

3D reflection maps from tetrahedron maps

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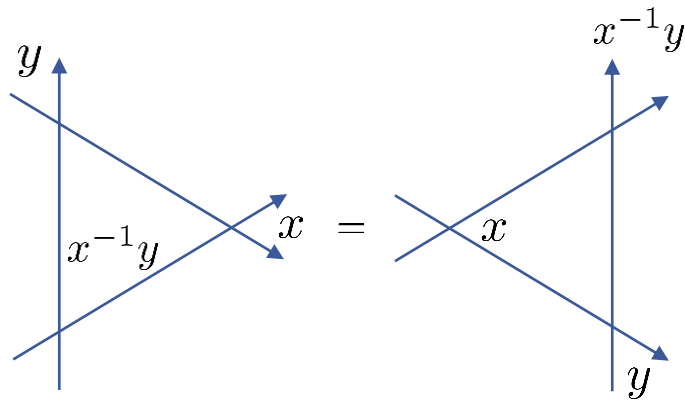
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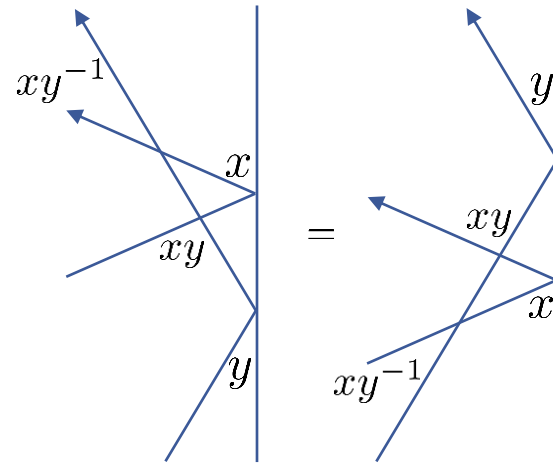
- Introduction: reflection maps from Yang-Baxter maps P.3~8
- Main part: 3D reflection maps from tetrahedron maps P.10~20
- Examples P.22~26
 - Birational transition map
 - Sergeev's electrical solution
 - Two-component solution associated with soliton equation

■ Bulk: Yang-Baxter equation



$$\begin{aligned} R_{12}(x)R_{13}(y)R_{23}(x^{-1}y) \\ = R_{23}(x^{-1}y)R_{13}(y)R_{12}(x) \end{aligned}$$

■ Boundary: reflection equation



$$\begin{aligned} R_{12}(xy^{-1})K_2(x)R_{21}(xy)K_1(y) \\ = K_1(y)R_{12}(xy)K_2(x)R_{21}(xy^{-1}) \end{aligned}$$

■ Setup1: R & K are matrices on linear spaces

■ Setup2: R & K are maps on sets (set-theoretical solution) [Drinfeld92]

- We call set-theoretical solutions to YBE *Yang-Baxter maps*, and so on.
- Yang-Baxter maps have been extensively studied in connection with various topics. [Veselov07]
- Several reflection maps are constructed
 - via directly solving the reflection equation [Kuniba-Okado-Yamada05]
 - Later, these solutions are *q-melted* by using coideal subalgebras of U_q . [Kuniba-Okado-Y19], [Kusano-Okado20]
 - from Yang-Baxter maps [Caudrelier-Zhang14], [Kuniba-Okado19]
 - by ring-theoretic methods [Smoktunowicz-Vendramin-Weston20]
- Here, we briefly review a result of [Kuniba-Okado19].

■ KR crystal for $A_{n-1}^{(1)}$

■ $B^{k,l} = \{\text{SST of } k \times l \text{ rectangular shape with letters from } \{1,2,\dots,n\}\}$

■ SST: semi-standard tableaux

■ $1 \leq k \leq n-1, l \geq 1$

■ $\tilde{e}_i, \tilde{f}_i: B^{k,l} \rightarrow B^{k,l} \cup \{0\}$ (Kashiwara operators)

■ For KR crystals B_1, B_2 , the actions of Kashiwara operators are also defined on $B_1 \otimes B_2 = \{b_1 \otimes b_2 | b_1 \in B_1, b_2 \in B_2\}$ and $B_1 \otimes B_2$ is connected.

e.g.

1	1	3
Λ	Λ	Λ
2	4	6

■ A map $R_{B_1, B_2}: B_1 \otimes B_2 \rightarrow B_2 \otimes B_1$ given by

$$R_{B_1, B_2}(\tilde{k}_i(b_1 \otimes b_2)) = \tilde{k}_i(R_{B_1, B_2}(b_1 \otimes b_2)) \quad (\tilde{k}_i = \tilde{e}_i, \tilde{f}_i)$$

is uniquely determined (combinatorial R matrices).

■ The action can be calculated by the *tableau product rule*.

■ Combinatorial R matrices satisfy YBE from $B_1 \otimes B_2 \otimes B_3$ to $B_3 \otimes B_2 \otimes B_1$:

$$(1 \otimes R_{B_1, B_2})(R_{B_1, B_3} \otimes 1)(1 \otimes R_{B_2, B_3}) = (R_{B_2, B_3} \otimes 1)(1 \otimes R_{B_1, B_3})(R_{B_1, B_2} \otimes 1)$$

- We define $v: B^{k,l} \rightarrow B^{n-k,l}$ by

$$b^v = \left[\begin{array}{|c|c|c|c|} \hline \overline{\lambda_l} & & \overline{\lambda_2} & \overline{\lambda_1} \\ \hline \end{array} \right] n-k \quad \text{for } b = \left[\begin{array}{|c|c|c|c|} \hline \lambda_1 & \lambda_2 & & \lambda_l \\ \hline \end{array} \right] k$$

- $\lambda_i \in \mathbb{N}^k, \overline{\lambda_i} \in \mathbb{N}^{n-k}$

- The complement is taken over $\{1, 2, \dots, n\}$.

- Proposition [Kuniba-Okado19]:

- For a KR crystal B , we have $R_{B^v, B}(b^v \otimes b) = c \otimes c^v$.

