

# 3D reflection maps from tetrahedron maps

Combinatorial Representation Theory  
and Connections with Related Fields@2021/10/19

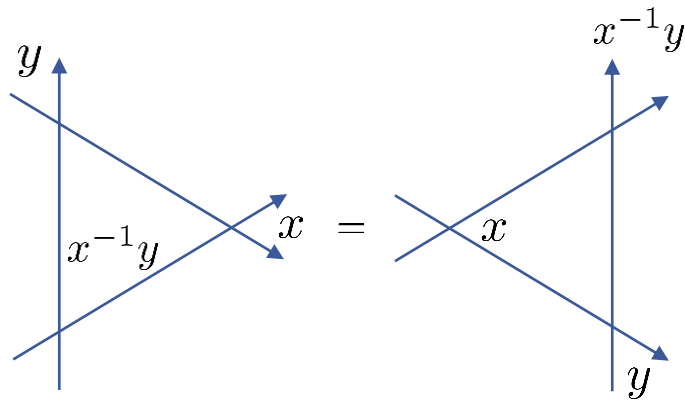
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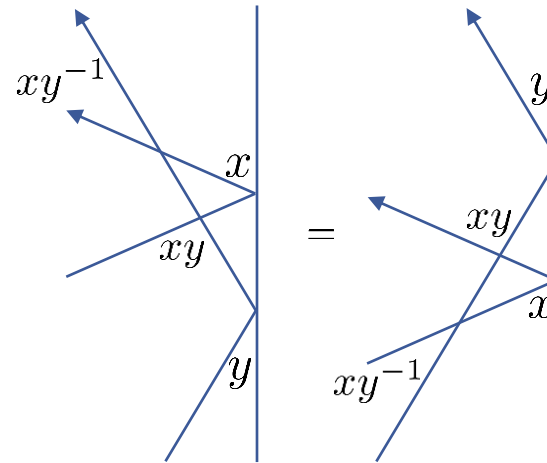
- Introduction: Reflection maps from Yang-Baxter maps P.3~8
- Main part: 3D reflection maps from tetrahedron maps P.10~23
- Examples P.25~29
  - Birational transition map
  - Sergeev's electrical solution
  - Two-component solution associated with soliton equation

## ■ Bulk: Yang-Baxter equation



$$\begin{aligned} R_{12}(x)R_{13}(y)R_{23}(x^{-1}y) \\ = R_{23}(x^{-1}y)R_{13}(y)R_{12}(x) \end{aligned}$$

## ■ Boundary: Reflection equation



$$\begin{aligned} R_{12}(xy^{-1})K_2(x)R_{21}(xy)K_1(y) \\ = K_1(y)R_{12}(xy)K_2(x)R_{21}(xy^{-1}) \end{aligned}$$

## ■ Problem settings:

- $R, K$  = matrices on linear spaces
- $R, K$  = maps on sets (set-theoretical solution) [Drinfeld92]

- We call set-theoretical solutions to Yang-Baxter equation *Yang-Baxter maps*, and so on.
- Yang-Baxter maps have been extensively studied in connection with various topics. [Veselov07]
- Several reflection maps are constructed
  - ▣ via directly solving the reflection equation [Kuniba-Okado-Yamada05]
  - ▣ Later, these solutions are *q-melted* by using coideal subalgebras of  $U_q$ .  
[Kuniba-Okado-Y19], [Kusano-Okado20]
  - ▣ from Yang-Baxter maps [Caudrelier-Zhang14], [Kuniba-Okado19]
  - ▣ by ring-theoretic methods [Smoktunowicz-Vendramin-Weston20]
- Here, we briefly review a result of [Kuniba-Okado19].

## ■ KR crystal for $A_{n-1}^{(1)}$

□  $B^{k,l} = \{\text{SST of } k \times l \text{ rectangular shape with letters from } \{1,2,\dots,n\}\}$

□ SST: semi-standard tableaux

□  $1 \leq k \leq n-1, l \geq 1$

□  $\tilde{e}_i, \tilde{f}_i: B^{k,l} \rightarrow B^{k,l} \cup \{0\}$  (Kashiwara operators)

□ For KR crystals  $B_1, B_2$ , the actions of Kashiwara operators are also defined on  $B_1 \otimes B_2 = \{b_1 \otimes b_2 | b_1 \in B_1, b_2 \in B_2\}$  and  $B_1 \otimes B_2$  is connected.

e.g.

1	1	3
Λ	Λ	Λ
2	4	6

## ■ Combinatorial $R$ matrices

□ A map  $R_{B_1, B_2}: B_1 \otimes B_2 \rightarrow B_2 \otimes B_1$  given by

$$R_{B_1, B_2}(\tilde{k}_i(b_1 \otimes b_2)) = \tilde{k}_i(R_{B_1, B_2}(b_1 \otimes b_2)) \quad (\tilde{k}_i = \tilde{e}_i, \tilde{f}_i)$$

is uniquely determined.

□ Combinatorial  $R$  matrices satisfy YBE from  $B_1 \otimes B_2 \otimes B_3$  to  $B_3 \otimes B_2 \otimes B_1$ :

$$\begin{aligned} & (1 \otimes R_{B_1, B_2})(R_{B_1, B_3} \otimes 1)(1 \otimes R_{B_2, B_3}) \\ &= (R_{B_2, B_3} \otimes 1)(1 \otimes R_{B_1, B_3})(R_{B_1, B_2} \otimes 1) \end{aligned}$$

□ The action can be calculated by the *tableau product rule*.

- We define  $v: B^{k,l} \rightarrow B^{n-k,l}$  by

$$b^v = \left[ \begin{array}{|c|c|c|c|} \hline \overline{\lambda_l} & \dots & \overline{\lambda_2} & \overline{\lambda_1} \\ \hline \end{array} \right] n-k \quad \text{for } b = \left[ \begin{array}{|c|c|c|c|} \hline \lambda_1 & \lambda_2 & \dots & \lambda_l \\ \hline \end{array} \right] k$$

- $\lambda_i \in \mathbb{N}^k, \overline{\lambda_i} \in \mathbb{N}^{n-k}$

- The complement is taken over  $\{1, 2, \dots, n\}$ .

- Proposition [Kuniba-Okado19]:

- For a KR crystal  $B$ , we have  $R_{B^v, B}(b^v \otimes b) = c \otimes c^v$ .

- Definition:

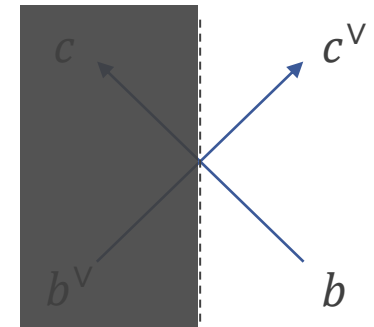
- Using  $b$  and  $c$  above, we define the combinatorial  $K$  matrix by

$$K_B: B \rightarrow B^v, b \rightarrow c^v$$

- Theorem: [Kuniba-Okado19]

