3D reflection maps from tetrahedron maps

Combinatorial Representation Theory
and Connections with Related Fields@2021/10/19

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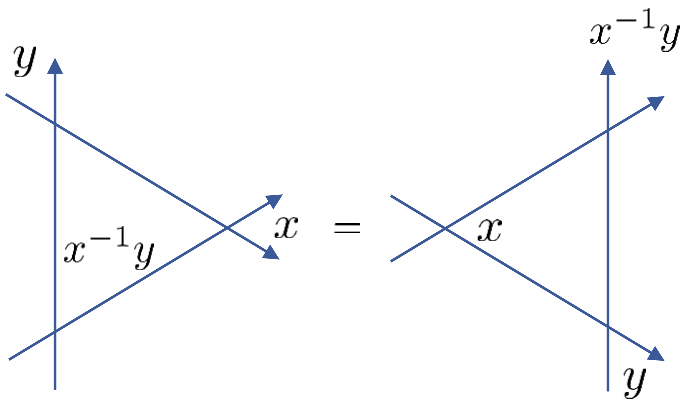
Based on: AY, Math. Phys. Anal. Geom. **24** 21 (2021)

Outline

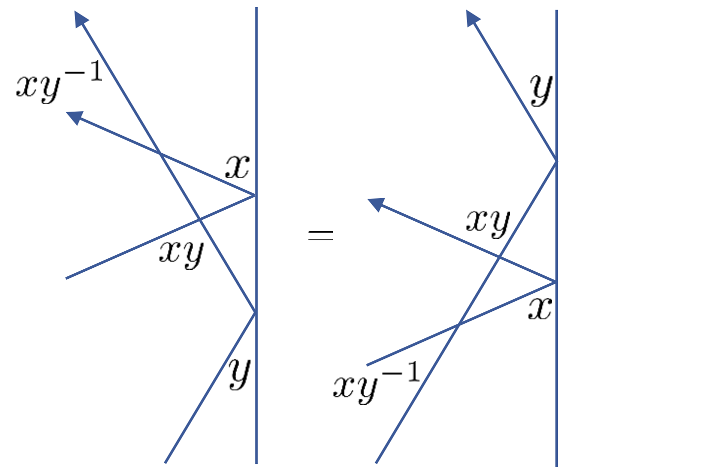
Integrability in 2D

- Bulk: Yang-Baxter equation



$$R_{12}(x)R_{13}(y)R_{23}(x^{-1}y)$$
$$= R_{23}(x^{-1}y)R_{13}(y)R_{12}(x)$$

- Boundary: Reflection equation



$$R_{\mathbf{12}}(xy^{-1})K_{\mathbf{2}}(x)R_{\mathbf{21}}(xy)K_{\mathbf{1}}(y)$$
$$= K_{\mathbf{1}}(y)R_{\mathbf{12}}(xy)K_{\mathbf{2}}(x)R_{\mathbf{21}}(xy^{-1})$$

- <u>Problem settings</u>:
  - R, K = matrices on linear spaces
  - R, K = maps on sets (set-theoretical solution) [Drinfeld92]

Yang-Baxter maps & reflection maps

- We call set-theoretical solutions to Yang-Baxter equation *Yang-Baxter maps*, and so on.
- Yang-Baxter maps have been extensively studied in connection with various topics. [Veselov07]
- Several reflection maps are constructed
  - via directly solving the reflection equation [Kuniba-Okado-Yamada05]
    - Later, these solutions are *q-melted* by using coideal subalgebras of U_q. [Kuniba-Okado-Y19], [Kusano-Okado20]
  - from Yang-Baxter maps [Caudrelier-Zhang14], [Kuniba-Okado19]
  - by ring-theoretic methods [Smoktunowicz-Vendramin-Weston20]
- Here, we briefly review a result of [Kuniba-Okado19].

Combinatorial R matrices

- KR crystal for $A_{n-1}^{(1)}$
  - $B^{k,l}$ = {SST of $k \times l$ rectangular shape with letters from {1,2,...,n}}
    - SST: semi-standard tableaux
    - $1 \leq k \leq n-1, l \geq 1$
  - $\tilde{e}_i, \tilde{f}_i : B^{k,l} \to B^{k,l} \cup \{0\}$ (Kashiwara operators)
  - For KR crystals B_1, B_2, the actions of Kashiwara operators are also defined on $B_1 \otimes B_2 = \{b_1 \otimes b_2 | b_1 \in B_1, b_2 \in B_2\}$ and $B_1 \otimes B_2$ is connected.

e.g.

$$\begin{array}{|c|c|c|}\hline 1 \leq & 1 \leq & 3 \\ \wedge & \wedge & \wedge \\ \hline 2 & 4 & 6 \\ \hline \end{array}$$

- Combinatorial R matrices
  - A map $R_{B_1,B_2} : B_1 \otimes B_2 \to B_2 \otimes B_1$ given by

$$R_{B_1,B_2}\left(\tilde{k}_i(b_1 \otimes b_2)\right) = \tilde{k}_i(R_{B_1,B_2}(b_1 \otimes b_2)) \quad (\tilde{k}_i = \tilde{e}_i, \tilde{f}_i)$$

    is uniquely determined.
  - Combinatorial R matrices satisfy YBE from $B_1 \otimes B_2 \otimes B_3$ to $B_3 \otimes B_2 \otimes B_1$:

$$\begin{aligned}&(1 \otimes R_{B_1,B_2})(R_{B_1,B_3} \otimes 1)(1 \otimes R_{B_2,B_3}) \\ &= (R_{B_2,B_3} \otimes 1)(1 \otimes R_{B_1,B_3})(R_{B_1,B_2} \otimes 1)\end{aligned}$$

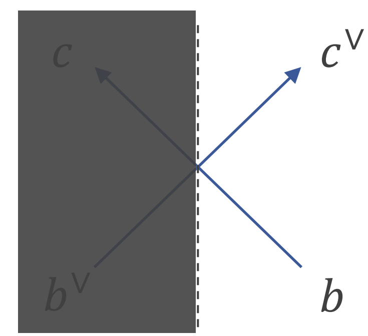
  - The action can be calculated by the *tableau product rule*.

Combinatorial K matrices

- We define $\vee: B^{k,l} \to B^{n-k,l}$ by

$$b^{\vee} = \underbrace{\begin{array}{|c|c|c|c|}\hline \overline{\lambda_l} & \cdots & \overline{\lambda_2} & \overline{\lambda_1} \\ \hline \end{array}}_{}\Big\}\, n-k \quad \text{for } b = \begin{array}{|c|c|c|c|}\hline \lambda_1 & \lambda_2 & \cdots & \lambda_l \\ \hline \end{array}\Big\}\, k$$

  - $\lambda_i \in \mathbb{N}^k$, $\overline{\lambda_i} \in \mathbb{N}^{n-k}$
  - The complement is taken over $\{1,2,\cdots,n\}$.



