

Tetrahedron and 3D reflection equation from PBW bases of the nilpotent subalgebra of quantum superalgebras

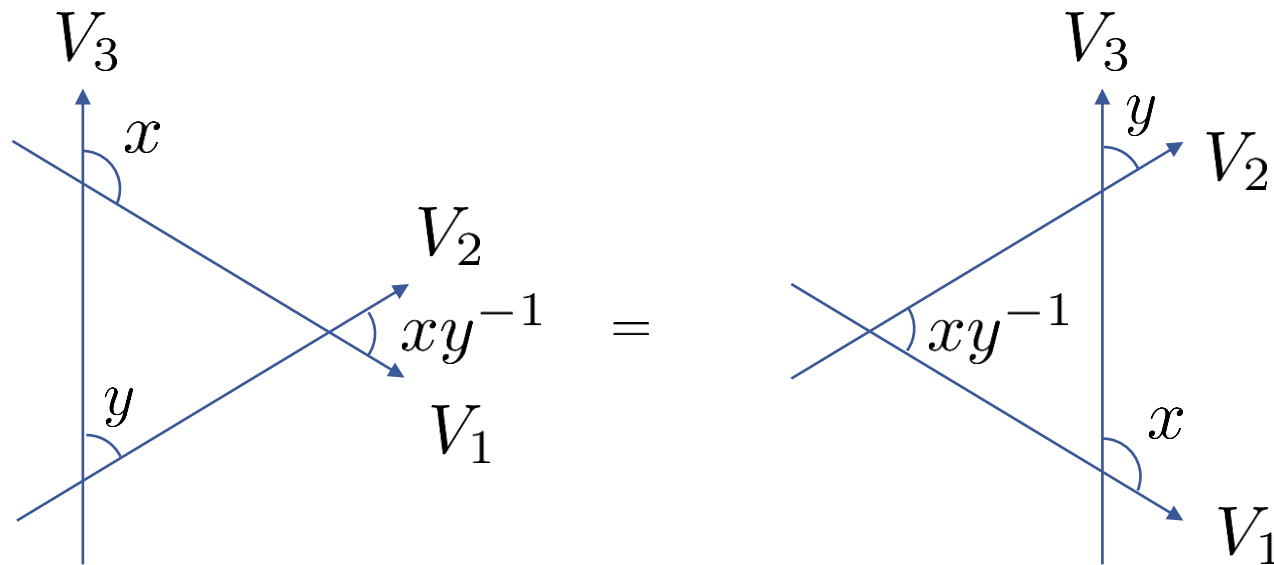
Discrete Mathematical Modelling Seminar@2021/01/13

Akihito Yoneyama (米山 瑛仁)

Institute of Physics, University of Tokyo, Komaba

Based on: AY, arXiv: 2012.13385

- Introduction: P.3~24
 - ▣ The 3D R and 3D L
 - ▣ Commuting transfer matrix
 - ▣ Matrix product solutions to the Yang-Baxter equation
- Setup: PBW bases of quantum superalgebras of type A P.26~37
- Main part:
 - ▣ Transition matrices of rank 2 P.39~43
 - ▣ Transition matrices of rank 3 and the tetrahedron equation P.45~66
- Concluding remarks



- Matrix equation on $V_1 \otimes V_2 \otimes V_3$ (V_i : linear space)

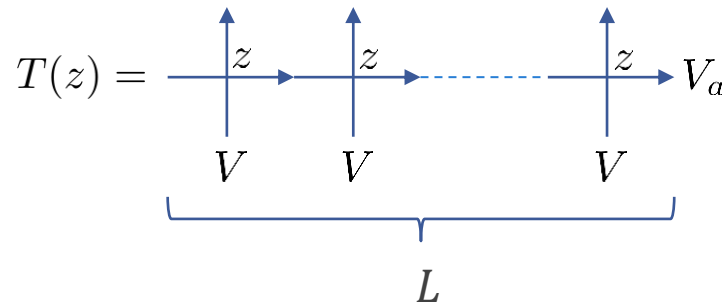
$$R_{12}(xy^{-1})R_{13}(x)R_{23}(y) = R_{23}(y)R_{13}(x)R_{12}(xy^{-1})$$

- $R_{ij}(z)$ acts non-trivially only on $V_i \otimes V_j$.
 - Solutions to the Yang-Baxter eq are called R matrices.
- R matrices are systematically (infinitely many) constructed via irreps of quantum affine algebra $U_q(g)$.

Monodromy matrix & RTT relation

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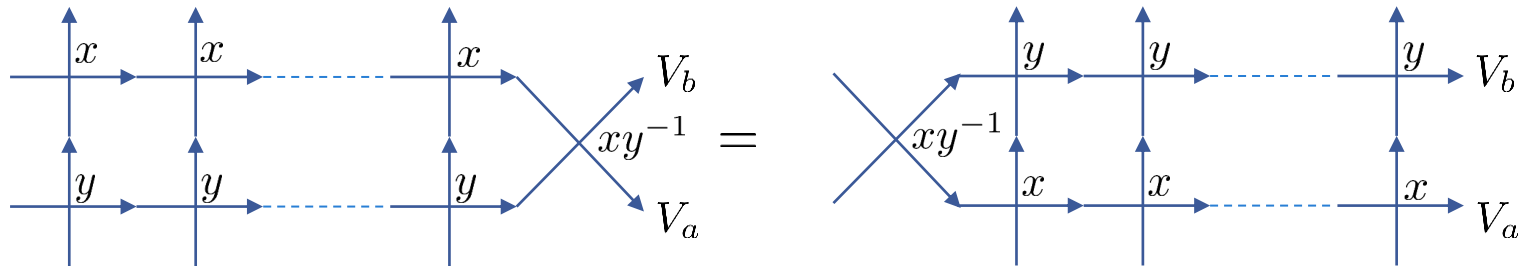
- Monodromy matrix $T_{a0}(z): V_a \otimes V_0 \rightarrow V_a \otimes V_0$



$V_0 = V^{\otimes L}$: physical sp.
 V_a : auxiliary sp.

- By repeated uses of the Yang-Baxter equation, we have

$$R_{ab}(xy^{-1})T_{a0}(x)T_{b0}(y) = T_{b0}(y)T_{a0}(x)R_{ab}(xy^{-1})$$



- Multiply $R_{ab}(xy^{-1})^{-1}$ from the left:

$$T_{a0}(x)T_{b0}(y) = [R_{ab}(xy^{-1})]^{-1}T_{b0}(y)T_{a0}(x)R_{ab}(xy^{-1})$$

- By taking the trace on $V_a \otimes V_b$, we have

$$[\tau(x), \tau(y)] = 0 \quad (\forall x, y \in \mathbb{C})$$

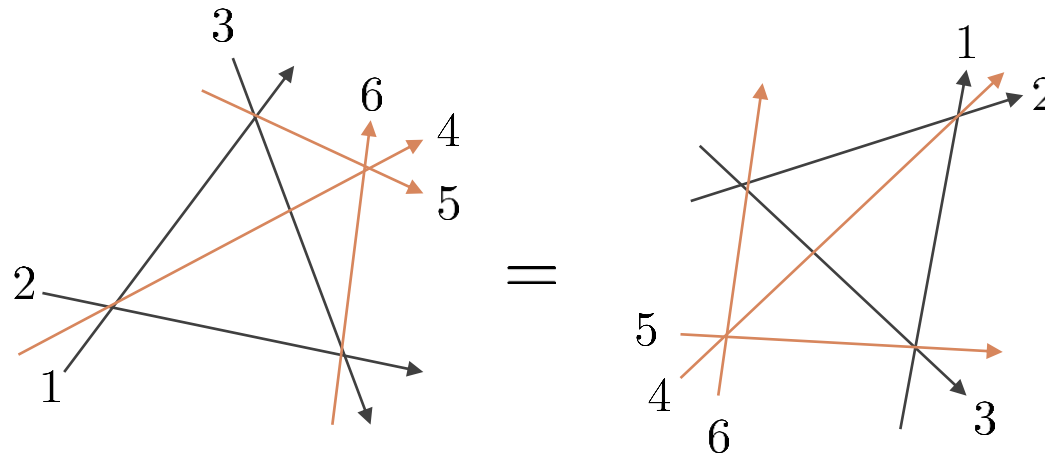
- Here we set the row-to-row transfer matrix by

$$\tau(z) = \text{Tr}_a(T_{a0}(z)) = \begin{array}{c} \begin{array}{c} \uparrow \\ z \\ \downarrow \\ V \end{array} \quad \begin{array}{c} \uparrow \\ z \\ \downarrow \\ V \end{array} \quad \cdots \quad \begin{array}{c} \uparrow \\ z \\ \downarrow \\ V \end{array} \end{array} \rightarrow V_a$$

- A lot of families of integrable one-dimensional quantum spin chains are constructed via commutativity of the transfer matrix.
 - The transfer matrix gives $O(L)$ conserved quantities.
 - Eigenvalues are obtained by the Bethe ansatz.

Tetrahedron equation

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- Matrix equation on $V_1 \otimes \cdots \otimes V_6$ (V_i : linear space)

$$\mathcal{R}_{124}\mathcal{R}_{135}\mathcal{R}_{236}\mathcal{R}_{456} = \mathcal{R}_{456}\mathcal{R}_{236}\mathcal{R}_{135}\mathcal{R}_{124}$$

- R_{ijk} acts non-trivially only on $V_i \otimes V_j \otimes V_k$.

- Tetrahedron equation = Yang-Baxter equation *up to conjugation*

- Unlike Yang-Baxter equation, a few families of solutions are known.
- We focus on solutions on the Fock spaces.

$$\text{boson Fock: } F = \bigoplus_{m=0,1,2,\dots} \mathbb{C} |m\rangle$$

$$\text{fermi Fock: } V = \bigoplus_{m=0,1} \mathbb{C} u_m$$

- Set $\mathcal{R} \in \text{End}(F^{\otimes 3})$ by

[Kapranov-Voevodsky94]

$$\mathcal{R} |i\rangle \otimes |j\rangle \otimes |k\rangle = \sum_{a,b,c} \mathcal{R}_{ijk}^{abc} |a\rangle \otimes |b\rangle \otimes |c\rangle$$

$$\mathcal{R}_{i,j,k}^{a,b,c} = \delta_{i+j}^{a+b} \delta_{j+k}^{b+c} \sum_{\lambda, \mu \geq 0, \lambda + \mu = b} (-1)^\lambda q^{i(c-j) + (k+1)\lambda + \mu(\mu-k)} \frac{(q^2)_{c+\mu}}{(q^2)_c} \binom{i}{\mu}_{q^2} \binom{j}{\lambda}_{q^2}$$

Here

$$(q)_k = \prod_{l=1}^k (1 - q^l) \quad \binom{a}{b}_q = \frac{(q)_a}{(q)_b (q)_{a-b}}$$

- The 3D R satisfies the following tetrahedron equation:

$$\mathcal{R}_{124} \mathcal{R}_{135} \mathcal{R}_{236} \mathcal{R}_{456} = \mathcal{R}_{456} \mathcal{R}_{236} \mathcal{R}_{135} \mathcal{R}_{124} \quad \dots (*)$$

- 3D R = intertwiner of irreps of quantum coordinate ring $A_q(A_2)$

$$\mathcal{R} \circ \pi_1 \otimes \pi_2 \otimes \pi_1(\Delta^{\text{op}}(g)) = \pi_2 \otimes \pi_1 \otimes \pi_2(\Delta(g)) \circ \mathcal{R} \quad \forall g \in A_q(A_2)$$

$$\pi_i : A_q(A_2) \rightarrow \text{End}(F)$$

- “121” and “212” are associated with the longest element of Weyl group.
- This gives a linearization method for tetrahedron equation.

- Set $\mathcal{L} \in \text{End}(V \otimes V \otimes F)$ by [Bazhanov-Sergeev06]

$$\mathcal{L}(u_i \otimes u_j \otimes |k\rangle) = \sum_{a,b \in \{0,1\}, c \in \mathbb{Z}_{\geq 0}} \mathcal{L}_{i,j,k}^{a,b,c} u_a \otimes u_b \otimes |c\rangle$$

$$\mathcal{L}_{0,0,k}^{0,0,c} = \mathcal{L}_{1,1,k}^{1,1,c} = \delta_{k,c}, \quad \mathcal{L}_{0,1,k}^{0,1,c} = -\delta_{k,c} q^{k+1}, \quad \mathcal{L}_{1,0,k}^{1,0,c} = \delta_{k,c} q^k,$$

$$\mathcal{L}_{1,0,k}^{0,1,c} = \delta_{k-1,c} (1 - q^{2k}), \quad \mathcal{L}_{0,1,k}^{1,0,c} = \delta_{k+1,c}$$

- The 3D L satisfies $\mathcal{L}_{124} \mathcal{L}_{135} \mathcal{L}_{236} \mathcal{R}_{456} = \mathcal{R}_{456} \mathcal{L}_{236} \mathcal{L}_{135} \mathcal{L}_{124} \dots (*)$
- BS obtained the 3D L by ansatz so that the tetrahedron equation of (*) type has a non-trivial solution, and solved (*) for the 3D R.
 - Later, (*) is “identified” with the intertwining relations for $A_q(A_2)$. [Kuniba-Okado12]
 - Algebraic origins of the 3D L has been still unclear.
- Recently, the classical limit of (*) is derived in relation to non-trivial transformations of a plabic network, which can be interpreted as cluster mutations. [Gavrylenko-Semenyakin-Zenkevich20]

- The 3D R and L satisfy the following weight conservation:

$$[x^{\mathbf{h}_1}(xy)^{\mathbf{h}_2}y^{\mathbf{h}_3}, \mathcal{R}] = [x^{\mathbf{h}_1}(xy)^{\mathbf{h}_2}y^{\mathbf{h}_3}, \mathcal{L}] = 0 \quad (\forall x, y \in \mathbb{C}) \quad \dots (*)$$

- Here, $\mathbf{h}_1 = \mathbf{h} \otimes 1 \otimes 1$ etc. and $\mathbf{h} |m\rangle = m |m\rangle$, $\mathbf{h}u_m = mu_m$.

