# Tetrahedron ~~and 3D reflection~~ equation from PBW bases of the nilpotent subalgebra of quantum superalgebras
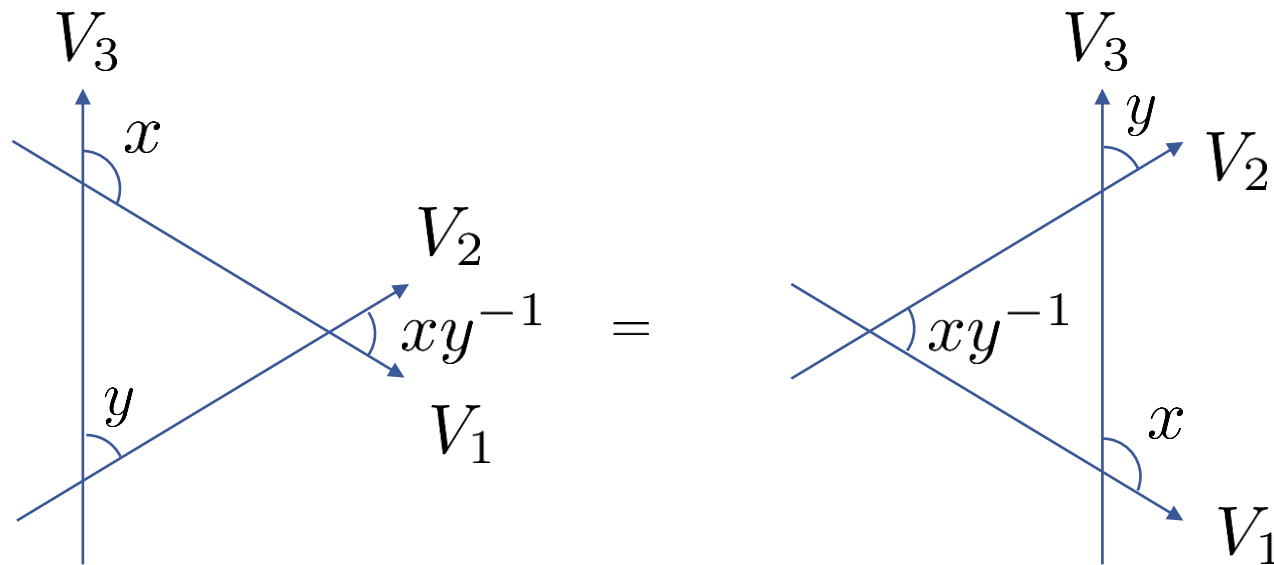
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# Outline

# Yang-Baxter equation

- Matrix equation on $V_1 \otimes V_2 \otimes V_3$ ($V_i$: linear space)

$$R_{12}(xy^{-1})R_{13}(x)R_{23}(y) = R_{23}(y)R_{13}(x)R_{12}(xy^{-1})$$

  □ $R_{ij}(z)$ acts non-trivially only on $V_i \otimes V_j$.

  □ Solutions to the Yang-Baxter eq are called $R$ matrices.

- $R$ matrices are systematically (infinitely many) constructed via irreps of quantum affine algebra $U_q(g)$.

■ Monodromy matrix $T_{a0}(z) \colon V_a \otimes V_0 \to V_a \otimes V_0$

$$T(z) = \quad \underset{V}{\overset{z}{\vert}} \quad \underset{V}{\overset{z}{\vert}} \quad \cdots \quad \underset{V}{\overset{z}{\vert}} \quad V_a$$

$$\underbrace{\phantom{V \quad V \quad V}}_{L}$$

$V_0 = V^{\otimes L}$ : physical sp.

$V_a$ : auxiliary sp.

■ By repeated uses of the Yang-Baxter equation, we have

$$R_{ab}(xy^{-1})T_{a0}(x)T_{b0}(y) = T_{b0}(y)T_{a0}(x)R_{ab}(xy^{-1})$$

# Commuting transfer matrix

- Multiply $R_{ab}(xy^{-1})^{-1}$ from the left:

$$T_{a0}(x)T_{b0}(y) = [R_{ab}(xy^{-1})]^{-1}T_{b0}(y)T_{a0}(x)R_{ab}(xy^{-1})$$

- By taking the trace on $V_a \otimes V_b$, we have

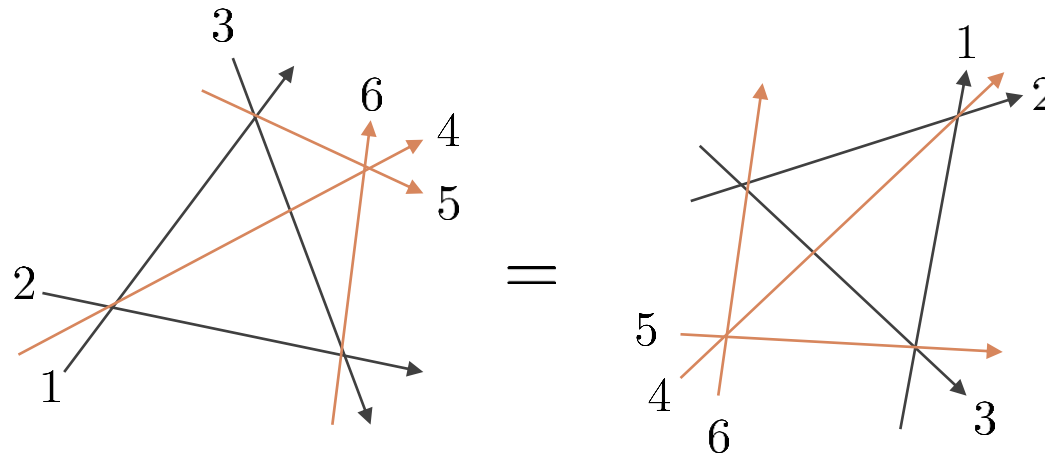$$[\tau(x), \tau(y)] = 0 \quad (^{\forall}x, y \in \mathbb{C})$$

  □ Here we set the row-to-row transfer matrix by

$$\tau(z) = \mathrm{Tr}_a(T_{a0}(z)) = $$



- A lot of families of <u>integrable</u> one-dimensional quantum spin chains are constructed via commutativty of the transfer matrix.
  □ The transfer matrix gives $O(L)$ conserved quantities.
  □ Eigenvalues are obtained by the Bethe ansatz.

- Matrix equation on $V_1 \otimes \cdots \otimes V_6$ ($V_i$: linear space)

$$\mathcal{R}_{124}\mathcal{R}_{135}\mathcal{R}_{236}\mathcal{R}_{456} = \mathcal{R}_{456}\mathcal{R}_{236}\mathcal{R}_{135}\mathcal{R}_{124}$$

  - $R_{ijk}$ acts non-trivially only on $V_i \otimes V_j \otimes V_k$.
  - Tetrahedron equation = Yang-Baxter equation *up to conjugation*

- Unlike Yang-Baxter equation, a few families of solutions are known.

- We focus on solutions on the Fock spaces.

  boson Fock: $F = \bigoplus_{m=0,1,2,\cdots} \mathbb{C}\,|m\rangle$

  fermi Fock: $V = \bigoplus_{m=0,1} \mathbb{C}u_m$

- Set $\mathcal{R} \in \mathrm{End}\left(F^{\otimes 3}\right)$ by                    [Kapranov-Voevodsky94]

$$\mathcal{R} \left|i\right\rangle \otimes \left|j\right\rangle \otimes \left|k\right\rangle = \sum_{a,b,c} \mathcal{R}_{ijk}^{abc} \left|a\right\rangle \otimes \left|b\right\rangle \otimes \left|c\right\rangle$$

$$\mathcal{R}_{i,j,k}^{a,b,c} = \delta_{i+j}^{a+b} \delta_{j+k}^{b+c} \sum_{\lambda,\mu \geq 0, \lambda+\mu=b} (-1)^{\lambda} q^{i(c-j)+(k+1)\lambda+\mu(\mu-k)} \frac{(q^2)_{c+\mu}}{(q^2)_c} \binom{i}{\mu}_{q^2} \binom{j}{\lambda}_{q^2}$$

Here

$$(q)_k = \prod_{l=1}^{k} (1 - q^l) \qquad \binom{a}{b}_q = \frac{(q)_a}{(q)_b (q)_{a-b}}$$

- The 3D R satisfies the following tetrahedron equation:

$$\mathcal{R}_{124} \mathcal{R}_{135} \mathcal{R}_{236} \mathcal{R}_{456} = \mathcal{R}_{456} \mathcal{R}_{236} \mathcal{R}_{135} \mathcal{R}_{124} \qquad \cdots (*)$$

- 3D R = intertwiner of irreps of quantum coordinate ring $A_q(A_2)$

$$\mathcal{R} \circ \pi_1 \otimes \pi_2 \otimes \pi_1 (\Delta^{\mathrm{op}}(g)) = \pi_2 \otimes \pi_1 \otimes \pi_2 (\Delta(g)) \circ \mathcal{R} \quad {}^{\forall} g \in A_q(A_2)$$

$$\pi_i : A_q(A_2) \to \mathrm{End}(F)$$

  □ ``121'' and ``212'' are associated with the longest element of Weyl group.
  □ This gives a linearization method for tetrahedron equation.

- Set $\mathcal{L} \in \mathrm{End}\,(V \otimes V \otimes F)$ by     [Bazhanov-Sergeev06]

$$\mathcal{L}(u_i \otimes u_j \otimes |k\rangle) = \sum_{a,b\in\{0,1\},c\in\mathbb{Z}_{\geq 0}} \mathcal{L}^{a,b,c}_{i,j,k} u_a \otimes u_b \otimes |c\rangle$$

$$\mathcal{L}^{0,0,c}_{0,0,k} = \mathcal{L}^{1,1,c}_{1,1,k} = \delta_{k,c}, \quad \mathcal{L}^{0,1,c}_{0,1,k} = -\delta_{k,c}q^{k+1}, \quad \mathcal{L}^{1,0,c}_{1,0,k} = \delta_{k,c}q^{k},$$

$$\mathcal{L}^{0,1,c}_{1,0,k} = \delta_{k-1,c}(1-q^{2k}), \quad \mathcal{L}^{1,0,c}_{0,1,k} = \delta_{k+1,c}$$

- The 3D L satisfies $\mathcal{L}_{124}\mathcal{L}_{135}\mathcal{L}_{236}\mathcal{R}_{456} = \mathcal{R}_{456}\mathcal{L}_{236}\mathcal{L}_{135}\mathcal{L}_{124}$     $\cdots(*)$

- BS obtained the 3D L by ansatz so that the tetrahedron equation of $(*)$ type has a non-trivial solution, and solved $(*)$ for the 3D R.
  □ Later, $(*)$ is ``identified'' with the intertwining relations for $A_q(A_2)$.
                                                                    [Kuniba-Okado12]
  □ Algebraic origins of the 3D L has been still unclear.

- Recently, the classical limit of $(*)$ is derived in relation to non-trivial transformations of a plabic network, which can be interpreted as cluster mutations.  [Gavrylenko-Semenyakin-Zenkevich20]

- The 3D R and L satisfy the following weight conservation:

$$[x^{\mathbf{h}_1}(xy)^{\mathbf{h}_2}y^{\mathbf{h}_3}, \mathcal{R}] = [x^{\mathbf{h}_1}(xy)^{\mathbf{h}_2}y^{\mathbf{h}_3}, \mathcal{L}] = 0 \quad (^{\forall}x, y \in \mathbb{C}) \quad \cdots (*)$$

  □ Here, $\mathbf{h}_1 = \mathbf{h} \otimes 1 \otimes 1$ etc. and $\mathbf{h}\,|m\rangle = m\,|m\rangle$, $\mathbf{h}u_m = mu_m$.

- The discussion below holds for solutions satisfying $(*)$.

- We use the following graphical notations:



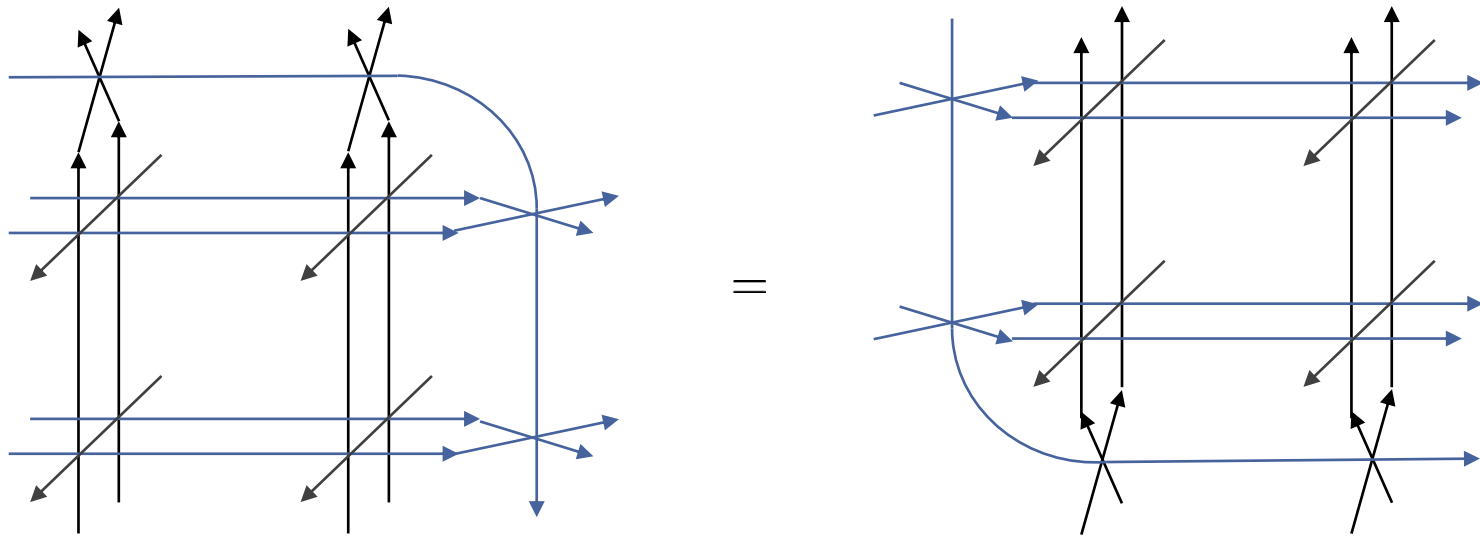$$= \mathcal{R}_{i,j,k}^{a,b,c} \text{ or } \mathcal{L}_{i,j,k}^{a,b,c}$$
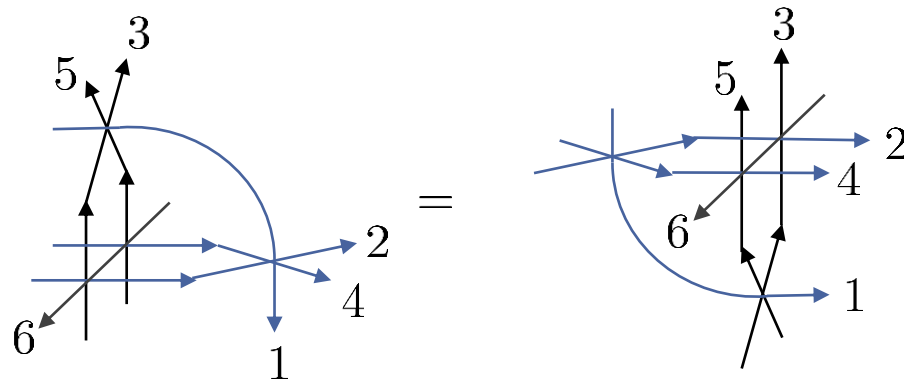
$$= \mathcal{R}_{i,j,k}^{a,b,c}$$