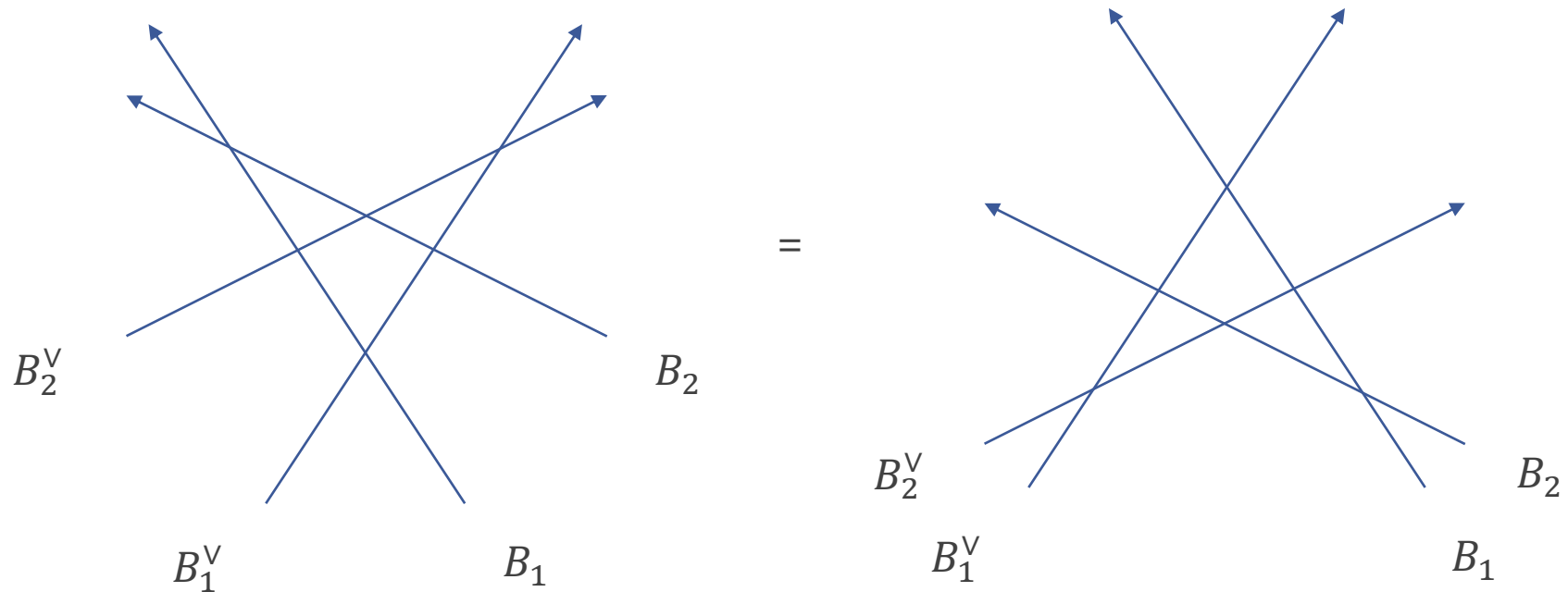
- The combinatorial  $R$  and  $K$  matrices satisfy RE from  $B_1 \otimes B_2 \rightarrow B_1^v \otimes B_2^v$ :

$$R_{B_2^v, B_1^v}(K_{B_2} \otimes 1)R_{B_1^v, B_2}(K_{B_1} \otimes 1) = (K_{B_1} \otimes 1)R_{B_2^v, B_1}(K_{B_2} \otimes 1)R_{B_1, B_2}$$



# Key lemma for reflection equation

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■ Lemma:

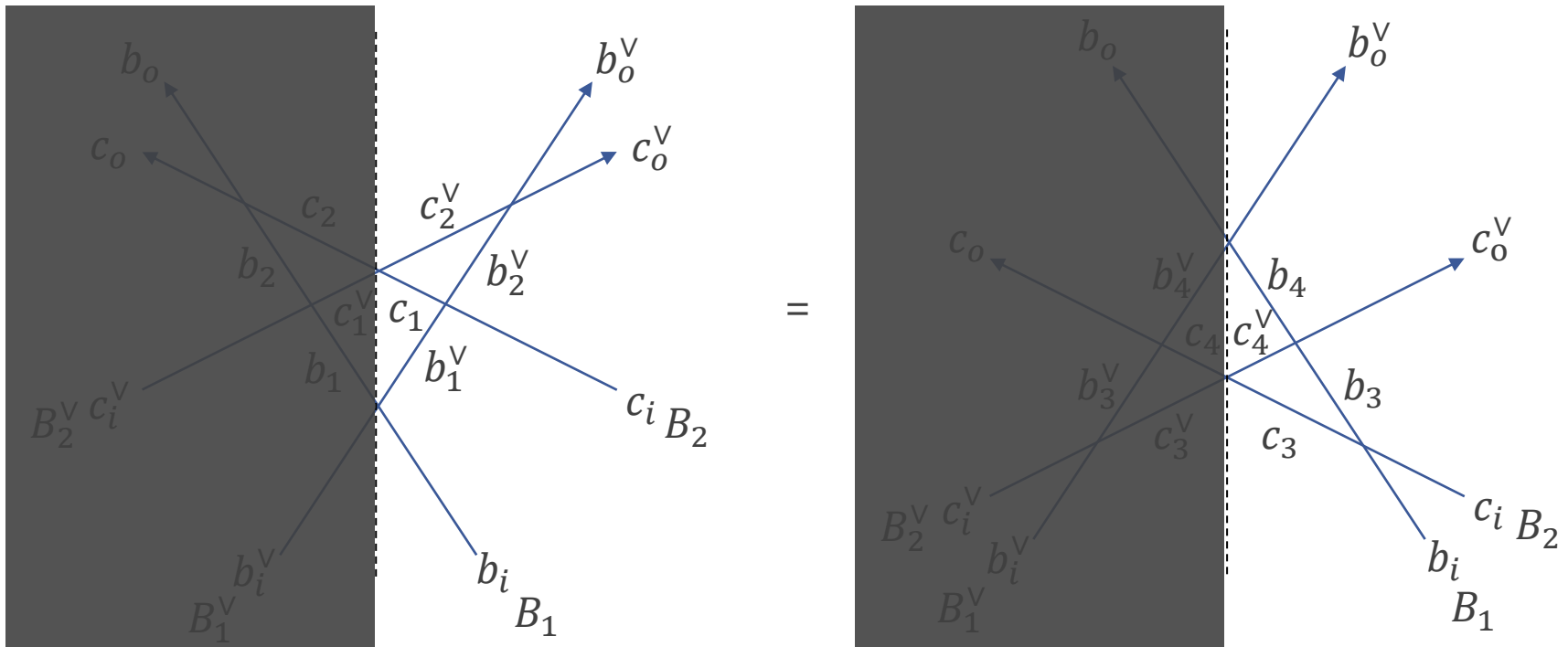
$$(R_{B_2^V, B_1^V} R_{B_1, B_2}) R_{B_2^V, B_2} (R_{B_1^V, B_2} R_{B_2^V, B_1}) R_{B_1^V, B_1} \\ = R_{B_1^V, B_1} (R_{B_2^V, B_1} R_{B_1^V, B_2}) R_{B_2^V, B_2} (R_{B_1, B_2} R_{B_2^V, B_1^V})$$

■ Proof:

□ By repeated uses of YBE, we have the above equation.

# Proof of reflection equation

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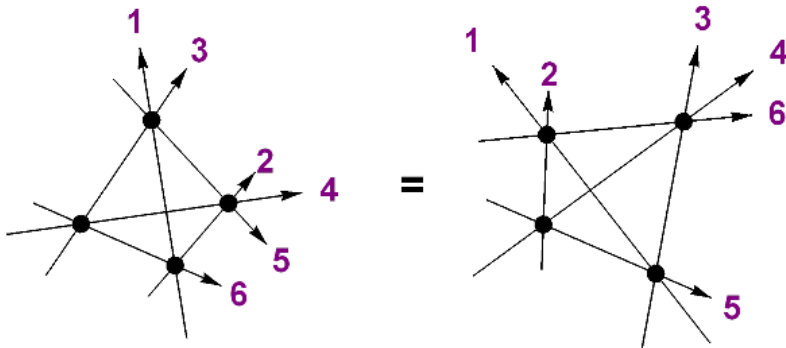
## ■ Sketch of proof:

- Input  $b_i, c_i$  on  $B_1, B_2$  and their duals on  $B_1^V, B_2^V$ .
- Use  $R_{B^V, B}(b^V \otimes b) = c \otimes c^V$ .
- Use  $R_{B_1, B_2}(b_1 \otimes b_2) = c_2 \otimes c_1 \Rightarrow R_{B_2^V, B_1^V}(b_2^V \otimes b_1^V) = c_1^V \otimes c_2^V$ .
- Just viewing the right parts of both sides, we then obtain RE.
  - because the above equation is equal to the direct product of RE for  ${}^V b_i, c_i$ .



