

Tetrahedron equation from PBW bases of the nilpotent subalgebra of quantum superalgebras

Mathsci Freshman Seminar@2021/02/13

Akihito Yoneyama (米山 瑛仁)

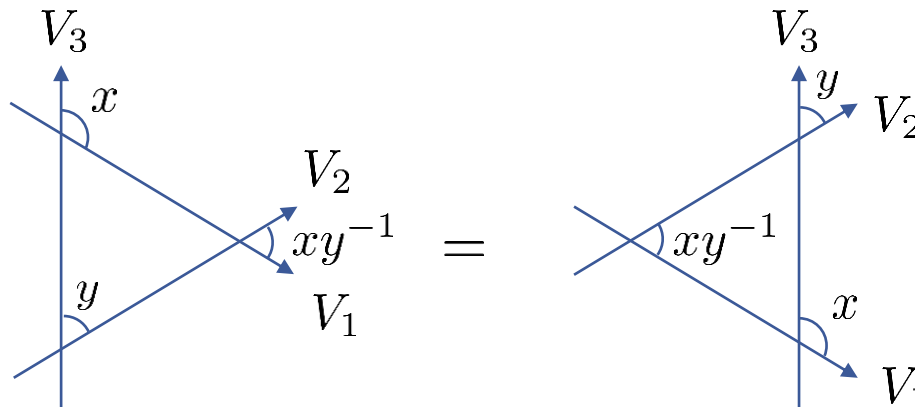
Institute of Physics, University of Tokyo, Komaba

Based on: AY, arXiv: 2012.13385

- Yang-Baxter equation in 2D integrability P.3~8
 - ▣ Commuting transfer matrix
 - ▣ Integrable hamiltonian
 - ▣ Drinfeld-Jimbo quantum affine algebra
- Tetrahedron equation in 3D integrability P.10~15
 - ▣ 3DR and 3DL
 - ▣ Matrix product solution to Yang-Baxter equation
 - ▣ Kuniba-Okado-Yamada theorem
- Main part: P.17~22
 - ▣ PBW bases of quantum superalgebras of type A of rank 2
- Concluding remarks

- Graphical notation:

$$R(z)_{i,j}^{a,b} = i \begin{array}{c} \begin{array}{c} \uparrow b \\ \nearrow z \\ \rightarrow a \\ \searrow \\ \downarrow j \end{array} \end{array}$$



$$\sum_{a_2} R(z)_{a_1, b_1}^{a_2, b_2} R(z)_{a_2, c_1}^{a_3, c_2}$$

Commuting transfer matrix

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■ From Yang-Baxter equation, we have the following:

■ Proposition:

▣ We set the *transfer matrix* $\tau(z): V^{\otimes L} \rightarrow V^{\otimes L}$ by

$$V_i = V$$

$$\tau(z) := \text{Tr}_a (R_{a1}(z)R_{a2}(z)\dots R_{aL}(z)) = \begin{array}{c} \begin{array}{ccccccc} & \uparrow & & \uparrow & & \cdots & \uparrow \\ & z & & z & & & z \\ & \downarrow & & \downarrow & & & \downarrow \\ V_1 & & V_2 & & & & V_L \end{array} \end{array} \rightarrow V_a$$

▣ Then, we have

$$[\tau(x), \tau(y)] = 0 \quad (\forall x, y \in \mathbb{C})$$

■ Proof:

▣ For the following eq, multiply $R_{ab}(xy^{-1})^{-1}$ and take trace on $V_a \otimes V_b$:

$$\begin{array}{c} \begin{array}{ccccccc} \uparrow & \uparrow & & \uparrow & & & \\ x & x & & x & & & \\ \rightarrow & \rightarrow & \cdots & \rightarrow & & & \\ \uparrow & \uparrow & & \uparrow & & & \\ y & y & & y & & & \\ \rightarrow & \rightarrow & \cdots & \rightarrow & & & \end{array} \begin{array}{c} \searrow \\ xy^{-1} \\ \swarrow \end{array} \begin{array}{c} V_b \\ V_a \end{array} = \begin{array}{c} \begin{array}{ccccccc} & \uparrow & & \uparrow & & \cdots & \uparrow \\ & y & & y & & & y \\ & \downarrow & & \downarrow & & & \downarrow \\ & x & & x & & & x \\ & \rightarrow & \rightarrow & \cdots & \rightarrow & & \end{array} \begin{array}{c} \swarrow \\ xy^{-1} \\ \searrow \end{array} \begin{array}{c} V_b \\ V_a \end{array} \end{array}$$

