MATH3075/3975 Financial Derivatives

Tutorial 7: Solutions

Exercise 1 (a) The cumulative distribution function of X reads

$$F_X(x) = \begin{cases} 0, & x < 1, \\ 0.1, & 1 \leq x < 2, \\ 0.2, & 2 \leq x < 3, \\ 0.5, & 3 \leq x < 4, \\ 0.7, & 4 \leq x < 5, \\ 1, & x \geq 5. \end{cases}$$

Equivalently, we may represent the probability distribution of X as follows:

x_j	1	2	3	4	5
p_j	0.1	0.1	0.3	0.2	0.3

(b) We now compute the conditional expectation $\mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{G})$ where the σ-field $\mathcal{G}$ is generated by the partition $\{A_1, A_2, A_3\}$ of Ω. We obtain

$$\mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{G}) = \begin{cases} \frac{1}{0.2}\,(0.1 \cdot 1 + 0.1 \cdot 2) = 1.5, & \omega \in A_1, \\ \frac{1}{0.3}\,(0.3 \cdot 3) = 3, & \omega \in A_2, \\ \frac{1}{0.5}\,(0.2 \cdot 4 + 0.3 \cdot 5) = 4.6, & \omega \in A_3. \end{cases}$$

(c) Let $Y := \mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{G})$. Then the cumulative distribution function of Y satisfies

$$F_Y(y) = \begin{cases} 0, & y < 1.5, \\ 0.2, & 1.5 \leq y < 3, \\ 0.5, & 3 \leq y < 4.6, \\ 1, & y \geq 4.6. \end{cases}$$

This means that the probability distribution of Y equals:

y_l	1.5	3	4.6
$\widehat{p}_l$	0.2	0.3	0.5

(d) We first compute the expectation of X

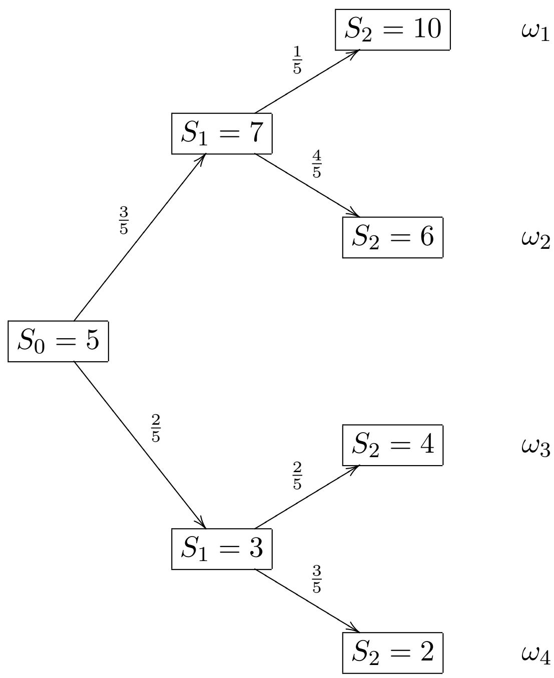
$$\mathbb{E}_{\mathbb{P}}(X) = \int_{-\infty}^{\infty} x\, dF_X(x) = \sum_{j=1}^{5} x_j p_j = 0.1 \cdot 1 + 0.1 \cdot 2 + 0.3 \cdot 3 + 0.2 \cdot 4 + 0.3 \cdot 5 = 3.5.$$

The expected value of Y equals

$$\mathbb{E}_{\mathbb{P}}(Y) = \int_{-\infty}^{\infty} y \, dF_Y(y) = \sum_{l=1}^{3} y_l \widehat{p}_l = 0.2 \cdot 1.5 + 0.3 \cdot 3 + 0.5 \cdot 4.6 = 3.5.$$

Hence the equality $\mathbb{E}_{\mathbb{P}}(X) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{G}))$ is satisfied.

