MATH3075/3975 Financial Derivatives

Tutorial 1: Solutions

Exercise 1 (a) The conditional distributions of X given $Y = j$ are:

$$\begin{aligned}
&\text{for } j = 1\colon\ (1/5, 3/5, 1/5),\\
&\text{for } j = 2\colon\ (2/3, 0, 1/3),\\
&\text{for } j = 3\colon\ (0, 3/5, 2/5),
\end{aligned}$$

and thus the conditional expectations $\mathbb{E}_{\mathbb{P}}(X|\,Y = j)$ are computed as follows:

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(X|\,Y = 1) &= 1/5 + 6/5 + 3/5 = 2,\\
\mathbb{E}_{\mathbb{P}}(X|\,Y = 2) &= 2/3 + 0 + 3/3 = 5/3,\\
\mathbb{E}_{\mathbb{P}}(X|\,Y = 3) &= 0 + 6/5 + 6/5 = 12/5.
\end{aligned}$$

(b) The marginal distributions of X and Y are:

$$\begin{aligned}
&\text{for } X\colon\ (4/18, 9/18, 5/18),\\
&\text{for } Y\colon\ (10/18, 3/18, 5/18).
\end{aligned}$$

On the one hand, we obtain

$$\mathbb{E}_{\mathbb{P}}(X) = \sum_{i=1}^{3} i\,\mathbb{P}(X = i) = \frac{4}{18} + 2\cdot\frac{9}{18} + 3\cdot\frac{5}{18} = \frac{37}{18}.$$

On the other hand, we get

$$\mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}(X|\,Y)\big) = \sum_{j=1}^{3} \mathbb{E}_{\mathbb{P}}(X|\,Y = j)\mathbb{P}(Y = j) = 2\cdot\frac{10}{18} + \frac{5}{3}\cdot\frac{3}{18} + \frac{12}{5}\cdot\frac{5}{18} = \frac{37}{18}.$$

Hence the equality $\mathbb{E}_{\mathbb{P}}(X) = \mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}(X|\,Y)\big)$ holds, as was expected.

(c) Random variables X and Y are not independent since the condition

$$\mathbb{P}(X = i, Y = j) = \mathbb{P}(X = i)\,\mathbb{P}(Y = j), \quad \forall\, i, j = 1, 2, 3,$$

is not satisfied. For instance, if we take $i = 1$ and $j = 2$, then

$$\mathbb{P}(X = 1, Y = 3) = 0 \neq \frac{4}{18}\cdot\frac{5}{18} = \mathbb{P}(X = 1)\,\mathbb{P}(Y = 3).$$

Exercise 2 (a) We need to show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{(X,Y)}(x, y)\,dxdy = \int_{0}^{\infty}\int_{0}^{\infty} \frac{1}{y}\,e^{-\frac{x}{y}}\,e^{-y}\,dxdy \stackrel{?}{=} 1.$$

We first compute the marginal density of Y. For $y \geq 0$, we obtain

$$f_Y(y) = \int_0^\infty \frac{1}{y}\, e^{-\frac{x}{y}}\, e^{-y}\, dx = \frac{1}{y}\, e^{-y} \int_0^\infty e^{-\frac{x}{y}}\, dx = \frac{1}{y}\, e^{-y} \left[-ye^{-\frac{x}{y}}\right]_0^\infty = e^{-y}.$$

Of course, we have $f_Y(y) = 0$ for all $y < 0$. Therefore,

$$\int_0^\infty \int_0^\infty \frac{1}{y}\, e^{-\frac{x}{y}}\, e^{-y}\, dxdy = \int_0^\infty f_Y(y)\, dy = \int_0^\infty e^{-y}\, dy = 1.$$

(b) For any fixed $y \geq 0$, the conditional density of X given $Y = y$ equals

$$f_{X|Y}(x|\, y) = \frac{f_{(X,Y)}(x, y)}{f_Y(y)} = \frac{1}{y}\, e^{-\frac{x}{y}}, \quad \forall\, x \geq 0,$$

and $f_{X|Y}(x|\, y) = 0$ for $x < 0$. Consequently, for every $y \geq 0$

$$\mathbb{E}_{\mathbb{P}}(X|\, Y = y) = \int_0^\infty x f_{X|Y}(x|\, y)\, dx = \int_0^\infty \frac{x}{y}\, e^{-\frac{x}{y}}\, dx = y \int_0^\infty z e^{-z}\, dz = y.$$

Exercise 3 We first compute the conditional cumulative distribution function of X given the event $\{X < 0.5\}$

$$F_{X|X<0.5}(x) := \mathbb{P}(X \leq x|\, X < 0.5), \quad \forall\, x \in \mathbb{R}.$$