- A lot of families of integrable one-dimensional quantum spin chains are constructed via commutativity of the transfer matrix.
- The following \mathcal{H} gives one-dimensional quantum spin chain

$$\mathcal{H} := \left. \frac{d}{dz} (\log \tau(z)) \right|_{z=1}$$

- This has a nearest-neighbor interaction (physical):

$$\mathcal{H} = \sum_i \mathcal{H}_{i,i+1} \quad \mathcal{H}_{i,i+1} : V_i \otimes V_{i+1} \rightarrow V_i \otimes V_{i+1}$$

- Integrability: the transfer matrix gives conserved quantities:

$$[\tau(x), \tau(y)] = 0 \quad (\forall x, y \in \mathbb{C}) \quad \longrightarrow \quad [\mathcal{H}, \tau(y)] = 0 \quad \forall y \in \mathbb{C}$$
$$\tau(y) = \underline{\tau(1)} + \underline{\tau'(1)y} + \dots$$

- All eigenvalues of $\tau(z)$ are obtained by Bethe ansatz for large classes of R matrices.

- R matrices are systematically (infinitely many) constructed via irreps of Drinfeld-Jimbo quantum affine algebra $U_q(g)$.
 - g : affine Lie algebra $A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, \dots$
 - $U_q(g)$ is one-parametric generalization of g , and reduces to g with $q \rightarrow 1$
- $(V_1, \pi_z^{(1)}), (V_2, \pi_z^{(2)})$: irreps of $U_q(g)$ z : spectral parameter
 - Example: Kirillov-Reshetikhin module
- If $\pi_x^{(1)} \otimes \pi_y^{(2)}, \pi_y^{(2)} \otimes \pi_x^{(1)}$ are irreducible, a linear map $\check{R}(z)$ given by
$$(\pi_y^{(2)} \otimes \pi_x^{(1)})(\Delta(g))\check{R}(x/y) = \check{R}(x/y)(\pi_x^{(1)} \otimes \pi_y^{(2)})(\Delta(g)) \quad \forall g \in U_q(X)$$
is uniquely determined by Schur's Lemma.
- $R(z) = P\check{R}(z)$ gives a solution to the Yang-Baxter equation.
 - "linearization" method for the Yang-Baxter equation. $P|i\rangle \otimes |j\rangle = |j\rangle \otimes |i\rangle$
- Point: characterization \neq explicit formula

Explicit formula for R matrices

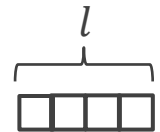
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- For some $U_q(g)$ and their representations, there exist explicit formulae for R matrices from *three dimensional integrability*.

- Example:

- $U_q(g) = U_q(A_{n-1}^{(1)})$

- $\pi_z^{(1)} = \pi_z^{(2)} =$ arbitrary (l -th) **symmetric tensor** representation

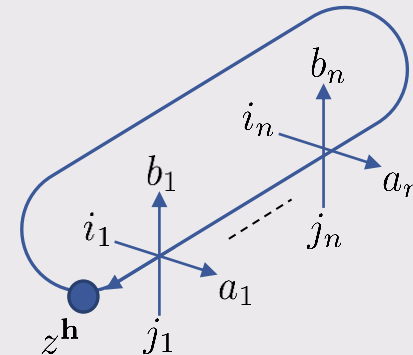


- Theorem:

$$R(z)_{i,j}^{a,b} = \text{Tr}_F(z^{\mathbf{h}} \mathcal{R}_{i_1, j_1}^{a_1, b_1} \cdots \mathcal{R}_{i_n, j_n}^{a_n, b_n}) =$$

satisfies the intertwining relation

$$(\pi_x^{(1)} \otimes \pi_y^{(2)})(\Delta^{\text{op}}(g))R(x/y) = R(x/y)(\pi_x^{(1)} \otimes \pi_y^{(2)})(\Delta(g)) \quad \forall g \in U_q(X)$$



- \mathcal{R} is the solution to the tetrahedron equation called **3DR**.

Explicit formula for R matrices

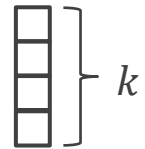
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- For some $U_q(g)$ and their representations, there exist explicit formulae for R matrices from *three dimensional integrability*.