- The discussion below holds for solutions satisfying (*).
- We use the following graphical notations:

$$= \mathcal{R}_{i,j,k}^{a,b,c} \text{ or } \mathcal{L}_{i,j,k}^{a,b,c}$$

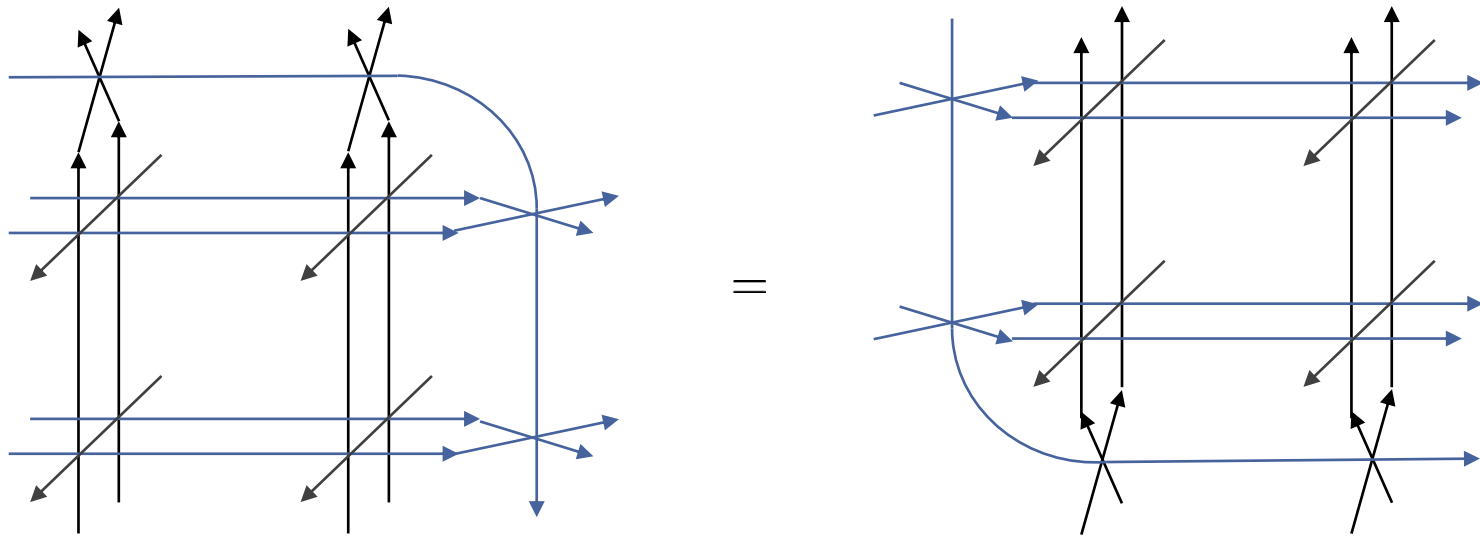
$$= \mathcal{R}_{i,j,k}^{a,b,c}$$

$$S^{(n)} =$$

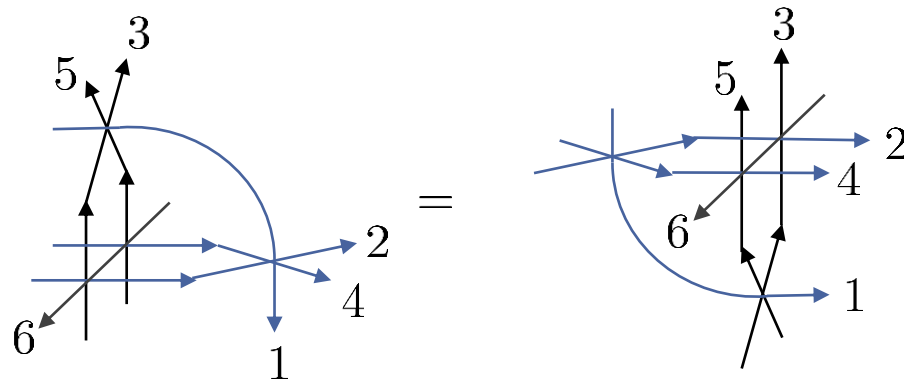
$$T^{(m,n)} =$$

STT=TTS: Commuting transfer matrix

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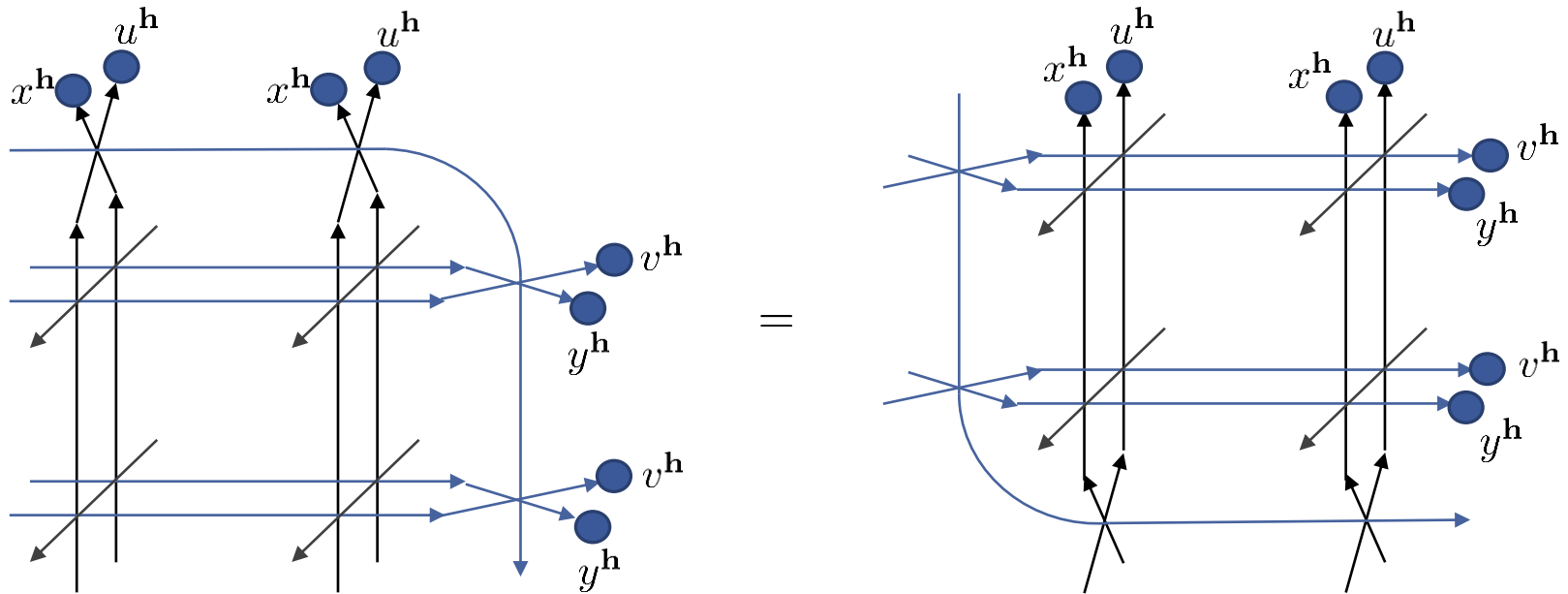
■ The above equation is obtained by repeated use of



STT=TTS: Commuting transfer matrix

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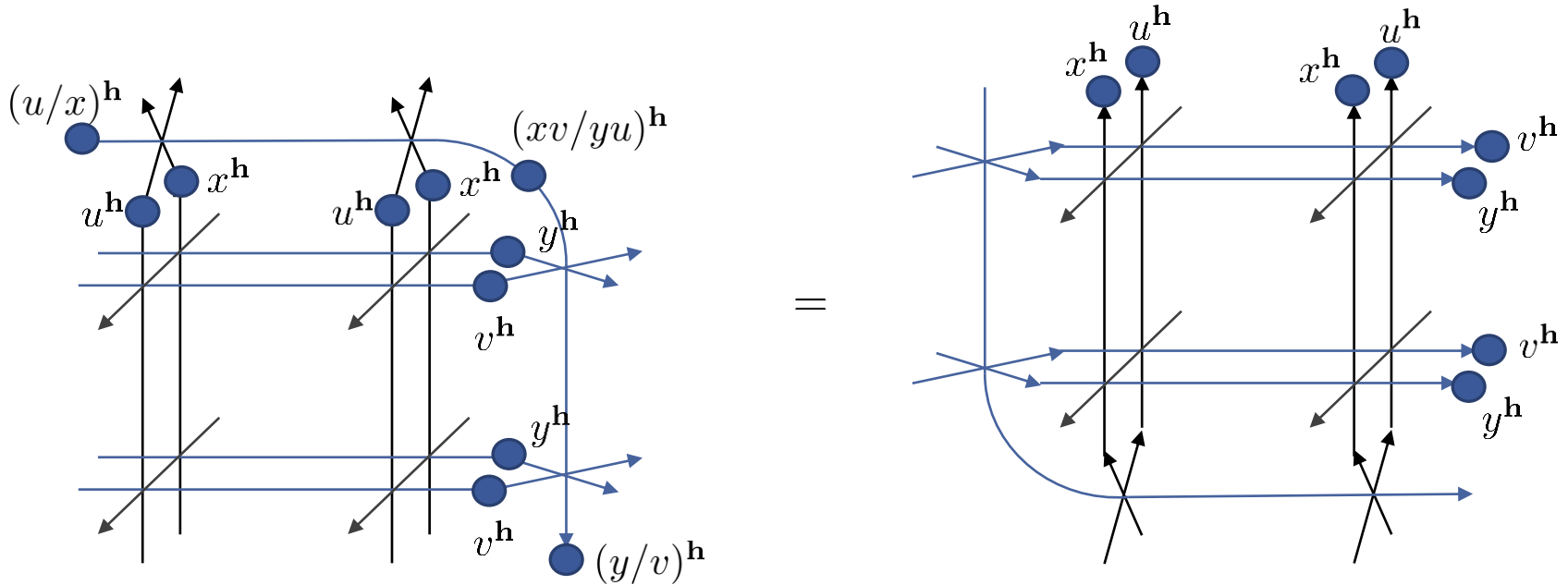
- Multiply x^h, y^h, u^h, v^h by several spaces:



STT=TTS: Commuting transfer matrix

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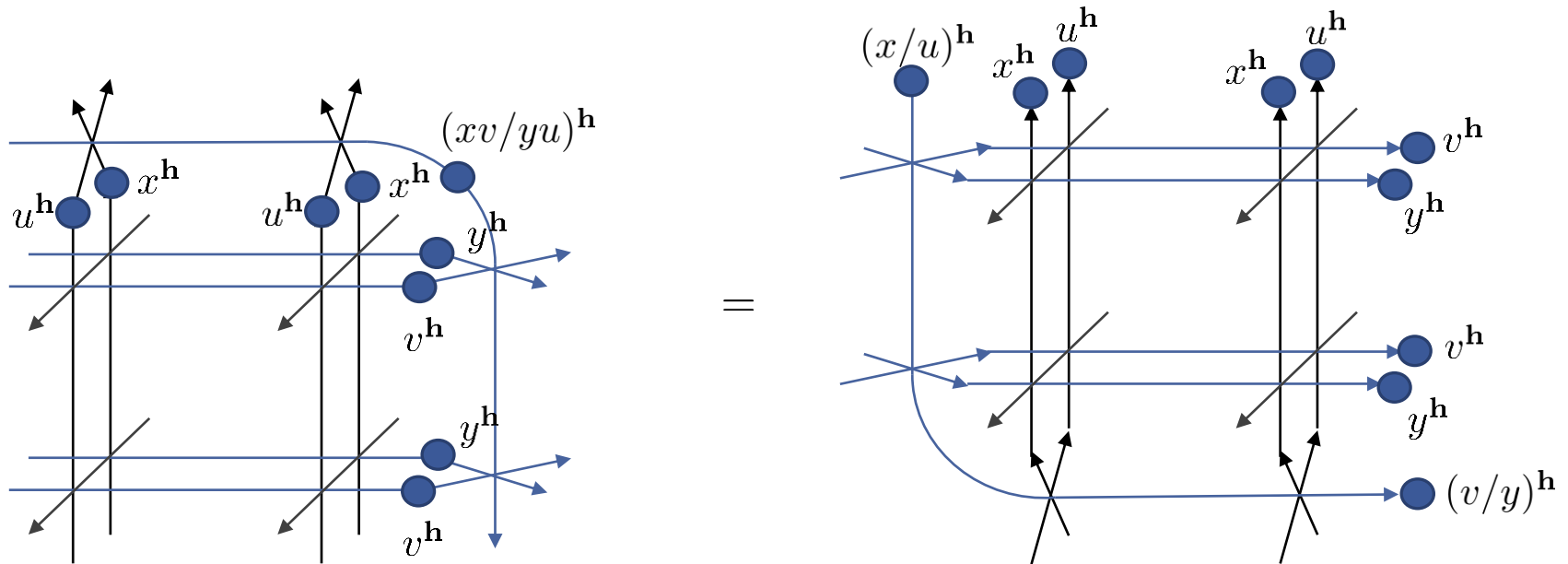
- Use the weight conservation $[x^{\mathbf{h}_1} (xy)^{\mathbf{h}_2} y^{\mathbf{h}_3}, \mathcal{R}] = [x^{\mathbf{h}_1} (xy)^{\mathbf{h}_2} y^{\mathbf{h}_3}, \mathcal{L}] = 0$