$$S^{(n)} =$$

$$T^{(m,n)} =$$

- The above equation is obtained by repreated use of

■ Multiply $x^{\mathbf{h}}, y^{\mathbf{h}}, u^{\mathbf{h}}, v^{\mathbf{h}}$ by several spaces:
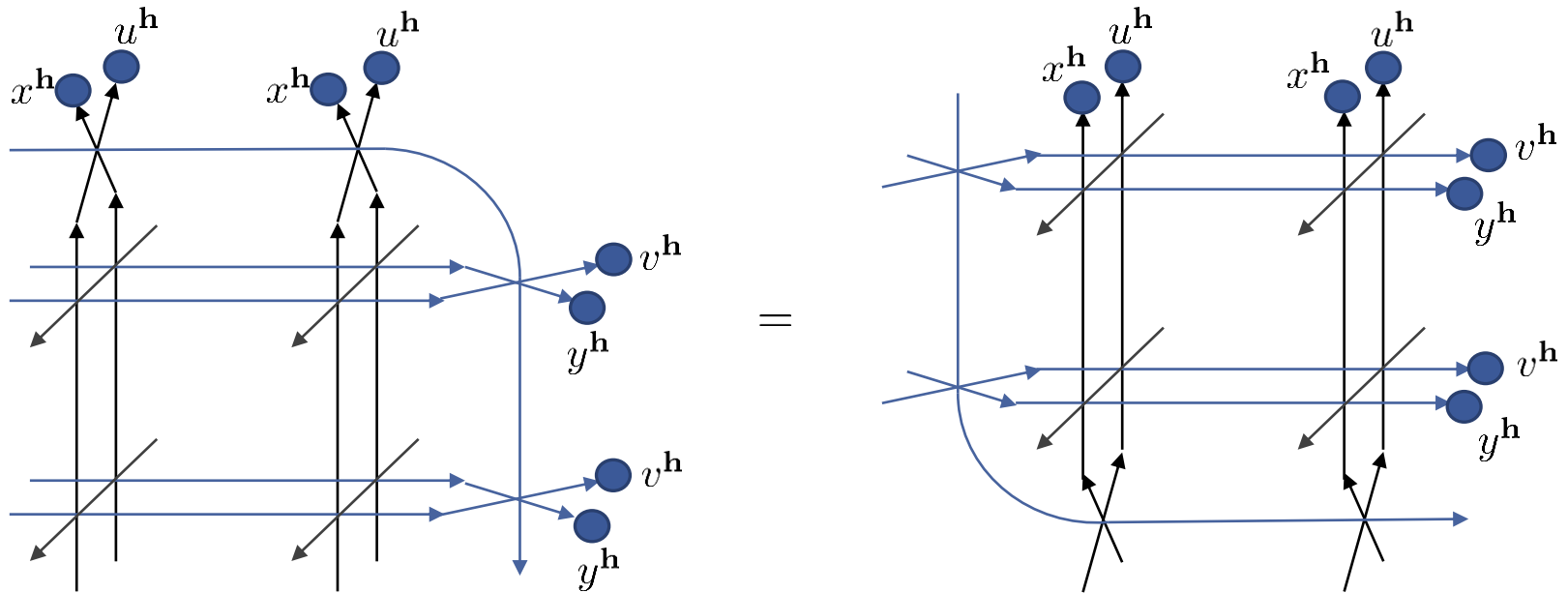
- Use the weight conservation $[x^{\mathbf{h}_1}(xy)^{\mathbf{h}_2}y^{\mathbf{h}_3}, \mathcal{R}] = [x^{\mathbf{h}_1}(xy)^{\mathbf{h}_2}y^{\mathbf{h}_3}, \mathcal{L}] = 0$
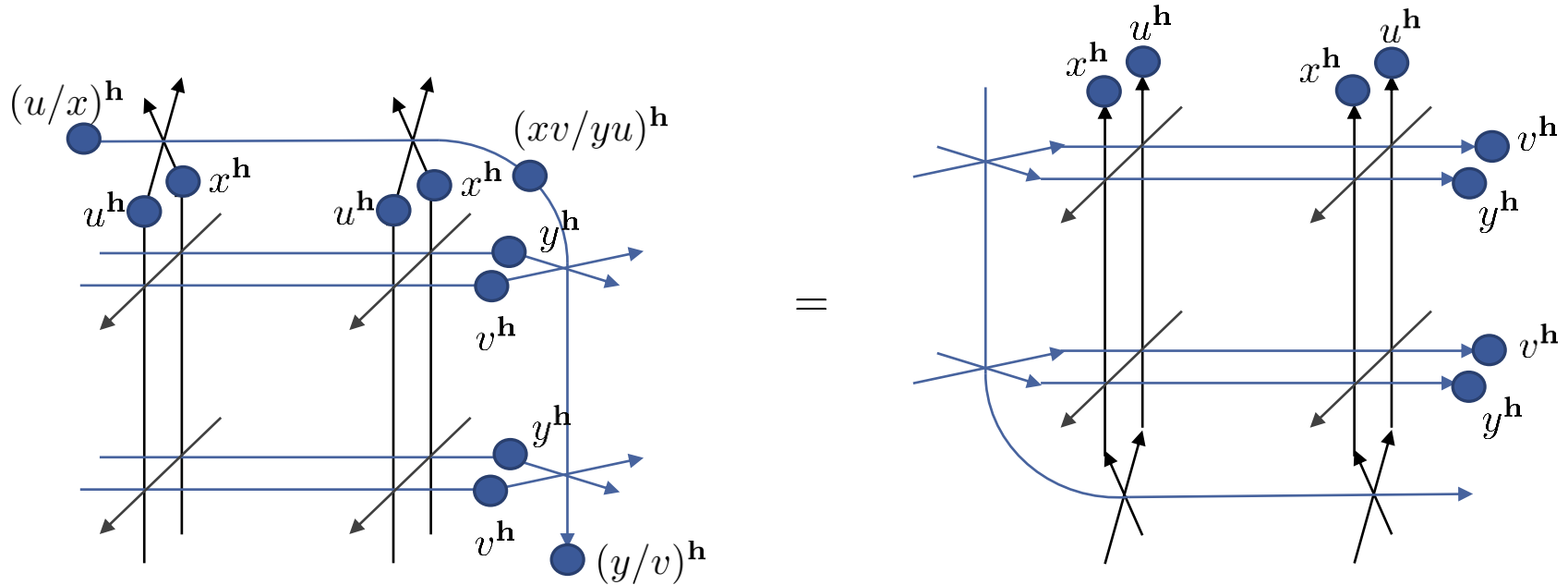
■ Move $(u/x)^{\mathbf{h}}$ and $(y/v)^{\mathbf{h}}$ to the right hand side:
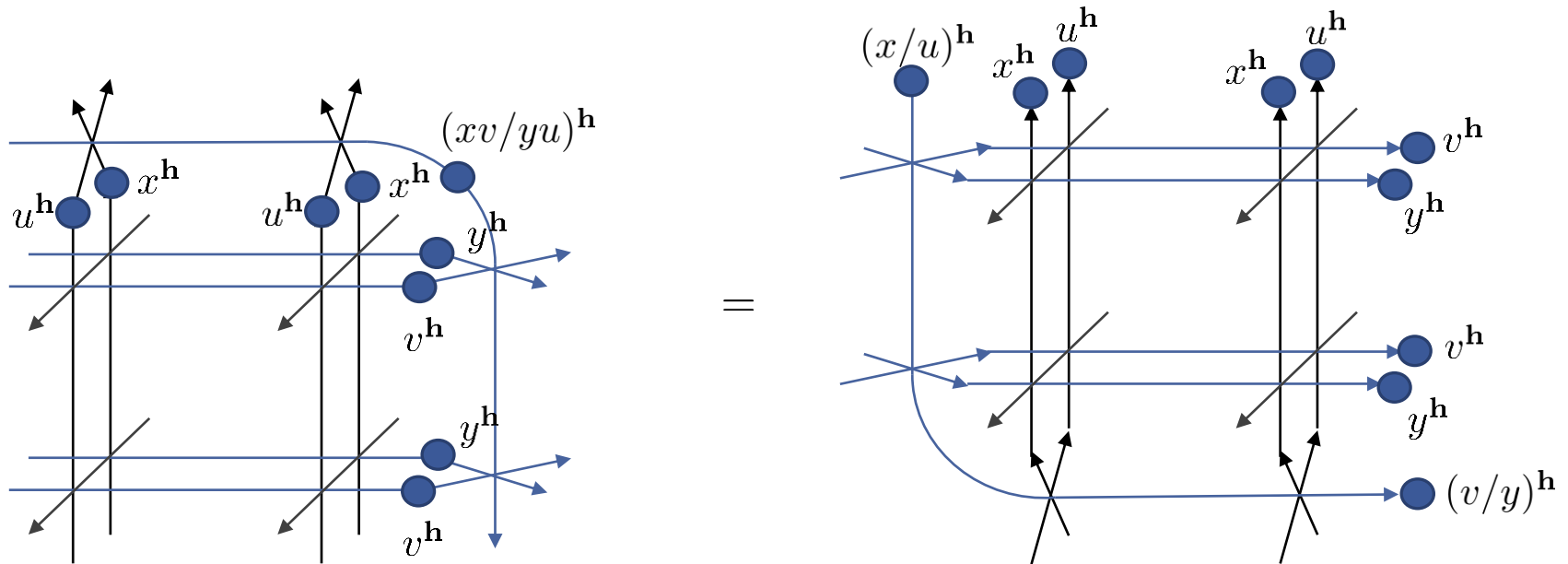
- Take the trace the auxiliary space:
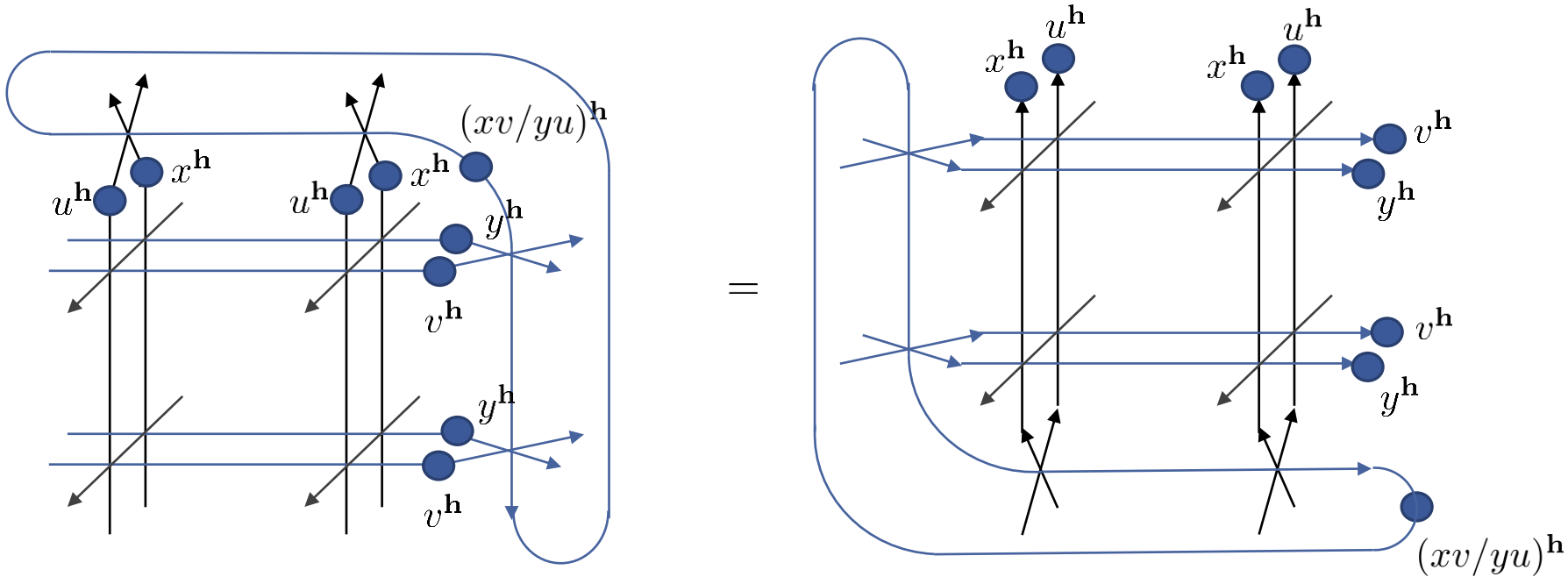
■ Note that the following parts are actually same matrices:

$(xv/yu)^{\mathbf{h}}$

$(xv/yu)^{\mathbf{h}}$
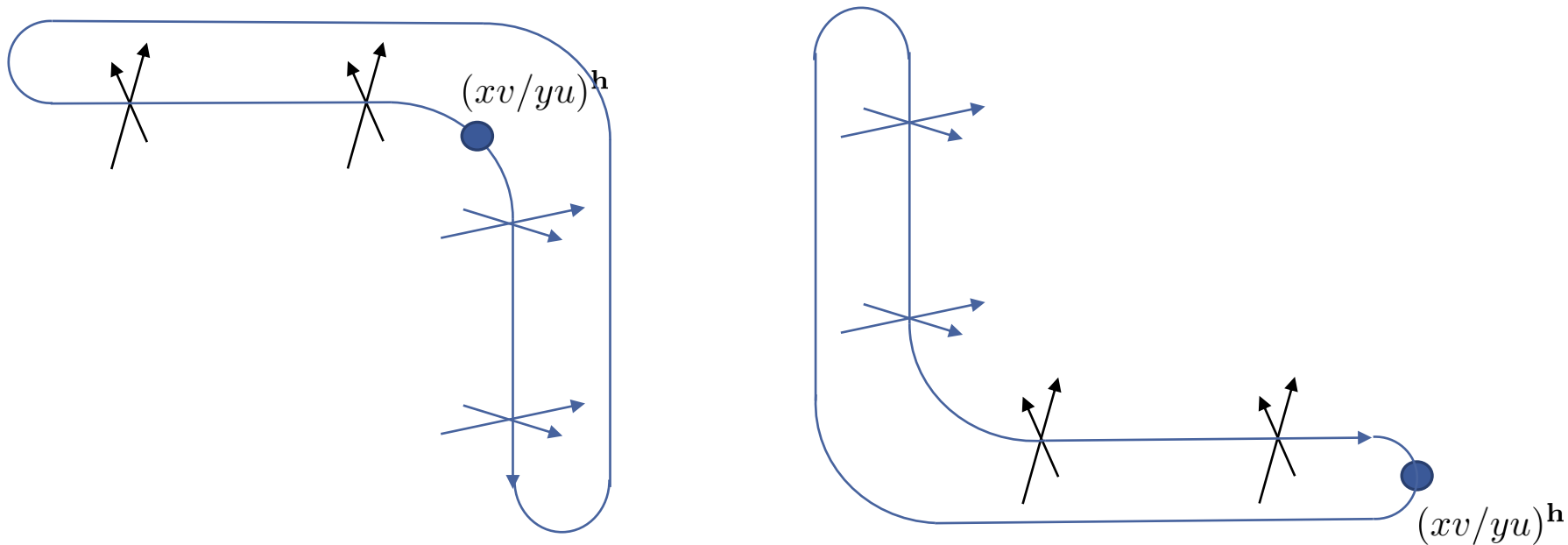
■ Then, if the above matrix is invertible, we can verify the commutativity by taking the trace on all auxiliary spaces.

■ Define the transfer matrix by

$$\tau^{(m,n)}(x, y) =$$



■ This satisfies the following commutativity:   [Sergeev06]

$$[\tau^{(m,n)}(x, y), \tau^{(m,n)}(x', y')] = 0 \quad (^\forall x, y, x', y' \in \mathbb{C})$$

■ This is often called the layer-to-layer transfer matrix.
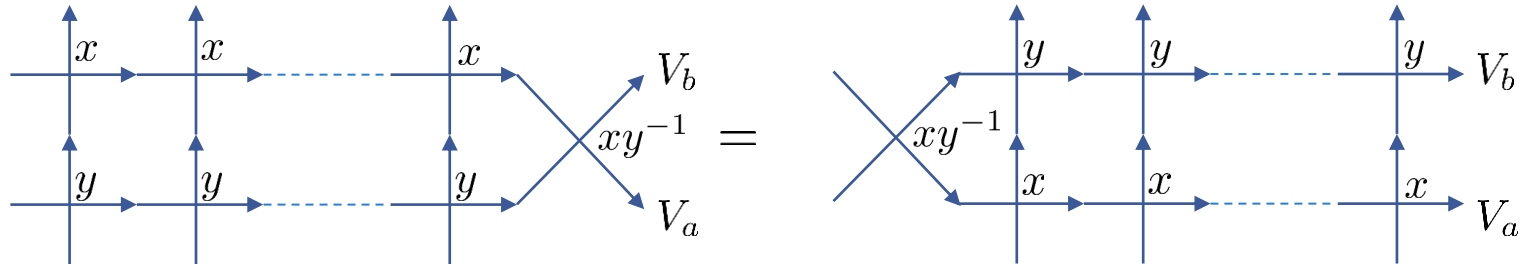
■ $1 + 1 + 0$ dimension in 2D



■ $2 + 2 + 1$ dimension in 3D

■ $1 + 1 + 1 + 0$ dimension in 3D



$= \qquad n$

■ The above equation is obtained by repreated use of



$=$

■ Multiply $x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5}y^{\mathbf{h}_6}$:
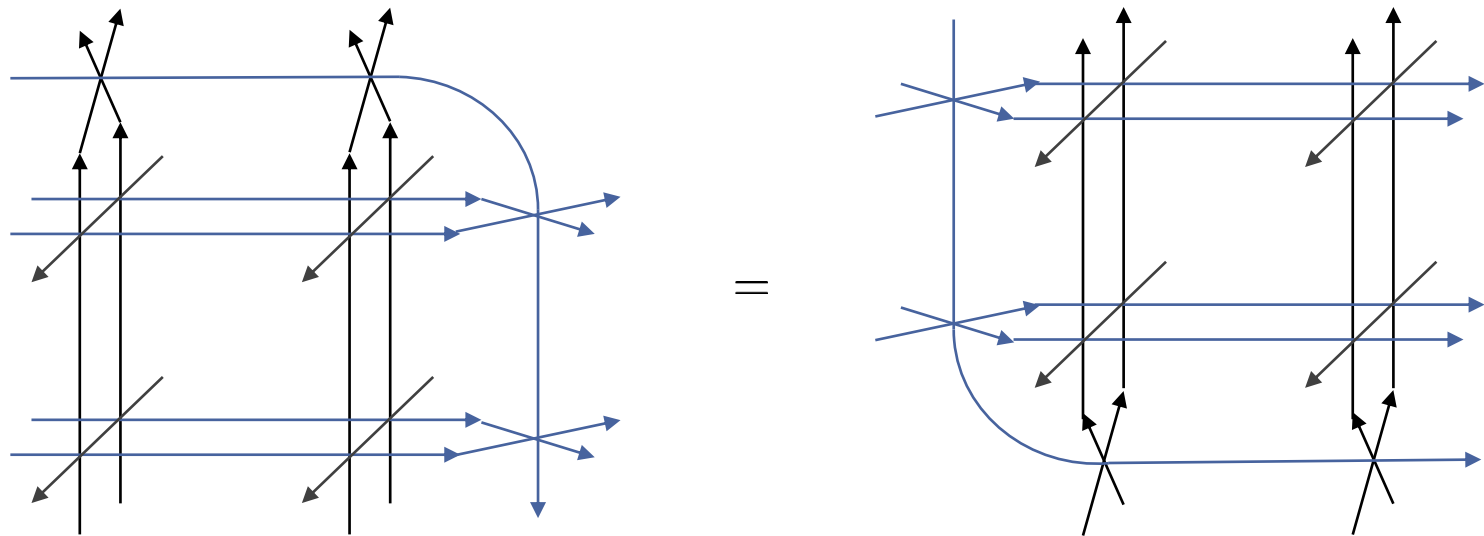
# RSSS=SSSR: Reduction to Yang-Baxter eq 20/68

■ Use the weight conservation $[x^{\mathbf{h_4}}(xy)^{\mathbf{h_5}}y^{\mathbf{h_6}}, \mathcal{R}] = 0$ :
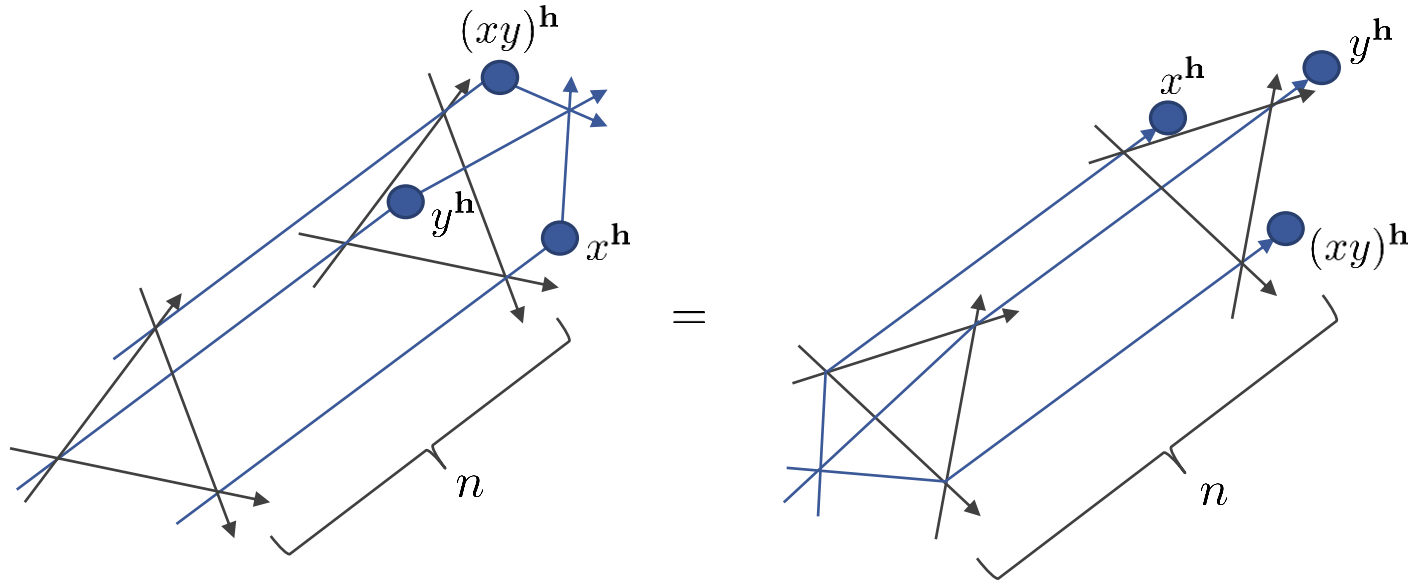


■ This gives the following identity:

$$\mathcal{R}_{456}(x^{\mathbf{h_4}}\mathcal{S}_{1_1 2_1 4}\cdots\mathcal{S}_{1_n 2_n 4})((xy)^{\mathbf{h_5}}\mathcal{S}_{1_1 3_1 5}\cdots\mathcal{S}_{1_n 3_n 5})(y^{\mathbf{h_6}}\mathcal{S}_{2_1 3_1 6}\cdots\mathcal{S}_{2_n 3_n 6})$$
$$= (y^{\mathbf{h_6}}\mathcal{S}_{2_1 3_1 6}\cdots\mathcal{S}_{2_n 3_n 6})((xy)^{\mathbf{h_5}}\mathcal{S}_{1_1 3_1 5}\cdots\mathcal{S}_{1_n 3_n 5})(x^{\mathbf{h_4}}\mathcal{S}_{1_1 2_1 4}\cdots\mathcal{S}_{1_n 2_n 4})\mathcal{R}_{456}$$

$$\mathcal{S} = \mathcal{R} \text{ or } \mathcal{L}$$

$$\mathcal{R}_{456}(x^{\mathbf{h_4}} \mathcal{S}_{1_1 2_1 4} \cdots \mathcal{S}_{1_n 2_n 4})((xy)^{\mathbf{h_5}} \mathcal{S}_{1_1 3_1 5} \cdots \mathcal{S}_{1_n 3_n 5})(y^{\mathbf{h_6}} \mathcal{S}_{2_1 3_1 6} \cdots \mathcal{S}_{2_n 3_n 6})$$

$$= (y^{\mathbf{h_6}} \mathcal{S}_{2_1 3_1 6} \cdots \mathcal{S}_{2_n 3_n 6})((xy)^{\mathbf{h_5}} \mathcal{S}_{1_1 3_1 5} \cdots \mathcal{S}_{1_n 3_n 5})(x^{\mathbf{h_4}} \mathcal{S}_{1_1 2_1 4} \cdots \mathcal{S}_{1_n 2_n 4})\mathcal{R}_{456}$$

■ From this identity, we can obtain solutions to Yang-Baxter eq by
1. multiplying $\mathcal{R}_{456}^{-1}$ and taking the trace on the spaces 456.
2. sandwiching between $\langle \chi_r | \otimes \langle \chi_r | \otimes \langle \chi_r |$ and $| \chi_{r'} \rangle \otimes | \chi_{r'} \rangle \otimes | \chi_{r'} \rangle$.