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## ■ Bulk: Tetrahedron equation

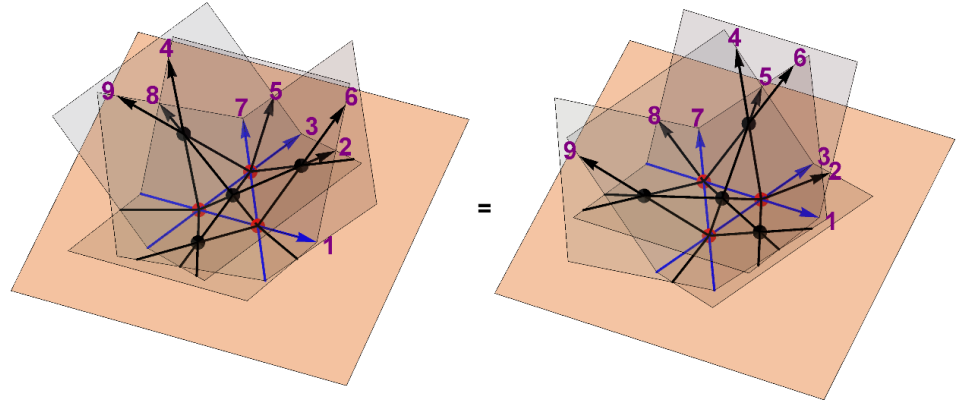


$$\mathbf{R}_{245}\mathbf{R}_{135}\mathbf{R}_{126}\mathbf{R}_{346}$$

$$= \mathbf{R}_{346}\mathbf{R}_{126}\mathbf{R}_{135}\mathbf{R}_{245}$$

[Zamolodchikov80]

## ■ Boundary: 3D reflection equation



$$\mathbf{R}_{489}\mathbf{J}_{3579}\mathbf{R}_{269}\mathbf{R}_{258}\mathbf{J}_{1678}\mathbf{J}_{1234}\mathbf{R}_{456}$$

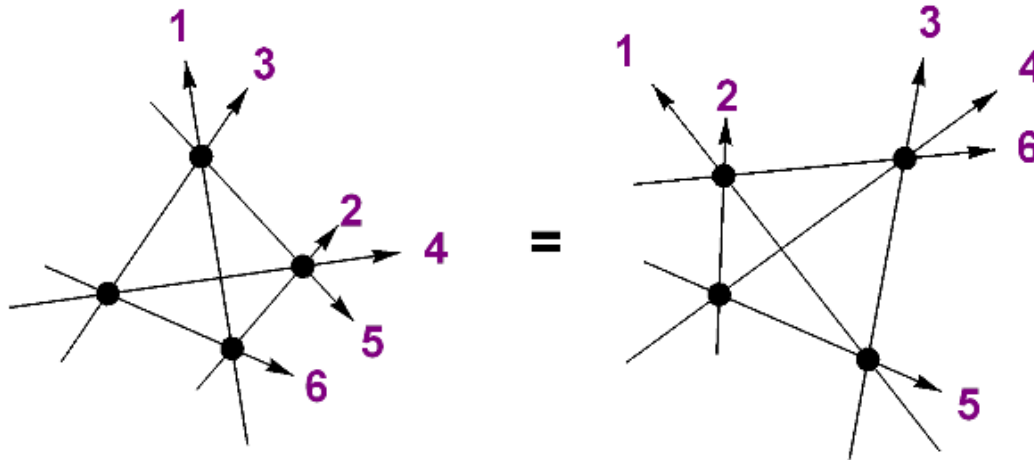
$$= \mathbf{R}_{456}\mathbf{J}_{1234}\mathbf{J}_{1678}\mathbf{R}_{258}\mathbf{R}_{269}\mathbf{J}_{3579}\mathbf{R}_{489}$$

[Isaev-Kulish97]

- Tetrahedron and 3D reflection equation are conditions for factorization of string scattering amplitude in 2+1D.

	Bulk	Boundary
2D	Yang-Baxter eq.	Reflection eq.
3D	Tetrahedron eq.	3D Reflection eq.

- Several tetrahedron maps are known although less systematically than Yang-Baxter maps.
  - In the context of the local YBE [Sergeev98]
  - Transition maps of Lusztig's parametrizations of the canonical basis of  $U_q(A_2)$  and their geometric liftings [Kuniba-Okado12] ... (1)
  - By using some KP tau functions [Kassotakis-Nieszporski-Papageorgiou-Tongas19]
- On the other hand, there are very few known 3D reflection maps.
  - Transition maps of Lusztig's parametrizations of the canonical basis of  $U_q(B_2)$  and  $U_q(C_2)$ , and their geometric liftings [Kuniba-Okado12] ... (2)
- Aim: Obtain 3D reflection maps from known tetrahedron maps
- Motivation:
  - Some 2D reflection maps are constructed from known Yang-Baxter maps.
  - It is known that (2) is constructed from (1) associated with folding the Dynkin diagram of  $A_3$  into one of  $B_2$ . [Berenstein-Zelevinsky01], [Lusztig11]



## ■ Definition:

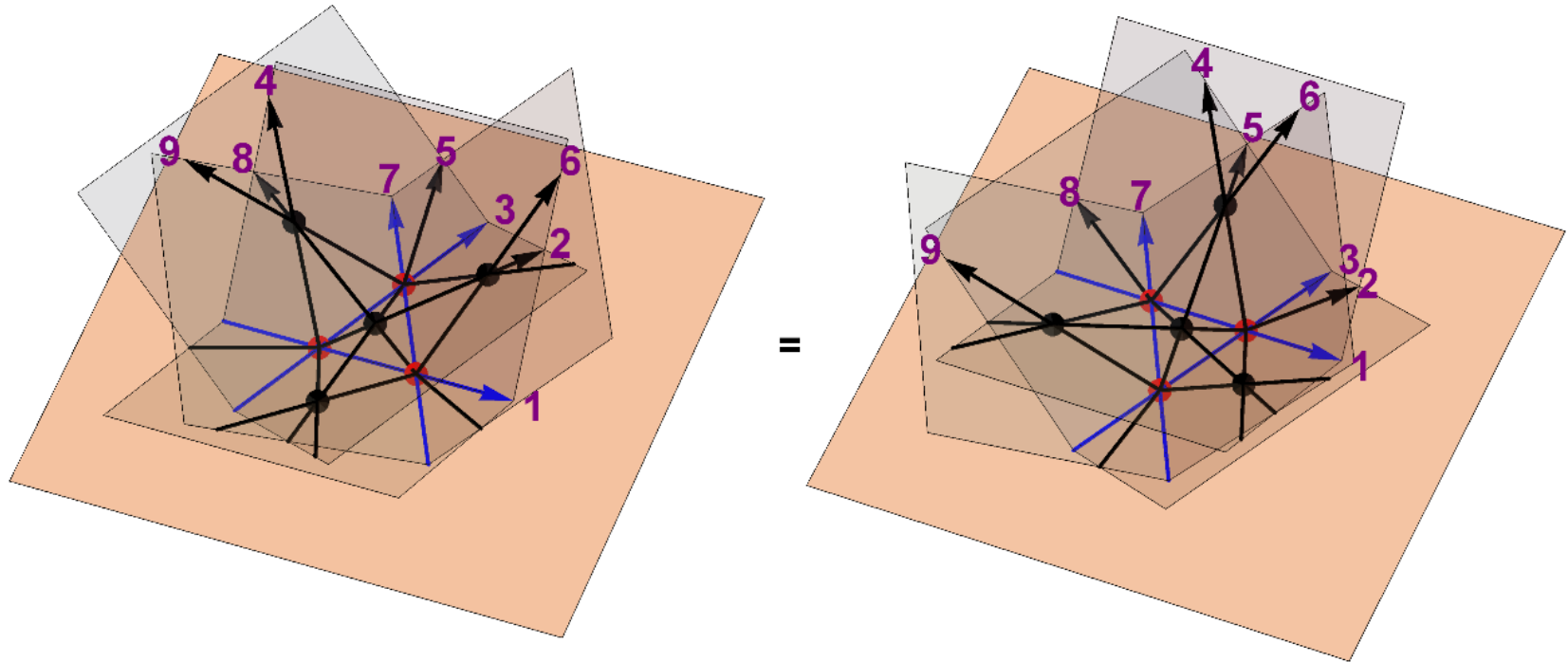
- Let  $\mathbf{R}: X^3 \rightarrow X^3$  ( $X$ : an arbitrary set) denote a map.
- We call  $\mathbf{R}$  *tetrahedron map* if it satisfies the tetrahedron equation on  $X^6$ :

$$\mathbf{R}_{245}\mathbf{R}_{135}\mathbf{R}_{126}\mathbf{R}_{346} = \mathbf{R}_{346}\mathbf{R}_{126}\mathbf{R}_{135}\mathbf{R}_{245} (=:\mathbf{T}_{123456}) \quad \cdots (*)$$

- We call  $\mathbf{T}$  the *tetrahedral composite* of the tetrahedron map  $\mathbf{R}$ .
- We call  $\mathbf{R}$  *involutive* if  $\mathbf{R}^2 = \text{id}$  and *symmetric* if  $\mathbf{R}_{123} = \mathbf{R}_{321}$ .

