Tetrahedron ~~and 3D reflection~~ equation from PBW bases of the nilpotent subalgebra of quantum superalgebras

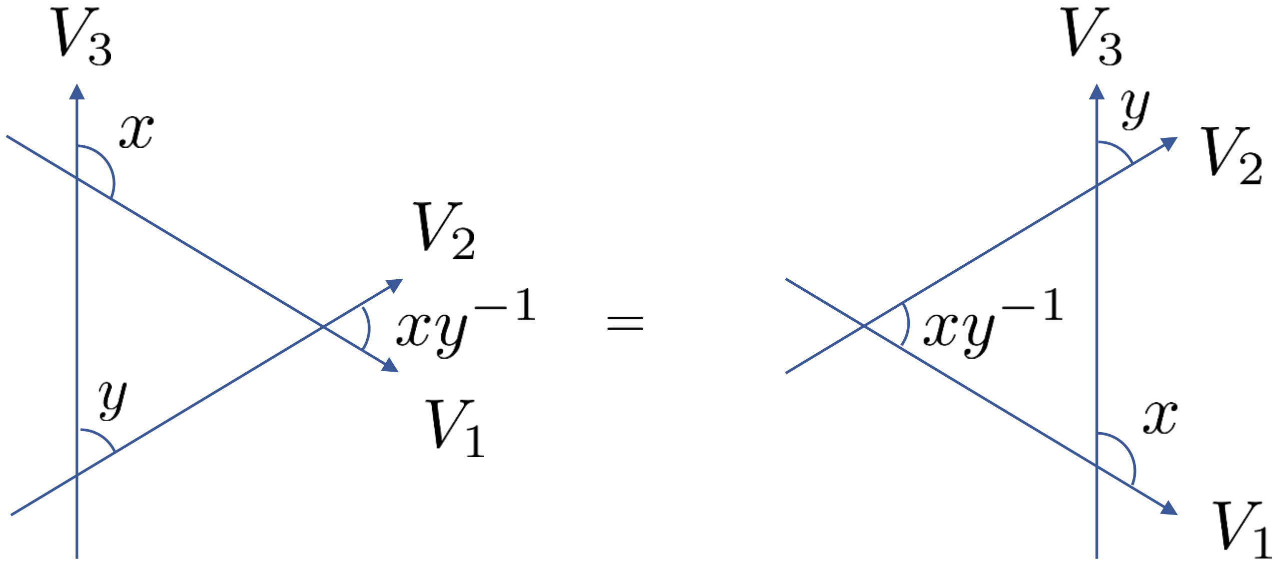
Discrete Mathematical Modelling Seminar@2021/01/13

Akihito Yoneyama (米山 瑛仁)
Institute of Physics, University of Tokyo, Komaba

Based on: AY, arXiv: 2012.13385

Outline

Yang-Baxter equation



- Matrix equation on $V_1 \otimes V_2 \otimes V_3$ (V_i: linear space)

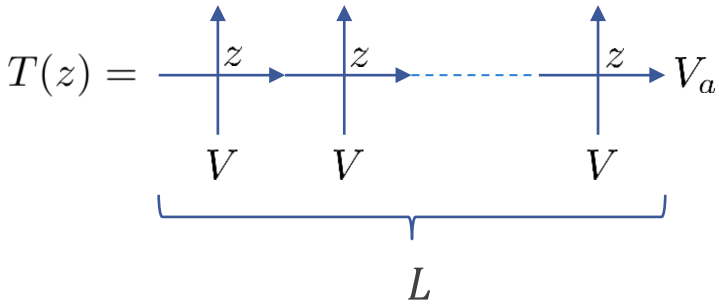
$$R_{12}(xy^{-1})R_{13}(x)R_{23}(y) = R_{23}(y)R_{13}(x)R_{12}(xy^{-1})$$

  - $R_{ij}(z)$ acts non-trivially only on $V_i \otimes V_j$.
  - Solutions to the Yang-Baxter eq are called R matrices.

- R matrices are systematically (infinitely many) constructed via irreps of quantum affine algebra $U_q(g)$.

Monodromy matrix & RTT relation

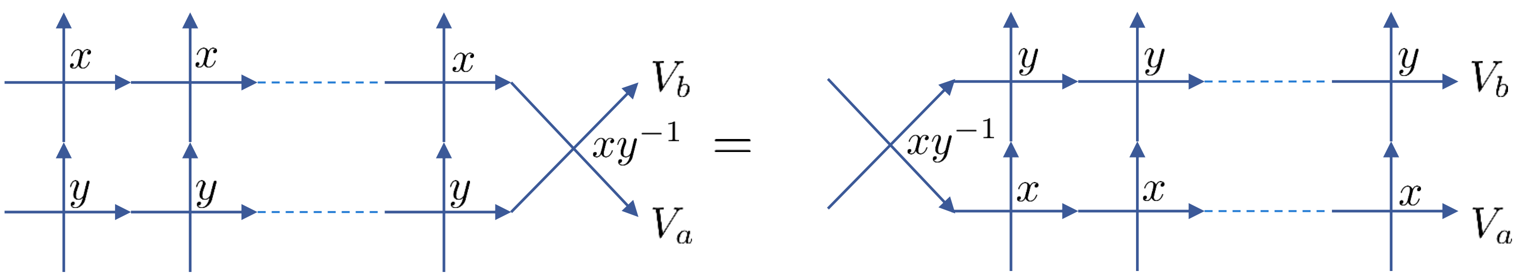
- Monodromy matrix $T_{a0}(z)$: $V_a \otimes V_0 \to V_a \otimes V_0$



$V_0 = V^{\otimes L}$: physical sp.
V_a : auxiliary sp.

- By repeated uses of the Yang-Baxter equation, we have

$$R_{ab}(xy^{-1})T_{a0}(x)T_{b0}(y) = T_{b0}(y)T_{a0}(x)R_{ab}(xy^{-1})$$



Commuting transfer matrix

- Multiply $R_{ab}(xy^{-1})^{-1}$ from the left:

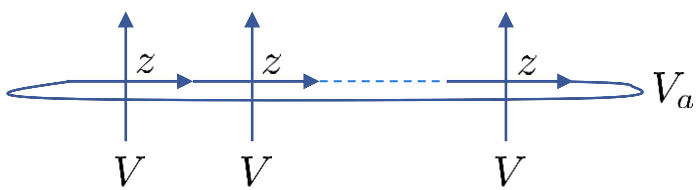
$$T_{a0}(x)T_{b0}(y) = [R_{ab}(xy^{-1})]^{-1}T_{b0}(y)T_{a0}(x)R_{ab}(xy^{-1})$$

- By taking the trace on $V_a \otimes V_b$, we have

$$[\tau(x), \tau(y)] = 0 \quad ({}^\forall x, y \in \mathbb{C})$$

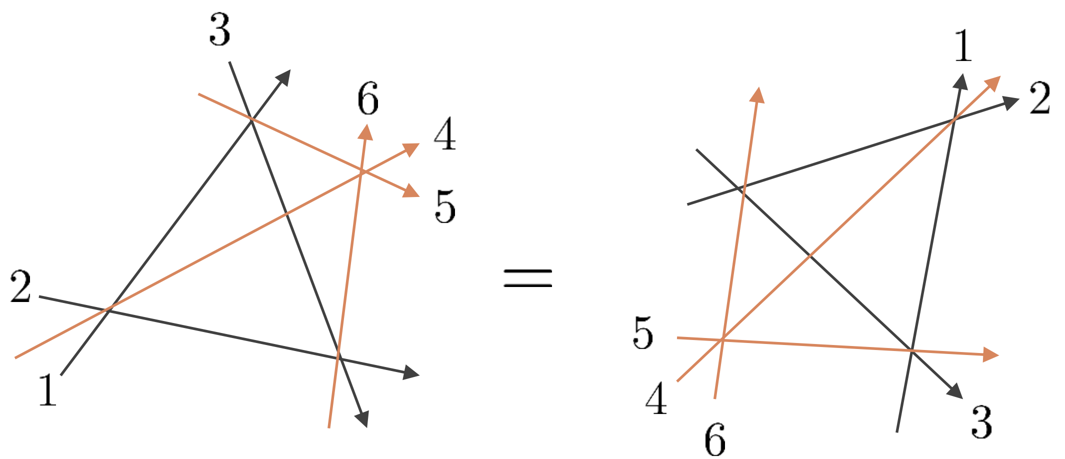
  - Here we set the row-to-row transfer matrix by

$$\tau(z) = \mathrm{Tr}_a(T_{a0}(z)) =$$



- A lot of families of <u>integrable</u> one-dimensional quantum spin chains are constructed via commutativty of the transfer matrix.
  - The transfer matrix gives $O(L)$ conserved quantities.
  - Eigenvalues are obtained by the Bethe ansatz.

Tetrahedron equation



- Matrix equation on $V_1 \otimes \cdots \otimes V_6$ (V_i: linear space)

$$\mathcal{R}_{124}\mathcal{R}_{135}\mathcal{R}_{236}\mathcal{R}_{456} = \mathcal{R}_{456}\mathcal{R}_{236}\mathcal{R}_{135}\mathcal{R}_{124}$$

  - R_{ijk} acts non-trivially only on $V_i \otimes V_j \otimes V_k$.
  - Tetrahedron equation $=$ Yang-Baxter equation *up to conjugation*

- Unlike Yang-Baxter equation, a few families of solutions are known.

- We focus on solutions on the Fock spaces.

boson Fock: $F = \bigoplus_{m=0,1,2,\cdots} \mathbb{C}\,|m\rangle$

fermi Fock: $V = \bigoplus_{m=0,1} \mathbb{C}u_m$

Solutions to tetrahedron equation: 3D R

- Set $\mathcal{R} \in \mathrm{End}\left(F^{\otimes 3}\right)$ by [Kapranov-Voevodsky94]

$$\mathcal{R}\,|i\rangle \otimes |j\rangle \otimes |k\rangle = \sum_{a,b,c} \mathcal{R}^{abc}_{ijk}\,|a\rangle \otimes |b\rangle \otimes |c\rangle$$

$$\mathcal{R}^{a,b,c}_{i,j,k} = \delta^{a+b}_{i+j}\delta^{b+c}_{j+k} \sum_{\lambda,\mu\geq 0,\lambda+\mu=b} (-1)^{\lambda} q^{i(c-j)+(k+1)\lambda+\mu(\mu-k)} \frac{(q^2)_{c+\mu}}{(q^2)_c} \binom{i}{\mu}_{q^2} \binom{j}{\lambda}_{q^2}$$

Here

$$(q)_k = \prod_{l=1}^{k}(1-q^l) \qquad \binom{a}{b}_q = \frac{(q)_a}{(q)_b(q)_{a-b}}$$

- The 3D R satisfies the following tetrahedron equation:

$$\mathcal{R}_{124}\mathcal{R}_{135}\mathcal{R}_{236}\mathcal{R}_{456} = \mathcal{R}_{456}\mathcal{R}_{236}\mathcal{R}_{135}\mathcal{R}_{124} \qquad \cdots (*)$$

- 3D R = intertwiner of irreps of quantum coordinate ring $A_q(A_2)$

$$\mathcal{R} \circ \pi_1 \otimes \pi_2 \otimes \pi_1(\Delta^{\mathrm{op}}(g)) = \pi_2 \otimes \pi_1 \otimes \pi_2(\Delta(g)) \circ \mathcal{R} \quad {}^{\forall} g \in A_q(A_2)$$

$$\pi_i : A_q(A_2) \to \mathrm{End}(F)$$

  - ``121″ and ``212″ are associated with the longest element of Weyl group.
  - This gives a linearization method for tetrahedron equation.

