6: MULTI-PERIOD MARKET MODELS

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Outline

We will examine the following issues within the framework of a general multi-period market model on a finite probability space:

1. Self-Financing Trading Strategies
2. Risk-Neutral Probability Measures and Martingales
3. Fundamental Theorem of Asset Pricing
4. Arbitrage Pricing of Attainable Claims
5. Risk-Neutral Valuation of Non-Attainable Claims
6. Completeness of Multi-Period Market Models

PART 1

SELF-FINANCING TRADING STRATEGIES

Primary Traded Assets

In a **multi-period market model** $\mathcal{M} = (B, S^1, \ldots, S^n)$, one needs to examine the concept of a dynamic trading strategy $\phi = (\phi^0, \phi^1, \ldots, \phi^n)$ and its wealth process $V(\phi)$.

We first define **primary traded assets** in $\mathcal{M}$:

- Let r be the interest rate over one period. The **money market account** is denoted by B_t for $t = 0, 1, \ldots, T$ and it satisfies

$$B_t = (1 + r)^t.$$

- There are n risky assets, called **stocks**, with price processes denoted by S_t^j for $t = 0, 1, \ldots, T$.
- We are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where the filtration $\mathbb{F}$ is generated by price processes of stocks so that $\mathbb{F} = \mathbb{F}^S$ where $S = (S^1, S^2, \ldots, S^n)$.

Dynamic Trading Strategy and Wealth Process

A dynamic **trading strategy** in a multi-period market model is defined as a stochastic process $\phi_t = \left(\phi_t^0, \phi_t^1, \ldots, \phi_t^n\right)$ for $t = 0, 1, \ldots, T$ where:

- ϕ_t^0 is the number of 'shares' of the money market account B held at time t.
- ϕ_t^j is the number of shares of the jth stock held at time t.

Definition (Value Process)

The **wealth process** (also known as the **value process**) of a trading strategy $\phi = \left(\phi^0, \phi^1, \ldots, \phi^n\right)$ is the stochastic process $V(\phi)$ given by

$$V_t\left(\phi\right) := \phi_t^0 B_t + \sum_{j=1}^{n} \phi_t^j S_t^j.$$

Self-Financing Trading Strategy

Definition (Self-Financing Trading Strategy)

A trading strategy ϕ is said to be **self-financing strategy** if for every $t = 0, 1, \ldots, T-1$,

$$\phi_t^0 B_{t+1} + \sum_{j=1}^{n} \phi_t^j S_{t+1}^j = \phi_{t+1}^0 B_{t+1} + \sum_{j=1}^{n} \phi_{t+1}^j S_{t+1}^j. \tag{1}$$

- The LHS of (1) represents the value of the portfolio at time $t+1$ before its revision, whereas the RHS represents the value at time $t+1$ after the portfolio was revised.
- Condition (1) says that these two values must be equal and this means that no cash was withdrawn or added.
- For $t = T-1$, both sides of (1) represent the wealth at time T, that is, $V_T(\phi)$. Portfolio ϕ is liquidated at time T.

Multi-Period Market Model

The concept of a multi-period market model is a natural extension of the notion of a single-period market model with only two dates: $t = 0$ and $t = T = 1$.

Definition (Market Model)

A **multi-period market model** $\mathcal{M} = (B, S^1, \dots, S^n)$ is given by the following data:

1. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$.
2. The money market account B given by $B_t = (1 + r)^t$.
3. A number of risky financial assets with prices $S^1, \dots, S^n$, which are assumed to be $\mathbb{F}$-adapted stochastic processes.
4. The class Φ of all self-financing trading strategies.

Gains Process

Definition (Gains Process)

The **gains process** $G(\phi) = (G_t(\phi))_{0 \le t \le T}$ of a trading strategy ϕ is given by

$$G_t(\phi) := V_t(\phi) - V_0(\phi).$$

Definition (Increment Processes)

- The **increment process** ΔS^j corresponding to the jth stock is defined by

$$\Delta S^j_{t+1} := S^j_{t+1} - S^j_t \quad \text{for} \quad t = 0, 1, \dots, T-1.$$

- The **increment process** ΔB of the money market account is given by

$$\Delta B_{t+1} := B_{t+1} - B_t = (1+r)^t r = B_t r \quad \text{for} \quad t = 0, 1, \dots, T-1.$$

Discounted Processes

Definition (Discounted Processes)

The **discounted stock prices** are given by

$$\widehat{S}_t^j := \frac{S_t^j}{B_t}$$

and the increments of discounted prices are $\Delta \widehat{S}_{t+1}^j := \widehat{S}_{t+1}^j - \widehat{S}_t^j$.
The **discounted wealth process** $\widehat{V}(\phi)$ of ϕ is given by

$$\widehat{V}_t(\phi) := \frac{V_t(\phi)}{B_t}.$$

The **discounted gains process** $\widehat{G}(\phi)$ of ϕ equals

$$\widehat{G}_t(\phi) := \widehat{V}_t(\phi) - \widehat{V}_0(\phi).$$

Discounted Processes

A trading strategy $\phi = (\phi_t)_{0 \le t \le T}$ is said to be $\mathbb{F}$-**adapted** is for every t and j, the random variable ϕ_t^j is $\mathcal{F}_t$-measurable.

