MATH3075/3975 Financial Mathematics

Tutorial 9: Solutions

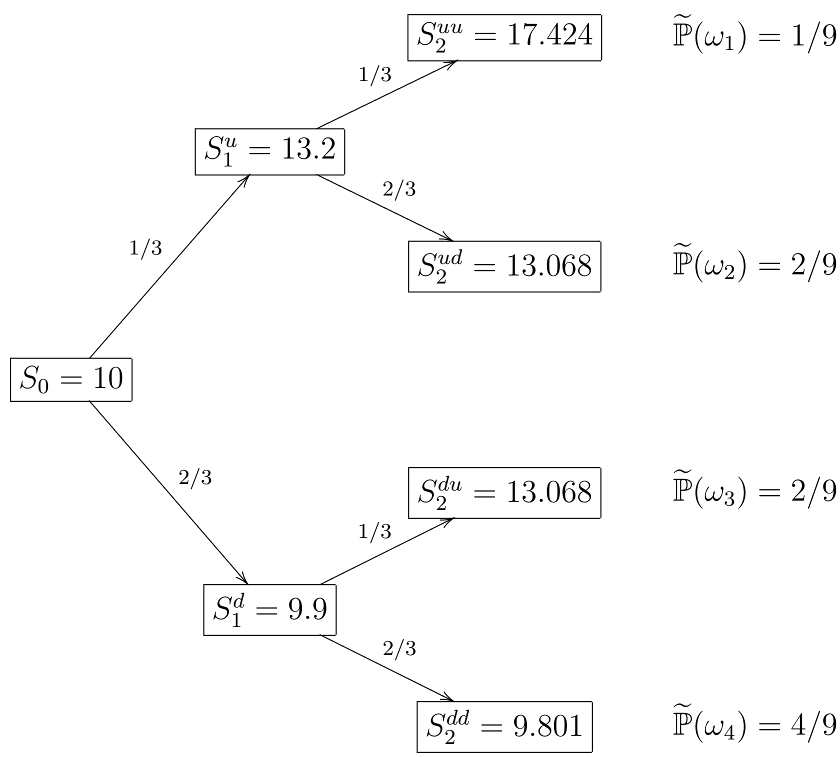
Exercise 1 Consider the CRR model $\mathcal{M} = (B, S)$ with the horizon date $T = 2$, the risk-free rate $r = 0.1$, and the following values of the stock price S at times $t = 0$ and $t = 1$:

$$S_0 = 10, \quad S_1^u = 13.2, \quad S_1^d = 9.9.$$

Let X be a European contingent claim with the maturity date $T = 2$ and the payoff

$$X = \big(\min(S_1, S_2) - 10\big)^+.$$

(a) The stock price and the martingale measure are given by the following diagram



(b) To show that the claim X is a path-dependent, it suffices to observe that

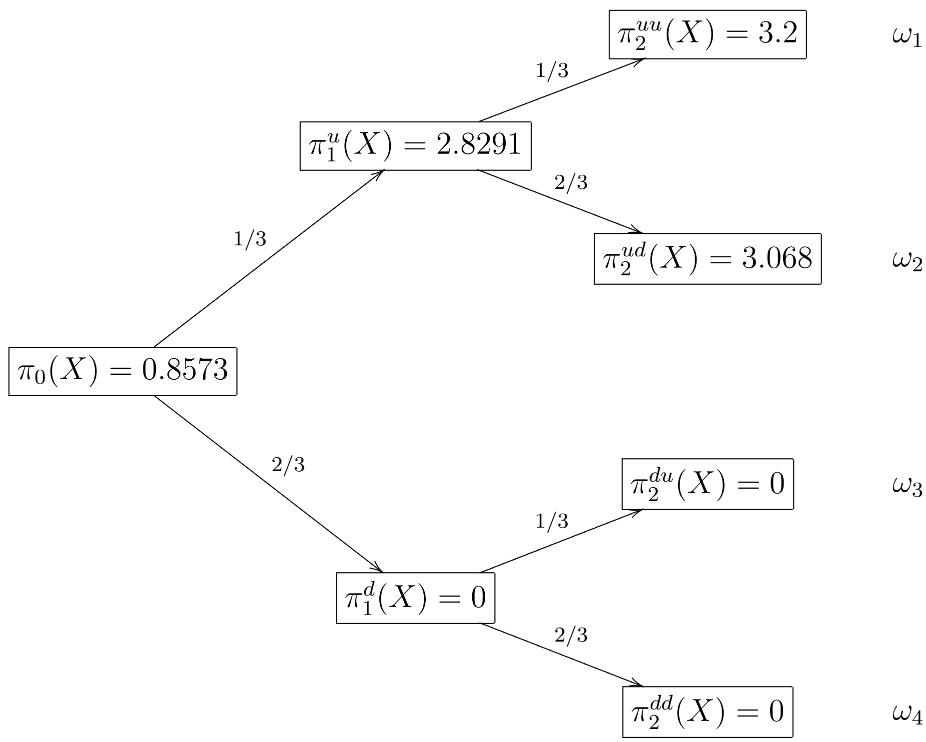
$$X = (X(\omega_1), X(\omega_2), X(\omega_3), X(\omega_4)) = (3.2, 3.068, 0, 0).$$