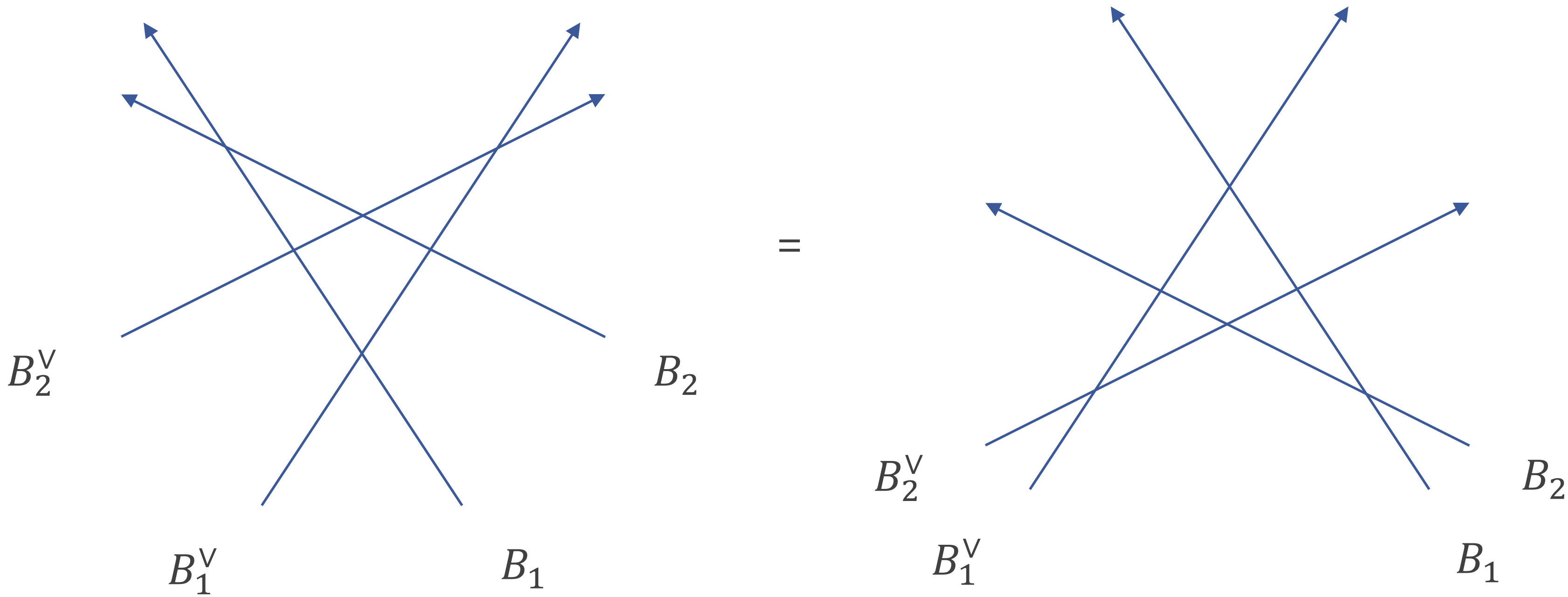
- Proposition [Kuniba-Okado19]:
  - For a KR crystal B, we have $R_{B^{\vee},B}(b^{\vee} \otimes b) = c \otimes c^{\vee}$.
- Definition:
  - Using b and c above, we define the combinatorial K matrix by

$$K_B: B \to B^{\vee},\ b \to c^{\vee}$$

- Theorem: [Kuniba-Okado19]
  - The combinatorial R and K matrices satisfy RE from $B_1 \otimes B_2 \to B_1^{\vee} \otimes B_2^{\vee}$:

$$R_{B_2^{\vee},B_1^{\vee}}(K_{B_2} \otimes 1)R_{B_1^{\vee},B_2}(K_{B_1} \otimes 1) = (K_{B_1} \otimes 1)\, R_{B_2^{\vee},B_1}(K_{B_2} \otimes 1)R_{B_1,B_2}$$

Key lemma for reflection equation

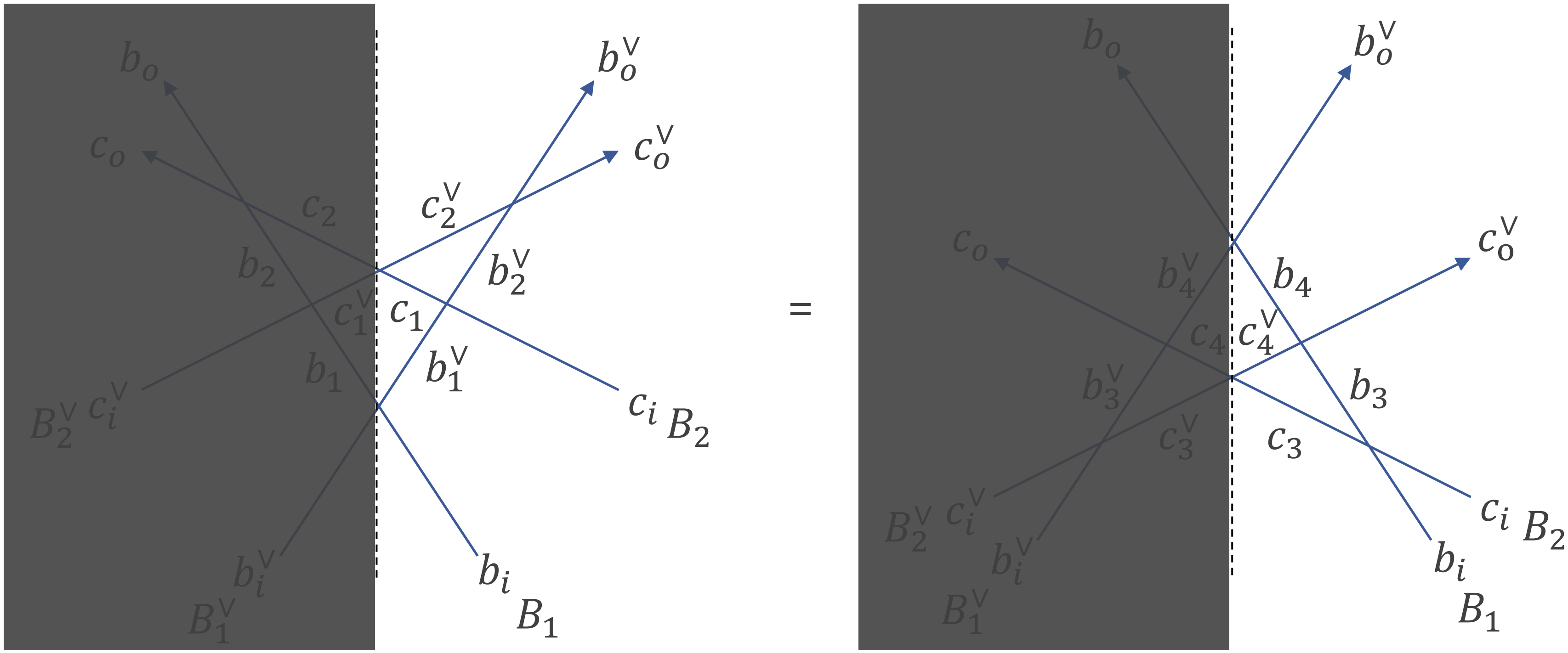


- Lemma:

$$(R_{B_2^\vee,B_1^\vee}R_{B_1,B_2})R_{B_2^\vee,B_2}(R_{B_1^\vee,B_2}R_{B_2^\vee,B_1})R_{B_1^\vee,B_1}$$
$$= R_{B_1^\vee,B_1}(R_{B_2^\vee,B_1}R_{B_1^\vee,B_2})R_{B_2^\vee,B_2}(R_{B_1,B_2}R_{B_2^\vee,B_1^\vee})$$

- Proof:
  - By repeated uses of YBE, we have the above equation.

Proof of reflection equation

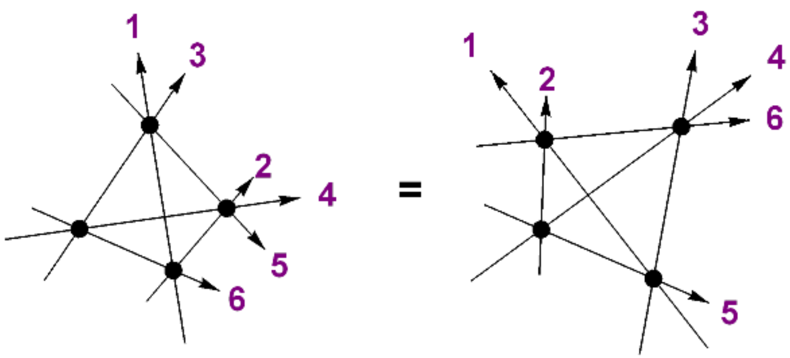


- Sketch of proof:
  - Input b_i, c_i on B_1, B_2 and their duals on $B_1^\vee, B_2^\vee$.
  - Use $R_{B^\vee,B}\,(b^\vee \otimes b) = c \otimes c^\vee$.
  - Use $R_{B_1,B_2}\,(b_1 \otimes b_2) = c_2 \otimes c_1 \Rightarrow R_{B_2^\vee,B_1^\vee}\,(b_2^\vee \otimes b_1^\vee) = c_1^\vee \otimes c_2^\vee$.
  - Just viewing the right parts of both sides, we then obtain RE.
    - because the above equation is equal to the direct product of RE for ${}^\forall b_i, c_i$.