STT=TTS: Commuting transfer matrix

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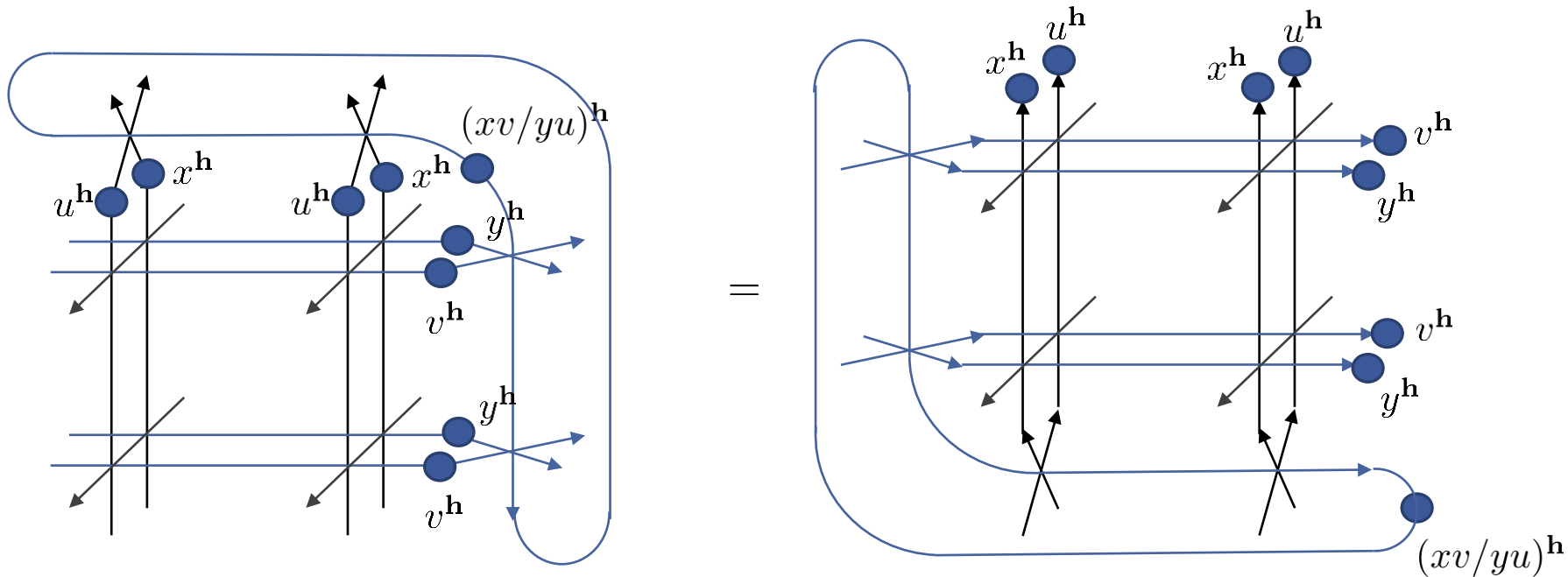
- Move $(u/x)^h$ and $(y/v)^h$ to the right hand side:



STT=TTS: Commuting transfer matrix

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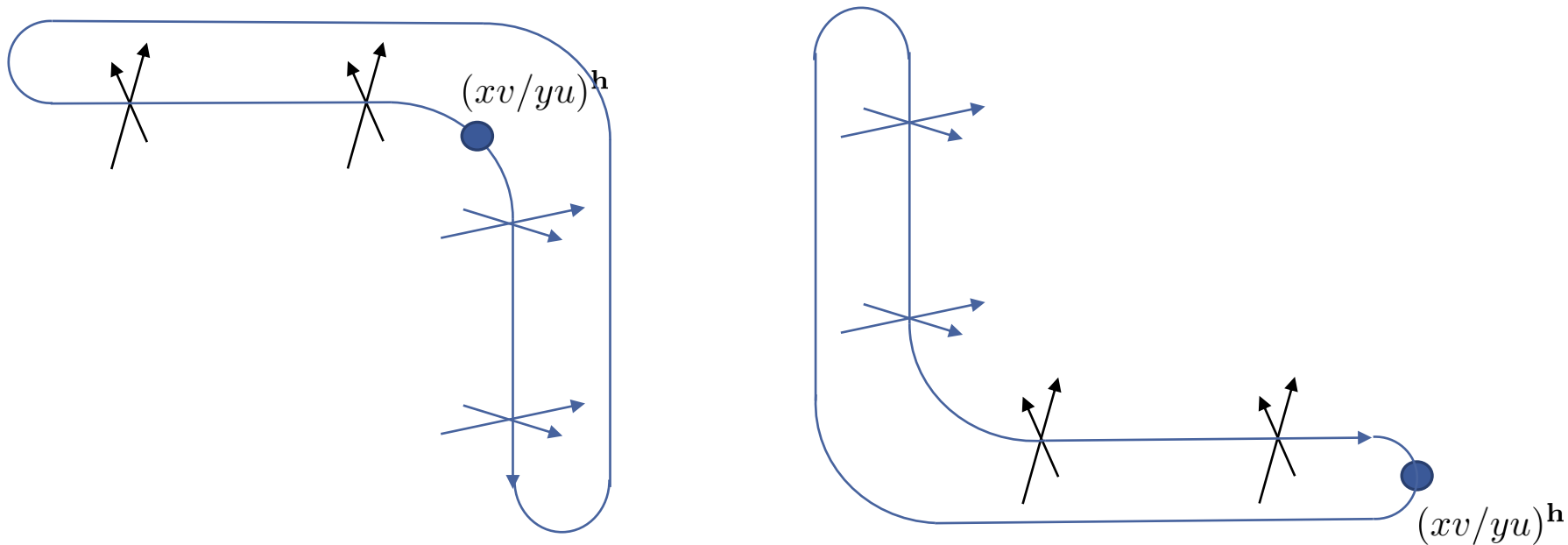
- Take the trace the auxiliary space:



STT=TTS: Commuting transfer matrix

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- Note that the following parts are actually same matrices:



- Then, if the above matrix is invertible, we can verify the commutativity by taking the trace on all auxiliary spaces.

- Define the transfer matrix by

$$\tau^{(m,n)}(x, y) =$$

- This satisfies the following commutativity: [Sergeev06]

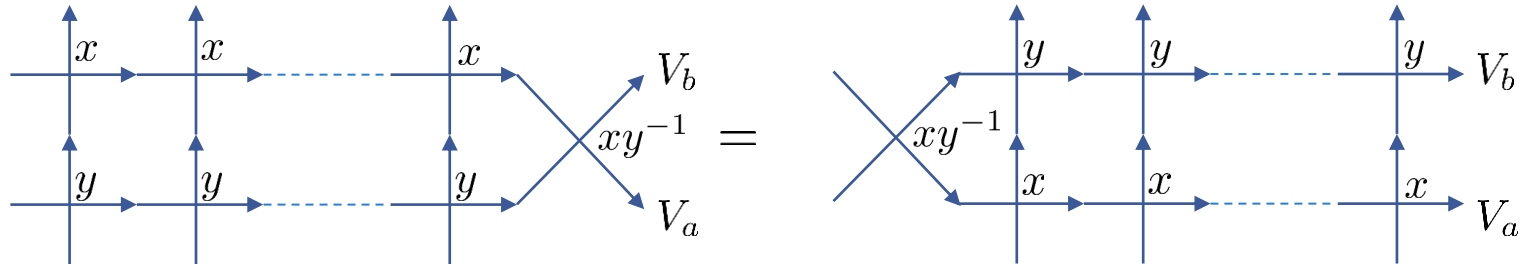
$$[\tau^{(m,n)}(x, y), \tau^{(m,n)}(x', y')] = 0 \quad (\forall x, y, x', y' \in \mathbb{C})$$

- This is often called the layer-to-layer transfer matrix.