Although we have the same value of the stock price $S_2(\omega_2) = 13.068 = S_2(\omega_3)$, we have different values of the payoff, depending on the level of the stock price at time 1 (note that $S_1(\omega_2) = 13.2 \neq 9.9 = S_1(\omega_3)$). Hence there is no function $f : \mathbb{R} \to \mathbb{R}$ such that $X = f(S_T)$ and thus X is a path-dependent claim (a special case of the *lookback option*).

(c) Our next goal is to compute the arbitrage price of X using the risk-neutral valuation formula, for $t = 0, 1, 2$,

$$\pi_t(X) = B_t \, \mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{X}{B_T} \,\middle|\, \mathcal{F}_t \right).$$

The price process $\pi_t(X)$ of the European claim X is given by



(d) We start by examining the replicating strategy $(\varphi_1^0, \varphi_1^1)$ at time $t = 1$.

- Assume first that the stock price has risen during the first period. Then we need to solve

$$\begin{aligned} 1.1\widetilde{\varphi}_1^0 + 17.424\varphi_1^1 &= 3.2, \\ 1.1\widetilde{\varphi}_1^0 + 13.068\varphi_1^1 &= 3.068. \end{aligned}$$

  Hence $(\widetilde{\varphi}_1^0, \varphi_1^1) = (2.4291, 0.0303)$ if the stock price has risen during the first period, that is, for $\omega \in \{\omega_1, \omega_2\}$. We check that

$$V_1^u(\varphi) = \widetilde{\varphi}_1^0 + \varphi_1^1 S_1^u = 2.4291 + 0.0303 \cdot 13.2 = 2.8291 = \pi_1^u(X).$$

- Let us now assume that the stock price has fallen during the first period. Then we need to solve

$$\begin{aligned} 1.1\widetilde{\varphi}_1^0 + 13.068\varphi_1^1 &= 0, \\ 1.1\widetilde{\varphi}_1^0 + 9.801\varphi_1^1 &= 0. \end{aligned}$$

  Hence $(\widetilde{\varphi}_1^0, \varphi_1^1) = (0, 0)$ if the stock price has fallen during the first period, that is, for $\omega \in \{\omega_3, \omega_4\}$. Obviously, $V_1^d(\varphi) = 0 = \pi_1^d(X)$.

Let us find the replicating portfolio at time $t = 0$. We need to solve

$$\begin{aligned} 1.1\varphi_0^0 + 13.2\varphi_0^1 &= 2.8291, \\ 1.1\varphi_0^0 + 9.9\varphi_0^1 &= 0. \end{aligned}$$

Hence $(\varphi_0^0, \varphi_0^1) = (-7.7157,\ 0.8573)$ for all ωs. We check that

$$V_0(\varphi) = \varphi_0^0 + \varphi_0^1 S_0 = -7.7157 + 0.8573 \cdot 10 = 0.8573 = \pi_0(X).$$

The dynamics of the portfolio are represented by the following diagram

$$\begin{tikzcd}
 & & \boxed{V_2^{uu}(\varphi) = 3.2} \\
 & \boxed{(\widetilde{\varphi}_1^0, \varphi_1^1) = (2.4291, 0.0303)} \arrow[ur] \arrow[dr] & \\
 & & \boxed{V_2^{ud}(\varphi) = 3.068} \\
\boxed{(\varphi_0^0, \varphi_0^1) = (-7.7157,\ 0.8573)} \arrow[uur] \arrow[ddr] & & \\
 & & \boxed{V_2^{du}(\varphi) = 0} \\
 & \boxed{(\widetilde{\varphi}_1^0, \varphi_1^1) = (0, 0)} \arrow[ur] \arrow[dr] & \\
 & & \boxed{V_2^{dd}(\varphi) = 0}
\end{tikzcd}$$

Exercise 2 We apply the CRR call option pricing formula

$$C_0 = S_0 \sum_{k=\widehat{k}}^{T} \binom{T}{k} \widehat{p}^k (1 - \widehat{p})^{T-k} - \frac{K}{(1+r)^T} \sum_{k=\widehat{k}}^{T} \binom{T}{k} \widetilde{p}^k (1 - \widetilde{p})^{T-k}$$

where $\widehat{k}$ is the smallest integer k such that

$$k > \frac{\ln\left(\frac{K}{S_0 d^T}\right)}{\ln\left(\frac{u}{d}\right)} =: \alpha_0.$$

