

Tetrahedron and 3D reflection equation from PBW bases of the nilpotent subalgebra of quantum superalgebras

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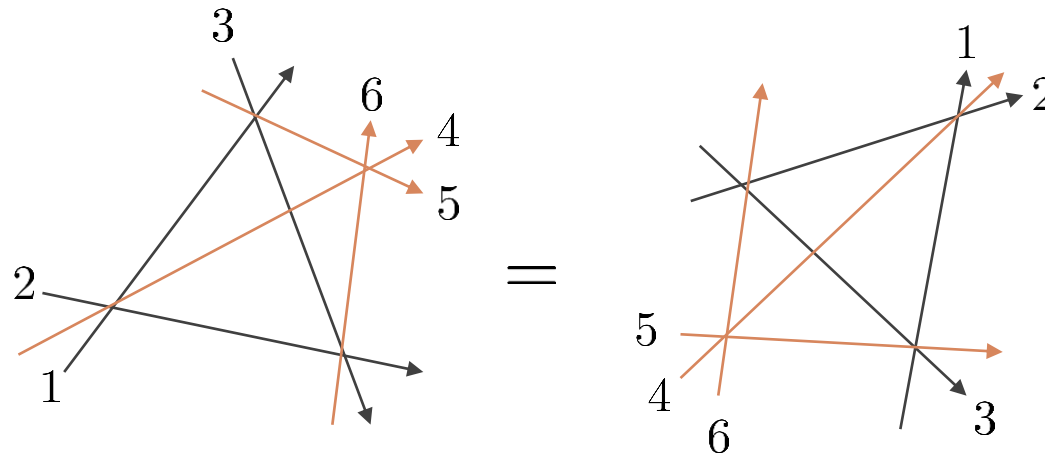
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Tetrahedron equation (TE)

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■ 3D analog of Yang-Baxter equation (YBE)

$$\mathcal{R}_{124}\mathcal{R}_{135}\mathcal{R}_{236}\mathcal{R}_{456} = \mathcal{R}_{456}\mathcal{R}_{236}\mathcal{R}_{135}\mathcal{R}_{124}$$

■ Several solutions are known although less systematically than YBE.


- ▣ Family of Zamolodchikov model: Zamolodchikov (81), Bazhanov-Baxter (92)
- ▣ Product of solution to YBE: Carter-Saito (96)
- ▣ Intertwiner for quantum coordinate ring $A_q(A_2)$: Kapranov-Voevodsky (94)
= Transition matrix of PBW bases of quantum group $U_q^+(A_2)$: Sergeev (07)
- ▣ solution to local YBE by ansatz: Bazhanov-Sergeev (06)

Two solutions on Fock spaces: 3DR & 3DL 3/9

- Fock spaces: $F = \bigoplus_{m=0,1,2,\dots} \mathbb{C} |m\rangle$ and $V = \bigoplus_{m=0,1} \mathbb{C} u_m$
- 3DR = transition matrix of PBW bases of quantum group $U_q^+(A_2)$
 - $\mathcal{R} \in \text{End}(F^{\otimes 3})$
 - $\mathcal{R} |i\rangle \otimes |j\rangle \otimes |k\rangle = \sum_{a,b,c} \mathcal{R}_{ijk}^{abc} |a\rangle \otimes |b\rangle \otimes |c\rangle$
 - $$\mathcal{R}_{i,j,k}^{a,b,c} = \delta_{i+j}^{a+b} \delta_{j+k}^{b+c} \sum_{\lambda, \mu \geq 0, \lambda + \mu = b} (-1)^\lambda q^{i(c-j) + (k+1)\lambda + \mu(\mu-k)} \frac{(q^2)_{c+\mu}}{(q^2)_c} \binom{i}{\mu}_{q^2} \binom{j}{\lambda}_{q^2}$$
 - $\mathcal{R}_{124} \mathcal{R}_{135} \mathcal{R}_{236} \mathcal{R}_{456} = \mathcal{R}_{456} \mathcal{R}_{236} \mathcal{R}_{135} \mathcal{R}_{124}$
- 3DL = solution to local YBE by ansatz
 - $\mathcal{L} \in \text{End}(V \otimes V \otimes F)$
 - $\mathcal{L}(u_i \otimes u_j \otimes |k\rangle) = \sum \mathcal{L}_{i,j,k}^{a,b,c} u_a \otimes u_b \otimes |c\rangle$
 - $$\begin{aligned} \mathcal{L}_{0,0,k}^{0,0,c} &= \mathcal{L}_{1,1,k}^{1,1,c} = \delta_{k,c}, & \mathcal{L}_{0,1,k}^{0,1,c} &= -\delta_{k,c} q^{k+1}, & \mathcal{L}_{1,0,k}^{1,0,c} &= \delta_{k,c} q^k, \\ \mathcal{L}_{1,0,k}^{0,1,c} &= \delta_{k-1,c} (1 - q^{2k}), & \mathcal{L}_{0,1,k}^{1,0,c} &= \delta_{k+1,c} \end{aligned}$$
 - $\mathcal{L}_{124} \mathcal{L}_{135} \mathcal{L}_{236} \mathcal{R}_{456} = \mathcal{R}_{456} \mathcal{L}_{236} \mathcal{L}_{135} \mathcal{L}_{124} \quad \mathcal{R} : \text{3DR}$