Outline

Integrability in 3D

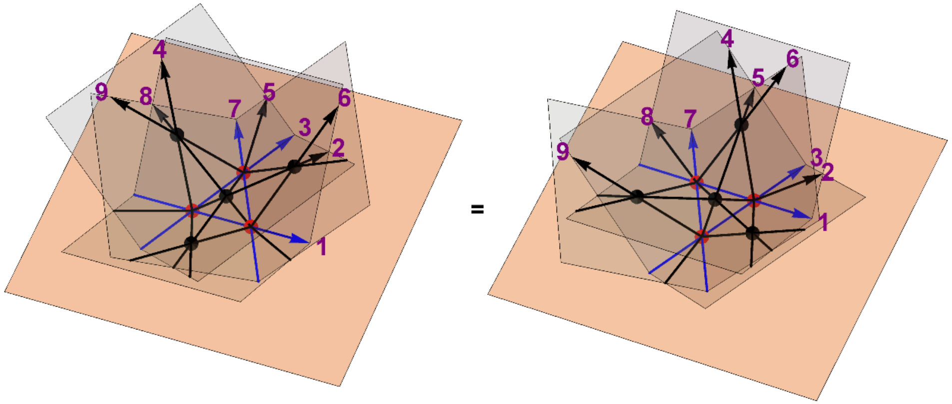
- Bulk: Tetrahedron equation



$$\mathbf{R}_{245}\mathbf{R}_{135}\mathbf{R}_{126}\mathbf{R}_{346}$$
$$= \mathbf{R}_{346}\mathbf{R}_{126}\mathbf{R}_{135}\mathbf{R}_{245}$$

[Zamolodchikov80]

- Boundary: 3D reflection equation



$$\mathbf{R}_{489}\mathbf{J}_{3579}\mathbf{R}_{269}\mathbf{R}_{258}\mathbf{J}_{1678}\mathbf{J}_{1234}\mathbf{R}_{456}$$
$$= \mathbf{R}_{456}\mathbf{J}_{1234}\mathbf{J}_{1678}\mathbf{R}_{258}\mathbf{R}_{269}\mathbf{J}_{3579}\mathbf{R}_{489}$$

[Isaev-Kulish97]

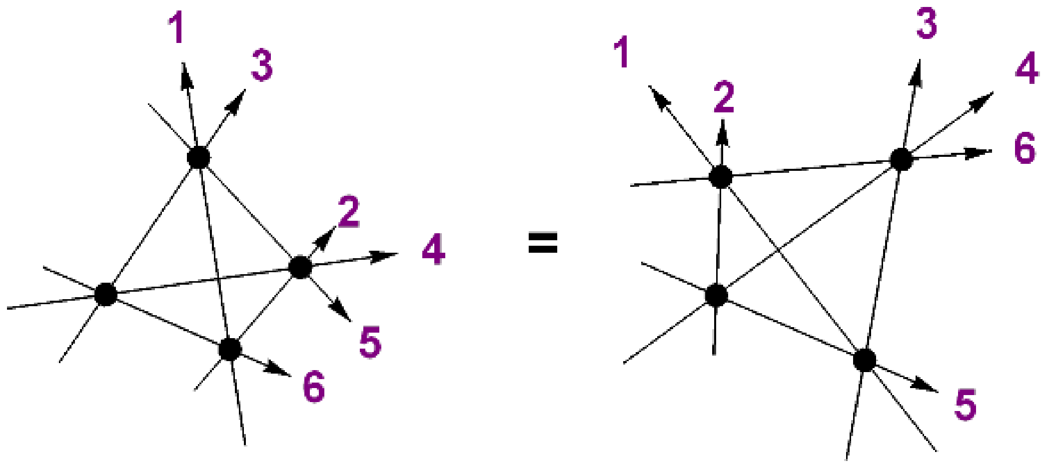
- Tetrahedron and 3D reflection equation are conditions for factorization of string scattering amplitude in 2+1D.

	Bulk	Boundary
2D	Yang-Baxter eq.	Reflection eq.
3D	Tetrahedron eq.	3D Reflection eq.

Motivation & Aim

- Several tetrahedron maps are known although less systematically than Yang-Baxter maps.
  - In the context of the local YBE [Sergeev98]
  - Transition maps of Lusztig's parametrizations of the canonical basis of $U_q(A_2)$ and their geometric liftings [Kuniba-Okado12] $\cdots(1)$
  - By using some KP tau functions [Kassotakis-Nieszporski-Papageorgiou-Tongas19]
- On the other hand, there are very few known 3D reflection maps.
  - Transition maps of Lusztig's parametrizations of the canonical basis of $U_q(B_2)$ and $U_q(C_2)$, and their geometric liftings [Kuniba-Okado12] $\cdots(2)$
- Aim: Obtain 3D reflection maps from known tetrahedron maps
- Motivation:
  - Some 2D reflection maps are constructed from known Yang-Baxter maps.
  - It is known that (2) is constructed from (1) associated with folding the Dynkin diagram of A_3 into one of B_2. [Berenstein-Zelevinsky01], [Lusztig11]

Tetrahedron maps



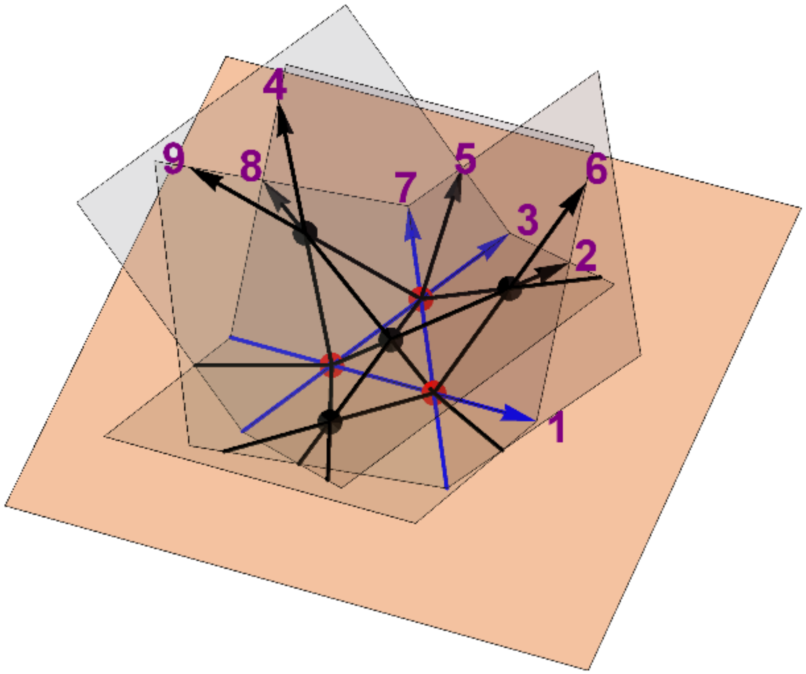
- Definition:
  - Let $\mathbf{R}\colon X^3 \to X^3$ (X: an arbitrary set) denote a map.
  - We call $\mathbf{R}$ *tetrahedron map* if it satisfies the tetrahedron equation on X^6:

$$\mathbf{R}_{245}\mathbf{R}_{135}\mathbf{R}_{126}\mathbf{R}_{346} = \mathbf{R}_{346}\mathbf{R}_{126}\mathbf{R}_{135}\mathbf{R}_{245} (=: \mathbf{T}_{123456}) \qquad \cdots (*)$$

  - We call $\mathbf{T}$ the *tetrahedral composite* of the tetrahedron map $\mathbf{R}$.
  - We call $\mathbf{R}$ *involutive* if $\mathbf{R}^2 = \mathrm{id}$ and *symmetric* if $\mathbf{R}_{123} = \mathbf{R}_{321}$.

- Remark:
  - For involutive and symmetric tetrahedron maps, $(*)$ corresponds to the usual tetrahedron equation.

