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Optimal management of DC pension plan under loss aversion and Value-at-Risk constraints

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HIGHLIGHTS

- Study the optimal management of DC pension plan based on prospect theory.
- Derive closed-forms of the optimal investment strategies under two criterions.
- The two criterions are loss aversion and VaR.
- Establish the risk financial model consisting of cash, bond and stock.
- Give a sensitivity analysis to clarify the behavior of the optimal strategies.

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ABSTRACT

This paper studies the risk management in a defined contribution (DC)pension plan. The financial market consists of cash, bond and stock. The interest rate in our model is assumed to follow an Ornstein–Uhlenbeck process while the contribution rate follows a geometric Brownian Motion. Thus, the pension manager has to hedge the risks of interest rate, stock and contribution rate. Different from most works in DC pension plan, the pension manger has to obtain the optimal allocations under loss aversion and Value-at-Risk(VaR) constraints. The loss aversion pension manager is sensitive to losses while the VaR pension manager has to ensure the quality of wealth at retirement. Since these problems are not standard concave optimization problems, martingale method is applied to derive the optimal investment strategies. Explicit solutions are obtained under these two optimization criterions. Moreover, sensitivity analysis is presented in the end to show the economic behaviors under these two criterions.
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1. Introduction

Pension fund arises as a popular subject in the retirement system since it helps to ensure personal life after retirement. The

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management of pension fund during its accumulation phase is very important. There are mainly two kinds of pension fund:defined benefit (DB) pension plan and defined contribution (DC) pension plan. The DC pension plan fixes the contribution in advance while the benefit is fixed in a DB pension plan. Since the risk of a DC plan is undertaken by the contributors, the plan sponsors pay more attentions to the DC pension plan in reality.

The benefit of a DC plan is mainly affected by the economic behavior of the plan before retirement. Thus, the risk management of a DC plan always tries to find the best portfolios during the accumulation phase. Many literatures concern about the optimal investment strategies for the DC pension plan. Vigna and Haberman (2001) firstly established a significant discrete model for the DC plan and derived the explicit solution. Later, Haberman and Vigna (2002) extended it to the case of stochastic interest rate and analyzed risk measures for the DC plan at retirement. The effects of contribution rate and the annuity at retirement are well studied in Cairns et al. (2006). The continuous model for DC plan with interest risk is firstly investigated in Boulier et al. (2001). In their model the terminal wealth is required to ensure personal life after retirement. In Deelstra et al. (2003), a more general model is formulated. The main assignments in the area of DC pension plan focus on managing the risks in the market. Since the accumulation phase in a DC plan may last long, Zhang et al. (2007) firstly takes into account the effect of inflation risk. Gao (2008) applied the Legendre transform in the DC plan to derive the explicit solution with logarithm utility. Gao (2009) concerned about the volatility risk from the stock. Guan and Liang (2014) considered the risks of interest and volatility of stock together and obtained the explicit solution to maximize the CRRA utility of terminal wealth over an annuity guarantee. Recently, Yao et al. (2013), Han and Hung (2012) solved problem of DC plan with inflation risk under CRRA utility maximization and mean–variance criterion, respectively. Yao et al. (2014) studied risks of mortality and contribution.

Most works in DC plan care about maximizing the expectation of a smooth utility of terminal wealth. So the pension manager is often assumed to be strictly risk averse towards the terminal wealth. The optimal investment strategies to hedge the risks in the market often consist of a substantial allocation in stock. Unfortunately, some people are unwilling to take the risks from stock. In particular, the risks at retirement in a DC plan may be very disgusting since the plan may not ensure personal life after retirement. Besides, some people may be risk seeking and thus invest more in the stock. Therefore, the optimal wealth at retirement in many literatures may lead to huge risk for a DC pension plan, which has been investigated in Haberman and Vigna (2002). However, Haberman and Vigna in Haberman and Vigna (2002) only estimated the probability of failing the target, the mean shortfall and a value at risk (VaR) measure, they did not provide a plummy portfolios to manage the risks.

Since the existing works in DC plan mainly care about the complexity in the market, we introduce here two different optimization criterions that are different from the smooth utility case. The first criterion belongs to prospect theory. The breakthrough in Kahneman and Tversky (1979) has been a cornerstone of the prospect theory, in which Kahneman and Tversky proposed reference point and distortion of probabilities in portfolio theory. These ideas have been proven to be of great use and can result in lowering risks for an investor. Because the prospect theory describes human behavior better, more and more literatures study the loss aversion utility and distortion of probability in portfolio selection. Berkelaar et al. (2004) firstly employed the martingale method to derive the optimal investment strategies with two utility functions under loss aversion in a continuous case. Later, Gomes (2005) considered the counterpart discrete model. Furthermore, Jarrow and Zhao (2006) introduced a mean–variance framework under loss aversion. The above works only concerned the loss aversion in prospect theory. The distortion of probability in portfolio selection can refer to Bernard and Ghossoub (2010), Jin and Zhou (2008), He and Zhou (2011) and references therein.

Besides the loss aversion, value-at-risk(VaR) also arises as an important subject for portfolio selection. VaR measures the probability of loss for a given account. Since people abhor losses, VaR has attracted much attention. Many literatures try to find the best allocations under VaR constraints. Basak and Shapiro (2001), and Yiu (2004) derived the optimal investment strategies to manage VaR risk by martingale method and stochastic dynamic programming, respectively. Alexander and Baptista (2002) analyzed the investment policy to maximize the expectation while minimize the VaR factor in a discrete model. Later, they (cf. Alexander and Baptista, 2004) considered CVaR constraint. Other breakthrough to employ VaR constraints in portfolio selection can refer to Basak et al. (2006), Cuoco et al. (2008), Chen et al. (2010) and references therein.

In this paper, we intend to investigate the optimal allocations for DC pension plan under the loss aversion and VaR constraints. The market in our model is similar to Boulier et al. (2001) and Guan and Liang (2014). The interest rate is modeled by an Ornstein–Uhlenbeck model while the stock price is assumed to follow a diffusion model. Moreover, we consider a stochastic contribution rate, which is related with the risks from interest rate and stock. So, we need to manage the risks of interest rate, the stock and the contribution rate. The work of Boulier et al. (2001) maximizes the CRRA expectation of terminal wealth over a guarantee. However, in this paper we consider the portfolio selection for DC plan under loss aversion and VaR constraints. The utility function under loss aversion we adopt is firstly studied in Kahneman and Tversky (1979). *The utility function is convex under a reference point while concave above the point*. This leads to a risk-seeking attitude towards losses. Besides, we also consider maximizing the expectation of CRRA utility function of terminal wealth under VaR constraints. Since the optimization problem is not a concave maximization problem, the optimal wealth is a piecewise function of the pricing kernel and so it seems that the stochastic programming method does not work here. We will apply the martingale method to derive the optimal investment strategies for loss aversion and VaR constraints. Moreover, the sensitivity analysis in the end shows that the pension manager under loss aversion has a complex behavior and may purchase more or less on the risk assets based on the reference point. When the reference point is very high, the pension manager judges the account by losses and acts as a risk seeking investor. However, the VaR constraint leads to a risk seeking attitude towards the terminal wealth. Thus, the manager with VaR constraints will purchase more risk assets.

The rest of this paper is organized as follows: Section 2 provides the model of financial market and the DC pension plan. In Section 3, the optimization problems of loss aversion and VaR constraints are formulated. The optimal investment strategies are derived via martingale method. The proofs in this section are presented in the Appendix. Section 4 presents a sensitivity analysis to show the impacts of the optimization criterions on the optimal strategies. Finally, Section 5 is a conclusion.

2. The financial market and pension plan

2.1. The financial model

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbf{P})$ be a filtered complete probability space and $\mathcal{F}_t$ is the information available before time t in the market. The pension fund starts at time 0 and the retirement time is T. The economic behavior of the pension manager occurs continuously within $[0, T]$ (cf. He and Liang, 2009). All the processes introduced below are supposed to be well-defined and adapted to $\{\mathcal{F}_t\}_{t\in[0,T]}$.

In this section, we introduce the financial market similar to Boulier et al. (2001) and Guan and Liang (2014). The model in Boulier et al. (2001) beautifully explores the optimal strategies in a pension fund. Since the pension plan takes a long time, we also consider a stochastic interest rate. The financial market in our model consists of cash, bond and a stock. The risk free asset (i.e., cash) $S_0(t)$ satisfies the following equation:

$$\mathrm{d}S_0(t) = S_0(t)r(t)\mathrm{d}t, \qquad S_0(0) = S_0. \tag{2.1}$$

The instantaneous interest rate is modeled by the following Ornstein–Uhlenbeck model:

$$\mathrm{d}r(t) = a(b - r(t))\mathrm{d}t - \sigma_r \mathrm{d}W_r(t), \qquad r(0) = r_0, \tag{2.2}$$

where a, b and σ_r are positive constants, $W_r(t)$ is a standard Brownian motion on the probability space. Therefore, $r(t)$ tends towards b.

We introduce here a zero-coupon bond with final payment of \$1 at maturity to manage the interest risk. Based on the general pricing theory, the price of a zero-coupon bond at t with maturity T is

$$B(t, T) = e^{C(t,T)-A(t,T)r(t)}, \tag{2.3}$$

where $A(t, T) = \frac{1-e^{-a(T-t)}}{a}$, $C(t, T) = -R(T - t) + A(t, T)[R - \frac{\sigma_r^2}{2a^2}] + \frac{\sigma_r^2}{4a^3}(1 - e^{-2a(T-t)})$, $R = b + \frac{\sigma_r\lambda_r}{a} - \frac{\sigma_r^2}{2a^2}$, and λ_r is the market price of risk of $W_r(t)$.

Moreover, $B(t, T)$ satisfies the following backward differential equation:

$$\begin{aligned} &\frac{\mathrm{d}B(t,T)}{B(t,T)} = r(t)\mathrm{d}t + \sigma_r A(t,T)(\lambda_r \mathrm{d}t + \mathrm{d}W_r(t)), \\ &B(T,T) = 1 \end{aligned} \tag{2.4}$$

where $\sigma_r A(t, T)$ represents the volatility of $B(t, T)$. We can observe that the volatility increases with the maturity.

Since the maturity of the bond in the real market is restricted, the investment in the $B(t, T)$ is unrealistic. Thus we introduce a rolling bond in the market to hedge the risk of interest rate. We are purchasing a bond with a constant maturity K. A rolling bond $B_K(t)$ with the constant maturity K is given by

$$\frac{\mathrm{d}B_K(t)}{B_K(t)} = r(t)\mathrm{d}t + \sigma_r A(t, t+K)(\lambda_r \mathrm{d}t + \mathrm{d}W_r(t)). \tag{2.5}$$

The risks in the rolling bond and the zero coupon bond are only related to the risk of interest rate. In the Ornstein–Uhlenbeck model, we demand the completeness of the risk of interest rate. So, the zero coupon bond in the market with any maturity in fact can be well replicated by cash and the rolling bond. The relationship between them is

$$\begin{aligned} \frac{\mathrm{d}B(t,s)}{B(t,s)} &= \left(1 - \frac{A(t,s)}{A(t,t+K)}\right)\frac{\mathrm{d}S_0(t)}{S_0(t)} \\ &\quad + \frac{A(t,s)}{A(t,t+K)}\frac{\mathrm{d}B_K(t)}{B_K(t)}, \qquad \forall s > t. \end{aligned} \tag{2.6}$$

In addition, we assume that the stock in the financial market satisfies

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{S(t)} = r(t)\mathrm{d}t + \sigma_1(\lambda_r \mathrm{d}t + \mathrm{d}W_r(t)) \\ \qquad\qquad + \sigma_2(\lambda_S \mathrm{d}t + \mathrm{d}W_S(t)), \\ S(0) = S_1, \end{cases} \tag{2.7}$$

where $W_S(t)$ is a standard Brownian motion on the filtered complete probability space, which is independent of $W_r(t)$. λ_S is the market price of risk of $W_S(t)$. λ_S denotes the earning to take the risk of the stock. The price of the stock is assumed to be closely connected to the risk of the interest rate. Since the market only characterizes risks of the interest rate and the stock, the market is complete.