■ Here, we use properties for the 3D R:
1. The 3D R is invertible.
2. The 3D R has two eigenvectors with eigenvalues = 1 given by

$$\mathcal{R} | \chi_r \rangle \otimes | \chi_r \rangle \otimes | \chi_r \rangle = | \chi_r \rangle \otimes | \chi_r \rangle \otimes | \chi_r \rangle \quad (r = 1, 2)$$

$$\langle \chi_r | \otimes \langle \chi_r | \otimes \langle \chi_r | \mathcal{R} = \langle \chi_r | \otimes \langle \chi_r | \otimes \langle \chi_r | \quad (r = 1, 2)$$

Here

$$| \chi_r \rangle = \sum_{m \geq 0} \frac{|rm\rangle}{(q^{r^2}; q^{r^2})_m}$$
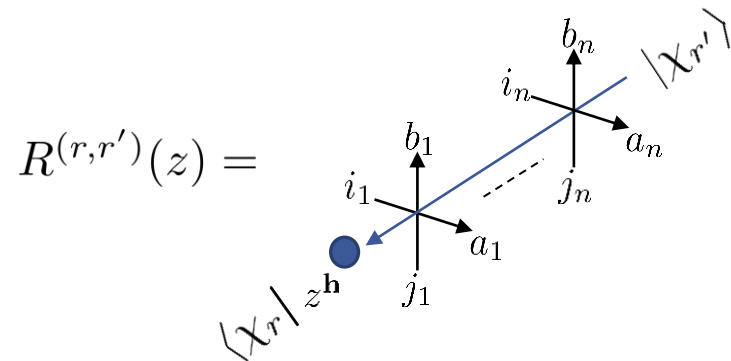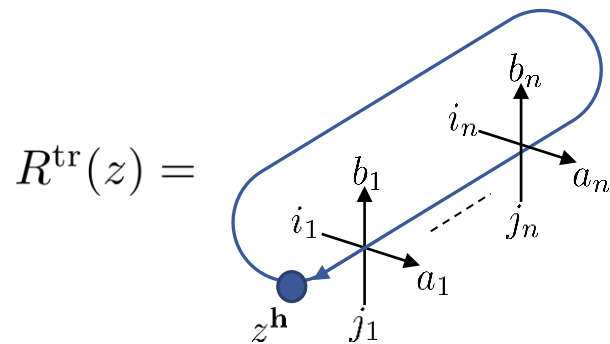
$$R^{\mathrm{tr}}(z) =$$



$$R^{(r,r')}(z) =$$



■ These solutions are characterized as the $R$ matrices associated with some quantum affine algebras. [Kuniba-Okado-Sergeev15]

| | $R^{\mathrm{tr}}(z)$ | $R^{(r,r')}(z)$ |
|---|---|---|
| $=$ 3D R | $U_q(A_{n-1}^{(1)})$ symmetric tensor rep. | $U_q(D_{n+1}^{(2)}), U_q(A_{2n}^{(2)}), U_q(C_n^{(1)})$ Fock rep. |
| $=$ 3D L | $U_q(A_{n-1}^{(1)})$ fundamental rep. | $U_q(D_{n+1}^{(2)}), U_q(B_n^{(1)}), U_q(D_n^{(1)})$ spin rep. |

■ Moreover, by mixing uses of the 3D R & L, we also obtain the $R$ matrices associated with generalized quantum groups.

# Rank-Size duality

- We consider the following transfer matrix where ⤢ are the 3D L.



- Let us consider projections from two directions (1) & (2).
  - Both of them give the row-to-row transfer matrix in two dimension.
  - This suggest the spectral duality between $sl(m)$ spin chain of size $n$ and $sl(n)$ spin chain of size $m$. [Bazhanov-Sergeev06]

- The duality also appears in the context of the five-dimensional gauge theory. [Mironov-Morozov-Runov-Zenkevich-Zotov13]

# Motivation & KOY theorem

- ■ <u>Motivation</u>
  - ❑ Why do the 3D R & L lead to such similar results, although they have totally different origins?

- ■ We study transition matrices of PBW bases of $U_q^+(sl(m|n))$ motivated by the KOY theorem which holds for non-super cases.

- ■ <u>Theorem</u> [Kuniba-Okado-Yamada13] (Rough Statement)
  - ❑ $g$: arbitrary finite-dimensional simple Lie algebra
  - ❑ $\Phi$: intertwiner of irreducible representations of $A_q(g)$
  - ❑ $\gamma$: transition matrix of PBW bases of the nilpotent subalgebra of $U_q(g)$
  - ❑ Then, we have $\Phi = \gamma$.

- ■ For ◯——◯, the 3D R gives the transition matrix for $U_q(A_2)$.

- ■ One of our result is that the 3D L is exactly the transition matrix for the quantum superalgebra associated with ◯——⊗.

# Outline

# Root system of Lie superalgebras $sl(m|n)$

- **Setup**
  - Weight lattice: $\mathcal{E}(m|n)_{\mathbb{Z}} = \sum_{i=1}^{m} \mathbb{Z}\epsilon_i \oplus \sum_{i=1}^{n} \mathbb{Z}\delta_i$
  
  $$(\epsilon_i, \epsilon_j) = (-1)^{\theta}\delta_{i,j}, \quad (\delta_i, \delta_j) = -(-1)^{\theta}\delta_{i,j}, \quad (\epsilon_i, \delta_j) = 0 \qquad \theta = 0, 1$$
  
  - Parity: $p(\lambda) = \sum_{i=1}^{n} b_i \pmod 2$ for $\lambda = \sum_{i=1}^{m} a_i\epsilon_i + \sum_{i=1}^{n} b_i\delta_i \in \mathcal{E}(m|n)_{\mathbb{Z}}$
  - We set $\{\bar{\epsilon}_i\}_{1 \leq i \leq m+n} = \{\epsilon_i\}_{1 \leq i \leq m} \cup \{\delta_i\}_{1 \leq i \leq n}$.

- **Lie superalgebra $sl(m|n)$**
  - Rank: $r = m + n - 1$
  - Simple roots: $\Pi = \{\alpha_1, \cdots, \alpha_r\}$, $\alpha_i = \bar{\epsilon}_i - \bar{\epsilon}_{i+1}$
  - Reduced positive roots:
  
  $$\tilde{\Phi}^+ = \{\bar{\epsilon}_i - \bar{\epsilon}_j \ (1 \leq i < j \leq r+1)\}$$
  
  - Even & odd parts: $\tilde{\Phi}^+ = \tilde{\Phi}^+_{\text{even}} \cup \tilde{\Phi}^+_{\text{iso}}$
  
  $$\tilde{\Phi}^+_{\text{even}} = \{\alpha \in \tilde{\Phi}^+ \mid p(\alpha) = 0\}$$
  $$\tilde{\Phi}^+_{\text{iso}} = \{\alpha \in \tilde{\Phi}^+ \mid p(\alpha) = 1\}$$

- **Cartan matrix for Lie superalgebra $sl(m|n)$**
  - ☐ $d_\alpha = (\alpha, \alpha)/2$ for $\alpha \in \tilde{\Phi}^+_{\text{even}}$, $d_\alpha = 1$ for $\alpha \in \tilde{\Phi}^+_{\text{iso}}$
  - ☐ $D = \text{diag}(d_1, \cdots, d_r)$, $d_i = d_{\alpha_i}$
  - ☐ Cartan matrix: $A = (a_{ij})_{i,j \in I}$, $a_{ij} = (\alpha_i, \alpha_j)/d_i$ $\qquad I = \{1, \cdots, r\}$
  - ☐ Simple coroots: $\{h_i\}_{i \in I}$, $\alpha_j(h_i) = a_{ij}$

- **Dynkin diagram for Cartan data $(A, p)$**
  - ☐ Prepare $r$ dots and decorate the $i$-th dot by

$$\bigcirc \text{ for } \alpha_i \in \tilde{\Phi}^+_{\text{even}} , \otimes \text{ for } \alpha_i \in \tilde{\Phi}^+_{\text{iso}}$$

  - ☐ Connect them if $a_{ij} \neq 0$ ($i \neq j$):

$$\overset{\bar{\epsilon}_1 - \bar{\epsilon}_2}{\times} \rule{1cm}{0.4pt} \overset{\bar{\epsilon}_2 - \bar{\epsilon}_3}{\times} \rule{1cm}{0.4pt} \cdots\cdots \rule{1cm}{0.4pt} \overset{\bar{\epsilon}_r - \bar{\epsilon}_{r+1}}{\times} \qquad \times: \bigcirc \text{ or } \otimes$$

- **Remark**
  - ☐ Dynkin diagrams do *not* correspond to Lie superalgebras themselves.

- $U_q(sl(m|n))$: quantum superalgebras associated with $(A, p)$
  - ☐ Generators: $e_i, f_i, k_i^{\pm 1} \ (i \in I)$
  - ☐ We use $q_i = q^{d_i}$.
  - ☐ Part of relations:

$$k_i^{\pm 1} k_i^{\mp 1} = 1, \quad k_i k_j = k_j k_i, \quad k_i e_j = q_i^{a_{ij}} e_j k_i, \quad k_i f_j = q_i^{-a_{ij}} f_j k_i,$$

$$e_i f_j - (-1)^{p(\alpha_i)p(\alpha_j)} f_j e_i = \delta_{i,j} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$$

- $U_q^+(sl(m|n))$: nilpotent subalgebra generated by $\{e_i\}_{i \in I}$
  - Root space decomposition:

$$U_q^+(\mathfrak{sl}(m|n))_\alpha = \{g \mid k_i g = q_i^{\alpha(h_i)} g k_i \ (i \in I)\}$$

  - $q$-commutator:

$$[x, y]_q = xy - (-1)^{p(\alpha)p(\beta)} q^{-(\alpha,\beta)} yx$$

$$x \in U_q^+(\mathfrak{sl}(m|n))_\alpha$$
$$y \in U_q^+(\mathfrak{sl}(m|n))_\beta$$

- Rest of relations:
  1. For $a_{ij} \neq 0$ $(i \neq j)$, $\alpha_i \in \tilde{\Phi}_{\text{even}}^+$:   $e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 = 0$
  2. For $a_{ij} = 0$:   $[e_i, e_j] = 0$
  3. For $\alpha_i \in \tilde{\Phi}_{\text{iso}}^+$:   $[[[e_{i-1}, e_i]_q, e_{i+1}]_q, e_i] = 0$
  4. $\{f_i\}_{i \in I}$ satisfies same relations as (1)~(3).

- <u>Remark</u>
  - For $a_{ii} = 0$ and $\alpha_i \in \widetilde{\Phi}_{\text{iso}}^+$, we have $e_i^2 = 0$ from (2).

# PBW bases of $U_q^+(sl(m|n))$

- We define two partial orders $O_1, O_2$ on $\widetilde{\Phi}^+$.
  - For $\alpha = \bar{\epsilon}_a - \bar{\epsilon}_b$, $\beta = \bar{\epsilon}_c - \bar{\epsilon}_d \in \widetilde{\Phi}^+$, we define

$$O_1 : \quad \alpha < \beta \quad \Longleftrightarrow \quad a < c \text{ or } (a = c \text{ and } b < d)$$
$$O_2 : \quad \alpha < \beta \quad \Longleftrightarrow \quad a > c \text{ or } (a = c \text{ and } b > d)$$

- <u>Definition</u> (quantum root vector)
  - For $\beta \in \widetilde{\Phi}^+$, we define $e_\beta \in U_q^+(\mathfrak{sl}(m|n))_\beta$ in two ways depending on $O_i$.
  - For $\beta = \alpha_i$, we set $e_\beta = e_i$.
  - For $\beta = \alpha + \alpha_i$ ($\alpha = \bar{\epsilon}_a - \bar{\epsilon}_b \in \widetilde{\Phi}^+$, $a < i$), we set

$$e_\beta = \begin{cases} [e_i, e_\alpha]_q & \text{for } O_1 \quad (\alpha < \alpha_i) \\ [e_\alpha, e_i]_q & \text{for } O_2 \quad (\alpha_i < \alpha) \end{cases}$$

- We consider the case of $O_1$.

$$\bar{\epsilon}_1 - \bar{\epsilon}_2 < \quad \bar{\epsilon}_1 - \bar{\epsilon}_3 \quad < \quad \bar{\epsilon}_1 - \bar{\epsilon}_4 \quad < \bar{\epsilon}_2 - \bar{\epsilon}_3 < \quad \bar{\epsilon}_2 - \bar{\epsilon}_4 \quad < \bar{\epsilon}_3 - \bar{\epsilon}_4$$

■ We consider the case of $O_1$.

$$\bar{\epsilon}_1 - \bar{\epsilon}_2 < \quad \bar{\epsilon}_1 - \bar{\epsilon}_3 \quad < \quad \bar{\epsilon}_1 - \bar{\epsilon}_4 \quad < \bar{\epsilon}_2 - \bar{\epsilon}_3 < \quad \bar{\epsilon}_2 - \bar{\epsilon}_4 \quad < \bar{\epsilon}_3 - \bar{\epsilon}_4$$

$$\alpha_1 \quad < \quad \alpha_1 + \alpha_2 \quad < \quad \alpha_1 + \alpha_2 + \alpha_3 \quad < \quad \alpha_2 \quad < \quad \alpha_2 + \alpha_3 \quad < \quad \alpha_3$$

- We consider the case of $O_1$.

$$\bar{\epsilon}_1 - \bar{\epsilon}_2 < \quad \bar{\epsilon}_1 - \bar{\epsilon}_3 \quad < \quad \bar{\epsilon}_1 - \bar{\epsilon}_4 \quad < \bar{\epsilon}_2 - \bar{\epsilon}_3 < \quad \bar{\epsilon}_2 - \bar{\epsilon}_4 \quad < \bar{\epsilon}_3 - \bar{\epsilon}_4$$

$$\alpha_1 \quad < \quad \alpha_1 + \alpha_2 \quad < \quad \alpha_1 + \alpha_2 + \alpha_3 \quad < \quad \alpha_2 \quad < \quad \alpha_2 + \alpha_3 \quad < \quad \alpha_3$$

$$e_1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad e_2 \qquad\qquad\qquad\qquad\qquad e_3$$

■ We consider the case of $O_1$.

$$\bar{\epsilon}_1 - \bar{\epsilon}_2 < \quad \bar{\epsilon}_1 - \bar{\epsilon}_3 \quad < \quad \bar{\epsilon}_1 - \bar{\epsilon}_4 \quad < \bar{\epsilon}_2 - \bar{\epsilon}_3 < \quad \bar{\epsilon}_2 - \bar{\epsilon}_4 \quad < \bar{\epsilon}_3 - \bar{\epsilon}_4$$

$$\alpha_1 \quad < \quad \alpha_1 + \alpha_2 \quad < \quad \alpha_1 + \alpha_2 + \alpha_3 \quad < \quad \alpha_2 \quad < \quad \alpha_2 + \alpha_3 \quad < \quad \alpha_3$$

$$e_1 \qquad\qquad [e_2, e_1]_q \qquad\qquad\qquad\qquad\qquad\qquad e_2 \qquad\qquad [e_3, e_2]_q \qquad\qquad e_3$$

