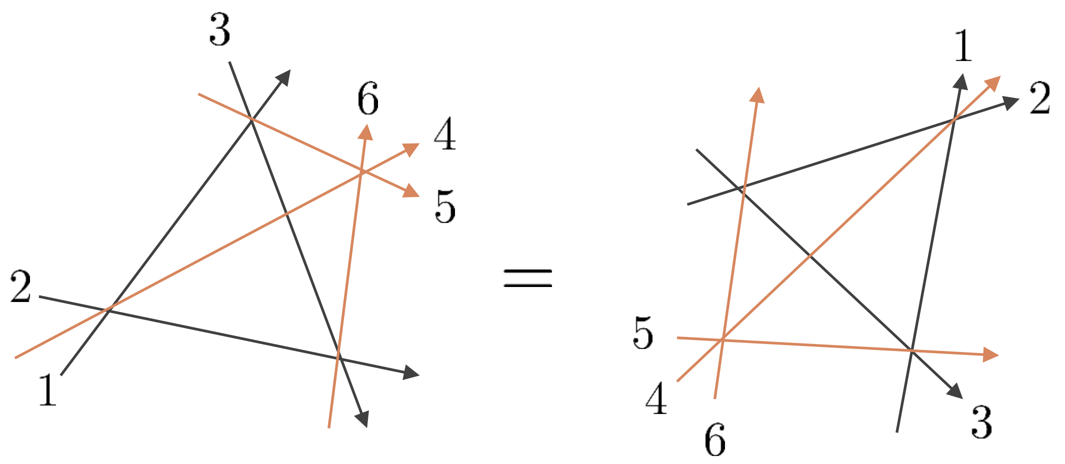
Tetrahedron and 3D reflection equation from PBW bases of the nilpotent subalgebra of quantum superalgebras

Mathematical Society of Japan Spring Meeting
Keio University@2021/03/17

Akihito Yoneyama (米山 瑛仁)
Institute of Physics, University of Tokyo, Komaba

Based on: AY, arXiv: 2012.13385

Tetrahedron equation (TE)



- 3D analog of Yang-Baxter equation (YBE)

$$\mathcal{R}_{124}\mathcal{R}_{135}\mathcal{R}_{236}\mathcal{R}_{456} = \mathcal{R}_{456}\mathcal{R}_{236}\mathcal{R}_{135}\mathcal{R}_{124}$$

- Several solutions are known although less systematically than YBE.
  - Family of Zamolodchikov model: Zamolodchikov (81), Bazhanov-Baxter (92)
  - Product of solution to YBE: Carter-Saito (96)
  - Interwtiner for quantum coordinate ring $A_q(A_2)$: Kapranov-Voevodsky (94) = Transition matrix of PBW bases of quantum group $U_q^+(A_2)$: Sergeev (07)
  - solution to local YBE by ansatz: Bazhanov-Sergeev (06)

Two solutions on Fock spaces: 3DR & 3DL

- Fock spaces: $F = \bigoplus_{m=0,1,2,\cdots} \mathbb{C}\,|m\rangle$ and $V = \bigoplus_{m=0,1} \mathbb{C}u_m$
- 3DR = transition matrix of PBW bases of quantum group $U_q^+(A_2)$
  - $\mathcal{R} \in \mathrm{End}\left(F^{\otimes 3}\right)$
  - $\mathcal{R}\,|i\rangle \otimes |j\rangle \otimes |k\rangle = \sum_{a,b,c} \mathcal{R}_{ijk}^{abc}\,|a\rangle \otimes |b\rangle \otimes |c\rangle$
  - $$\mathcal{R}_{i,j,k}^{a,b,c} = \delta_{i+j}^{a+b}\delta_{j+k}^{b+c} \sum_{\lambda,\mu\geq 0,\lambda+\mu=b} (-1)^\lambda q^{i(c-j)+(k+1)\lambda+\mu(\mu-k)} \frac{(q^2)_{c+\mu}}{(q^2)_c} \binom{i}{\mu}_{q^2} \binom{j}{\lambda}_{q^2}$$
  - $\mathcal{R}_{124}\mathcal{R}_{135}\mathcal{R}_{236}\mathcal{R}_{456} = \mathcal{R}_{456}\mathcal{R}_{236}\mathcal{R}_{135}\mathcal{R}_{124}$
- 3DL = solution to local YBE by ansatz
  - $\mathcal{L} \in \mathrm{End}\,(V \otimes V \otimes F)$
  - $\mathcal{L}(u_i \otimes u_j \otimes |k\rangle) = \sum \mathcal{L}_{i,j,k}^{a,b,c} u_a \otimes u_b \otimes |c\rangle$
  - $$\mathcal{L}_{0,0,k}^{0,0,c} = \mathcal{L}_{1,1,k}^{1,1,c} = \delta_{k,c}, \quad \mathcal{L}_{0,1,k}^{0,1,c} = -\delta_{k,c}q^{k+1}, \quad \mathcal{L}_{1,0,k}^{1,0,c} = \delta_{k,c}q^k,$$ $$\mathcal{L}_{1,0,k}^{0,1,c} = \delta_{k-1,c}(1-q^{2k}), \quad \mathcal{L}_{0,1,k}^{1,0,c} = \delta_{k+1,c}$$
  - $\mathcal{L}_{124}\mathcal{L}_{135}\mathcal{L}_{236}\mathcal{R}_{456} = \mathcal{R}_{456}\mathcal{L}_{236}\mathcal{L}_{135}\mathcal{L}_{124} \qquad \mathcal{R}: \text{3DR}$