3D reflection maps



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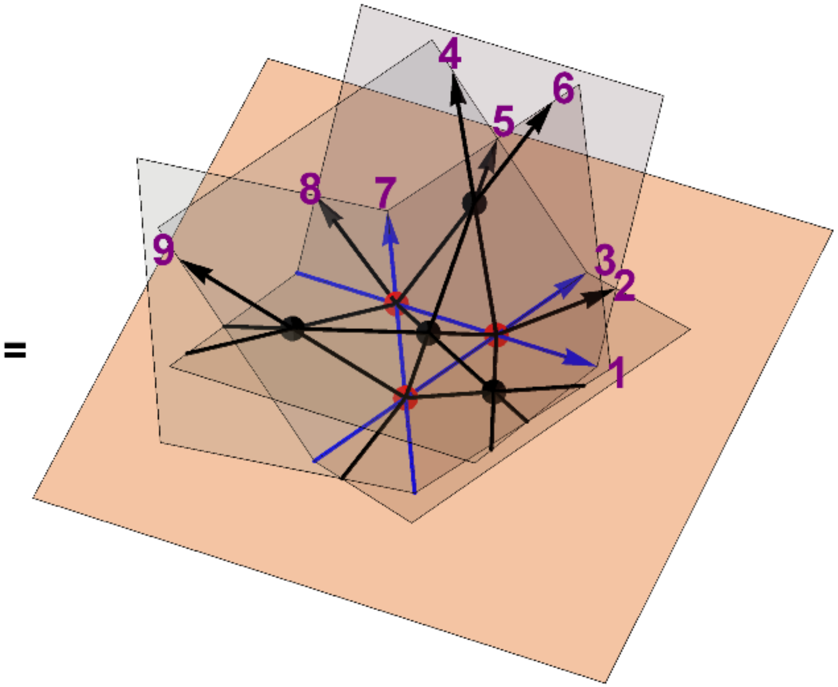


- <u>Definition</u>:
  - Let $\mathbf{J}: X^4 \to X^4$ denote a map.
  - We set a tetrahedron map by $\mathbf{R}: X^3 \to X^3$.
  - We call $\mathbf{J}$ *3D reflection map* if it satisfies the 3D reflection equation on X^9:

$$\mathbf{R}_{489}\mathbf{J}_{3579}\mathbf{R}_{269}\mathbf{R}_{258}\mathbf{J}_{1678}\mathbf{J}_{1234}\mathbf{R}_{456} = \mathbf{R}_{456}\mathbf{J}_{1234}\mathbf{J}_{1678}\mathbf{R}_{258}\mathbf{R}_{269}\mathbf{J}_{3579}\mathbf{R}_{489}$$

[Isaev-Kulish97]

Boundarization

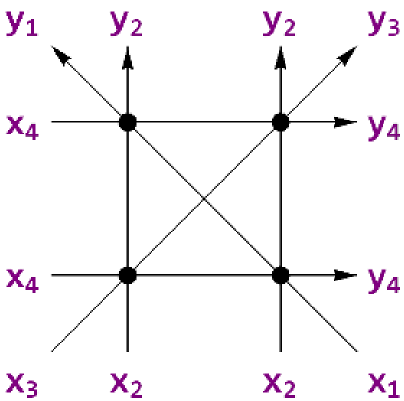
- We set the subset of X^6 by $Y = \{(x_1, \cdots, x_6) \mid x_2 = x_3, x_5 = x_6\}$.
  - We set $\phi: X^4 \to Y$ by $\phi(x_1, x_2, x_3, x_4) = (x_1, x_2, x_2, x_3, x_4, x_4)$ (embedding)
  - We set $\varphi: Y \to X^4$ by $\varphi(x_1, x_2, x_2, x_3, x_4, x_4) = (x_1, x_2, x_3, x_4)$ (projection)

- <u>Definition</u>:
  - Let $\mathbf{R}: X^3 \to X^3$ denote a tetrahedron map and $\mathbf{T}$ its tetrahedral composite.
  - We call $\mathbf{R}$ *boundarizable* if the following condition is satisfied:

$$\mathbf{x} \in Y \ \Rightarrow \ \mathbf{T}(\mathbf{x}) \in Y$$

  - In that case, we define the *boundarization* $\mathbf{J}: X^4 \to X^4$ of $\mathbf{R}$ by

$$\mathbf{J}(\mathbf{x}) = \varphi(\mathbf{T}(\phi(\mathbf{x})))$$



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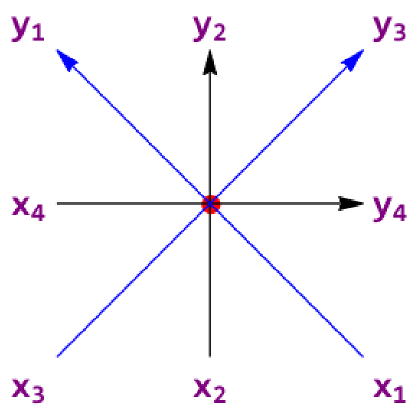


$$y_i = \varphi(\mathbf{T}(\phi(\mathbf{x})))_i$$

Example for boundarization

- We set the tetrahedron map $\mathbf{R}\colon \mathbb{R}^3_{>0} \to \mathbb{R}^3_{>0}$ by

$$\mathbf{R} : (x_1, x_2, x_3) \mapsto (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \left(\frac{x_1 x_2}{x_1 + x_3}, x_1 + x_3, \frac{x_2 x_3}{x_1 + x_3} \right)$$

- Let's act tetrahedral composite $\mathbf{T} = \mathbf{R}_{245}\mathbf{R}_{135}\mathbf{R}_{126}\mathbf{R}_{346}$ on $\boldsymbol{x} \in Y$.

$$(x_1, x_2, \underline{x_2}, \underline{x_3}, x_4, \underline{x_4})$$

$$\overset{\mathbf{R}_{346}}{\mapsto} \left(\underline{x_1}, \underline{x_2}, \frac{x_2 x_3}{x_2 + x_4}, x_2 + x_4, x_4, \underline{\frac{x_3 x_4}{x_2 + x_4}} \right)$$

$$\overset{\mathbf{R}_{126}}{\mapsto} \left(\underline{\frac{x_1 x_2 (x_2 + x_4)}{x_3 x_4 + x_1 (x_2 + x_4)}}, x_1 + \frac{x_3 x_4}{x_2 + x_4}, \underline{\frac{x_2 x_3}{x_2 + x_4}}, x_2 + x_4, \underline{x_4}, \frac{x_2 x_3 x_4}{x_3 x_4 + x_1 (x_2 + x_4)} \right)$$

Example for boundarization

$$\stackrel{\mathbf{R}_{135}}{\mapsto}\left(\frac{x_1x_2^2x_3}{x_3x_4^2+x_1(x_2+x_4)^2}, \underline{x_1+\frac{x_3x_4}{x_2+x_4}}, \frac{x_3x_4^2+x_1(x_2+x_4)^2}{x_3x_4+x_1(x_2+x_4)}\right.$$

$$\left., \underline{x_2+x_4}, \underline{\frac{x_2x_3x_4(x_3x_4+x_1(x_2+x_4))}{(x_2+x_4)(x_3x_4^2+x_1(x_2+x_4)^2)}}, \frac{x_2x_3x_4}{x_3x_4+x_1(x_2+x_4)}\right)$$

$$\stackrel{\mathbf{R}_{245}}{\mapsto}\left(\frac{x_1x_2^2x_3}{x_3x_4^2+x_1(x_2+x_4)^2}, \frac{x_3x_4^2+x_1(x_2+x_4)^2}{x_3x_4+x_1(x_2+x_4)}, \frac{x_3x_4^2+x_1(x_2+x_4)^2}{x_3x_4+x_1(x_2+x_4)}\right.$$

$$\left., \frac{(x_3x_4+x_1(x_2+x_4))^2}{x_3x_4^2+x_1(x_2+x_4)^2}, \frac{x_2x_3x_4}{x_3x_4+x_1(x_2+x_4)}, \frac{x_2x_3x_4}{x_3x_4+x_1(x_2+x_4)}\right) \in Y$$