- Example:

- $U_q(g) = U_q(A_{n-1}^{(1)})$

- $\pi_z^{(1)} = \pi_z^{(2)} = \text{arbitrary } (k\text{-th}) \text{ fundamental representation}$



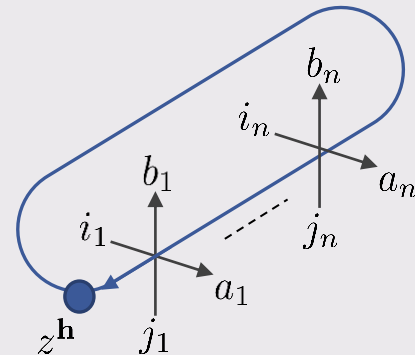
$$1 \leq k \leq n - 1$$

- Theorem:

$$R(z)_{i,j}^{a,b} = \text{Tr}_F(z^{\mathbf{h}} \mathcal{L}_{i_1, j_1}^{a_1, b_1} \cdots \mathcal{L}_{i_n, j_n}^{a_n, b_n}) =$$

satisfies the intertwining relation

$$(\pi_x^{(1)} \otimes \pi_y^{(2)})(\Delta^{\text{op}}(g))R(x/y) = R(x/y)(\pi_x^{(1)} \otimes \pi_y^{(2)})(\Delta(g)) \quad \forall g \in U_q(X)$$

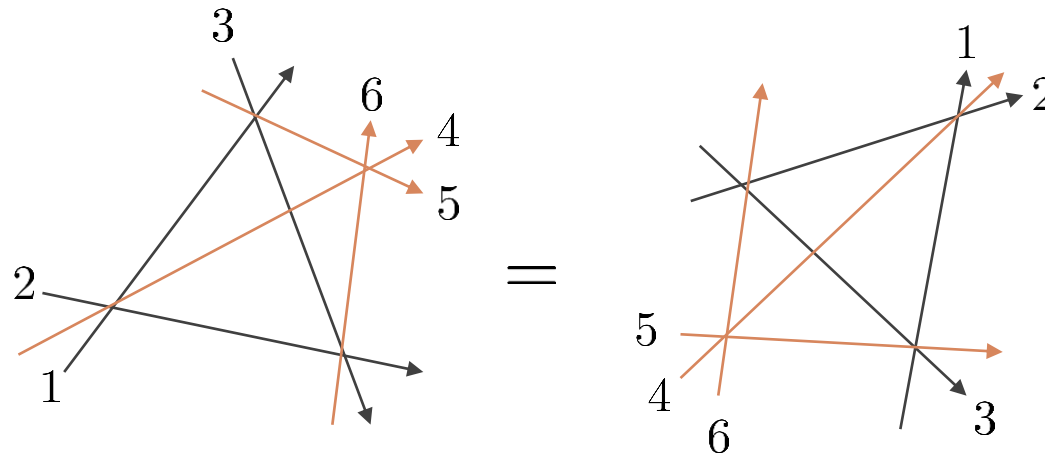


- \mathcal{L} is the solution to the tetrahedron equation called **3DL**.

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Tetrahedron equation

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- Matrix equation on $V_1 \otimes \cdots \otimes V_6$ (V_i : linear space)

$$\mathcal{R}_{124}\mathcal{R}_{135}\mathcal{R}_{236}\mathcal{R}_{456} = \mathcal{R}_{456}\mathcal{R}_{236}\mathcal{R}_{135}\mathcal{R}_{124}$$

- R_{ijk} acts non-trivially only on $V_i \otimes V_j \otimes V_k$.

- Tetrahedron equation = Yang-Baxter equation *up to conjugation*

- Unlike Yang-Baxter equation, a few families of solutions are known.