Solutions to tetrahedron equation: 3D L

- Set $\mathcal{L} \in \mathrm{End}\,(V \otimes V \otimes F)$ by [Bazhanov-Sergeev06]

$$\mathcal{L}(u_i \otimes u_j \otimes |k\rangle) = \sum_{a,b \in \{0,1\}, c \in \mathbb{Z}_{\geq 0}} \mathcal{L}_{i,j,k}^{a,b,c} u_a \otimes u_b \otimes |c\rangle$$

$$\mathcal{L}_{0,0,k}^{0,0,c} = \mathcal{L}_{1,1,k}^{1,1,c} = \delta_{k,c}, \quad \mathcal{L}_{0,1,k}^{0,1,c} = -\delta_{k,c} q^{k+1}, \quad \mathcal{L}_{1,0,k}^{1,0,c} = \delta_{k,c} q^{k},$$
$$\mathcal{L}_{1,0,k}^{0,1,c} = \delta_{k-1,c}(1-q^{2k}), \quad \mathcal{L}_{0,1,k}^{1,0,c} = \delta_{k+1,c}$$

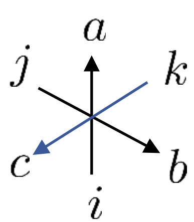
- The 3D L satisfies $\mathcal{L}_{124}\mathcal{L}_{135}\mathcal{L}_{236}\mathcal{R}_{456} = \mathcal{R}_{456}\mathcal{L}_{236}\mathcal{L}_{135}\mathcal{L}_{124} \quad \cdots (*)$
- BS obtained the 3D L by ansatz so that the tetrahedron equation of $(*)$ type has a non-trivial solution, and solved $(*)$ for the 3D R.
  - Later, $(*)$ is ``identified" with the intertwining relations for $A_q(A_2)$. [Kuniba-Okado12]
  - Algebraic origins of the 3D L has been still unclear.
- Recently, the classical limit of $(*)$ is derived in relation to non-trivial transformations of a plabic network, which can be interpreted as cluster mutations. [Gavrylenko-Semenyakin-Zenkevich20]

Monodromy matrices

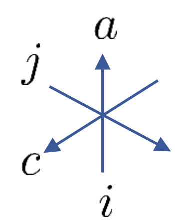
- The 3D R and L satisfy the following weight conservation:

$$[x^{\mathbf{h}_1}(xy)^{\mathbf{h}_2}y^{\mathbf{h}_3}, \mathcal{R}] = [x^{\mathbf{h}_1}(xy)^{\mathbf{h}_2}y^{\mathbf{h}_3}, \mathcal{L}] = 0 \quad ({}^{\forall}x, y \in \mathbb{C}) \quad \cdots (*)$$

  - Here, $\mathbf{h}_1 = \mathbf{h} \otimes 1 \otimes 1$ etc. and $\mathbf{h}\,|m\rangle = m\,|m\rangle$, $\mathbf{h}u_m = mu_m$.
- The discussion below holds for solutions satisfying $(*)$.
- We use the following graphical notations:

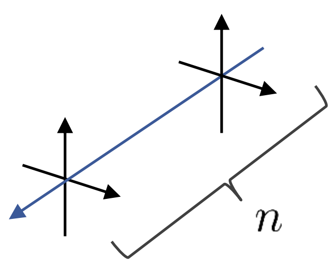


$= \mathcal{R}^{a,b,c}_{i,j,k}$ or $\mathcal{L}^{a,b,c}_{i,j,k}$

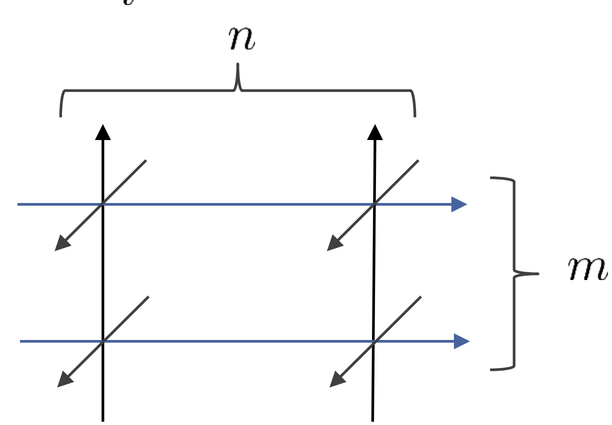


$= \mathcal{R}^{a,b,c}_{i,j,k}$

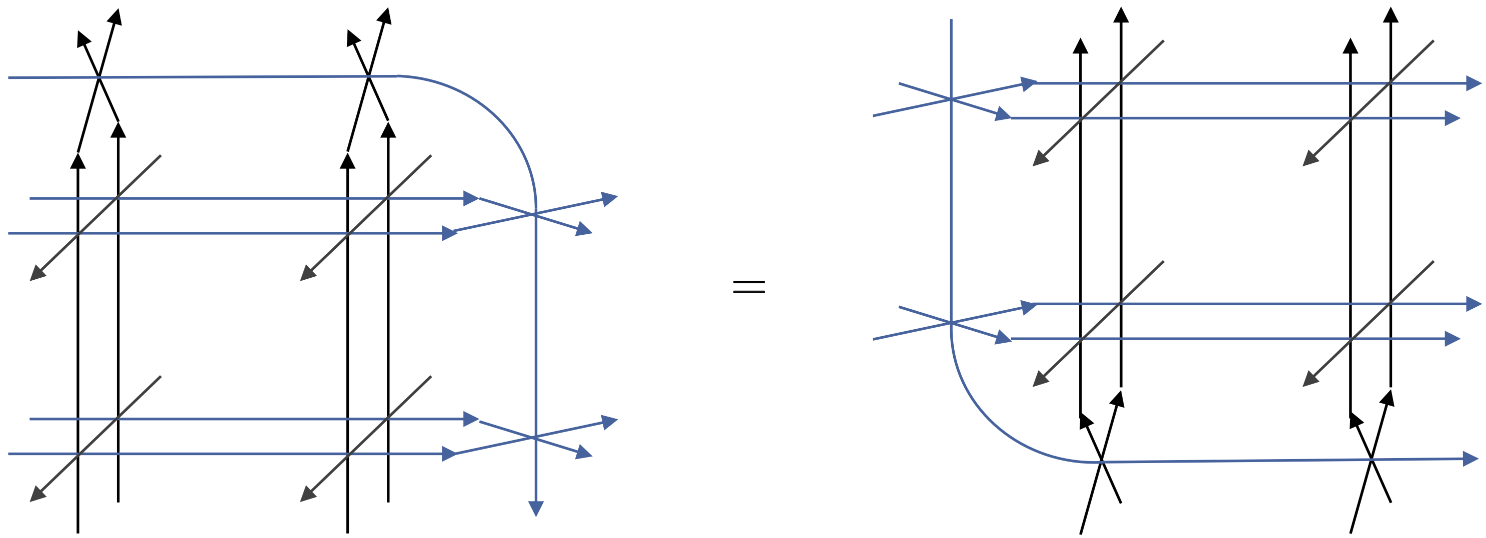
$S^{(n)} =$



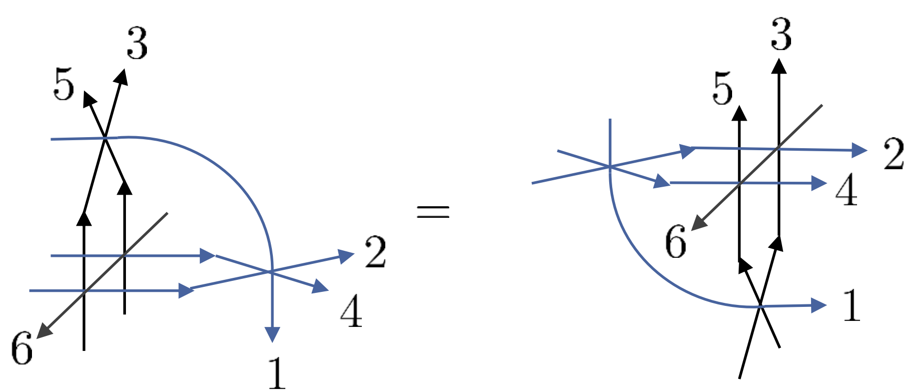
$T^{(m,n)} =$



STT=TTS: Commuting transfer matrix

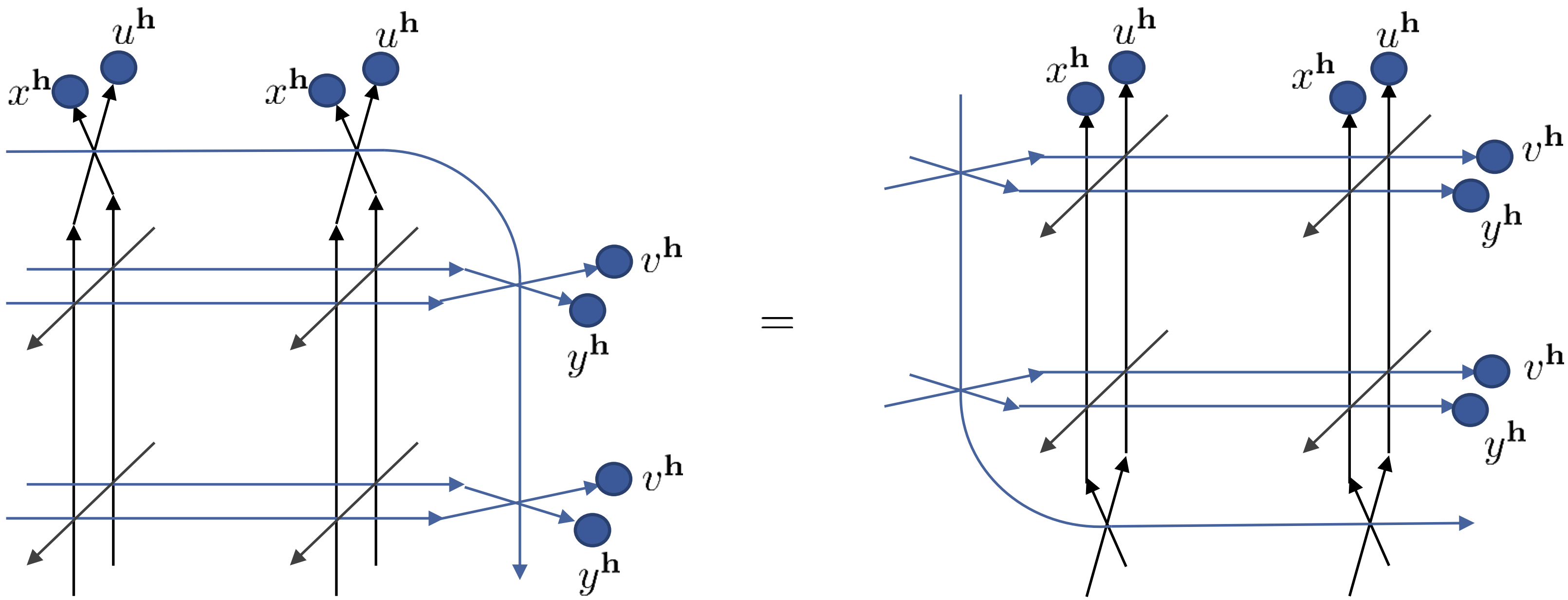


- The above equation is obtained by repreated use of



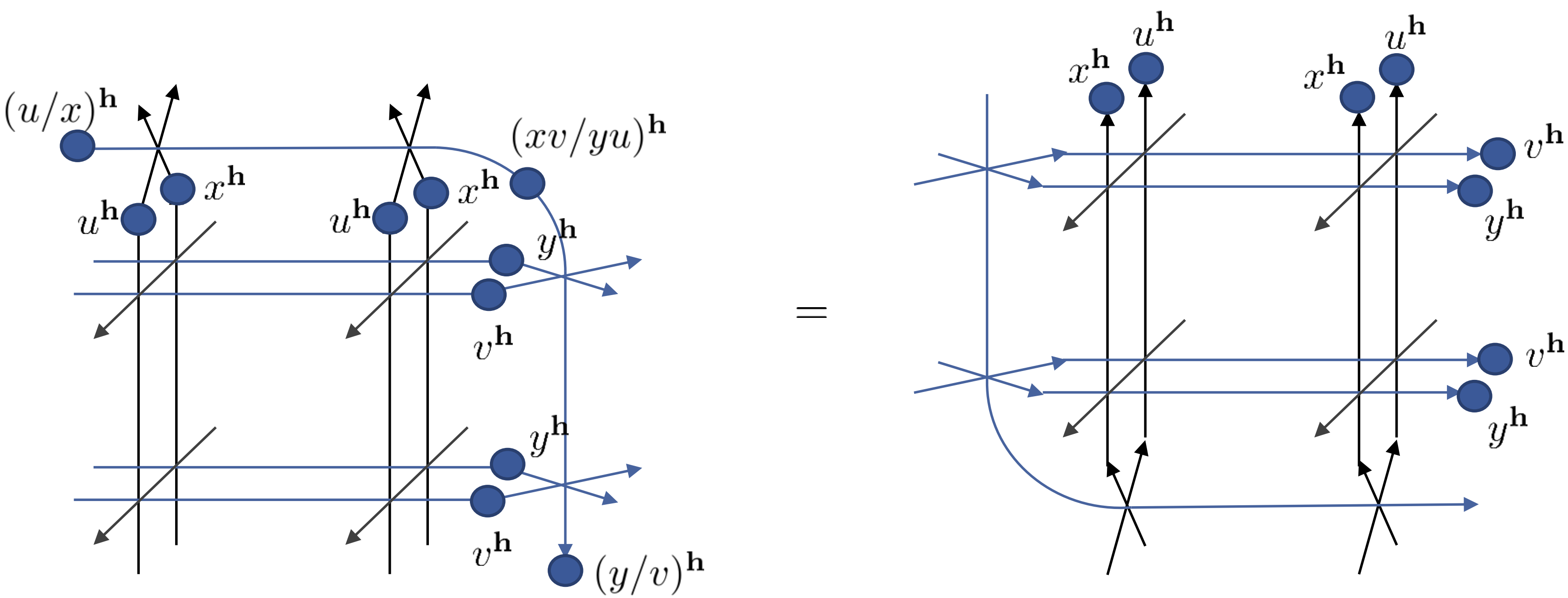
STT=TTS: Commuting transfer matrix

- Multiply $x^{\mathbf{h}}, y^{\mathbf{h}}, u^{\mathbf{h}}, v^{\mathbf{h}}$ by several spaces:



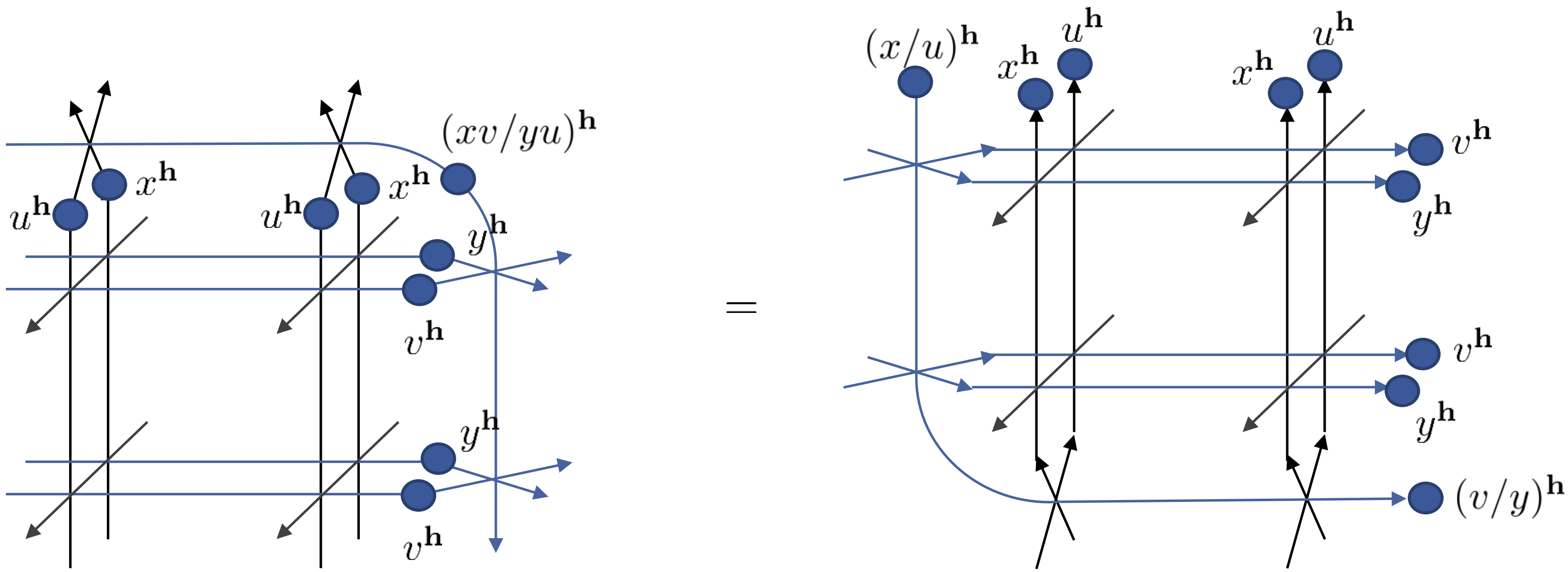
STT=TTS: Commuting transfer matrix

- Use the weight conservation $[x^{\mathbf{h}_1}(xy)^{\mathbf{h}_2}y^{\mathbf{h}_3}, \mathcal{R}] = [x^{\mathbf{h}_1}(xy)^{\mathbf{h}_2}y^{\mathbf{h}_3}, \mathcal{L}] = 0$



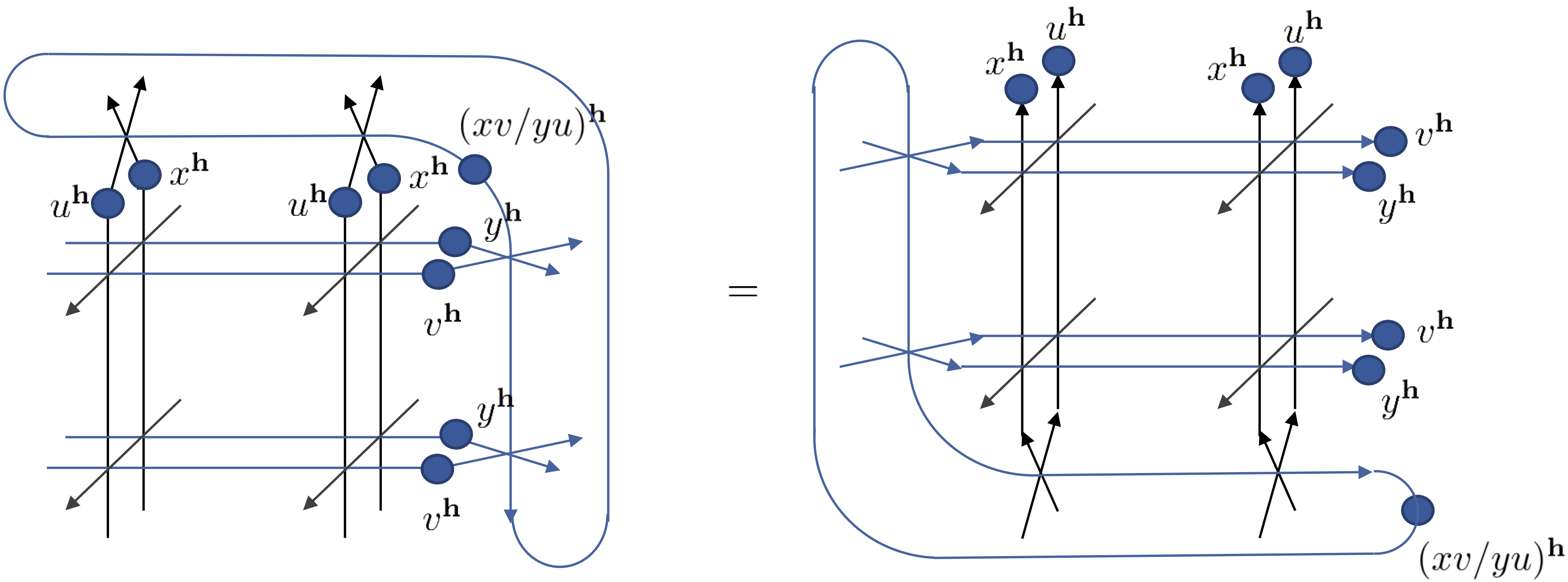
STT=TTS: Commuting transfer matrix

- Move $(u/x)^{\mathbf{h}}$ and $(y/v)^{\mathbf{h}}$ to the right hand side:



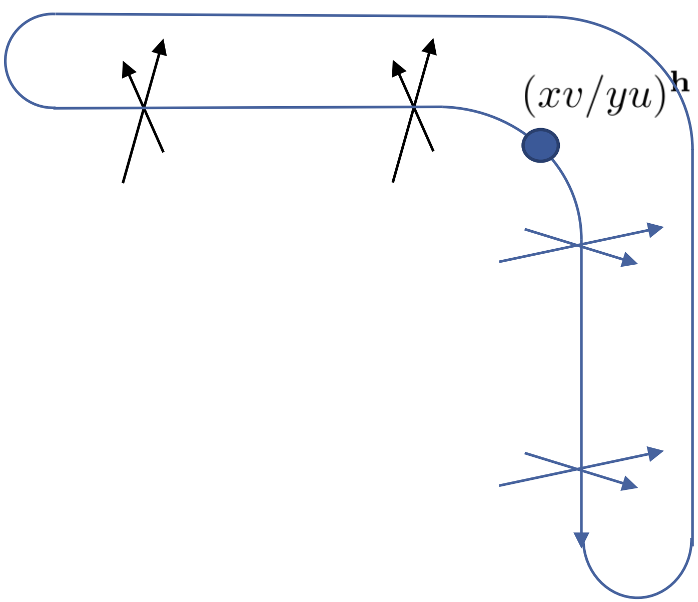
STT=TTS: Commuting transfer matrix

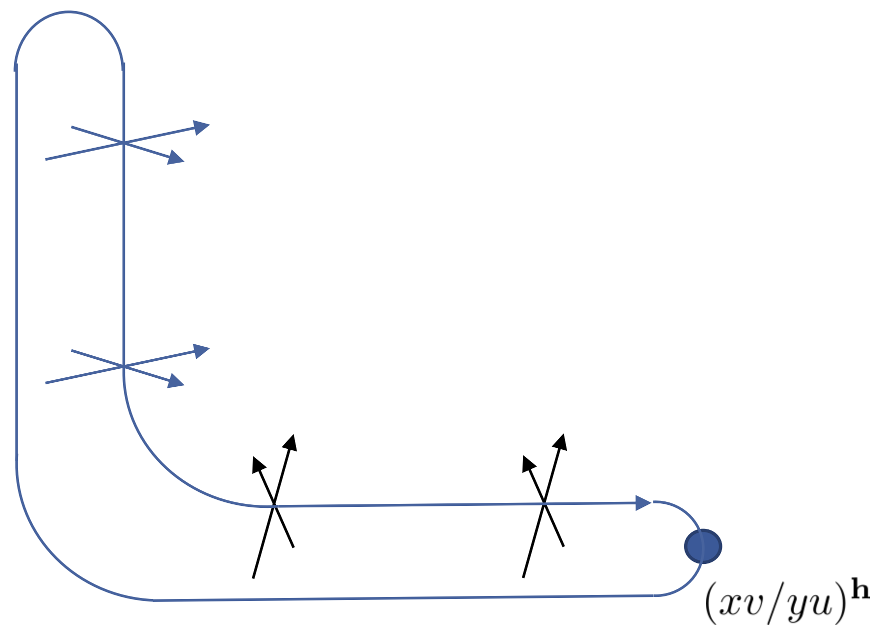
- Take the trace the auxiliary space:



STT=TTS: Commuting transfer matrix

- Note that the following parts are actually same matrices:



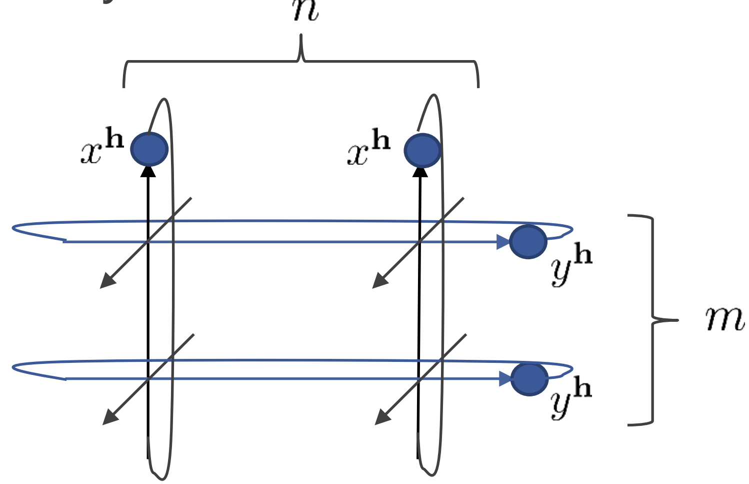


- Then, if the above matrix is invertible, we can verify the commutativity by taking the trace on all auxiliary spaces.

STT=TTS: Commuting transfer matrix

- Define the transfer matrix by

$$\tau^{(m,n)}(x,y) =$$



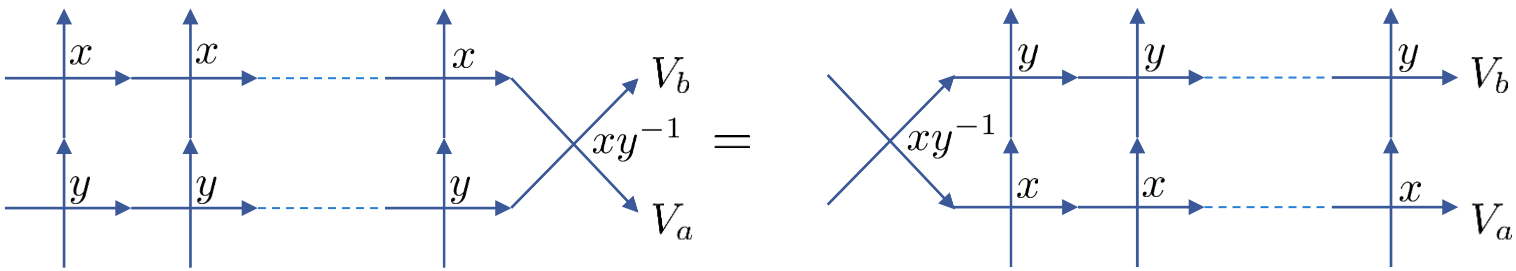
- This satisfies the following commutativity: [Sergeev06]

$$[\tau^{(m,n)}(x,y), \tau^{(m,n)}(x',y')] = 0 \quad ({}^{\forall}x,y,x',y' \in \mathbb{C})$$

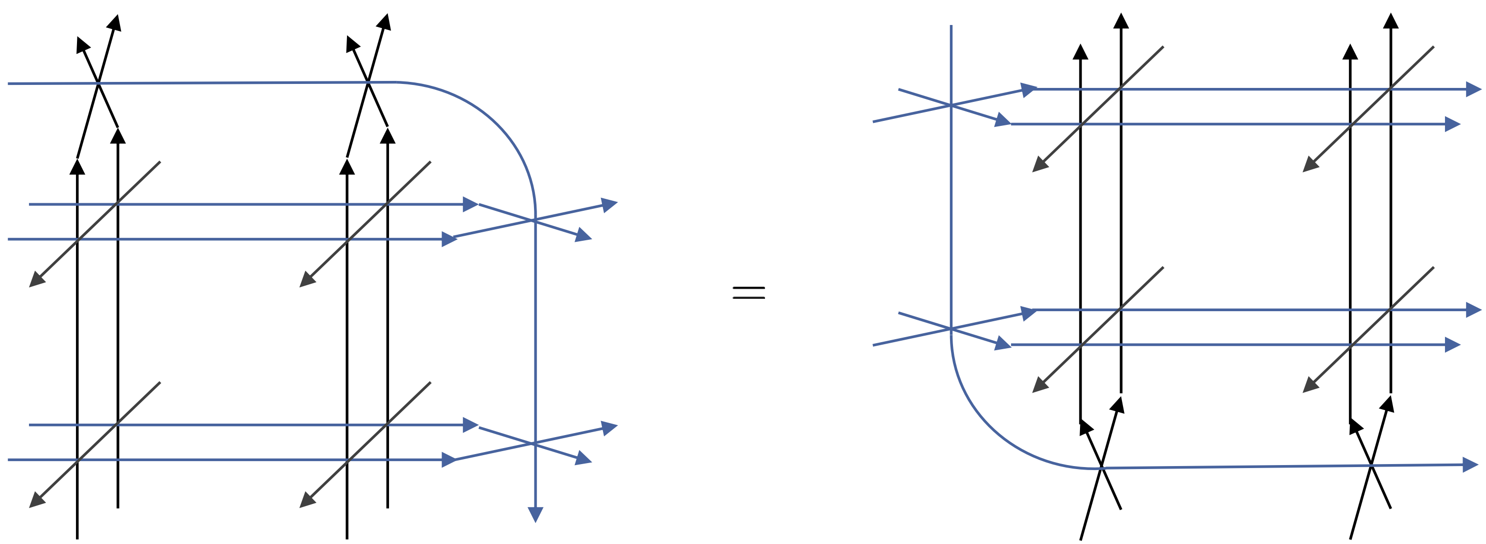
- This is often called the layer-to-layer transfer matrix.