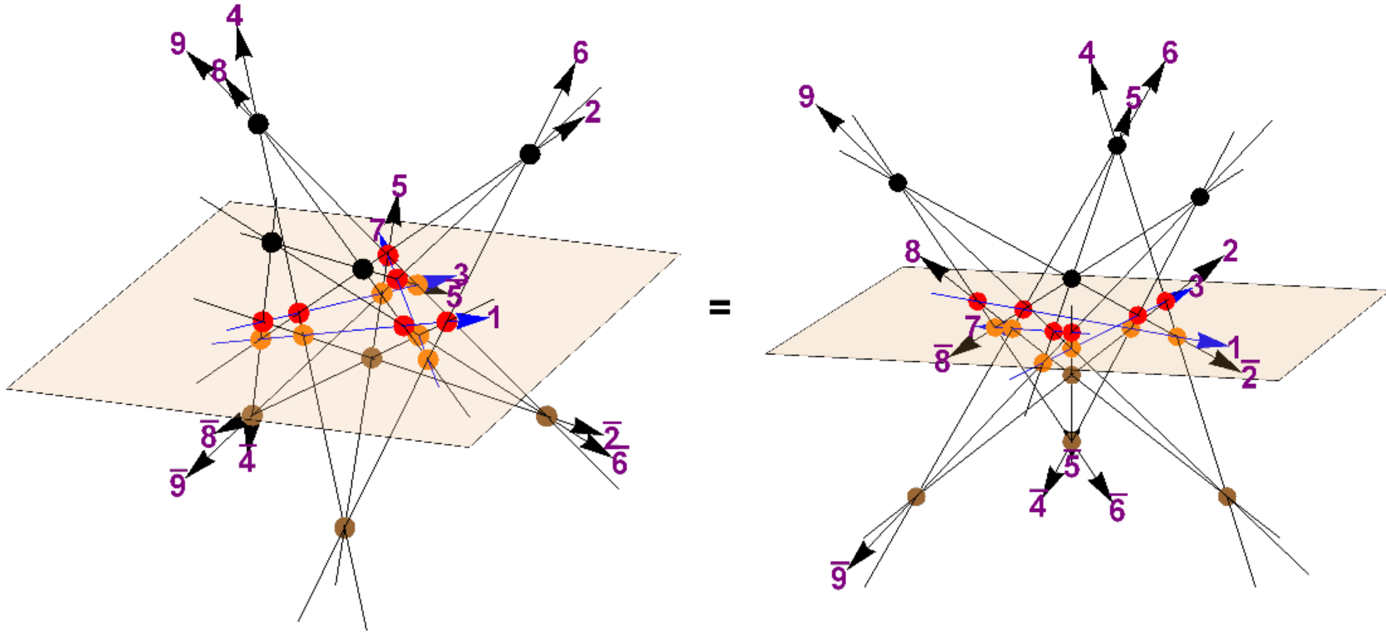
- Then, the boundarization $\mathbf{J}\colon \mathbb{R}^4_{>0} \to \mathbb{R}^4_{>0}$ is obtained as follows:

$$\mathbf{J} : (x_1, x_2, x_3, x_4) \mapsto \left(\frac{x_1x_2^2x_3}{y_1}, \frac{y_1}{y_2}, \frac{y_2^2}{y_1}, \frac{x_2x_3x_4}{y_2}\right)$$

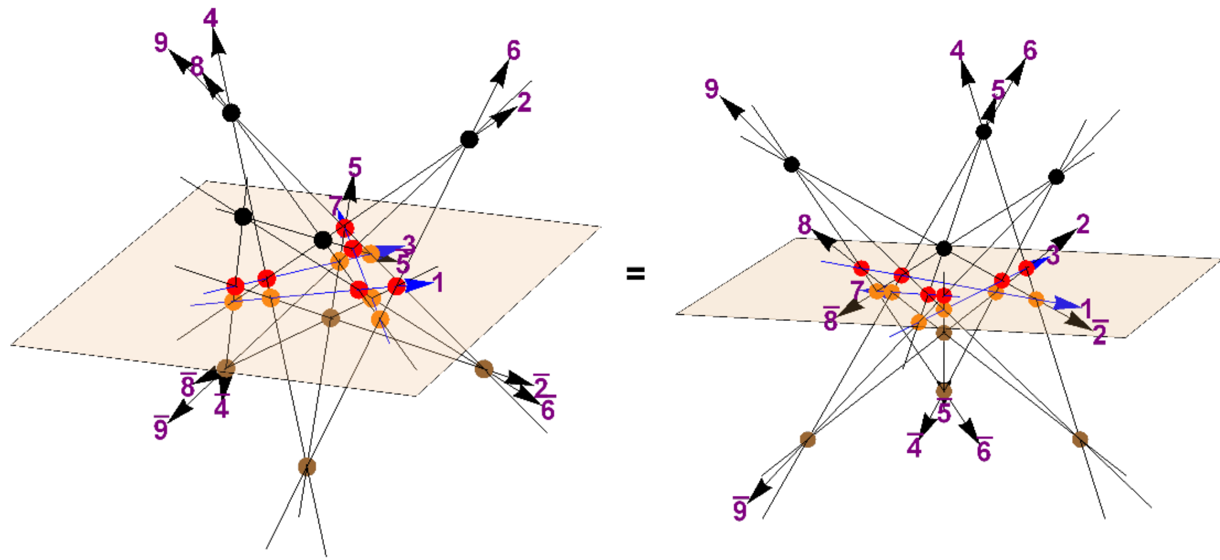
$$y_1 = x_1(x_2+x_4)^2 + x_3x_4^2, \quad y_2 = x_1(x_2+x_4) + x_3x_4$$

Main theorem

- Theorem:
  - Let $\mathbf{R}: X^3 \to X^3$ denote an involutive, symmetric and boundarizable tetrahedron map, and $\mathbf{J}: X^4 \to X^4$ its boundarization.
  - Then they satisfy 3D reflection equation.
- Sketch of Proof:
  - Cut the following identiy on X^{15} into half:



Key lemma for main theorem



- Lemma:
  - Let $\mathbf{R}: X^3 \to X^3$ denote an involutive and symmetric tetrahedron map. Then we have the following identiy on X^{15}:

$$
\begin{aligned}
&(\mathbf{R}_{489}\mathbf{R}_{\bar{4}\bar{8}\bar{9}})(\mathbf{R}_{579}\mathbf{R}_{3\bar{5}9}\mathbf{R}_{35\bar{9}}\mathbf{R}_{\bar{5}7\bar{9}})(\mathbf{R}_{26\bar{9}}\mathbf{R}_{\bar{2}69})(\mathbf{R}_{2\bar{5}8}\mathbf{R}_{\bar{2}5\bar{8}}) \\
&\qquad \times (\mathbf{R}_{\bar{6}7\bar{8}}\mathbf{R}_{16\bar{8}}\mathbf{R}_{1\bar{6}8}\mathbf{R}_{678})(\mathbf{R}_{234}\mathbf{R}_{1\bar{2}4}\mathbf{R}_{12\bar{4}}\mathbf{R}_{\bar{2}3\bar{4}})(\mathbf{R}_{\bar{4}\bar{5}\bar{6}}\mathbf{R}_{456}) \\
&= (\mathbf{R}_{456}\mathbf{R}_{\bar{4}\bar{5}\bar{6}})(\mathbf{R}_{234}\mathbf{R}_{1\bar{2}4}\mathbf{R}_{12\bar{4}}\mathbf{R}_{\bar{2}3\bar{4}})(\mathbf{R}_{\bar{6}7\bar{8}}\mathbf{R}_{16\bar{8}}\mathbf{R}_{1\bar{6}8}\mathbf{R}_{678}) \\
&\qquad \times (\mathbf{R}_{\bar{2}5\bar{8}}\mathbf{R}_{2\bar{5}8})(\mathbf{R}_{\bar{2}69}\mathbf{R}_{26\bar{9}})(\mathbf{R}_{579}\mathbf{R}_{3\bar{5}9}\mathbf{R}_{35\bar{9}}\mathbf{R}_{\bar{5}7\bar{9}})(\mathbf{R}_{\bar{4}\bar{8}\bar{9}}\mathbf{R}_{489})
\end{aligned}
$$

$\bar{\ }$: mirrored space

- Proof:
  - By repeated uses of TE, we have the above equation.