Proposition (6.1)

An $\mathbb{F}$-adapted trading strategy $\phi = (\phi_t)_{0 \le t \le T}$ is self-financing if and only if any of the two equivalent statements hold:

1. *for every $t = 1, 2, \ldots, T$*

$$G_t(\phi) = \sum_{u=0}^{t-1} \phi_u^0 \, \Delta B_{u+1} + \sum_{u=0}^{t-1} \sum_{j=1}^{n} \phi_u^j \, \Delta S_{u+1}^j.$$

2. *for every $t = 1, 2, \ldots, T$*

$$\widehat{G}_t(\phi) = \sum_{u=0}^{t-1} \sum_{j=1}^{n} \phi_u^j \, \Delta \widehat{S}_{u+1}^j.$$

Properties of Discounted Gains and Wealth

Proof of Proposition 6.1.

Since $\widehat{G}_{t+1}(\phi) - \widehat{G}_t(\phi) = \widehat{V}_{t+1}(\phi) - \widehat{V}_t(\phi)$, it suffices to show that

$$\widehat{V}_{t+1}(\phi) - \widehat{V}_t(\phi) = \sum_{j=1}^{n} \phi_t^j \, \Delta \widehat{S}_{t+1}^j.$$

Using the self-financing condition

$$\phi_t^0 B_{t+1} + \sum_{j=1}^{n} \phi_t^j S_{t+1}^j = \phi_{t+1}^0 B_{t+1} + \sum_{j=1}^{n} \phi_{t+1}^j S_{t+1}^j.$$

we obtain

$$\widehat{V}_{t+1}(\phi) - \widehat{V}_t(\phi) = \frac{V_{t+1}(\phi)}{B_{t+1}} - \frac{V_t(\phi)}{B_t} = \frac{\phi_{t+1}^0 B_{t+1} + \sum_{j=1}^{n} \phi_{t+1}^j S_{t+1}^j}{B_{t+1}} - \frac{\phi_t^0 B_t + \sum_{j=1}^{n} \phi_t^j S_t^j}{B_t}$$

$$= \frac{\phi_t^0 B_{t+1} + \sum_{j=1}^{n} \phi_t^j S_{t+1}^j}{B_{t+1}} - \frac{\phi_t^0 B_t + \sum_{j=1}^{n} \phi_t^j S_t^j}{B_t} = \sum_{j=1}^{n} \phi_t^j \Delta \widehat{S}_{t+1}^j.$$

Properties of Discounted Gains and Wealth

- It is important to note that the process $\widehat{G}(\phi)$ given by condition (2) does not depend on the component ϕ^0 of a trading strategy $\phi \in \Phi$.
- In view of Proposition 6.1, the discounted wealth process of any $\phi \in \Phi$ satisfies

$$\widehat{V}_t(\phi) = \widehat{V}_0(\phi) + \sum_{u=0}^{t-1} \sum_{j=1}^{n} \phi_u^j \, \Delta \widehat{S}_{u+1}^j .$$

- Equivalently, for every $t = 0, \dots, T-1$

$$\Delta \widehat{V}_{t+1}(\phi) = \Delta \widehat{G}_{t+1}(\phi) = \sum_{j=1}^{n} \phi_t^j \, \Delta \widehat{S}_{t+1}^j .$$

- In a single-period model, we have $t = 0$ and we may write $\phi^j = \phi_0^j$ so that

$$\Delta \widehat{V}_1(\phi) = \Delta \widehat{G}_1(\phi) = \sum_{j=1}^{n} \phi_0^j \, \Delta \widehat{S}_1^j = \sum_{j=1}^{n} \phi^j \, \Delta \widehat{S}_1^j .$$

PART 2

RISK-NEUTRAL PROBABILITY MEASURES AND MARTINGALES

Arbitrage Opportunity

We work under the standing assumption that the sample space Ω is finite.

Definition (Arbitrage Opportunity)

A trading strategy $\phi \in \Phi$ is called an **arbitrage opportunity** in $\mathcal{M}$ if

1. $V_0(\phi) = 0$,
2. $V_T(\phi)(\omega) \geq 0$ for all $\omega \in \Omega$,
3. $V_T(\phi)(\omega) > 0$ for some $\omega \in \Omega$ or, equivalently, $\mathbb{E}_{\mathbb{P}}(V_T(\phi)) > 0$.

Definition (Arbitrage-Free Market)

We say that a multi-period market model $\mathcal{M}$ is **arbitrage-free** if there are no arbitrage opportunities in the class Φ of all self-financing trading strategies.