We consider the European call option with strike price $K = 10$ and maturity date $T = 5$ years. We assume that the initial stock price $S_0 = 9$, the risk-free simple interest rate is $r = 0.01$ and the stock price volatility equals $\sigma = 0.1$ per annum. The CRR parametrization for u and d with $\Delta t = 1$ gives

$$u = e^{\sigma\sqrt{\Delta t}} = e^{\sigma} = 1.105171, \quad d = \frac{1}{u} = 0.904837.$$

Consequently,

$$\widetilde{p} = \frac{1+r-d}{u-d} = 0.524938, \quad \widehat{p} = \frac{\widetilde{p}u}{1+r} = 0.574402.$$

(a) We first compute the call price at time 0. It is easy to check that $\alpha_0 = 3.0268...$ and thus $\widehat{k} = \widehat{k}(S_0, T) = 4$ and thus

$$C_0 = 9\sum_{k=4}^{5}\binom{5}{k}\widehat{p}^k(1-\widehat{p})^{5-k} - \frac{10}{(1.01)^5}\sum_{k=4}^{5}\binom{5}{k}\widetilde{p}^k(1-\widetilde{p})^{5-k} = 0.552247.$$

Hence the price of the option at time 0 equals $C_0 = 0.552247$.

(b) We will now compute the prices of the option at time $t = 1$. Notice that the time to maturity is now equal to $T - 1 = 4$.

- If $S_1 = uS_0 = 9.9465$ then we obtain the new value of $\widehat{k}(uS_0, T-1) = 3$ since $\alpha_1^u = 2.0268...$ and thus in that case

$$C_1^u = 9.9465\sum_{k=3}^{4}\binom{4}{k}\widehat{p}^k(1-\widehat{p})^{4-k} - \frac{10}{(1.01)^4}\sum_{k=3}^{4}\binom{4}{k}\widetilde{p}^k(1-\widetilde{p})^{4-k} = 0.920649.$$

- If $S_1 = dS_0 = 8.1435$ then we obtain the value of $\widehat{k}(dS_0, T-1) = 4$ since $\alpha_1^d = 2.026...$ and thus

$$\begin{aligned} C_1^d &= 8.1435\sum_{k=4}^{4}\binom{4}{k}\widehat{p}^k(1-\widehat{p})^{4-k} - \frac{10}{(1.01)^4}\sum_{k=4}^{4}\binom{4}{k}\widetilde{p}^k(1-\widetilde{p})^{4-k} \\ &= 8.1435\,(0.574402)^4 - \frac{10}{(1.01)^4}\,(0.524938)^4 = 0.156793. \end{aligned}$$

In Week 10, the price of the same option at any time t (in particular, for $t = 1$) will be computed using the backward induction method.

(c) The hedge ratio at time 0 equals

$$\frac{C_1^u - C_1^d}{S_1^u - S_1^d} = \frac{0.920649 - 0.156793}{9.9465 - 8.1435} = 0.423659 = \varphi_0^1$$

and thus φ_0^0 satisfies

$$C_0 = 0.552247 = \varphi_0^0 + \varphi_0^1 S_0 = \varphi_0^0 + 0.423659 \times 9.$$

Hence $\varphi_0^0 = -3.260657$.

Exercise 3 **(MATH3975)** (a) It is clear that $L_0 = 1$ and L is strictly positive. The martingale property of L under under $\widetilde{\mathbb{Q}}$ follows from the definition of $\widetilde{\mathbb{Q}}$ since $L = cS/B$ where c is a constant and, by assumption, the process S/B is a martingale respect to the filtration $\mathbb{F}$ under the probability measure $\widetilde{\mathbb{Q}}$

(b) Let us denote $M = B/S$. It suffices to observe that the product $LM = L(B/S) = B_0/S_0$ is constant and thus the process LM is clearly is a martingale with respect to the filtration $\mathbb{F}$ under the probability measure $\widehat{\mathbb{Q}}$. From part (c) in Exercise 4 in Week 7, we deduce that the process $M = B/S$ is a martingale with respect to the filtration $\mathbb{F}$ under $\widetilde{\mathbb{Q}}$.

(c) Once again, from part (b) in Exercise 4 in Week 7, we know that the equality

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}(Y \,|\, \mathcal{F}_t) = (L_t)^{-1}\, \mathbb{E}_{\widehat{\mathbb{Q}}}(Y L_s \,|\, \mathcal{F}_t). \tag{1}$$

holds for any $\mathcal{F}_s$-measurable random variable Y. Recall that the Radon-Nikodým density L satisfies

$$L_t = \frac{B_0}{S_0}\frac{S_t}{B_t}, \quad L_s = \frac{B_0}{S_0}\frac{S_s}{B_s}.$$

Since S_s is $\mathcal{F}_s$-measurable, if X is any $\mathcal{F}_s$-measurable random variable X, then the random variable X/S_s is $\mathcal{F}_s$-measurable as well.