- We focus on solutions on the Fock spaces.

$$\text{boson Fock: } F = \bigoplus_{m=0,1,2,\dots} \mathbb{C} |m\rangle$$

$$\text{fermi Fock: } V = \bigoplus_{m=0,1} \mathbb{C} u_m$$

Solutions to tetrahedron equation: 3DR

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- Set $\mathcal{R} \in \text{End}(F^{\otimes 3})$ by

[Kapranov-Voevodsky94]

$$\mathcal{R} |i\rangle \otimes |j\rangle \otimes |k\rangle = \sum_{a,b,c} \mathcal{R}_{ijk}^{abc} |a\rangle \otimes |b\rangle \otimes |c\rangle$$

$$\mathcal{R}_{i,j,k}^{a,b,c} = \delta_{i+j}^{a+b} \delta_{j+k}^{b+c} \sum_{\lambda, \mu \geq 0, \lambda + \mu = b} (-1)^\lambda q^{i(c-j) + (k+1)\lambda + \mu(\mu-k)} \frac{(q^2)_{c+\mu}}{(q^2)_c} \binom{i}{\mu}_{q^2} \binom{j}{\lambda}_{q^2}$$

Here

$$(q)_k = \prod_{l=1}^k (1 - q^l) \quad \binom{a}{b}_q = \frac{(q)_a}{(q)_b (q)_{a-b}}$$

- The 3DR satisfies the following tetrahedron equation:

$$\mathcal{R}_{124} \mathcal{R}_{135} \mathcal{R}_{236} \mathcal{R}_{456} = \mathcal{R}_{456} \mathcal{R}_{236} \mathcal{R}_{135} \mathcal{R}_{124}$$

- 3DR = intertwiner of irreps of quantum coordinate ring $A_q(A_2)$

$$\mathcal{R} \circ \pi_1 \otimes \pi_2 \otimes \pi_1(\Delta^{\text{op}}(g)) = \pi_2 \otimes \pi_1 \otimes \pi_2(\Delta(g)) \circ \mathcal{R} \quad \forall g \in A_q(A_2)$$

$$\pi_i : A_q(A_2) \rightarrow \text{End}(F)$$

- This gives a linearization method for tetrahedron equation.
- “121” and “212” are associated with the longest element of Weyl group.

- Set $\mathcal{L} \in \text{End}(V \otimes V \otimes F)$ by [Bazhanov-Sergeev06]

$$\mathcal{L}(u_i \otimes u_j \otimes |k\rangle) = \sum_{a,b \in \{0,1\}, c \in \mathbb{Z}_{\geq 0}} \mathcal{L}_{i,j,k}^{a,b,c} u_a \otimes u_b \otimes |c\rangle$$

$$\mathcal{L}_{0,0,k}^{0,0,c} = \mathcal{L}_{1,1,k}^{1,1,c} = \delta_{k,c}, \quad \mathcal{L}_{0,1,k}^{0,1,c} = -\delta_{k,c} q^{k+1}, \quad \mathcal{L}_{1,0,k}^{1,0,c} = \delta_{k,c} q^k,$$

$$\mathcal{L}_{1,0,k}^{0,1,c} = \delta_{k-1,c} (1 - q^{2k}), \quad \mathcal{L}_{0,1,k}^{1,0,c} = \delta_{k+1,c}$$

- The 3DL satisfies the following tetrahedron equation:

$$\mathcal{L}_{124} \mathcal{L}_{135} \mathcal{L}_{236} \mathcal{R}_{456} = \mathcal{R}_{456} \mathcal{L}_{236} \mathcal{L}_{135} \mathcal{L}_{124} \cdots (*)$$

- BS obtained the 3DL by ansatz so that the tetrahedron equation of (*) type has a non-trivial solution, and solved (*) for the 3DR.
 - They gives an alternative derivation for the 3DR.
 - Algebraic origins of the 3DL has been still unclear.

Reduction to Yang-Baxter equation

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- We use the following graphical notations:

$$= \mathcal{R}_{i,j,k}^{a,b,c} \text{ or } \mathcal{L}_{i,j,k}^{a,b,c}$$

$$= \mathcal{R}_{i,j,k}^{a,b,c}$$

- Fact:

$$=$$

- \mathbf{h} : number counting operator of the Fock spaces

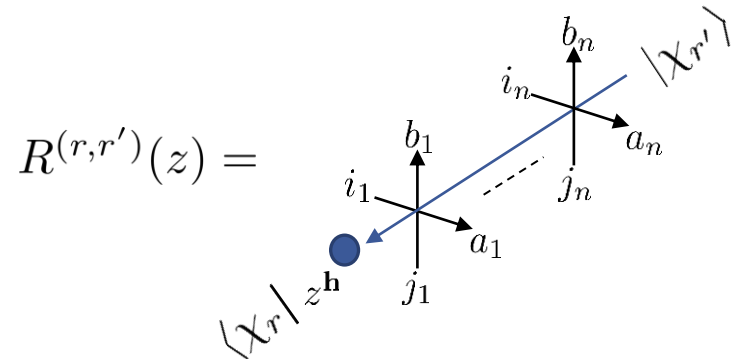
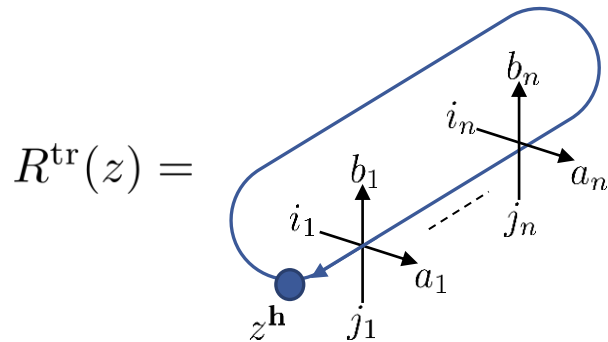
$$\mathbf{h} |m\rangle = m |m\rangle$$

$$\mathbf{h} u_m = m u_m$$

- By multiplying \mathcal{R}^{-1} and taking trace on the auxiliary spaces, we obtain the Yang-Baxter equation.

Family of matrix product solutions

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- These solutions are characterized as the R matrices associated with some quantum affine algebras. [Kuniba-Okado-Sergeev15]

	$R^{\text{tr}}(z)$	$R^{(r,r')}(z)$
= 3DR	$U_q(A_{n-1}^{(1)})$ symmetric tensor rep.	$U_q(D_{n+1}^{(2)}), U_q(A_{2n}^{(2)}), U_q(C_n^{(1)})$ Fock rep.
= 3DL	$U_q(A_{n-1}^{(1)})$ fundamental rep.	$U_q(D_{n+1}^{(2)}), U_q(B_n^{(1)}), U_q(D_n^{(1)})$ spin rep.

- Moreover, by mixing uses of the 3DR & 3DL, we also obtain the R matrices associated with *generalized quantum groups*.

■ Motivation:

- Why do the 3DR & 3DL lead to such similar results, although they have different origins?

- We study transition matrices of PBW bases of $U_q^+(sl(m|n))$ motivated by the KOY theorem which holds for non-super cases.

■ Theorem: [Kuniba-Okado-Yamada13] (Rough Statement)

- g : arbitrary finite-dimensional simple Lie algebra
- Φ : intertwiner of irreducible representations of $A_q(g)$
- γ : transition matrix of PBW bases of the nilpotent subalgebra of $U_q(g)$
- Then, we have $\Phi = \gamma$.

- For $\bigcirc \text{---} \bigcirc$, the 3DR gives the transition matrix for $U_q(A_2)$.

- One of our result is that the 3DL is exactly the transition matrix for the quantum superalgebra associated with $\bigcirc \text{---} \bigotimes$.

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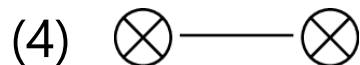
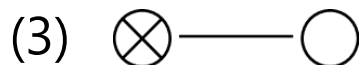
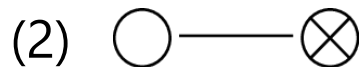
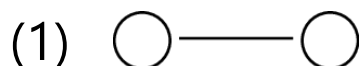
Root data of Lie superalgebra of type A

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- We only consider rank 2 cases.
- Lie superalgebra $sl(m|n)$ ($m+n=3$)
 - ▣ Simple roots: $\Pi = \{\alpha_1, \alpha_2\}$
 - ▣ Parity: $p: \Pi \rightarrow \{0,1\}$
 - ▣ Positive roots: $\Phi = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ $p(\alpha_1 + \alpha_2) = p(\alpha_1) + p(\alpha_2) \pmod{2}$
 - ▣ Even & odd parts of positive roots: $\Phi = \Phi_{\text{even}} \cup \Phi_{\text{odd}}$
- Dynkin diagram for $sl(m|n)$ ($m+n=3$)
 - ▣ Prepare connected 2 dots and decorate the i -th dot by

\bigcirc for $\alpha_i \in \Phi_{\text{even}}$, \bigotimes for $\alpha_i \in \Phi_{\text{odd}}$

- ▣ All diagrams are listed as follows



PBW bases for $U_q^+(sl(m|n))$ of rank 2

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- $U_q^+ = U_q^+(sl(m|n))$: nilpotent subalgebra of $U_q(sl(m|n))$ ($m+n=3$)