2.2. The wealth process

The contributor contributes a continuous part of salary to the pension plan. This contribution continuously increases the wealth of the pension account. When the pension manager designs the strategies, he should take into account the contributions from the contributor. Since the salary of an employee is often correlated with the financial market, it is realistic to consider a stochastic contribution rate. We assume that the contribution rate satisfies

$$\frac{\mathrm{d}C(t)}{C(t)} = \mu \mathrm{d}t + \sigma_{C_1}\mathrm{d}W_r(t) + \sigma_{C_2}\mathrm{d}W_S(t), \qquad C(0) = C_0 \tag{2.8}$$

where μ, σ_{C_1}, C_0 and σ_{C_2} are positive constants. The contribution rate is deterministic when $\sigma_{C_1} = \sigma_{C_2} = 0$.

The pension account is endowed with an initial money X_0. The pension manager can invest in the financial market to avoid risk. Suppose that there are no transaction costs or taxes in the market and short buying is also allowed. Denote the money invested in the cash, bond and stock by $u_0(t)$, $u_B(t)$ and $u_B(t)$, respectively. The wealth $X(t)$ of the pension fund satisfies

$$\mathrm{d}X(t) = u_0(t)\frac{\mathrm{d}S_0(t)}{S_0(t)} + u_B(t)\frac{\mathrm{d}B_K(t)}{B_K(t)} + u_S(t)\frac{\mathrm{d}S(t)}{S(t)} + C(t)\mathrm{d}t. \tag{2.9}$$

Substituting Eqs. (2.1), (2.5) and (2.7) into the above equation, we can derive the explicit form of the wealth by $X(t) = u_0(t)+u_B(t)+u_B(t)$ as follows:

$$\begin{cases} \mathrm{d}X(t) = r(t)X(t)\mathrm{d}t + [u_B(t)\sigma_r A(t, t+K) + u_S(t)\sigma_1] \\ \qquad\qquad \times (\lambda_r \mathrm{d}t + \mathrm{d}W_r(t)) \\ \qquad\qquad + u_S(t)\sigma_2(\lambda_S \mathrm{d}t + \mathrm{d}W_S(t)) + C(t)\mathrm{d}t, \\ X(0) = X_0 \geq 0. \end{cases} \tag{2.10}$$

The pension manager has to find the best allocations in the bond and stock. The allocations will reduce the risks of the pension fund. The rest money of the account is naturally allocated in the cash.

3. Risk management

The goal of the pension manager is to find the optimal investment strategies within $[0, T]$ under some optimization criterion. Boulier et al. (2001), Deelstra et al. (2003) considered the concave utility function maximization in a DC pension fund. The martingale method is applied in their work. They invest a plenty of money in the stock. Therefore, the wealth at retirement brings a lot of risk for the pension fund. The pension manager may experience a large loss at retirement. But, in the real world, some people may be unwilling to take the large risk from the stock. They may be more interested to allocate money in the bond. Besides, others may be risk seeking and invest more in the stock. The general optimization problem only characterizes a risk averse investor and cannot reflect others' behavior towards risk. Moreover, the utility function adopted for the concave utility problem is not connected with the contribution of the pension fund there. However, it makes more sense to take the effect of contribution in the optimization problem. We formulate two different optimization problems below for the DC pension plan, which better manage the risks at retirement.

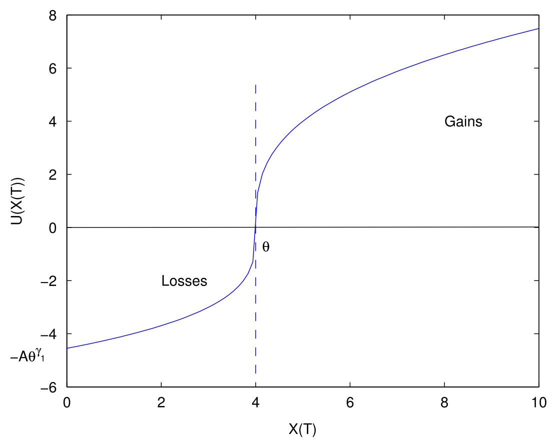

Fig. 1. Representation of utility function (3.1) with parameters $A = 3$, $B = 4$, $\theta = 4$, $\gamma_1 = 0.3$, $\gamma_2 = 0.35$.

3.1. *Loss aversion risk management*

3.1.1. *Formulation of loss aversion*

This section formulates the optimization problem under loss aversion. Kahneman and Tversky (1979) firstly established the theory of prospect theory. They stated that people always make decisions relative to some reference levels. The reference levels may be different for different people. They judge the amount of the wealth over(under) the reference as gain(losses). People often act differently towards gains and losses. In fact, people are more sensitive to losses than gains. They also demonstrated their idea based on the following utility function:

$$U(X(T)) = \begin{cases} -A(\theta - X(T))^{\gamma_1}, & X(T) \le \theta \\ B(X(T) - \theta)^{\gamma_2}, & X(T) > \theta, \end{cases} \tag{3.1}$$

where A and B are positive constants, $0 < \gamma_1 \le 1$, $0 < \gamma_2 < 1$. Statistics are showed in Kahneman and Tversky (1979) to support the above utility function. The investor is risk-averse towards gains while risk-seeking towards losses. The reference point θ is chosen in advance. In the DC pension fund, the reference point θ can be chosen to be connected with the contribution rate and initial wealth in the pension account. Fig. 1 illustrates the properties of the loss aversion function (3.1). The utility function is convex when the wealth is less than θ and concave when the wealth is bigger than θ.

3.1.2. *Solution under loss aversion*

Denote $u(t) = (u_B(t), u_S(t))$. We call $u(t)$ an admissible strategy if Eq. (2.10) admits an unique solution and $u(t)$ is progressively measurable w.r.t. $\{\mathcal{F}_t\}_{t\in[0,T]}$. We search the optimal strategy in the admissible space Π. The goal of the pension manager is to find the optimal investment strategies to maximize the following expectation of the loss-averse utility:

$$\begin{aligned} &\max_{u(\cdot)} \quad \mathbf{E}[U(X(T))] \\ &\text{s.t.} \quad \begin{cases} (X(t), u(t)) \text{ satisfies (2.10) and} \\ u(t) \in \Pi. \end{cases} \end{aligned} \tag{3.2}$$

In order to solve problem (3.2), we first introduce the pricing kernel in the complete market. The price of a given random variable can be obtained by multiplying by the pricing kernel. The pricing kernel $H(t)$ satisfies

$$\frac{\mathrm{d}H(t)}{H(t)} = -r(t)\mathrm{d}t - \lambda_r \mathrm{d}W_r(t) - \lambda_S \mathrm{d}W_S(t), \qquad H(0) = 1. \tag{3.3}$$

The pricing kernel also represents the Arrow Debreu price per unit probability $\mathbf{P}$ of one unit of consumption good in state $\omega \in \Omega$ at T. In the market, we can derive the solution by martingale method. Since the utility function is not concave, stochastic programming method may not work. Based on the Cox and Huang (1989), the problem (3.2) can be transformed into the following equivalent problem with budget constraint:

$$\begin{aligned} &\max_{X(T)} \quad \mathbf{E}[U(X(T))] \\ &\text{s.t.} \quad \begin{cases} \mathbf{E}\left[H(T)X(T) - \int_0^T H(s)C(s)\mathrm{d}s\right] \le X(0), \\ X(T) \ge 0. \end{cases} \end{aligned} \tag{3.4}$$

The budget constraint is derived based on Eq. (2.10). The constraint is equivalent to Eq. (2.10) since the financial market is complete in our model. The budget constraint in problem (3.4) has good economic interpretation. In the DC pension fund, the money allocated in the account has two sources: the initial wealth and the contribution rate. The constraint in problem (3.4) reveals that the price of the terminal wealth is not bigger than the initial value and the present value of the aggregated contribution form t to T. In the martingale problem, we firstly focus on the optimal terminal wealth. Since the terminal wealth satisfies the budget constraint, the optimal investment strategy can be well obtained by the completeness of the market. We have the following theorem about the terminal wealth.

Proposition 3.1. *The optimal terminal wealth for the loss aversion utility is*

$$X^{*,\lambda^*}(T) = \begin{cases} \theta + \left(\dfrac{\lambda^* H(T)}{B\gamma_2}\right)^{\frac{1}{\gamma_2 - 1}} & \text{if } H(T) < \bar{H}, \\ 0 & \text{if } H(T) \ge \bar{H}, \end{cases} \tag{3.5}$$

where $\bar{H}$ satisfies $f(\bar{H}) = 0$ with

$$f(x) = B(1-\gamma_2)\left(\frac{B\gamma_2}{\lambda^* x}\right)^{\frac{\gamma_2}{1-\gamma_2}} + A\theta^{\gamma_1} - \lambda^*\theta x,$$

$\lambda^* > 0$ is a lagrange multiplier w.r.t. the budget constraint and satisfies $\mathbf{E}[H(T)X^{*,\lambda^*}(T) - \int_0^T H(s)C(s)\mathrm{d}s] = X(0)$.

Proof. See Appendix. □

In Proposition 3.1, the optimal terminal wealth is divided into two parts: the wealth is similar to the smooth CRRA utility when the pricing kernel is low, while the wealth drops to zero when the pricing kernel is relatively high. The optimal terminal wealth with smooth utility is in fact a decreasing function of pricing kernel. When the pricing kernel is small, the optimal wealth with smooth utility lies above θ. Thus, the optimal wealth in these states under loss aversion equals the optimal wealth with smooth utility. However, the optimal wealth is smaller than the reference point θ when the pricing kernel is relatively high. The loss aversion states a risk-seeking preference under θ. Therefore, we let the terminal wealth equal to 0 in this case.

After obtaining the optimal terminal wealth at retirement, we can derive the wealth and investment strategies during $[0, T]$. Since the market is complete, the optimal wealth at time t for the terminal wealth (3.5) is as follows:

$$\begin{aligned} X^{*,\lambda^*}(t) &= \frac{1}{H(t)}\mathbf{E}[H(T)X^{*,\lambda^*}(T)|\mathcal{F}_t] \\ &\quad - \int_t^T \mathbf{E}\left[\frac{H(s)}{H(t)}C(s)|\mathcal{F}_t\right]\mathrm{d}s. \end{aligned} \tag{3.6}$$

The optimal wealth at t consists of two components. The first part in (3.6) is the risk-neutral price of the random variable $X^{*,\lambda^*}(T)$ at time t. The second part represents the price of the aggregated contribution from t to T. Since the contribution increases the wealth within $[t,T]$, $X^{*,\lambda^*}(t)$ is smaller than the price of $X^{*,\lambda^*}(T)$ at t. Now we derive the explicit form of $X^{*,\lambda^*}(t)$. We have the following two lemmas dealing with the two parts of $X^{*,\lambda^*}(t)$ in (3.6).