- Definition:

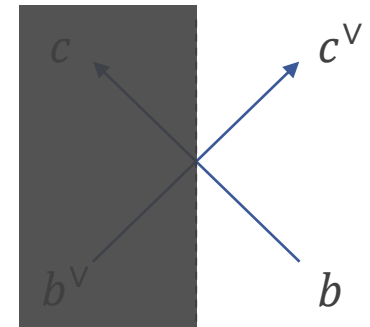
- Using b and c above, we define the combinatorial K matrix by

$$K_B: B \rightarrow B^v, b \rightarrow c^v$$

- Theorem: [Kuniba-Okado19]

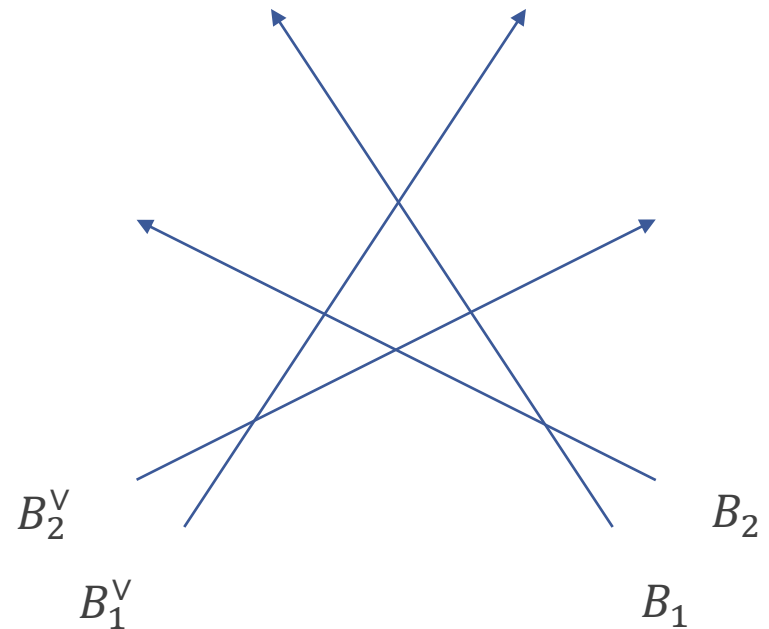
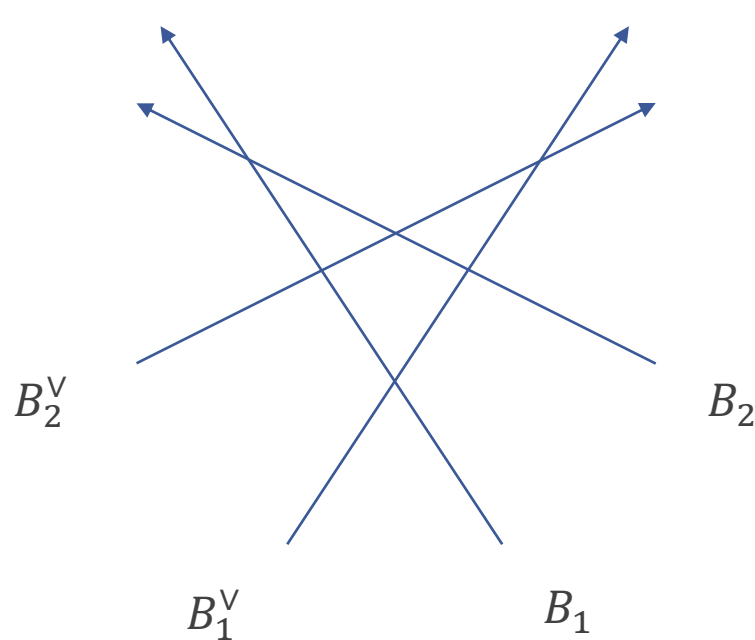
- The combinatorial R and K matrices satisfy RE from $B_1 \otimes B_2 \rightarrow B_1^v \otimes B_2^v$:

$$R_{B_2^v, B_1^v}(K_{B_2} \otimes 1)R_{B_1^v, B_2}(K_{B_1} \otimes 1) = (K_{B_1} \otimes 1)R_{B_2^v, B_1}(K_{B_2} \otimes 1)R_{B_1, B_2}$$



Key lemma for reflection equation

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■ Lemma:

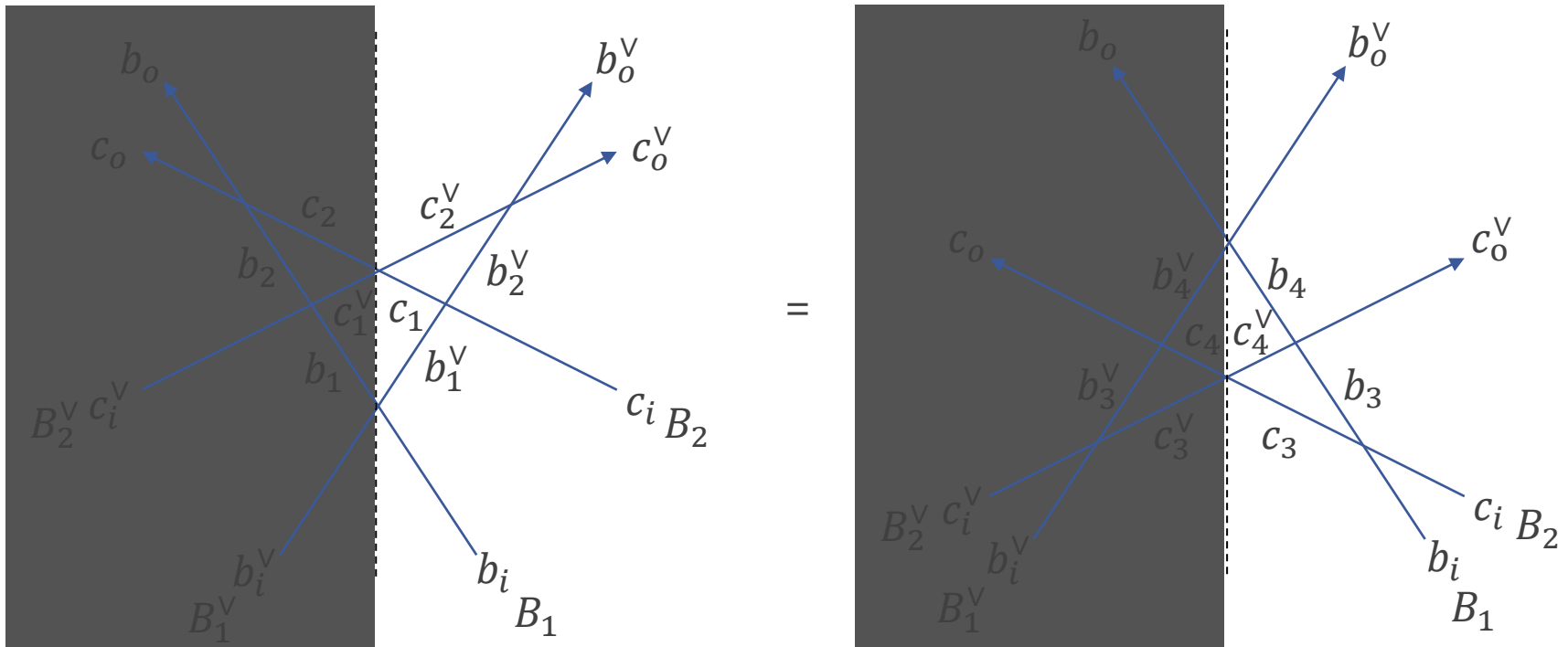
$$\begin{aligned} & (R_{B_2^V, B_1^V} R_{B_1, B_2}) R_{B_2^V, B_2} (R_{B_1^V, B_2} R_{B_2^V, B_1}) R_{B_1^V, B_1} \\ &= R_{B_1^V, B_1} (R_{B_2^V, B_1} R_{B_1^V, B_2}) R_{B_2^V, B_2} (R_{B_1, B_2} R_{B_2^V, B_1^V}) \end{aligned}$$

■ Proof:

□ By repeated uses of YBE, we have the above equation.