- Aim: Find a parallel origin for 3DR and 3DL
- Motivation:
 - Generally, infinitely many solutions to YBE are obtained from one solution to TE.
 - For 3DR and 3DL, we can obtain R matrices with *spectral parameters*.
 - They are identified with R matrices associated with some quantum affine algebras $U_q(\hat{g})$.

Solution to TE	Solution to YBE
3DR	$U_q(A_{n-1}^{(1)})$, symmetric tensor rep.
3DL	$U_q(A_{n-1}^{(1)})$, fundamental rep.

 look similar

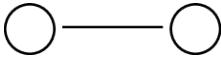

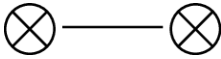
c.f. Kuniba-Okado-Sergeev (15)

Main result for TE (1/2)

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■ Theorem [Y20]:

- ▣ Transition matrices of PBW bases of the nilpotent subalgebra of quantum superalgebras of type A of rank 2 are given as follows:

Dynkin diagram	Transition matrix
	3DR
	3DL
	3DN

$$s_1 s_2 s_1 = s_2 s_1 s_2$$

- ▣ Here, 3DN is a new object as $\mathcal{N} \in \text{End}(V \otimes F \otimes V)$
- ▣ By constructing transition matrices for rank 3 in two ways, we obtain several solutions to TE, which 3DR, L and N satisfy.

Main result for TE (2/2)

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■ For $\bigcirc \text{---} \bigcirc \text{---} \bigotimes$

$$\mathcal{L}_{123}\mathcal{L}_{145}\mathcal{L}_{246}\mathcal{R}_{356} = \mathcal{R}_{356}\mathcal{L}_{246}\mathcal{L}_{145}\mathcal{L}_{123}$$

□ Sketch of Proof:

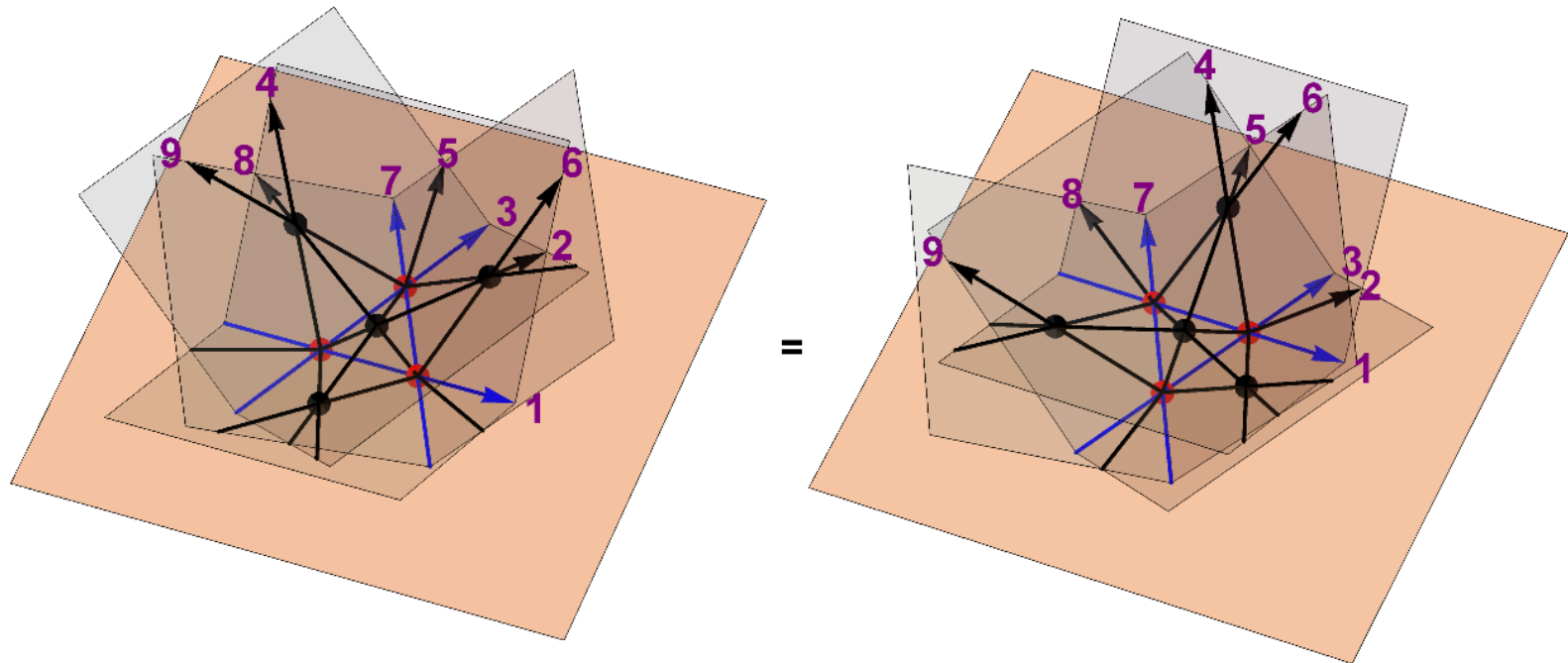
$$\begin{aligned} & \frac{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)}}{e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}} \quad e_i : \text{generator of } U_q \\ &= \sum \mathcal{L}_{x_1, x_2, x_3}^{o_1, o_2, o_3} e_2^{(x_3)} e_{32}^{(x_2)} \frac{e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}}{e_1^{(o_6)}} \\ &= \sum \mathcal{L}_{x_1, x_2, x_3}^{o_1, o_2, o_3} \mathcal{L}_{i_1, x_4, x_5}^{x_1, o_4, o_5} e_2^{(x_3)} \frac{e_{32}^{(x_2)} e_{12}^{(x_5)} e_{3(12)}^{(x_4)} e_3^{(i_1)} e_1^{(o_6)}}{e_1^{(o_6)}} \\ &= \sum \mathcal{L}_{x_1, x_2, x_3}^{o_1, o_2, o_3} \mathcal{L}_{i_1, x_4, x_5}^{x_1, o_4, o_5} e_2^{(x_3)} \frac{e_{12}^{(x_5)} e_{32}^{(x_2)} e_{1(32)}^{(x_4)} e_1^{(o_6)} e_3^{(i_1)}}{e_1^{(o_6)}} \\ &= \sum \mathcal{L}_{x_1, x_2, x_3}^{o_1, o_2, o_3} \mathcal{L}_{i_1, x_4, x_5}^{x_1, o_4, o_5} \mathcal{L}_{i_2, i_4, x_6}^{x_2, x_4, o_6} \frac{e_2^{(x_3)} e_{12}^{(x_5)} e_1^{(x_6)} e_{321}^{(i_4)} e_{32}^{(i_2)} e_3^{(i_1)}}{e_1^{(x_6)}} \\ &= \sum \mathcal{L}_{x_1, x_2, x_3}^{o_1, o_2, o_3} \mathcal{L}_{i_1, x_4, x_5}^{x_1, o_4, o_5} \mathcal{L}_{i_2, i_4, x_6}^{x_2, x_4, o_6} \mathcal{R}_{i_3, i_5, i_6}^{x_3, x_5, x_6} \frac{e_1^{(i_6)} e_{21}^{(i_5)} e_2^{(i_3)} e_{321}^{(i_4)} e_{32}^{(i_2)} e_3^{(i_1)}}{e_1^{(i_6)}} \\ &= \sum \mathcal{L}_{x_1, x_2, x_3}^{o_1, o_2, o_3} \mathcal{L}_{i_1, x_4, x_5}^{x_1, o_4, o_5} \mathcal{L}_{i_2, i_4, x_6}^{x_2, x_4, o_6} \mathcal{R}_{i_3, i_5, i_6}^{x_3, x_5, x_6} \frac{e_1^{(i_6)} e_{21}^{(i_5)} e_{321}^{(i_4)} e_2^{(i_3)} e_{32}^{(i_2)} e_3^{(i_1)}}{e_1^{(i_6)}} \end{aligned}$$