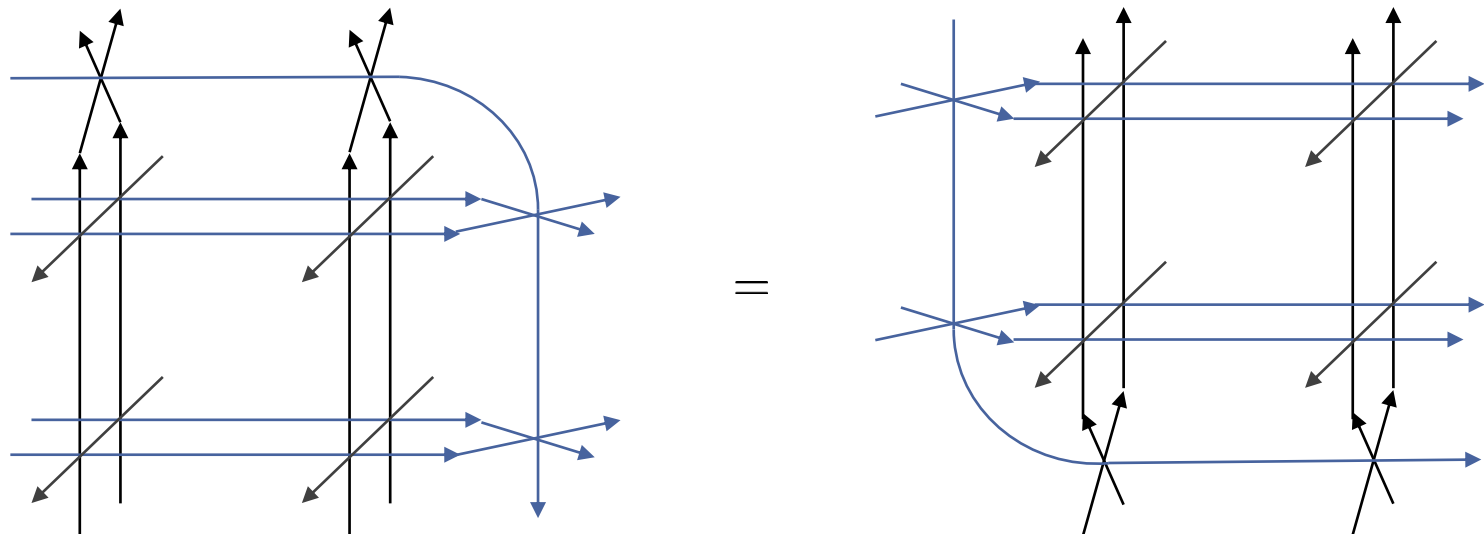
Comparing RTT relations in 2D and 3D

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■ 1 + 1 + 0 dimension in 2D

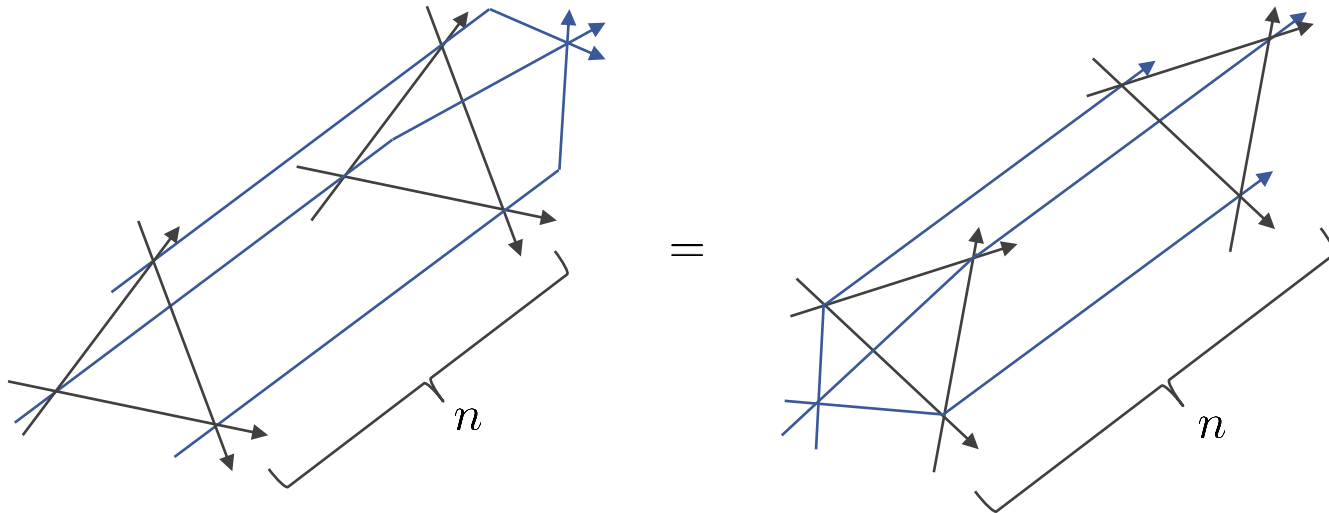


■ 2 + 2 + 1 dimension in 3D

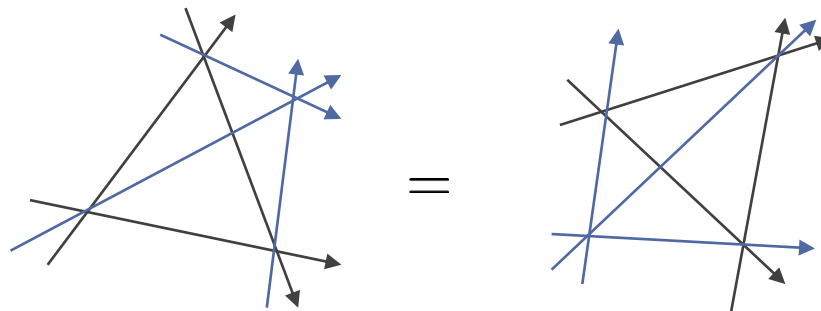


RSSS=SSSR: Reduction to Yang-Baxter eq 18/68

- $1 + 1 + 1 + 0$ dimension in 3D

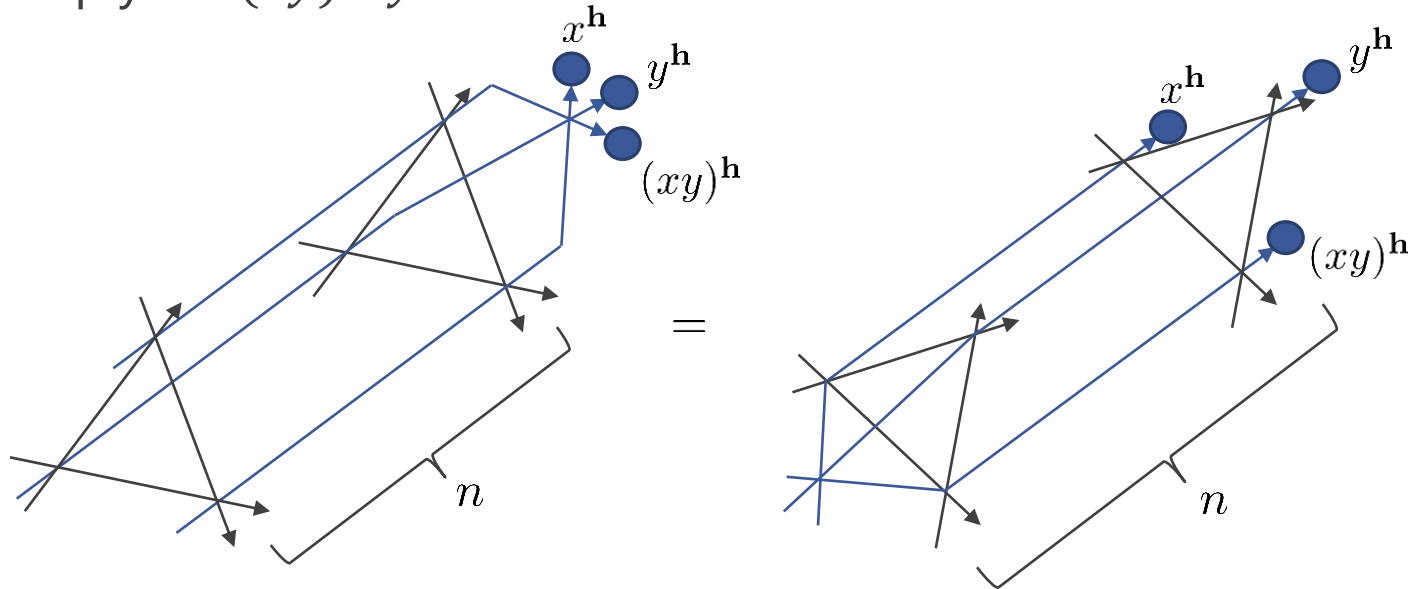


- The above equation is obtained by repeated use of



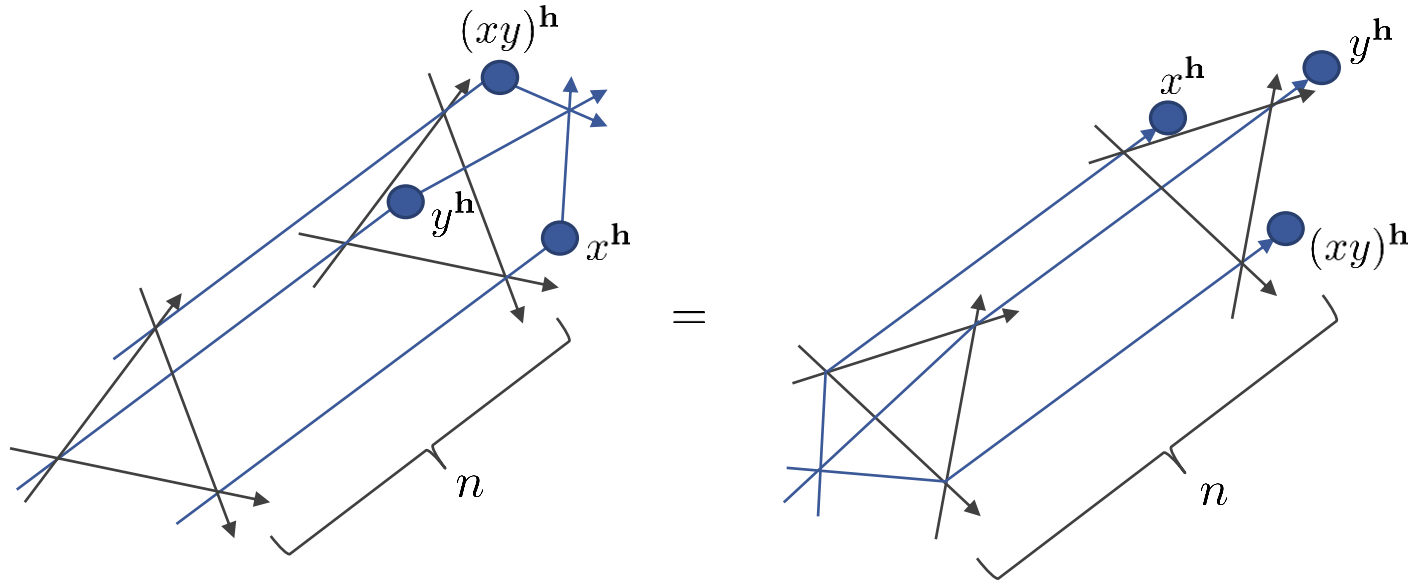
RSSS=SSSR: Reduction to Yang-Baxter eq 19/68

■ Multiply $x^{h_4}(xy)^{h_5}y^{h_6}$:



RSSS=SSSR: Reduction to Yang-Baxter eq 20/68

- Use the weight conservation $[x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5}y^{\mathbf{h}_6}, \mathcal{R}] = 0$:



- This gives the following identity:

$$\begin{aligned}
 & (x^{\mathbf{h}_4} \mathcal{R}_{1_1 2_1 4} \cdots \mathcal{R}_{1_n 2_n 4}) ((xy)^{\mathbf{h}_5} \mathcal{R}_{1_1 3_1 5} \cdots \mathcal{R}_{1_n 3_n 5}) (y^{\mathbf{h}_6} \mathcal{R}_{2_1 3_1 6} \cdots \mathcal{R}_{2_n 3_n 6}) \mathcal{R}_{456} \\
 &= \mathcal{R}_{456} (y^{\mathbf{h}_6} \mathcal{R}_{2_1 3_1 6} \cdots \mathcal{R}_{2_n 3_n 6}) ((xy)^{\mathbf{h}_5} \mathcal{R}_{1_1 3_1 5} \cdots \mathcal{R}_{1_n 3_n 5}) (x^{\mathbf{h}_4} \mathcal{R}_{1_1 2_1 4} \cdots \mathcal{R}_{1_n 2_n 4})
 \end{aligned}$$

RSSS=SSSR: Reduction to Yang-Baxter eq 21/68

$$\begin{aligned} & (x^{\mathbf{h}_4} \mathcal{R}_{1_1 2_1 4} \cdots \mathcal{R}_{1_n 2_n 4}) ((xy)^{\mathbf{h}_5} \mathcal{R}_{1_1 3_1 5} \cdots \mathcal{R}_{1_n 3_n 5}) (y^{\mathbf{h}_6} \mathcal{R}_{2_1 3_1 6} \cdots \mathcal{R}_{2_n 3_n 6}) \mathcal{R}_{456} \\ &= \mathcal{R}_{456} (y^{\mathbf{h}_6} \mathcal{R}_{2_1 3_1 6} \cdots \mathcal{R}_{2_n 3_n 6}) ((xy)^{\mathbf{h}_5} \mathcal{R}_{1_1 3_1 5} \cdots \mathcal{R}_{1_n 3_n 5}) (x^{\mathbf{h}_4} \mathcal{R}_{1_1 2_1 4} \cdots \mathcal{R}_{1_n 2_n 4}) \end{aligned}$$

■ From this identity, we can obtain solutions to Yang-Baxter eq by

1. multiplying \mathcal{R}_{456}^{-1} and taking the trace on the spaces 456.
2. sandwiching between $\langle \chi_r | \otimes \langle \chi_r | \otimes \langle \chi_r |$ and $|\chi_{r'}\rangle \otimes |\chi_{r'}\rangle \otimes |\chi_{r'}\rangle$.

■ Here, we use properties for the 3D R:

1. The 3D R is invertible.
2. The 3D R has two eigenvectors with eigenvalues = 1 given by

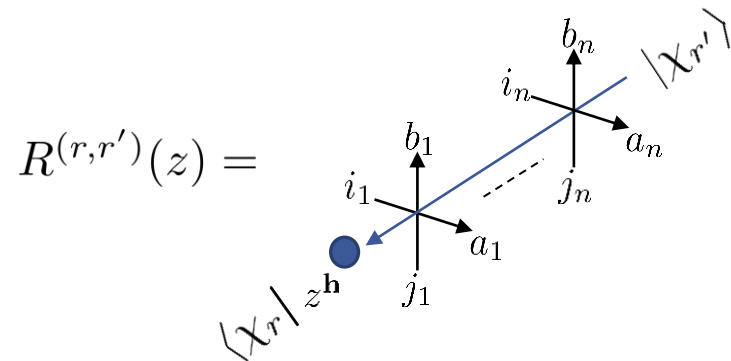
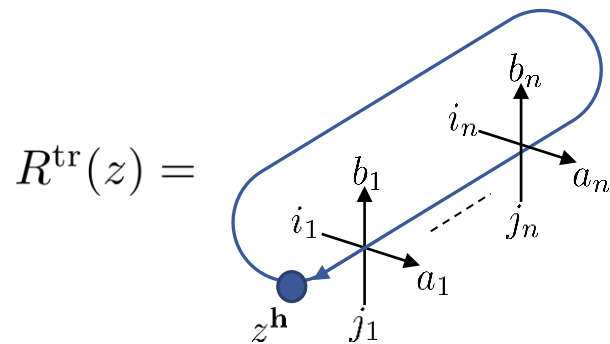
$$\mathcal{R} |\chi_r\rangle \otimes |\chi_r\rangle \otimes |\chi_r\rangle = |\chi_r\rangle \otimes |\chi_r\rangle \otimes |\chi_r\rangle \quad (r = 1, 2)$$

$$\langle \chi_r | \otimes \langle \chi_r | \otimes \langle \chi_r | \mathcal{R} = \langle \chi_r | \otimes \langle \chi_r | \otimes \langle \chi_r | \quad (r = 1, 2)$$

Here

$$|\chi_r\rangle = \sum_{m \geq 0} \frac{|rm\rangle}{(q^{r^2}; q^{r^2})_m}$$

Matrix product solution to Yang-Baxter eq^{22/68}

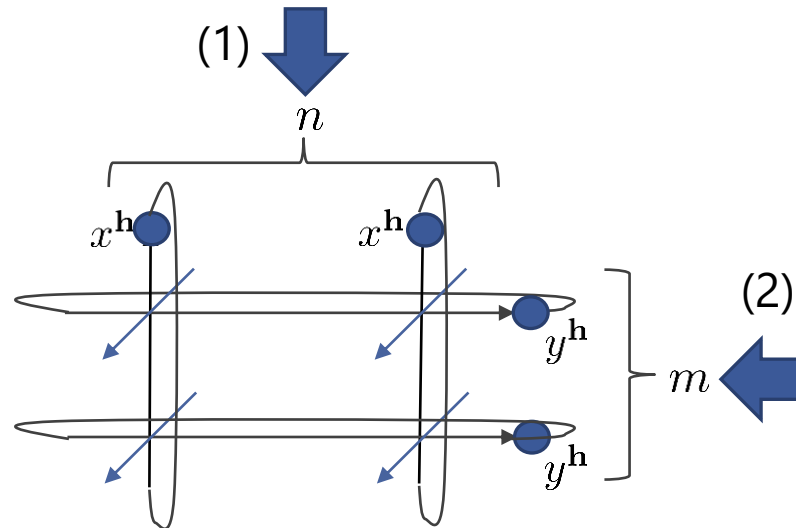


- These solutions are characterized as the R matrices associated with some quantum affine algebras. [Kuniba-Okado-Sergeev15]

	$R^{\text{tr}}(z)$	$R^{(r,r')}(z)$
= 3D R	$U_q(A_{n-1}^{(1)})$ symmetric tensor rep.	$U_q(D_{n+1}^{(2)}), U_q(A_{2n}^{(2)}), U_q(C_n^{(1)})$ Fock rep.
= 3D L	$U_q(A_{n-1}^{(1)})$ fundamental rep.	$U_q(D_{n+1}^{(2)}), U_q(B_n^{(1)}), U_q(D_n^{(1)})$ spin rep.

- Moreover, by mixing uses of the 3D R & L, we also obtain the R matrices associated with generalized quantum groups.

- We consider the following transfer matrix where  are the 3D L.



- Let us consider projections from two directions (1) & (2).
 - Both of them give the row-to-row transfer matrix in two dimension.
 - This suggest the spectral duality between $sl(m)$ spin chain of size n and $sl(n)$ spin chain of size m . [Bazhanov-Sergeev06]
- The duality also appears in the context of the five-dimensional gauge theory. [Mironov-Morozov-Runov-Zenkevich-Zotov13]

■ Motivation

- Why do the 3D R & L lead to such similar results, although they have totally different origins?

- We study transition matrices of PBW bases of $U_q^+(sl(m|n))$ motivated by the KOY theorem which holds for non-super cases.

■ Theorem [Kuniba-Okado-Yamada13] (Rough Statement)

- g : arbitrary finite-dimensional simple Lie algebra
- Φ : intertwiner of irreducible representations of $A_q(g)$
- γ : transition matrix of PBW bases of the nilpotent subalgebra of $U_q(g)$
- Then, we have $\Phi = \gamma$.

- For $\bigcirc \text{---} \bigcirc$, the 3D R gives the transition matrix for $U_q(A_2)$.

- One of our result is that the 3D L is exactly the transition matrix for the quantum superalgebra associated with $\bigcirc \text{---} \bigotimes$.

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Root system of Lie superalgebras $sl(m|n)$ 26/68

■ Setup

□ Weight lattice: $\mathcal{E}(m|n)_{\mathbb{Z}} = \sum_{i=1}^m \mathbb{Z}\epsilon_i \oplus \sum_{i=1}^n \mathbb{Z}\delta_i$

$$(\epsilon_i, \epsilon_j) = (-1)^\theta \delta_{i,j}, \quad (\delta_i, \delta_j) = -(-1)^\theta \delta_{i,j}, \quad (\epsilon_i, \delta_j) = 0 \quad \theta = 0, 1$$

□ Parity: $p(\lambda) = \sum_{i=1}^n b_i \pmod{2}$ for $\lambda = \sum_{i=1}^m a_i \epsilon_i + \sum_{i=1}^n b_i \delta_i \in \mathcal{E}(m|n)_{\mathbb{Z}}$

□ We set $\{\bar{\epsilon}_i\}_{1 \leq i \leq m+n} = \{\epsilon_i\}_{1 \leq i \leq m} \cup \{\delta_i\}_{1 \leq i \leq n}$.