Comparing RTT relations in 2D and 3D

- $1 + 1 + 0$ dimension in 2D

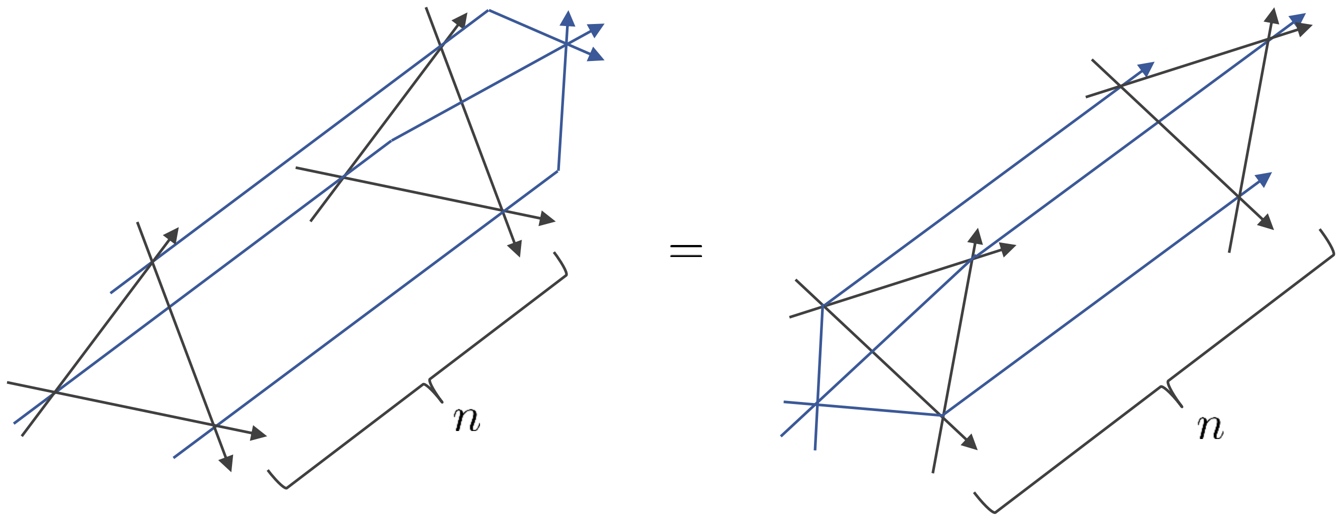


- $2 + 2 + 1$ dimension in 3D

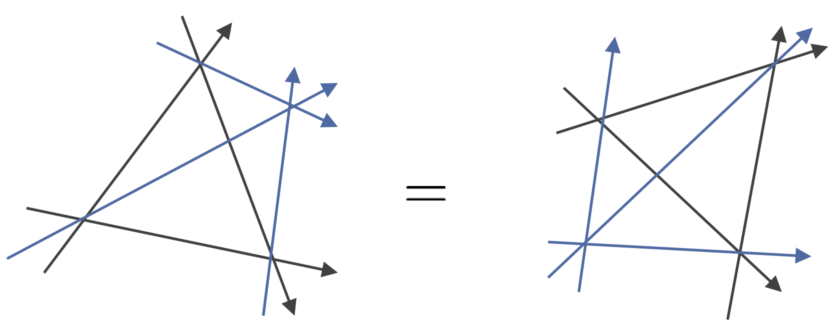


RSSS=SSSR: Reduction to Yang-Baxter eq

- $1+1+1+0$ dimension in 3D

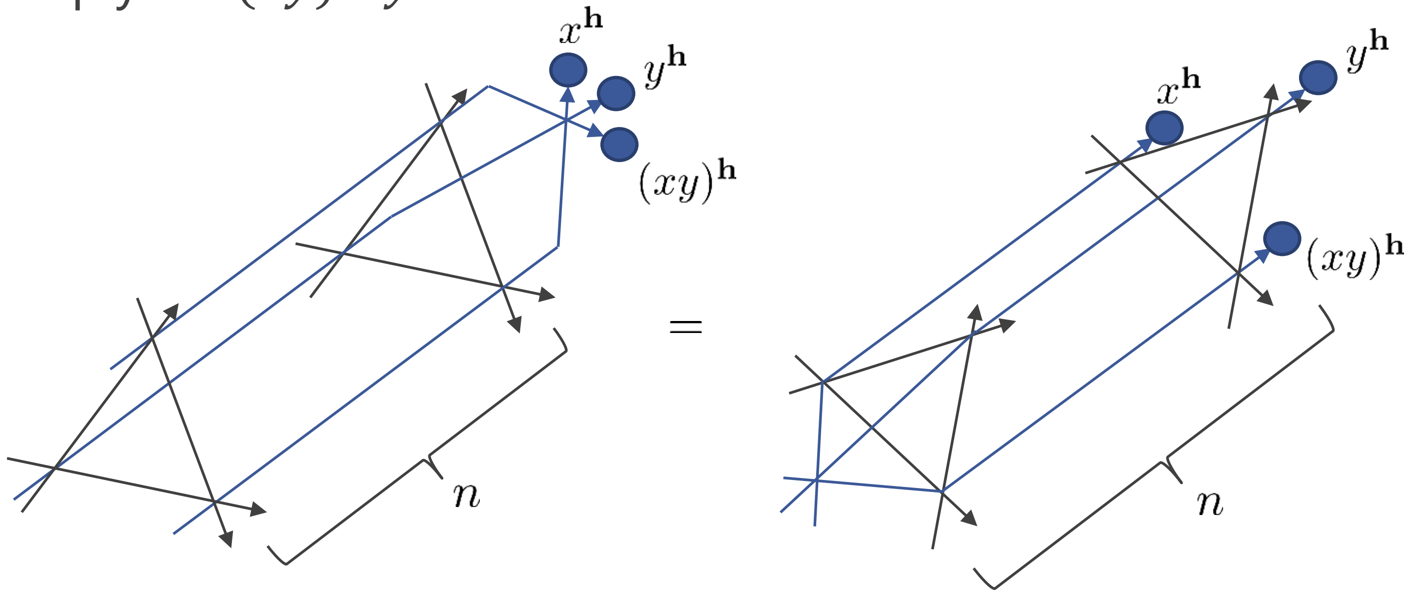


- The above equation is obtained by repreated use of



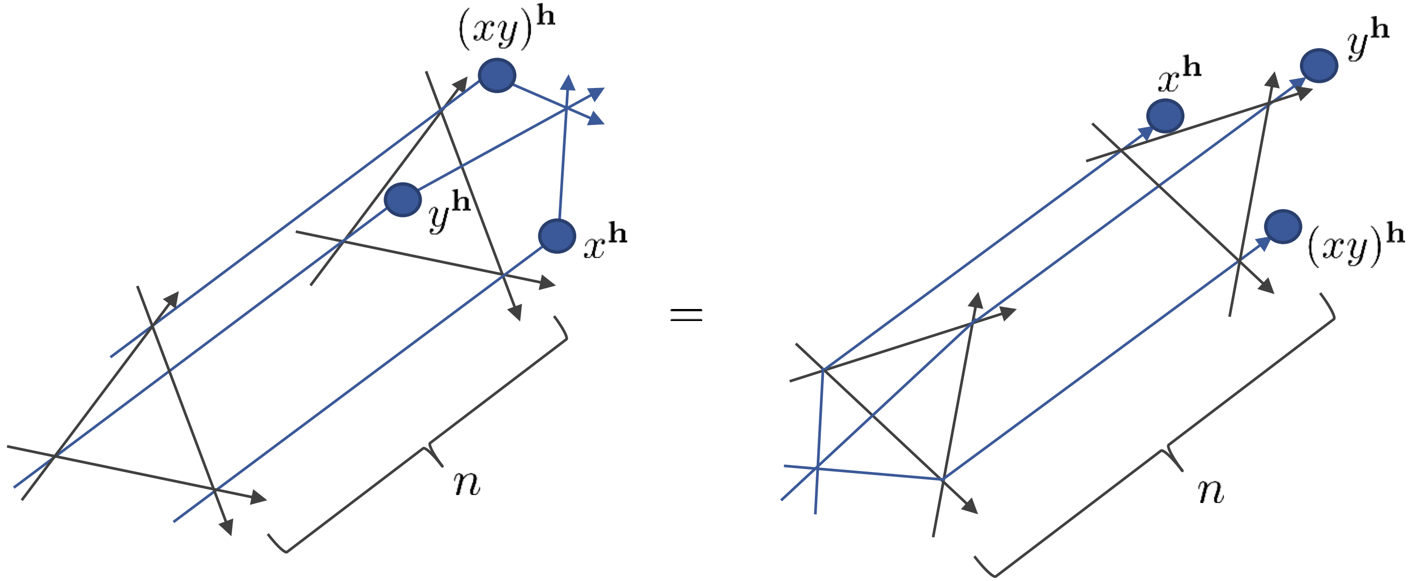
RSSS=SSSR: Reduction to Yang-Baxter eq

- Multiply $x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5}y^{\mathbf{h}_6}$:



RSSS=SSSR: Reduction to Yang-Baxter eq

- Use the weight conservation $[x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5}y^{\mathbf{h}_6}, \mathcal{R}] = 0$:



- This gives the following identity:

$$
\begin{aligned}
&(x^{\mathbf{h_4}}\mathcal{R}_{1_1 2_1 4}\cdots\mathcal{R}_{1_n 2_n 4})((xy)^{\mathbf{h_5}}\mathcal{R}_{1_1 3_1 5}\cdots\mathcal{R}_{1_n 3_n 5})(y^{\mathbf{h_6}}\mathcal{R}_{2_1 3_1 6}\cdots\mathcal{R}_{2_n 3_n 6})\mathcal{R}_{456} \\
&= \mathcal{R}_{456}(y^{\mathbf{h_6}}\mathcal{R}_{2_1 3_1 6}\cdots\mathcal{R}_{2_n 3_n 6})((xy)^{\mathbf{h_5}}\mathcal{R}_{1_1 3_1 5}\cdots\mathcal{R}_{1_n 3_n 5})(x^{\mathbf{h_4}}\mathcal{R}_{1_1 2_1 4}\cdots\mathcal{R}_{1_n 2_n 4})
\end{aligned}
$$

RSSS=SSSR: Reduction to Yang-Baxter eq

$$(x^{\mathbf{h_4}}\mathcal{R}_{1_1 2_1 4}\cdots\mathcal{R}_{1_n 2_n 4})((xy)^{\mathbf{h_5}}\mathcal{R}_{1_1 3_1 5}\cdots\mathcal{R}_{1_n 3_n 5})(y^{\mathbf{h_6}}\mathcal{R}_{2_1 3_1 6}\cdots\mathcal{R}_{2_n 3_n 6})\mathcal{R}_{456}$$
$$= \mathcal{R}_{456}(y^{\mathbf{h_6}}\mathcal{R}_{2_1 3_1 6}\cdots\mathcal{R}_{2_n 3_n 6})((xy)^{\mathbf{h_5}}\mathcal{R}_{1_1 3_1 5}\cdots\mathcal{R}_{1_n 3_n 5})(x^{\mathbf{h_4}}\mathcal{R}_{1_1 2_1 4}\cdots\mathcal{R}_{1_n 2_n 4})$$

- From this identity, we can obtain solutions to Yang-Baxter eq by
  1. multiplying $\mathcal{R}_{456}^{-1}$ and taking the trace on the spaces 456.
  2. sandwiching between $\langle\chi_r|\otimes\langle\chi_r|\otimes\langle\chi_r|$ and $|\chi_{r'}\rangle\otimes|\chi_{r'}\rangle\otimes|\chi_{r'}\rangle$.