## ■ Remark:

- For involutive and symmetric tetrahedron maps,  $(*)$  corresponds to the usual tetrahedron equation.



## ■ Definition:

- Let  $J: X^4 \rightarrow X^4$  denote a map.
- We set a tetrahedron map by  $R: X^3 \rightarrow X^3$ .
- We call  $J$  *3D reflection map* if it satisfies the 3D reflection equation on  $X^9$ :

$$R_{489} J_{3579} R_{269} R_{258} J_{1678} J_{1234} R_{456} = R_{456} J_{1234} J_{1678} R_{258} R_{269} J_{3579} R_{489}$$

[Isaev-Kulish97]

- We set the subset of  $X^6$  by  $Y = \{(x_1, \dots, x_6) \mid x_2 = x_3, x_5 = x_6\}$ .
  - ▣ We set  $\phi: X^4 \rightarrow Y$  by  $\phi(x_1, x_2, x_3, x_4) = (x_1, x_2, x_2, x_3, x_4, x_4)$  (embedding)
  - ▣ We set  $\varphi: Y \rightarrow X^4$  by  $\varphi(x_1, x_2, x_2, x_3, x_4, x_4) = (x_1, x_2, x_3, x_4)$  (projection)

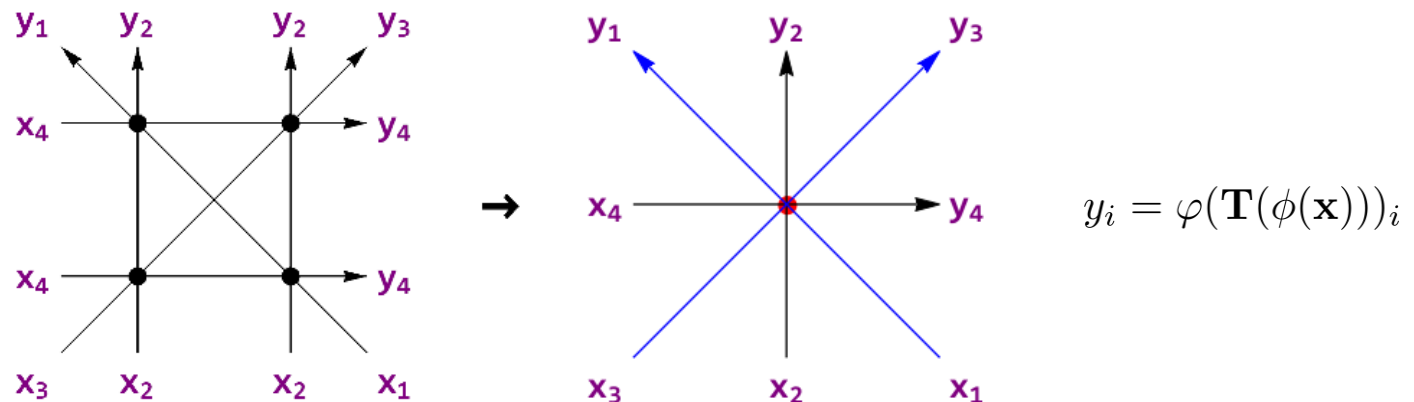
## ■ Definition:

- ▣ Let  $\mathbf{R}: X^3 \rightarrow X^3$  denote a tetrahedron map and  $\mathbf{T}$  its tetrahedral composite.
- ▣ We call  $\mathbf{R}$  *boundarizable* if the following condition is satisfied:

$$\mathbf{x} \in Y \Rightarrow \mathbf{T}(\mathbf{x}) \in Y$$

- ▣ In that case, we define the *boundarization*  $\mathbf{J}: X^4 \rightarrow X^4$  of  $\mathbf{R}$  by

$$\mathbf{J}(\mathbf{x}) = \varphi(\mathbf{T}(\phi(\mathbf{x})))$$



# Example for boundarization

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- We set the tetrahedron map  $\mathbf{R}: \mathbb{R}_{>0}^3 \rightarrow \mathbb{R}_{>0}^3$  by

$$\mathbf{R} : (x_1, x_2, x_3) \mapsto (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \left( \frac{x_1 x_2}{x_1 + x_3}, x_1 + x_3, \frac{x_2 x_3}{x_1 + x_3} \right)$$

- Let's act tetrahedral composite  $\mathbf{T} = \mathbf{R}_{245} \mathbf{R}_{135} \mathbf{R}_{126} \mathbf{R}_{346}$  on  $\mathbf{x} \in Y$ .

$$(x_1, x_2, \underline{x_2}, \underline{x_3}, x_4, \underline{x_4})$$

$$\mathbf{R}_{346} \mapsto \left( \underline{x_1}, \underline{x_2}, \frac{x_2 x_3}{x_2 + x_4}, x_2 + x_4, x_4, \frac{x_3 x_4}{\underline{x_2 + x_4}} \right)$$

$$\mathbf{R}_{126} \mapsto \left( \frac{x_1 x_2 (x_2 + x_4)}{\underline{x_3 x_4 + x_1 (x_2 + x_4)}}, x_1 + \frac{x_3 x_4}{x_2 + x_4}, \frac{x_2 x_3}{\underline{x_2 + x_4}}, x_2 + x_4, \underline{x_4}, \frac{x_2 x_3 x_4}{x_3 x_4 + x_1 (x_2 + x_4)} \right)$$

# Example for boundarization

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$$\begin{aligned}
 \mathbf{R}_{135} \mapsto & \left( \frac{x_1 x_2^2 x_3}{x_3 x_4^2 + x_1 (x_2 + x_4)^2}, \underline{x_1 + \frac{x_3 x_4}{x_2 + x_4}}, \frac{x_3 x_4^2 + x_1 (x_2 + x_4)^2}{x_3 x_4 + x_1 (x_2 + x_4)} \right. \\
 & \left. , \underline{x_2 + x_4}, \frac{x_2 x_3 x_4 (x_3 x_4 + x_1 (x_2 + x_4))}{(x_2 + x_4) (x_3 x_4^2 + x_1 (x_2 + x_4)^2)}, \frac{x_2 x_3 x_4}{x_3 x_4 + x_1 (x_2 + x_4)} \right) \\
 \mathbf{R}_{245} \mapsto & \left( \frac{x_1 x_2^2 x_3}{x_3 x_4^2 + x_1 (x_2 + x_4)^2}, \frac{x_3 x_4^2 + x_1 (x_2 + x_4)^2}{x_3 x_4 + x_1 (x_2 + x_4)}, \frac{x_3 x_4^2 + x_1 (x_2 + x_4)^2}{x_3 x_4 + x_1 (x_2 + x_4)} \right. \\
 & \left. , \frac{(x_3 x_4 + x_1 (x_2 + x_4))^2}{x_3 x_4^2 + x_1 (x_2 + x_4)^2}, \frac{x_2 x_3 x_4}{x_3 x_4 + x_1 (x_2 + x_4)}, \frac{x_2 x_3 x_4}{x_3 x_4 + x_1 (x_2 + x_4)} \right) \in Y
 \end{aligned}$$