- Generator: $\{e_1, e_2\}$

- If $\alpha_i \in \Phi_{\text{even}}$, $e_i^2 e_{3-i} - (q+q^{-1})e_i e_{3-i} e_i + e_{3-i} e_i^2 = 0$

- If $\alpha_i \in \Phi_{\text{odd}}$, $e_i^2 = 0$

- PBW bases for U_q^+

- We set $e_{ij} = e_i e_j - q e_j e_i$. Then, the following B_1 & B_2 give bases of U_q^+ :

$$B_1 = \left\{ e_1^{(a_{\alpha_1})} e_{21}^{(a_{\alpha_1+\alpha_2})} e_2^{(a_{\alpha_2})} \mid a_{\alpha} \in \mathbb{Z}_{\geq 0} \ (\alpha \in \Phi_{\text{even}}), \ a_{\alpha} \in \{0, 1\} \ (\alpha \in \Phi_{\text{odd}}) \right\}$$

$$B_2 = \left\{ e_2^{(a_{\alpha_2})} e_{12}^{(a_{\alpha_1+\alpha_2})} e_1^{(a_{\alpha_1})} \mid a_{\alpha} \in \mathbb{Z}_{\geq 0} \ (\alpha \in \Phi_{\text{even}}), \ a_{\alpha} \in \{0, 1\} \ (\alpha \in \Phi_{\text{odd}}) \right\}$$

- $e_i^{(a)}$: divided power given by $e_i^{(a)} = e_i^a / [a]!$ $[m]! = \prod_{k=1}^m [k]$

- Transition matrix γ

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}}$$

$$e_2^{(a)} e_{12}^{(b)} e_1^{(c)} = \sum_{i,j,k} \gamma_{i,j,k}^{a,b,c} e_1^{(k)} e_{21}^{(j)} e_2^{(i)}$$

The case $\bigcirc \text{---} \bigcirc$

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■ Root system

$$\Phi_{\text{even}} = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} \quad \Phi_{\text{odd}} = \{\}$$

■ Transition matrix

$$e_2^{(a)} e_{12}^{(b)} e_1^{(c)} = \sum_{i,j,k} \gamma_{i,j,k}^{a,b,c} e_1^{(k)} e_{21}^{(j)} e_2^{(i)} \cdots (*) \quad i, j, k, a, b, c \in \mathbb{Z}_{\geq 0}$$

■ Theorem: [Kuniba-Okado-Yamada13]

$$\gamma_{i,j,k}^{a,b,c} = \mathcal{R}_{i,j,k}^{a,b,c}$$

■ Example: For $(a, b, c) = (0, 1, 1)$, $(*)$ becomes

$$e_{12} e_1 = \mathcal{R}_{0,1,1}^{0,1,1} e_1 e_{21} + \mathcal{R}_{1,0,2}^{0,1,1} \frac{e_1^2}{q + q^{-1}} e_2$$

$$-q e_2 e_1^2 + (1 + q^2) e_1 e_2 e_1 - q e_1^2 e_2 = 0 \quad \because \mathcal{R}_{0,1,1}^{0,1,1} = -q^2, \mathcal{R}_{1,0,2}^{0,1,1} = 1 - q^4$$

This is a relation of $\bigcirc \text{---} \bigcirc$.

The case $\bigcirc \text{---} \bigotimes$

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■ Root system

$$\Phi_{\text{even}} = \{\alpha_1\} \quad \Phi_{\text{odd}} = \{\alpha_2, \alpha_1 + \alpha_2\}$$

■ Transition matrix

$$e_2^{(a)} e_{12}^{(b)} e_1^{(c)} = \sum_{i,j,k} \gamma_{i,j,k}^{a,b,c} e_1^{(k)} e_{21}^{(j)} e_2^{(i)} \cdots (*) \quad \begin{array}{l} k, c \in \mathbb{Z}_{\geq 0} \\ i, j, a, b \in \{0, 1\} \end{array}$$

■ Theorem: [Y20]

$$\gamma_{i,j,k}^{a,b,c} = \mathcal{L}_{i,j,k}^{a,b,c}$$

■ Example: For $(a, b, c) = (0, 1, 1)$, $(*)$ becomes

$$e_{12} e_1 = \mathcal{L}_{0,1,1}^{0,1,1} e_1 e_{21} + \mathcal{L}_{1,0,2}^{0,1,1} \frac{e_1^2}{q + q^{-1}} e_2$$

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This is a relation of $\bigcirc \text{---} \bigotimes$.