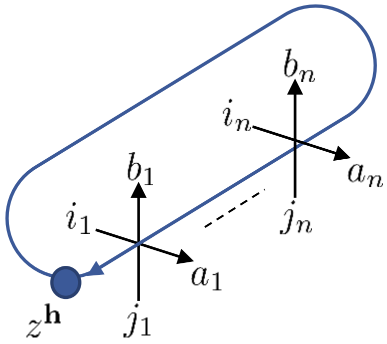
- Here, we use properties for the 3D R:
  1. The 3D R is invertible.
  2. The 3D R has two eigenvectors with eigenvalues = 1 given by

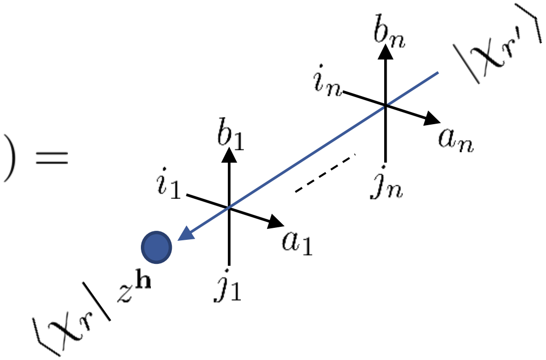
$$\mathcal{R}\,|\chi_r\rangle\otimes|\chi_r\rangle\otimes|\chi_r\rangle = |\chi_r\rangle\otimes|\chi_r\rangle\otimes|\chi_r\rangle \quad (r=1,2)$$
$$\langle\chi_r|\otimes\langle\chi_r|\otimes\langle\chi_r|\,\mathcal{R} = \langle\chi_r|\otimes\langle\chi_r|\otimes\langle\chi_r| \quad (r=1,2)$$

Here

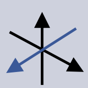
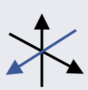
$$|\chi_r\rangle = \sum_{m\geq 0}\frac{|rm\rangle}{(q^{r^2};q^{r^2})_m}$$

Matrix product solution to Yang-Baxter eq

$R^{\mathrm{tr}}(z) =$ 

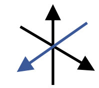
$R^{(r,r')}(z) =$ 

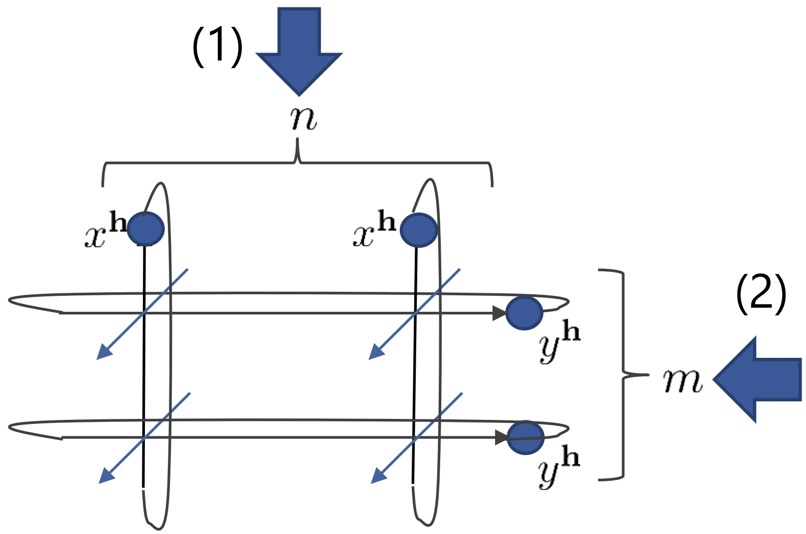
- These solutions are characterized as the R matrices associated with some quantum affine algebras. [Kuniba-Okado-Sergeev15]

	$R^{\mathrm{tr}}(z)$	$R^{(r,r')}(z)$
 = 3D R	$U_q(A_{n-1}^{(1)})$ symmetric tensor rep.	$U_q(D_{n+1}^{(2)})$, $U_q(A_{2n}^{(2)})$, $U_q(C_n^{(1)})$ Fock rep.
 = 3D L	$U_q(A_{n-1}^{(1)})$ fundamental rep.	$U_q(D_{n+1}^{(2)})$, $U_q(B_n^{(1)})$, $U_q(D_n^{(1)})$ spin rep.

- Moreover, by mixing uses of the 3D R & L, we also obtain the R matrices associated with generalized quantum groups.

Rank-Size duality

- We consider the following transfer matrix where  are the 3D L.



- Let us consider projections from two directions (1) & (2).
  - Both of them give the row-to-row transfer matrix in two dimension.
  - This suggest the spectral duality between $sl(m)$ spin chain of size n and $sl(n)$ spin chain of size m. [Bazhanov-Sergeev06]
- The duality also appears in the context of the five-dimensional gauge theory. [Mironov-Morozov-Runov-Zenkevich-Zotov13]

Motivation & KOY theorem

- Motivation
  - Why do the 3D R & L lead to such similar results, although they have totally different origins?
- We study transition matrices of PBW bases of $U_q^+(sl(m|n))$ motivated by the KOY theorem which holds for non-super cases.
- Theorem [Kuniba-Okado-Yamada13] (Rough Statement)
  - g: arbitrary finite-dimensional simple Lie algebra
  - Φ: intertwiner of irreducible representations of $A_q(g)$
  - γ: transition matrix of PBW bases of the nilpotent subalgebra of $U_q(g)$
  - Then, we have $\Phi = \gamma$.
- For ○—○, the 3D R gives the transition matrix for $U_q(A_2)$.
- One of our result is that the 3D L is exactly the transition matrix for the quantum superalgebra associated with ○—⊗.

Outline

Root system of Lie superalgebras $sl(m|n)$

- Setup
  - Weight lattice: $\mathcal{E}(m|n)_{\mathbb{Z}} = \sum_{i=1}^{m} \mathbb{Z}\epsilon_i \oplus \sum_{i=1}^{n} \mathbb{Z}\delta_i$

$$(\epsilon_i, \epsilon_j) = (-1)^{\theta}\delta_{i,j}, \quad (\delta_i, \delta_j) = -(-1)^{\theta}\delta_{i,j}, \quad (\epsilon_i, \delta_j) = 0 \qquad \theta = 0, 1$$

  - Parity: $p(\lambda) = \sum_{i=1}^{n} b_i \pmod 2$ for $\lambda = \sum_{i=1}^{m} a_i\epsilon_i + \sum_{i=1}^{n} b_i\delta_i \in \mathcal{E}(m|n)_{\mathbb{Z}}$
  - We set $\{\bar{\epsilon}_i\}_{1\le i\le m+n} = \{\epsilon_i\}_{1\le i\le m} \cup \{\delta_i\}_{1\le i\le n}$.

- Lie superalgebra $sl(m|n)$
  - Rank: $r = m + n - 1$
  - Simple roots: $\Pi = \{\alpha_1, \cdots, \alpha_r\}$, $\alpha_i = \bar{\epsilon}_i - \bar{\epsilon}_{i+1}$
  - Reduced positive roots:

$$\tilde{\Phi}^+ = \{\bar{\epsilon}_i - \bar{\epsilon}_j \ (1 \le i < j \le r+1)\}$$

  - Even & odd parts: $\tilde{\Phi}^+ = \tilde{\Phi}^+_{\mathrm{even}} \cup \tilde{\Phi}^+_{\mathrm{iso}}$

$$\tilde{\Phi}^+_{\mathrm{even}} = \{\alpha \in \tilde{\Phi}^+ \mid p(\alpha) = 0\}$$
$$\tilde{\Phi}^+_{\mathrm{iso}} = \{\alpha \in \tilde{\Phi}^+ \mid p(\alpha) = 1\}$$

Cartan matrix and Dynkin diagram

- Cartan matrix for Lie superalgebra $sl(m|n)$
  - $d_\alpha = (\alpha, \alpha)/2$ for $\alpha \in \tilde{\Phi}^+_{\text{even}}$, $d_\alpha = 1$ for $\alpha \in \tilde{\Phi}^+_{\text{iso}}$
  - $D = \text{diag}(d_1, \cdots, d_r)$, $d_i = d_{\alpha_i}$
  - Cartan matrix: $A = \left(a_{ij}\right)_{i,j \in I}$, $a_{ij} = (\alpha_i, \alpha_j)/d_i$ $\qquad I = \{1, \cdots, r\}$
  - Simple coroots: $\{h_i\}_{i \in I}$, $\alpha_j(h_i) = a_{ij}$
- Dynkin diagram for Cartan data (A, p)
  - Prepare r dots and decorate the i-th dot by

$$\bigcirc \text{ for } \alpha_i \in \tilde{\Phi}^+_{\text{even}}, \quad \bigotimes \text{ for } \alpha_i \in \tilde{\Phi}^+_{\text{iso}}$$

  - Connect them if $a_{ij} \neq 0$ $(i \neq j)$:

$$\overset{\bar{\epsilon}_1 - \bar{\epsilon}_2}{\times} \text{———} \overset{\bar{\epsilon}_2 - \bar{\epsilon}_3}{\times} \text{———} \cdots\cdots \text{———} \overset{\bar{\epsilon}_r - \bar{\epsilon}_{r+1}}{\times} \qquad \times\text{: } \bigcirc \text{ or } \bigotimes$$

- <u>Remark</u>
  - Dynkin diagrams do *not* correspond to Lie superalgebras themselves.

Quantum superalgebra $U_q(g(m|n))$

- $U_q(sl(m|n))$: quantum superalgebras associated with (A, p)
  - Generators: $e_i, f_i, k_i^{\pm 1} \ (i \in I)$
  - We use $q_i = q^{d_i}$.
  - Part of relations:

$$k_i^{\pm 1} k_i^{\mp 1} = 1, \quad k_i k_j = k_j k_i, \quad k_i e_j = q_i^{a_{ij}} e_j k_i, \quad k_i f_j = q_i^{-a_{ij}} f_j k_i,$$

$$e_i f_j - (-1)^{p(\alpha_i)p(\alpha_j)} f_j e_i = \delta_{i,j} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$$

Nilpotent subalgebra $U_q^+(sl(m|n))$

- $U_q^+(sl(m|n))$: nilpotent subalgebra generated by $\{e_i\}_{i\in I}$
  - Root space decomposition:

$$U_q^+(\mathfrak{sl}(m|n))_\alpha = \{g \mid k_i g = q_i^{\alpha(h_i)} g k_i \ (i \in I)\}$$

  - q-commutator:

$$[x,y]_q = xy - (-1)^{p(\alpha)p(\beta)} q^{-(\alpha,\beta)} yx \qquad \begin{array}{l} x \in U_q^+(\mathfrak{sl}(m|n))_\alpha \\ y \in U_q^+(\mathfrak{sl}(m|n))_\beta \end{array}$$

- Rest of relations:
  1. For $a_{ij} \neq 0 \ (i \neq j),\ \alpha_i \in \tilde{\Phi}^+_{\text{even}}$: $\quad e_i^2 e_j - (q+q^{-1}) e_i e_j e_i + e_j e_i^2 = 0$
  2. For $a_{ij} = 0$: $\quad [e_i, e_j] = 0$
  3. For $\alpha_i \in \tilde{\Phi}^+_{\text{iso}}$: $\quad [[[e_{i-1}, e_i]_q, e_{i+1}]_q, e_i] = 0$
  4. $\{f_i\}_{i\in I}$ satisfies same relations as (1)~(3).

- Remark
  - For $a_{ii} = 0$ and $\alpha_i \in \tilde{\Phi}^+_{\text{iso}}$, we have $e_i^2 = 0$ from (2).