■ We consider the case of $O_1$.

$$\bar{\epsilon}_1 - \bar{\epsilon}_2 < \quad \bar{\epsilon}_1 - \bar{\epsilon}_3 \quad < \quad \bar{\epsilon}_1 - \bar{\epsilon}_4 \quad < \bar{\epsilon}_2 - \bar{\epsilon}_3 < \quad \bar{\epsilon}_2 - \bar{\epsilon}_4 \quad < \bar{\epsilon}_3 - \bar{\epsilon}_4$$

$$\alpha_1 \quad < \quad \alpha_1 + \alpha_2 \quad < \quad \alpha_1 + \alpha_2 + \alpha_3 \quad < \quad \alpha_2 \quad < \quad \alpha_2 + \alpha_3 \quad < \quad \alpha_3$$

$$e_1 \qquad [e_2, e_1]_q \qquad [e_3, [e_2, e_1]_q]_q \qquad e_2 \qquad [e_3, e_2]_q \qquad e_3$$

- **Theorem** [Yamane94]
  - ☐ Let $\beta_1 < \cdots < \beta_l$ denote elements of $\tilde{\Phi}^+$ under $O_i$     $l = |\tilde{\Phi}^+|$
  - ☐ For $A = (a_1, \cdots, a_l)$ where $a_t$ are given by

    $$a_t \in \mathbb{Z}_{\geq 0} \text{ for } \beta_t \in \tilde{\Phi}^+_{\mathrm{even}} \qquad a_t \in \{0,1\} \text{ for } \beta_t \in \tilde{\Phi}^+_{\mathrm{iso}}$$

    we define $E_i^A$ by

    $$E_i^A = e_{\beta_1}^{(a_1)} e_{\beta_2}^{(a_2)} \cdots e_{\beta_l}^{(a_l)}$$

  - ☐ Then

    $$B_i = \{ E_i^A \mid a_t \in \mathbb{Z}_{\geq 0} \ (\beta_t \in \tilde{\Phi}^+_{\mathrm{even}}), \ a_t \in \{0,1\} \ (\beta_t \in \tilde{\Phi}^+_{\mathrm{iso}}) \}$$

    gives a basis of $U_q^+$.

  - ☐ $e_{\beta_t}^{(a_t)}$: divided power given by $e_{\beta_t}^{(a_t)} = e_{\beta_t}^{a_t} / [a_t]_{p_t}!$    $p_t = q^{d_{\beta_t}}$
  - ☐ $[k]_q!$: factorial of $q$-number given by

    $$[m]_q! = \prod_{k=1}^m [k]_q \qquad [k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}$$

# Transition matrix of PBW bases

- We define the transition matrix $\gamma$ of PBW bases as follows:

$$E_2^A = \sum_B \gamma_B^A E_1^{B^{\mathrm{op}}}$$

  □ Here, we set $X^{\mathrm{op}} = (x_l, \cdots, x_1)$ for $X = (x_1, \cdots, x_l)$.

- From now on, we only consider the cases of rank 2 & 3.
  □ Transition matrices for higher rank cases are constructed as compositions of ones of rank 2.
  □ For rank 3 cases, compositions take the form of the left/right hand side of the tetrahedron equation.

# Outline

- We set $e_{ij} = [e_i, e_j]_q$

- Quantum root vectors of rank 2

$$
\begin{aligned}
B_1 : \quad & e_{\beta_1} = e_1, \quad e_{\beta_2} = e_{21}, \quad e_{\beta_3} = e_2 \\
B_2 : \quad & e_{\beta_1} = e_2, \quad e_{\beta_2} = e_{12}, \quad e_{\beta_3} = e_1
\end{aligned}
\qquad \beta_1 < \beta_2 < \beta_3
$$

- Transition matrix

$$
e_2^{(a)} e_{12}^{(b)} e_1^{(c)} = \sum_{i,j,k} \gamma_{i,j,k}^{a,b,c} e_1^{(k)} e_{21}^{(j)} e_2^{(i)}
$$

- Dynkin diagrams of rank 2

# The case ◯———◯

$$\epsilon_1 - \epsilon_2 \qquad \epsilon_2 - \epsilon_3$$

◯————————◯
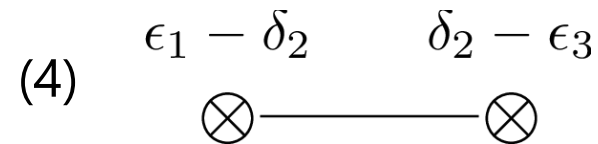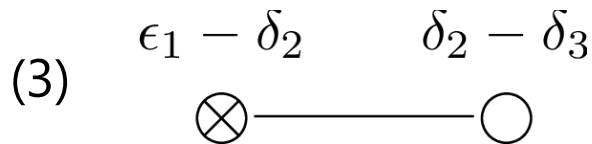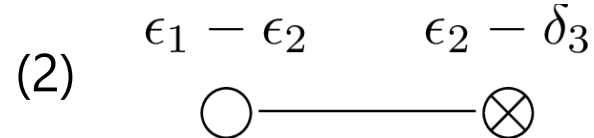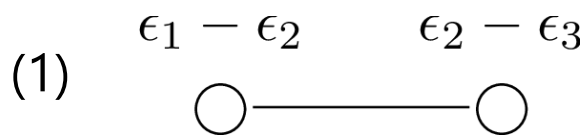
$$(\epsilon_i, \epsilon_i) = 1 \qquad DA = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

- Root system

$$\tilde{\Phi}^+_{\text{even}} = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} \quad \tilde{\Phi}^+_{\text{iso}} = \{\}$$

- Transition matrix

$$e_2^{(a)} e_{12}^{(b)} e_1^{(c)} = \sum_{i,j,k} \gamma_{i,j,k}^{a,b,c} e_1^{(k)} e_{21}^{(j)} e_2^{(i)} \quad \cdots (*) \quad i, j, k, a, b, c \in \mathbb{Z}_{\geq 0}$$

- <u>Theorem</u> [Kuniba-Okado-Yamada13] $\quad \gamma_{i,j,k}^{a,b,c} = \mathcal{R}_{i,j,k}^{a,b,c}$

- <u>Example</u> For $(a, b, c) = (0,1,1)$, $(*)$ becomes

$$e_{12} e_1 = \mathcal{R}_{0,1,1}^{0,1,1} e_1 e_{21} + \mathcal{R}_{1,0,2}^{0,1,1} \frac{e_1^2}{q + q^{-1}} e_2$$

$$-q e_2 e_1^2 + (1 + q^2) e_1 e_2 e_1 - q e_1^2 e_2 = 0 \qquad \because \mathcal{R}_{0,1,1}^{0,1,1} = -q^2, \ \mathcal{R}_{1,0,2}^{0,1,1} = 1 - q^4$$

This is a relation of ◯———◯.

$$\epsilon_1 - \epsilon_2 \qquad \epsilon_2 - \delta_3 \qquad (\epsilon_i, \epsilon_i) = 1 \qquad DA = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$$
$$\underset{\bigcirc}{\phantom{a}}\text{———}\underset{\otimes}{\phantom{a}} \qquad (\delta_i, \delta_i) = -1$$

- Root system

$$\tilde{\Phi}^+_{\text{even}} = \{\alpha_1\} \quad \tilde{\Phi}^+_{\text{iso}} = \{\alpha_2, \alpha_1 + \alpha_2\}$$

- Transition matrix

$$e_2^a e_{12}^b e_1^{(c)} = \sum_{i,j,k} \gamma_{i,j,k}^{a,b,c} e_1^{(k)} e_{21}^j e_2^i \quad \cdots (*) \qquad \begin{array}{l} k, c \in \mathbb{Z}_{\geq 0} \\ i, j, a, b \in \{0, 1\} \end{array}$$

- <u>Theorem</u> [Y20] $\quad \gamma_{i,j,k}^{a,b,c} = \mathcal{L}_{i,j,k}^{a,b,c}$

- <u>Example</u> For $(a, b, c) = (0,1,1)$, $(*)$ becomes

$$e_{12}e_1 = \mathcal{L}_{0,1,1}^{0,1,1} e_1 e_{21} + \mathcal{L}_{1,0,2}^{0,1,1} \frac{e_1^2}{q + q^{-1}} e_2$$

$$-qe_2 e_1^2 + (1 + q^2)e_1 e_2 e_1 - q e_1^2 e_2 = 0 \qquad \because \mathcal{L}_{0,1,1}^{0,1,1} = -q^2, \ \mathcal{L}_{1,0,2}^{0,1,1} = 1 - q^4$$

This is a relation of ◯———⊗.

# The case $\otimes\!\!-\!\!-\!\!\bigcirc$

$$\epsilon_1 - \delta_2 \qquad \delta_2 - \delta_3 \qquad (\epsilon_i, \epsilon_i) = -1 \qquad DA = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$$
$$\otimes\!\!-\!\!-\!\!\bigcirc \qquad (\delta_i, \delta_i) = 1$$

- Root system

$$\tilde{\Phi}^+_{\text{even}} = \{\alpha_2\} \quad \tilde{\Phi}^+_{\text{iso}} = \{\alpha_1, \alpha_1 + \alpha_2\}$$

- Transition matrix

$$e_2^{(a)} e_{12}^b e_1^c = \sum_{i,j,k} \gamma_{i,j,k}^{a,b,c} e_1^k e_{21}^j e_2^{(i)} \qquad \cdots (*) \qquad \begin{array}{l} i, a \in \mathbb{Z}_{\geq 0} \\ j, k, b, c \in \{0, 1\} \end{array}$$

- <u>Corollary</u>   $\gamma_{i,j,k}^{a,b,c} = \mathcal{M}_{i,j,k}^{a,b,c}$

  □ Here, we define $\mathcal{M} \in \mathrm{End}\,(F \otimes V \otimes V)$ by $\mathcal{M}_{i,j,k}^{a,b,c} = \mathcal{L}_{k,j,i}^{c,b,a}$.

# The case $\otimes\!\!-\!\!\otimes$

$$\epsilon_1 - \delta_2 \qquad \delta_2 - \epsilon_3 \qquad (\epsilon_i, \epsilon_i) = -1 \qquad DA = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$
$$\otimes\!\!-\!\!\otimes \qquad\qquad (\delta_i, \delta_i) = 1$$

- Root system

$$\tilde{\Phi}^+_{\text{even}} = \{\alpha_1 + \alpha_2\} \quad \tilde{\Phi}^+_{\text{iso}} = \{\alpha_1, \alpha_2\}$$

- Transition matrix

$$e_2^a e_{12}^{(b)} e_1^c = \sum_{i,j,k} \gamma_{i,j,k}^{a,b,c} e_1^k e_{21}^{(j)} e_2^i \qquad\qquad j, b \in \mathbb{Z}_{\geq 0}$$
$$i, k, a, c \in \{0, 1\}$$

- <u>Theorem</u> [Y20] $\quad \gamma_{i,j,k}^{a,b,c} = \mathcal{N}_{i,j,k}^{a,b,c}$
  - □ Here, we define $\mathcal{N} \in \mathrm{End}\,(V \otimes F \otimes V)$ as follows:

$$\mathcal{N}(u_i \otimes |j\rangle \otimes u_k) = \sum_{a,c \in \{0,1\}, b \in \mathbb{Z}_{\geq 0}} \mathcal{N}_{i,j,k}^{a,b,c} u_a \otimes |b\rangle \otimes u_c$$

$$\mathcal{N}_{0,j,0}^{0,b,0} = \delta_{j,b} q^j, \quad \mathcal{N}_{1,j,1}^{1,b,1} = -\delta_{j,b} q^{j+1}, \quad \mathcal{N}_{0,j,1}^{0,b,1} = \mathcal{N}_{1,j,0}^{1,b,0} = \delta_{j,b},$$

$$\mathcal{N}_{1,j,1}^{0,b,0} = \delta_{j+1,b} q^j (1 - q^2), \quad \mathcal{N}_{0,j,0}^{1,b,1} = \delta_{j-1,b} [j]_q,$$

# Outline

# Type A of rank 3 cases

- We set $e_{(ij)k}, e_{i(jk)} \in U_q^+(\mathfrak{sl}(m|n))$ as follows:

$$e_{(ij)k} = [e_{ij}, e_k]_q, \quad e_{i(jk)} = [e_i, e_{jk}]_q \qquad e_{ij} = [e_i, e_j]_q$$

  - □ For $|i - k| > 1$, we have $e_{(ij)k} = e_{i(jk)} =: e_{ijk}$.