Lemma 3.2. *The price of $X^{*,\lambda^*}(T)$ at time t $(0 \le t < T)$ is given by*

$$\begin{aligned}
&\frac{1}{H(t)}\mathbf{E}[H(T)X^{*,\lambda^*}(T)|\mathcal{F}_t] \\
&= \theta \exp\left[\frac{1}{2}\mathrm{Var}\{N_t\} + \mathbf{E}\{N_t\}\right]\Phi(d_1(\bar{H})) + \left(\frac{\lambda^* H(t)}{B\gamma_2}\right)^{\frac{1}{\gamma_2-1}} \\
&\times \exp\left(\frac{\gamma_2^2}{2(\gamma_2-1)^2}\mathrm{Var}\{N_t\} + \frac{\gamma_2}{\gamma_2-1}\mathbf{E}\{N_t\}\right)\Phi(d_2(\bar{H})), \quad (3.7)
\end{aligned}$$

where $\Phi(\cdot)$ represents the cumulative distribution function of a standard normal variable, and

$$\begin{aligned}
N_t &= -\int_t^T r(s)\mathrm{d}s - \frac{1}{2}\lambda_r^2(T-t) - \frac{1}{2}\lambda_S^2(T-t) \\
&\quad - \lambda_r[W_r(T) - W_r(t)] - \lambda_S[W_S(T) - W_S(t)],
\end{aligned}$$

$$\begin{aligned}
\mathbf{E}\{N_t\} &= -(r(t)-b)\frac{1-\exp(-a(T-t))}{a} - b(T-t) \\
&\quad - \frac{1}{2}(\lambda_r^2 + \lambda_S^2)(T-t),
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}\{N_t\} &= \frac{\sigma_r^2}{a^2}\left[(T-t) + \frac{2\exp(-a(T-t))}{a}\right. \\
&\quad \left. - \frac{\exp(-2a(T-t))}{2a} - \frac{3}{2a}\right] + (\lambda_r^2 + \lambda_S^2)(T-t) \\
&\quad - 2\frac{\lambda_r}{a}(\sigma_r(T-t) - \sigma_r A(t,T)),
\end{aligned}$$

$$d_1(\bar{H}) = \frac{\ln\left(\frac{\bar{H}}{H(t)}\right) - \mathbf{E}\{N_t\} - \mathrm{Var}\{N_t\}}{\sqrt{\mathrm{Var}\{N_t\}}},$$

$$d_2(\bar{H}) = \frac{\ln\left(\frac{\bar{H}}{H(t)}\right) - \mathbf{E}\{N_t\} - \frac{\gamma_2}{\gamma_2-1}\mathrm{Var}\{N_t\}}{\sqrt{\mathrm{Var}\{N_t\}}}.$$

Proof. See Appendix. □

We see that $\frac{1}{H(t)}\mathbf{E}[H(T)X^{*,\lambda^*}(T)|\mathcal{F}_t]$ is a complicate function of $H(t), r(t)$. The functions in $X^{*,\lambda^*}(t)$ are continuous of $(H(t), r(t))$. However, $X^{*,\lambda^*}(T)$ is discontinuous at the point $\bar{H}$ w.r.t. $H(T)$.

In order to calculate the price of aggregated contribution rate from t to T. Denote $D(t,s) = \mathbf{E}\left[\frac{H(s)}{H(t)}C(s)|\mathcal{F}_t\right]$. $D(t,s)$ represents the value of an asset with payment of $C(s)$ at maturity s. Integrating $D(t,s)$ w.r.t. s over $[t,T]$, we can get the price of the aggregated contribution. Let $F(t,T) = \int_t^T D(t,s)\mathrm{d}s$, which is the second part in $X^{*,\lambda^*}(t)$. The following lemma presents the explicit forms of $D(t,s)$ and $F(t,T)$.

Lemma 3.3. *The explicit form of $D(t,s)$ is given by*

$$\begin{aligned}
D(t,s) &= C(t)\exp\left[\left(\mu - \frac{1}{2}\sigma_{C_1}^2 - \frac{1}{2}\sigma_{C_2}^2 - \frac{1}{2}\lambda_r^2 - \frac{1}{2}\lambda_S^2\right)\right. \\
&\quad \left. \times (s-t)\right]\exp\left[\mathbf{E}\{Q(t,s)\} + \frac{1}{2}\mathrm{Var}\{Q(t,s)\}\right], \quad (3.8)
\end{aligned}$$

where

$$\mathbf{E}\{Q(t,s)\} = -(r(t)-b)\frac{1-\exp(-a(s-t))}{a} - b(s-t),$$

$$\begin{aligned}
\mathrm{Var}\{Q(s,t)\} &= \int_t^s \sigma_r^2 A(u,s)^2\mathrm{d}u + (\sigma_{C_1} - \lambda_r)^2(s-t) \\
&\quad + (\sigma_{C_2} - \lambda_S)^2(s-t) + 2(\sigma_{C_1} - \lambda_r)\int_t^s \sigma_r A(u,s)\mathrm{d}u.
\end{aligned}$$

Besides, $D(t,s)$ satisfies the following backward stochastic differential equation:

$$\begin{cases}
\dfrac{\mathrm{d}D(t,s)}{D(t,s)} = r(t)\mathrm{d}t + (\sigma_{C_1} + \sigma_r A(t,s)) \\
\qquad \times[\lambda_r\mathrm{d}t + \mathrm{d}W_r(t)] + \sigma_{C_2}[\lambda_S\mathrm{d}t + \mathrm{d}W_S(t)], \\
D(s,s) = C(s), \quad s \ge t.
\end{cases} \quad (3.9)$$

So $F(t,T) = \int_t^T D(t,s)\mathrm{d}s$ can be explicitly derived and $F(t,T)$ also satisfies the following backward stochastic differential equation:

$$\begin{cases}
\mathrm{d}F(t,T) = -C(t)\mathrm{d}t + r(t)F(t,T)\mathrm{d}t \\
\quad + F_1(t)[\lambda_r\mathrm{d}t + \mathrm{d}W_r(t)] + F_2(t)[\lambda_S\mathrm{d}t + \mathrm{d}W_S(t)], \\
F(T,T) = 0,
\end{cases} \quad (3.10)$$

where $F_1(t) = \sigma_{C_1}F(t,T) + \int_t^T D(t,s)\sigma_r A(t,s)\mathrm{d}s$, $F_2(t) = \sigma_{C_2}F(t,T)$.

Proof. See Appendix. □

After obtaining the explicit forms of $\frac{1}{H(t)}\mathbf{E}[H(T)X^{*,\lambda^*}(T)|\mathcal{F}_t]$ and $F(t,T)$, we sort the above two lemmas and calculate the optimal wealth at time t as follows.

Proposition 3.4. *The optimal wealth for the loss aversion manager at time t $(0 \le t < T)$ is*

$$\begin{aligned}
X^{*,\lambda^*}(t) &= \frac{1}{H(t)}\mathbf{E}[H(T)X^{*,\lambda^*}(T)|\mathcal{F}_t] \\
&\quad - \int_t^T \mathbf{E}\left[\frac{H(s)}{H(t)}C(s)\middle|\mathcal{F}_t\right]\mathrm{d}s, \quad (3.11)
\end{aligned}$$

where $\frac{1}{H(t)}\mathbf{E}[H(T)X^{*,\lambda^*}(T)|\mathcal{F}_t]$ and $D(t,s) = \mathbf{E}\left[\frac{H(s)}{H(t)}C(s)|\mathcal{F}_t\right]$ are calculated as in Lemmas 3.2 and 3.3, respectively.

The optimal wealth at time t is obtained by the pricing theory. We have not considered the stochastic differential equation (2.10) until now. In fact, after deriving the explicit form of $X^{*,\lambda^*}(t)$, we can also get the differential form of $X^{*,\lambda^*}(t)$. Comparing the differential form of $X^{*,\lambda^*}(t)$ with Eq. (2.10), we can obtain the optimal investment strategies.

Since Lemma 3.2 shows that $\frac{1}{H(t)}\mathbf{E}[H(T)X^{*,\lambda^*}(T)|\mathcal{F}_t]$ is a function of $(t, r(t), H(t))$, we denote $G(t, r(t), H(t)) = \frac{1}{H(t)}\mathbf{E}[H(T)X^{*,\lambda^*}(T)|\mathcal{F}_t]$ as the price of $X^{*,\lambda^*}(T)$ at time t. So the differential of $X^{*,\lambda^*}(t)$ is given by

$$\mathrm{d}X^{*,\lambda^*}(t) = \mathrm{d}G(t, r(t), H(t)) - \mathrm{d}F(t,T). \quad (3.12)$$

By Eqs. (2.2), (3.3) and (3.10), we can express $\mathrm{d}X^{*,\lambda^*}(t)$ in the explicit form as

$$\begin{aligned}
\mathrm{d}X^{*,\lambda^*}(t) &= \Box\mathrm{d}t - \left[\sigma_r\frac{\partial G(t,r(t),H(t))}{\partial r(t)}\right. \\
&\quad \left. + \lambda_r\frac{\partial G(t,r(t),H(t))}{\partial H(t)}H(t) + F_1(t)\right]\mathrm{d}W_r(t) \\
&\quad - \left[\lambda_S\frac{\partial G(t,r(t),H(t))}{\partial H(t)}H(t) + F_2(t)\right]\mathrm{d}W_S(t), \quad (3.13)
\end{aligned}$$

where $\Box$ denotes some function which is not related to deriving the optimal investment strategies. Since we can obtain the optimal investment strategies by singly comparing the coefficients in the diffusion parts, we only calculate the diffusion parts of $dX^{*,\lambda^*}(t)$ in the explicit form. Comparing Eqs. (2.10) and (3.13), we derive the optimal money invested in the bond and stock as follows.

Proposition 3.5. *The optimal money invested in the bond and stock is*

$$\begin{aligned} u_B^*(t) &= -\frac{\partial G(t,r(t),H(t))}{\partial r(t)}\frac{1}{A(t,t+K)} \\ &\quad -\lambda_r\frac{\partial G(t,r(t),H(t))}{\partial H(t)}\frac{H(t)}{\sigma_r A(t,t+K)} \\ &\quad -\frac{F_1(t)}{\sigma_r A(t,t+K)}+\lambda_S\frac{\partial G(t,r(t),H(t))}{\partial H(t)} \\ &\quad \times\frac{\sigma_1 H(t)}{\sigma_2\sigma_r A(t,t+K)}+\frac{\sigma_1 F_2(t)}{\sigma_2\sigma_r A(t,t+K)}, \\ u_S^*(t) &= -\lambda_S\frac{\partial G(t,r(t),H(t))}{\partial H(t)}\frac{H(t)}{\sigma_2}-\frac{F_2(t)}{\sigma_2}, \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} &\frac{\partial G(t,r(t),H(t))}{\partial H(t)} \\ &= -\frac{\theta}{H(t)\sqrt{\mathrm{Var}\{N_t\}}}\times\exp\Big[\frac{1}{2}\mathrm{Var}\{N_t\}+\mathbf{E}\{N_t\}\Big]\varphi(d_1(\bar H)) \\ &\quad +\frac{1}{(\gamma_2-1)H(t)}\left(\frac{\lambda^*H(t)}{B\gamma_2}\right)^{\frac{1}{\gamma_2-1}} \\ &\quad \times\exp\left(\frac{\gamma_2^2}{2(\gamma_2-1)^2}\mathrm{Var}\{N_t\}+\frac{\gamma_2}{\gamma_2-1}\mathbf{E}\{N_t\}\right)\Phi(d_2(\bar H)) \\ &\quad -\frac{1}{H(t)\sqrt{\mathrm{Var}\{N_t\}}}\left(\frac{\lambda^*H(t)}{B\gamma_2}\right)^{\frac{1}{\gamma_2-1}} \\ &\quad \times\exp\left(\frac{\gamma_2^2}{2(\gamma_2-1)^2}\mathrm{Var}\{N_t\}+\frac{\gamma_2}{\gamma_2-1}\mathbf{E}\{N_t\}\right)\varphi(d_2(\bar H)), \end{aligned} \tag{3.15}$$

$$\begin{aligned} &\frac{\partial G(t,r(t),H(t))}{\partial r(t)} \\ &= -\theta A(t,T)\exp\left[\frac{1}{2}\mathrm{Var}\{N_t\}+\mathbf{E}\{N_t\}\right]\Phi(d_1(\bar H)) \\ &\quad +\theta\frac{A(t,T)}{\sqrt{\mathrm{Var}\{N_t\}}}\exp\left[\frac{1}{2}\mathrm{Var}\{N_t\}+\mathbf{E}\{N_t\}\right]\varphi(d_1(\bar H)) \\ &\quad -\frac{\gamma_2 A(t,T)}{(\gamma_2-1)}\left(\frac{\lambda^*H(t)}{B\gamma_2}\right)^{\frac{1}{\gamma_2-1}}\exp\left(\frac{\gamma_2^2}{2(\gamma_2-1)^2}\mathrm{Var}\{N_t\}\right. \\ &\quad \left.+\frac{\gamma_2}{\gamma_2-1}\mathbf{E}\{N_t\}\right)\Phi(d_2(\bar H))+\frac{A(t,T)}{\sqrt{\mathrm{Var}\{N_t\}}}\left(\frac{\lambda^*H(t)}{B\gamma_2}\right)^{\frac{1}{\gamma_2-1}} \\ &\quad \times\exp\left(\frac{\gamma_2^2}{2(\gamma_2-1)^2}\mathrm{Var}\{N_t\}+\frac{\gamma_2}{\gamma_2-1}\mathbf{E}\{N_t\}\right)\varphi(d_2(\bar H)), \end{aligned} \tag{3.16}$$

and $\varphi(\cdot)$ denotes the standard normal density function: $\varphi(x)=\exp(-\frac{x^2}{2})$.