Proof of reflection equation

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■ Sketch of proof:

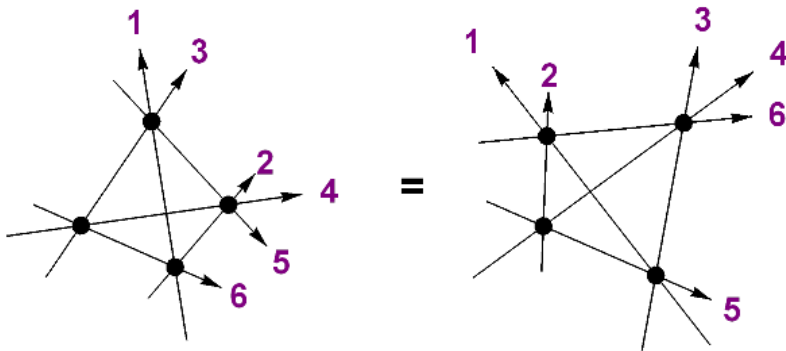
- ▣ Input b_i, c_i on B_1, B_2 and their duals on B_1^V, B_2^V .
- ▣ Use $R_{B^V, B}(b^V \otimes b) = c \otimes c^V$.
- ▣ Use $R_{B_1, B_2}(b_1 \otimes b_2) = c_2 \otimes c_1 \Rightarrow R_{B_2^V, B_1^V}(b_2^V \otimes b_1^V) = c_1^V \otimes c_2^V$.
- ▣ Just viewing the right parts of both sides, we then obtain RE.

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- Tetrahedron and 3D reflection equation are conditions for factorization of string scattering amplitude in 2+1D.

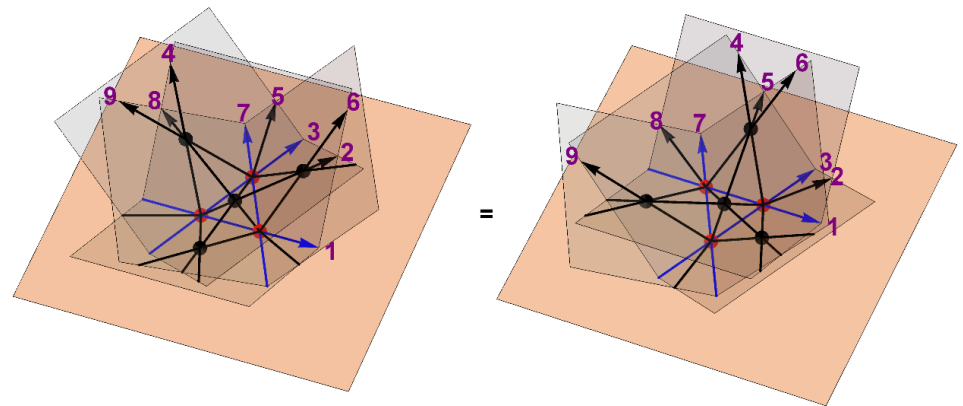
	Bulk	Boundary
2D	Yang-Baxter eq.	Reflection eq.
3D	Tetrahedron eq.	3D Reflection eq.

- Tetrahedron equation



$$\mathbf{R}_{245}\mathbf{R}_{135}\mathbf{R}_{126}\mathbf{R}_{346} \\ = \mathbf{R}_{346}\mathbf{R}_{126}\mathbf{R}_{135}\mathbf{R}_{245}$$

- 3D reflection equation



$$\mathbf{R}_{489}\mathbf{J}_{3579}\mathbf{R}_{269}\mathbf{R}_{258}\mathbf{J}_{1678}\mathbf{J}_{1234}\mathbf{R}_{456} \\ = \mathbf{R}_{456}\mathbf{J}_{1234}\mathbf{J}_{1678}\mathbf{R}_{258}\mathbf{R}_{269}\mathbf{J}_{3579}\mathbf{R}_{489}$$

- Tetrahedron and 3D reflection equation are conditions for factorization of string scattering amplitude in 2+1D.

	Bulk	Boundary
2D	Yang-Baxter eq.	Reflection eq.
3D	Tetrahedron eq.	3D Reflection eq.

- Several tetrahedron maps are known although less systematically than Yang-Baxter maps.
 - ▣ Solutions to the local YBE [Sergeev98]
 - ▣ Transition maps of Lusztig's parametrizations of the canonical basis of $U_q(A_2)$ and their geometric liftings [Kuniba-Okado12]
 - ▣ Solutions for relations which semi-invariants for some discrete KP equations satisfy [Kassotakis-Nieszporski-Papageorgiou-Tongas19]
- On the other hand, there are very few known 3D reflection maps.
 - ▣ Transition maps of Lusztig's parametrizations of the canonical basis of $U_q(B_2)$ and $U_q(C_2)$, and their geometric liftings [Kuniba-Okado12]

■ Let X denote an arbitrary set and x_i its elements.

■ We set the transposition $P_{ij}: X^n \rightarrow X^n$ by

$$P_{ij}(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, \overbrace{x_j}, x_{i+1}, \dots, x_{j-1}, \overbrace{x_i}, x_{j+1}, \dots, x_n)$$

■ Let $\mathbf{R}: X^3 \rightarrow X^3$ denote a map given by

$$\mathbf{R}(x, y, z) = (f(x, y, z), g(x, y, z), h(x, y, z)) \quad (f, g, h: X^3 \rightarrow X)$$