Figure for the identity on X^{15}

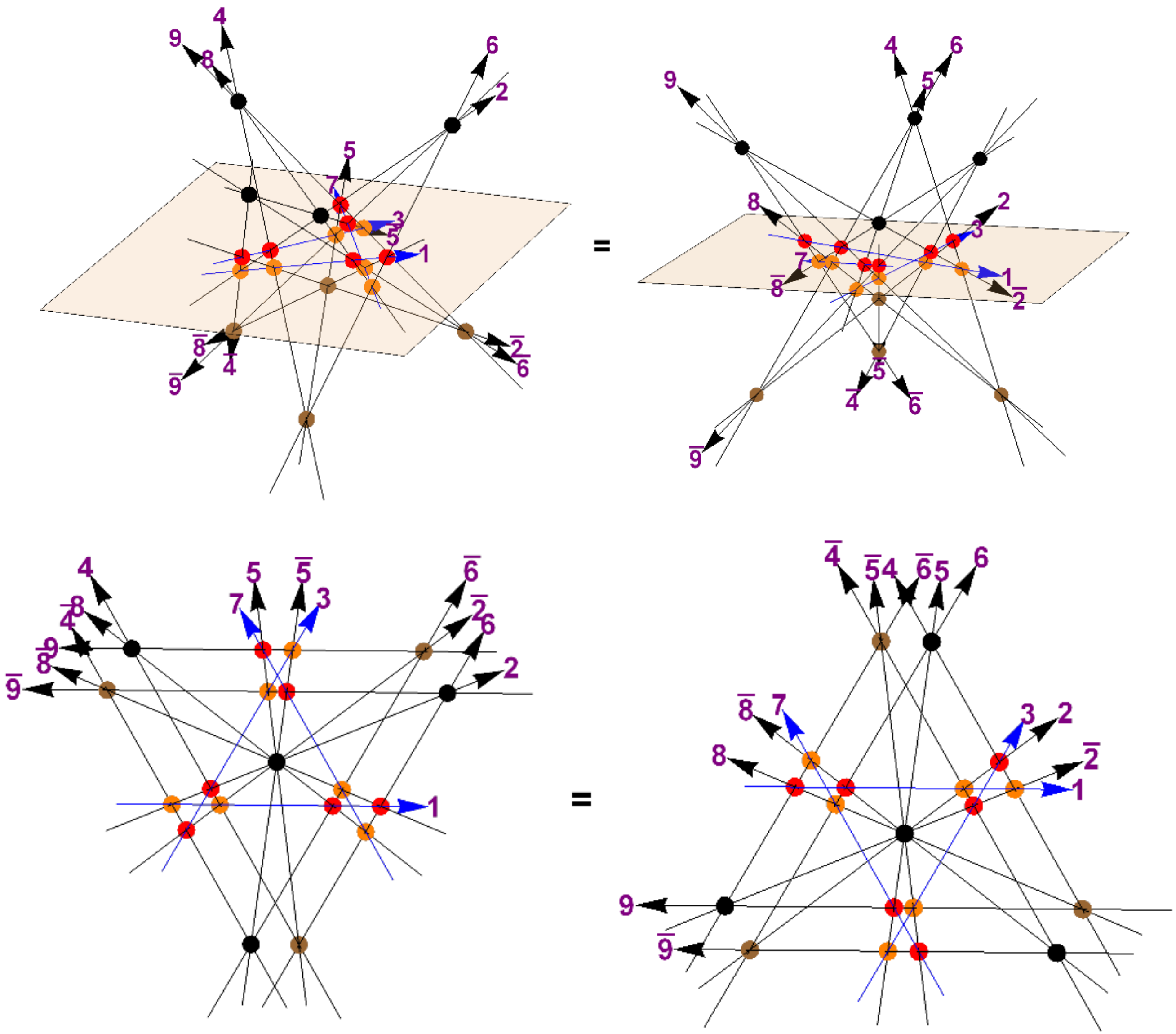
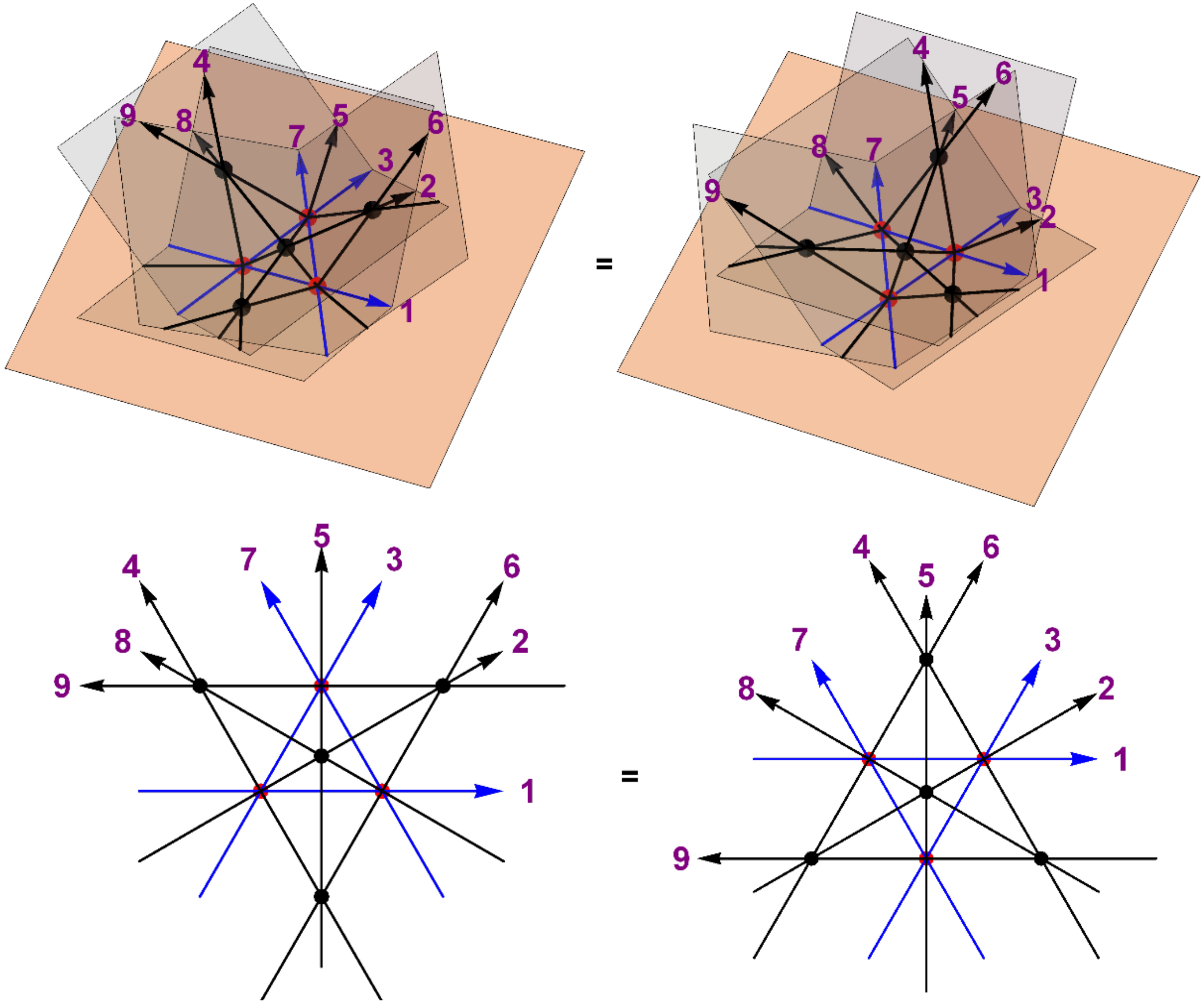


Figure for 3D reflection equation



Proof of main theorem

- <u>Sketch of proof</u>:
  - Let $\mathbf{T}$ denote the tetrahedral composite of $\mathbf{R}$.
  - By using $\mathbf{T}$, the identify in the previous lemma is written as follows:

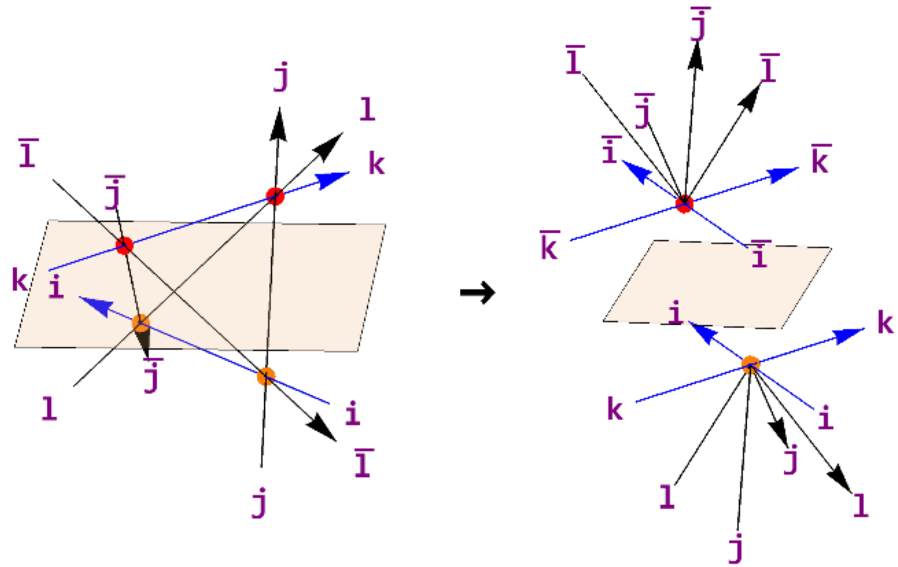
$$
\begin{aligned}
&(\mathbf{R}_{489}\mathbf{R}_{\bar{4}\bar{8}\bar{9}})\mathbf{T}_{35\bar{5}79\bar{9}}(\mathbf{R}_{269}\mathbf{R}_{\bar{2}\bar{6}\bar{9}})(\mathbf{R}_{258}\mathbf{R}_{\bar{2}\bar{5}\bar{8}})\mathbf{T}_{16\bar{6}78\bar{8}}\underline{\mathbf{T}_{12\bar{2}34\bar{4}}(\mathbf{R}_{\bar{4}\bar{5}\bar{6}}\mathbf{R}_{456})} \\
&= (\mathbf{R}_{456}\mathbf{R}_{\bar{4}\bar{5}\bar{6}})\mathbf{T}_{12\bar{2}34\bar{4}}\mathbf{T}_{16\bar{6}78\bar{8}}(\mathbf{R}_{\bar{2}\bar{5}\bar{8}}\mathbf{R}_{258})(\mathbf{R}_{\bar{2}\bar{6}\bar{9}}\mathbf{R}_{269})\mathbf{T}_{35\bar{5}79\bar{9}}(\mathbf{R}_{\bar{4}\bar{8}\bar{9}}\mathbf{R}_{489})
\end{aligned}
\qquad \cdots (*)
$$