Proof. We can obtain the proposition by directly comparing Eqs. (2.10) and (3.13) and differentiating w.r.t. $G(t,r(t),H(t))$. □

3.2. VaR risk management

3.2.1. Formulation of VaR

This section introduces the concept of VaR risk management for the DC pension plan. In the financial market, the concept of VaR has been widely used in risk management. The VaR describes the loss given a confidence level. VaR is judged as the risk of a given account. We assume that the probability that the terminal wealth exceeds a given level should be higher than a given probability. The VaR constraint for the pension fund is formulated by

$$\mathbf{P}\big(X(T)\ge\widetilde{X}\big)\ge 1-\alpha,\quad \alpha\in[0,1]. \tag{3.17}$$

The VaR constraint requires that the probability of the terminal wealth should not be low than $\widetilde{X}$ is higher than $1-\alpha$. Especially when $\alpha=1$, the constraint is not binding. When $\alpha=0$, the terminal wealth should always be higher than the level $\widetilde{X}$, which has been studied in many literatures.

So we consider the following optimization problem with the VaR constraint:

$$\begin{aligned} &\max_{u(\cdot)}\quad \mathbf{E}[U(X(T))] \\ &\text{s.t.}\quad \begin{cases}(X(t),u(t))\ \text{satisfy (2.10)}\\ u(t)\in\Pi,\\ \mathbf{P}\big(X(T)\ge\widetilde{X}\big)\ge 1-\alpha.\end{cases} \end{aligned} \tag{3.18}$$

In order to simplify the above problem, the utility function in (3.18) is the general smooth concave increasing function, such as the CRRA and CARA utility functions.

3.2.2. Solution under VaR constraints

We firstly formulated the martingale problem of problem (3.18). The problem can be rewritten as follows:

$$\begin{aligned} &\max_{X(T)}\quad \mathbf{E}[U(X(T))] \\ &\text{s.t.}\quad \begin{cases}\mathbf{E}[H(T)X(T)-\int_0^T H(s)C(s)ds]\le X(0),\\ \mathbf{P}\big(X(T)\ge\widetilde{X}\big)\ge 1-\alpha,\\ X(T)\ge 0.\end{cases} \end{aligned} \tag{3.19}$$

The VaR constraint in the problem can be removed by the Lagrange dual theory. This procedure results in a change in the utility function. Thus the VaR constraint can be in fact viewed as a change over the original utility function. The utility function is different in the cases: $X(T)\ge\widetilde{X}$ and $X(T)<\widetilde{X}$. Therefore, the concept of VaR constraint is similar to the idea of prospect theory. If we judge $X(T)\ge\widetilde{X}$ by gains while $X(T)<\widetilde{X}$ by losses, the VaR constraint also leads to a different attitude towards gains and losses.

Proposition 3.6. *The optimal terminal wealth for the VaR constraint is*

$$X^{VaR}(T)=\begin{cases}I(yH(T)) & \text{if } H(T)\le\underline{H},\\ \widetilde{X} & \text{if } \underline{H}<H(T)\le\overline{H},\\ I(yH(T)) & \text{if } H(T)>\overline{H},\end{cases} \tag{3.20}$$

where $\underline{H}=U'(\widetilde{X})/y$, $I:[0,U'(0)]\to[0,+\infty]$ is the strictly decreasing inverse function of $U':[0,+\infty]\to[0,U'(0)]$. $\overline{H}$ solves $\mathbf{P}\big(H(T)>\overline{H}\big)=\alpha$. $y>0$ satisfies $\mathbf{E}[H(T)X^{VaR}(T)-\int_0^T H(s)C(s)ds]=X(0)$. In the case $\underline{H}\ge\overline{H}$, the VaR constraint is not binding.

Proof. See Appendix. □

We can see that, in the VaR risk management, the optimal wealth at retirement is divided into three parts. The optimal wealth equals the reference of VaR constraint when the pricing kernel is in the intermediate states. However, the optimal wealth follows the smooth utility function case when the pricing kernel is relatively high or low. The main difference between the optimal wealth with VaR constraint and no constraint lies in the middle states of the pricing kernel. The modification in the middle states ensures that the wealth satisfies the VaR constraint. However, in other states the optimal wealth is not modified in order to maximize the expectation of its utility. The above proposition also states that when $\underline{H} \geq \overline{H}$, the optimal wealth is the same as the case of no constraint. This is because the optimal wealth with no constraint naturally satisfies the VaR constraint in this case. Hence, no modification is made in the optimal wealth.

When $\alpha = 1$, the VaR constraint states nothing and $\overline{H} = 0$. So the optimal wealth is the same as the smooth utility case. When $\alpha = 0$, the terminal wealth is restricted to be higher than $\tilde{X}$. In this situation, $\overline{H} = +\infty$ and the optimal wealth is only divided into two parts. The higher states of the optimal wealth is revised to be equal to $\tilde{X}$.

As an example, we solve the VaR constraint with CRRA utility maximization problem for the utility function $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$, where $\gamma > 0$ represents the risk aversion of $U(x)$. So the inverse of $U'(x)$ is given by $I(x) = x^{-\frac{1}{\gamma}}$. We firstly derive the price of $X^{VaR}(T)$ at t.

Lemma 3.7. *The risk-neutral price of the random variable $X^{VaR}(t)$ is*

$$\left\{\begin{aligned} &\frac{1}{H(t)}\mathbf{E}\big[H(T)X^{VaR}(T)|\mathcal{F}_t\big] \\ &\quad = \tilde{X}\exp\Big[\frac{1}{2}\mathrm{Var}\{N_t\} + \mathbf{E}\{N_t\}\Big]\big[\Phi(d_1(\overline{H})) \\ &\quad - \Phi(d_1(\underline{H}))\big] + \left(\frac{1}{yH(t)}\right)^{\frac{1}{\gamma}}\exp\left(\frac{(1-\gamma)^2}{2\gamma^2}\mathrm{Var}\{N_t\}\right. \\ &\quad \left.+\frac{\gamma-1}{\gamma}\mathbf{E}\{N_t\}\right)[1+\Phi(d_2(\underline{H})) - \Phi(d_2(\overline{H}))], \end{aligned}\right. \tag{3.21}$$

where $\Phi(\cdot)$ represents the cumulative distribution function of a standard normal variable. $\mathbf{E}\{N_t\}$ and $\mathrm{Var}\{N_t\}$ are respectively defined in Lemma 3.2,

$$d_1(x) = \frac{\ln\left(\frac{x}{H(t)}\right) - \mathbf{E}\{N_t\} - \mathrm{Var}\{N_t\}}{\sqrt{\mathrm{Var}\{N_t\}}}$$

and

$$d_2(x) = \frac{\ln\left(\frac{x}{H(t)}\right) - \mathbf{E}\{N_t\} - \frac{\gamma-1}{\gamma}\mathrm{Var}\{N_t\}}{\sqrt{\mathrm{Var}\{N_t\}}}.$$

Proof. The proof is the same as in Lemma 3.2 and we omit it here. □

Since we do not change our contribution rate in this case, the value of the aggregated contribution from t to T can also be presented in Lemma 3.3. By the completeness of the market, we have the following proposition about the optimal wealth.

Proposition 3.8. *The optimal wealth for the pension manager at time $t (0 \leq t < T)$ with VaR constraint is*

$$\begin{aligned} X^{VaR}(t) = &\frac{1}{H(t)}\mathbf{E}[H(T)X^{VaR}(T)|\mathcal{F}_t] \\ &- \int_t^T \mathbf{E}\left[\frac{H(s)}{H(t)}C(s)|\mathcal{F}_t\right]\mathrm{d}s \end{aligned} \tag{3.22}$$

where $\frac{1}{H(t)}\mathbf{E}[H(T)X^{VaR}(T)|\mathcal{F}_t]$ and $D(t,s) = \mathbf{E}\big[\frac{H(s)}{H(t)}C(s)|\mathcal{F}_t\big]$ are calculated in Lemmas 3.7 and 3.3, respectively.

Denote $G^{VaR}(t, r(t), H(t)) = \frac{1}{H(t)}\mathbf{E}[H(T)X^{VaR}(T)|\mathcal{F}_t]$. We can derive the optimal investment strategies similar to the work for loss aversion. We present a counterpart of Proposition 3.5.

Proposition 3.9. *The optimal money invested in the bond and stock under VaR constraint is*

$$\left\{\begin{aligned} u_B^{VaR}(t) = &-\frac{\partial G^{VaR}(t,r(t),H(t))}{\partial r(t)}\frac{1}{A(t,t+K)} \\ &-\lambda_r\frac{\partial G^{VaR}(t,r(t),H(t))}{\partial H(t)}\frac{H(t)}{\sigma_r A(t,T)} \\ &-\frac{F_1(t)}{h(K)} + \lambda_S\frac{\partial G^{VaR}(t,r(t),H(t))}{\partial H(t)} \\ &\times\frac{\sigma_1 H(t)}{\sigma_2\sigma_r A(t,T)} + \frac{\sigma_1 F_2(t)}{\sigma_2\sigma_r A(t,T)}, \\ u_S^{VaR}(t) = &-\lambda_S\frac{\partial G^{VaR}(t,r(t),H(t))}{\partial H(t)}\frac{H(t)}{\sigma_2} - \frac{F_2(t)}{\sigma_2}, \end{aligned}\right. \tag{3.23}$$

where

$$\begin{aligned} &\frac{\partial G^{VaR}(t,r(t),H(t))}{\partial H(t)} \\ &= \frac{\tilde{X}}{H(t)\sqrt{\mathrm{Var}\{N_t\}}}\exp\left(\frac{1}{2}\mathrm{Var}\{N_t\} + \mathbf{E}\{N_t\}\right) \\ &\times[\varphi(d_1(\underline{H})) - \varphi(d_1(\overline{H}))] - \frac{1}{\gamma H(t)}\left(\frac{1}{yH(t)}\right)^{\frac{1}{\gamma}} \\ &\times\exp\left(\frac{(1-\gamma)^2}{2\gamma^2}\mathrm{Var}\{N_t\} + \frac{\gamma-1}{\gamma}\mathbf{E}\{N_t\}\right) \\ &\times[1+\Phi(d_2(\underline{H})) - \Phi(d_2(\overline{H}))] - \frac{1}{H(t)\sqrt{\mathrm{Var}\{N_t\}}} \\ &\times\left(\frac{1}{yH(t)}\right)^{\frac{1}{\gamma}}\exp\left(\frac{(1-\gamma)^2}{2\gamma^2}\mathrm{Var}\{N_t\} + \frac{\gamma-1}{\gamma}\mathbf{E}\{N_t\}\right) \\ &\times[\varphi(d_2(\underline{H})) - \varphi(d_2(\overline{H}))] \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} &\frac{\partial G^{VaR}(t,r(t),H(t))}{\partial r(t)} \\ &= -\tilde{X}A(t,T)\exp\left(\frac{1}{2}\mathrm{Var}\{N_t\} + \mathbf{E}\{N_t\}\right) \\ &\times[\Phi(d_1(\overline{H})) - \Phi(d_1(\underline{H}))] + \tilde{X}\frac{A(t,T)}{\sqrt{\mathrm{Var}\{N_t\}}} \\ &\times\exp\left(\frac{1}{2}\mathrm{Var}\{N_t\} + \mathbf{E}\{N_t\}\right)[\varphi(d_1(\overline{H})) - \varphi(d_1(\underline{H}))] \\ &-\frac{(\gamma-1)A(t,T)}{\gamma}\left(\frac{1}{yH(t)}\right)^{\frac{1}{\gamma}}\exp\left(\frac{(1-\gamma)^2}{2\gamma^2}\mathrm{Var}\{N_t\}\right. \\ &\left.+\frac{\gamma-1}{\gamma}\mathbf{E}\{N_t\}\right)[1+\Phi(d_2(\underline{H})) - \Phi(d_2(\overline{H}))] \\ &+\frac{A(t,T)}{\sqrt{\mathrm{Var}\{N_t\}}}\left(\frac{1}{yH(t)}\right)^{\frac{1}{\gamma}}\exp\left(\frac{(1-\gamma)^2}{2\gamma^2}\mathrm{Var}\{N_t\}\right. \\ &\left.+\frac{\gamma-1}{\gamma}\mathbf{E}\{N_t\}\right)[\varphi(d_2(\underline{H})) - \varphi(d_2(\overline{H}))], \end{aligned} \tag{3.25}$$

where $\varphi(\cdot)$ denotes the standard normal density function $\varphi(x) = \exp(-\frac{x^2}{2})$.