Therefore, by applying (1) to the random variable $Y = X/S_s$, we obtain for

$$\mathbb{E}_{\widehat{\mathbb{Q}}}\left(\frac{X}{S_s} \,\middle|\, \mathcal{F}_t\right) = (L_t)^{-1}\mathbb{E}_{\widetilde{\mathbb{Q}}}\left(\frac{L_s X}{S_s} \,\middle|\, \mathcal{F}_t\right)$$

or, more explicitly,

$$\mathbb{E}_{\widehat{\mathbb{Q}}}\left(\frac{X}{S_s} \,\middle|\, \mathcal{F}_t\right) = \frac{S_0}{B_0}\frac{B_t}{S_t}\, \mathbb{E}_{\widetilde{\mathbb{Q}}}\left(\frac{B_0}{S_0}\frac{S_s}{B_s}\frac{X}{S_s} \,\middle|\, \mathcal{F}_t\right).$$

This in turn yields

$$S_t\, \mathbb{E}_{\widehat{\mathbb{Q}}}\left(\frac{X}{S_s} \,\middle|\, \mathcal{F}_t\right) = B_t\, \mathbb{E}_{\widetilde{\mathbb{Q}}}\left(\frac{X}{B_s} \,\middle|\, \mathcal{F}_t\right). \tag{2}$$

as was required to show.

(d) The payoff C_T of the call option can be decomposed as follows

$$C_T = S_T \mathbb{1}_D - K \mathbb{1}_D = X_1 - X_2.$$

Since C_T is assumed to be an attainable claim, its arbitrage price can be computed using any martingale measure for the process S/B, that is, for the model $\mathcal{M}$. Hence we may take the martingale measure $\widetilde{\mathbb{Q}} \in \mathbb{M}$ and compute the arbitrage price of the option using the risk-neutral valuation formula

$$C_t = B_t\, \mathbb{E}_{\widetilde{\mathbb{Q}}}\left(\frac{C_T}{B_T} \,\middle|\, \mathcal{F}_t\right).$$

Using the additivity property of conditional expectation and part (c) (specifically, equation (2) applied to X_1), we obtain

$$
\begin{aligned}
C_t &= B_t\,\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(\frac{C_T}{B_T}\,\Big|\,\mathcal{F}_t\Big) = B_t\,\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(\frac{X_1}{B_T}\,\Big|\,\mathcal{F}_t\Big) - B_t\,\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(\frac{X_2}{B_T}\,\Big|\,\mathcal{F}_t\Big)\\
&\overset{\text{(c)}}{=} S_t\,\mathbb{E}_{\widehat{\mathbb{Q}}}\Big(\frac{X_1}{S_T}\,\Big|\,\mathcal{F}_t\Big) - B_t\,\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(\frac{X_2}{B_T}\,\Big|\,\mathcal{F}_t\Big)\\
&= S_t\,\mathbb{E}_{\widehat{\mathbb{Q}}}\Big(\frac{S_T\mathbb{1}_D}{S_T}\,\Big|\,\mathcal{F}_t\Big) - B_t\,\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(\frac{K\mathbb{1}_D}{B_T}\,\Big|\,\mathcal{F}_t\Big)\\
&= S_t\,\widehat{\mathbb{Q}}(D\,|\,\mathcal{F}_t) - KB_t(B_T)^{-1}\,\widetilde{\mathbb{Q}}(D\,|\,\mathcal{F}_t)\\
&= S_t\,\widehat{\mathbb{Q}}(D\,|\,\mathcal{F}_t) - KB(t,T)\,\widetilde{\mathbb{Q}}(D\,|\,\mathcal{F}_t)
\end{aligned}
$$

since K and B_T are deterministic and, obviously,

$$
\mathbb{E}_{\widehat{\mathbb{Q}}}(\mathbb{1}_D\,|\,\mathcal{F}_t) = \widehat{\mathbb{Q}}(D\,|\,\mathcal{F}_t), \quad \mathbb{E}_{\widetilde{\mathbb{Q}}}(\mathbb{1}_D\,|\,\mathcal{F}_t) = \widetilde{\mathbb{Q}}(D\,|\,\mathcal{F}_t).
$$

(e) It suffices to argue that the payoff $X = 1$ at time T can be replicated by investing $\alpha = B_t/B_T$ units of cash at time t in the asset B and keeping the portfolio constant till time T. Then the wealth of the portfolio at time T will be $\alpha B_T/B_t = 1$. This means that the arbitrage price $B(t,T)$ of the unit zero-coupon bond at time t satisfies $B(t,T) = B_t/B_T$.