  - Let's act both sides of $(*)$ on ``mirrored" inputs:

$$
\begin{aligned}
&(x_1, x_2, x_2, x_3, x_4, x_4, x_5, x_5, x_6, x_6, x_7, x_8, x_8, x_9, x_9) \\
&\in \underline{1} \times 2 \times \bar{2} \times \underline{3} \times 4 \times \bar{4} \times 5 \times \bar{5} \times 6 \times \bar{6} \times \underline{7} \times 8 \times \bar{8} \times 9 \times \bar{9}
\end{aligned}
$$

  - We can verify that all $\mathbf{T}$ in $(*)$ receive elements of Y.
  - For example, $\mathbf{R}_{456}$ and $\mathbf{R}_{\bar{4}\bar{5}\bar{6}}$ in ___ output the same, so $\mathbf{T}_{12\bar{2}34\bar{4}}$ receive an element of Y.
  - Then, all $\mathbf{T}$ in $(*)$ satisfy $\mathbf{T} = \phi \circ \mathbf{J} \circ \varphi$ by the definition of $\mathbf{J}$.
  - As a matter of fact, we expect $\mathbf{T} \sim \mathbf{JJ}$ rather than $\mathbf{T} \sim \mathbf{J}$ in view of the previous figures.

Proof of main theorem



- <u>Sketch of proof</u>:
  - By applying $\mathbf{T}_{ij\bar{j}kl\bar{l}} \mapsto P_{j\bar{j}}P_{l\bar{l}}\mathbf{J}_{ijkl}\mathbf{J}_{\bar{i}\bar{j}\bar{k}\bar{l}}$ (cutting and reconnection), to the previous equation, we obtain

$$
\begin{aligned}
&(\mathbf{R}_{489}\mathbf{R}_{\bar{4}\bar{8}\bar{9}})(P_{5\bar{5}}P_{9\bar{9}}\mathbf{J}_{3579}\mathbf{J}_{\bar{3}\bar{5}\bar{7}\bar{9}})(\mathbf{R}_{26\bar{9}}\mathbf{R}_{\bar{2}\bar{6}9})(\mathbf{R}_{2\bar{5}8}\mathbf{R}_{\bar{2}5\bar{8}}) \\
&\qquad \times (P_{6\bar{6}}P_{8\bar{8}}\mathbf{J}_{1678}\mathbf{J}_{\bar{1}\bar{6}\bar{7}\bar{8}})(P_{2\bar{2}}P_{4\bar{4}}\mathbf{J}_{1234}\mathbf{J}_{\bar{1}\bar{2}\bar{3}\bar{4}})(\mathbf{R}_{\bar{4}\bar{5}\bar{6}}\mathbf{R}_{456}) \\
&= (\mathbf{R}_{456}\mathbf{R}_{\bar{4}\bar{5}\bar{6}})(P_{2\bar{2}}P_{4\bar{4}}\mathbf{J}_{1234}\mathbf{J}_{\bar{1}\bar{2}\bar{3}\bar{4}})(P_{6\bar{6}}P_{8\bar{8}}\mathbf{J}_{1678}\mathbf{J}_{\bar{1}\bar{6}\bar{7}\bar{8}}) \\
&\qquad \times (\mathbf{R}_{\bar{2}5\bar{8}}\mathbf{R}_{2\bar{5}8})(\mathbf{R}_{\bar{2}\bar{6}9}\mathbf{R}_{26\bar{9}})(P_{5\bar{5}}P_{9\bar{9}}\mathbf{J}_{3579}\mathbf{J}_{\bar{3}\bar{5}\bar{7}\bar{9}})(\mathbf{R}_{\bar{4}\bar{8}\bar{9}}\mathbf{R}_{489}).
\end{aligned}
$$

  - By eliminating $P_{i\bar{i}}$, we obtain the direct product of 3DRE. (15+1+1+1=9+9)

Overview of analogous results

- 2D case
  - K_B from $R_{B^\vee,B}$:

$$K_B : B \to B^\vee,\ b \to c^\vee \qquad R_{B^\vee,B}\,(b^\vee \otimes b) = c \otimes c^\vee$$

  - Key lemma:

$$\begin{aligned}&(R_{B_2^\vee,B_1^\vee} R_{\ B_1,B_2}) R_{B_2^\vee,B_2} (R_{B_1^\vee,B_2} R_{B_2^\vee,B_1}) R_{B_1^\vee,B_1} \\ &= R_{B_1^\vee,B_1} (R_{B_2^\vee,B_1} R_{B_1^\vee,B_2}) R_{B_2^\vee,B_2} (R_{\ B_1,B_2} R_{B_2^\vee,B_1^\vee})\end{aligned}$$

- 3D case
  - $\mathbf{J}$ from $\mathbf{R}$:

$$\mathbf{J}(\mathbf{x}) = \varphi(\mathbf{T}(\phi(\mathbf{x}))) \qquad \mathbf{T} = \mathbf{R}_{245}\mathbf{R}_{135}\mathbf{R}_{126}\mathbf{R}_{346}$$

  - Key lemma:

$$\begin{aligned}&(\mathbf{R}_{489}\mathbf{R}_{\bar{4}\bar{8}\bar{9}})(\mathbf{R}_{579}\mathbf{R}_{3\bar{5}9}\mathbf{R}_{35\bar{9}}\mathbf{R}_{\bar{5}7\bar{9}})(\mathbf{R}_{26\bar{9}}\mathbf{R}_{\bar{2}\bar{6}9})(\mathbf{R}_{2\bar{5}8}\mathbf{R}_{\bar{2}5\bar{8}}) \\ &\qquad \times (\mathbf{R}_{\bar{6}7\bar{8}}\mathbf{R}_{16\bar{8}}\mathbf{R}_{1\bar{6}8}\mathbf{R}_{678})(\mathbf{R}_{234}\mathbf{R}_{1\bar{2}4}\mathbf{R}_{12\bar{4}}\mathbf{R}_{\bar{2}3\bar{4}})(\mathbf{R}_{\bar{4}\bar{5}\bar{6}}\mathbf{R}_{456}) \\ &= (\mathbf{R}_{456}\mathbf{R}_{\bar{4}\bar{5}\bar{6}})(\mathbf{R}_{234}\mathbf{R}_{1\bar{2}4}\mathbf{R}_{12\bar{4}}\mathbf{R}_{\bar{2}3\bar{4}})(\mathbf{R}_{\bar{6}7\bar{8}}\mathbf{R}_{16\bar{8}}\mathbf{R}_{1\bar{6}8}\mathbf{R}_{678}) \\ &\qquad \times (\mathbf{R}_{\bar{2}5\bar{8}}\mathbf{R}_{2\bar{5}8})(\mathbf{R}_{\bar{2}\bar{6}9}\mathbf{R}_{26\bar{9}})(\mathbf{R}_{579}\mathbf{R}_{3\bar{5}9}\mathbf{R}_{35\bar{9}}\mathbf{R}_{\bar{5}7\bar{9}})(\mathbf{R}_{\bar{4}\bar{8}\bar{9}}\mathbf{R}_{489})\end{aligned}$$

Outline

Example1: birational transition map

- We set $\mathbf{R}$: $\mathbb{R}^3_{>0} \to \mathbb{R}^3_{>0}$ by

$$\mathbf{R} : (x_1, x_2, x_3) \mapsto (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \left(\frac{x_1 x_2}{y}, y, \frac{x_2 x_3}{y}\right) \qquad y = x_1 + x_3$$