Aim & Motivation

- Aim: Find a parallel origin for 3DR and 3DL
- Motivation:
  - Generally, infinitely many solutions to YBE are obtained from one solution to TE.
  - For 3DR and 3DL, we can obtain R matrices with *spectral parameters*.
  - They are identified with R matrices associated with some quantum affine algebras $U_q(\hat{g})$.

Solution to TE	Solution to YBE
3DR	$U_q(A_{n-1}^{(1)})$, symmetric tensor rep.
3DL	$U_q(A_{n-1}^{(1)})$, fundamental rep.

look similar

e.g. Kuniba-Okado-Sergeev (15)

Main result for TE (1/2)

- Theorem [Y20]:
  - Transition matrices of PBW bases of the nilpotent subalgebra of quantum superalgebras of type A of rank 2 are given as follows:

Dynkin diagram	Transition matrix
$\bigcirc - \bigcirc$	3DR
$\bigcirc - \otimes$ $\quad$ $\otimes - \bigcirc$	3DL
$\otimes - \otimes$	3DN

$s_1 s_2 s_1 = s_2 s_1 s_2$

  - Here, 3DN is a new object as $\mathcal{N} \in \mathrm{End}\,(V \otimes F \otimes V)$
  - By constructing transition matrices for rank 3 in two ways, we obtain several solutions to TE, which 3DR, L and N satisfy.

Main result for TE (2/2)

- For $\bigcirc\!-\!\!-\!\!-\bigcirc\!-\!\!-\!\!-\otimes$

$$\mathcal{L}_{123}\mathcal{L}_{145}\mathcal{L}_{246}\mathcal{R}_{356} = \mathcal{R}_{356}\mathcal{L}_{246}\mathcal{L}_{145}\mathcal{L}_{123}$$

  - <u>Sketch of Proof</u>:

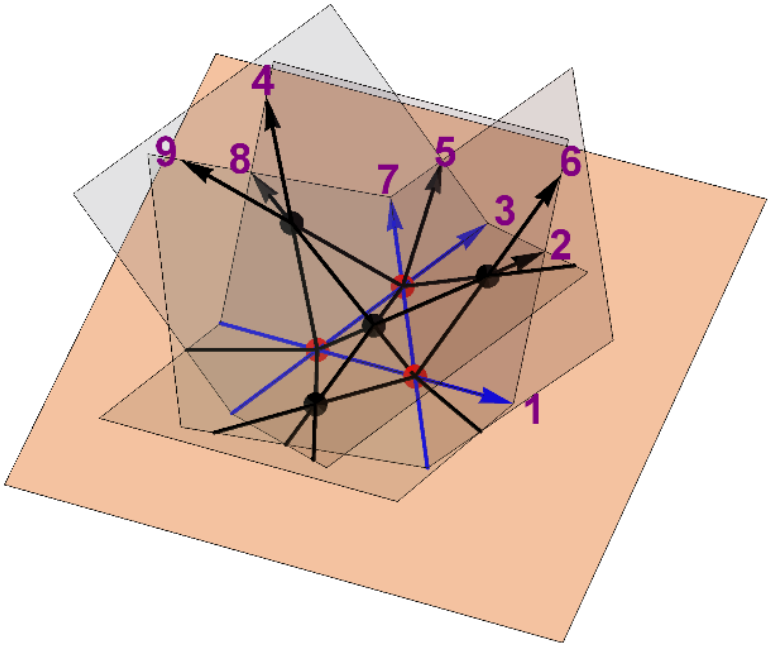
e_i : generator of U_q

$$
\begin{aligned}
&\underline{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)}}\, e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)} \\
&= \sum \mathcal{L}^{o_1,o_2,o_3}_{x_1,x_2,x_3}\, e_2^{(x_3)} e_{32}^{(x_2)}\, \underline{e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)}}\, e_1^{(o_6)} \\
&= \sum \mathcal{L}^{o_1,o_2,o_3}_{x_1,x_2,x_3} \mathcal{L}^{x_1,o_4,o_5}_{i_1,x_4,x_5}\, e_2^{(x_3)}\, \underline{e_{32}^{(x_2)} e_{12}^{(x_5)}}\; \underline{e_{3(12)}^{(x_4)}}\; \underline{e_3^{(i_1)} e_1^{(o_6)}} \\
&= \sum \mathcal{L}^{o_1,o_2,o_3}_{x_1,x_2,x_3} \mathcal{L}^{x_1,o_4,o_5}_{i_1,x_4,x_5}\, e_2^{(x_3)} e_{12}^{(x_5)}\, \underline{e_{32}^{(x_2)} e_{1(32)}^{(x_4)} e_1^{(o_6)}}\, e_3^{(i_1)} \\
&= \sum \mathcal{L}^{o_1,o_2,o_3}_{x_1,x_2,x_3} \mathcal{L}^{x_1,o_4,o_5}_{i_1,x_4,x_5} \mathcal{L}^{x_2,x_4,o_6}_{i_2,i_4,x_6}\, \underline{e_2^{(x_3)} e_{12}^{(x_5)} e_1^{(x_6)}}\, e_{321}^{(i_4)} e_{32}^{(i_2)} e_3^{(i_1)} \\
&= \sum \mathcal{L}^{o_1,o_2,o_3}_{x_1,x_2,x_3} \mathcal{L}^{x_1,o_4,o_5}_{i_1,x_4,x_5} \mathcal{L}^{x_2,x_4,o_6}_{i_2,i_4,x_6} \mathcal{R}^{x_3,x_5,x_6}_{i_3,i_5,i_6}\, e_1^{(i_6)} e_{21}^{(i_5)}\, \underline{e_2^{(i_3)} e_{321}^{(i_4)}}\, e_{32}^{(i_2)} e_3^{(i_1)} \\
&= \sum \mathcal{L}^{o_1,o_2,o_3}_{x_1,x_2,x_3} \mathcal{L}^{x_1,o_4,o_5}_{i_1,x_4,x_5} \mathcal{L}^{x_2,x_4,o_6}_{i_2,i_4,x_6} \mathcal{R}^{x_3,x_5,x_6}_{i_3,i_5,i_6}\, e_1^{(i_6)} e_{21}^{(i_5)} e_{321}^{(i_4)} e_2^{(i_3)} e_{32}^{(i_2)} e_3^{(i_1)}
\end{aligned}
$$

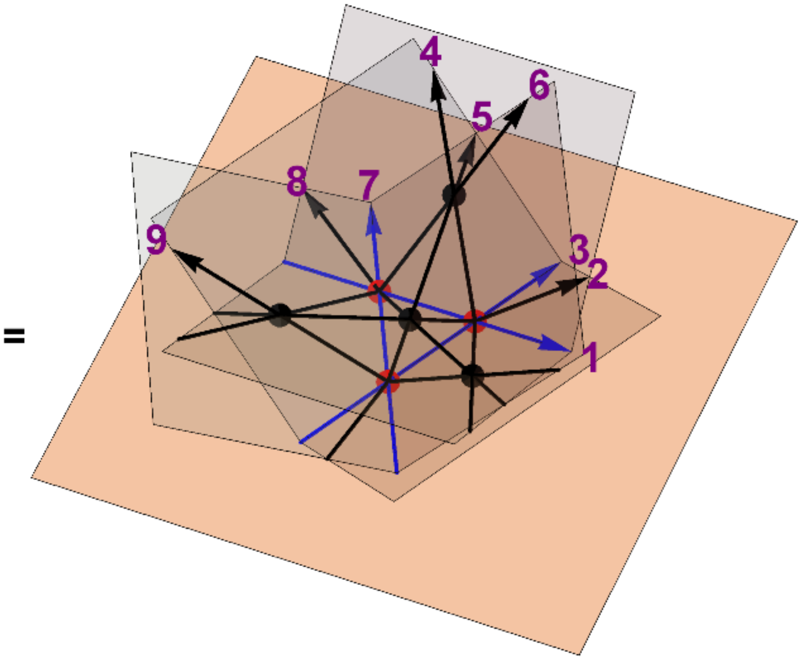
- For $\bigcirc\!-\!\!-\!\!-\otimes\!-\!\!-\!\!-\otimes$ (new solution)

$$\mathcal{N}(q^{-1})_{123}\mathcal{N}(q^{-1})_{145}\mathcal{R}_{246}\mathcal{L}_{356} = \mathcal{L}_{356}\mathcal{R}_{246}\mathcal{N}(q^{-1})_{145}\mathcal{N}(q^{-1})_{123}$$