PBW bases of $U_q^+(sl(m|n))$

- We define two partial orders O_1, O_2 on $\widetilde{\Phi}^+$.
  - For $\alpha = \bar{\epsilon}_a - \bar{\epsilon}_b,\ \beta = \bar{\epsilon}_c - \bar{\epsilon}_d \in \widetilde{\Phi}^+$, we define

$$\begin{array}{llcl} O_1: & \alpha < \beta & \Longleftrightarrow & a < c \text{ or } (a = c \text{ and } b < d) \\ O_2: & \alpha < \beta & \Longleftrightarrow & a > c \text{ or } (a = c \text{ and } b > d) \end{array}$$

- <u>Definition</u> (quantum root vector)
  - For $\beta \in \widetilde{\Phi}^+$, we define $e_\beta \in U_q^+(\mathfrak{sl}(m|n))_\beta$ in two ways depending on O_i.
  - For $\beta = \alpha_i$, we set $e_\beta = e_i$.
  - For $\beta = \alpha + \alpha_i$ ($\alpha = \bar{\epsilon}_a - \bar{\epsilon}_b \in \widetilde{\Phi}^+,\ a < i$), we set

$$e_\beta = \begin{cases} [e_i, e_\alpha]_q & \text{for } O_1 \quad (\alpha < \alpha_i) \\ [e_\alpha, e_i]_q & \text{for } O_2 \quad (\alpha_i < \alpha) \end{cases}$$

Example: Rank 3

- We consider the case of O_1.

$$\bar{\epsilon}_1 - \bar{\epsilon}_2 < \bar{\epsilon}_1 - \bar{\epsilon}_3 < \bar{\epsilon}_1 - \bar{\epsilon}_4 < \bar{\epsilon}_2 - \bar{\epsilon}_3 < \bar{\epsilon}_2 - \bar{\epsilon}_4 < \bar{\epsilon}_3 - \bar{\epsilon}_4$$

Example: Rank 3

- We consider the case of O_1.

$$\bar{\epsilon}_1 - \bar{\epsilon}_2 < \bar{\epsilon}_1 - \bar{\epsilon}_3 < \bar{\epsilon}_1 - \bar{\epsilon}_4 < \bar{\epsilon}_2 - \bar{\epsilon}_3 < \bar{\epsilon}_2 - \bar{\epsilon}_4 < \bar{\epsilon}_3 - \bar{\epsilon}_4$$

$$\alpha_1 < \alpha_1 + \alpha_2 < \alpha_1 + \alpha_2 + \alpha_3 < \alpha_2 < \alpha_2 + \alpha_3 < \alpha_3$$

Example: Rank 3

- We consider the case of O_1.

$$\bar{\epsilon}_1 - \bar{\epsilon}_2 < \bar{\epsilon}_1 - \bar{\epsilon}_3 < \bar{\epsilon}_1 - \bar{\epsilon}_4 < \bar{\epsilon}_2 - \bar{\epsilon}_3 < \bar{\epsilon}_2 - \bar{\epsilon}_4 < \bar{\epsilon}_3 - \bar{\epsilon}_4$$

$$\alpha_1 < \alpha_1 + \alpha_2 < \alpha_1 + \alpha_2 + \alpha_3 < \alpha_2 < \alpha_2 + \alpha_3 < \alpha_3$$

$$e_1 \qquad\qquad\qquad e_2 \qquad\qquad e_3$$

Example: Rank 3

- We consider the case of O_1.

$$
\begin{array}{ccccccccccc}
\bar{\epsilon}_1 - \bar{\epsilon}_2 & < & \bar{\epsilon}_1 - \bar{\epsilon}_3 & < & \bar{\epsilon}_1 - \bar{\epsilon}_4 & < & \bar{\epsilon}_2 - \bar{\epsilon}_3 & < & \bar{\epsilon}_2 - \bar{\epsilon}_4 & < & \bar{\epsilon}_3 - \bar{\epsilon}_4 \\
\\
\alpha_1 & < & \alpha_1 + \alpha_2 & < & \alpha_1 + \alpha_2 + \alpha_3 & < & \alpha_2 & < & \alpha_2 + \alpha_3 & < & \alpha_3 \\
\\
e_1 & & [e_2, e_1]_q & & & & e_2 & & [e_3, e_2]_q & & e_3
\end{array}
$$

Example: Rank 3

- We consider the case of O_1.

$$
\begin{array}{ccccccccccc}
\bar{\epsilon}_1 - \bar{\epsilon}_2 & < & \bar{\epsilon}_1 - \bar{\epsilon}_3 & < & \bar{\epsilon}_1 - \bar{\epsilon}_4 & < & \bar{\epsilon}_2 - \bar{\epsilon}_3 & < & \bar{\epsilon}_2 - \bar{\epsilon}_4 & < & \bar{\epsilon}_3 - \bar{\epsilon}_4 \\
\alpha_1 & < & \alpha_1 + \alpha_2 & < & \alpha_1 + \alpha_2 + \alpha_3 & < & \alpha_2 & < & \alpha_2 + \alpha_3 & < & \alpha_3 \\
e_1 & & [e_2, e_1]_q & & [e_3, [e_2, e_1]_q]_q & & e_2 & & [e_3, e_2]_q & & e_3
\end{array}
$$

PBW bases of $U_q^+(sl(m|n))$

- Theorem [Yamane94]
  - Let $\beta_1 < \cdots < \beta_l$ denote elements of $\tilde{\Phi}^+$ under O_i $\quad l = |\tilde{\Phi}^+|$
  - For $A = (a_1, \cdots, a_l)$ where a_t are given by

$$a_t \in \mathbb{Z}_{\geq 0} \text{ for } \beta_t \in \tilde{\Phi}^+_{\text{even}} \qquad a_t \in \{0, 1\} \text{ for } \beta_t \in \tilde{\Phi}^+_{\text{iso}}$$

  we define E_i^A by

$$E_i^A = e_{\beta_1}^{(a_1)} e_{\beta_2}^{(a_2)} \cdots e_{\beta_l}^{(a_l)}$$

  - Then

$$B_i = \{E_i^A \mid a_t \in \mathbb{Z}_{\geq 0} \ (\beta_t \in \tilde{\Phi}^+_{\text{even}}),\ a_t \in \{0,1\} \ (\beta_t \in \tilde{\Phi}^+_{\text{iso}})\}$$

  gives a basis of U_q.
  - $e_{\beta_t}^{(a_t)}$: divided power given by $e_{\beta_t}^{(a_t)} = e_{\beta_t}^{a_t} / [a_t]_{p_t}!$ $\quad p_t = q^{d_{\beta_t}}$
  - $[k]_q!$: factorial of q-number given by

$$[m]_q! = \prod_{k=1}^{m} [k]_q \qquad [k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}$$

Transition matrix of PBW bases

- We define the transition matrix γ of PBW bases as follows:

$$E_2^A = \sum_B \gamma_B^A E_1^{B^{\mathrm{op}}}$$

  - Here, we set $X^{\mathrm{op}} = (x_l, \cdots, x_1)$ for $X = (x_1, \cdots, x_l)$.
- From now on, we only consider the cases of rank 2 & 3.
  - The transition matrix for higher rank cases are constructed as a composite of one of rank 2.
  - For rank 3 cases, the composite takes the form of the tetrahedron eq.

Outline

Type A of rank 2 cases

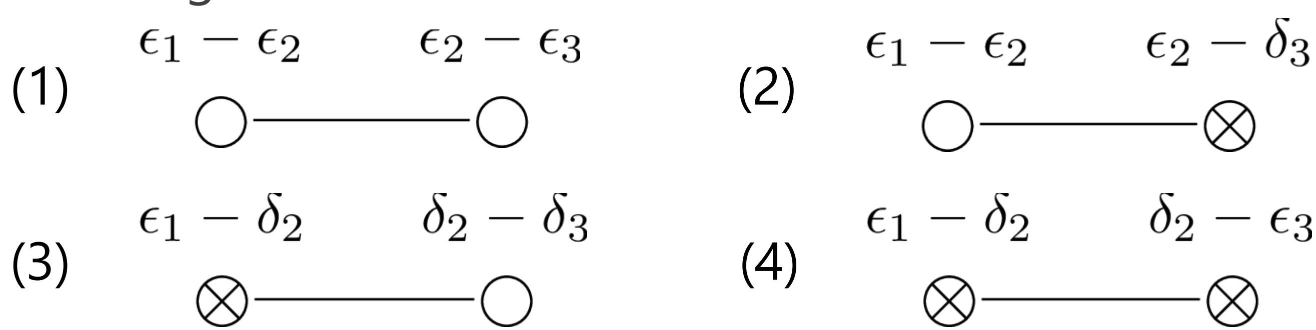
- We set $e_{ij} = [e_i, e_j]_q$
- Quantum root vectors of rank 2

$$\begin{aligned} B_1 :&\quad e_{\beta_1} = e_1, \quad e_{\beta_2} = e_{21}, \quad e_{\beta_3} = e_2 \\ B_2 :&\quad e_{\beta_1} = e_2, \quad e_{\beta_2} = e_{12}, \quad e_{\beta_3} = e_1 \end{aligned} \qquad \beta_1 < \beta_2 < \beta_3$$

- Transition matrix

$$e_2^{(a)} e_{12}^{(b)} e_1^{(c)} = \sum_{i,j,k} \gamma_{i,j,k}^{a,b,c} e_1^{(k)} e_{21}^{(j)} e_2^{(i)}$$

- Dynkin diagrams of rank 2



The case ○——○

$$\begin{array}{ccc} \epsilon_1 - \epsilon_2 & & \epsilon_2 - \epsilon_3 \\ \bigcirc & \text{———} & \bigcirc \end{array} \qquad (\epsilon_i, \epsilon_i) = 1 \qquad DA = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

- Root system

$$\tilde{\Phi}^{+}_{\text{even}} = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} \quad \tilde{\Phi}^{+}_{\text{iso}} = \{\}$$

- Transition matrix

$$e_2^{(a)} e_{12}^{(b)} e_1^{(c)} = \sum_{i,j,k} \gamma^{a,b,c}_{i,j,k} e_1^{(k)} e_{21}^{(j)} e_2^{(i)} \quad \cdots (*) \qquad i, j, k, a, b, c \in \mathbb{Z}$$

- Theorem [Kuniba-Okado-Yamada13] $\gamma^{a,b,c}_{i,j,k} = \mathcal{R}^{a,b,c}_{i,j,k}$

- Example For $(a, b, c) = (0,1,1)$, $(*)$ becomes

$$e_{12} e_1 = \mathcal{R}^{0,1,1}_{0,1,1} e_1 e_{21} + \mathcal{R}^{0,1,1}_{1,0,2} \frac{e_1^2}{q + q^{-1}} e_2$$

$$-q e_2 e_1^2 + (1 + q^2) e_1 e_2 e_1 - q e_1^2 e_2 = 0 \qquad \because \mathcal{R}^{0,1,1}_{0,1,1} = -q^2, \ \mathcal{R}^{0,1,1}_{1,0,2} = 1 - q^4$$

This is a relation of ○——○.

The case $\bigcirc\!\!-\!\!\!-\!\!\!-\!\!\otimes$

$$\begin{array}{ccc} \epsilon_1-\epsilon_2 & & \epsilon_2-\delta_3 \\ \bigcirc & \text{———} & \otimes \end{array} \qquad \begin{array}{l} (\epsilon_i,\epsilon_i)=1 \\ (\delta_i,\delta_i)=-1 \end{array} \qquad DA=\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$$

- Root system

$$\tilde{\Phi}^+_{\text{even}}=\{\alpha_1\} \quad \tilde{\Phi}^+_{\text{iso}}=\{\alpha_2,\alpha_1+\alpha_2\}$$

- Transition matrix

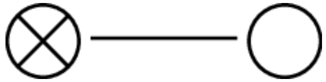
$$e_2^a e_{12}^b e_1^{(c)} = \sum_{i,j,k} \gamma^{a,b,c}_{i,j,k} e_1^{(k)} e_{21}^j e_2^i \quad \cdots(*) \qquad \begin{array}{l} k,c\in\mathbb{Z} \\ i,j,a,b\in\{0,1\} \end{array}$$

- Theorem [Y20] $\quad \gamma^{a,b,c}_{i,j,k}=\mathcal{L}^{a,b,c}_{i,j,k}$
- Example For $(a,b,c)=(0,1,1)$, $(*)$ becomes

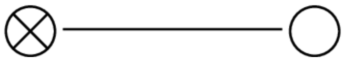
$$e_{12}e_1=\mathcal{L}^{0,1,1}_{0,1,1}e_1e_{21}+\mathcal{L}^{0,1,1}_{1,0,2}\frac{e_1^2}{q+q^{-1}}e_2$$

$$-qe_2e_1^2+(1+q^2)e_1e_2e_1-qe_1^2e_2=0 \qquad \because \mathcal{L}^{0,1,1}_{0,1,1}=-q^2,\ \mathcal{L}^{0,1,1}_{1,0,2}=1-q^4$$

This is a relation of $\bigcirc\!\!-\!\!\!-\!\!\!-\!\!\otimes$.

The case 

$\epsilon_1 - \delta_2 \qquad \delta_2 - \delta_3$



$$(\epsilon_i, \epsilon_i) = -1 \qquad (\delta_i, \delta_i) = 1$$

$$DA = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$$

- Root system

$$\tilde{\Phi}^+_{\mathrm{even}} = \{\alpha_2\} \quad \tilde{\Phi}^+_{\mathrm{iso}} = \{\alpha_1, \alpha_1 + \alpha_2\}$$

- Transition matrix

$$e_2^{(a)} e_{12}^b e_1^c = \sum_{i,j,k} \gamma^{a,b,c}_{i,j,k} e_1^k e_{21}^j e_2^{(i)} \qquad \cdots (*) \qquad \begin{array}{l} i, a \in \mathbb{Z} \\ j,k,b,c \in \{0,1\} \end{array}$$

- <u>Corollary</u> $\gamma^{a,b,c}_{i,j,k} = \mathcal{M}^{a,b,c}_{i,j,k}$
  - Here, we define $\mathcal{M} \in \mathrm{End}\,(F \otimes V \otimes V)$ by $\mathcal{M}^{a,b,c}_{i,j,k} = \mathcal{L}^{c,b,a}_{k,j,i}$.