■ Lie superalgebra $sl(m|n)$

□ Rank: $r = m + n - 1$

□ Simple roots: $\Pi = \{\alpha_1, \dots, \alpha_r\}$, $\alpha_i = \bar{\epsilon}_i - \bar{\epsilon}_{i+1}$

□ Reduced positive roots:

$$\tilde{\Phi}^+ = \{\bar{\epsilon}_i - \bar{\epsilon}_j \mid (1 \leq i < j \leq r+1)\}$$

□ Even & odd parts: $\tilde{\Phi}^+ = \tilde{\Phi}_{\text{even}}^+ \cup \tilde{\Phi}_{\text{iso}}^+$

$$\tilde{\Phi}_{\text{even}}^+ = \{\alpha \in \tilde{\Phi}^+ \mid p(\alpha) = 0\}$$

$$\tilde{\Phi}_{\text{iso}}^+ = \{\alpha \in \tilde{\Phi}^+ \mid p(\alpha) = 1\}$$

■ Cartan matrix for Lie superalgebra $sl(m|n)$

$$\square d_\alpha = (\alpha, \alpha)/2 \text{ for } \alpha \in \tilde{\Phi}_{\text{even}}^+, d_\alpha = 1 \text{ for } \alpha \in \tilde{\Phi}_{\text{iso}}^+$$

$$\square D = \text{diag}(d_1, \dots, d_r), d_i = d_{\alpha_i}$$

$$\square \text{Cartan matrix: } A = (a_{ij})_{i,j \in I}, a_{ij} = (\alpha_i, \alpha_j)/d_i \quad I = \{1, \dots, r\}$$

$$\square \text{Simple coroots: } \{h_i\}_{i \in I}, \alpha_j(h_i) = a_{ij}$$

■ Dynkin diagram for Cartan data (A, p)

$$\square \text{Prepare } r \text{ dots and decorate the } i\text{-th dot by}$$

$$\bigcirc \text{ for } \alpha_i \in \tilde{\Phi}_{\text{even}}^+, \bigotimes \text{ for } \alpha_i \in \tilde{\Phi}_{\text{iso}}^+$$

$$\square \text{Connect them if } a_{ij} \neq 0 \ (i \neq j):$$

$$\begin{array}{ccccccc} \bar{\epsilon}_1 - \bar{\epsilon}_2 & \bar{\epsilon}_2 - \bar{\epsilon}_3 & & & \bar{\epsilon}_r - \bar{\epsilon}_{r+1} & & \\ \times \text{---} \times \text{---} & \dots & \text{---} & \times & & & \end{array} \quad \times: \bigcirc \text{ or } \bigotimes$$

■ Remark

$$\square \text{Dynkin diagrams do } \textit{not} \text{ correspond to Lie superalgebras themselves.}$$

■ $U_q(sl(m|n))$: quantum superalgebras associated with (A, p)

▣ Generators: $e_i, f_i, k_i^{\pm 1}$ ($i \in I$)

▣ We use $q_i = q^{d_i}$.

▣ Part of relations:

$$k_i^{\pm 1} k_i^{\mp 1} = 1, \quad k_i k_j = k_j k_i, \quad k_i e_j = q_i^{a_{ij}} e_j k_i, \quad k_i f_j = q_i^{-a_{ij}} f_j k_i,$$

$$e_i f_j - (-1)^{p(\alpha_i)p(\alpha_j)} f_j e_i = \delta_{i,j} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$$

- $U_q^+(sl(m|n))$: nilpotent subalgebra generated by $\{e_i\}_{i \in I}$

- ▣ Root space decomposition:

$$U_q^+(\mathfrak{sl}(m|n))_\alpha = \{g \mid k_i g = q_i^{\alpha(h_i)} g k_i \ (i \in I)\}$$

- ▣ q -commutator:

$$[x, y]_q = xy - (-1)^{p(\alpha)p(\beta)} q^{-(\alpha, \beta)} yx$$

$$x \in U_q^+(\mathfrak{sl}(m|n))_\alpha$$

$$y \in U_q^+(\mathfrak{sl}(m|n))_\beta$$

- Rest of relations:

1. For $a_{ij} \neq 0$ ($i \neq j$), $\alpha_i \in \tilde{\Phi}_{\text{even}}^+$: $e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 = 0$
2. For $a_{ij} = 0$: $[e_i, e_j] = 0$
3. For $\alpha_i \in \tilde{\Phi}_{\text{iso}}^+$: $[[[e_{i-1}, e_i]_q, e_{i+1}]_q, e_i] = 0$
4. $\{f_i\}_{i \in I}$ satisfies same relations as (1)~(3).

- Remark

- ▣ For $a_{ii} = 0$ and $\alpha_i \in \tilde{\Phi}_{\text{iso}}^+$, we have $e_i^2 = 0$ from (2).

- We define two partial orders O_1, O_2 on $\tilde{\Phi}^+$.

- For $\alpha = \bar{\epsilon}_a - \bar{\epsilon}_b, \beta = \bar{\epsilon}_c - \bar{\epsilon}_d \in \tilde{\Phi}^+$, we define

$$O_1 : \quad \alpha < \beta \quad \Longleftrightarrow \quad a < c \text{ or } (a = c \text{ and } b < d)$$

$$O_2 : \quad \alpha < \beta \quad \Longleftrightarrow \quad a > c \text{ or } (a = c \text{ and } b > d)$$

- Definition (quantum root vector)

- For $\beta \in \tilde{\Phi}^+$, we define $e_\beta \in U_q^+(\mathfrak{sl}(m|n))_\beta$ in two ways depending on O_i .
 - For $\beta = \alpha_i$, we set $e_\beta = e_i$.
 - For $\beta = \alpha + \alpha_i$ ($\alpha = \bar{\epsilon}_a - \bar{\epsilon}_b \in \tilde{\Phi}^+, a < i$), we set

$$e_\beta = \begin{cases} [e_i, e_\alpha]_q & \text{for } O_1 \quad (\alpha < \alpha_i) \\ [e_\alpha, e_i]_q & \text{for } O_2 \quad (\alpha_i < \alpha) \end{cases}$$

Example: Rank 3

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- We consider the case of \mathcal{O}_1 .

$$\epsilon_1 - \epsilon_2 < \epsilon_1 - \epsilon_3 < \epsilon_1 - \epsilon_4 < \epsilon_2 - \epsilon_3 < \epsilon_2 - \epsilon_4 < \epsilon_3 - \epsilon_4$$

Example: Rank 3

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■ We consider the case of \mathcal{O}_1 .

$$\epsilon_1 - \epsilon_2 < \epsilon_1 - \epsilon_3 < \epsilon_1 - \epsilon_4 < \epsilon_2 - \epsilon_3 < \epsilon_2 - \epsilon_4 < \epsilon_3 - \epsilon_4$$

$$\alpha_1 < \alpha_1 + \alpha_2 < \alpha_1 + \alpha_2 + \alpha_3 < \alpha_2 < \alpha_2 + \alpha_3 < \alpha_3$$

Example: Rank 3

33/68

■ We consider the case of \mathcal{O}_1 .

$$\epsilon_1 - \epsilon_2 < \epsilon_1 - \epsilon_3 < \epsilon_1 - \epsilon_4 < \epsilon_2 - \epsilon_3 < \epsilon_2 - \epsilon_4 < \epsilon_3 - \epsilon_4$$

$$\alpha_1 < \alpha_1 + \alpha_2 < \alpha_1 + \alpha_2 + \alpha_3 < \alpha_2 < \alpha_2 + \alpha_3 < \alpha_3$$

e_1

e_2

e_3

Example: Rank 3

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■ We consider the case of \mathcal{O}_1 .

$$\epsilon_1 - \epsilon_2 < \epsilon_1 - \epsilon_3 < \epsilon_1 - \epsilon_4 < \epsilon_2 - \epsilon_3 < \epsilon_2 - \epsilon_4 < \epsilon_3 - \epsilon_4$$

$$\alpha_1 < \alpha_1 + \alpha_2 < \alpha_1 + \alpha_2 + \alpha_3 < \alpha_2 < \alpha_2 + \alpha_3 < \alpha_3$$

$$e_1 \quad [e_2, e_1]_q \quad e_2 \quad [e_3, e_2]_q \quad e_3$$

Example: Rank 3

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■ We consider the case of \mathcal{O}_1 .

$$\epsilon_1 - \epsilon_2 < \epsilon_1 - \epsilon_3 < \epsilon_1 - \epsilon_4 < \epsilon_2 - \epsilon_3 < \epsilon_2 - \epsilon_4 < \epsilon_3 - \epsilon_4$$

$$\alpha_1 < \alpha_1 + \alpha_2 < \alpha_1 + \alpha_2 + \alpha_3 < \alpha_2 < \alpha_2 + \alpha_3 < \alpha_3$$

$$e_1 \quad [e_2, e_1]_q \quad [e_3, [e_2, e_1]_q]_q \quad e_2 \quad [e_3, e_2]_q \quad e_3$$

■ Theorem [Yamane94]

□ Let $\beta_1 < \dots < \beta_l$ denote elements of $\tilde{\Phi}^+$ under O_i $l = |\tilde{\Phi}^+|$

□ For $A = (a_1, \dots, a_l)$ where a_t are given by

$$a_t \in \mathbb{Z}_{\geq 0} \text{ for } \beta_t \in \tilde{\Phi}_{\text{even}}^+ \quad a_t \in \{0, 1\} \text{ for } \beta_t \in \tilde{\Phi}_{\text{iso}}^+$$

we define E_i^A by

$$E_i^A = e_{\beta_1}^{(a_1)} e_{\beta_2}^{(a_2)} \dots e_{\beta_l}^{(a_l)}$$

□ Then

$$B_i = \{E_i^A \mid a_t \in \mathbb{Z}_{\geq 0} (\beta_t \in \tilde{\Phi}_{\text{even}}^+), a_t \in \{0, 1\} (\beta_t \in \tilde{\Phi}_{\text{iso}}^+)\}$$

gives a basis of U_q .

□ $e_{\beta_t}^{(a_t)}$: divided power given by $e_{\beta_t}^{(a_t)} = e_{\beta_t}^{a_t} / [a_t]_{p_t}!$ $p_t = q^{d_{\beta_t}}$

□ $[k]_q!$: factorial of q -number given by

$$[m]_q! = \prod_{k=1}^m [k]_q \quad [k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}$$

- We define the transition matrix γ of PBW bases as follows:

$$E_2^A = \sum_B \gamma_B^A E_1^{B^{\text{op}}}$$

- Here, we set $X^{\text{op}} = (x_l, \dots, x_1)$ for $X = (x_1, \dots, x_l)$.
- From now on, we only consider the cases of rank 2 & 3.
 - The transition matrix for higher rank cases are constructed as a composite of one of rank 2.
 - For rank 3 cases, the composite takes the form of the tetrahedron eq.

- Introduction: P.3~24
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Type A of rank 2 cases

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■ We set $e_{ij} = [e_i, e_j]_q$

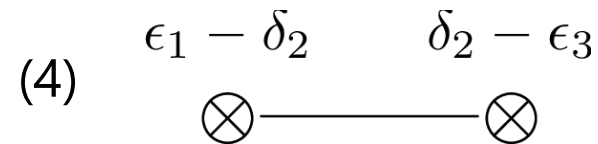
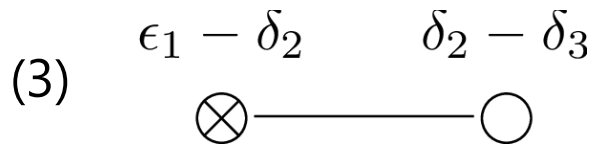
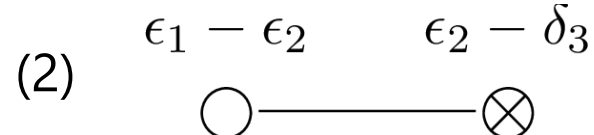
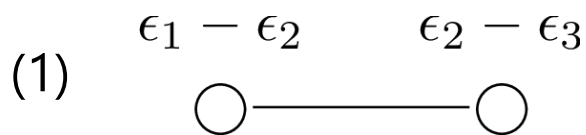
■ Quantum root vectors of rank 2

$$\begin{aligned} B_1 : \quad e_{\beta_1} &= e_1, & e_{\beta_2} &= e_{21}, & e_{\beta_3} &= e_2 \\ B_2 : \quad e_{\beta_1} &= e_2, & e_{\beta_2} &= e_{12}, & e_{\beta_3} &= e_1 \end{aligned} \quad \beta_1 < \beta_2 < \beta_3$$

■ Transition matrix

$$e_2^{(a)} e_{12}^{(b)} e_1^{(c)} = \sum_{i,j,k} \gamma_{i,j,k}^{a,b,c} e_1^{(k)} e_{21}^{(j)} e_2^{(i)}$$

■ Dynkin diagrams of rank 2



The case $\bigcirc \text{---} \bigcirc$

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$$\begin{array}{ccc} \epsilon_1 - \epsilon_2 & \epsilon_2 - \epsilon_3 & \\ \bigcirc \text{---} \bigcirc & & \end{array} \quad (\epsilon_i, \epsilon_i) = 1 \quad DA = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

■ Root system

$$\tilde{\Phi}_{\text{even}}^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} \quad \tilde{\Phi}_{\text{iso}}^+ = \{\}$$

■ Transition matrix

$$e_2^{(a)} e_{12}^{(b)} e_1^{(c)} = \sum_{i,j,k} \gamma_{i,j,k}^{a,b,c} e_1^{(k)} e_{21}^{(j)} e_2^{(i)} \cdots (*) \quad i, j, k, a, b, c \in \mathbb{Z}$$

■ Theorem [Kuniba-Okado-Yamada13] $\gamma_{i,j,k}^{a,b,c} = \mathcal{R}_{i,j,k}^{a,b,c}$

■ Example For $(a, b, c) = (0, 1, 1)$, $(*)$ becomes

$$e_{12} e_1 = \mathcal{R}_{0,1,1}^{0,1,1} e_1 e_{21} + \mathcal{R}_{1,0,2}^{0,1,1} \frac{e_1^2}{q + q^{-1}} e_2$$

$$-q e_2 e_1^2 + (1 + q^2) e_1 e_2 e_1 - q e_1^2 e_2 = 0 \quad \because \mathcal{R}_{0,1,1}^{0,1,1} = -q^2, \mathcal{R}_{1,0,2}^{0,1,1} = 1 - q^4$$

This is a relation of $\bigcirc \text{---} \bigcirc$.

The case $\bigcirc \text{---} \bigotimes$

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$$\begin{array}{ccc} \epsilon_1 - \epsilon_2 & \epsilon_2 - \delta_3 & (\epsilon_i, \epsilon_i) = 1 \\ \bigcirc \text{---} \bigotimes & & (\delta_i, \delta_i) = -1 \end{array} \quad DA = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$$

■ Root system

$$\tilde{\Phi}_{\text{even}}^+ = \{\alpha_1\} \quad \tilde{\Phi}_{\text{iso}}^+ = \{\alpha_2, \alpha_1 + \alpha_2\}$$

■ Transition matrix

$$e_2^a e_{12}^b e_1^{(c)} = \sum_{i,j,k} \gamma_{i,j,k}^{a,b,c} e_1^{(k)} e_{21}^j e_2^i \quad \dots (*) \quad \begin{array}{l} k, c \in \mathbb{Z} \\ i, j, a, b \in \{0, 1\} \end{array}$$

■ Theorem [Y20] $\gamma_{i,j,k}^{a,b,c} = \mathcal{L}_{i,j,k}^{a,b,c}$

■ Example For $(a, b, c) = (0, 1, 1)$, $(*)$ becomes

$$e_{12}e_1 = \mathcal{L}_{0,1,1}^{0,1,1} e_1 e_{21} + \mathcal{L}_{1,0,2}^{0,1,1} \frac{e_1^2}{q + q^{-1}} e_2$$

$$-q e_2 e_1^2 + (1 + q^2) e_1 e_2 e_1 - q e_1^2 e_2 = 0 \quad \because \mathcal{L}_{0,1,1}^{0,1,1} = -q^2, \mathcal{L}_{1,0,2}^{0,1,1} = 1 - q^4$$

This is a relation of $\bigcirc \text{---} \bigotimes$.