- Quantum root vectors of rank 3

$$B_1 : \quad e_{\beta_1} = e_1, \quad e_{\beta_2} = e_{21}, \quad e_{\beta_3} = e_{321},$$
$$e_{\beta_4} = e_2, \quad e_{\beta_5} = e_{32}, \quad e_{\beta_6} = e_3$$

$$\beta_1 < \cdots < \beta_6$$

$$B_2 : \quad e_{\beta_1} = e_3, \quad e_{\beta_2} = e_{23}, \quad e_{\beta_3} = e_2,$$
$$e_{\beta_4} = e_{123}, \quad e_{\beta_5} = e_{12}, \quad e_{\beta_6} = e_1$$

- Transition matrix

$$e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)} = \sum_{i_1,i_2,i_3,i_4,i_5,i_6} \gamma_{i_1,i_2,i_3,i_4,i_5,i_6}^{o_1,o_2,o_3,o_4,o_5,o_6} e_1^{(i_6)} e_{21}^{(i_5)} e_{321}^{(i_4)} e_2^{(i_3)} e_{32}^{(i_2)} e_3^{(i_1)}$$

■ We use the following matrices:

$$e_2^{(a)} e_{12}^{(b)} e_1^{(c)} = \sum_{i,j,k} \Gamma^{(2|1)\,a,b,c}_{\qquad i,j,k} e_1^{(k)} e_{21}^{(j)} e_2^{(i)}$$

$$e_3^{(a)} e_{23}^{(b)} e_2^{(c)} = \sum_{i,j,k} \Gamma^{(3|2)\,a,b,c}_{\qquad i,j,k} e_2^{(k)} e_{32}^{(j)} e_3^{(i)}$$

$$e_{23}^{(a)} e_{123}^{(b)} e_1^{(c)} = \sum_{i,j,k} \Gamma^{(23|1)\,a,b,c}_{\qquad i,j,k} e_1^{(k)} e_{(23)1}^{(j)} e_{23}^{(i)}$$

$$e_{32}^{(a)} e_{1(32)}^{(b)} e_1^{(c)} = \sum_{i,j,k} \Gamma^{(32|1)\,a,b,c}_{\qquad i,j,k} e_1^{(k)} e_{321}^{(j)} e_{32}^{(i)}$$

$$e_3^{(a)} e_{123}^{(b)} e_{12}^{(c)} = \sum_{i,j,k} \Gamma^{(3|12)\,a,b,c}_{\qquad i,j,k} e_{12}^{(k)} e_{3(12)}^{(j)} e_3^{(i)}$$

$$e_3^{(a)} e_{(21)3}^{(b)} e_{21}^{(c)} = \sum_{i,j,k} \Gamma^{(3|21)\,a,b,c}_{\qquad i,j,k} e_{21}^{(k)} e_{321}^{(j)} e_3^{(i)}$$

☐ Given a Dynkin diagram, $\Gamma^{(x)}$ is specified as the 3D R, L, M or N.

$$e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}$$

$$e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}$$

$$\underline{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}}$$
$$= \sum \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3} e_2^{(x_3)} e_{32}^{(x_2)} e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}$$

■ For the underlined part, we used

$$e_3^{(a)} e_{23}^{(b)} e_2^{(c)} = \sum_{i,j,k} \Gamma^{(3|2)}{}^{a,b,c}_{i,j,k} e_2^{(k)} e_{32}^{(j)} e_3^{(i)}$$

$$\underline{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)}} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}$$
$$= \sum \Gamma^{(3|2) o_1, o_2, o_3}_{\quad\ \ x_1, x_2, x_3} e_2^{(x_3)} e_{32}^{(x_2)} \underline{e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)}} e_1^{(o_6)}$$

$$\underline{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)}} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}$$

$$= \sum \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3} e_2^{(x_3)} e_{32}^{(x_2)} \underline{e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)}} e_1^{(o_6)}$$

$$= \sum \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3} \Gamma^{(3|12)}{}^{x_1,o_4,o_5}_{i_1,x_4,x_5} e_2^{(x_3)} e_{32}^{(x_2)} e_{12}^{(x_5)} e_{3(12)}^{(x_4)} e_3^{(i_1)} e_1^{(o_6)}$$

■ For the underlined part, we used

$$e_3^{(a)} e_{123}^{(b)} e_{12}^{(c)} = \sum_{i,j,k} \Gamma^{(3|12)}{}^{a,b,c}_{i,j,k} e_{12}^{(k)} e_{3(12)}^{(j)} e_3^{(i)}$$

$$e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}$$

$$= \sum \Gamma^{(3|2)\, o_1,o_2,o_3}_{\phantom{(3|2)}\, x_1,x_2,x_3} e_2^{(x_3)} e_{32}^{(x_2)} \underline{e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)}} e_1^{(o_6)}$$

$$= \sum \Gamma^{(3|2)\, o_1,o_2,o_3}_{\phantom{(3|2)}\, x_1,x_2,x_3} \Gamma^{(3|12)\, x_1,o_4,o_5}_{\phantom{(3|12)}\, i_1,x_4,x_5} e_2^{(x_3)} \underline{e_{32}^{(x_2)} e_{12}^{(x_5)}}\ \underline{e_{3(12)}^{(x_4)}}\ \underline{e_3^{(i_1)} e_1^{(o_6)}}$$

$$\rho_1 = p(\alpha_1)p(\alpha_3)$$
$$\rho_2 = p(\alpha_1 + \alpha_2)p(\alpha_2 + \alpha_3)$$

$$\underline{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}}$$

$$= \sum \Gamma^{(3|2)\, o_1, o_2, o_3}_{\phantom{(3|2)}\, x_1, x_2, x_3} e_2^{(x_3)} e_{32}^{(x_2)} \underline{e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)}} e_1^{(o_6)}$$

$$= \sum \Gamma^{(3|2)\, o_1, o_2, o_3}_{\phantom{(3|2)}\, x_1, x_2, x_3} \Gamma^{(3|12)\, x_1, o_4, o_5}_{\phantom{(3|12)}\, i_1, x_4, x_5} e_2^{(x_3)} \underline{e_{32}^{(x_2)} e_{12}^{(x_5)}} \; \underline{e_{3(12)}^{(x_4)}} \; \underline{e_3^{(i_1)} e_1^{(o_6)}}$$

$$= \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5} \Gamma^{(3|2)\, o_1, o_2, o_3}_{\phantom{(3|2)}\, x_1, x_2, x_3} \Gamma^{(3|12)\, x_1, o_4, o_5}_{\phantom{(3|12)}\, i_1, x_4, x_5}$$

$$\times \; e_2^{(x_3)} e_{12}^{(x_5)} e_{32}^{(x_2)} e_{1(32)}^{(x_4)} e_1^{(o_6)} e_3^{(i_1)}$$

■ For the underlined part, we used

$$[e_{32}, e_{12}] = [e_3, e_1] = 0 \qquad e_{3(12)} = (-1)^{p(\alpha_1)p(\alpha_3)} e_{1(32)}$$

$$\rho_1 = p(\alpha_1)p(\alpha_3)$$
$$\rho_2 = p(\alpha_1 + \alpha_2)p(\alpha_2 + \alpha_3)$$

$$\underline{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}}$$

$$= \sum \Gamma^{(3|2)\, o_1, o_2, o_3}_{\quad\quad x_1, x_2, x_3} e_2^{(x_3)} e_{32}^{(x_2)} \underline{e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)}} e_1^{(o_6)}$$

$$= \sum \Gamma^{(3|2)\, o_1, o_2, o_3}_{\quad\quad x_1, x_2, x_3} \Gamma^{(3|12)\, x_1, o_4, o_5}_{\quad\quad i_1, x_4, x_5} e_2^{(x_3)} \underline{e_{32}^{(x_2)} e_{12}^{(x_5)}} \; \underline{e_{3(12)}^{(x_4)}} \; \underline{e_3^{(i_1)} e_1^{(o_6)}}$$

$$= \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5} \Gamma^{(3|2)\, o_1, o_2, o_3}_{\quad\quad x_1, x_2, x_3} \Gamma^{(3|12)\, x_1, o_4, o_5}_{\quad\quad i_1, x_4, x_5}$$

$$\times e_2^{(x_3)} e_{12}^{(x_5)} \underline{e_{32}^{(x_2)} e_{1(32)}^{(x_4)} e_1^{(o_6)}} e_3^{(i_1)}$$

$$\rho_1 = p(\alpha_1)p(\alpha_3)$$
$$\rho_2 = p(\alpha_1 + \alpha_2)p(\alpha_2 + \alpha_3)$$

$$\underline{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}}$$

$$= \sum \Gamma^{(3|2)\,o_1,o_2,o_3}_{\quad\quad x_1,x_2,x_3} e_2^{(x_3)} e_{32}^{(x_2)} \underline{e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}}$$

$$= \sum \Gamma^{(3|2)\,o_1,o_2,o_3}_{\quad\quad x_1,x_2,x_3} \Gamma^{(3|12)\,x_1,o_4,o_5}_{\quad\quad\quad i_1,x_4,x_5} e_2^{(x_3)} \underline{e_{32}^{(x_2)} e_{12}^{(x_5)}} \underline{e_{3(12)}^{(x_4)}} \underline{e_3^{(i_1)} e_1^{(o_6)}}$$

$$= \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5} \Gamma^{(3|2)\,o_1,o_2,o_3}_{\quad\quad x_1,x_2,x_3} \Gamma^{(3|12)\,x_1,o_4,o_5}_{\quad\quad\quad i_1,x_4,x_5}$$

$$\times\, e_2^{(x_3)} e_{12}^{(x_5)} \underline{e_{32}^{(x_2)} e_{1(32)}^{(x_4)} e_1^{(o_6)}} e_3^{(i_1)}$$

$$= \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5} \Gamma^{(3|2)\,o_1,o_2,o_3}_{\quad\quad x_1,x_2,x_3} \Gamma^{(3|12)\,x_1,o_4,o_5}_{\quad\quad\quad i_1,x_4,x_5} \Gamma^{(32|1)\,x_2,x_4,o_6}_{\quad\quad\quad i_2,i_4,x_6}$$

$$\times\, e_2^{(x_3)} e_{12}^{(x_5)} e_1^{(x_6)} e_{321}^{(i_4)} e_{32}^{(i_2)} e_3^{(i_1)}$$

- ■ For the underlined part, we used

$$e_{32}^{(a)} e_{1(32)}^{(b)} e_1^{(c)} = \sum_{i,j,k} \Gamma^{(32|1)\,a,b,c}_{\quad\quad i,j,k} e_1^{(k)} e_{321}^{(j)} e_{32}^{(i)}$$

# Two ways construction of $\gamma$: first way

$$\rho_1 = p(\alpha_1)p(\alpha_3)$$
$$\rho_2 = p(\alpha_1 + \alpha_2)p(\alpha_2 + \alpha_3)$$
$$\rho_3 = p(\alpha_2)p(\alpha_1 + \alpha_2 + \alpha_3)$$

$$\underline{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)}} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}$$

$$= \sum \Gamma^{(3|2)\,o_1,o_2,o_3}_{\phantom{(3|2)}x_1,x_2,x_3} e_2^{(x_3)} e_{32}^{(x_2)} \underline{e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)}} e_1^{(o_6)}$$

$$= \sum \Gamma^{(3|2)\,o_1,o_2,o_3}_{\phantom{(3|2)}x_1,x_2,x_3} \Gamma^{(3|12)\,x_1,o_4,o_5}_{\phantom{(3|12)}i_1,x_4,x_5} e_2^{(x_3)} \underline{e_{32}^{(x_2)} e_{12}^{(x_5)}} \; \underline{e_{3(12)}^{(x_4)}} \; \underline{e_3^{(i_1)} e_1^{(o_6)}}$$

$$= \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5} \Gamma^{(3|2)\,o_1,o_2,o_3}_{\phantom{(3|2)}x_1,x_2,x_3} \Gamma^{(3|12)\,x_1,o_4,o_5}_{\phantom{(3|12)}i_1,x_4,x_5}$$

$$\times \; e_2^{(x_3)} e_{12}^{(x_5)} \underline{e_{32}^{(x_2)} e_{1(32)}^{(x_4)} e_1^{(o_6)}} e_3^{(i_1)}$$

$$= \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5} \Gamma^{(3|2)\,o_1,o_2,o_3}_{\phantom{(3|2)}x_1,x_2,x_3} \Gamma^{(3|12)\,x_1,o_4,o_5}_{\phantom{(3|12)}i_1,x_4,x_5} \Gamma^{(32|1)\,x_2,x_4,o_6}_{\phantom{(32|1)}i_2,i_4,x_6}$$

$$\times \; \underline{e_2^{(x_3)} e_{12}^{(x_5)} e_1^{(x_6)}} e_{321}^{(i_4)} e_{32}^{(i_2)} e_3^{(i_1)}$$

$$= \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5} \Gamma^{(3|2)\,o_1,o_2,o_3}_{\phantom{(3|2)}x_1,x_2,x_3} \Gamma^{(3|12)\,x_1,o_4,o_5}_{\phantom{(3|12)}i_1,x_4,x_5} \Gamma^{(32|1)\,x_2,x_4,o_6}_{\phantom{(32|1)}i_2,i_4,x_6} \Gamma^{(2|1)\,x_3,x_5,x_6}_{\phantom{(2|1)}i_3,i_5,i_6}$$

$$\times \; e_1^{(i_6)} e_{21}^{(i_5)} \underline{e_2^{(i_3)} e_{321}^{(i_4)}} e_{32}^{(i_2)} e_3^{(i_1)}$$

$$= \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 i_3 i_4} \Gamma^{(3|2)\,o_1,o_2,o_3}_{\phantom{(3|2)}x_1,x_2,x_3} \Gamma^{(3|12)\,x_1,o_4,o_5}_{\phantom{(3|12)}i_1,x_4,x_5} \Gamma^{(32|1)\,x_2,x_4,o_6}_{\phantom{(32|1)}i_2,i_4,x_6} \Gamma^{(2|1)\,x_3,x_5,x_6}_{\phantom{(2|1)}i_3,i_5,i_6}$$

$$\times \; e_1^{(i_6)} e_{21}^{(i_5)} e_{321}^{(i_4)} e_2^{(i_3)} e_{32}^{(i_2)} e_3^{(i_1)}$$

$$\rho_1 = p(\alpha_1)p(\alpha_3)$$
$$\rho_2 = p(\alpha_1 + \alpha_2)p(\alpha_2 + \alpha_3)$$
$$\rho_3 = p(\alpha_2)p(\alpha_1 + \alpha_2 + \alpha_3)$$

$$e_3^{(o_1)} e_{23}^{(o_2)} \underline{e_2^{(o_3)} e_{123}^{(o_4)}} e_{12}^{(o_5)} e_1^{(o_6)}$$

$$= (-1)^{\rho_3 o_3 o_4} e_3^{(o_1)} e_{23}^{(o_2)} e_{123}^{(o_4)} \underline{e_2^{(o_3)} e_{12}^{(o_5)} e_1^{(o_6)}}$$