Arbitrage Conditions

- Observe that in arbitrage conditions (1)–(3), instead of the wealth V, one can use either the discounted wealth process $\widehat{V}$ or, equivalently, the discounted gains process $\widehat{G}$.
- It is also important to note that conditions (1)–(3) hold under $\mathbb{P}$ if and only if they are satisfied under some probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$.
- The next step is to introduce the notion of a **martingale measure** for a multi-period market model.
- The martingale measures (also known as the **risk-neutral probability measures**) are very closely related to the question of arbitrage-free property and completeness of a multi-period market model.

Martingale Measure

Definition (Martingale Measure)

A probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$ is called a **martingale measure** for a multi-period market model $\mathcal{M} = (B, S^1, \dots, S^n)$ if

1. $\mathbb{Q}(\omega) > 0$ for all $\omega \in \Omega$,
2. $\mathbb{E}_{\mathbb{Q}}(\Delta \widehat{S}^j_{t+1} \,|\, \mathcal{F}_t) = 0$ for all $j = 1, 2, \dots, n$ and $t = 0, 1, \dots, T-1$.

We denote by $\mathbb{M}$ the class of all martingale measures for $\mathcal{M}$.

Observe that condition (2) is equivalent to the following equality for every $t = 0, 1, \dots, T-1$

$$\mathbb{E}_{\mathbb{Q}}\big(\widehat{S}^j_{t+1} \,|\, \mathcal{F}_t\big) = \widehat{S}^j_t .$$

since using the 'taking out what is known' property, we obtain

$$\mathbb{E}_{\mathbb{Q}}(\Delta \widehat{S}^j_{t+1} \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(\widehat{S}^j_{t+1} - \widehat{S}^j_t \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(\widehat{S}^j_{t+1} \,|\, \mathcal{F}_t) - \mathbb{E}_{\mathbb{Q}}(\widehat{S}^j_t \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(\widehat{S}^j_{t+1} \,|\, \mathcal{F}_t) - \widehat{S}^j_t .$$

Martingales (MATH3975)

Martingales are stochastic processes representing **fair games**.

Definition (Martingale)

An $\mathbb{F}$-adapted process $X = (X_t)_{0 \leq t \leq T}$ on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **martingale** whenever for all $s < t$

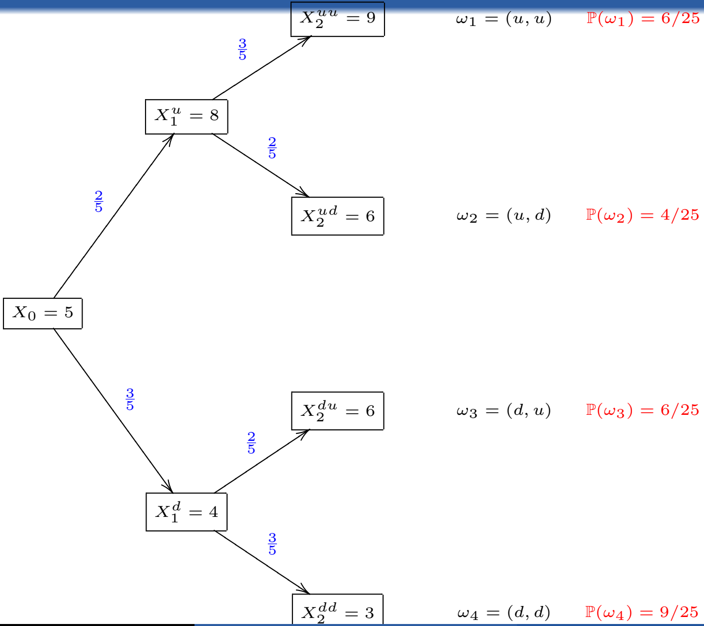
$$\mathbb{E}_{\mathbb{P}}(X_t \,|\, \mathcal{F}_s) = X_s.$$

In fact, to establish the martingale property, it suffices to check that the following equality holds for every $t = 0, 1, \ldots, T-1$

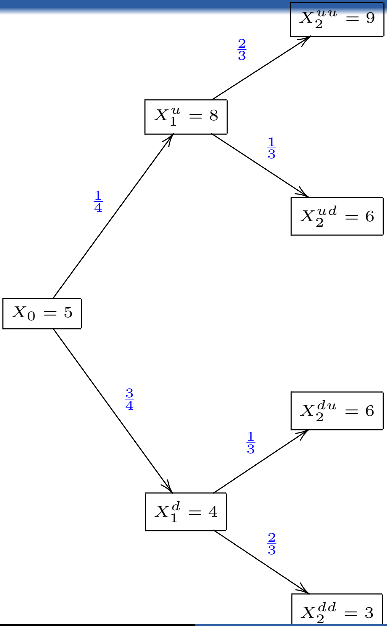
$$\mathbb{E}_{\mathbb{P}}(X_{t+1} \,|\, \mathcal{F}_t) = X_t.$$

Notice that the discounted stock prices $\widehat{S}^1, \widehat{S}^2, \ldots, \widehat{S}^n$ are all martingales under any martingale measure $\mathbb{Q} \in \mathbb{M}$.