The case \otimes — \circ

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$$\begin{array}{ccc} \epsilon_1 - \delta_2 & \delta_2 - \delta_3 & (\epsilon_i, \epsilon_i) = -1 \\ \otimes \text{---} \circ & & (\delta_i, \delta_i) = 1 \end{array} \quad DA = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$$

■ Root system

$$\tilde{\Phi}_{\text{even}}^+ = \{\alpha_2\} \quad \tilde{\Phi}_{\text{iso}}^+ = \{\alpha_1, \alpha_1 + \alpha_2\}$$

■ Transition matrix

$$e_2^{(a)} e_{12}^b e_1^c = \sum_{i,j,k} \gamma_{i,j,k}^{a,b,c} e_1^k e_{21}^j e_2^{(i)} \quad \dots (*) \quad \begin{array}{l} i, a \in \mathbb{Z} \\ j, k, b, c \in \{0, 1\} \end{array}$$

■ Corollary $\gamma_{i,j,k}^{a,b,c} = \mathcal{M}_{i,j,k}^{a,b,c}$

□ Here, we define $\mathcal{M} \in \text{End}(F \otimes V \otimes V)$ by $\mathcal{M}_{i,j,k}^{a,b,c} = \mathcal{L}_{k,j,i}^{c,b,a}$.

The case $\otimes \text{---} \otimes$

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$$\begin{array}{ccc} \epsilon_1 - \delta_2 & \delta_2 - \epsilon_3 & (\epsilon_i, \epsilon_i) = -1 \\ \otimes \text{---} \otimes & & (\delta_i, \delta_i) = 1 \end{array} \quad DA = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

■ Root system

$$\tilde{\Phi}_{\text{even}}^+ = \{\alpha_1 + \alpha_2\} \quad \tilde{\Phi}_{\text{iso}}^+ = \{\alpha_1, \alpha_2\}$$

■ Transition matrix

$$e_2^a e_{12}^{(b)} e_1^c = \sum_{i,j,k} \gamma_{i,j,k}^{a,b,c} e_1^k e_{21}^{(j)} e_2^i \quad \begin{array}{l} j, b \in \mathbb{Z} \\ i, k, a, c \in \{0, 1\} \end{array}$$

■ Theorem [Y20] $\gamma_{i,j,k}^{a,b,c} = \mathcal{N}_{i,j,k}^{a,b,c}$

□ Here, we define $\mathcal{N} \in \text{End}(V \otimes F \otimes V)$ as follows:

$$\mathcal{N}(u_i \otimes |j\rangle \otimes u_k) = \sum_{a,c \in \{0,1\}, b \in \mathbb{Z}_{\geq 0}} \mathcal{N}_{i,j,k}^{a,b,c} u_a \otimes |b\rangle \otimes u_c$$

$$\mathcal{N}_{0,j,0}^{0,b,0} = \delta_{j,b} q^j, \quad \mathcal{N}_{1,j,1}^{1,b,1} = -\delta_{j,b} q^{j+1}, \quad \mathcal{N}_{0,j,1}^{0,b,1} = \mathcal{N}_{1,j,0}^{1,b,0} = \delta_{j,b},$$

$$\mathcal{N}_{1,j,1}^{0,b,0} = \delta_{j+1,b} q^j (1 - q^2), \quad \mathcal{N}_{0,j,0}^{1,b,1} = \delta_{j-1,b} [j]_q,$$

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- We set $e_{(ij)k}, e_{i(jk)} \in U_q^+(\mathfrak{sl}(m|n))$ as follows:

$$e_{(ij)k} = [e_{ij}, e_k]_q, \quad e_{i(jk)} = [e_i, e_{jk}]_q \quad e_{ij} = [e_i, e_j]_q$$

- For $|i - k| > 1$, we have $e_{(ij)k} = e_{i(jk)} =: e_{ijk}$.

- Quantum root vectors of rank 3

$$B_1 : \quad e_{\beta_1} = e_1, \quad e_{\beta_2} = e_{21}, \quad e_{\beta_3} = e_{321}, \\ e_{\beta_4} = e_2, \quad e_{\beta_5} = e_{32}, \quad e_{\beta_6} = e_3$$

$$\beta_1 < \cdots < \beta_6$$

$$B_2 : \quad e_{\beta_1} = e_3, \quad e_{\beta_2} = e_{23}, \quad e_{\beta_3} = e_2, \\ e_{\beta_4} = e_{123}, \quad e_{\beta_5} = e_{12}, \quad e_{\beta_6} = e_1$$

- Transition matrix

$$e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)} = \sum_{i_1, i_2, i_3, i_4, i_5, i_6} \gamma_{i_1, i_2, i_3, i_4, i_5, i_6}^{o_1, o_2, o_3, o_4, o_5, o_6} e_1^{(i_6)} e_{21}^{(i_5)} e_{321}^{(i_4)} e_2^{(i_3)} e_{32}^{(i_2)} e_3^{(i_1)}$$

- We use the following matrices:

$$e_2^{(a)} e_{12}^{(b)} e_1^{(c)} = \sum_{i,j,k} \Gamma^{(2|1)}_{i,j,k} e_1^{(k)} e_{21}^{(j)} e_2^{(i)}$$

$$e_3^{(a)} e_{23}^{(b)} e_2^{(c)} = \sum_{i,j,k} \Gamma^{(3|2)}_{i,j,k} e_2^{(k)} e_{32}^{(j)} e_3^{(i)}$$

$$e_{23}^{(a)} e_{123}^{(b)} e_1^{(c)} = \sum_{i,j,k} \Gamma^{(23|1)}_{i,j,k} e_1^{(k)} e_{(23)1}^{(j)} e_{23}^{(i)}$$

$$e_{32}^{(a)} e_{1(32)}^{(b)} e_1^{(c)} = \sum_{i,j,k} \Gamma^{(32|1)}_{i,j,k} e_1^{(k)} e_{321}^{(j)} e_{32}^{(i)}$$

$$e_3^{(a)} e_{123}^{(b)} e_{12}^{(c)} = \sum_{i,j,k} \Gamma^{(3|12)}_{i,j,k} e_{12}^{(k)} e_{3(12)}^{(j)} e_3^{(i)}$$

$$e_3^{(a)} e_{(21)3}^{(b)} e_{21}^{(c)} = \sum_{i,j,k} \Gamma^{(3|21)}_{i,j,k} e_{21}^{(k)} e_{321}^{(j)} e_3^{(i)}$$

- Given a Dynkin diagram, $\Gamma^{(x)}$ is specified as the 3D R, L, M or N.

Two ways construction of γ : first way

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$$e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}$$

Two ways construction of γ : first way

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$$\underline{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)}} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}$$

Two ways construction of γ : first way

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$$\frac{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}}{=} \sum \Gamma^{(3|2)}_{o_1, o_2, o_3, x_1, x_2, x_3} e_2^{(x_3)} e_{32}^{(x_2)} e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}$$

■ For the underlined part, we used

$$e_3^{(a)} e_{23}^{(b)} e_2^{(c)} = \sum_{i,j,k} \Gamma^{(3|2)}_{a,b,c,i,j,k} e_2^{(k)} e_{32}^{(j)} e_3^{(i)}$$

Two ways construction of γ : first way

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$$\frac{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}}{= \sum \Gamma^{(3|2)}_{o_1, o_2, o_3} e_2^{(x_3)} e_{32}^{(x_2)} \underline{e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}}}$$

Two ways construction of γ : first way

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$$\begin{aligned}
 & \frac{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}}{=} \\
 &= \sum \Gamma(3|2)_{o_1, o_2, o_3}^{x_1, x_2, x_3} e_2^{(x_3)} e_{32}^{(x_2)} \underline{e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}} \\
 &= \sum \Gamma(3|2)_{o_1, o_2, o_3}^{x_1, x_2, x_3} \Gamma(3|12)_{i_1, x_4, x_5}^{x_1, o_4, o_5} e_2^{(x_3)} e_{32}^{(x_2)} e_{12}^{(x_5)} \underline{e_{3(12)}^{(x_4)} e_3^{(i_1)} e_1^{(o_6)}}
 \end{aligned}$$

■ For the underlined part, we used

$$e_3^{(a)} e_{123}^{(b)} e_{12}^{(c)} = \sum_{i,j,k} \Gamma(3|12)_{i,j,k}^{a,b,c} e_{12}^{(k)} e_{3(12)}^{(j)} e_3^{(i)}$$

Two ways construction of γ : first way

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$$\begin{aligned}
 & \frac{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}}{=} \sum \Gamma^{(3|2)}_{o_1, o_2, o_3} e_2^{(x_3)} e_{32}^{(x_2)} \frac{e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}}{=} \\
 & = \sum \Gamma^{(3|2)}_{o_1, o_2, o_3} \Gamma^{(3|12)}_{i_1, x_4, x_5} e_2^{(x_3)} \frac{e_{32}^{(x_2)} e_{12}^{(x_5)}}{=} \frac{e_{3(12)}^{(x_4)}}{=} \frac{e_3^{(i_1)} e_1^{(o_6)}}{=}
 \end{aligned}$$

Two ways construction of γ : first way

53/68

$$\begin{aligned}
 & \frac{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}}{=} \\
 &= \sum \Gamma(3|2)_{o_1, o_2, o_3}^{x_1, x_2, x_3} e_2^{(x_3)} e_{32}^{(x_2)} \underline{e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}} \\
 &= \sum \Gamma(3|2)_{o_1, o_2, o_3}^{x_1, x_2, x_3} \Gamma(3|12)_{i_1, x_4, x_5}^{x_1, o_4, o_5} e_2^{(x_3)} e_{32}^{(x_2)} e_{12}^{(x_5)} \underline{e_{3(12)}^{(x_4)}} \underline{e_3^{(i_1)} e_1^{(o_6)}} \\
 &= \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5} \Gamma(3|2)_{o_1, o_2, o_3}^{x_1, x_2, x_3} \Gamma(3|12)_{i_1, x_4, x_5}^{x_1, o_4, o_5} \\
 &\quad \times e_2^{(x_3)} e_{12}^{(x_5)} e_{32}^{(x_2)} e_{1(32)}^{(x_4)} e_1^{(o_6)} e_3^{(i_1)}
 \end{aligned}$$

$$\rho_1 = p(\alpha_1)p(\alpha_3)$$

$$\rho_2 = p(\alpha_1 + \alpha_2)p(\alpha_2 + \alpha_3)$$

■ For the underlined part, we used

$$[e_{32}, e_{12}] = [e_3, e_1] = 0 \quad e_{3(12)} = (-1)^{p(\alpha_1)p(\alpha_3)} e_{1(32)}$$

Two ways construction of γ : first way

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$$\begin{aligned}
 & \frac{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}}{=} \\
 &= \sum \Gamma(3|2)_{x_1, x_2, x_3}^{o_1, o_2, o_3} e_2^{(x_3)} e_{32}^{(x_2)} \frac{e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}}{=} \\
 &= \sum \Gamma(3|2)_{x_1, x_2, x_3}^{o_1, o_2, o_3} \Gamma(3|12)_{i_1, x_4, x_5}^{x_1, o_4, o_5} e_2^{(x_3)} \frac{e_{32}^{(x_2)} e_{12}^{(x_5)} e_{3(12)}^{(x_4)}}{=} \frac{e_3^{(i_1)} e_1^{(o_6)}}{=} \\
 &= \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5} \Gamma(3|2)_{x_1, x_2, x_3}^{o_1, o_2, o_3} \Gamma(3|12)_{i_1, x_4, x_5}^{x_1, o_4, o_5} \\
 & \quad \times e_2^{(x_3)} e_{12}^{(x_5)} \frac{e_{32}^{(x_2)} e_{1(32)}^{(x_4)} e_1^{(o_6)} e_3^{(i_1)}}{=}
 \end{aligned}$$

$$\rho_1 = p(\alpha_1)p(\alpha_3)$$

$$\rho_2 = p(\alpha_1 + \alpha_2)p(\alpha_2 + \alpha_3)$$

Two ways construction of γ : first way

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$$\rho_1 = p(\alpha_1)p(\alpha_3)$$

$$\rho_2 = p(\alpha_1 + \alpha_2)p(\alpha_2 + \alpha_3)$$

$$\begin{aligned}
 & \frac{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}}{=} \\
 &= \sum \Gamma(3|2)_{o_1, o_2, o_3}^{x_1, x_2, x_3} e_2^{(x_3)} e_{32}^{(x_2)} \underline{e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}} \\
 &= \sum \Gamma(3|2)_{o_1, o_2, o_3}^{x_1, x_2, x_3} \Gamma(3|12)_{i_1, x_4, x_5}^{x_1, o_4, o_5} e_2^{(x_3)} e_{32}^{(x_2)} e_{12}^{(x_5)} \underline{e_{3(12)}^{(x_4)} e_3^{(i_1)} e_1^{(o_6)}} \\
 &= \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5} \Gamma(3|2)_{o_1, o_2, o_3}^{x_1, x_2, x_3} \Gamma(3|12)_{i_1, x_4, x_5}^{x_1, o_4, o_5} \\
 &\quad \times e_2^{(x_3)} e_{12}^{(x_5)} e_{32}^{(x_2)} e_{1(32)}^{(x_4)} e_1^{(o_6)} e_3^{(i_1)} \\
 &= \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5} \Gamma(3|2)_{o_1, o_2, o_3}^{x_1, x_2, x_3} \Gamma(3|12)_{i_1, x_4, x_5}^{x_1, o_4, o_5} \Gamma(32|1)_{i_2, i_4, x_6}^{x_2, x_4, o_6} \\
 &\quad \times e_2^{(x_3)} e_{12}^{(x_5)} e_1^{(x_6)} e_{321}^{(i_4)} e_{32}^{(i_2)} e_3^{(i_1)}
 \end{aligned}$$

■ For the underlined part, we used

$$e_{32}^{(a)} e_{1(32)}^{(b)} e_1^{(c)} = \sum_{i, j, k} \Gamma(32|1)_{i, j, k}^{a, b, c} e_1^{(k)} e_{321}^{(j)} e_{32}^{(i)}$$