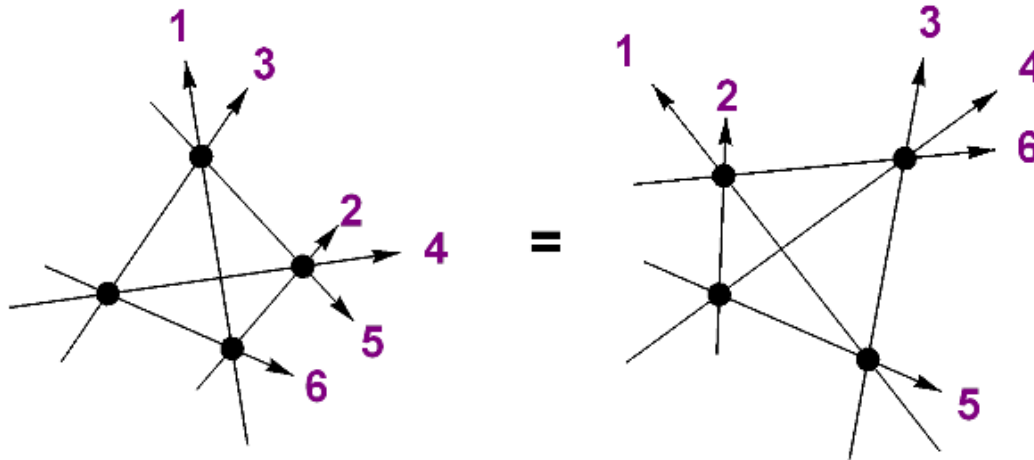
□ For $i < j < k$, we define $\mathbf{R}_{ijk}: X^n \rightarrow X^n$ by

$$\begin{aligned} \mathbf{R}_{ijk}(x_1, \dots, x_n) = & (x_1, \dots, x_{i-1}, f(x_i, x_j, x_k), x_{i+1}, \\ & \dots, x_{j-1}, g(x_i, x_j, x_k), x_{j+1}, \\ & \dots, x_{k-1}, h(x_i, x_j, x_k), x_{k+1}, \dots, x_n). \end{aligned}$$

□ Otherwise, we define $\mathbf{R}_{ijk}: X^n \rightarrow X^n$ by sandwiching R between the permutations which sort (i, j, k) in ascending order.

□ For example, if $i > j > k$, we define $\mathbf{R}_{ijk} = P_{ik}\mathbf{R}_{kji}P_{ik}$.

■ We use the same notation for $\mathbf{J}: X^4 \rightarrow X^4$.



■ Definition:

□ Let $\mathbf{R}: X^3 \rightarrow X^3$ denote a map.

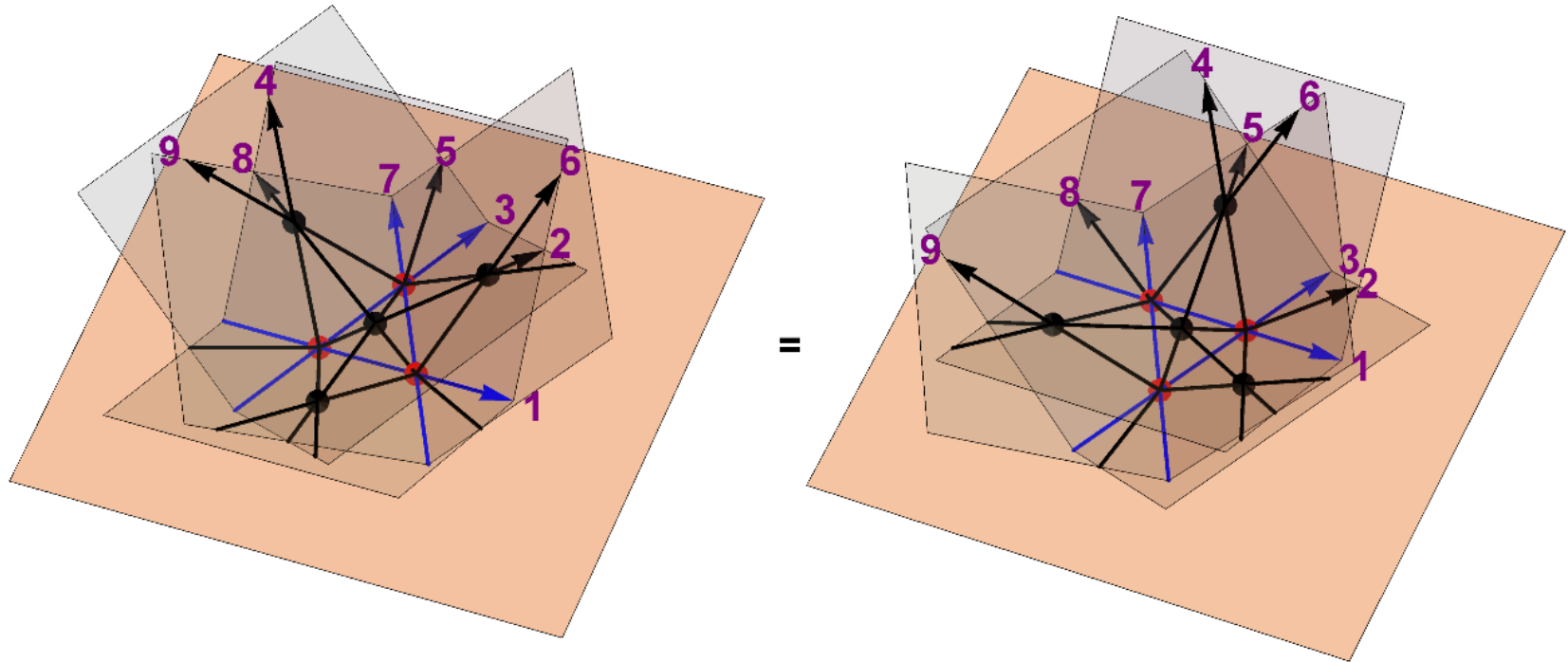
□ We call \mathbf{R} *tetrahedron map* if it satisfies the tetrahedron equation on X^6 :

$$\mathbf{R}_{245}\mathbf{R}_{135}\mathbf{R}_{126}\mathbf{R}_{346} = \mathbf{R}_{346}\mathbf{R}_{126}\mathbf{R}_{135}\mathbf{R}_{245} (=:\mathbf{T}_{123456}) \quad \cdots (*)$$

□ We call \mathbf{T} the *tetrahedral composite* of the tetrahedron map \mathbf{R} .

■ We call \mathbf{R} *involutive* if $\mathbf{R}^2 = \text{id}$ and *symmetric* if $\mathbf{R}_{123} = \mathbf{R}_{321}$.

□ For involutive and symmetric tetrahedron maps, $(*)$ corresponds to the usual tetrahedron equation.