X is not a martingale under $\mathbb{P}$ since $\mathbb{E}_{\mathbb{P}}(X_1 \,|\, \mathcal{F}_0) = 5.6 \neq 5 = X_0$.



X is a martingale under $\mathbb{Q}$ since $\mathbb{E}_{\mathbb{Q}}(X_1 \,|\, \mathcal{F}_0) = X_0$ and $\mathbb{E}_{\mathbb{Q}}(X_2 \,|\, \mathcal{F}_1) = X_1$



$\omega_1 = (u, u)$ $\quad \mathbb{Q}(\omega_1) = 2/12$

$\omega_2 = (u, d)$ $\quad \mathbb{Q}(\omega_2) = 1/12$

$\omega_3 = (d, u)$ $\quad \mathbb{Q}(\omega_3) = 3/12$

$\omega_4 = (d, d)$ $\quad \mathbb{Q}(\omega_4) = 6/12$

Discounted Wealth as a Martingale

Proposition (6.2)

Let $\phi \in \Phi$ be a self-financing trading strategy. Then the discounted wealth process $\widehat{V}(\phi)$ and the discounted gains process $\widehat{G}(\phi)$ are martingales under any martingale measure $\mathbb{Q} \in \mathbb{M}$.

Proof of Proposition 6.2 (MATH3975).

- Recall that $\widehat{V}_t(\phi) = \widehat{V}_0(\phi) + \widehat{G}_t(\phi)$ for every $t = 0, 1, \dots, T$.
- Since $\widehat{V}_0(\phi)$ (the initial endowment) is a constant, it suffices to show that the process $\widehat{G}(\phi)$ is a martingale under any $\mathbb{Q} \in \mathbb{M}$.
- From Proposition 6.1, we obtain

$$\widehat{G}_{t+1}(\phi) = \widehat{G}_t(\phi) + \sum_{j=1}^{n} \phi_t^j \, \Delta \widehat{S}_{t+1}^j .$$

□

Proof of Proposition 6.2 (MATH3975)

Proof of Proposition 6.2 (Continued).

- Hence

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(\widehat{G}_{t+1}(\phi)|\mathcal{F}_t) &= \widehat{G}_t(\phi) + \sum_{j=1}^{n} \mathbb{E}_{\mathbb{Q}}\left(\phi_t^j \Delta \widehat{S}_{t+1}^j \,\middle|\, \mathcal{F}_t\right) \\
&= \widehat{G}_t(\phi) + \sum_{j=1}^{n} \phi_t^j \underbrace{\mathbb{E}_{\mathbb{Q}}\big(\Delta \widehat{S}_{t+1}^j \big| \mathcal{F}_t\big)}_{=0} \\
&= \widehat{G}_t(\phi)
\end{aligned}$$

- We used the fact that ϕ_t^j is $\mathcal{F}_t$-measurable and the "taking out what is known" property of the conditional expectation.
- We conclude that $\widehat{G}(\phi)$ is a martingale under any $\mathbb{Q} \in \mathbb{M}$.

$\square$

PART 3

FUNDAMENTAL THEOREM OF ASSET PRICING

Fundamental Theorem of Asset Pricing

- We will show that the **F**undamental **T**heorem of **A**sset **P**ricing (FTAP) can be extended to a multi-period market model.
- Recall that the class of admissible trading strategies Φ in a multi-period market model is assumed to be the **full** set of all self-financing trading strategies.
- It possible to show that in that case, the relationship between the existence of a martingale measure $\mathbb{Q}$ and no arbitrage for the market model $\mathcal{M}$ is "if and only if".
- We will only prove here the following implication:

$$\textbf{Existence of } \mathbb{Q} \in \mathbb{M} \;\Rightarrow\; \textbf{Model } \mathcal{M} \textbf{ is arbitrage-free}$$

Fundamental Theorem of Asset Pricing

Theorem (FTAP)

Consider a multi-period market model $\mathcal{M} = (B, S^1, \ldots, S^n)$*.*
The following statements hold:

1. *If the class* $\mathbb{M}$ *of martingale measures for* $\mathcal{M}$ *is non-empty, then there are no arbitrage opportunities in the class* Φ *of all self-financing trading strategies and thus the model* $\mathcal{M}$ *is arbitrage-free.*
2. *If there are no arbitrage opportunities in the class* Φ *of all self-financing trading strategies, then there exists a martingale measure for* $\mathcal{M}$ *so that the class* $\mathbb{M}$ *is non-empty.*

To sum up:

Class $\mathbb{M}$ is non-empty $\Leftrightarrow$ Market model $\mathcal{M}$ is arbitrage-free

Proof of the FTAP (MATH3975)

Proof of the FTAP ($\Rightarrow$).

Let us assume that a martingale measure $\mathbb{Q}$ for $\mathcal{M}$ exists. Our goal it to show that the model $\mathcal{M}$ is arbitrage-free.