Proof. The proof is the same as Proposition 3.5, we omit here. □

4. Sensitivity analysis

In this section we present a sensitivity analysis to explore the economic behavior of the optimal strategies. Since the optimal

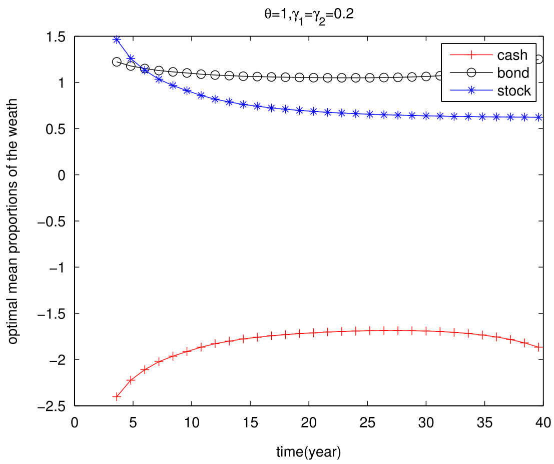

Fig. 2. $\theta = 1, \gamma_1 = \gamma_2 = 0.2$.

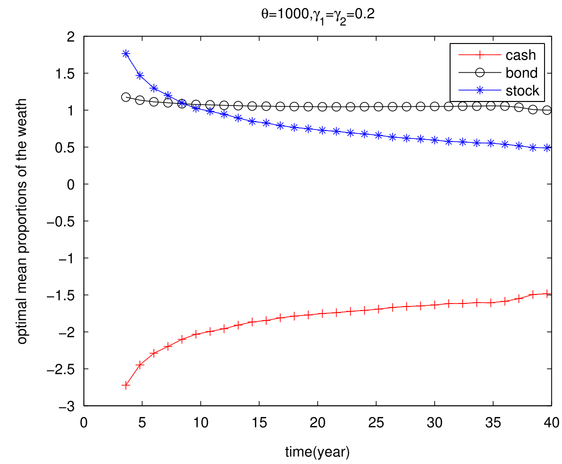

Fig. 3. $\theta = 1000, \gamma_1 = \gamma_2 = 0.2$.

strategies or the optimal proportions are stochastic, we apply the Monte Carlo Methods(MCM) to find the mean optimal proportions of cash, bond and stock. The analyses of optimal investment strategies under loss aversion and VaR constraints are showed, respectively. Since the contribution rate is usually fixed in the initial time, we consider a deterministic contribution rate here.

4.1. Optimal strategies under loss aversion

We show the economic analysis of the optimal proportions of assets under loss aversion in this section. Unless otherwise stated, the parameters we used are as follows: $a = 0.2$; $b = 0.02$; $\sigma_r = 0.02$; $r_0 = 0.04$; $A = 2.25$; $B = 1$; $K = 20$; $T = 40$; $\lambda_r = 0.15$; $\lambda_S = 0.2$; $\sigma_1 = 0.2$; $\sigma_2 = 0.4$; $C_0 = 0.15$; $\mu = 0.02$; $\sigma_{C_1} = \sigma_{C_2} = 0$; $X_0 = 1$; $\theta = 200$. Since the impacts of the economic parameters on the optimal strategies have been studied in many literatures, see Deelstra et al. (2003), He and Liang (2009) and etc., we mainly investigate the influence of the loss aversion on the optimal strategies. Different loss aversion functions are corresponding to different people. This section in fact describes the optimal strategies for different people. Figs. 2–5 show the optimal investment strategies under different loss aversion function. The reference point for Fig. 2 is $\theta = 1$. Since the pension manager is endowed with an initial wealth 1 and also receives a continuous contribution, the wealth of the pension account is surely to be higher than 1. Thus, the pension manager is risk averse during the accumulation phase and acts as a general CRRA utility maximization manager. Besides, since the risk aversion $\gamma_1 = 0.2$ is small, the pension manager is less risk averse towards the wealth. Thus, a high proportion of money is allocated in the stock. As is shown in Fig. 2, the pension manager puts about 70% of money in the stock during the accumulation phase. The proportion in the stock starts at 150% and decrease slowly along the time horizon. The optimal proportion in the bond stays stably at about 110%. In order to maximize the loss aversion utility of terminal wealth, the pension manager always keep a short position in the cash, which increases from −250% to −170% slowly.

Fig. 3 describes an investor who could be risk averse or risk seeking towards the terminal wealth. The pension manager is not sure to obtain a wealth higher than $\theta = 1000$. Moreover, since the reference point is not enough high, the pension manager has access to achieve a terminal wealth higher than 1000 at some states. Therefore, the manager has a complex behavior towards the terminal wealth. The pension manager acts differently especially at initial time and near retirement. Since the risk is big at initial time, the risk averse property leads to a higher proportion of money in the stock at beginning, i.e., about 180%. However, the money in the

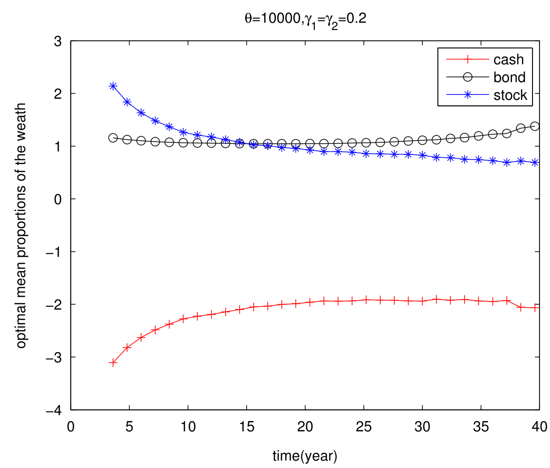

Fig. 4. $\theta = 10\,000, \gamma_1 = \gamma_2 = 0.2$.

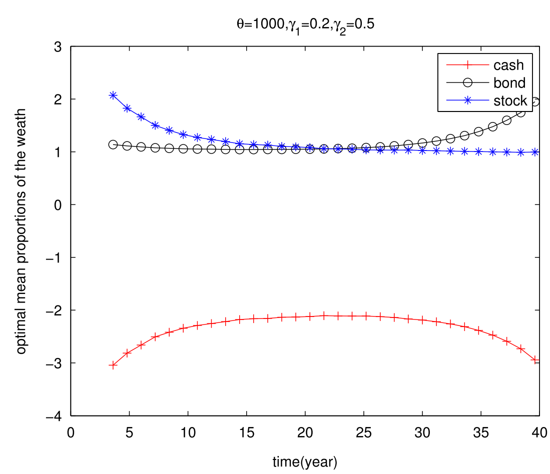

Fig. 5. $\theta = 1000, \gamma_1 = 0.2, \gamma_2 = 0.5$.

account increases and thus the proportion in the stock decreases. Near the retirement time, the pension manager achieves a wealth higher than 1000 at some states and thus allocates fewer in the stock compared with Fig. 2. The proportion in the bond also moves around about 100% in this case.

The optimal investment strategies for a risk averse manager is illustrated in Fig. 4. The reference point in Fig. 4 is $\theta = 10\,000$, which is hardly achieved by the pension manager. So the pension manager is nearly a risk averse investor. Therefore, the pension

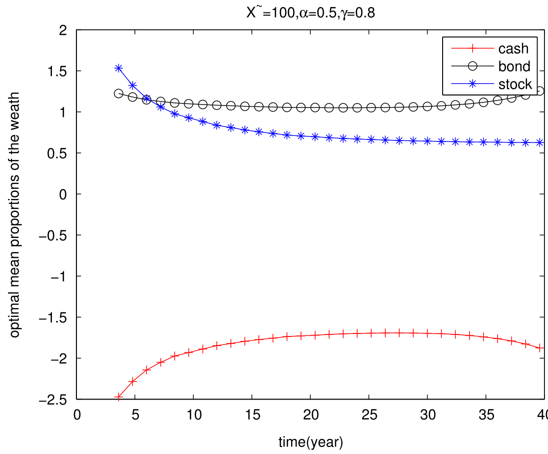

Fig. 6. $\tilde{X} = 100, \alpha = 0.5, \gamma = 0.8$.

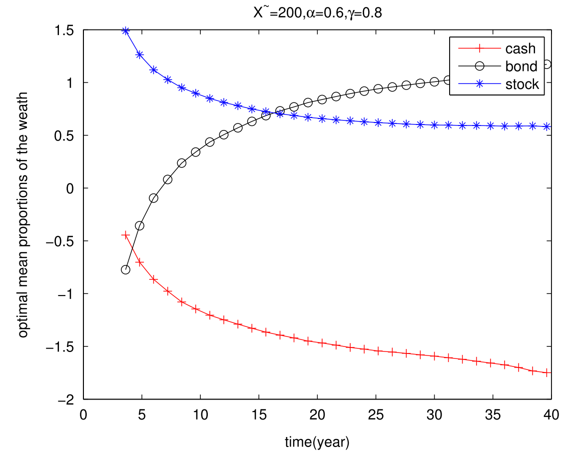

Fig. 8. $\tilde{X} = 200, \alpha = 0.6, \gamma = 0.8$.

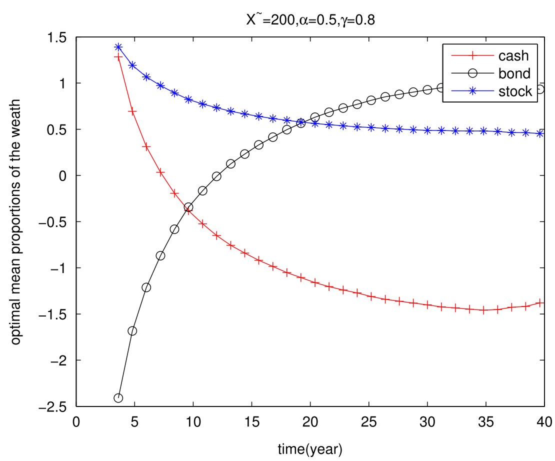

Fig. 7. $\tilde{X} = 200, \alpha = 0.5, \gamma = 0.8$.

manager invests a massive proportion of money in the stock, which decreases from 200% to about 100% at retirement time. The proportion of money in the cash starts at about −300% and increases to stay at −200% after time 15. Besides, the proportion money allocated in the bond is also about 100% during the accumulation phase. However, the proportion of money in the bond has an increasing tendency near the retirement time.

We are also cared about the influence of risk aversion γ_2 on the optimal investment strategies. The case when $\gamma_2 = 0.5$ is shown in Fig. 5. Compared with Fig. 3, γ_2 increases and thus the manager is less risk averse in this case. Therefore, the manager maintains a higher proportion of money in the stock, which decreases from 200% to about 100% at retirement. However, the proportion of money varies on the opposite direction. The proportion of money increases firstly and stays at about −220% from time 10 to time 30.