We obtain

$$F_{X|X<0.5}(x) = \frac{\mathbb{P}(X \leq x,\, X < 0.5)}{\mathbb{P}(X < 0.5)} = \begin{cases} 0, & \text{if } x \leq 0, \\ 2\,\mathbb{P}(X \leq x) = 2x, & \text{if } x \in (0, 0.5), \\ 1, & \text{if } x \geq 0.5. \end{cases}$$

so that the conditional density of X given the event $\{X < 0.5\}$ equals

$$f_{X|X<0.5}(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 2, & \text{if } x \in (0, 0.5), \\ 0, & \text{if } x \geq 0.5. \end{cases}$$

Therefore,

$$\mathbb{E}_{\mathbb{P}}(X|\, X < 0.5) = \int_{-\infty}^\infty x f_{X|X<0.5}(x)\, dx = \int_0^{0.5} 2x\, dx = 0.25.$$

Exercise 4 We have

$$f_X(x) = \frac{1}{\lambda}\, e^{-\frac{x}{\lambda}}, \quad \forall\, x > 0,$$

and thus

$$\mathbb{P}(X > x) = 1 - F_X(x) = \int_x^\infty f_X(u)\, du = e^{-\frac{x}{\lambda}}, \quad \forall\, x > 0.$$

Consequently,

$$\mathbb{P}(X > x|\, X > 1) = \frac{\mathbb{P}(X > x, X > 1)}{\mathbb{P}(X > 1)} = \begin{cases} \frac{\mathbb{P}(X>x)}{\mathbb{P}(X>1)} = e^{-\frac{x-1}{\lambda}}, & \text{if } x \geq 1, \\ 1, & \text{if } x < 1. \end{cases}$$

Hence the conditional density equals

$$f_{X|X>1}(x) = \begin{cases} \frac{1}{\lambda}\, e^{-\frac{x-1}{\lambda}}, & \text{if } x \geq 1, \\ 0, & \text{if } x < 1, \end{cases}$$

and thus

$$\mathbb{E}_{\mathbb{P}}(X|\, X > 1) = \int_1^\infty \frac{x}{\lambda}\, e^{-\frac{x-1}{\lambda}}\, dx = 1 + \lambda = 1 + \mathbb{E}_{\mathbb{P}}(X).$$

Exercise 5 Since

$$\mathrm{Cov}(X, Y) = \mathbb{E}_{\mathbb{P}}(XY) - \mathbb{E}_{\mathbb{P}}(X)\, \mathbb{E}_{\mathbb{P}}(Y) = \mathbb{E}_{\mathbb{P}}(X^3) - \mathbb{E}_{\mathbb{P}}(X)\, \mathbb{E}_{\mathbb{P}}(X^2) = 0$$

since $\mathbb{E}_{\mathbb{P}}(X) = \mathbb{E}_{\mathbb{P}}(X^3) = 0$. Therefore, the random variables X and Y are uncorrelated. They are not independent, however, since, for instance

$$\mathbb{E}_{\mathbb{P}}(Y|\, X = 1) = 1 \neq 4 = \mathbb{E}_{\mathbb{P}}(Y|\, X = 2).$$

Recall that under independence of X and Y we have $\mathbb{E}_{\mathbb{P}}(Y|\, X) = \mathbb{E}_{\mathbb{P}}(Y)$ and $\mathbb{E}_{\mathbb{P}}(X|\, Y) = \mathbb{E}_{\mathbb{P}}(X)$.

Exercise 6 (MATH3975) We have

$$\begin{aligned} \mathrm{Cov}(X, Y) &= \mathbb{E}_{\mathbb{P}}(XY) - \mathbb{E}_{\mathbb{P}}(X)\, \mathbb{E}_{\mathbb{P}}(Y) \\ &= \mathbb{E}_{\mathbb{P}}\big((U+V)(U-V)\big) - \mathbb{E}_{\mathbb{P}}(U+V)\mathbb{E}_{\mathbb{P}}(U-V) \\ &= \mathbb{E}_{\mathbb{P}}\big(U^2 - V^2\big) - \mathbb{E}_{\mathbb{P}}(U+V)\big(\mathbb{E}_{\mathbb{P}}(U) - \mathbb{E}_{\mathbb{P}}(V)\big) = 0. \end{aligned}$$

since $\mathbb{E}_{\mathbb{P}}(U) = \mathbb{E}_{\mathbb{P}}(V)$ and $\mathbb{E}_{\mathbb{P}}(U^2) = \mathbb{E}_{\mathbb{P}}(V^2)$. Hence the random variables X and Y are uncorrelated. To check whether X and Y are independent, we need first to specify the joint distribution of X and Y. For instance, if we take $U = V$, then $X = 2U$ and $Y = 0$ so that X and Y are independent. However, if we take U and V independent (but not deterministic), then X and Y are not necessarily independent. For instance, we may take as U and V the i.i.d. Bernoulli random variables with $\mathbb{P}(U = 1) = p = 1 - \mathbb{P}(U = 0)$. It is then easy to check that X and Y are not independent.