{V}_1(0,\varphi) = \widehat{G}_1(0,\varphi)$ and thus the second part in Proposition 2.2.1 is true as well.