To this end, we argue by contradiction. Let us thus assume that there exists an arbitrage opportunity $\phi \in \Phi$. Such a strategy would satisfy the following conditions:

1. the initial endowment $\widehat{V}_0(\phi) = 0$,
2. the discounted gains process $\widehat{G}_T(\phi) \geq 0$,
3. there exists at least one $\omega \in \Omega$ such that $\widehat{G}_T(\phi)(\omega) > 0$.

On the one hand, from conditions (2) and (3) above, we deduce that

$$\mathbb{E}_{\mathbb{Q}}\big(\widehat{G}_T(\phi)\big) > 0.$$

□

Proof of the FTAP (MATH3975)

Proof of the FTAP ($\Rightarrow$).

On the other hand, it is already known from Proposition 6.2 that the discounted gains process $\widehat{G}(\phi)$ is a martingale under $\mathbb{Q}$.

Hence

$$\mathbb{E}_{\mathbb{Q}}(\widehat{G}_T(\phi)) = \mathbb{E}_{\mathbb{Q}}(\widehat{G}_T(\phi) \,|\, \mathcal{F}_0) = \widehat{G}_0(\phi) = 0$$

where we used the properties of the conditional expectation.

This clearly contradicts the inequality obtained in the first step. Hence the market model $\mathcal{M}$ is arbitrage-free if a martingale measure $\mathbb{Q}$ for $\mathcal{M}$ exists. $\square$

Proof of the FTAP ($\Leftarrow$).

The proof of the implication ($\Leftarrow$) in the FTAP requires a good familiarity with the theory of martingales, although it hinges on the same idea as the proof for the single-period case. $\square$

PART 4

ARBITRAGE PRICING OF ATTAINABLE CLAIMS

Replicating Strategy

- Note that a contingent claim of European style can only be exercised at its maturity date T (as opposed to contingent claims of American style).
- A **European contingent claim** in a multi-period market model is defined as an $\mathcal{F}_T$-measurable random variable X on Ω, which is interpreted as the random payoff at the terminal date T.
- For the sake of brevity, European contingent claims will also be referred to as **contingent claims** or simply **claims**.

Definition (Replicating Strategy)

A **replicating strategy** (or a **hedging strategy**) for a contingent claim X is a trading strategy $\phi \in \Phi$ such that $V_T(\phi) = X$, that is, the terminal wealth of the trading strategy ϕ matches the claim's payoff for all ω, that is,

$$V_T(\phi)(\omega_i) = X(\omega_i), \quad \forall\, i = 1, 2, \ldots, k.$$

Principle of No-Arbitrage

Definition (Principle of No-Arbitrage)

A stochastic process $(\pi_t(X))_{0 \leq t \leq T}$ is a price process for the contingent claim X that **complies with the principle of no-arbitrage** if there are no arbitrage opportunities in the extended model $\widetilde{\mathcal{M}} = (B, S^1, \dots, S^n, S^{n+1})$ where the asset S^{n+1} is given by $S_t^{n+1} = \pi_t(X)$ for $0 \leq t \leq T-1$ and $S_T^{n+1} = X$.

- The standard method to price a contingent claim is to use the replication principle, provided that it can be applied.
- The price of a claim X will now depend on time t and thus one has to specify a whole price process $\pi(X)$, rather than just its initial price $\pi_0(X)$, as in the single-period market model.
- Obviously, the equality $\pi_T(X) = X$ holds for any contingent claim X.

Arbitrage Pricing of Attainable Claims

- In the next result, we deal with an **attainable** claim, meaning that we assume a priori that a replicating strategy for X exists.

Proposition (6.3)

Let X be a contingent claim in an arbitrage-free multi-period market model $\mathcal{M}$ and let $\phi \in \Phi$ be any replicating strategy for X. Then the only price process for X that complies with the principle of no-arbitrage is the wealth process $V(\phi)$.

- The arbitrage price at time t of an attainable claim X is unique and it is also denoted as $\pi_t(X)$.
- Hence the equality $\pi_t(X) = V_t(\phi)$ holds for any replicating strategy $\phi \in \Phi$ for X.
- The price at time $t = 0$ is the initial wealth of any replicating strategy for X, that is, $\pi_0(X) = V_0(\phi)$ for any $\phi \in \Phi$ such that $V_T(\phi) = X$.

Example: Replication of a Digital Call Option

Example (6.1)

- We will now examine replication of a contingent claim in a two-period market model using the so-called **backward induction** method.
- We consider a two-period market model consisting of the savings account B and one risky stock S.
- The interest rate equals $r = \frac{1}{9}$ so that $B_t = (1+r)^t = \left(\frac{10}{9}\right)^t$.
- The price of the stock is represented in the following exhibit in which the real-world probability $\mathbb{P}$ is also specified.
- It is easy to check that the model $\mathcal{M} = (B, S)$ is arbitrage-free.
- Our goal is to price a particular contingent claim X using replication.