The case $\otimes$——$\otimes$

$$\begin{array}{ccc} \epsilon_1 - \delta_2 & & \delta_2 - \epsilon_3 \\ \otimes & \text{———} & \otimes \end{array} \qquad \begin{array}{l} (\epsilon_i, \epsilon_i) = -1 \\ (\delta_i, \delta_i) = 1 \end{array} \qquad DA = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

- Root system

$$\tilde{\Phi}^+_{\mathrm{even}} = \{\alpha_1 + \alpha_2\} \quad \tilde{\Phi}^+_{\mathrm{iso}} = \{\alpha_1, \alpha_2\}$$

- Transition matrix

$$e_2^a e_{12}^{(b)} e_1^c = \sum_{i,j,k} \gamma_{i,j,k}^{a,b,c} e_1^k e_{21}^{(j)} e_2^i \qquad \begin{array}{l} j, b \in \mathbb{Z} \\ i, k, a, c \in \{0, 1\} \end{array}$$

- Theorem [Y20] $\gamma_{i,j,k}^{a,b,c} = \mathcal{N}_{i,j,k}^{a,b,c}$
  - Here, we define $\mathcal{N} \in \mathrm{End}\,(V \otimes F \otimes V)$ as follows:

$$\mathcal{N}(u_i \otimes |j\rangle \otimes u_k) = \sum_{a,c \in \{0,1\}, b \in \mathbb{Z}_{\geq 0}} \mathcal{N}_{i,j,k}^{a,b,c} u_a \otimes |b\rangle \otimes u_c$$

$$\mathcal{N}_{0,j,0}^{0,b,0} = \delta_{j,b} q^j, \quad \mathcal{N}_{1,j,1}^{1,b,1} = -\delta_{j,b} q^{j+1}, \quad \mathcal{N}_{0,j,1}^{0,b,1} = \mathcal{N}_{1,j,0}^{1,b,0} = \delta_{j,b},$$

$$\mathcal{N}_{1,j,1}^{0,b,0} = \delta_{j+1,b} q^j (1 - q^2), \quad \mathcal{N}_{0,j,0}^{1,b,1} = \delta_{j-1,b} [j]_q,$$

Outline

Type A of rank 3 cases

- We set $e_{(ij)k}, e_{i(jk)} \in U_q^+(\mathfrak{sl}(m|n))$ as follows:

$$e_{(ij)k} = [e_{ij}, e_k]_q, \quad e_{i(jk)} = [e_i, e_{jk}]_q \qquad\qquad e_{ij} = [e_i, e_j]_q$$

  - For $|i - k| > 1$, we have $e_{(ij)k} = e_{i(jk)} =: e_{ijk}$.

- Quantum root vectors of rank 3

$$\begin{aligned} B_1 :\quad & e_{\beta_1} = e_1, \quad e_{\beta_2} = e_{21}, \quad e_{\beta_3} = e_{321}, \\ & e_{\beta_4} = e_2, \quad e_{\beta_5} = e_{32}, \quad e_{\beta_6} = e_3 \\ B_2 :\quad & e_{\beta_1} = e_3, \quad e_{\beta_2} = e_{23}, \quad e_{\beta_3} = e_2, \\ & e_{\beta_4} = e_{123}, \quad e_{\beta_5} = e_{12}, \quad e_{\beta_6} = e_1 \end{aligned} \qquad \beta_1 < \cdots < \beta_6$$

- Transition matrix

$$e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)} = \sum_{i_1,i_2,i_3,i_4,i_5,i_6} \gamma_{i_1,i_2,i_3,i_4,i_5,i_6}^{o_1,o_2,o_3,o_4,o_5,o_6} e_1^{(i_6)} e_{21}^{(i_5)} e_{321}^{(i_4)} e_2^{(i_3)} e_{32}^{(i_2)} e_3^{(i_1)}$$

Transition matrix of rank 2 in rank 3

- We use the following matrices:

$$e_2^{(a)} e_{12}^{(b)} e_1^{(c)} = \sum_{i,j,k} \Gamma^{(2|1)}{}^{a,b,c}_{i,j,k} e_1^{(k)} e_{21}^{(j)} e_2^{(i)}$$

$$e_3^{(a)} e_{23}^{(b)} e_2^{(c)} = \sum_{i,j,k} \Gamma^{(3|2)}{}^{a,b,c}_{i,j,k} e_2^{(k)} e_{32}^{(j)} e_3^{(i)}$$

$$e_{23}^{(a)} e_{123}^{(b)} e_1^{(c)} = \sum_{i,j,k} \Gamma^{(23|1)}{}^{a,b,c}_{i,j,k} e_1^{(k)} e_{(23)1}^{(j)} e_{23}^{(i)}$$

$$e_{32}^{(a)} e_{1(32)}^{(b)} e_1^{(c)} = \sum_{i,j,k} \Gamma^{(32|1)}{}^{a,b,c}_{i,j,k} e_1^{(k)} e_{321}^{(j)} e_{32}^{(i)}$$

$$e_3^{(a)} e_{123}^{(b)} e_{12}^{(c)} = \sum_{i,j,k} \Gamma^{(3|12)}{}^{a,b,c}_{i,j,k} e_{12}^{(k)} e_{3(12)}^{(j)} e_3^{(i)}$$

$$e_3^{(a)} e_{(21)3}^{(b)} e_{21}^{(c)} = \sum_{i,j,k} \Gamma^{(3|21)}{}^{a,b,c}_{i,j,k} e_{21}^{(k)} e_{321}^{(j)} e_3^{(i)}$$

  - Given a Dynkin diagram, $\Gamma^{(x)}$ is specified as the 3D R, L, M or N.

Two ways construction of γ: first way

$$e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}$$

Two ways construction of γ: first way

$$\underline{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)}} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}$$

Two ways construction of γ: first way

$$\underline{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)}} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}$$
$$= \sum \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3} e_2^{(x_3)} e_{32}^{(x_2)} e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}$$

- For the underlined part, we used

$$e_3^{(a)} e_{23}^{(b)} e_2^{(c)} = \sum_{i,j,k} \Gamma^{(3|2)}{}^{a,b,c}_{i,j,k} e_2^{(k)} e_{32}^{(j)} e_3^{(i)}$$

Two ways construction of γ: first way

$$
\underline{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)}} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)}
$$

$$
= \sum \Gamma^{(3|2)}{}_{x_1,x_2,x_3}^{o_1,o_2,o_3} e_2^{(x_3)} e_{32}^{(x_2)} \underline{e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)}} e_1^{(o_6)}
$$

Two ways construction of γ: first way

$$
\begin{aligned}
&\underline{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)}}\, e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)} \\
&= \sum \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3}\, e_2^{(x_3)} e_{32}^{(x_2)}\, \underline{e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)}}\, e_1^{(o_6)} \\
&= \sum \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3}\, \Gamma^{(3|12)}{}^{x_1,o_4,o_5}_{i_1,x_4,x_5}\, e_2^{(x_3)} e_{32}^{(x_2)} e_{12}^{(x_5)}\; e_{3(12)}^{(x_4)}\; e_3^{(i_1)} e_1^{(o_6)}
\end{aligned}
$$

- For the underlined part, we used

$$
e_3^{(a)} e_{123}^{(b)} e_{12}^{(c)} = \sum_{i,j,k} \Gamma^{(3|12)}{}^{a,b,c}_{i,j,k}\, e_{12}^{(k)} e_{3(12)}^{(j)} e_3^{(i)}
$$

Two ways construction of γ: first way

$$
\begin{aligned}
&\underline{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)}}\, e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)} \\
&= \sum \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3}\, e_2^{(x_3)} e_{32}^{(x_2)}\, \underline{e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)}}\, e_1^{(o_6)} \\
&= \sum \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3}\, \Gamma^{(3|12)}{}^{x_1,o_4,o_5}_{i_1,x_4,x_5}\, e_2^{(x_3)}\, \underline{e_{32}^{(x_2)} e_{12}^{(x_5)}}\ \underline{e_{3(12)}^{(x_4)}}\ \underline{e_3^{(i_1)} e_1^{(o_6)}}
\end{aligned}
$$

Two ways construction of γ: first way

$$\rho_1 = p(\alpha_1)p(\alpha_3)$$

$$\rho_2 = p(\alpha_1+\alpha_2)p(\alpha_2+\alpha_3)$$

$$\begin{aligned}
&\underline{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)}}\, e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)} \\
&= \sum \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3}\, e_2^{(x_3)} e_{32}^{(x_2)}\, \underline{e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)}}\, e_1^{(o_6)} \\
&= \sum \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3}\, \Gamma^{(3|12)}{}^{x_1,o_4,o_5}_{i_1,x_4,x_5}\, e_2^{(x_3)}\, \underline{e_{32}^{(x_2)} e_{12}^{(x_5)}}\; \underline{e_{3(12)}^{(x_4)}}\; \underline{e_3^{(i_1)} e_1^{(o_6)}} \\
&= \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5}\, \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3}\, \Gamma^{(3|12)}{}^{x_1,o_4,o_5}_{i_1,x_4,x_5} \\
&\qquad \times e_2^{(x_3)} e_{12}^{(x_5)} e_{32}^{(x_2)} e_{1(32)}^{(x_4)} e_1^{(o_6)} e_3^{(i_1)}
\end{aligned}$$

- For the underlined part, we used

$$[e_{32}, e_{12}] = [e_3, e_1] = 0 \qquad e_{3(12)} = (-1)^{p(\alpha_1)p(\alpha_3)} e_{1(32)}$$

Two ways construction of γ: first way

$$\rho_1 = p(\alpha_1)p(\alpha_3)$$
$$\rho_2 = p(\alpha_1+\alpha_2)p(\alpha_2+\alpha_3)$$

$$\begin{aligned}
&\underline{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)}}\, e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)} \\
&= \sum \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3} e_2^{(x_3)} e_{32}^{(x_2)} \underline{e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)}}\, e_1^{(o_6)} \\
&= \sum \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3} \Gamma^{(3|12)}{}^{x_1,o_4,o_5}_{i_1,x_4,x_5} e_2^{(x_3)} \underline{e_{32}^{(x_2)} e_{12}^{(x_5)}}\ \underline{e_{3(12)}^{(x_4)}}\ \underline{e_3^{(i_1)} e_1^{(o_6)}} \\
&= \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5} \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3} \Gamma^{(3|12)}{}^{x_1,o_4,o_5}_{i_1,x_4,x_5} \\
&\qquad \times e_2^{(x_3)} e_{12}^{(x_5)} \underline{e_{32}^{(x_2)} e_{1(32)}^{(x_4)} e_1^{(o_6)}}\, e_3^{(i_1)}
\end{aligned}$$

Two ways construction of γ: first way

$$\rho_1 = p(\alpha_1)p(\alpha_3)$$

$$\rho_2 = p(\alpha_1+\alpha_2)p(\alpha_2+\alpha_3)$$

$$\begin{aligned}
&\underline{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)}} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)} \\
&= \sum \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3} e_2^{(x_3)} e_{32}^{(x_2)} \underline{e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)}} e_1^{(o_6)} \\
&= \sum \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3} \Gamma^{(3|12)}{}^{x_1,o_4,o_5}_{i_1,x_4,x_5} e_2^{(x_3)} \underline{e_{32}^{(x_2)} e_{12}^{(x_5)}}\ \underline{e_{3(12)}^{(x_4)}}\ \underline{e_3^{(i_1)} e_1^{(o_6)}} \\
&= \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5} \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3} \Gamma^{(3|12)}{}^{x_1,o_4,o_5}_{i_1,x_4,x_5} \\
&\qquad \times e_2^{(x_3)} e_{12}^{(x_5)} \underline{e_{32}^{(x_2)} e_{1(32)}^{(x_4)} e_1^{(o_6)}} e_3^{(i_1)} \\
&= \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5} \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3} \Gamma^{(3|12)}{}^{x_1,o_4,o_5}_{i_1,x_4,x_5} \Gamma^{(32|1)}{}^{x_2,x_4,o_6}_{i_2,i_4,x_6} \\
&\qquad \times e_2^{(x_3)} e_{12}^{(x_5)} e_1^{(x_6)} e_{321}^{(i_4)} e_{32}^{(i_2)} e_3^{(i_1)}
\end{aligned}$$

- For the underlined part, we used

$$e_{32}^{(a)} e_{1(32)}^{(b)} e_1^{(c)} = \sum_{i,j,k} \Gamma^{(32|1)}{}^{a,b,c}_{i,j,k} e_1^{(k)} e_{321}^{(j)} e_{32}^{(i)}$$