4.2. Optimal strategies under value-at-risk constraints

We study the optimal investment strategies of DC pension fund under VaR constraints in this section. The economic parameters we adopt are the same as in the previous section. We are interested in the influence of the VaR constraint on the optimal strategies. So the sensitivity analysis mainly studies the optimal investment strategies under different VaR constraints.

The case when $\tilde{X} = 100, \alpha = 0.5, \gamma = 0.8$ is illustrated in Fig. 6. The terminal wealth is required to be higher than 100 with probability 0.5. A simple calculation can show that $\underline{H} > \overline{H}$ and thus the VaR constraint is not binding. Therefore, the optimal investment strategies coincide with the optimal strategies with no constraint. Since Fig. 2 presents the behavior of a risk averse investor, we can see that Fig. 6 is almost the same as Fig. 2 under loss aversion. The pension manager invests about 100% in the bond and a decreasing proportion from 150% to 60% in the stock. The rest of money is invested in the cash.

Fig. 7 shows the optimal portfolios with constraint $\mathbf{P}\big(X(T) \geq 200\big) \geq 0.5$. Although most literature fixes the level α at a small number, we choose $\alpha = 0.5$ in order to investigate the impacts of VaR constraint better. In this case, the VaR constraint is binding. Thus a modification is made on the terminal wealth to satisfy the VaR constraint. The VaR constraint indeed leads to a preference over risk assets. As is shown in Fig. 7, the pension manager invest fewer in the stock compared with Fig. 6. However, the pension manager invests heavily in the cash, which is about 130% at initial time. The proportion of money invested in cash then decreases fast to about −150% at retirement. Besides, the manager holds a short position of money in the bond before time 12. The money in the bond increases during the accumulation phase. We can conclude that even the manager purchases less stock, he holds more cash and less bond. Since cash is in fact risk, the manager buy more risk assets in order to satisfy the VaR constraint.

We discuss the influence of α by comparing Figs. 8 and 7. Fig. 8 binds the terminal wealth with lower probability to be higher than 200. Thus the manager can achieve the VaR constraint easier. So the pension manager invests a lower proportion of money in the risk assets. As is shown in Fig. 8, the proportion of money in the bonds starts at about −80% and increases to about 100%. Besides, the proportion of money in the stock is almost the same as in Fig. 6. Moreover, in this case, the manager always holds a short position in the cash, which has a negative relationship with time.

Fig. 9 presents the optimal investment strategies with risk aversion $\gamma = 4$. A higher risk aversion leads to a preference over risk-less asset. However, the VaR constraint is hardly satisfied by singly investing more in the risk-less asset. So the risk aversion has a complex effect on the optimal strategies. Comparing Fig. 9 with Fig. 7, the pension manager purchases more risk assets at initial time and less risk assets at retirement. The proportion of money in the stock is less than Fig. 7 all the time. However, the manager shorts about −300% of bond in the beginning of the pension fund and gradually increases the proportion of money in the bond. The proportion of bond increases to about 50% at retirement. Besides, the proportion of money in the cash starts with a high proportion of 200% and decreases to −100% in the end. We can see that a higher risk aversion leads to a higher proportions in the risk assets. This behavior ensures the VaR constraint at retirement. However, the impact of risk aversion arises near the retirement time. The manager invests less in the risk assets at retirement.

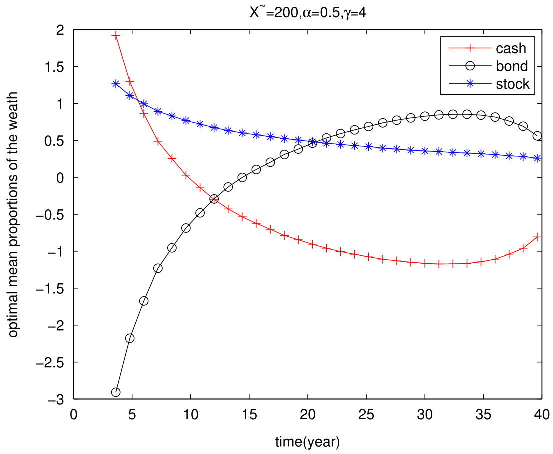

Fig. 9. $\tilde{X} = 200, \alpha = 0.5, \gamma = 4$.

5. Conclusion

In this paper, we consider the risk management for DC pension plan under two different optimization criterions. The pension manager can invest in the cash, bond and stock to avoid risk. The interest rate and the contribution rate in our model are stochastic. So, the goal of the pension plan manager is to find the best allocations in the assets in the financial market. In general, stochastic programming method and the martingale method can be applied in the work to maximize the expectation of a smooth utility function of terminal wealth. However, in our work, only martingale method is suitable since the optimization problem is not strictly concave.

Based on the technique of martingale method, our work presents the optimal terminal wealth and optimal investment strategies under loss aversion and VaR constraints. The optimal terminal wealths are piecewise smooth functions determined by the pricing kernel in the financial market. We can see that in most states of the pricing kernel, the optimal wealths are the same as the smooth utility function case. However, since the terminal wealth under loss aversion is risk-seeking towards losses, the optimal wealth is modified in some states. So the optimal investment strategies under loss aversion may allocate fewer money in the stock. The sensitivity analysis in the end also shows that a risk averse investor invests more in the stock. However, the loss aversion investor holds less risk assets at retirement. In the case of VaR constraints, the optimal terminal wealth is also modified in some states to satisfy the VaR constraint. Particularly when the optimal wealth without VaR constraint satisfies the VaR constraint, the VaR constraint is not binding. Since the VaR constraint manager expects a large probability of terminal wealth over a level, the manager may allocate more in the stock and thus suffer larger losses. Therefore, these two different criterions can characterize the human behavior better and provide more efficient strategies for the pension manager.

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Appendix

In this section we present the proofs of the propositions and lemmas in the previous text.

Proof of Proposition 3.1. Problem (3.4) can be solved by the lagrange dual theory. Firstly we define the Lagrangian of problem (3.4) as follows:

$$\mathcal{L}(X(T),\lambda) = \mathbf{E}[U(X(T))] - \lambda\mathbf{E}[H(T)X(T)] + \lambda X(0) + \lambda\mathbf{E}\left[\int_0^T H(s)C(s)\mathrm{d}s\right]. \tag{A.1}$$

Thus, the original problem (3.4) is equivalent to the following problem:

$$\begin{cases} \inf\limits_{\lambda\geq 0}\sup\limits_{X(T)} \mathcal{L}(X(T),\lambda), \\ X(T)\geq 0. \end{cases} \tag{A.2}$$

So the procedure to solve problem (3.4) can be divided into two parts. Firstly, for fixed lagrange multiplier λ, we find the optimal terminal wealth $X^{*,\lambda}(T)$ to maximize $\mathcal{L}(X(T),\lambda)$, i.e., solving the following problem.

$$\begin{cases} \sup\limits_{X(T)} \mathcal{L}(X(T),\lambda), \\ X(T)\geq 0. \end{cases} \tag{A.3}$$

The above problem is a pointwise optimization problem and can be solved explicitly. Then, we can obtain an optimization problem w.r.t. λ:

$$\inf_{\lambda\geq 0} \mathcal{L}(X^{*,\lambda}(T),\lambda). \tag{A.4}$$

In fact, we do not need to solve problem (A.4) directly, we only need the complementary slackness condition in dual theory. Denote the optimal solution of problem (A.4) by λ^*. By the Lagrange dual theory, the optimal expectation of the utility of terminal wealth in problem (3.4) is given by $\mathcal{L}(X^{*,\lambda^*}(T),\lambda^*)$ and the optimal terminal wealth is given by $X^{*,\lambda^*}(T)$.

So, the original problem (3.4) can be well solved by introducing problem (A.3) and problem (A.4). Next, we try to find the optimal terminal wealth for problem (A.3). Since problem (A.3) is an optimization problem w.r.t. $X(T)$, we can omit the part not related with $X(T)$ in problem (A.3) and obtain the following simplified problem with fixed λ:

$$\begin{cases} \max\limits_{X(T)}\left\{\mathbf{E}[U(X(T))] - \lambda\mathbf{E}[H(T)X(T)]\right\}, \\ X(T)\geq 0. \end{cases} \tag{A.5}$$

Denote $U_1(x) = -A(\theta - x)^{\gamma_1}$ and $U_2(x) = B(x-\theta)^{\gamma_2}$. We first get the pointwise optimizer in the lagrange dual problem. When $U(x) = U_2(x)$, the optimization problem (A.5) is a concave optimization problem and the optimizer is derived by the first order condition as the following expression:

$$X_2^{*,\lambda} = \theta + \left(\frac{B\gamma_2}{\lambda H(T)}\right)^{\frac{1}{1-\gamma_2}}.$$

When $U(x) = U_1(x)$, the problem (A.5) is a convex maximization problem and the optimizer is on the boundaries $X_1^{*,\lambda} = 0$ or θ.

Next we point out whether $X^{*,\lambda}(T) = X_1^{*,\lambda}$ or $X(T) = X_2^{*,\lambda}$. If $f(H(T)) = U(X_2^{*,\lambda}) - \lambda H(T)X_2^{*,\lambda} - [U(X_1^{*,\lambda}) - \lambda H(T)X_1^{*,\lambda}] > 0$, then $X^{*,\lambda}(T) = X_2^{*,\lambda}$, otherwise $X^{*,\lambda}(T) = X_1^{*,\lambda}$.

When $X_1^{*,\lambda} = \theta$, $f(H(T)) = B(1-\gamma_2)(\frac{B\gamma_2}{\lambda H(T)})^{\frac{\gamma_2}{1-\gamma_2}} > 0$. So θ can never be the global optimizer. When $X_1^{*,\lambda} = 0$, $f(H(T)) = B(1-\gamma_2)(\frac{B\gamma_2}{\lambda H(T)})^{\frac{\gamma_2}{1-\gamma_2}} + A\theta^{\gamma_1} - \lambda\theta H(T)$.

A simple calculation can show that $f(H(T)) > 0$ in the case $H(T) < \frac{A}{\lambda}\theta^{\gamma_1-1}$. Moreover, $\lim_{H(T)\to+\infty} f(H(T)) = -\infty$ and $f'(H(T)) > 0$, thus $f(H(T))$ admits an unique root on $(\frac{A}{\lambda}\theta^{\gamma_1-1}, +\infty)$. Denote the root of $f(H(T))$ by $\bar{H}$. Summarizing the above analysis we have

$$f(H(T)) \le 0, \qquad H(T) \ge \bar{H},$$
$$f(H(T)) > 0, \qquad H(T) < \bar{H}.$$

So the global optimizer of problem (A.5) can be written as

$$X^{*,\lambda}(T) = \begin{cases} \theta + \left(\dfrac{\lambda H(T)}{B\gamma_2}\right)^{\frac{1}{\gamma_2-1}} & \text{if } H(T) < \bar{H}, \\ 0 & \text{if } H(T) \ge \bar{H}. \end{cases} \tag{A.6}$$

Thus, $X^{*,\lambda}$ also solves problem (A.3). Next we derive the optimal λ^* to solve problem (A.4). We introduce the complementary slackness condition here. The optimal pair $(X^{*,\lambda *}(T), \lambda^*)$ satisfies the following complementary slackness condition in dual theory:

$$\lambda^* \mathbf{E}\left[\left(\theta + \left(\frac{\lambda^* H(T)}{B\gamma_2}\right)^{\frac{1}{\gamma_2-1}}\right) 1_{\{H(T)<\bar{H}\}} - \int_0^T H(s)C(s)\mathrm{d}s\right] = \lambda^* X(0). \tag{A.7}$$

i. If $\lambda^* = 0$, i.e. the budget constraint is not binding.