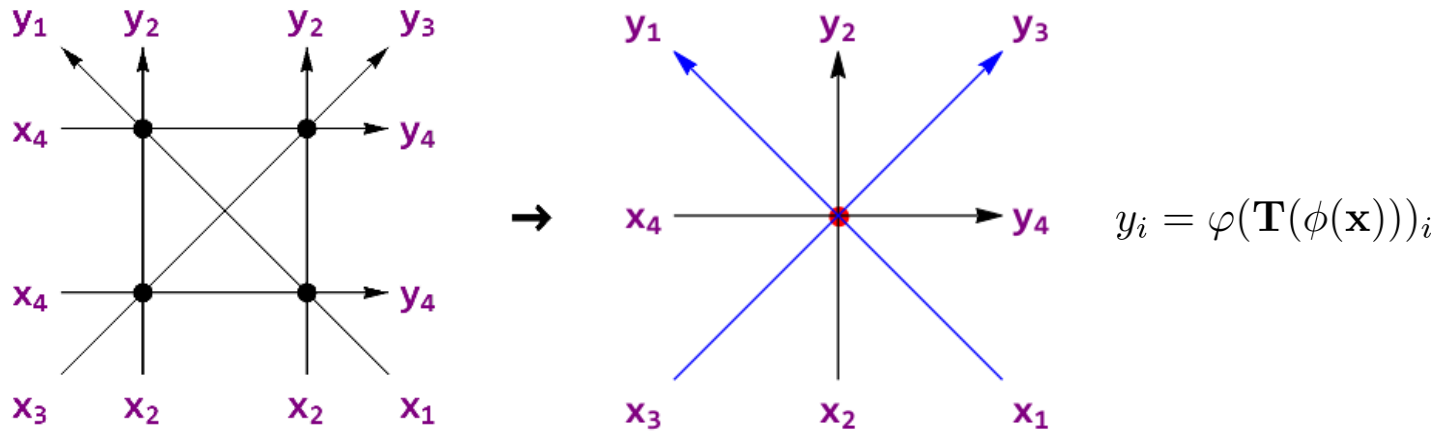


■ Definition:

- Let $J: X^4 \rightarrow X^4$ denote a map.
- We set a tetrahedron map by $R: X^3 \rightarrow X^3$.
- We call J *3D reflection map* if it satisfies the 3D reflection equation on X^9 :

$$R_{489} J_{3579} R_{269} R_{258} J_{1678} J_{1234} R_{456} = R_{456} J_{1234} J_{1678} R_{258} R_{269} J_{3579} R_{489}$$

[Isaev-Kulish97]



- We set the subset of X^6 by $Y = \{(x_1, \dots, x_6) \mid x_2 = x_3, x_5 = x_6\}$.
 - ▣ We set $\phi: X^4 \rightarrow Y$ by $\phi(x_1, x_2, x_3, x_4) = (x_1, x_2, x_2, x_3, x_4, x_4)$ (embedding)
 - ▣ We set $\varphi: Y \rightarrow X^4$ by $\varphi(x_1, x_2, x_2, x_3, x_4, x_4) = (x_1, x_2, x_3, x_4)$ (projection)
- Definition:
 - ▣ Let $\mathbf{R}: X^3 \rightarrow X^3$ denote a tetrahedron map and \mathbf{T} its tetrahedral composite.
 - ▣ We call \mathbf{R} *boundarizable* if the following condition is satisfied:

$$x \in Y \Rightarrow \mathbf{T}(x) \in Y$$
 - ▣ In that case, we define the *boundarization* $\mathbf{J}: X^4 \rightarrow X^4$ of \mathbf{R} by

$$\mathbf{J}(\mathbf{x}) = \varphi(\mathbf{T}(\phi(\mathbf{x})))$$

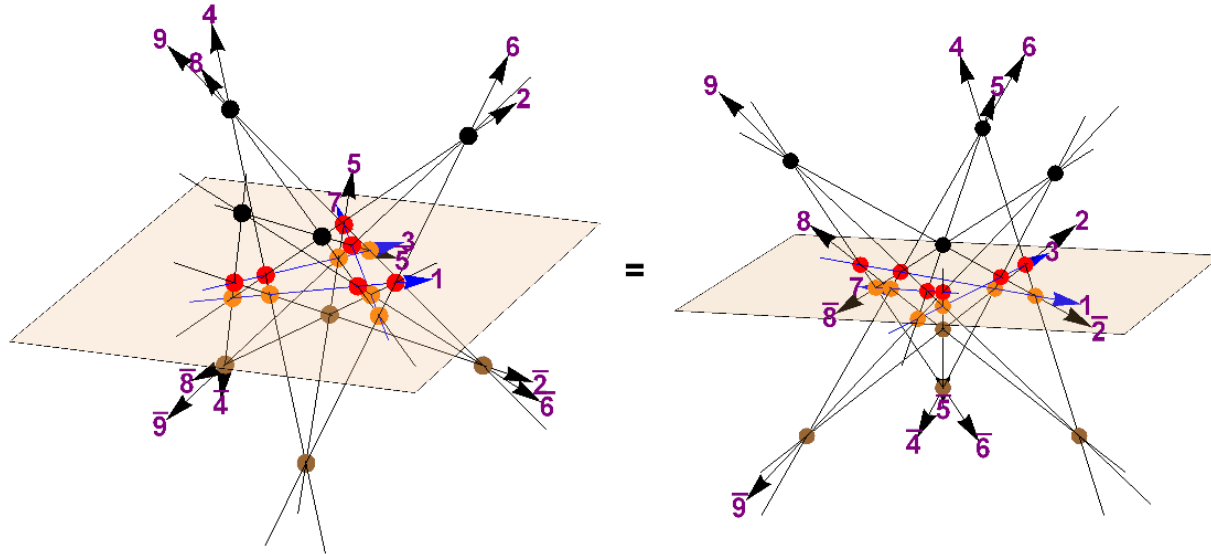
■ Theorem:

- Let $\mathbf{R}: X^3 \rightarrow X^3$ denote an involutive, symmetric and boundarizable tetrahedron map, and $\mathbf{J}: X^4 \rightarrow X^4$ its boundarization.
- Then they satisfy 3D reflection equation:

$$\mathbf{R}_{489} \mathbf{J}_{3579} \mathbf{R}_{269} \mathbf{R}_{258} \mathbf{J}_{1678} \mathbf{J}_{1234} \mathbf{R}_{456} = \mathbf{R}_{456} \mathbf{J}_{1234} \mathbf{J}_{1678} \mathbf{R}_{258} \mathbf{R}_{269} \mathbf{J}_{3579} \mathbf{R}_{489}$$

Key lemma for main theorem

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■ Lemma:

□ Let $\mathbf{R}: X^3 \rightarrow X^3$ denote an involutive and symmetric tetrahedron map. Then we have the following identity on X^{15} :

$$\begin{aligned}
 & (\mathbf{R}_{489} \mathbf{R}_{\bar{4}\bar{8}\bar{9}})(\mathbf{R}_{579} \mathbf{R}_{359} \mathbf{R}_{359} \mathbf{R}_{\bar{5}\bar{7}\bar{9}})(\mathbf{R}_{269} \mathbf{R}_{\bar{2}\bar{6}\bar{9}})(\mathbf{R}_{258} \mathbf{R}_{\bar{2}\bar{5}\bar{8}}) \\
 & \quad \times (\mathbf{R}_{\bar{6}\bar{7}\bar{8}} \mathbf{R}_{168} \mathbf{R}_{168} \mathbf{R}_{678})(\mathbf{R}_{234} \mathbf{R}_{124} \mathbf{R}_{124} \mathbf{R}_{\bar{2}\bar{3}\bar{4}})(\mathbf{R}_{\bar{4}\bar{5}\bar{6}} \mathbf{R}_{456}) \\
 & = (\mathbf{R}_{456} \mathbf{R}_{\bar{4}\bar{5}\bar{6}})(\mathbf{R}_{234} \mathbf{R}_{124} \mathbf{R}_{124} \mathbf{R}_{\bar{2}\bar{3}\bar{4}})(\mathbf{R}_{\bar{6}\bar{7}\bar{8}} \mathbf{R}_{168} \mathbf{R}_{168} \mathbf{R}_{678}) \\
 & \quad \times (\mathbf{R}_{\bar{2}\bar{5}\bar{8}} \mathbf{R}_{258})(\mathbf{R}_{\bar{2}\bar{6}\bar{9}} \mathbf{R}_{269})(\mathbf{R}_{579} \mathbf{R}_{359} \mathbf{R}_{359} \mathbf{R}_{\bar{5}\bar{7}\bar{9}})(\mathbf{R}_{\bar{4}\bar{8}\bar{9}} \mathbf{R}_{489})
 \end{aligned}$$

■ Proof:

$\bar{}$: mirrored space

□ By repeated uses of TE, we have the above equation.