The case \otimes — \otimes

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■ Root system

$$\Phi_{\text{even}} = \{\alpha_1 + \alpha_2\} \quad \Phi_{\text{odd}} = \{\alpha_1, \alpha_2\}$$

■ Transition matrix

$$e_2^{(a)} e_{12}^{(b)} e_1^{(c)} = \sum_{i,j,k} \gamma_{i,j,k}^{a,b,c} e_1^{(k)} e_{21}^{(j)} e_2^{(i)} \cdots (*) \quad \begin{array}{l} j, b \in \mathbb{Z}_{\geq 0} \\ i, k, a, c \in \{0, 1\} \end{array}$$

■ Theorem: [Y20]

$$\gamma_{i,j,k}^{a,b,c} = \mathcal{N}_{i,j,k}^{a,b,c}$$

□ Here, we define $\mathcal{N} \in \text{End}(V \otimes F \otimes V)$ as follows:

$$\mathcal{N}(u_i \otimes |j\rangle \otimes u_k) = \sum_{a,c \in \{0,1\}, b \in \mathbb{Z}_{\geq 0}} \mathcal{N}_{i,j,k}^{a,b,c} u_a \otimes |b\rangle \otimes u_c$$

$$\mathcal{N}_{0,j,0}^{0,b,0} = \delta_{j,b} q^j, \quad \mathcal{N}_{1,j,1}^{1,b,1} = -\delta_{j,b} q^{j+1}, \quad \mathcal{N}_{0,j,1}^{0,b,1} = \mathcal{N}_{1,j,0}^{1,b,0} = \delta_{j,b},$$

$$\mathcal{N}_{1,j,1}^{0,b,0} = \delta_{j+1,b} q^j (1 - q^2), \quad \mathcal{N}_{0,j,0}^{1,b,1} = \delta_{j-1,b} [j]_q,$$

- Transition matrices γ for higher rank cases are constructed as compositions of ones for rank 2.
- By constructing γ for rank 3 in two ways, we obtain the tetrahedron equation.

$$e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)} = \sum_{i_1, i_2, i_3, i_4, i_5, i_6} \gamma_{i_1, i_2, i_3, i_4, i_5, i_6}^{o_1, o_2, o_3, o_4, o_5, o_6} e_1^{(i_6)} e_{21}^{(i_5)} e_{321}^{(i_4)} e_2^{(i_3)} e_{32}^{(i_2)} e_3^{(i_1)}$$

■ For $\bigcirc \text{---} \bigcirc \text{---} \bigcirc$

$$\mathcal{R}_{123} \mathcal{R}_{145} \mathcal{R}_{246} \mathcal{R}_{356} = \mathcal{R}_{356} \mathcal{R}_{246} \mathcal{R}_{145} \mathcal{R}_{123}$$

■ For $\bigcirc \text{---} \bigcirc \text{---} \bigotimes$

$$\mathcal{L}_{123} \mathcal{L}_{145} \mathcal{L}_{246} \mathcal{R}_{356} = \mathcal{R}_{356} \mathcal{L}_{246} \mathcal{L}_{145} \mathcal{L}_{123}$$

■ For $\bigcirc \text{---} \bigotimes \text{---} \bigotimes$ (new solution to the tetrahedron equation)

$$\mathcal{N}(q^{-1})_{123} \mathcal{N}(q^{-1})_{145} \mathcal{R}_{246} \mathcal{L}_{356} = \mathcal{L}_{356} \mathcal{R}_{246} \mathcal{N}(q^{-1})_{145} \mathcal{N}(q^{-1})_{123}$$

■ Remark:

1. Type B cases give new solutions to the 3D reflection equations, which describes boundary integrability in three dimensions.
2. The crystal limit for \mathcal{L}, \mathcal{N} gives a super analog of transition maps of Lusztig's parametrizations of the canonical basis of quantum algebras.

■ Summary:

1. The 3DL is characterized as the transition matrix for $\bigcirc \longrightarrow \bigotimes$.
2. A new solution to the tetrahedron equation the 3DN is obtained by considering the transition matrix for $\bigotimes \longrightarrow \bigotimes$.