Since $0 < \gamma_2 < 1$, we can observe from Eq. (A.6) that $X^{*,\lambda^*}(T) = +\infty$ and $\bar{H} = +\infty$. So this case is not realistic and we reject it. In fact, we can also observe from problem (A.3) that when $\lambda = 0$, we intend to maximize $U(X(T))$. However, the utility function is strictly increasing and we do not have a maximize point.

ii. If $\lambda^* > 0$, i.e., the budget constraint is binding.

In this case, the optimal pair $(X^{*,\lambda *}(T), \lambda^*)$ satisfies the budget condition:

$$\mathbf{E}\left[\left(\theta + \left(\frac{\lambda^* H(T)}{B\gamma_2}\right)^{\frac{1}{\gamma_2-1}}\right) 1_{\{H(T)<\bar{H}\}} - \int_0^T H(s)C(s)\mathrm{d}s\right] = X(0). \tag{A.8}$$

The optimal lagrange multiplier is solved by Eq. (A.8). Thus the optimal lagrange multiplier is given by λ^* and the optimal terminal wealth is given by $X^{*,\lambda^*}(T)$. Next we show that $X^{*,\lambda^*}(T)$ indeed solves the original optimization problem (3.4). Choose an arbitrary random variable $X(T)$ satisfying the budget constraint in problem (3.4).

$$\begin{aligned} &\mathbf{E}[U(X^{*,\lambda^*}(T))] - \mathbf{E}[U(X(T))] \\ &= \mathbf{E}[U(X^{*,\lambda^*}(T))] - \mathbf{E}[U(X(T))] - \lambda^* X(0) \\ &\quad - \lambda^* \mathbf{E}\left[\int_0^T H(s)C(s)\mathrm{d}s\right] \\ &\quad + \lambda^* X(0) + \lambda^* \mathbf{E}\left[\int_0^T H(s)C(s)\mathrm{d}s\right] \\ &\ge \mathbf{E}[U(X^{*,\lambda^*}(T))] - \mathbf{E}[U(X(T))] \\ &\quad - \lambda^* \mathbf{E}[H(T)X^{*,\lambda^*}(T)] + \lambda^* \mathbf{E}[H(T)X(T)] \\ &\ge 0. \end{aligned} \tag{A.9}$$

The first inequality holds since the budget constraint holds with equality for $X^{*,\lambda^*}(T)$ while with inequality for $X(T)$. The second inequality holds since $X^{*,\lambda^*}(T)$ is the global optimizer for problem (A.5) when the lagrange multiplier $\lambda = \lambda^*$. So $X^{*,\lambda^*}(T)$ indeed solves problem (3.4). □

Proof of Lemma 3.2. The optimal wealth at retirement is shown in Eq. (3.5). Since the market is complete, the wealth at t can be derived based on the pricing kernel. The price of $X^{*,\lambda^*}(T)$ at t is calculated as follows.

$$\begin{aligned} &\frac{1}{H(t)}\mathbf{E}[H(T)X^{*,\lambda^*}(T)|\mathcal{F}_t] \\ &= \frac{1}{H(t)}\mathbf{E}[H(T)\theta 1_{\{H(T)<\bar{H}\}}|\mathcal{F}_t] \\ &\quad + \frac{1}{H(t)}\mathbf{E}\left[H(T)\left(\frac{\lambda^* H(T)}{B\gamma_2}\right)^{\frac{1}{\gamma_2-1}} 1_{\{H(T)<\bar{H}\}}|\mathcal{F}_t\right]. \end{aligned} \tag{A.10}$$

In order to calculate $\frac{1}{H(t)}\mathbf{E}[H(T)X^{*,\lambda^*}(T)|\mathcal{F}_t]$, we need to reveal the properties of the interest rate $r(t)$. We have the following lemma.

Lemma A.1. *The stochastic interest rate $r(t)$ satisfies the following equations:*

$$r(t) = (r_0 - b)\exp(-at) + b - \sigma_r \exp(-at) \times \int_0^t \exp(as)\mathrm{d}W_r(s) \tag{A.11}$$

and

$$\int_t^T r(s)\mathrm{d}s = (r(t) - b)\frac{1-\exp(-a(T-t))}{a} + b(T-t) - \int_t^T \sigma_r A(s,T)\mathrm{d}W_r(s). \tag{A.12}$$

Moreover, $\int_t^T r(s)\mathrm{d}s$ is a normal distributed random variable, i.e., $\int_t^T r(s)\mathrm{d}s \sim N\big((r(t)-b)\frac{1-\exp(-a(T-t))}{a} + b(T-t),\ \frac{\sigma_r^2}{a^2}[(T-t) + \frac{2\exp(-a(T-t))}{a} - \frac{\exp(-2a(T-t))}{2a} - \frac{3}{2a}]\big)$.

Proof. We can easily find that the solution of the Ornstein–Uhlenbeck process is the form of the first formula. In order to derive the second formula, we have

$$\begin{aligned} \int_t^T r(s)\mathrm{d}(s) &= \int_t^T \Big[(r(t)-b)\exp(-a(s-t)) + b \\ &\quad - \sigma_r \exp(-as)\int_t^s \exp(au)\mathrm{d}W_r(u)\Big]\mathrm{d}s \\ &= (r(t)-b)\frac{1-\exp(-a(T-t))}{a} + b(T-t) \\ &\quad - \sigma_r \int_t^T \exp(-as)\int_t^s \exp(au)\mathrm{d}W_r(u)\mathrm{d}s \\ &= (r(t)-b)\frac{1-\exp(-a(T-t))}{a} + b(T-t) \\ &\quad - \sigma_r \int_0^t \frac{1-\exp(-a(T-s))}{a}\mathrm{d}W_r(s) \\ &= (r(t)-b)\frac{1-\exp(-a(T-t))}{a} + b(T-t) \\ &\quad - \int_t^T \sigma_r A(s,T)\mathrm{d}W_r(s). \end{aligned} \tag{A.13}$$

And the distribution of $\int_t^T r(s)\mathrm{d}s$ follows. □

By Eq. (3.3) for $H(t)$ we have

$$H(T) = H(t)\exp(N_t), \tag{A.14}$$

where $N_t = -\int_t^T r(s)\mathrm{d}s - \frac{1}{2}\lambda_r^2(T-t) - \frac{1}{2}\lambda_S^2(T-t) - \lambda_r[W_r(T) - W_r(t)] - \lambda_S[W_S(T) - W_S(t)]$. We also present a lemma to characterize N_t.

Lemma A.2. *N_t is normally distributed random variable, and the expectation and variance of N_t are, respectively,*

$$\begin{aligned}\mathbf{E}\{N_t\} &= -(r(t)-b)\frac{1-\exp(-a(T-t))}{a} \\ &\quad - b(T-t) - \frac{1}{2}(\lambda_r^2+\lambda_S^2)(T-t), \\ \mathrm{Var}\{N_t\} &= \frac{\sigma_r^2}{a^2}\left[(T-t)+\frac{2\exp(-a(T-t))}{a}\right. \\ &\quad \left. -\frac{\exp(-2a(T-t))}{2a}-\frac{3}{2a}\right]+(\lambda_r^2+\lambda_S^2)(T-t) \\ &\quad -2\frac{\lambda_r}{a}(\sigma_r(T-t)-\sigma_r A(t,T)).\end{aligned} \tag{A.15}$$

Proof. Firstly,

$$\begin{aligned}&\mathrm{Cov}\left(\int_t^T r(s)\mathrm{d}s, W_r(T)-W_r(t)\right) \\ &= \mathbf{E}\left[\int_t^T r(s)\mathrm{d}s[W_r(T)-W_r(t)]\right] \\ &= -\sigma_r\mathbf{E}\left[\int_t^T \frac{1-\exp(-a(T-s))}{a}\mathrm{d}W_r(s)\int_t^T \mathrm{d}W_r(s)\right] \\ &= -\sigma_r\int_t^T \frac{1-\exp(-a(T-s))}{a}\mathrm{d}s \\ &= -\frac{1}{a}(\sigma_r(T-t)-\sigma_r A(t,T)).\end{aligned} \tag{A.16}$$

The expectation of N_t can be easily obtained. By (A.16) we have

$$\begin{aligned}\mathrm{Var}\{N_t\} &= \mathrm{Var}\left(\int_t^T r(s)\mathrm{d}s\right)+\mathrm{Var}(\lambda_r[W_r(T)-W_r(s)] \\ &\quad +\lambda_S[W_S(T)-W_S(s)]) \\ &\quad +2\mathrm{Cov}\left(\int_t^T r(s)\mathrm{d}s, \lambda_r[W_r(T)-W_r(s)]\right) \\ &= \frac{\sigma_r^2}{a^2}\left[(T-t)+\frac{2\exp(-a(T-t))}{a}\right. \\ &\quad \left. -\frac{\exp(-2a(T-t))}{2a}-\frac{3}{2a}\right]+(\lambda_r^2+\lambda_S^2)(T-t) \\ &\quad -2\frac{\lambda_r}{a}(\sigma_r(T-t)-\sigma_r A(t,T)). \quad \square\end{aligned} \tag{A.17}$$

Thus

$$\begin{aligned}&\mathbf{E}[\theta H(T)1_{\{H(T)<\bar{H}\}}|\mathcal{F}_t] \\ &= \theta H(t)\mathbf{E}\left[\exp(N_t)1_{\left\{N_t<\ln\left(\frac{\bar{H}}{H(t)}\right)\right\}}\right] \\ &= \theta H(t)\int_{-\infty}^{\ln\left(\frac{\bar{H}}{H(t)}\right)}\exp\left[-\left(\frac{(x-\mathbf{E}\{N_t\})^2}{2\mathrm{Var}\{N_t\}}\right)\right]\frac{e^x}{\sqrt{2\pi\mathrm{Var}\{N_t\}}}\mathrm{d}x \\ &= \frac{\theta H(t)}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\ln\left(\frac{\bar{H}}{H(t)}\right)-\mathbf{E}\{N_t\}-\mathrm{Var}\{N_t\}}{\sqrt{\mathrm{Var}\{N_t\}}}}\exp\left(-\frac{y^2}{2}\right) \\ &\quad \times\exp\left(\mathbf{E}\{N_t\}+\frac{1}{2}\mathrm{Var}\{N_t\}\right)\mathrm{d}y \\ &= \theta H(t)\exp\left(\mathbf{E}\{N_t\}+\frac{1}{2}\mathrm{Var}\{N_t\}\right) \\ &\quad \times\Phi\left[\frac{\ln\left(\frac{\bar{H}}{H(t)}\right)-\mathbf{E}\{N_t\}-\mathrm{Var}\{N_t\}}{\sqrt{\mathrm{Var}\{N_t\}}}\right].\end{aligned} \tag{A.18}$$

The same statement can also show that

$$\begin{aligned}&\frac{1}{H(t)}\mathbf{E}\left[H(T)\left(\frac{\lambda^* H(T)}{B\gamma_2}\right)^{\frac{1}{\gamma_2-1}}1_{\{H(T)<\bar{H}\}}|\mathcal{F}_t\right] \\ &= \left(\frac{\lambda^* H(t)}{B\gamma_2}\right)^{\frac{1}{\gamma_2-1}}\exp\left(\frac{\gamma_2^2}{2(\gamma_2-1)^2}\mathrm{Var}\{N_t\}+\frac{\gamma_2}{\gamma_2-1}\mathbf{E}\{N_t\}\right) \\ &\quad \times\Phi(d_2(\bar{H})),\end{aligned} \tag{A.19}$$

where $d_2(\bar{H}) = \frac{\ln\left(\frac{\bar{H}}{H(t)}\right)-\mathbf{E}\{N_t\}-\frac{\gamma_2}{\gamma_2-1}\mathrm{Var}\{N_t\}}{\sqrt{\mathrm{Var}\{N_t\}}}$. $\square$

Proof of Lemma 3.3. We calculate the value of the continuous contribution at time t. The contribution rate follows a geometric Brownian motion and can be solved explicitly by

$$\begin{aligned}C(s) &= C(t)\exp\left[\left(\mu-\frac{1}{2}\sigma_{C_1}^2-\frac{1}{2}\sigma_{C_2}^2\right)(s-t)\right. \\ &\quad +\sigma_{C_1}(W_r(s)-W_r(t)) \\ &\quad \left.+\sigma_{C_2}(W_S(s)-W_S(t))\right], \quad \forall s\geq t.\end{aligned} \tag{A.20}$$

Therefore

$$\begin{aligned}&\mathbf{E}\left[\frac{H(s)}{H(t)}C(s)|\mathcal{F}_t\right] \\ &= \mathbf{E}\left[C(t)\exp\left(\left(\mu-\frac{1}{2}\sigma_{C_1}^2-\frac{1}{2}\sigma_{C_2}^2-\frac{1}{2}\lambda_r^2-\frac{1}{2}\lambda_S^2\right)\right.\right. \\ &\quad \left.\left.\times(s-t)\right)\exp(Q(t,s))\right]. \\ &= C(t)\exp\left(\left(\mu-\frac{1}{2}\sigma_{C_1}^2-\frac{1}{2}\sigma_{C_2}^2-\frac{1}{2}\lambda_r^2-\frac{1}{2}\lambda_S^2\right)\right. \\ &\quad \left.\times(s-t)\right)\exp\left(\mathbf{E}\{Q(t,s)\}+\frac{1}{2}\mathrm{Var}\{Q(t,s)\}\right),\end{aligned} \tag{A.21}$$

where $Q(t,s) = -\int_t^s r(u)\mathrm{d}u+(\sigma_{C_1}-\lambda_r)[W_r(s)-W_r(t)]+(\sigma_{C_2}-\lambda_S)[W_S(s)-W_S(t)]$.