■ Then, the boundarization  $\mathbf{J}: \mathbb{R}_{>0}^4 \rightarrow \mathbb{R}_{>0}^4$  is obtained as follows:

$$\begin{aligned}
 \mathbf{J} : (x_1, x_2, x_3, x_4) & \mapsto \left( \frac{x_1 x_2^2 x_3}{y_1}, \frac{y_1}{y_2}, \frac{y_2^2}{y_1}, \frac{x_2 x_3 x_4}{y_2} \right) \\
 y_1 &= x_1 (x_2 + x_4)^2 + x_3 x_4^2, \quad y_2 = x_1 (x_2 + x_4) + x_3 x_4
 \end{aligned}$$

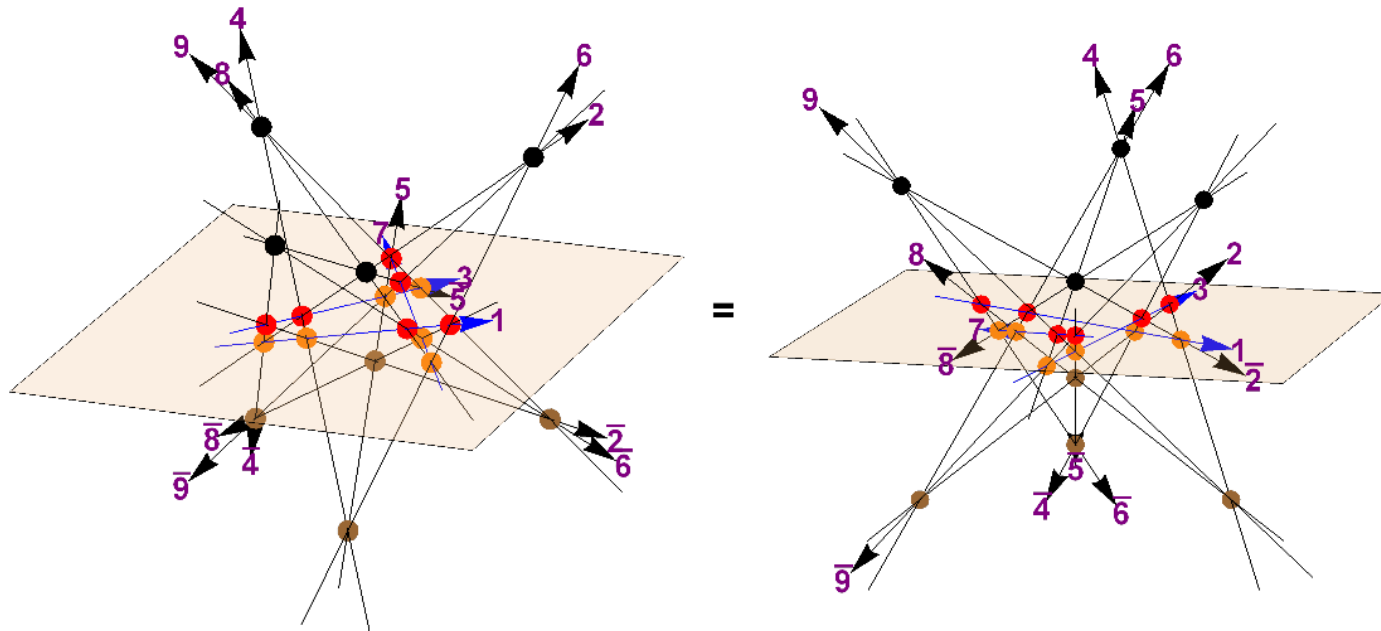


## ■ Theorem:

- Let  $\mathbf{R}: X^3 \rightarrow X^3$  denote an involutive, symmetric and boundarizable tetrahedron map, and  $\mathbf{J}: X^4 \rightarrow X^4$  its boundarization.
- Then they satisfy 3D reflection equation.

## ■ Sketch of Proof:

- Cut the following identity on  $X^{15}$  into half:





■ Lemma:

■ Let  $\mathbf{R}: X^3 \rightarrow X^3$  denote an involutive and symmetric tetrahedron map. Then we have the following identity on  $X^{15}$ :

$$\begin{aligned}
& (\mathbf{R}_{489}\mathbf{R}_{\bar{4}8\bar{9}})(\mathbf{R}_{579}\mathbf{R}_{3\bar{5}9}\mathbf{R}_{35\bar{9}}\mathbf{R}_{\bar{5}7\bar{9}})(\mathbf{R}_{26\bar{9}}\mathbf{R}_{\bar{2}6\bar{9}})(\mathbf{R}_{25\bar{8}}\mathbf{R}_{\bar{2}5\bar{8}}) \\
& \quad \times (\mathbf{R}_{\bar{6}7\bar{8}}\mathbf{R}_{16\bar{8}}\mathbf{R}_{1\bar{6}8}\mathbf{R}_{678})(\mathbf{R}_{234}\mathbf{R}_{1\bar{2}4}\mathbf{R}_{12\bar{4}}\mathbf{R}_{\bar{2}3\bar{4}})(\mathbf{R}_{\bar{4}5\bar{6}}\mathbf{R}_{456}) \\
& = (\mathbf{R}_{456}\mathbf{R}_{\bar{4}5\bar{6}})(\mathbf{R}_{234}\mathbf{R}_{1\bar{2}4}\mathbf{R}_{12\bar{4}}\mathbf{R}_{\bar{2}3\bar{4}})(\mathbf{R}_{\bar{6}7\bar{8}}\mathbf{R}_{16\bar{8}}\mathbf{R}_{1\bar{6}8}\mathbf{R}_{678}) \\
& \quad \times (\mathbf{R}_{\bar{2}5\bar{8}}\mathbf{R}_{25\bar{8}})(\mathbf{R}_{\bar{2}6\bar{9}}\mathbf{R}_{26\bar{9}})(\mathbf{R}_{579}\mathbf{R}_{3\bar{5}9}\mathbf{R}_{35\bar{9}}\mathbf{R}_{\bar{5}7\bar{9}})(\mathbf{R}_{\bar{4}8\bar{9}}\mathbf{R}_{489})
\end{aligned}$$

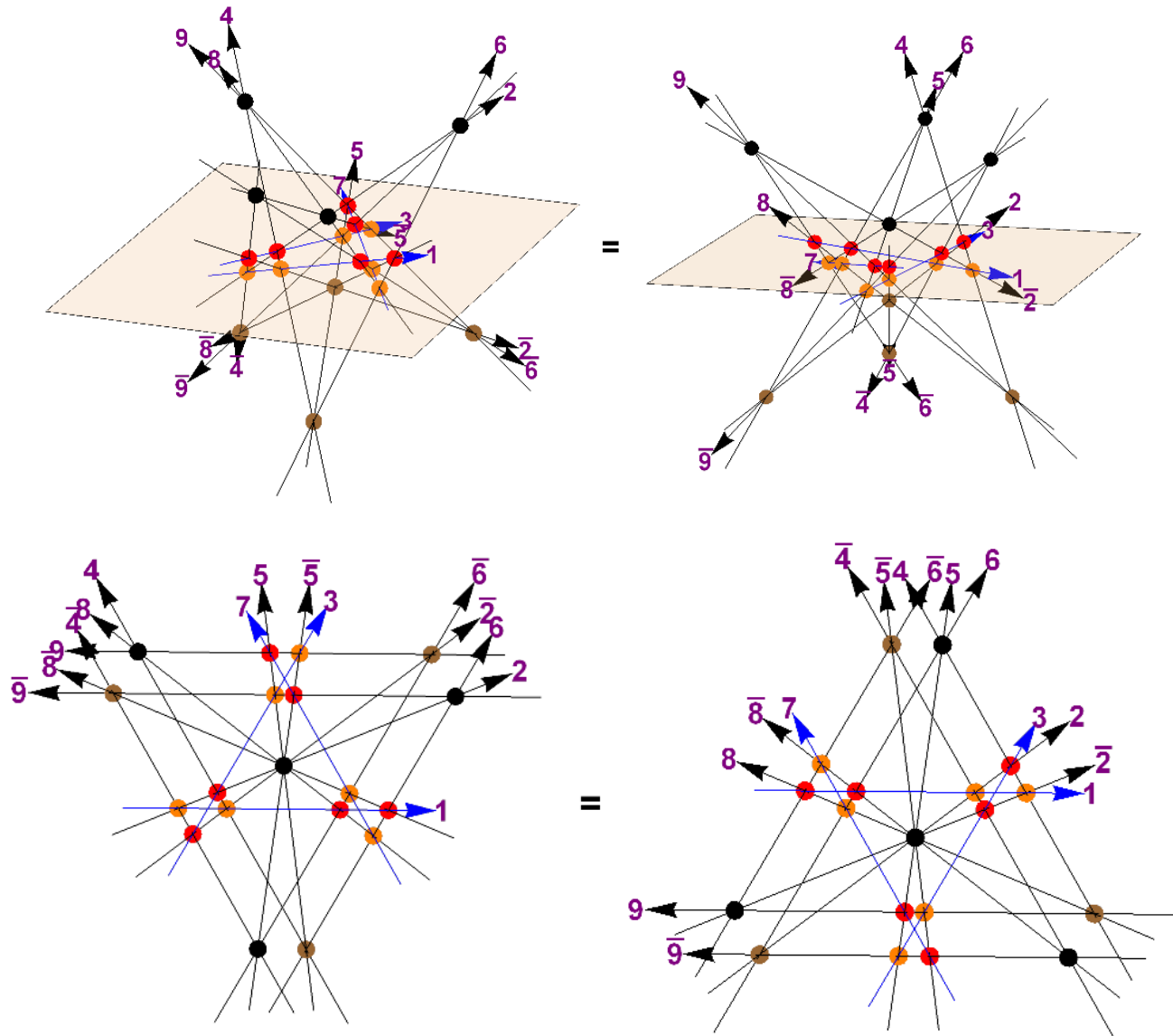
■ Proof:

$^-$ : mirrored space

■ By repeated uses of TE, we have the above equation.

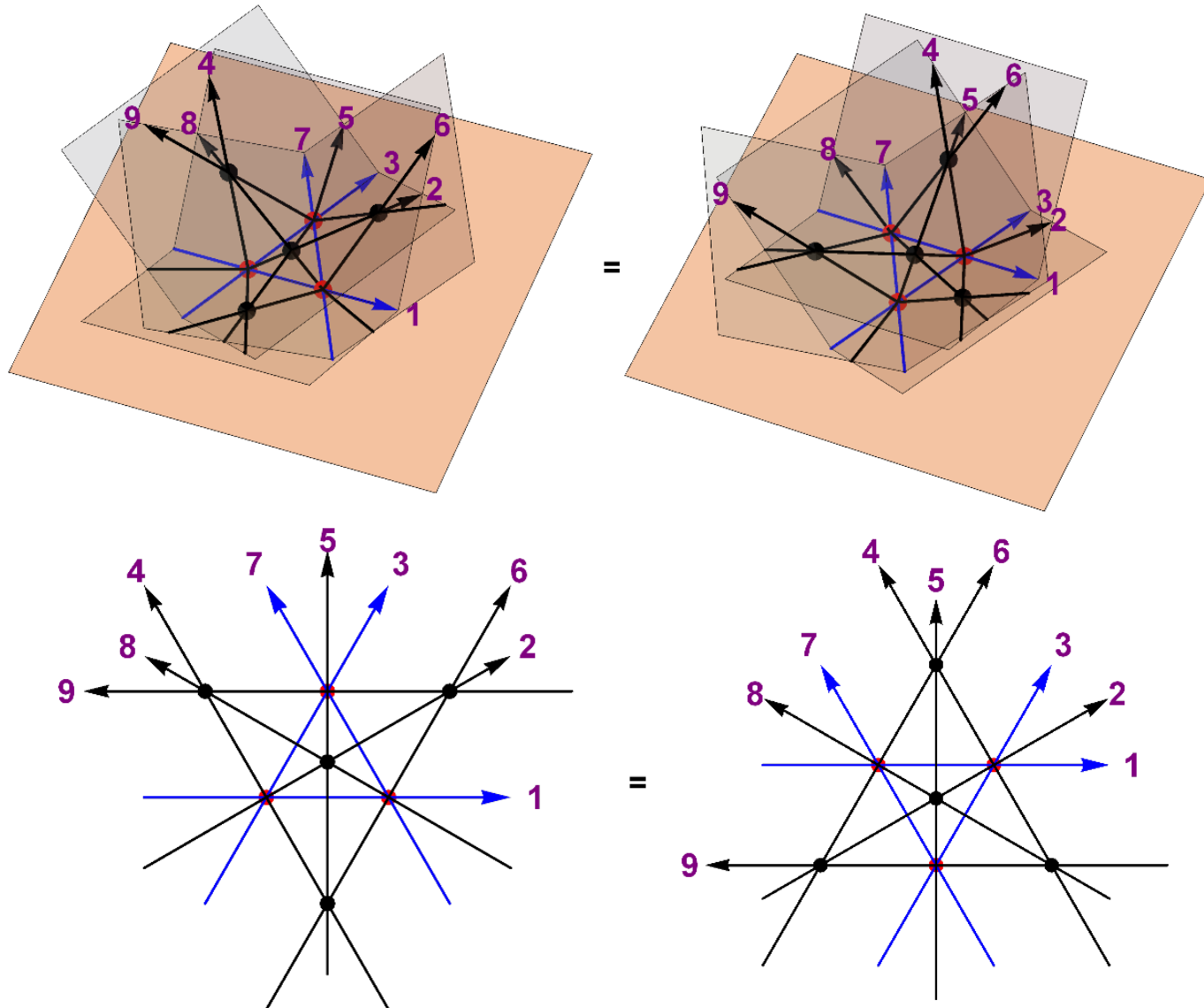
# Figure for the identity on $X^{15}$

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# Figure for 3D reflection equation

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## ■ Sketch of proof:

□ Let  $\mathbf{T}$  denote the tetrahedral composite of  $\mathbf{R}$ .

□ By using  $\mathbf{T}$ , the identify in the previous lemma is written as follows:

$$\begin{aligned} & (\mathbf{R}_{489}\mathbf{R}_{\bar{4}\bar{8}\bar{9}})\mathbf{T}_{35\bar{5}79\bar{9}}(\mathbf{R}_{26\bar{9}}\mathbf{R}_{\bar{2}\bar{6}\bar{9}})(\mathbf{R}_{2\bar{5}8}\mathbf{R}_{\bar{2}\bar{5}\bar{8}})\mathbf{T}_{16\bar{6}78\bar{8}}\mathbf{T}_{12\bar{2}34\bar{4}}(\mathbf{R}_{\bar{4}\bar{5}\bar{6}}\mathbf{R}_{456}) \\ & = (\mathbf{R}_{456}\mathbf{R}_{\bar{4}\bar{5}\bar{6}})\mathbf{T}_{12\bar{2}34\bar{4}}\mathbf{T}_{16\bar{6}78\bar{8}}(\mathbf{R}_{\bar{2}\bar{5}\bar{8}}\mathbf{R}_{258})(\mathbf{R}_{\bar{2}\bar{6}\bar{9}}\mathbf{R}_{269})\mathbf{T}_{35\bar{5}79\bar{9}}(\mathbf{R}_{\bar{4}\bar{8}\bar{9}}\mathbf{R}_{489}) \quad \dots (*) \end{aligned}$$

□ Let's act both sides of  $(*)$  on "mirrored" inputs:

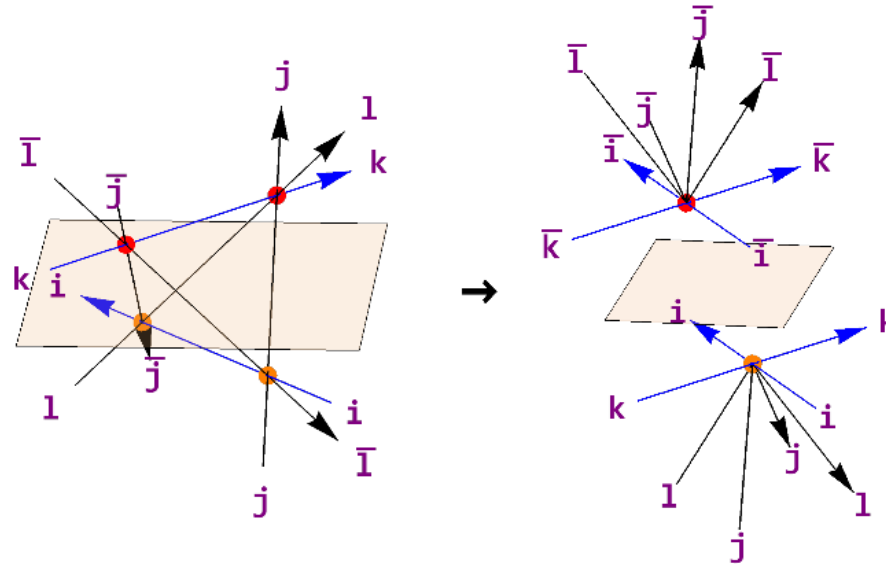
$$\begin{aligned} & (x_1, x_2, x_2, x_3, x_4, x_4, x_5, x_5, x_6, x_6, x_7, x_8, x_8, x_9, x_9) \\ & \in \underline{1} \times 2 \times \bar{2} \times \underline{3} \times 4 \times \bar{4} \times 5 \times \bar{5} \times 6 \times \bar{6} \times \underline{7} \times 8 \times \bar{8} \times 9 \times \bar{9} \end{aligned}$$

□ We can verify that all  $\mathbf{T}$  in  $(*)$  receive elements of  $Y$ .

□ For example,  $\mathbf{R}_{456}$  and  $\mathbf{R}_{\bar{4}\bar{5}\bar{6}}$  in      output the same, so  $\mathbf{T}_{12\bar{2}34\bar{4}}$  receive an element of  $Y$ .

□ Then, all  $\mathbf{T}$  in  $(*)$  satisfy  $\mathbf{T} = \phi \circ \mathbf{J} \circ \varphi$  by the definition of  $\mathbf{J}$ .

□ As a matter of fact, we expect  $\mathbf{T} \sim \mathbf{J}\mathbf{J}$  rather than  $\mathbf{T} \sim \mathbf{J}$  in view of the previous figures.



## ■ Sketch of proof:

- By applying  $\mathbf{T}_{ij\bar{j}kl\bar{l}} \mapsto P_{j\bar{j}}P_{l\bar{l}}\mathbf{J}_{ijkl}\mathbf{J}_{i\bar{j}\bar{k}\bar{l}}$  (cutting and reconnection), to the previous equation, we obtain

$$\begin{aligned}
 & (\mathbf{R}_{489}\mathbf{R}_{\bar{4}\bar{8}\bar{9}})(P_{5\bar{5}}P_{9\bar{9}}\mathbf{J}_{3579}\mathbf{J}_{\bar{3}\bar{5}\bar{7}\bar{9}})(\mathbf{R}_{269}\mathbf{R}_{\bar{2}\bar{6}\bar{9}})(\mathbf{R}_{258}\mathbf{R}_{\bar{2}\bar{5}\bar{8}}) \\
 & \quad \times (P_{6\bar{6}}P_{8\bar{8}}\mathbf{J}_{1678}\mathbf{J}_{\bar{1}\bar{6}\bar{7}\bar{8}})(P_{2\bar{2}}P_{4\bar{4}}\mathbf{J}_{1234}\mathbf{J}_{\bar{1}\bar{2}\bar{3}\bar{4}})(\mathbf{R}_{\bar{4}\bar{5}\bar{6}}\mathbf{R}_{456}) \\
 & = (\mathbf{R}_{456}\mathbf{R}_{\bar{4}\bar{5}\bar{6}})(P_{2\bar{2}}P_{4\bar{4}}\mathbf{J}_{1234}\mathbf{J}_{\bar{1}\bar{2}\bar{3}\bar{4}})(P_{6\bar{6}}P_{8\bar{8}}\mathbf{J}_{1678}\mathbf{J}_{\bar{1}\bar{6}\bar{7}\bar{8}}) \\
 & \quad \times (\mathbf{R}_{\bar{2}\bar{5}\bar{8}}\mathbf{R}_{258})(\mathbf{R}_{\bar{2}\bar{6}\bar{9}}\mathbf{R}_{269})(P_{5\bar{5}}P_{9\bar{9}}\mathbf{J}_{3579}\mathbf{J}_{\bar{3}\bar{5}\bar{7}\bar{9}})(\mathbf{R}_{\bar{4}\bar{8}\bar{9}}\mathbf{R}_{489}).
 \end{aligned}$$

- By eliminating  $P_{i\bar{i}}$ , we obtain the direct product of 3DRE. (15+1+1+1=9+9)

## ■ 2D case

□  $K_B$  from  $R_{B^\vee, B}$ :

$$K_B: B \rightarrow B^\vee, b \rightarrow c^\vee \quad R_{B^\vee, B} (b^\vee \otimes b) = c \otimes c^\vee$$

□ Key lemma:

$$\begin{aligned} & (R_{B_2^\vee, B_1^\vee} R_{B_1, B_2}) R_{B_2^\vee, B_2} (R_{B_1^\vee, B_2} R_{B_2^\vee, B_1}) R_{B_1^\vee, B_1} \\ &= R_{B_1^\vee, B_1} (R_{B_2^\vee, B_1} R_{B_1^\vee, B_2}) R_{B_2^\vee, B_2} (R_{B_1, B_2} R_{B_2^\vee, B_1^\vee}) \end{aligned}$$

## ■ 3D case

□  $\mathbf{J}$  from  $\mathbf{R}$ :

$$\mathbf{J}(\mathbf{x}) = \varphi(\mathbf{T}(\phi(\mathbf{x}))) \quad \mathbf{T} = \mathbf{R}_{245} \mathbf{R}_{135} \mathbf{R}_{126} \mathbf{R}_{346}$$

□ Key lemma:

$$\begin{aligned} & (\mathbf{R}_{489} \mathbf{R}_{\bar{4}8\bar{9}}) (\mathbf{R}_{579} \mathbf{R}_{3\bar{5}9} \mathbf{R}_{35\bar{9}} \mathbf{R}_{\bar{5}7\bar{9}}) (\mathbf{R}_{269} \mathbf{R}_{\bar{2}6\bar{9}}) (\mathbf{R}_{2\bar{5}8} \mathbf{R}_{\bar{2}5\bar{8}}) \\ & \quad \times (\mathbf{R}_{\bar{6}7\bar{8}} \mathbf{R}_{16\bar{8}} \mathbf{R}_{1\bar{6}8} \mathbf{R}_{678}) (\mathbf{R}_{234} \mathbf{R}_{1\bar{2}4} \mathbf{R}_{12\bar{4}} \mathbf{R}_{\bar{2}3\bar{4}}) (\mathbf{R}_{\bar{4}\bar{5}\bar{6}} \mathbf{R}_{456}) \\ &= (\mathbf{R}_{456} \mathbf{R}_{\bar{4}\bar{5}\bar{6}}) (\mathbf{R}_{234} \mathbf{R}_{1\bar{2}4} \mathbf{R}_{12\bar{4}} \mathbf{R}_{\bar{2}3\bar{4}}) (\mathbf{R}_{\bar{6}7\bar{8}} \mathbf{R}_{16\bar{8}} \mathbf{R}_{1\bar{6}8} \mathbf{R}_{678}) \\ & \quad \times (\mathbf{R}_{\bar{2}5\bar{8}} \mathbf{R}_{2\bar{5}8}) (\mathbf{R}_{\bar{2}6\bar{9}} \mathbf{R}_{269}) (\mathbf{R}_{579} \mathbf{R}_{3\bar{5}9} \mathbf{R}_{35\bar{9}} \mathbf{R}_{\bar{5}7\bar{9}}) (\mathbf{R}_{\bar{4}8\bar{9}} \mathbf{R}_{489}) \end{aligned}$$

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  - Sergeev's electrical solution
  - Two-component solution associated with soliton equation



# Example1: birational transition map

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- We set  $\mathbf{R}: \mathbb{R}_{>0}^3 \rightarrow \mathbb{R}_{>0}^3$  by

$$\mathbf{R}: (x_1, x_2, x_3) \mapsto (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \left( \frac{x_1 x_2}{y}, y, \frac{x_2 x_3}{y} \right) \quad y = x_1 + x_3$$