Example: Replication of a Digital Call Option

Example (6.1 Continued)

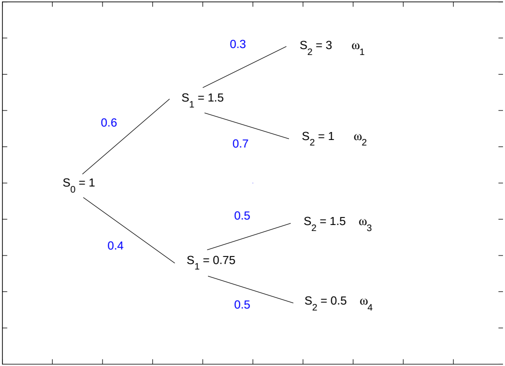


Figure: Stock Price Dynamics

Example: Replication of a Digital Call Option

Example (6.1 Continued)

- Consider a **digital call option** with the payoff function

$$X(\omega) = g\left(S_2(\omega)\right) = \begin{cases} 1 & \text{if } S_2(\omega) > 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$X = \big(X(\omega_1), X(\omega_2), X(\omega_3), X(\omega_4)\big) = (1, 0, 0, 0).$$

- By the definition of replication, we have $V_2(\phi) = X$ or, more explicitly,

$$V_2(\phi) = \phi_1^0 B_2 + \phi_1^1 S_2 = g(S_2) = (1, 0, 0, 0).$$

- Observe that $\phi_1^i(\omega_1) = \phi_1^i(\omega_2)$ and $\phi_1^i(\omega_3) = \phi_1^i(\omega_4)$ for $i = 0, 1$ since ϕ is an $\mathbb{F}$-adapted process.

Example: Replication of a Digital Call Option

Example (6.1 Continued)

At time $t = 1$ we obtain two linear systems for $\phi_1 = (\phi_1^0, \phi_1^1)$

- For $\omega \in A_1 = \{\omega_1, \omega_2\}$

$$\left(\frac{10}{9}\right)^2 \phi_1^0 + 3\phi_1^1 = \left(\frac{10}{9}\right) \widetilde{\phi}_1^0 + 3\phi_1^1 = 1,$$
$$\left(\frac{10}{9}\right)^2 \phi_1^0 + \phi_1^1 = \left(\frac{10}{9}\right) \widetilde{\phi}_1^0 + \phi_1^1 = 0.$$

- For $\omega \in A_2 = \{\omega_3, \omega_4\}$

$$\left(\frac{10}{9}\right)^2 \phi_1^0 + 1.5\phi_1^1 = \left(\frac{10}{9}\right) \widetilde{\phi}_1^0 + 1.5\phi_1^1 = 0,$$
$$\left(\frac{10}{9}\right)^2 \phi_1^0 + 0.5\phi_1^1 = \left(\frac{10}{9}\right) \widetilde{\phi}_1^0 + 0.5\phi_1^1 = 0.$$

Example: Replication of a Digital Call Option

Example (6.1 Continued)

- We obtain the replicating strategy at $t = 1$:

$$(\phi_1^0, \phi_1^1) = \left(-\frac{81}{200}, \frac{1}{2}\right) \quad \text{if } \omega \in \{\omega_1, \omega_2\},$$
$$(\phi_1^0, \phi_1^1) = (0, 0) \qquad \text{if } \omega \in \{\omega_3, \omega_4\}.$$

- If $S_1 = 1.5$ then the price of the digital call at $t = 1$ equals

$$V_1(\phi) = \phi_1^0 B_1 + \phi_1^1 S_1 = \widetilde{\phi}_1^0 + \phi_1^1 S_1 = -\frac{81}{200} \cdot \frac{10}{9} + \frac{1}{2} \cdot \frac{3}{2} = 0.3.$$

- If $S_1 = 0.75$ then the price of the digital call at $t = 1$ equals 0.
- Hence the price of X at time 1 equals $\pi_1(X) = 0.3\,\mathbb{1}_{\{S_1 = 1.5\}}$.

Example: Replication of a Digital Call Option

Example (6.1 Continued)

- We now compute the price of the digital call at $t = 0$. The replicating strategy at time $t = 0$ should satisfy $\phi_0^0 B_1 + \phi_0^1 S_1 = \pi_1(X)$, that is,

$$\frac{10}{9}\phi_0^0 + 1.5\phi_0^1 = 0.3,$$
$$\frac{10}{9}\phi_0^0 + 0.75\phi_0^1 = 0.$$

- Then $\left(\phi_0^0, \phi_0^1\right) = (-0.27, 0.4)$. Since $S_0 = 1$, the price of the digital call at time $t = 0$ equals

$$V_0(\phi) = \phi_0^0 B_0 + \phi_0^1 S_0 = -0.27 + 0.4 = 0.13.$$

- Hence the arbitrage price of the claim X at time 0 equals $\pi_0(X) = 0.13$.

Example: Replication of a Digital Call Option

Example (6.1 Continued)

Summary of pricing and hedging results for a digital call option.