Exercise 2 We consider the two-period market model $\mathcal{M} = (B, S)$ with the savings account B given by $B_0 = 1$, $B_1 = 1 + r$, $B_2 = (1 + r)^2$ with $r = 0.25$ and the stock price S represented by



(a) The probabilities of the states $\omega_1, \omega_2, \omega_3, \omega_4$ are:

$$\mathbb{P}(\omega_i) = \begin{cases} \frac{3}{5} \cdot \frac{1}{5} = \frac{3}{25}, & i = 1, \\ \frac{3}{5} \cdot \frac{4}{5} = \frac{12}{25}, & i = 2, \\ \frac{2}{5} \cdot \frac{2}{5} = \frac{4}{25}, & i = 3, \\ \frac{2}{5} \cdot \frac{3}{5} = \frac{6}{25}, & i = 4. \end{cases}$$

For instance, $\mathbb{P}(\omega_1)$ is computed as follows

$$\begin{aligned} \mathbb{P}(\omega_1) &= \mathbb{P}(S_0 = 5,\ S_1 = 7,\ S_2 = 10) \\ &= \mathbb{P}(S_2 = 10 \,|\, S_1 = 7)\, \mathbb{P}(S_1 = 7 \,|\, S_0 = 5) = \tfrac{3}{5} \cdot \tfrac{1}{5} = \tfrac{3}{25} \end{aligned}$$

and $\mathbb{P}(\omega_2)$ satisfies

$$\begin{aligned} \mathbb{P}(\omega_2) &= \mathbb{P}(S_0 = 5,\ S_1 = 7,\ S_2 = 6) \\ &= \mathbb{P}(S_2 = 6 \,|\, S_1 = 7)\, \mathbb{P}(S_1 = 7 \,|\, S_0 = 5) = \tfrac{3}{5} \cdot \tfrac{4}{5} = \tfrac{12}{25}. \end{aligned}$$

(b1) We observe that $\mathcal{F}_1 = \{\emptyset, A_1, A_2, \Omega\}$ where $A_1 = \{\omega_1, \omega_2\}$ and $A_2 = \{\omega_3, \omega_4\}$. On the event A_1, we obtain

$$\frac{1}{\mathbb{P}(A_1)} \sum_{\omega \in A_1} S_2(\omega) = \tfrac{25}{15} \left(\tfrac{3}{25} \cdot 10 + \tfrac{12}{25} \cdot 6 \right) = \tfrac{34}{5}$$

and on A_2, we get

$$\frac{1}{\mathbb{P}(A_2)} \sum_{\omega \in A_2} S_2(\omega) = \tfrac{25}{10} \left(\tfrac{4}{25} \cdot 4 + \tfrac{6}{25} \cdot 2 \right) = \tfrac{14}{5}.$$

Hence

$$\mathbb{E}_{\mathbb{P}}(S_2 \,|\, \mathcal{F}_1) = \tfrac{34}{5} \mathbb{1}_{A_1} + \tfrac{14}{5} \mathbb{1}_{A_2} = \tfrac{34}{5} \mathbb{1}_{\{S_1 = 7\}} + \tfrac{14}{5} \mathbb{1}_{\{S_1 = 3\}}.$$

(b2) We will now make use of the conditional probabilities $\mathbb{P}(S_2 = s_j \,|\, S_1 = 7)$ and $\mathbb{P}(S_2 = s_j \,|\, S_1 = 3)$. We obtain

$$\mathbb{E}_{\mathbb{P}}(S_2 \,|\, S_1 = 7) = \tfrac{1}{5} \cdot 10 + \tfrac{4}{5} \cdot 6 = \tfrac{34}{5}$$

and

$$\mathbb{E}_{\mathbb{P}}(S_2 \,|\, S_1 = 3) = \tfrac{2}{5} \cdot 4 + \tfrac{3}{5} \cdot 2 = \tfrac{14}{5}$$

(c) We first compute $\mathbb{E}_{\mathbb{P}}(S_2)$ directly

$$\mathbb{E}_{\mathbb{P}}(S_2) = \tfrac{2}{25} \cdot 10 + \tfrac{12}{25} \cdot 6 + \tfrac{4}{25} \cdot 4 + \tfrac{6}{25} \cdot 2 = \tfrac{130}{25}.$$

Next we compute

$$\mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}(S_2 | \mathcal{F}_1)\big) = \tfrac{34}{5} \cdot \tfrac{3}{5} + \tfrac{14}{5} \cdot \tfrac{2}{5} = \tfrac{130}{25}.$$

Exercise 3 **(MATH3975)** Let $\{A_1, A_2, \dots, A_m\}$ be a partition of the space $\Omega = \{\omega_1, \omega_2, \dots, \omega_k\}$. Note that since Ω is finite any σ-field $\mathcal{G}$ is generated by some finite partition of Ω.