■ For $\bigcirc \text{---} \bigotimes \text{---} \bigotimes$ (new solution)

$$\mathcal{N}(q^{-1})_{123}\mathcal{N}(q^{-1})_{145}\mathcal{R}_{246}\mathcal{L}_{356} = \mathcal{L}_{356}\mathcal{R}_{246}\mathcal{N}(q^{-1})_{145}\mathcal{N}(q^{-1})_{123}$$

3D reflection equation (3DRE)

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- Boundary analog of TE proposed by Isaev-Kulish (97)

$$\mathcal{R}_{489} \mathcal{J}_{3579} \mathcal{R}_{269} \mathcal{R}_{258} \mathcal{J}_{1678} \mathcal{J}_{1234} \mathcal{R}_{456} = \mathcal{R}_{456} \mathcal{J}_{1234} \mathcal{J}_{1678} \mathcal{R}_{258} \mathcal{R}_{269} \mathcal{J}_{3579} \mathcal{R}_{489}$$

- Only two solutions are known.

- Interwtiner of quantum coordinate ring $A_q(B_2)$ or $A_q(C_2)$
= Transition matrix of PBW bases of quantum group $U_q(B_2)$ or $U_q(C_2)$

Kuniba-Okado (12, 13), Kuniba-Okado-Yamada (13)

Main result for 3DRE

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■ Theorem [Y20]:

- ▣ Transition matrices of PBW bases of the nilpotent subalgebra of quantum superalgebras of type B of rank 2 are given as follows:

Dynkin diagram	Transition matrix	Space
$\bigcirc \Longrightarrow \bigcirc$	3DJ	$F^{\otimes 4}$
$\bigotimes \Longrightarrow \bigcirc$	3DX (new)	$F \otimes V \otimes F \otimes V$
$\bigotimes \Longrightarrow \bullet$	3DY (new)	$F \otimes V \otimes F \otimes V$
$\bigcirc \Longrightarrow \bullet$	3DZ (new)	$F^{\otimes 4}$

- ▣ We obtained explicit formulae for 3DX and 3DY.
- ▣ Any matrix elements for 3DZ can be calculated by recurrence equations.
- ▣ The following cases give new solutions to 3DRE.



$$\begin{aligned}
 & \mathcal{L}_{456} \mathcal{N}(q^{-1})_{489} \mathcal{Y}(q^{-1})_{3579} \mathcal{N}(q^{-1})_{269} \mathcal{L}_{258} \mathcal{J}_{1678} \mathcal{X}_{1234} \\
 &= \mathcal{X}_{1234} \mathcal{J}_{1678} \mathcal{L}_{258} \mathcal{N}(q^{-1})_{269} \mathcal{Y}(q^{-1})_{3579} \mathcal{N}(q^{-1})_{489} \mathcal{L}_{456}
 \end{aligned}$$

■ Remark:

- The crystal limit for obtained transition matrices gives a super analog of transition maps of Lusztig's parametrizations of the canonical basis of U_q .

■ Proposition [Y20]:

- $\mathcal{L}_{i,j,k}^{a,b,c} = \lim_{q \rightarrow 0} \mathcal{L}(q)_{i,j,k}^{a,b,c}$ gives a non-trivial bijection on $\{0,1\}^2 \times \mathbb{Z}_{\geq 0}$.

- Non-zero elements are given by

$$\mathcal{L}_{0,0,k}^{0,0,c} = \mathcal{L}_{1,1,k}^{1,1,c} = \delta_{k,c}, \quad \mathcal{L}_{0,1,k}^{1,0,c} = \delta_{k+1,c}, \quad \mathcal{L}_{1,0,0}^{1,0,0} = 1, \quad \mathcal{L}_{1,0,k}^{0,1,c} = \delta_{k-1,c}$$

- $\mathcal{N}_{i,j,k}^{a,b,c} = \lim_{q \rightarrow 0} \left(\frac{[b]_q!}{[j]_q!} \mathcal{N}(q)_{i,j,k}^{a,b,c} \right)$ also gives a non-trivial bijection.

■ Summary:

- The 3DL is characterized as the transition matrix for $\bigcirc \text{---} \bigotimes$.
- The 3DN is obtained as a new solution to the tetrahedron equation by considering the transition matrix for $\bigotimes \text{---} \bigotimes$.

■ Outlook: Obtained transition matrices are characterized as intertwiners for quantum super coordinate rings?