  - This map is characterized as the transition map of parametrizations of the positive part of SL_3: [Lusztig94]

$$G_1(x_3)G_2(x_2)G_1(x_1) = G_2(\tilde{x}_1)G_1(\tilde{x}_2)G_2(\tilde{x}_3) \qquad G_i(x) = 1 + xE_{i,i+1}$$

  - Actually, the transition map for SL_4 is the tetrahedral composite of $\mathbf{R}$ and we have TE for $\mathbf{R}$ by considering $\mathbf{T}$ in two ways. [Kuniba-Okado12]

$$G_1(x_6)G_3(x_5)G_2(x_4)G_1(x_3)G_3(x_2)G_2(x_1)$$
$$= G_2(\tilde{x}_1)G_1(\tilde{x}_2)G_3(\tilde{x}_3)G_2(\tilde{x}_4)G_1(\tilde{x}_5)G_3(\tilde{x}_6)$$
$$\mathbf{T} : (x_1, \cdots, x_6) \mapsto (\tilde{x}_1, \cdots, \tilde{x}_6)$$

  - This tetrahedron map was also obtained in the context of the local YBE and by considering AKP tau function.
[Sergeev98], [Kassotakis-Nieszporski-Papageorgiou-Tongas19]
  - We can verify $\mathbf{R}$ is involutive, symmetric and boundarizable.

Example1: birational transition map

- The associated 3D reflection map $\mathbf{J}$: $\mathbb{R}^4_{>0} \to \mathbb{R}^4_{>0}$ is calculated as follows:

$$\mathbf{J} : (x_1, x_2, x_3, x_4) \mapsto \left(\frac{x_1 x_2^2 x_3}{y_1}, \frac{y_1}{y_2}, \frac{y_2^2}{y_1}, \frac{x_2 x_3 x_4}{y_2} \right)$$

$$y_1 = x_1(x_2 + x_4)^2 + x_3 x_4^2, \quad y_2 = x_1(x_2 + x_4) + x_3 x_4$$

  - Actually, $(\mathbf{R}, \mathbf{J})$ is a known solution to 3D reflection equation. [Kuniba-Okado12]

  - $\mathbf{J}$ is characterized as the transition map of parametrizations of the positive part of SP_4:

$$H_1(x_4)H_2(x_3)H_1(x_2)H_2(x_1) = H_2(\tilde{x_1})H_1(\tilde{x_2})H_2(\tilde{x_3})H_1(\tilde{x_4})$$

$$H_1(x) = \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & x \\ & & & 1 \end{pmatrix} \quad H_2(x) = \begin{pmatrix} 1 & & & \\ & 1 & x & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad \begin{aligned} H_1(x) &= G_1(x)G_3(x) \\ H_2(x) &= G_2(x) \end{aligned}$$

  - This correspondence is a consequence from folding the Dynkin diagram of A_3 into one of C_2. [Berenstein-Zelevinsky01], [Lusztig11]

Example2: Sergeev's electrical solution

- For $\lambda \in \mathbb{C}$, we set the tetrahedron map $\mathbf{R}(\lambda): \mathbb{C}^3 \to \mathbb{C}^3$ by

$$\mathbf{R}(\lambda) : (x_1, x_2, x_3) \mapsto (\tilde{x_1}, \tilde{x_2}, \tilde{x_3}) = \left(\frac{x_1 x_2}{y}, y, \frac{x_2 x_3}{y}\right) \quad y = x_1 + x_3 + \lambda x_1 x_2 x_3$$

  - This solution was obtained in the context of the local YBE and by considering BKP tau function.
  - $\mathbf{R}(\lambda)$ is also characterized as the relation which elements of the positive part of electorical Lie groups of type A satisfy: [Lam-Pylyavskyy15]

$$\begin{aligned}&\hat{G}_1(x_3)\hat{G}_2(x_2)\hat{G}_1(x_1)\\ &= \hat{G}_2(\tilde{x_1})\hat{G}_1(\tilde{x_2})\hat{G}_2(\tilde{x_3})\end{aligned} \qquad \hat{G}_1(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \quad \hat{G}_2(x) = \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}$$

  - We can verify $\mathbf{R}(\lambda)$ is involutive, symmetric and boundarizable.

- The associated 3D reflection map $\mathbf{J}(\lambda): \mathbb{C}^4 \to \mathbb{C}^4$ is calculated as follows:

$$\mathbf{J}(\lambda) : (x_1, x_2, x_3, x_4) \mapsto \left(\frac{x_1 x_2^2 x_3}{y_1}, \frac{y_1}{y_2}, \frac{y_2^2}{y_1}, \frac{x_2 x_3 x_4}{y_2}\right)$$

$$y_1 = x_1(x_2 + x_4)(x_2 + x_4 + 2\lambda x_2 x_3 x_4) + x_3 x_4^2$$

$$y_2 = x_1(x_2 + x_4 + 2\lambda x_2 x_3 x_4) + x_3 x_4$$

Example3: Two component solution

- We set the tetrahedron map $\mathbf{R}$: $(\mathbb{C}^2)^3 \to (\mathbb{C}^2)^3$ by

$$\mathbf{R}:\left(\begin{pmatrix}x_1\\y_1\end{pmatrix},\begin{pmatrix}x_2\\y_2\end{pmatrix},\begin{pmatrix}x_3\\y_3\end{pmatrix}\right)$$
$$\mapsto\left(\begin{pmatrix}\dfrac{x_1x_2}{x_1+x_3}\\[2ex]\dfrac{(x_1+x_3)y_1y_2}{x_1y_1+x_3y_3}\end{pmatrix},\begin{pmatrix}x_1+x_3\\[2ex]\dfrac{x_1y_1+x_3y_3}{x_1+x_3}\end{pmatrix},\begin{pmatrix}\dfrac{x_2x_3}{x_1+x_3}\\[2ex]\dfrac{(x_1+x_3)y_2y_3}{x_1y_1+x_3y_3}\end{pmatrix}\right)$$

  - This solution was obtained by considering modified KP tau function.
    [Kassotakis-Nieszporski-Papageorgiou-Tongas19]
  - This gives the previous birational transition map when we set $y_i = 1$.
  - We can verify $\mathbf{R}$ is involutive, symmetric and boundarizable.

Example3: Two component solution

- The associated 3D reflection map **J**: $(\mathbb{C}^2)^4 \to (\mathbb{C}^2)^4$ is calculated as follows:

$$\mathbf{J}:\left(\begin{pmatrix}x_1\\y_1\end{pmatrix},\begin{pmatrix}x_2\\y_2\end{pmatrix},\begin{pmatrix}x_3\\y_3\end{pmatrix},\begin{pmatrix}x_4\\y_4\end{pmatrix}\right)$$

$$\mapsto\left(\begin{pmatrix}\dfrac{x_1x_2^2x_3}{z_1}\\ \\ \dfrac{y_1y_2^2y_3z_1}{w_1}\end{pmatrix},\begin{pmatrix}\dfrac{z_1}{z_2}\\ \\ \dfrac{z_2w_1}{z_1w_2}\end{pmatrix},\begin{pmatrix}\dfrac{z_2^2}{z_1}\\ \\ \dfrac{z_1w_2^2}{z_2^2w_1}\end{pmatrix},\begin{pmatrix}\dfrac{x_2x_3x_4}{z_2}\\ \\ \dfrac{y_2y_3y_4z_2}{w_2}\end{pmatrix}\right)$$

$$z_1 = x_1(x_2+x_4)^2 + x_3x_4^2$$
$$z_2 = x_1(x_2+x_4) + x_3x_4$$
$$w_1 = x_1y_1(x_2y_2+x_4y_4)^2 + x_3x_4^2y_3y_4^2$$
$$w_2 = x_1y_1(x_2y_2+x_4y_4) + x_3x_4y_3y_4$$

Concluding remarks

- Summary:
  - We present a method for obtaining 3D reflection maps by using known tetrahedron maps. The theorem is an analog of the results in 2D.
  - Our method is a kind of generalization of the relation by Berenstein and Zelevinsky and gives 3D interpretation to their relation.
  - By applying our theorem to known tetrahedron maps, we obtain several 3D reflection maps which include new solutions.

- Remark:
  - Our theorem can be extended to *inhomogeneous* cases, that is, the case tetrahedron maps are defined on direct product of different sets.