(a) Let G be an arbitrary event from the σ-field $\mathcal{G}$ generated by the partition $\{A_1, A_2, \dots, A_m\}$. Then there exists a set $L \subset \{1, 2, \dots, m\}$ such that

$$G = \cup_{l \in L} A_l. \tag{1}$$

Moreover, we know that for every $l \in L$ on the event A_l we have that

$$\mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{G}) = \frac{1}{\mathbb{P}(A_l)} \sum_{\omega \in A_l} X(\omega)\, \mathbb{P}(\omega). \tag{2}$$

Consequently,

$$\begin{aligned}
\sum_{\omega \in G} X(\omega)\, \mathbb{P}(\omega) &\overset{(1)}{=} \sum_{l \in L} \sum_{\omega \in A_l} X(\omega)\, \mathbb{P}(\omega) \overset{(2)}{=} \sum_{l \in L} \mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{G}) \mathbb{P}(A_l) \\
&\overset{(3)}{=} \sum_{l \in L} \sum_{\omega \in A_l} \mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{G})(\omega)\, \mathbb{P}(\omega) \overset{(1)}{=} \sum_{\omega \in G} \mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{G})(\omega)\, \mathbb{P}(\omega)
\end{aligned}$$

where equality (3) holds since the conditional expectation $\mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{G})$ is constant on each event A_l, as can be seen from equation (2). If we take $G = \Omega$, then we obtain

$$\mathbb{E}_{\mathbb{P}}(X) = \sum_{\omega \in \Omega} X(\omega)\, \mathbb{P}(\omega) = \sum_{\omega \in \Omega} \mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{G})(\omega)\, \mathbb{P}(\omega) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{G})).$$

This shows that the equality $\mathbb{E}_{\mathbb{P}}(X) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{G}))$ is always valid when Ω is finite and $\mathcal{G}$ is an arbitrary σ-field.

(b) If η is $\mathcal{G}$-measurable, then it is constant on each event A_l and thus $\eta = \sum_{l=1}^{m} b_l \mathbb{1}_{A_l}$ for some real numbers $b_1, b_2, \ldots, b_m$. If we take $G = A_l$, then the postulated equality gives

$$\sum_{\omega \in A_l} X(\omega)\mathbb{P}(\omega) = \sum_{\omega \in A_l} \eta(\omega)\mathbb{P}(\omega) = b_l \, \mathbb{P}(A_l)$$

which implies that

$$\eta = \sum_{l=1}^{m} \frac{1}{\mathbb{P}(A_l)} \sum_{\omega \in A_l} X(\omega)\mathbb{P}(\omega)\mathbb{1}_{A_l} = \mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})$$

where the last equality follows from the definition of the conditional expectation $\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})$.

Exercise 4 **(MATH3975)** Notice that the $\mathcal{F}_t$-measurable random variable L_t (respectively, $\mathcal{F}_s$-measurable random variable L_s) is the Radon-Nikodym density of $\mathbb{Q}$ with respect to $\mathbb{P}$ on the space $(\Omega, \mathcal{F}_t)$ (respectively on the space $(\Omega, \mathcal{F}_s)$).

(a) We assume that Ω is finite. The random variable L_s is $\mathcal{F}_s$-measurable and, by definition, the following equality holds for every B_l from the partition generating $\mathcal{F}_s$

$$\mathbb{Q}(B_l) = \sum_{\omega \in B_l} L_s(\omega)\mathbb{P}(\omega). \tag{3}$$

Similarly, the random variable L_t is $\mathcal{F}_t$-measurable and thus constant on every A_j from the partition generating $\mathcal{F}_t$. Moreover, for every A_j from the partition generating $\mathcal{F}_t$

$$\mathbb{Q}(A_j) = \sum_{\omega \in A_j} L_t(\omega)\mathbb{P}(\omega). \tag{4}$$