- This map is characterized as the transition map of parametrizations of the positive part of  $SL_3$ : [Lusztig94]

$$G_1(x_3)G_2(x_2)G_1(x_1) = G_2(\tilde{x}_1)G_1(\tilde{x}_2)G_2(\tilde{x}_3) \quad G_i(x) = 1 + xE_{i,i+1}$$

- Actually, the transition map for  $SL_4$  is the tetrahedral composite of  $\mathbf{R}$  and we have TE for  $\mathbf{R}$  by considering  $\mathbf{T}$  in two ways. [Kuniba-Okado12]

$$\begin{aligned} & \frac{G_1(x_6)G_3(x_5)G_2(x_4)G_1(x_3)G_3(x_2)G_2(x_1)}{=} \\ & = G_2(\tilde{x}_1)G_1(\tilde{x}_2)G_3(\tilde{x}_3)G_2(\tilde{x}_4)G_1(\tilde{x}_5)G_3(\tilde{x}_6) \end{aligned}$$

$$\mathbf{T}: (x_1, \dots, x_6) \mapsto (\tilde{x}_1, \dots, \tilde{x}_6)$$

- This tetrahedron map was also obtained in the context of the local YBE and by considering AKP tau function.

[Sergeev98], [Kassotakis-Nieszporski-Papageorgiou-Tongas19]

- We can verify  $\mathbf{R}$  is involutive, symmetric and boundarizable.

# Example1: birational transition map

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- The associated 3D reflection map  $\mathbf{J}: \mathbb{R}_{>0}^4 \rightarrow \mathbb{R}_{>0}^4$  is calculated as follows:

$$\mathbf{J} : (x_1, x_2, x_3, x_4) \mapsto \left( \frac{x_1 x_2^2 x_3}{y_1}, \frac{y_1}{y_2}, \frac{y_2^2}{y_1}, \frac{x_2 x_3 x_4}{y_2} \right)$$

$$y_1 = x_1(x_2 + x_4)^2 + x_3 x_4^2, \quad y_2 = x_1(x_2 + x_4) + x_3 x_4$$

- Actually,  $(\mathbf{R}, \mathbf{J})$  is a known solution to 3D reflection equation.  
[Kuniba-Okado12]

- $\mathbf{J}$  is characterized as the transition map of parametrizations of the positive part of  $SP_4$ :

$$H_1(x_4)H_2(x_3)H_1(x_2)H_2(x_1) = H_2(\tilde{x}_1)H_1(\tilde{x}_2)H_2(\tilde{x}_3)H_1(\tilde{x}_4)$$

$$H_1(x) = \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & x \\ & & & 1 \end{pmatrix} \quad H_2(x) = \begin{pmatrix} 1 & & & \\ & 1 & x & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad \begin{aligned} H_1(x) &= G_1(x)G_3(x) \\ H_2(x) &= G_2(x) \end{aligned}$$

- This correspondence is a consequence from folding the Dynkin diagram of  $A_3$  into one of  $C_2$ .  
[Berenstein-Zelevinsky01], [Lusztig11]

# Example2: Sergeev's electrical solution

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- For  $\lambda \in \mathbb{C}$ , we set the tetrahedron map  $\mathbf{R}(\lambda): \mathbb{C}^3 \rightarrow \mathbb{C}^3$  by

$$\mathbf{R}(\lambda) : (x_1, x_2, x_3) \mapsto (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \left( \frac{x_1 x_2}{y}, y, \frac{x_2 x_3}{y} \right) \quad y = x_1 + x_3 + \lambda x_1 x_2 x_3$$

- This solution was obtained in the context of the local YBE and by considering BKP tau function.
- $\mathbf{R}(\lambda = 1)$  is also characterized as the relation which elements of the positive part of electorical Lie groups of type A satisfy: [Lam-Pylyavskyy15]

$$\begin{aligned} \hat{G}_1(x_3) \hat{G}_2(x_2) \hat{G}_1(x_1) &= \hat{G}_2(\tilde{x}_1) \hat{G}_1(\tilde{x}_2) \hat{G}_2(\tilde{x}_3) \end{aligned} \quad \hat{G}_1(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \quad \hat{G}_2(x) = \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}$$

- We can verify  $\mathbf{R}(\lambda)$  is involutive, symmetric and boundarizable.

- The associated 3D reflection map  $\mathbf{J}(\lambda): \mathbb{C}^4 \rightarrow \mathbb{C}^4$  is calculated as follows:

$$\mathbf{J}(\lambda) : (x_1, x_2, x_3, x_4) \mapsto \left( \frac{x_1 x_2^2 x_3}{y_1}, \frac{y_1}{y_2}, \frac{y_2^2}{y_1}, \frac{x_2 x_3 x_4}{y_2} \right)$$

$$y_1 = x_1(x_2 + x_4)(x_2 + x_4 + 2\lambda x_2 x_3 x_4) + x_3 x_4^2$$

$$y_2 = x_1(x_2 + x_4 + 2\lambda x_2 x_3 x_4) + x_3 x_4$$

# Example3: Two component solution

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- We set the tetrahedron map  $\mathbf{R}: (\mathbb{C}^2)^3 \rightarrow (\mathbb{C}^2)^3$  by

$$\mathbf{R} : \left( \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \right) \mapsto \left( \begin{pmatrix} \frac{x_1 x_2}{x_1 + x_3} \\ \frac{(x_1 + x_3) y_1 y_2}{x_1 y_1 + x_3 y_3} \end{pmatrix}, \begin{pmatrix} x_1 + x_3 \\ \frac{x_1 y_1 + x_3 y_3}{x_1 + x_3} \end{pmatrix}, \begin{pmatrix} \frac{x_2 x_3}{x_1 + x_3} \\ \frac{(x_1 + x_3) y_2 y_3}{x_1 y_1 + x_3 y_3} \end{pmatrix} \right)$$

- This solution was obtained by considering modified KP tau function.  
[Kassotakis-Nieszporski-Papageorgiou-Tongas19]
- This gives the previous birational transition map when we set  $y_i = 1$ .
- We can verify  $\mathbf{R}$  is involutive, symmetric and boundarizable.

# Example3: Two component solution

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- The associated 3D reflection map  $\mathbf{J}: (\mathbb{C}^2)^4 \rightarrow (\mathbb{C}^2)^4$  is calculated as follows:

$$\mathbf{J} : \left( \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}, \begin{pmatrix} x_4 \\ y_4 \end{pmatrix} \right) \\ \mapsto \left( \begin{pmatrix} \frac{x_1 x_2^2 x_3}{z_1} \\ \frac{y_1 y_2^2 y_3 z_1}{w_1} \end{pmatrix}, \begin{pmatrix} \frac{z_1}{z_2} \\ \frac{z_2 w_1}{z_1 w_2} \end{pmatrix}, \begin{pmatrix} \frac{z_2^2}{z_1} \\ \frac{z_1 w_2^2}{z_2^2 w_1} \end{pmatrix}, \begin{pmatrix} \frac{x_2 x_3 x_4}{z_2} \\ \frac{y_2 y_3 y_4 z_2}{w_2} \end{pmatrix} \right)$$

$$z_1 = x_1(x_2 + x_4)^2 + x_3 x_4^2$$

$$z_2 = x_1(x_2 + x_4) + x_3 x_4$$

$$w_1 = x_1 y_1 (x_2 y_2 + x_4 y_4)^2 + x_3 x_4^2 y_3 y_4^2$$

$$w_2 = x_1 y_1 (x_2 y_2 + x_4 y_4) + x_3 x_4 y_3 y_4$$

## ■ Summary:

- ▣ We present a method for obtaining 3D reflection maps by using known tetrahedron maps. The theorem is an analog of the results in 2D.
- ▣ Our method is a kind of generalization of the relation by Berenstein and Zelevinsky and gives 3D interpretation to their relation.
- ▣ By applying our theorem to known tetrahedron maps, we obtain several 3D reflection maps which include new solutions.

## ■ Remark:

- ▣ Our theorem can be extended to *inhomogeneous* cases, that is, the case tetrahedron maps are defined on direct product of different sets.