$$= \sum (-1)^{\rho_3 o_3 o_4} \Gamma^{(2|1)\,o_3,o_5,o_6}_{\quad\ x_3,x_5,x_6} e_3^{(o_1)} \underline{e_{23}^{(o_2)} e_{123}^{(o_4)} e_1^{(x_6)}} e_{21}^{(x_5)} e_2^{(x_3)}$$

$$= \sum (-1)^{\rho_3 o_3 o_4} \Gamma^{(2|1)\,o_3,o_5,o_6}_{\quad\ x_3,x_5,x_6} \Gamma^{(23|1)\,o_2,o_4,x_6}_{\quad\ x_2,x_4,i_6} \underline{e_3^{(o_1)} e_1^{(i_6)}} \ \underline{e_{(23)1}^{(x_4)}} \ \underline{e_{23}^{(x_2)} e_{21}^{(x_5)}} e_2^{(x_3)}$$

$$= \sum (-1)^{\rho_1(o_1 i_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 o_3 o_4} \Gamma^{(2|1)\,o_3,o_5,o_6}_{\quad\ x_3,x_5,x_6} \Gamma^{(23|1)\,o_2,o_4,x_6}_{\quad\ x_2,x_4,i_6}$$
$$\times\ e_1^{(i_6)} \underline{e_3^{(o_1)} e_{(21)3}^{(x_4)} e_{21}^{(x_5)}} e_{23}^{(x_2)} e_2^{(x_3)}$$

$$= \sum (-1)^{\rho_1(o_1 i_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 o_3 o_4} \Gamma^{(2|1)\,o_3,o_5,o_6}_{\quad\ x_3,x_5,x_6} \Gamma^{(23|1)\,o_2,o_4,x_6}_{\quad\ x_2,x_4,i_6} \Gamma^{(3|21)\,o_1,x_4,x_5}_{\quad\ x_1,i_4,i_5}$$
$$\times\ e_1^{(i_6)} e_{21}^{(i_5)} e_{321}^{(i_4)} \underline{e_3^{(x_1)} e_{23}^{(x_2)} e_2^{(x_3)}}$$

$$= \sum (-1)^{\rho_1(o_1 i_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 o_3 o_4} \Gamma^{(2|1)\,o_3,o_5,o_6}_{\quad\ x_3,x_5,x_6} \Gamma^{(23|1)\,o_2,o_4,x_6}_{\quad\ x_2,x_4,i_6} \Gamma^{(3|21)\,o_1,x_4,x_5}_{\quad\ x_1,i_4,i_5} \Gamma^{(3|2)\,x_1,x_2,x_3}_{\quad\ i_1,i_2,i_3}$$
$$\times\ e_1^{(i_6)} e_{21}^{(i_5)} e_{321}^{(i_4)} e_2^{(i_3)} e_{23}^{(i_2)} e_3^{(i_1)}$$

# Mother of tetrahedron equation

- Now, $\{e_1^{(i_6)} e_{21}^{(i_5)} e_{321}^{(i_4)} e_2^{(i_3)} e_{23}^{(i_2)} e_3^{(i_1)}\}$ is linearly independent. Then, we have the following result.

- <u>Theorem</u> [Y20]

$$\sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 i_3 i_4}$$

$$\times \Gamma^{(3|2)\, o_1, o_2, o_3}_{\phantom{(3|2)}\, x_1, x_2, x_3} \Gamma^{(3|12)\, x_1, o_4, o_5}_{\phantom{(3|12)}\, i_1, x_4, x_5} \Gamma^{(32|1)\, x_2, x_4, o_6}_{\phantom{(32|1)}\, i_2, i_4, x_6} \Gamma^{(2|1)\, x_3, x_5, x_6}_{\phantom{(2|1)}\, i_3, i_5, i_6}$$

$$= \sum (-1)^{\rho_1(o_1 i_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 o_3 o_4}$$

$$\times \Gamma^{(2|1)\, o_3, o_5, o_6}_{\phantom{(2|1)}\, x_3, x_5, x_6} \Gamma^{(23|1)\, o_2, o_4, x_6}_{\phantom{(23|1)}\, x_2, x_4, i_6} \Gamma^{(3|21)\, o_1, x_4, x_5}_{\phantom{(3|21)}\, x_1, i_4, i_5} \Gamma^{(3|2)\, x_1, x_2, x_3}_{\phantom{(3|2)}\, i_1, i_2, i_3}$$

  - Here, sign factors are given by

$$\rho_1 = p(\alpha_1)p(\alpha_3), \quad \rho_2 = p(\alpha_1 + \alpha_2)p(\alpha_2 + \alpha_3), \quad \rho_3 = p(\alpha_2)p(\alpha_1 + \alpha_2 + \alpha_3)$$

■ Without exchanges $\epsilon \leftrightarrow \delta$, all Dynkin diagrams of rank 3 are given by

(1)
$$\epsilon_1 - \epsilon_2 \qquad \epsilon_2 - \epsilon_3 \qquad \epsilon_3 - \epsilon_4$$
○————○————○

(2)
$$\epsilon_1 - \epsilon_2 \qquad \epsilon_2 - \epsilon_3 \qquad \epsilon_3 - \delta_4$$
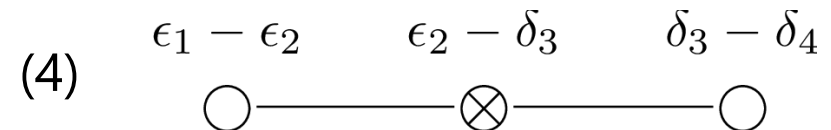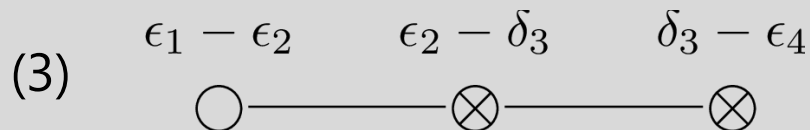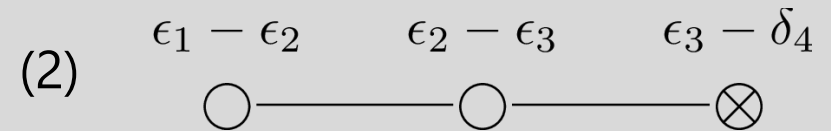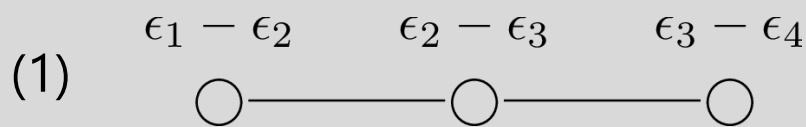○————○————⊗

(3)
$$\epsilon_1 - \epsilon_2 \qquad \epsilon_2 - \delta_3 \qquad \delta_3 - \epsilon_4$$
○————⊗————⊗

(4)
$$\epsilon_1 - \epsilon_2 \qquad \epsilon_2 - \delta_3 \qquad \delta_3 - \delta_4$$
○————⊗————○

(5)
$$\epsilon_1 - \delta_2 \qquad \delta_2 - \delta_3 \qquad \delta_3 - \epsilon_4$$
⊗————○————⊗

(6)
$$\epsilon_1 - \delta_2 \qquad \delta_2 - \epsilon_3 \qquad \epsilon_3 - \delta_4$$
⊗————⊗————⊗

■ The followings are easily attributed to (2) and (3), respectively.

(7)
$$\epsilon_1 - \delta_2 \qquad \delta_2 - \delta_3 \qquad \delta_3 - \delta_4$$
⊗————○————○

(8)
$$\epsilon_1 - \delta_2 \qquad \delta_2 - \epsilon_3 \qquad \epsilon_3 - \epsilon_4$$
⊗————⊗————○

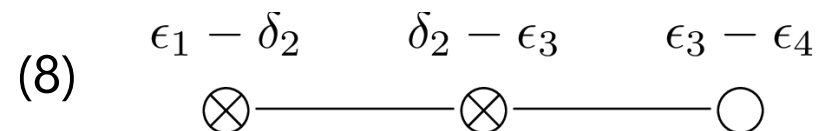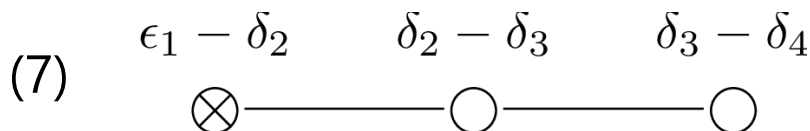■ For (1),(2),(3), we obtain tetrahedron eq because $\rho_1 = \rho_2 = \rho_3 = 0$.

# Dynkin diagrams of type A of rank 3

- Without exchanges $\epsilon \leftrightarrow \delta$, all Dynkin diagrams of rank 3 are given by

(1)
$$\epsilon_1 - \epsilon_2 \qquad \epsilon_2 - \epsilon_3 \qquad \epsilon_3 - \epsilon_4$$
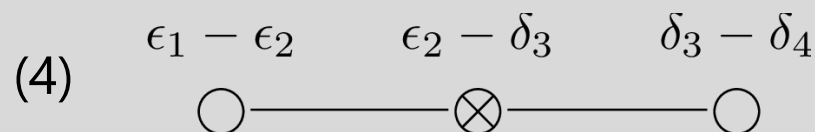○————————○————————○

(2)
$$\epsilon_1 - \epsilon_2 \qquad \epsilon_2 - \epsilon_3 \qquad \epsilon_3 - \delta_4$$
○————————○————————⊗

(3)
$$\epsilon_1 - \epsilon_2 \qquad \epsilon_2 - \delta_3 \qquad \delta_3 - \epsilon_4$$
○————————⊗————————⊗

(4)
$$\epsilon_1 - \epsilon_2 \qquad \epsilon_2 - \delta_3 \qquad \delta_3 - \delta_4$$
○————————⊗————————○

(5)
$$\epsilon_1 - \delta_2 \qquad \delta_2 - \delta_3 \qquad \delta_3 - \epsilon_4$$
⊗————————○————————⊗

(6)
$$\epsilon_1 - \delta_2 \qquad \delta_2 - \epsilon_3 \qquad \epsilon_3 - \delta_4$$
⊗————————⊗————————⊗
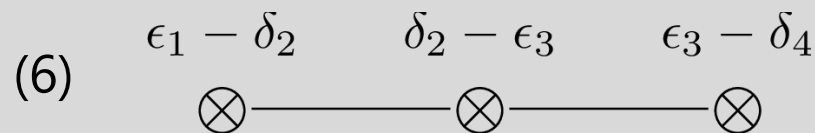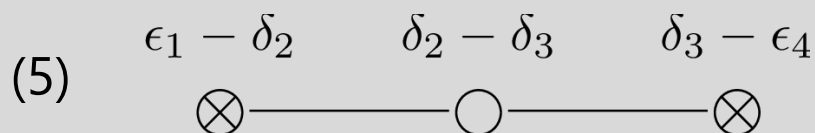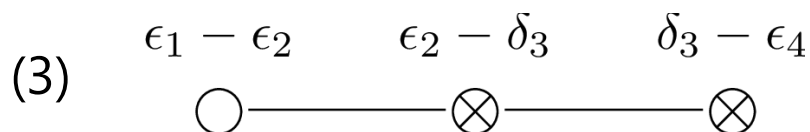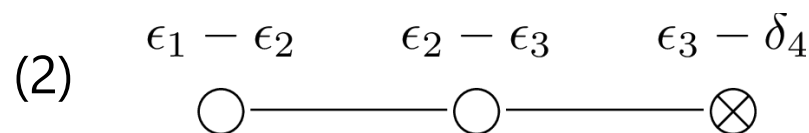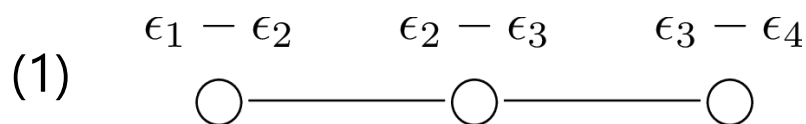
- For (4),(5),(6), some $\rho_i$ are non-zero. Then, associated equations become the tetrahedron equation *up to sign factors*.

- Here, we only consider (4) for them.

$$\epsilon_1 - \epsilon_2 \qquad \epsilon_2 - \epsilon_3 \qquad \epsilon_3 - \epsilon_4$$

◯———————◯————————◯

$$(\epsilon_i, \epsilon_i) = 1 \qquad DA = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

- **Root system**

$$\tilde{\Phi}^+_{\text{even}} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\} \quad \tilde{\Phi}^+_{\text{iso}} = \{\}$$

- <u>**Lemma**</u>

$$\Gamma^{(2|1)} = \Gamma^{(3|2)} = \Gamma^{(23|1)} = \Gamma^{(32|1)} = \Gamma^{(3|12)} = \Gamma^{(3|21)} = \mathcal{R}$$

- <u>Proof</u> (The case $\Gamma^{(23|1)}$)

  □ $h\colon$ ◯———◯ → ◯———◯———◯ defined by $e_1 \mapsto e_1$, $e_2 \mapsto e_{23}$ is an algebra hom by higher-order relations:

  $$e_1^2 e_{23} - (q + q^{-1})e_1 e_{23} e_1 + e_{23} e_1^2 = e_{23}^2 e_1 - (q + q^{-1})e_{23} e_1 e_{23} + e_1 e_{23}^2 = 0$$

  □ $[m]_{q^{d\alpha_2 + \alpha_3}}! = [m]_{q^{d\alpha_2}}!$ and $[m]_{q^{d\alpha_1 + \alpha_2 + \alpha_3}}! = [m]_{q^{d\alpha_1 + \alpha_2}}!$ hold.