- Recall that the price at time $t = 2$ is given by $\pi_2(X) = X$.
- Replicating strategy ϕ should satisfy $V_2(\phi) = X$. It was computed by recursion using the **backward induction** method.
- The arbitrage price process of X equals $\pi(X) = V(\phi)$.

	$A_1 = \{\omega_1, \omega_2\}$	$A_2 = \{\omega_3, \omega_4\}$
$t = 0$	$(\phi_0^0, \phi_0^1) = (-0.27, 0.4)$	$(\phi_0^0, \phi_0^1) = (-0.27, 0.4)$
$t = 0$	$\pi_0(X) = 0.13$	$\pi_0(X) = 0.13$
$t = 1$	$(\phi_1^0, \phi_1^1) = \left(-\frac{81}{200}, \frac{1}{2}\right)$	$(\phi_1^0, \phi_1^1) = (0, 0)$
$t = 1$	$\pi_1(X) = 0.3$	$\pi_1(X) = 0$
$t = 2$	$X(\omega_1) = 1,\ X(\omega_2) = 0$	$X(\omega_3) = 0,\ X(\omega_4) = 0$

PART 5

RISK-NEUTRAL VALUATION OF NON-ATTAINABLE CONTINGENT CLAIMS

Attainability of Contingent Claims and Completeness

Definition (Attainable Contingent Claim)

A contingent claim X is called to be **attainable** if there exists a trading strategy $\phi \in \Phi$, which replicates X, i.e., $V_T(\phi) = X$.

- For attainable contingent claims, it is clear how to price them by the initial investment needed for a replicating strategy.
- As in single-period market models, for some contingent claims a hedging strategy may fail to exist.

Definition (Completeness)

A multi-period market model is said to be **complete** if and only if all contingent claims have replicating strategies. If a multi period market model is not complete, it is said to be **incomplete**.

Risk-Neutral Valuation Formula

Proposition (6.4)

Let X be a contingent claim (possibly non-attainable) and $\mathbb{Q}$ any martingale measure for the multi-period market model $\mathcal{M}$. Then the **risk-neutral valuation formula**

$$\pi_t(X) = B_t \, \mathbb{E}_{\mathbb{Q}} \left(\frac{X}{B_T} \,\middle|\, \mathcal{F}_t \right)$$

defines a price process $\pi(X) = (\pi_t(X))_{0 \le t \le T}$ for X that complies with the principle of no-arbitrage.

Proof.

If X is attainable then we observe that $\widehat{V}(\phi)$ is a martingale under $\mathbb{Q}$ and we apply the definition of a martingale: for any $t \le T$

$$\widehat{V}_t(\phi) = \frac{V_t(\phi)}{B_t} = \frac{\pi_t(X)}{B_t} = \mathbb{E}_{\mathbb{Q}} \left(\frac{V_T(\phi)}{B_T} \,\middle|\, \mathcal{F}_t \right) = \mathbb{E}_{\mathbb{Q}} \left(\frac{X}{B_T} \,\middle|\, \mathcal{F}_t \right).$$

Example: Backward Induction

Example (6.2)

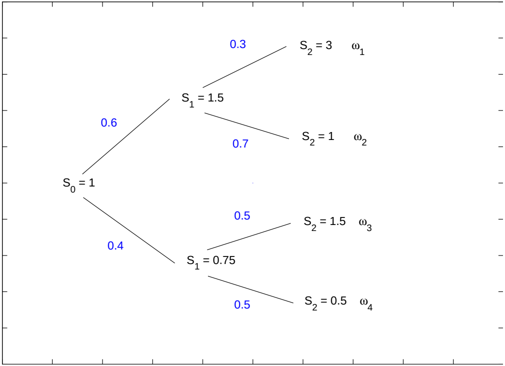


Figure: Stock Price Dynamics

Example: Backward Induction

Example (6.2 Continued)

- We consider the market model introduced in Example 6.1.
- The two-period market model is composed of the following single-period market models:
    1. $S_1 = 1.5$, $S_2 = 3$ and $S_2 = 1$.
    2. $S_1 = 0.75$, $S_2 = 1.5$ and $S_2 = 0.5$.
    3. $S_0 = 1$, $S_1 = 1.5$ and $S_1 = 0.75$.
- Note that these sub-models are elementary market models.
- Hence the unique martingale measure can be computed using the well known equalities

$$\tilde{p} = \frac{1 + r - d}{u - d}, \quad \tilde{q} = 1 - \tilde{p} = \frac{u - (1 + r)}{u - d}.$$

Example: Backward Induction

Example (6.2 Continued)

- Recall that if the elementary market model is arbitrage-free then it is also complete and all contingent claims can be priced using the risk-neutral probability.
- **M**odel 1. In the first sub-model, the martingale measure is

$$\begin{aligned}\tilde{p}_1 &= \frac{1+r-d_1}{u_1-d_1} = \frac{1+\frac{1}{9}-\frac{2}{3}}{2-\frac{2}{3}} = \frac{1}{3} \\ \tilde{q}_1 &= 1-\tilde{p}_1 = \frac{2}{3}\end{aligned}$$