Since $\mathcal{F}_t \subset \mathcal{F}_s$ (recall that $t \le s$) and the conditional expectation $\mathbb{E}_{\mathbb{P}}(L_s \,|\, \mathcal{F}_t)$ is constant on every event A_j and satisfies for every j

$$\begin{aligned}\sum_{\omega \in A_j} \mathbb{E}_{\mathbb{P}}(L_s \,|\, \mathcal{F}_t)(\omega)\mathbb{P}(\omega) &= \sum_{\omega \in A_j} L_s(\omega)\mathbb{P}(\omega) = \sum_{\omega \in B_l,\, B_l \subset A_j} L_s(\omega)\mathbb{P}(\omega) \\ &\overset{(3)}{=} \sum_{B_l \subset A_j} \mathbb{Q}(B_l) = \mathbb{Q}(A_j) \overset{(4)}{=} \sum_{\omega \in A_j} L_t(\omega)\mathbb{P}(\omega).\end{aligned}$$

Using part (b) in Exercise 3, we conclude that $\mathbb{E}_{\mathbb{P}}(L_s \,|\, \mathcal{F}_t) = L_t$.

(b) By applying the abstract Bayes formula to an arbitrary $\mathcal{F}_s$-measurable random variable Y, we obtain for every $0 \le t \le s$

$$\mathbb{E}_{\mathbb{Q}}(Y \,|\, \mathcal{F}_t) = \frac{\mathbb{E}_{\mathbb{P}}(YL_s \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(L_s \,|\, \mathcal{F}_t)} \overset{(a)}{\Leftrightarrow} \mathbb{E}_{\mathbb{Q}}(Y \,|\, \mathcal{F}_t) = (L_t)^{-1} \, \mathbb{E}_{\mathbb{P}}(YL_s \,|\, \mathcal{F}_t).$$

(c) It suffices to observe that, for every $0 \le t \le s \le T$

$$\mathbb{E}_{\mathbb{Q}}(M_s \,|\, \mathcal{F}_t) = M_t \overset{(b)}{\Leftrightarrow} (L_t)^{-1} \, \mathbb{E}_{\mathbb{P}}(M_s L_s \,|\, \mathcal{F}_t) = M_t \;\Leftrightarrow\; \mathbb{E}_{\mathbb{P}}(M_s L_s \,|\, \mathcal{F}_t) = M_t L_t.$$

Exercise 5 **(MATH3975)** It is clear that (i) implies (ii). To derive (iii) from (ii) we use the tower property (TP)

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}(M_T\,|\,\mathcal{F}_t) &\overset{\text{(TP)}}{=} \mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}(M_T\,|\,\mathcal{F}_{T-1})\,|\,\mathcal{F}_t\big) \overset{\text{(ii)}}{=} \mathbb{E}_{\mathbb{P}}(M_{T-1}\,|\,\mathcal{F}_t)\\ &\overset{\text{(TP)}}{=} \mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}(M_{T-1}\,|\,\mathcal{F}_{T-2})\,|\,\mathcal{F}_t\big) \overset{\text{(ii)}}{=} \mathbb{E}_{\mathbb{P}}(M_{T-2}\,|\,\mathcal{F}_t) = \cdots = M_t.\end{aligned}$$

Finally, we show that (iii) implies (i). We have that, for all $0 \le t \le s \le T$,

$$\mathbb{E}_{\mathbb{P}}(M_T\,|\,\mathcal{F}_t) = M_t, \quad \mathbb{E}_{\mathbb{P}}(M_T\,|\,\mathcal{F}_s) = M_s.$$

Hence

$$\mathbb{E}_{\mathbb{P}}(M_s\,|\,\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}(M_T\,|\,\mathcal{F}_s)\,|\,\mathcal{F}_t\big) \overset{\text{(TP)}}{=} \mathbb{E}_{\mathbb{P}}(M_T\,|\,\mathcal{F}_t) = M_t.$$

If X is an $\mathcal{F}_T$-measurable random variable and $X = M_T$ where M is a martingale, then it is clear that

$$M_t = \mathbb{E}_{\mathbb{P}}(M_T\,|\,\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(X\,|\,\mathcal{F}_t)$$