Figure for the identity on X^{15}

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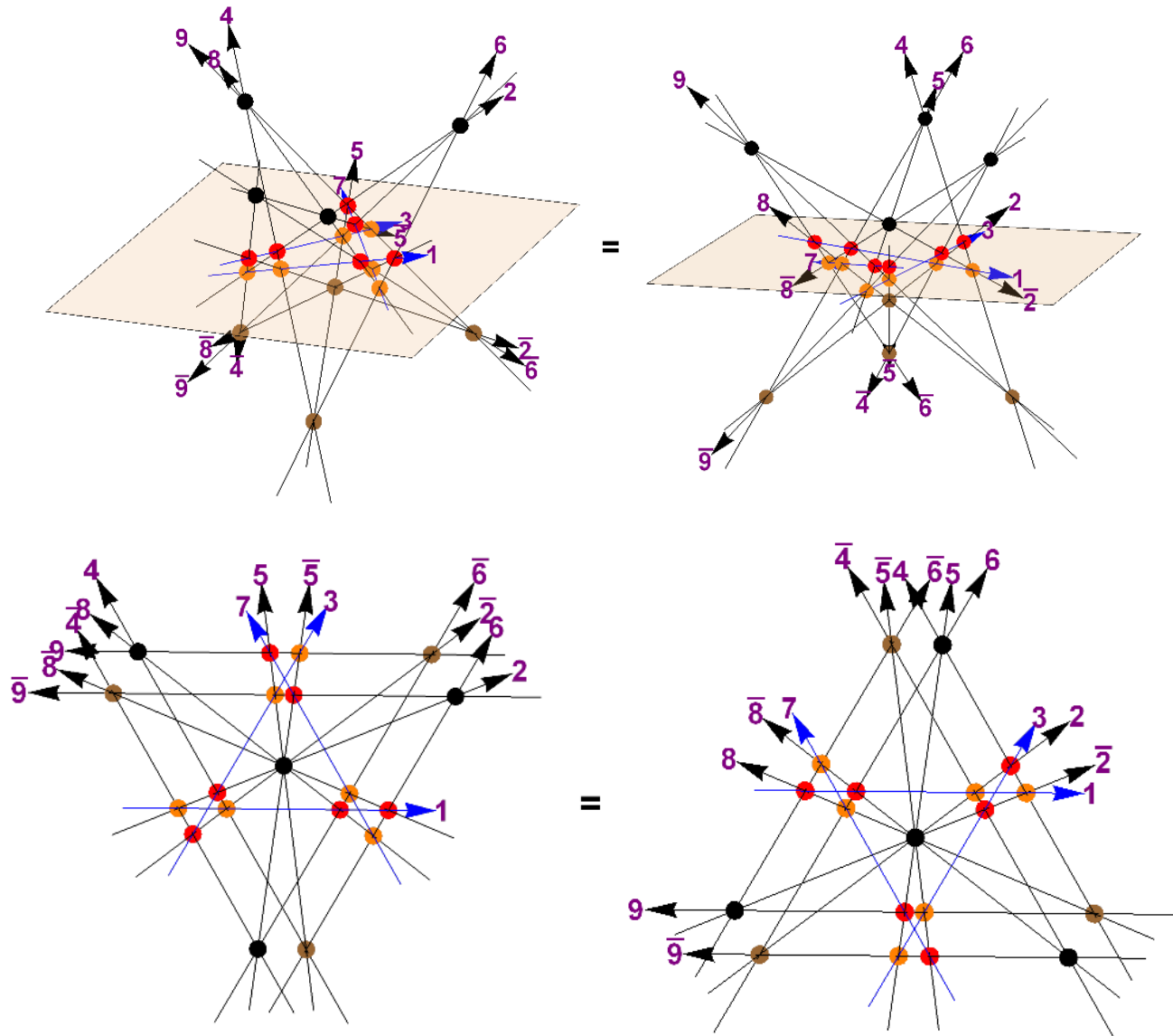
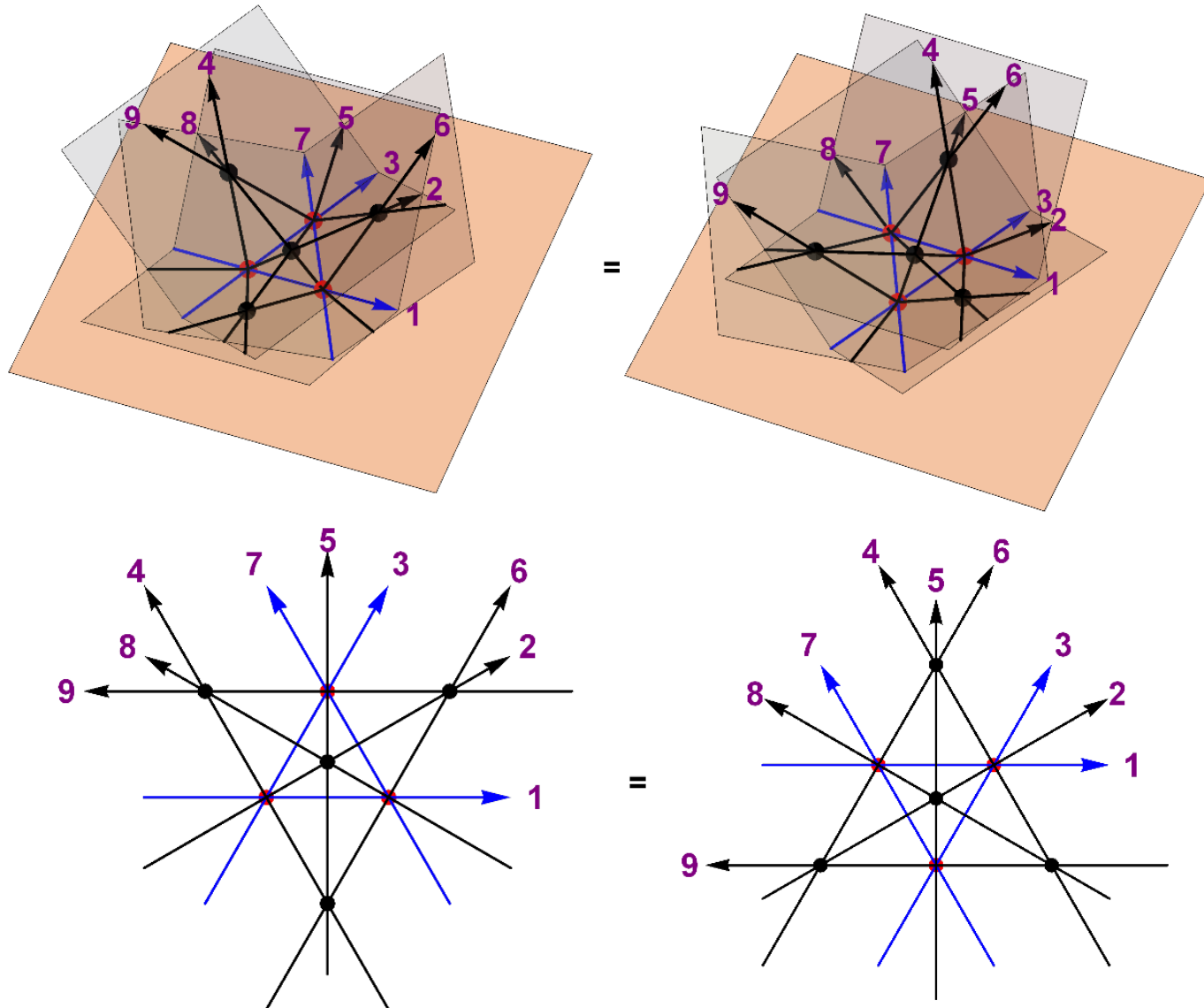