Two ways construction of γ : first way

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$$\begin{aligned}
 & \frac{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}}{=} \\
 &= \sum \Gamma(3|2)_{x_1, x_2, x_3}^{o_1, o_2, o_3} e_2^{(x_3)} e_{32}^{(x_2)} \underline{e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}} \\
 &= \sum \Gamma(3|2)_{x_1, x_2, x_3}^{o_1, o_2, o_3} \Gamma(3|12)_{i_1, x_4, x_5}^{x_1, o_4, o_5} e_2^{(x_3)} \underline{e_{32}^{(x_2)} e_{12}^{(x_5)} e_{3(12)}^{(x_4)} e_3^{(i_1)} e_1^{(o_6)}} \\
 &= \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5} \Gamma(3|2)_{x_1, x_2, x_3}^{o_1, o_2, o_3} \Gamma(3|12)_{i_1, x_4, x_5}^{x_1, o_4, o_5} \\
 &\quad \times \underline{e_2^{(x_3)} e_{12}^{(x_5)} e_{32}^{(x_2)} e_{1(32)}^{(x_4)} e_1^{(o_6)} e_3^{(i_1)}} \\
 &= \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5} \Gamma(3|2)_{x_1, x_2, x_3}^{o_1, o_2, o_3} \Gamma(3|12)_{i_1, x_4, x_5}^{x_1, o_4, o_5} \Gamma(32|1)_{i_2, i_4, x_6}^{x_2, x_4, o_6} \\
 &\quad \times \underline{e_2^{(x_3)} e_{12}^{(x_5)} e_1^{(x_6)} e_{321}^{(i_4)} e_{32}^{(i_2)} e_3^{(i_1)}} \\
 &= \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5} \Gamma(3|2)_{x_1, x_2, x_3}^{o_1, o_2, o_3} \Gamma(3|12)_{i_1, x_4, x_5}^{x_1, o_4, o_5} \Gamma(32|1)_{i_2, i_4, x_6}^{x_2, x_4, o_6} \Gamma(2|1)_{i_3, i_5, i_6}^{x_3, x_5, x_6} \\
 &\quad \times \underline{e_1^{(i_6)} e_{21}^{(i_5)} e_2^{(i_3)} e_{321}^{(i_4)} e_{32}^{(i_2)} e_3^{(i_1)}} \\
 &= \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 i_3 i_4} \Gamma(3|2)_{x_1, x_2, x_3}^{o_1, o_2, o_3} \Gamma(3|12)_{i_1, x_4, x_5}^{x_1, o_4, o_5} \Gamma(32|1)_{i_2, i_4, x_6}^{x_2, x_4, o_6} \Gamma(2|1)_{i_3, i_5, i_6}^{x_3, x_5, x_6} \\
 &\quad \times \underline{e_1^{(i_6)} e_{21}^{(i_5)} e_{321}^{(i_4)} e_2^{(i_3)} e_{32}^{(i_2)} e_3^{(i_1)}}
 \end{aligned}$$

$$\rho_1 = p(\alpha_1)p(\alpha_3)$$

$$\rho_2 = p(\alpha_1 + \alpha_2)p(\alpha_2 + \alpha_3)$$

$$\rho_3 = p(\alpha_2)p(\alpha_1 + \alpha_2 + \alpha_3)$$

Two ways construction of γ : second way

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$$\begin{aligned}
 & e_3^{(o_1)} e_{23}^{(o_2)} \underline{e_2^{(o_3)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}} \\
 &= (-1)^{\rho_3 o_3 o_4} e_3^{(o_1)} e_{23}^{(o_2)} e_{123}^{(o_4)} \underline{e_2^{(o_3)} e_{12}^{(o_5)} e_1^{(o_6)}} \\
 &= \sum (-1)^{\rho_3 o_3 o_4} \Gamma(2|1)_{x_3, x_5, x_6}^{o_3, o_5, o_6} e_3^{(o_1)} \underline{e_{23}^{(o_2)} e_{123}^{(o_4)} e_1^{(x_6)} e_{21}^{(x_5)} e_2^{(x_3)}} \\
 &= \sum (-1)^{\rho_3 o_3 o_4} \Gamma(2|1)_{x_3, x_5, x_6}^{o_3, o_5, o_6} \Gamma(23|1)_{x_2, x_4, i_6}^{o_2, o_4, x_6} \underline{e_3^{(o_1)} e_1^{(i_6)} e_{(23)1}^{(x_4)} e_{23}^{(x_2)} e_{21}^{(x_5)} e_2^{(x_3)}} \\
 &= \sum (-1)^{\rho_1(o_1 i_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 o_3 o_4} \Gamma(2|1)_{x_3, x_5, x_6}^{o_3, o_5, o_6} \Gamma(23|1)_{x_2, x_4, i_6}^{o_2, o_4, x_6} \\
 &\quad \times \underline{e_1^{(i_6)} e_3^{(o_1)} e_{(21)3}^{(x_4)} e_{21}^{(x_5)} e_{23}^{(x_2)} e_2^{(x_3)}} \\
 &= \sum (-1)^{\rho_1(o_1 i_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 o_3 o_4} \Gamma(2|1)_{x_3, x_5, x_6}^{o_3, o_5, o_6} \Gamma(23|1)_{x_2, x_4, i_6}^{o_2, o_4, x_6} \Gamma(3|21)_{x_1, i_4, i_5}^{o_1, x_4, x_5} \\
 &\quad \times \underline{e_1^{(i_6)} e_{21}^{(i_5)} e_{321}^{(i_4)} e_3^{(x_1)} e_{23}^{(x_2)} e_2^{(x_3)}} \\
 &= \sum (-1)^{\rho_1(o_1 i_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 o_3 o_4} \Gamma(2|1)_{x_3, x_5, x_6}^{o_3, o_5, o_6} \Gamma(23|1)_{x_2, x_4, i_6}^{o_2, o_4, x_6} \Gamma(3|21)_{x_1, i_4, i_5}^{o_1, x_4, x_5} \Gamma(3|2)_{i_1, i_2, i_3}^{x_1, x_2, x_3} \\
 &\quad \times \underline{e_1^{(i_6)} e_{21}^{(i_5)} e_{321}^{(i_4)} e_2^{(i_3)} e_{23}^{(i_2)} e_3^{(i_1)}}
 \end{aligned}$$

$$\rho_1 = p(\alpha_1)p(\alpha_3)$$

$$\rho_2 = p(\alpha_1 + \alpha_2)p(\alpha_2 + \alpha_3)$$

$$\rho_3 = p(\alpha_2)p(\alpha_1 + \alpha_2 + \alpha_3)$$

- Now, $\{e_1^{(i_6)} e_{21}^{(i_5)} e_{321}^{(i_4)} e_2^{(i_3)} e_{23}^{(i_2)} e_3^{(i_1)}\}$ is linearly independent.
Then, we have the following result.

- Theorem [Y20]

$$\begin{aligned}
 & \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 i_3 i_4} \\
 & \quad \times \Gamma(3|2)_{o_1, o_2, o_3}^{x_1, x_2, x_3} \Gamma(3|12)_{i_1, x_4, x_5}^{x_1, o_4, o_5} \Gamma(32|1)_{i_2, i_4, x_6}^{x_2, x_4, o_6} \Gamma(2|1)_{i_3, i_5, i_6}^{x_3, x_5, x_6} \\
 & = \sum (-1)^{\rho_1(o_1 i_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 o_3 o_4} \\
 & \quad \times \Gamma(2|1)_{x_3, x_5, x_6}^{o_3, o_5, o_6} \Gamma(23|1)_{x_2, x_4, i_6}^{o_2, o_4, x_6} \Gamma(3|21)_{x_1, i_4, i_5}^{o_1, x_4, x_5} \Gamma(3|2)_{i_1, i_2, i_3}^{x_1, x_2, x_3}
 \end{aligned}$$

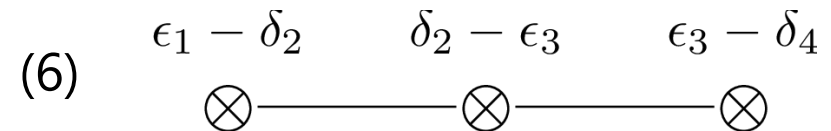
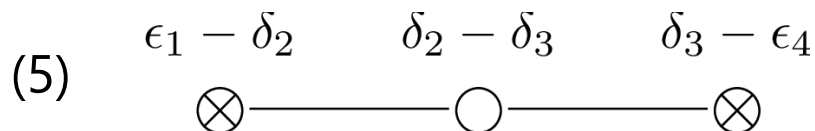
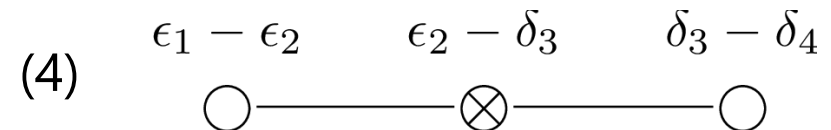
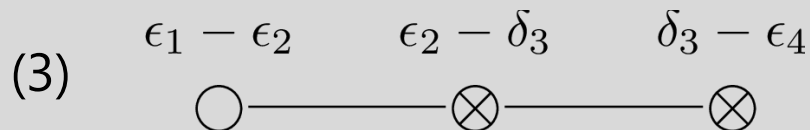
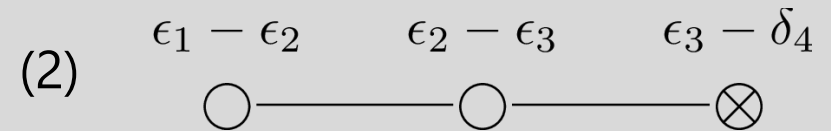
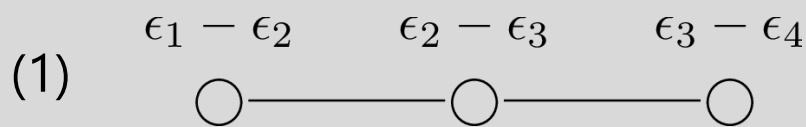
- Here, sign factors are given by

$$\rho_1 = p(\alpha_1)p(\alpha_3), \quad \rho_2 = p(\alpha_1 + \alpha_2)p(\alpha_2 + \alpha_3), \quad \rho_3 = p(\alpha_2)p(\alpha_1 + \alpha_2 + \alpha_3)$$

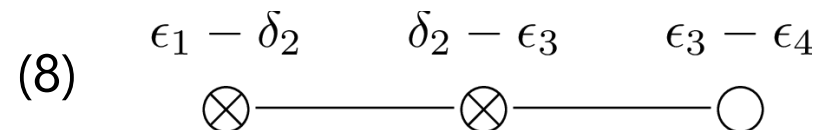
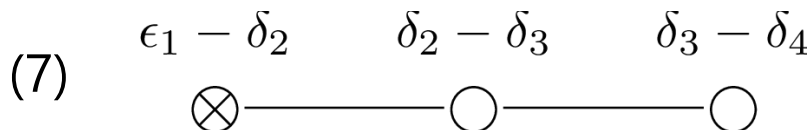
Dynkin diagrams of type A of rank 3

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■ Without exchanges $\epsilon \leftrightarrow \delta$, all Dynkin diagrams of rank 3 are given by



■ The followings are easily attributed to (2) and (3), respectively.





■ For (1),(2),(3), we obtain tetrahedron eq because $\rho_1 = \rho_2 = \rho_3 = 0$.

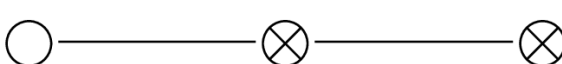
Dynkin diagrams of type A of rank 3

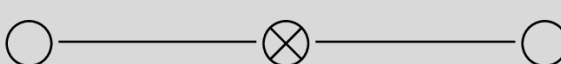
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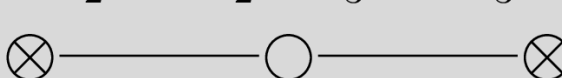
- Without exchanges $\epsilon \leftrightarrow \delta$, all Dynkin diagrams of rank 3 are given by

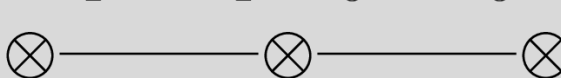
(1) $\epsilon_1 - \epsilon_2 \quad \epsilon_2 - \epsilon_3 \quad \epsilon_3 - \epsilon_4$


(2) $\epsilon_1 - \epsilon_2 \quad \epsilon_2 - \epsilon_3 \quad \epsilon_3 - \delta_4$


(3) $\epsilon_1 - \epsilon_2 \quad \epsilon_2 - \delta_3 \quad \delta_3 - \epsilon_4$


(4) $\epsilon_1 - \epsilon_2 \quad \epsilon_2 - \delta_3 \quad \delta_3 - \delta_4$


(5) $\epsilon_1 - \delta_2 \quad \delta_2 - \delta_3 \quad \delta_3 - \epsilon_4$


(6) $\epsilon_1 - \delta_2 \quad \delta_2 - \epsilon_3 \quad \epsilon_3 - \delta_4$


- For (4),(5),(6), some ρ_i are non-zero. Then, associated equations become the tetrahedron equation *up to sign factors*.
- Here, we only consider (4) for them.

The case $\bigcirc \text{---} \bigcirc \text{---} \bigcirc$

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$$\begin{array}{ccc} \epsilon_1 - \epsilon_2 & \epsilon_2 - \epsilon_3 & \epsilon_3 - \epsilon_4 \\ \bigcirc \text{---} \bigcirc \text{---} \bigcirc & (\epsilon_i, \epsilon_i) = 1 & DA = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \end{array}$$

■ Root system

$$\tilde{\Phi}_{\text{even}}^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\} \quad \tilde{\Phi}_{\text{iso}}^+ = \{\}$$

■ Lemma

$$\Gamma^{(2|1)} = \Gamma^{(3|2)} = \Gamma^{(23|1)} = \Gamma^{(32|1)} = \Gamma^{(3|12)} = \Gamma^{(3|21)} = \mathcal{R}$$

■ Proof (The case $\Gamma^{(23|1)}$)

□ $h: \bigcirc \text{---} \bigcirc \rightarrow \bigcirc \text{---} \bigcirc \text{---} \bigcirc$ defined by $e_1 \mapsto e_1$, $e_2 \mapsto e_{23}$ is an algebra hom by higher-order relations:

$$e_1^2 e_{23} - (q + q^{-1}) e_1 e_{23} e_1 + e_{23} e_1^2 = e_{23}^2 e_1 - (q + q^{-1}) e_{23} e_1 e_{23} + e_1 e_{23}^2 = 0$$

□ $[m]_{q^{d_{\alpha_2+\alpha_3}}}! = [m]_{q^{d_{\alpha_2}}}!$ and $[m]_{q^{d_{\alpha_1+\alpha_2+\alpha_3}}}! = [m]_{q^{d_{\alpha_1+\alpha_2}}}!$ hold.