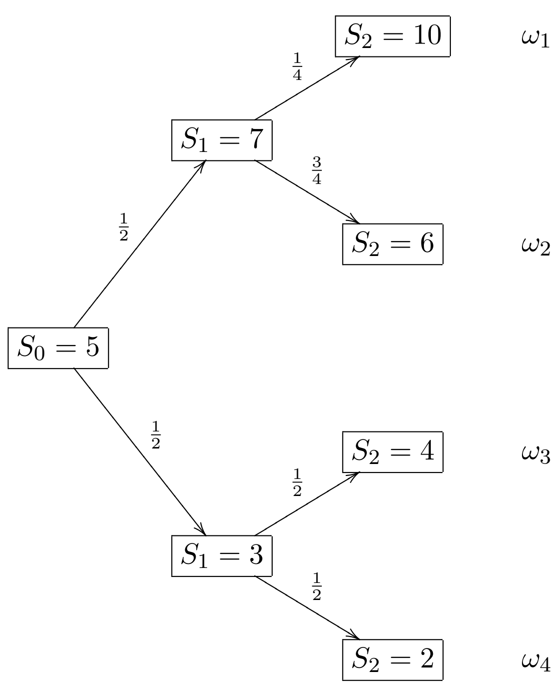
and thus it is unique.

Exercise 6 **(MATH3975)** (a) It suffices to observe that

$$\mathbb{E}_{\mathbb{P}}(S_1\,|\,\mathcal{F}_0) = \mathbb{E}_{\mathbb{P}}(S_1\,|\,\mathcal{F}_0) = 27/5 \neq 5 = S_0.$$

Similarly, one can check that $\mathbb{E}_{\mathbb{P}}(S_2\,|\,\mathcal{F}_1) \neq S_1$ but, of course, this is not needed to conclude that S is not a martingale under $\mathbb{P}$.

(b) The unique martingale measure $\mathbb{Q}$ for the process S can be represented as follows



This means that $\mathbb{Q} = (\mathbb{Q}(\omega_1), \mathbb{Q}(\omega_2), \mathbb{Q}(\omega_3), \mathbb{Q}(\omega_4)) = (1/8, 3/8, 1/4, 1/4)$.

(c) This is easy to check using part (b). In particular, on the σ-field $\mathcal{F}_2$

$$\mathbb{P} = (\mathbb{P}(\omega_1), \mathbb{P}(\omega_2), \mathbb{P}(\omega_3), \mathbb{P}(\omega_4)) = (3/25, 12/25, 4/25, 6/25)$$

and thus on $(\Omega, \mathcal{F}_2)$ we obtain

$$L_2 = (L_2(\omega_1), L_2(\omega_2), L_2(\omega_3), L_2(\omega_4)) = 25(1/24, 1/32, 1/16, 1/24).$$

For $(\Omega, \mathcal{F}_1)$, we denote $A_1 = \{\omega_1, \omega_2\}$ and $A_2 = \{\omega_3, \omega_4\}$ and we obtain

$$\mathbb{P}_{|\mathcal{F}_1} = (\mathbb{P}(A_1), \mathbb{P}(A_2)) = (3/5, 2/5)$$

and

$$\mathbb{Q}_{|\mathcal{F}_1} = (\mathbb{Q}(A_1), \mathbb{Q}(A_2)) = (1/2, 1/2).$$

Hence

$$L_1 = (L_1(\omega_1), L_1(\omega_2), L_1(\omega_3), L_1(\omega_4)) = (5/6, 5/6, 5/4, 5/4) = (5/6)\mathbb{1}_{A_1} + (5/4)\mathbb{1}_{A_2}.$$

It is easy to check that

$$\mathbb{E}_{\mathbb{P}}(L_2 \,|\, \mathcal{F}_1) = L_1$$

and

$$\mathbb{E}_{\mathbb{P}}(L_1 \,|\, \mathcal{F}_0) = \mathbb{E}_{\mathbb{P}}(L_1) = 1 = L_0$$

so that the Radon-Nikodym density process L is a martingale under $\mathbb{P}$.