Figure for 3D reflection equation

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■ Sketch of proof:

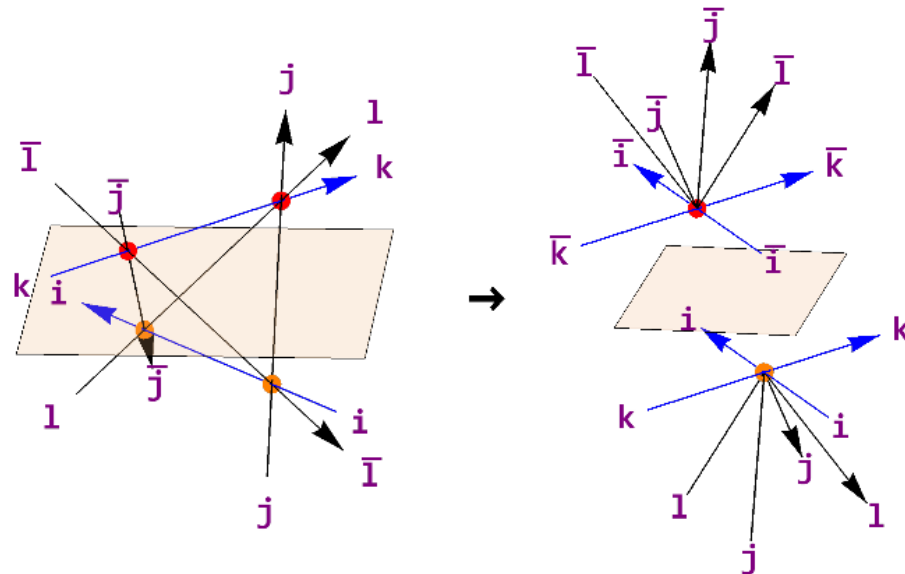
▣ Let \mathbf{T} denote the tetrahedral composite of \mathbf{R} .

▣ By using \mathbf{T} , the identify in the previous lemma is written as follows:

$$\begin{aligned} & (\mathbf{R}_{489}\mathbf{R}_{\bar{4}\bar{8}\bar{9}})\mathbf{T}_{35\bar{5}79\bar{9}}(\mathbf{R}_{26\bar{9}}\mathbf{R}_{\bar{2}\bar{6}\bar{9}})(\mathbf{R}_{2\bar{5}\bar{8}}\mathbf{R}_{\bar{2}\bar{5}\bar{8}})\mathbf{T}_{16\bar{6}78\bar{8}}\mathbf{T}_{12\bar{2}34\bar{4}}(\mathbf{R}_{\bar{4}\bar{5}\bar{6}}\mathbf{R}_{456}) \\ & = (\mathbf{R}_{456}\mathbf{R}_{\bar{4}\bar{5}\bar{6}})\mathbf{T}_{12\bar{2}34\bar{4}}\mathbf{T}_{16\bar{6}78\bar{8}}(\mathbf{R}_{\bar{2}\bar{5}\bar{8}}\mathbf{R}_{2\bar{5}8})(\mathbf{R}_{\bar{2}\bar{6}\bar{9}}\mathbf{R}_{26\bar{9}})\mathbf{T}_{35\bar{5}79\bar{9}}(\mathbf{R}_{\bar{4}\bar{8}\bar{9}}\mathbf{R}_{489}) \quad \dots (*) \end{aligned}$$

▣ Let us act both sides of $(*)$ on "mirrored" inputs (e.g. input x_5 for both $5, \bar{5}$).

▣ By applying $\mathbf{T}_{ij\bar{j}kl\bar{l}} \mapsto P_{j\bar{j}}P_{l\bar{l}}\mathbf{J}_{ijkl}\mathbf{J}_{\bar{i}\bar{j}\bar{k}\bar{l}}$ (cutting and reconnection), we then obtain the direct product of 3DRE.



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Example1: birational transition map

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- We set $\mathbf{R}: \mathbb{R}_{>0}^3 \rightarrow \mathbb{R}_{>0}^3$ by

$$\mathbf{R}: (x_1, x_2, x_3) \mapsto (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \left(\frac{x_1 x_2}{y}, y, \frac{x_2 x_3}{y} \right) \quad y = x_1 + x_3$$

- This map is characterized as the transition map of parametrizations of the positive part of SL_3 : [Lusztig94]

$$G_1(x_3)G_2(x_2)G_1(x_1) = G_2(\tilde{x}_1)G_1(\tilde{x}_2)G_2(\tilde{x}_3) \quad G_i(x) = 1 + xE_{i,i+1}$$

- Actually, the transition map for SL_4 is the tetrahedral composite of \mathbf{R} and we have TE for \mathbf{R} by considering \mathbf{T} in two ways. [Kuniba-Okado12]

$$\begin{aligned} & G_1(x_6)G_3(x_5)G_2(x_4)G_1(x_3)G_3(x_2)G_2(x_1) \\ &= G_2(\tilde{x}_1)G_1(\tilde{x}_2)G_3(\tilde{x}_3)G_2(\tilde{x}_4)G_1(\tilde{x}_5)G_3(\tilde{x}_6) \end{aligned}$$

$$\mathbf{T}: (x_1, \dots, x_6) \mapsto (\tilde{x}_1, \dots, \tilde{x}_6)$$

- This tetrahedron map was also derived in the context of the local YBE and by considering semi-invariants for discrete AKP equation. [Sergeev98], [Kassotakis-Nieszporski-Papageorgiou-Tongas19]

- We can verify \mathbf{R} is involutive, symmetric and boundarizable.