3D reflection equation (3DRE)



=



- Boundary analog of TE proposed by Isaev-Kulish (97)

$$\mathcal{R}_{489}\mathcal{J}_{3579}\mathcal{R}_{269}\mathcal{R}_{258}\mathcal{J}_{1678}\mathcal{J}_{1234}\mathcal{R}_{456} = \mathcal{R}_{456}\mathcal{J}_{1234}\mathcal{J}_{1678}\mathcal{R}_{258}\mathcal{R}_{269}\mathcal{J}_{3579}\mathcal{R}_{489}$$

- Only two solutions are known.
  - Interwtiner of quantum coordinate ring $A_q(B_2)$ or $A_q(C_2)$
    = Transition matrix of PBW bases of quantum group $U_q(B_2)$ or $U_q(C_2)$

Kuniba-Okado (12, 13), Kuniba-Okado-Yamada (13)

Main result for 3DRE

- Theorem [Y20]:
  - Transition matrices of PBW bases of the nilpotent subalgebra of quantum superalgebras of type B of rank 2 are given as follows:

Dynkin diagram	Transition matrix	Space
$\bigcirc \Rightarrow \bigcirc$	3DJ	$F^{\otimes 4}$
$\otimes \Rightarrow \bigcirc$	3DX (new)	$F \otimes V \otimes F \otimes V$
$\otimes \Rightarrow \bullet$	3DY (new)	$F \otimes V \otimes F \otimes V$
$\bigcirc \Rightarrow \bullet$	3DZ (new)	$F^{\otimes 4}$

  - We obtained explicit formulae for 3DX and 3DY.
  - Any matrix elements for 3DZ can be calculated by recurrence equations.
  - The following cases give new solutions to 3DRE.

$\otimes - \bigcirc \Rightarrow \bigcirc$ $\otimes - \otimes \Rightarrow \bigcirc$ $\bigcirc - \bigcirc \Rightarrow \bullet$ $\bigcirc - \otimes \Rightarrow \bullet$

$$
\begin{aligned}
&\mathcal{L}_{456}\mathcal{N}(q^{-1})_{489}\mathcal{Y}(q^{-1})_{3579}\mathcal{N}(q^{-1})_{269}\mathcal{L}_{258}\mathcal{J}_{1678}\mathcal{X}_{1234} \\
&= \mathcal{X}_{1234}\mathcal{J}_{1678}\mathcal{L}_{258}\mathcal{N}(q^{-1})_{269}\mathcal{Y}(q^{-1})_{3579}\mathcal{N}(q^{-1})_{489}\mathcal{L}_{456}
\end{aligned}
$$

Concluding remarks

- Remark:
  - The crystal limit for obtained transition matrices gives a super analog of transition maps of Lusztig's parametrizations of the canonical basis of U_q.
- Proposition [Y20]:
  - $\mathcal{L}_{i,j,k}^{a,b,c} = \lim_{q\to 0} \mathcal{L}(q)_{i,j,k}^{a,b,c}$ gives a non-trivial bijection on $\{0,1\}^2 \times \mathbb{Z}_{\geq 0}$.
  - Non-zero elements are given by

$$\mathcal{L}_{0,0,k}^{0,0,c} = \mathcal{L}_{1,1,k}^{1,1,c} = \delta_{k,c}, \quad \mathcal{L}_{0,1,k}^{1,0,c} = \delta_{k+1,c}, \quad \mathcal{L}_{1,0,0}^{1,0,0} = 1, \quad \mathcal{L}_{1,0,k}^{0,1,c} = \delta_{k-1,c}$$

  - $\mathcal{N}_{i,j,k}^{a,b,c} = \lim_{q\to 0} \left(\frac{[b]_q!}{[j]_q!} \mathcal{N}(q)_{i,j,k}^{a,b,c} \right)$ also gives a non-trivial bijection.
- Summary:
  - The 3DL is characterized as the transition matrix for $\bigcirc$——$\bigotimes$.
  - The 3DN is obtained as a new solution to the tetrahedron equation by considering the transition matrix for $\bigotimes$——$\bigotimes$.
- Outlook: Obtained transition matrices are characterized as intertwiners for quantum super coordinate rings?