  □ Then we have

  $$e_2^{(a)} e_{12}^{(b)} e_1^{(c)} = \sum_{i,j,k} \mathcal{R}_{i,j,k}^{a,b,c} e_1^{(k)} e_{21}^{(j)} e_2^{(i)} \mapsto e_{23}^{(a)} e_{123}^{(b)} e_1^{(c)} = \sum_{i,j,k} \mathcal{R}_{i,j,k}^{a,b,c} e_1^{(k)} e_{(23)1}^{(j)} e_{23}^{(i)}$$

# The case $\bigcirc\!\!-\!\!\!-\!\!\bigcirc\!\!-\!\!\!-\!\!\bigcirc$

$$\rho_1 = p(\alpha_1)p(\alpha_3)$$
$$\rho_2 = p(\alpha_1 + \alpha_2)p(\alpha_2 + \alpha_3)$$
$$\rho_3 = p(\alpha_2)p(\alpha_1 + \alpha_2 + \alpha_3)$$

■ <u>Previous Theorem</u> [Y20]

$$\sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 i_3 i_4}$$

$$\times \Gamma^{(3|2)\,o_1,o_2,o_3}_{\quad x_1,x_2,x_3} \Gamma^{(3|12)\,x_1,o_4,o_5}_{\quad i_1,x_4,x_5} \Gamma^{(32|1)\,x_2,x_4,o_6}_{\quad i_2,i_4,x_6} \Gamma^{(2|1)\,x_3,x_5,x_6}_{\quad i_3,i_5,i_6}$$

$$= \sum (-1)^{\rho_1(o_1 i_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 o_3 o_4}$$

$$\times \Gamma^{(2|1)\,o_3,o_5,o_6}_{\quad x_3,x_5,x_6} \Gamma^{(23|1)\,o_2,o_4,x_6}_{\quad x_2,x_4,i_6} \Gamma^{(3|21)\,o_1,x_4,x_5}_{\quad x_1,i_4,i_5} \Gamma^{(3|2)\,x_1,x_2,x_3}_{\quad i_1,i_2,i_3}$$

■ <u>Previous Lemma</u>

$$\Gamma^{(2|1)} = \Gamma^{(3|2)} = \Gamma^{(23|1)} = \Gamma^{(32|1)} = \Gamma^{(3|12)} = \Gamma^{(3|21)} = \mathcal{R}$$

■ The theorem is specialized as follows:

$$\sum \mathcal{R}^{o_1,o_2,o_3}_{x_1,x_2,x_3} \mathcal{R}^{x_1,o_4,o_5}_{i_1,x_4,x_5} \mathcal{R}^{x_2,x_4,o_6}_{i_2,i_4,x_6} \mathcal{R}^{x_3,x_5,x_6}_{i_3,i_5,i_6} = \sum \mathcal{R}^{o_3,o_5,o_6}_{x_3,x_5,x_6} \mathcal{R}^{o_2,o_4,x_6}_{x_2,x_4,i_6} \mathcal{R}^{o_1,x_4,x_5}_{x_1,i_4,i_5} \mathcal{R}^{x_1,x_2,x_3}_{i_1,i_2,i_3}$$

☐ This is exactly the tetrahedron equation of [Kapranov-Voevodsky94]:

$$\mathcal{R}_{123}\mathcal{R}_{145}\mathcal{R}_{246}\mathcal{R}_{356} = \mathcal{R}_{356}\mathcal{R}_{246}\mathcal{R}_{145}\mathcal{R}_{123}$$

# The case ◯━━◯━━⊗

$$\epsilon_1 - \epsilon_2 \qquad \epsilon_2 - \epsilon_3 \qquad \epsilon_3 - \delta_4 \qquad (\epsilon_i, \epsilon_i) = 1$$

◯━━━━━━◯━━━━━━⊗

$$(\delta_i, \delta_i) = -1$$

$$DA = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

- **Root system**

$$\tilde{\Phi}^+_{\text{even}} = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} \quad \tilde{\Phi}^+_{\text{iso}} = \{\alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$$

- **Lemma**

$$\Gamma^{(2|1)} = \mathcal{R} \quad \Gamma^{(3|2)} = \Gamma^{(23|1)} = \Gamma^{(32|1)} = \Gamma^{(3|12)} = \Gamma^{(3|21)} = \mathcal{L}$$

- **The theorem is specialized as follows:**

$$\sum \mathcal{L}^{o_1,o_2,o_3}_{x_1,x_2,x_3} \mathcal{L}^{x_1,o_4,o_5}_{i_1,x_4,x_5} \mathcal{L}^{x_2,x_4,o_6}_{i_2,i_4,x_6} \mathcal{R}^{x_3,x_5,x_6}_{i_3,i_5,i_6} = \sum \mathcal{R}^{o_3,o_5,o_6}_{x_3,x_5,x_6} \mathcal{L}^{o_2,o_4,x_6}_{x_2,x_4,i_6} \mathcal{L}^{o_1,x_4,x_5}_{x_1,i_4,i_5} \mathcal{L}^{x_1,x_2,x_3}_{i_1,i_2,i_3}$$

  ☐ This is exactly the tetrahedron equation of [Bazhanov-Sergeev06]:

$$\mathcal{L}_{123}\mathcal{L}_{145}\mathcal{L}_{246}\mathcal{R}_{356} = \mathcal{R}_{356}\mathcal{L}_{246}\mathcal{L}_{145}\mathcal{L}_{123}$$

$$\epsilon_1 - \epsilon_2 \qquad \epsilon_2 - \delta_3 \qquad \delta_3 - \epsilon_4 \qquad (\epsilon_i, \epsilon_i) = 1$$

◯————————⊗————————⊗   $(\delta_i, \delta_i) = -1$

$$DA = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

- **Root system**

$$\tilde{\Phi}^+_{\text{even}} = \{\alpha_1, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\} \quad \tilde{\Phi}^+_{\text{iso}} = \{\alpha_2, \alpha_3, \alpha_1 + \alpha_2\}$$

- <u>Lemma</u>

$$\Gamma^{(2|1)} = \mathcal{L}, \quad \Gamma^{(3|2)} = \Gamma^{(3|12)} = \Gamma^{(3|21)} = \mathcal{N}(q^{-1}), \quad = \Gamma^{(23|1)} = \Gamma^{(32|1)} = \mathcal{R}$$

- **The theorem is specialized as follows:**

$$\sum \mathcal{N}(q^{-1})^{o_1,o_2,o_3}_{x_1,x_2,x_3} \mathcal{N}(q^{-1})^{x_1,o_4,o_5}_{i_1,x_4,x_5} \mathcal{R}^{x_2,x_4,o_6}_{i_2,i_4,x_6} \mathcal{L}^{x_3,x_5,x_6}_{i_3,i_5,i_6}$$

$$= \sum \mathcal{L}^{o_3,o_5,o_6}_{x_3,x_5,x_6} \mathcal{R}^{o_2,o_4,x_6}_{x_2,x_4,i_6} \mathcal{N}(q^{-1})^{o_1,x_4,x_5}_{x_1,i_4,i_5} \mathcal{N}(q^{-1})^{x_1,x_2,x_3}_{i_1,i_2,i_3}$$

  ☐ This gives a new solution to the tetrahedron equation:

$$\mathcal{N}(q^{-1})_{123}\mathcal{N}(q^{-1})_{145}\mathcal{R}_{246}\mathcal{L}_{356} = \mathcal{L}_{356}\mathcal{R}_{246}\mathcal{N}(q^{-1})_{145}\mathcal{N}(q^{-1})_{123}$$

$$\epsilon_1 - \epsilon_2 \qquad \epsilon_2 - \delta_3 \qquad \delta_3 - \delta_4 \qquad (\epsilon_i, \epsilon_i) = 1$$

◯————————⊗————————◯

$$(\delta_i, \delta_i) = -1 \qquad DA = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

■ Root system

$$\tilde{\Phi}^+_{\text{even}} = \{\alpha_1, \alpha_3\} \quad \tilde{\Phi}^+_{\text{iso}} = \{\alpha_2, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$$

■ <u>Lemma</u>

$$\Gamma^{(2|1)} = \Gamma^{(23|1)} = \Gamma^{(32|1)} = \mathcal{L}, \quad \Gamma^{(3|2)} = \Gamma^{(3|12)} = \Gamma^{(3|21)} = \mathcal{M}(q^{-1})$$

# The case $\bigcirc\!\!-\!\!\!-\!\!\otimes\!\!-\!\!\!-\!\!\bigcirc$

$$\rho_1 = p(\alpha_1)p(\alpha_3)$$
$$\rho_2 = p(\alpha_1 + \alpha_2)p(\alpha_2 + \alpha_3)$$
$$\rho_3 = p(\alpha_2)p(\alpha_1 + \alpha_2 + \alpha_3)$$

- <u>Previous Theorem</u> [Y20]

$$\sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 i_3 i_4}$$

$$\times \Gamma^{(3|2)\,o_1,o_2,o_3}_{\phantom{(3|2)}x_1,x_2,x_3} \Gamma^{(3|12)\,x_1,o_4,o_5}_{\phantom{(3|12)}i_1,x_4,x_5} \Gamma^{(32|1)\,x_2,x_4,o_6}_{\phantom{(32|1)}i_2,i_4,x_6} \Gamma^{(2|1)\,x_3,x_5,x_6}_{\phantom{(2|1)}i_3,i_5,i_6}$$

$$= \sum (-1)^{\rho_1(o_1 i_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 o_3 o_4}$$

$$\times \Gamma^{(2|1)\,o_3,o_5,o_6}_{\phantom{(2|1)}x_3,x_5,x_6} \Gamma^{(23|1)\,o_2,o_4,x_6}_{\phantom{(23|1)}x_2,x_4,i_6} \Gamma^{(3|21)\,o_1,x_4,x_5}_{\phantom{(3|21)}x_1,i_4,i_5} \Gamma^{(3|2)\,x_1,x_2,x_3}_{\phantom{(3|2)}i_1,i_2,i_3}$$

- <u>Previous Lemma</u>

$$\Gamma^{(2|1)} = \Gamma^{(23|1)} = \Gamma^{(32|1)} = \mathcal{L}, \quad \Gamma^{(3|2)} = \Gamma^{(3|12)} = \Gamma^{(3|21)} = \mathcal{M}(q^{-1})$$

- By using $\rho_1 = 0, \rho_2 = \rho_3 = 1$, the theorem is specialized as follows:

$$\sum (-1)^{x_2 x_5 + i_3 i_4} \mathcal{M}(q^{-1})^{o_1,o_2,o_3}_{x_1,x_2,x_3} \mathcal{M}(q^{-1})^{x_1,o_4,o_5}_{i_1,x_4,x_5} \mathcal{L}^{x_2,x_4,o_6}_{i_2,i_4,x_6} \mathcal{L}^{x_3,x_5,x_6}_{i_3,i_5,i_6}$$

$$= \sum (-1)^{x_2 x_5 + o_3 o_4} \mathcal{L}^{o_3,o_5,o_6}_{x_3,x_5,x_6} \mathcal{L}^{o_2,o_4,x_6}_{x_2,x_4,i_6} \mathcal{M}(q^{-1})^{o_1,x_4,x_5}_{x_1,i_4,i_5} \mathcal{M}(q^{-1})^{x_1,x_2,x_3}_{i_1,i_2,i_3},$$

- □ There are ``nonlocal'' sign factors which can not be eliminated at present.

# Concluding remarks: Crystal limit

- Consider non-super cases
  - $B$: canonical basis
  - $\mathbf{i}$: indices of reduced expression of the longest element of Weyl group
  - Lusztig's parametrization: a bijection $b_{\mathbf{i}} \colon \mathbb{Z}_{\geq 0}^l \to B$ associated with $\mathbf{i}$
  - Transition map: $R_{\mathbf{i}}^{\mathbf{i}'} = (b_{\mathbf{i}'})^{-1} \circ b_{\mathbf{i}} \colon \mathbb{Z}_{\geq 0}^l \to \mathbb{Z}_{\geq 0}^l$

- Transition maps are obtained by transition matrices with $q \to 0$:

$$\lim_{q \to 0} \mathcal{R}(q)_{i,j,k}^{a,b,c} = \delta_{a,i+j-\min(i,k)} \delta_{b,\min(i,k)} \delta_{c,j+k-\min(i,k)}$$

- A super analog of transition maps is obtained in a similar way.

- <u>Prop</u> [Y20]
  - $\mathcal{L}_{i,j,k}^{a,b,c} = \lim_{q \to 0} \mathcal{L}(q)_{i,j,k}^{a,b,c}$ gives a non-trivial bijection on $\{0,1\}^2 \times \mathbb{Z}_{\geq 0}$.
  - Non-zero elements are given by

$$\mathcal{L}_{0,0,k}^{0,0,c} = \mathcal{L}_{1,1,k}^{1,1,c} = \delta_{k,c}, \quad \mathcal{L}_{0,1,k}^{1,0,c} = \delta_{k+1,c}, \quad \mathcal{L}_{1,0,0}^{1,0,0} = 1, \quad \mathcal{L}_{1,0,k}^{0,1,c} = \delta_{k-1,c}$$

  - $\mathcal{N}_{i,j,k}^{a,b,c} = \lim_{q \to 0} \left( \dfrac{[b]_q!}{[j]_q!} \mathcal{N}(q)_{i,j,k}^{a,b,c} \right)$ also gives a non-trivial bijection.

# Concluding remarks

■ <u>Remark</u>

1. Type B cases give new solutions to the 3D reflection equations.
2. The crystal limit for $\bigcirc\!\!\Longrightarrow\!\!\bullet$ and $\otimes\!\!\Longrightarrow\!\!\bullet$ take values $0, \pm 1$.

■ <u>Summary</u>

1. The 3D L is characterized as the transition matrix for $\bigcirc\!\!-\!\!\otimes$.
2. A new solution to the tetrahedron equation the 3D N is obtained by considering the transition matrix for $\otimes\!\!-\!\!\otimes$.
3. Several solutions to the tetrahedron equations are obtained without using any result for quantum coordinate rings (c.f. KOY theorem).

■ <u>Outlook</u>

1. Eliminating nonlocal sign factors for $\bigcirc\!\!-\!\!\otimes\!\!-\!\!\bigcirc$
2. A super analog of KOY theorem
3. Geometric lifting of a super analog of transition maps