Example1: birational transition map

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- The associated 3D reflection map $\mathbf{J}: \mathbb{R}_{>0}^4 \rightarrow \mathbb{R}_{>0}^4$ is calculated as follows:

$$\mathbf{J} : (x_1, x_2, x_3, x_4) \mapsto \left(\frac{x_1 x_2^2 x_3}{y_1}, \frac{y_1}{y_2}, \frac{y_2^2}{y_1}, \frac{x_2 x_3 x_4}{y_2} \right)$$

$$y_1 = x_1(x_2 + x_4)^2 + x_3 x_4^2, \quad y_2 = x_1(x_2 + x_4) + x_3 x_4$$

- Actually, (\mathbf{R}, \mathbf{J}) is a known solution to 3D reflection equation.
[Kuniba-Okado12]

- \mathbf{J} is characterized as the transition map of parametrizations of the positive part of SP_4 :

$$H_1(x_4)H_2(x_3)H_1(x_2)H_2(x_1) = H_2(\tilde{x}_1)H_1(\tilde{x}_2)H_2(\tilde{x}_3)H_1(\tilde{x}_4)$$

$$H_1(x) = \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{pmatrix} \quad H_2(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & x \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}$$

- This correspondence is a consequence from folding the Dynkin diagram of A_3 into one of C_2 .
[Berenstein-Zelevinsky01], [Lusztig11]

Example2: Sergeev's electrical solution

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- For $\lambda \in \mathbb{C}$, we set the tetrahedron map $\mathbf{R}(\lambda): \mathbb{C}^3 \rightarrow \mathbb{C}^3$ by

$$\mathbf{R}(\lambda) : (x_1, x_2, x_3) \mapsto (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \left(\frac{x_1 x_2}{y}, y, \frac{x_2 x_3}{y} \right) \quad y = x_1 + x_3 + \lambda x_1 x_2 x_3$$

- This solution was derived in the context of the local YBE and by considering semi-invariants for discrete BKP equation.

[Sergeev98], [Kassotakis-Nieszporski-Papageorgiou-Tongas19]

- $\mathbf{R}(\lambda)$ is also characterized as the relation which elements of the positive part of electorical Lie groups of type A satisfy: [Lam-Pylyavskyy15]

$$\hat{G}_1(x_3) \hat{G}_2(x_2) \hat{G}_1(x_1) = \hat{G}_2(\tilde{x}_1) \hat{G}_1(\tilde{x}_2) \hat{G}_2(\tilde{x}_3)$$

- We can verify $\mathbf{R}(\lambda)$ is involutive, symmetric and boundarizable.

- The associated 3D reflection map $\mathbf{J}(\lambda): \mathbb{C}^4 \rightarrow \mathbb{C}^4$ is calculated as follows:

$$\mathbf{J}(\lambda) : (x_1, x_2, x_3, x_4) \mapsto \left(\frac{x_1 x_2^2 x_3}{y_1}, \frac{y_1}{y_2}, \frac{y_2^2}{y_1}, \frac{x_2 x_3 x_4}{y_2} \right)$$

$$y_1 = x_1(x_2 + x_4)(x_2 + x_4 + 2\lambda x_2 x_3 x_4) + x_3 x_4^2$$

$$y_2 = x_1(x_2 + x_4 + 2\lambda x_2 x_3 x_4) + x_3 x_4$$

Example3: Two component solution

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- We set the tetrahedron map $\mathbf{R}: (\mathbb{C}^2)^3 \rightarrow (\mathbb{C}^2)^3$ by

$$\mathbf{R} : \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \right) \mapsto \left(\begin{pmatrix} \frac{x_1 x_2}{x_1 + x_3} \\ \frac{(x_1 + x_3) y_1 y_2}{x_1 y_1 + x_3 y_3} \end{pmatrix}, \begin{pmatrix} x_1 + x_3 \\ \frac{x_1 y_1 + x_3 y_3}{x_1 + x_3} \end{pmatrix}, \begin{pmatrix} \frac{x_2 x_3}{x_1 + x_3} \\ \frac{(x_1 + x_3) y_2 y_3}{x_1 y_1 + x_3 y_3} \end{pmatrix} \right)$$

- This solution was derived by considering semi-invariants for discrete modified KP equation. [Kassotakis-Nieszporski-Papageorgiou-Tongas19]
- This gives the previous birational transition map when we set $y_i = 1$.
- We can verify \mathbf{R} is involutive, symmetric and boundarizable.

Example3: Two component solution

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- The associated 3D reflection map $\mathbf{J}: (\mathbb{C}^2)^4 \rightarrow (\mathbb{C}^2)^4$ is calculated as follows:

$$\mathbf{J} : \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}, \begin{pmatrix} x_4 \\ y_4 \end{pmatrix} \right) \\ \mapsto \left(\begin{pmatrix} \frac{x_1 x_2^2 x_3}{z_1} \\ \frac{y_1 y_2^2 y_3 z_1}{w_1} \end{pmatrix}, \begin{pmatrix} \frac{z_1}{z_2} \\ \frac{z_2 w_1}{z_1 w_2} \end{pmatrix}, \begin{pmatrix} \frac{z_2^2}{z_1} \\ \frac{z_1 w_2^2}{z_2^2 w_1} \end{pmatrix}, \begin{pmatrix} \frac{x_2 x_3 x_4}{z_2} \\ \frac{y_2 y_3 y_4 z_2}{w_2} \end{pmatrix} \right)$$

$$z_1 = x_1(x_2 + x_4)^2 + x_3 x_4^2$$

$$z_2 = x_1(x_2 + x_4) + x_3 x_4$$

$$w_1 = x_1 y_1 (x_2 y_2 + x_4 y_4)^2 + x_3 x_4^2 y_3 y_4^2$$

$$w_2 = x_1 y_1 (x_2 y_2 + x_4 y_4) + x_3 x_4 y_3 y_4$$

■ Summary:

- We present a method for obtaining 3D reflection maps by using known tetrahedron maps. The theorem is an analog of the results in 2D.
- By applying our theorem to known tetrahedron maps, we obtain several 3D reflection maps which include new solutions.

■ Remark:

- Our theorem can be extended to *inhomogeneous* cases, that is, the case tetrahedron maps are defined on direct product of different sets.