$Q(t,s)$ is also a normally distributed random variable, and using the same procedure as in N_t, the expectation and variance of $Q(t,s)$ can be derived by

$$\begin{aligned}\mathbf{E}\{Q(t,s)\} &= -(r(t)-b)\frac{1-\exp(-a(s-t))}{a}-b(s-t), \\ \mathrm{Var}\{Q(s,t)\} &= \int_t^s \sigma_r^2 A(u,s)^2\mathrm{d}u+(\sigma_{C_1}-\lambda_r)^2(s-t) \\ &\quad +(\sigma_{C_2}-\lambda_S)^2(s-t) \\ &\quad +2(\sigma_{C_1}-\lambda_r)\int_t^s \sigma_r A(u,s)\mathrm{d}u.\end{aligned} \tag{A.22}$$

Denote $D(t,s) = \mathbf{E}[\frac{H(s)}{H(t)}C(s)|\mathcal{F}_t]$. $D(t,s)$ satisfies the following backward stochastic differential equation:

$$\begin{cases}\dfrac{\mathrm{d}D(t,s)}{D(t,s)} = r(t)\mathrm{d}t+(\sigma_{C_1}+\sigma_r A(t,s))[\lambda_r\mathrm{d}t+\mathrm{d}W_r(t)] \\ \quad +\sigma_{C_2}[\lambda_S\mathrm{d}t+\mathrm{d}W_S(t)], \\ D(s,s) = C(s), \quad s\geq t.\end{cases} \tag{A.23}$$

Since $D(t,s)$ is solved explicitly, the explicit form of $F(t,T)$ can be obtained by integrating w.r.t. $D(t,s)$. $\square$

Proof of Proposition 3.6. The case where $\overline{H} \leq \underline{H}$ implies that $\mathbf{P}(I(yH(T)) \geq \tilde{X}) = P(H(T) \leq \underline{H}) \geq 1-\alpha$. So the optimal wealth without constraint naturally satisfies the VaR constraint and also maximizes the problem with VaR constraint. In other words, the VaR constraint is not binding in this case.

We require $\overline{H} > \underline{H}$ in the following paragraph.

Problem (3.19) is an optimization problem with wealth constraints. So, similar to problem (3.4), we can solve problem (3.19) by the Lagrange dual theory. The Lagrangian of problem (3.19) is:

$$\begin{aligned}\mathcal{L}(X(T), y, y_2) = {} & \mathbf{E}[U(X(T))] - y\mathbf{E}[H(T)X(T)] + yX(0) \\ & + y\mathbf{E}\left[\int_0^T H(s)C(s)\mathrm{d}s\right] \\ & + \mathbf{E}[y_2 1_{\{X(T)>\tilde{X}\}} + y_2(\alpha-1)]. \end{aligned} \tag{A.24}$$

Similar to problem (3.4), we can maximize the Lagrangian (A.24) for fixed lagrange multipliers y, y_2. We can obtain the optimal wealth $X^{VaR,y,y_2}(T)$ w.r.t. the fixed y, y_2. Thus we introduce the following problem:

$$\begin{cases} \max\limits_{X(T)} \mathcal{L}(X(T), y, y_2), \\ X(T) > 0. \end{cases} \tag{A.25}$$

Problem (A.25) is an optimization problem w.r.t. $X(T)$ and y, y_2 are fixed. So we can omit the parts not related with $X(T)$ and obtain the following optimization problem:

$$\max_X \mathbf{E}[U(X) - yH(T)X + y_2 1_{\{X\geq\tilde{X}\}}]. \tag{A.26}$$

After solving problem, we can obtain X^{VaR,y,y_2}. Then similar to problem (3.4), we can use the complementary slackness conditions to derive the optimal pairs (y^*, y_2^*). Therefore, X^{VaR,y^*,y_2^*} solves the original problem (3.19).

The procedure to solve the lagrange dual problem is a little computational and we omit the computational process here. We just prove that the solution (3.20) we obtain solves the original optimization problem (3.19). In the following paragraph, we present the proof.

We disentangle the VaR constraint problem by the following steps. Firstly, we show that Eq. (3.20) solves the following problem:

$$\max_X \{U(X) - y^*H(T)X + y_2^* 1_{\{X\geq\tilde{X}\}}\}, \tag{A.27}$$

where $y_2^* = U(I(y^*\overline{H})) - y^*\overline{H}I(y^*\overline{H}) + y^*\tilde{X}\overline{H} - U(\tilde{X}) \geq 0$ and $y^* \geq 0$ are the lagrange multipliers w.r.t. the VaR constraint and the budget constraint, respectively. The above problem is equivalent to problem (A.26). So we are in fact proving that X^{VaR,y^*,y_2^*} maximizes $\mathcal{L}(X(T), y^*, y_2^*)$ for fixed lagrange multipliers y^*, y_2^*. As explained before, we only need to prove that X^{VaR,y^*,y_2^*} is the maximize point of $\mathcal{L}(X(T), y^*, y_2^*)$ for fixed lagrange multipliers y^*, y_2^*. So the terms not related with $X(T)$ in $\mathcal{L}(X(T), y^*, y_2^*)$ can be omitted. The terms not related with $X(T)$ only influences the values of the lagrange multipliers.

In fact, the global optimizer of problem (A.27) is $I(yH(T))$ or $\tilde{X}$. $\overline{H}$ and $\underline{H}$ are defined in Proposition 3.6. We investigate two cases: $X^* = I(y^*H(T))$ or $X^* = \tilde{X}$.

(i) $H(T) \leq \underline{H}$. In this case, $I(yH(T)) \geq \tilde{X}$ since $I(\cdot)$ is a strictly decreasing function. This results in

$$\begin{aligned} & U(I(y^*H(T))) - y^*H(T)I(y^*H(T)) + y_2^* \\ & \quad \geq U(\tilde{X}) - y^*H(T)\tilde{X} + y_2^*. \end{aligned}$$

The above inequality follows from that $U(x) - y^*H(T)I(x)$ increases with x. So $X^{VaR,y^*,y_2^*} = I(y^*H(T))$.

(ii) $\overline{H} \geq H(T) > \underline{H}$. We have $I(y^*H(T)) < \tilde{X}$. So,

$$\begin{aligned} & U(\tilde{X}) - y^*H(T)\tilde{X} + y_2^* \\ & \quad = U(I(y^*\overline{H})) - y^*\overline{H}I(y^*\overline{H}) + y^*\tilde{X}(\overline{H} - H(T)) \\ & \quad \geq U(I(y^*H(T))) - y^*H(T)I(y^*H(T)). \end{aligned}$$

The first equality holds by substituting the form of y_2^* in it. In fact, we have $\frac{\partial}{\partial x}\{U(I(y^*x)) - y^*xI(y^*x) + y^*\tilde{X}x\} = y^*[\tilde{X} - I(y^*x)] > 0$ for $x > \underline{H}$. Since $\overline{H} \geq H(T)$, $\overline{H}$ maximizes $\{U(I(y^*x)) - y^*xI(y^*x) + y^*\tilde{X}x\}$ and the inequality holds. Thus, $X^{VaR} = \tilde{X}$. Besides, y_2^* can be rewritten as

$$\begin{aligned} y_2^* = {} & [U(I(y^*\overline{H})) - y\overline{H}I(y^*\overline{H}) + y^*\tilde{X}\overline{H}] \\ & - [U(I(y^*\underline{H})) - y^*\underline{H}I(y^*\underline{H}) + y^*\tilde{X}\underline{H}]. \end{aligned}$$

Since we require $\overline{H} > \underline{H}$ in this case, $y_2^* > 0$ is satisfied.

(iii) $H(T) > \underline{H}$.

$$\begin{aligned} & U(\tilde{X}) - y^*H(T)\tilde{X} + y_2^* \\ & \quad = U(I(y^*\overline{H})) - y^*\overline{H}I(y^*\overline{H}) + y^*\tilde{X}(\overline{H} - H(T)) \\ & \quad < U(I(y^*H(T))) - y^*H(T)I(y^*H(T)). \end{aligned}$$

The proof is similar to the case of $\overline{H} \geq H(T) > \underline{H}$. Noticing that $H(T) > \underline{H}$, the inequality holds in the opposite direction. Therefore we have $X^{VaR,y^*,y_2^*} = I(y^*H(T))$ in this case.

Next, we show $X^{VaR,y^*,y_2^*}(T)$ solves problem (3.19). Let y^* satisfy $\mathbf{E}[H(T)X^{VaR,y^*,y_2^*}(T) - \int_0^T H(s)C(s)\mathrm{d}s] = X(0)$. Choose an arbitrary $X(T)$ satisfying the budget constraint of problem (3.19). The following equations hold.

$$\begin{aligned} & \mathbf{E}[U(X^{VaR,y^*,y_2^*}(T))] - \mathbf{E}[U(X(T))] \\ & \quad = \mathbf{E}[U(X^{VaR,y^*,y_2^*}(T))] - \mathbf{E}[U(X(T))] \\ & \qquad - y^*X(0) - y^*\mathbf{E}\left[\int_0^T H(t)C(t)\mathrm{d}t\right] + y_2^*(1-\alpha) \\ & \qquad + y^*X(0) + y^*\mathbf{E}\left[\int_0^T H(t)C(t)\mathrm{d}t\right] - y_2^*(1-\alpha) \\ & \quad \geq \mathbf{E}[U(X^{VaR,y^*,y_2^*}(T))] - \mathbf{E}[y^*H(T)X^{VaR,y^*,y_2^*}(T)] \\ & \qquad + \mathbf{E}[y_2^* 1_{\{X^{VaR,y^*,y_2^*}(T)\geq\tilde{X}\}}] \\ & \qquad - \{\mathbf{E}[U(X(T))] - \mathbf{E}[y^*H(T)X(T)] + \mathbf{E}[y_2^* 1_{\{X(T)\geq\tilde{X}\}}]\} \\ & \quad \geq 0. \end{aligned} \tag{A.28}$$

The first inequality holds since the budget constraint holds with equality for $X^{VaR,y^*,y_2^*}(T)$ while with inequality for $X(T)$. Because $X^{VaR,y^*,y_2^*}(T)$ maximizes problem (A.27), the second inequality follows. So $X^{VaR,y^*,y_2^*}(T)$ solves problem (3.19). For simplicity, we present our result in Proposition 3.6 without superscripts for $X^{VaR}(T)$ and y. □

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