- The price of the digital call option in the considered model equals

$$\pi_1^u(X) = \frac{1}{1+\frac{1}{9}}\left(1\cdot\frac{1}{3}+0\cdot\frac{2}{3}\right) = 0.3$$

Example: Backward Induction

Example (6.2 Continued)

- **M**odel 2. In the second sub-model, the martingale measure is

$$\tilde{p}_2 = \frac{1 + r - d_2}{u_2 - d_2} = \frac{1 + \frac{1}{9} - \frac{2}{3}}{2 - \frac{2}{3}} = \frac{1}{3}$$

$$\tilde{q}_2 = 1 - \tilde{p}_2 = \frac{2}{3}$$

- The price of the digital call option in this model equals 0 since its payoff is 0. Formally,

$$\pi_1^d(X) = \frac{1}{1 + \frac{1}{9}} \left(0 \cdot \frac{1}{3} + 0 \cdot \frac{2}{3} \right) = 0$$

Example: Backward Induction

Example (6.2 Continued)

- **M**odel 3. In the last sub-model, the martingale measure is

$$\tilde{p}_3 = \frac{1 + r - d_3}{u_3 - d_3} = \frac{1 + \frac{1}{9} - \frac{3}{4}}{\frac{3}{2} - \frac{3}{4}} = \frac{13}{27}$$

$$\tilde{q}_3 = 1 - \tilde{p}_3 = \frac{14}{27}$$

We consider the contingent claim $\pi_1(X)$ with the payoff at time $t = 1$ given by

$$\pi_1(X) = \begin{cases} \pi_1^u(X) = 0.3 & \text{if } S_1 = u_3 S_0 = 1.5 \\ \pi_1^d(X) = 0 & \text{if } S_1 = d_3 S_0 = 0.75 \end{cases}$$

Example: Backward Induction

Example (6.2 Continued)

- The price of X at time 0 equals

$$\pi_0(X) = \frac{1}{1+r}\,\mathbb{E}_{\mathbb{Q}}(\pi_1(X)) = \frac{1}{1+\frac{1}{9}}\left(0.3 \cdot \frac{13}{27} + 0 \cdot \frac{14}{27}\right) = 0.13$$

- Hence the price $\pi_0(X) = 0.13$, $\pi_1(X)(\omega) = 0.3$ if $\omega \in \{\omega_1, \omega_2\}$ and $\pi_1(X)(\omega) = 0$ if $\omega \in \{\omega_3, \omega_4\}$.
- The unique martingale measure $\mathbb{Q}$ in the two-period model equals:

$$\begin{aligned}
\mathbb{Q}(\omega_1) &= \tilde{p}_3 \tilde{p}_1 = \frac{13}{27} \cdot \frac{1}{3} = \frac{13}{81} \\
\mathbb{Q}(\omega_2) &= \tilde{p}_3 (1 - \tilde{p}_1) = \frac{13}{27} \cdot \frac{2}{3} = \frac{26}{81} \\
\mathbb{Q}(\omega_3) &= (1 - \tilde{p}_3)\, \tilde{p}_2 = \frac{14}{27} \cdot \frac{1}{3} = \frac{14}{81} \\
\mathbb{Q}(\omega_4) &= (1 - \tilde{p}_3)(1 - \tilde{p}_2) = \frac{14}{27} \cdot \frac{2}{3} = \frac{28}{81}
\end{aligned}$$

PART 6

COMPLETENESS OF MULTI-PERIOD MARKET MODELS

Completeness

As a handy criterion for the market completeness, we have the theorem, which extends the known result for the single-period case.

Theorem (6.1)

Assume that a multi-period market model $\mathcal{M} = (B, S^1, \dots, S^n)$ is arbitrage-free. Then $\mathcal{M}$ is complete if and only if there is only one martingale measure, that is, $\mathbb{M} = \{\mathbb{Q}\}$ is a singleton.

In the context of a model decomposition, the following statements are known to hold:

- If all single-period models which compose a multi-period model are arbitrage-free then the multi-period model is also arbitrage-free.
- If they are also complete then the multi-period model is also complete.
- The converse of the above statement is also correct.

Summary

We examined two related pricing and hedging approaches:

1. The first method is based on the idea of **replication** of a contingent claim. It can only be applied to attainable contingent claim in either a complete or incomplete model and it yields the hedging strategy and arbitrage price process.
2. The second method relies on the concept of an **martingale measure** and thus it can be used in either a complete or an incomplete model. It furnishes the **unique** arbitrage price process for any attainable claim and a **possible** arbitrage price process for any non-attainable contingent claim. In the latter case the price process depends on the choice of a particular martingale measure from the class $\mathbb{M}$.
3. In the **backward induction** approach, the pricing and hedging are performed in a recursive way starting from the date T and moving step-by-step towards the initial date 0.