□ Then we have

$$e_2^{(a)} e_{12}^{(b)} e_1^{(c)} = \sum_{i,j,k} \mathcal{R}_{i,j,k}^{a,b,c} e_1^{(k)} e_{21}^{(j)} e_2^{(i)} \mapsto e_{23}^{(a)} e_{123}^{(b)} e_1^{(c)} = \sum_{i,j,k} \mathcal{R}_{i,j,k}^{a,b,c} e_1^{(k)} e_{(23)1}^{(j)} e_{23}^{(i)}$$

■ Previous Theorem [Y20]

$$\begin{aligned} & \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 i_3 i_4} \\ & \quad \times \Gamma^{(3|2)}_{x_1, x_2, x_3}^{o_1, o_2, o_3} \Gamma^{(3|12)}_{i_1, x_4, x_5}^{x_1, o_4, o_5} \Gamma^{(32|1)}_{i_2, i_4, x_6}^{x_2, x_4, o_6} \Gamma^{(2|1)}_{i_3, i_5, i_6}^{x_3, x_5, x_6} \\ & = \sum (-1)^{\rho_1(o_1 i_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 o_3 o_4} \\ & \quad \times \Gamma^{(2|1)}_{x_3, x_5, x_6}^{o_3, o_5, o_6} \Gamma^{(23|1)}_{x_2, x_4, i_6}^{o_2, o_4, x_6} \Gamma^{(3|21)}_{x_1, i_4, i_5}^{o_1, x_4, x_5} \Gamma^{(3|2)}_{i_1, i_2, i_3}^{x_1, x_2, x_3} \end{aligned}$$

$$\rho_1 = p(\alpha_1)p(\alpha_3)$$

$$\rho_2 = p(\alpha_1 + \alpha_2)p(\alpha_2 + \alpha_3)$$

$$\rho_3 = p(\alpha_2)p(\alpha_1 + \alpha_2 + \alpha_3)$$

■ Previous Lemma

$$\Gamma^{(2|1)} = \Gamma^{(3|2)} = \Gamma^{(23|1)} = \Gamma^{(32|1)} = \Gamma^{(3|12)} = \Gamma^{(3|21)} = \mathcal{R}$$

■ The theorem is specialized as follows:

$$\sum \mathcal{R}_{x_1, x_2, x_3}^{o_1, o_2, o_3} \mathcal{R}_{i_1, x_4, x_5}^{x_1, o_4, o_5} \mathcal{R}_{i_2, i_4, x_6}^{x_2, x_4, o_6} \mathcal{R}_{i_3, i_5, i_6}^{x_3, x_5, x_6} = \sum \mathcal{R}_{x_3, x_5, x_6}^{o_3, o_5, o_6} \mathcal{R}_{x_2, x_4, i_6}^{o_2, o_4, x_6} \mathcal{R}_{x_1, i_4, i_5}^{o_1, x_4, x_5} \mathcal{R}_{i_1, i_2, i_3}^{x_1, x_2, x_3}$$

□ This is exactly the tetrahedron equation of [Kapranov-Voevodsky94]:

$$\mathcal{R}_{123} \mathcal{R}_{145} \mathcal{R}_{246} \mathcal{R}_{356} = \mathcal{R}_{356} \mathcal{R}_{246} \mathcal{R}_{145} \mathcal{R}_{123}$$

The case $\bigcirc \text{---} \bigcirc \text{---} \bigotimes$

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$$\begin{array}{ccc} \epsilon_1 - \epsilon_2 & \epsilon_2 - \epsilon_3 & \epsilon_3 - \delta_4 \\ \bigcirc \text{---} \bigcirc \text{---} \bigotimes & (\epsilon_i, \epsilon_i) = 1 & (\delta_i, \delta_i) = -1 \end{array} \quad DA = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

■ Root system

$$\tilde{\Phi}_{\text{even}}^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} \quad \tilde{\Phi}_{\text{iso}}^+ = \{\alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$$

■ Lemma

$$\Gamma^{(2|1)} = \mathcal{R} \quad \Gamma^{(3|2)} = \Gamma^{(23|1)} = \Gamma^{(32|1)} = \Gamma^{(3|12)} = \Gamma^{(3|21)} = \mathcal{L}$$

■ The theorem is specialized as follows:

$$\sum \mathcal{L}_{x_1, x_2, x_3}^{o_1, o_2, o_3} \mathcal{L}_{i_1, x_4, x_5}^{x_1, o_4, o_5} \mathcal{L}_{i_2, i_4, x_6}^{x_2, x_4, o_6} \mathcal{R}_{i_3, i_5, i_6}^{x_3, x_5, x_6} = \sum \mathcal{R}_{x_3, x_5, x_6}^{o_3, o_5, o_6} \mathcal{L}_{x_2, x_4, i_6}^{o_2, o_4, x_6} \mathcal{L}_{x_1, i_4, i_5}^{o_1, x_4, x_5} \mathcal{L}_{i_1, i_2, i_3}^{x_1, x_2, x_3}$$

□ This is exactly the tetrahedron equation of [Bazhanov-Sergeev06]:

$$\mathcal{L}_{123} \mathcal{L}_{145} \mathcal{L}_{246} \mathcal{R}_{356} = \mathcal{R}_{356} \mathcal{L}_{246} \mathcal{L}_{145} \mathcal{L}_{123}$$

The case $\bigcirc \text{---} \bigotimes \text{---} \bigotimes$

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$$\begin{array}{ccccc} \epsilon_1 - \epsilon_2 & \epsilon_2 - \delta_3 & \delta_3 - \epsilon_4 & (\epsilon_i, \epsilon_i) = 1 & \\ \bigcirc \text{---} \bigotimes \text{---} \bigotimes & (\delta_i, \delta_i) = -1 & & & DA = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

■ Root system

$$\tilde{\Phi}_{\text{even}}^+ = \{\alpha_1, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\} \quad \tilde{\Phi}_{\text{iso}}^+ = \{\alpha_2, \alpha_3, \alpha_1 + \alpha_2\}$$

■ Lemma

$$\Gamma^{(2|1)} = \mathcal{L}, \quad \Gamma^{(3|2)} = \Gamma^{(3|12)} = \Gamma^{(3|21)} = \mathcal{N}(q^{-1}), \quad \Gamma^{(23|1)} = \Gamma^{(32|1)} = \mathcal{R}$$

■ The theorem is specialized as follows:

$$\begin{aligned} & \sum \mathcal{N}(q^{-1})_{x_1, x_2, x_3}^{o_1, o_2, o_3} \mathcal{N}(q^{-1})_{i_1, x_4, x_5}^{x_1, o_4, o_5} \mathcal{R}_{i_2, i_4, x_6}^{x_2, x_4, o_6} \mathcal{L}_{i_3, i_5, i_6}^{x_3, x_5, x_6} \\ &= \sum \mathcal{L}_{x_3, x_5, x_6}^{o_3, o_5, o_6} \mathcal{R}_{x_2, x_4, i_6}^{o_2, o_4, x_6} \mathcal{N}(q^{-1})_{x_1, i_4, i_5}^{o_1, x_4, x_5} \mathcal{N}(q^{-1})_{i_1, i_2, i_3}^{x_1, x_2, x_3} \end{aligned}$$

□ This gives a new solution to the tetrahedron equation:

$$\mathcal{N}(q^{-1})_{123} \mathcal{N}(q^{-1})_{145} \mathcal{R}_{246} \mathcal{L}_{356} = \mathcal{L}_{356} \mathcal{R}_{246} \mathcal{N}(q^{-1})_{145} \mathcal{N}(q^{-1})_{123}$$

The case $\bigcirc \text{---} \bigotimes \text{---} \bigcirc$

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$$\begin{array}{ccccccc}
 \epsilon_1 - \epsilon_2 & \epsilon_2 - \delta_3 & \delta_3 - \delta_4 & (\epsilon_i, \epsilon_i) = 1 & & & \\
 \bigcirc \text{---} \bigotimes \text{---} \bigcirc & & & (\delta_i, \delta_i) = -1 & & DA = & \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}
 \end{array}$$

■ Root system

$$\tilde{\Phi}_{\text{even}}^+ = \{\alpha_1, \alpha_3\} \quad \tilde{\Phi}_{\text{iso}}^+ = \{\alpha_2, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$$

■ Lemma

$$\Gamma^{(2|1)} = \Gamma^{(23|1)} = \Gamma^{(32|1)} = \mathcal{L}, \quad \Gamma^{(3|2)} = \Gamma^{(3|12)} = \Gamma^{(3|21)} = \mathcal{M}(q^{-1})$$

■ Previous Theorem [Y20]

$$\begin{aligned} & \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 i_3 i_4} \\ & \quad \times \Gamma^{(3|2)}_{x_1, x_2, x_3}^{o_1, o_2, o_3} \Gamma^{(3|12)}_{i_1, x_4, x_5}^{x_1, o_4, o_5} \Gamma^{(32|1)}_{i_2, i_4, x_6}^{x_2, x_4, o_6} \Gamma^{(2|1)}_{i_3, i_5, i_6}^{x_3, x_5, x_6} \\ & = \sum (-1)^{\rho_1(o_1 i_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 o_3 o_4} \\ & \quad \times \Gamma^{(2|1)}_{x_3, x_5, x_6}^{o_3, o_5, o_6} \Gamma^{(23|1)}_{x_2, x_4, i_6}^{o_2, o_4, x_6} \Gamma^{(3|21)}_{x_1, i_4, i_5}^{o_1, x_4, x_5} \Gamma^{(3|2)}_{i_1, i_2, i_3}^{x_1, x_2, x_3} \end{aligned}$$

$$\rho_1 = p(\alpha_1)p(\alpha_3)$$

$$\rho_2 = p(\alpha_1 + \alpha_2)p(\alpha_2 + \alpha_3)$$

$$\rho_3 = p(\alpha_2)p(\alpha_1 + \alpha_2 + \alpha_3)$$

■ Previous Lemma

$$\Gamma^{(2|1)} = \Gamma^{(23|1)} = \Gamma^{(32|1)} = \mathcal{L}, \quad \Gamma^{(3|2)} = \Gamma^{(3|12)} = \Gamma^{(3|21)} = \mathcal{M}(q^{-1})$$

■ By using $\rho_1 = 0, \rho_2 = \rho_3 = 1$, the theorem is specialized as follows:

$$\begin{aligned} & \sum (-1)^{x_2 x_5 + i_3 i_4} \mathcal{M}(q^{-1})_{x_1, x_2, x_3}^{o_1, o_2, o_3} \mathcal{M}(q^{-1})_{i_1, x_4, x_5}^{x_1, o_4, o_5} \mathcal{L}_{i_2, i_4, x_6}^{x_2, x_4, o_6} \mathcal{L}_{i_3, i_5, i_6}^{x_3, x_5, x_6} \\ & = \sum (-1)^{x_2 x_5 + o_3 o_4} \mathcal{L}_{x_3, x_5, x_6}^{o_3, o_5, o_6} \mathcal{L}_{x_2, x_4, i_6}^{o_2, o_4, x_6} \mathcal{M}(q^{-1})_{x_1, i_4, i_5}^{o_1, x_4, x_5} \mathcal{M}(q^{-1})_{i_1, i_2, i_3}^{x_1, x_2, x_3}, \end{aligned}$$

□ There are “nonlocal” sign factors which can not be eliminated at present.

■ Consider non-super cases

- \mathbf{B} : canonical basis
- \mathbf{i} : indices of reduced expression of the longest element of Weyl group
- Lusztig's parametrization: a bijection $b_{\mathbf{i}}: \mathbb{Z}_{\geq 0}^l \rightarrow \mathbf{B}$ associated with \mathbf{i}
- Transition map: $R_{\mathbf{i}}^{\mathbf{i}'} = (b_{\mathbf{i}'})^{-1} \circ b_{\mathbf{i}} : \mathbb{Z}_{\geq 0}^l \rightarrow \mathbb{Z}_{\geq 0}^l$

■ Transition maps are obtained by transition matrices with $q \rightarrow 0$:

$$\lim_{q \rightarrow 0} \mathcal{R}(q)_{i,j,k}^{a,b,c} = \delta_{a,i+j-\min(i,k)} \delta_{b,\min(i,k)} \delta_{c,j+k-\min(i,k)}$$

■ A super analog of transition maps is obtained in a similar way.

■ Prop [Y20]

- $\mathcal{L}_{i,j,k}^{a,b,c} = \lim_{q \rightarrow 0} \mathcal{L}(q)_{i,j,k}^{a,b,c}$ gives a non-trivial bijection on $\{0,1\}^2 \times \mathbb{Z}_{\geq 0}$.
- Non-zero elements are given by

$$\mathcal{L}_{0,0,k}^{0,0,c} = \mathcal{L}_{1,1,k}^{1,1,c} = \delta_{k,c}, \quad \mathcal{L}_{0,1,k}^{1,0,c} = \delta_{k+1,c}, \quad \mathcal{L}_{1,0,0}^{1,0,0} = 1, \quad \mathcal{L}_{1,0,k}^{0,1,c} = \delta_{k-1,c}$$

- $\mathcal{N}_{i,j,k}^{a,b,c} = \lim_{q \rightarrow 0} \left(\frac{[b]_q!}{[j]_q!} \mathcal{N}(q)_{i,j,k}^{a,b,c} \right)$ also gives a non-trivial bijection.

■ Remark

1. Type B cases give new solutions to the 3D reflection equations.
2. The crystal limit for $\bigcirc \Longrightarrow \bullet$ and $\bigotimes \Longrightarrow \bullet$ take values $0, \pm 1$.

■ Summary

1. The 3D L is characterized as the transition matrix for $\bigcirc \text{---} \bigotimes$.
2. A new solution to the tetrahedron equation the 3D N is obtained by considering the transition matrix for $\bigotimes \text{---} \bigotimes$.
3. Several solutions to the tetrahedron equations are obtained without using any result for quantum coordinate rings (c.f. KOY theorem).

■ Outlook

1. Eliminating nonlocal sign factors for $\bigcirc \text{---} \bigotimes \text{---} \bigcirc$
2. A super analog of KOY theorem
3. Geometric lifting of a super analog of transition maps