Two ways construction of γ: first way

$$\rho_1 = p(\alpha_1)p(\alpha_3)$$
$$\rho_2 = p(\alpha_1+\alpha_2)p(\alpha_2+\alpha_3)$$
$$\rho_3 = p(\alpha_2)p(\alpha_1+\alpha_2+\alpha_3)$$

$$\begin{aligned}
&\underline{e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)}}\, e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)} \\
&= \sum \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3}\, e_2^{(x_3)} e_{32}^{(x_2)}\, \underline{e_3^{(x_1)} e_{123}^{(o_4)} e_{12}^{(o_5)}}\, e_1^{(o_6)} \\
&= \sum \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3} \Gamma^{(3|12)}{}^{x_1,o_4,o_5}_{i_1,x_4,x_5}\, e_2^{(x_3)}\, \underline{e_{32}^{(x_2)} e_{12}^{(x_5)}}\; \underline{e_{3(12)}^{(x_4)}}\; \underline{e_3^{(i_1)} e_1^{(o_6)}} \\
&= \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5} \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3} \Gamma^{(3|12)}{}^{x_1,o_4,o_5}_{i_1,x_4,x_5} \\
&\qquad \times e_2^{(x_3)} e_{12}^{(x_5)}\, \underline{e_{32}^{(x_2)} e_{1(32)}^{(x_4)} e_1^{(o_6)}}\, e_3^{(i_1)} \\
&= \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5} \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3} \Gamma^{(3|12)}{}^{x_1,o_4,o_5}_{i_1,x_4,x_5} \Gamma^{(32|1)}{}^{x_2,x_4,o_6}_{i_2,i_4,x_6} \\
&\qquad \times \underline{e_2^{(x_3)} e_{12}^{(x_5)} e_1^{(x_6)}}\, e_{321}^{(i_4)} e_{32}^{(i_2)} e_3^{(i_1)} \\
&= \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5} \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3} \Gamma^{(3|12)}{}^{x_1,o_4,o_5}_{i_1,x_4,x_5} \Gamma^{(32|1)}{}^{x_2,x_4,o_6}_{i_2,i_4,x_6} \Gamma^{(2|1)}{}^{x_3,x_5,x_6}_{i_3,i_5,i_6} \\
&\qquad \times e_1^{(i_6)} e_{21}^{(i_5)}\, \underline{e_2^{(i_3)} e_{321}^{(i_4)}}\, e_{32}^{(i_2)} e_3^{(i_1)} \\
&= \sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 i_3 i_4} \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3} \Gamma^{(3|12)}{}^{x_1,o_4,o_5}_{i_1,x_4,x_5} \Gamma^{(32|1)}{}^{x_2,x_4,o_6}_{i_2,i_4,x_6} \Gamma^{(2|1)}{}^{x_3,x_5,x_6}_{i_3,i_5,i_6} \\
&\qquad \times e_1^{(i_6)} e_{21}^{(i_5)} e_{321}^{(i_4)} e_2^{(i_3)} e_{32}^{(i_2)} e_3^{(i_1)}
\end{aligned}$$

Two ways construction of γ: second way

$$\rho_1 = p(\alpha_1)p(\alpha_3)$$
$$\rho_2 = p(\alpha_1+\alpha_2)p(\alpha_2+\alpha_3)$$
$$\rho_3 = p(\alpha_2)p(\alpha_1+\alpha_2+\alpha_3)$$

$$
\begin{aligned}
& e_3^{(o_1)} e_{23}^{(o_2)} \underline{e_2^{(o_3)} e_{123}^{(o_4)}} e_{12}^{(o_5)} e_1^{(o_6)} \\
&= (-1)^{\rho_3 o_3 o_4} e_3^{(o_1)} e_{23}^{(o_2)} e_{123}^{(o_4)} \underline{e_2^{(o_3)} e_{12}^{(o_5)} e_1^{(o_6)}} \\
&= \sum (-1)^{\rho_3 o_3 o_4} \Gamma^{(2|1)}{}^{o_3,o_5,o_6}_{x_3,x_5,x_6} e_3^{(o_1)} \underline{e_{23}^{(o_2)} e_{123}^{(o_4)} e_1^{(x_6)}} e_{21}^{(x_5)} e_2^{(x_3)} \\
&= \sum (-1)^{\rho_3 o_3 o_4} \Gamma^{(2|1)}{}^{o_3,o_5,o_6}_{x_3,x_5,x_6} \Gamma^{(23|1)}{}^{o_2,o_4,x_6}_{x_2,x_4,i_6} \underline{e_3^{(o_1)} e_1^{(i_6)}}\ \underline{e_{(23)1}^{(x_4)}}\ \underline{e_{23}^{(x_2)} e_{21}^{(x_5)}} e_2^{(x_3)} \\
&= \sum (-1)^{\rho_1(o_1 i_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 o_3 o_4} \Gamma^{(2|1)}{}^{o_3,o_5,o_6}_{x_3,x_5,x_6} \Gamma^{(23|1)}{}^{o_2,o_4,x_6}_{x_2,x_4,i_6} \\
&\qquad \times e_1^{(i_6)} \underline{e_3^{(o_1)} e_{(21)3}^{(x_4)} e_{21}^{(x_5)}} e_{23}^{(x_2)} e_2^{(x_3)} \\
&= \sum (-1)^{\rho_1(o_1 i_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 o_3 o_4} \Gamma^{(2|1)}{}^{o_3,o_5,o_6}_{x_3,x_5,x_6} \Gamma^{(23|1)}{}^{o_2,o_4,x_6}_{x_2,x_4,i_6} \Gamma^{(3|21)}{}^{o_1,x_4,x_5}_{x_1,i_4,i_5} \\
&\qquad \times e_1^{(i_6)} e_{21}^{(i_5)} e_{321}^{(i_4)} \underline{e_3^{(x_1)} e_{23}^{(x_2)} e_2^{(x_3)}} \\
&= \sum (-1)^{\rho_1(o_1 i_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 o_3 o_4} \Gamma^{(2|1)}{}^{o_3,o_5,o_6}_{x_3,x_5,x_6} \Gamma^{(23|1)}{}^{o_2,o_4,x_6}_{x_2,x_4,i_6} \Gamma^{(3|21)}{}^{o_1,x_4,x_5}_{x_1,i_4,i_5} \Gamma^{(3|2)}{}^{x_1,x_2,x_3}_{i_1,i_2,i_3} \\
&\qquad \times e_1^{(i_6)} e_{21}^{(i_5)} e_{321}^{(i_4)} e_2^{(i_3)} e_{23}^{(i_2)} e_3^{(i_1)}
\end{aligned}
$$

Mother of tetrahedron equation

- Now, $\{e_1^{(i_6)} e_{21}^{(i_5)} e_{321}^{(i_4)} e_2^{(i_3)} e_{23}^{(i_2)} e_3^{(i_1)}\}$ is linearly independent. Then, we have the following result.
- Theorem [Y20]

$$\begin{aligned}
&\sum (-1)^{\rho_1(i_1 o_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 i_3 i_4} \\
&\qquad \times \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3} \Gamma^{(3|12)}{}^{x_1,o_4,o_5}_{i_1,x_4,x_5} \Gamma^{(32|1)}{}^{x_2,x_4,o_6}_{i_2,i_4,x_6} \Gamma^{(2|1)}{}^{x_3,x_5,x_6}_{i_3,i_5,i_6} \\
&= \sum (-1)^{\rho_1(o_1 i_6 + x_4) + \rho_2 x_2 x_5 + \rho_3 o_3 o_4} \\
&\qquad \times \Gamma^{(2|1)}{}^{o_3,o_5,o_6}_{x_3,x_5,x_6} \Gamma^{(23|1)}{}^{o_2,o_4,x_6}_{x_2,x_4,i_6} \Gamma^{(3|21)}{}^{o_1,x_4,x_5}_{x_1,i_4,i_5} \Gamma^{(3|2)}{}^{x_1,x_2,x_3}_{i_1,i_2,i_3}
\end{aligned}$$

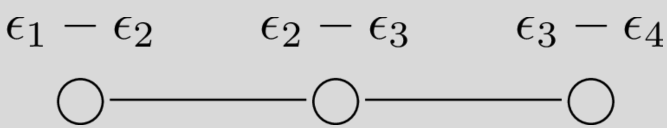
  - Here, sign factors are given by

$$\rho_1 = p(\alpha_1)p(\alpha_3), \quad \rho_2 = p(\alpha_1+\alpha_2)p(\alpha_2+\alpha_3), \quad \rho_3 = p(\alpha_2)p(\alpha_1+\alpha_2+\alpha_3)$$

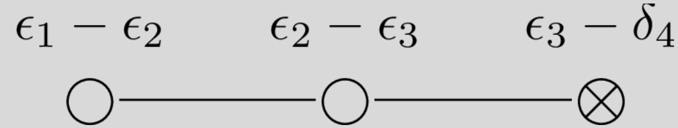
Dynkin diagrams of type A of rank 3

- Without exchanges $\epsilon \leftrightarrow \delta$, all Dynkin diagrams of rank 3 are given by

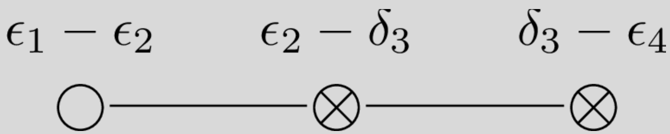
(1)



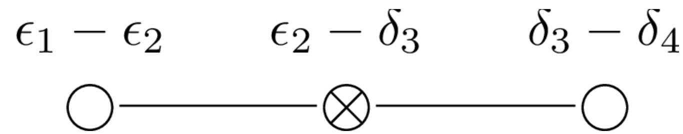
(2)



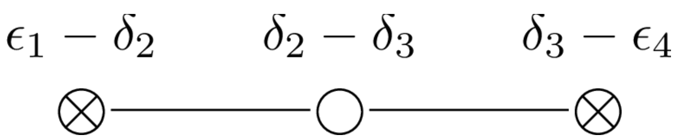
(3)



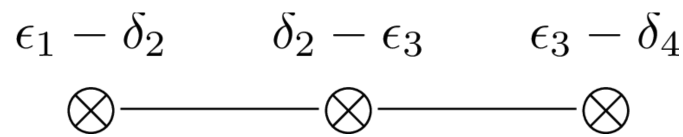
(4)



(5)

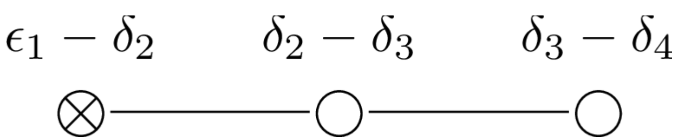


(6)

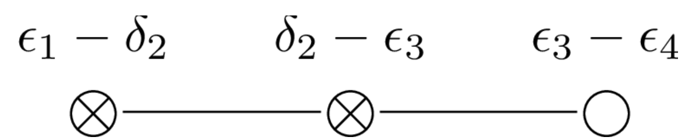


- The followings are easily attributed to (2) and (3), respectively.

(7)



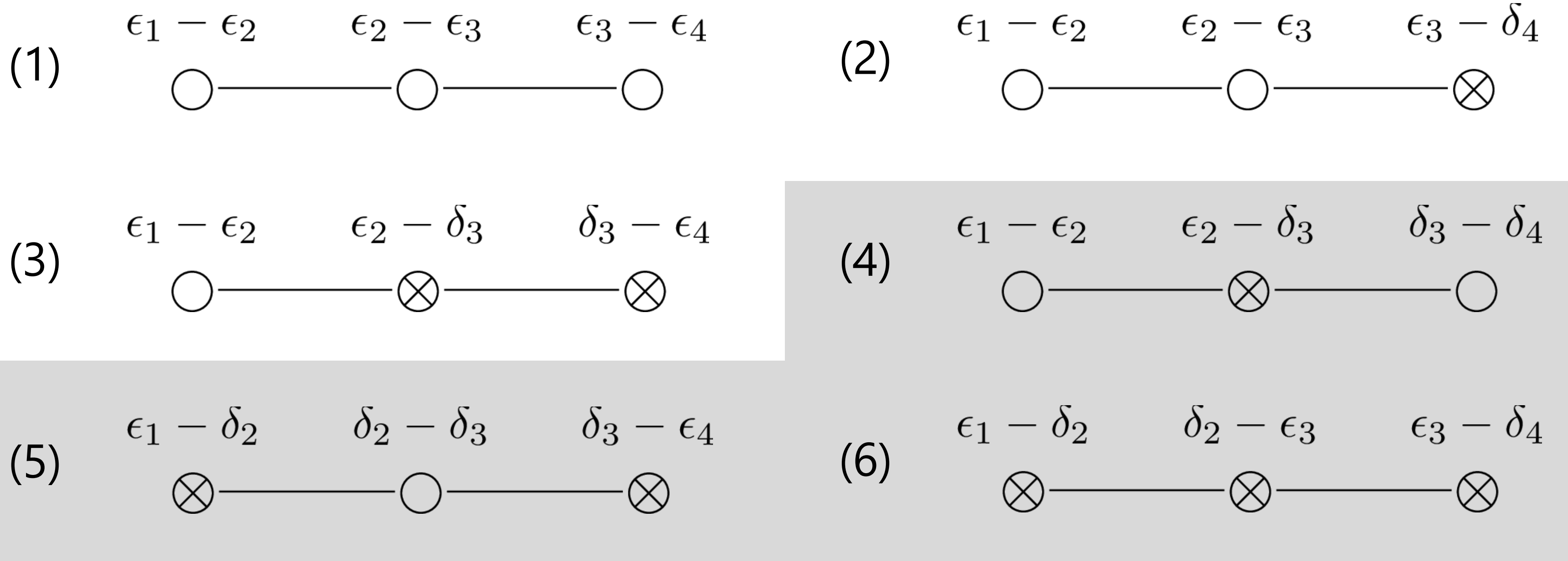
(8)



- For (1),(2),(3), we obtain tetrahedron eq because $\rho_1 = \rho_2 = \rho_3 = 0$.

Dynkin diagrams of type A of rank 3

- Without exchanges $\epsilon \leftrightarrow \delta$, all Dynkin diagrams of rank 3 are given by



- For (4),(5),(6), some ρ_i are non-zero. Then, associated equations become the tetrahedron equation *up to sign factors*.
- Here, we only consider (4) for them.

The case ○——○——○

$\epsilon_1 - \epsilon_2$ $\quad$ $\epsilon_2 - \epsilon_3$ $\quad$ $\epsilon_3 - \epsilon_4$

○——○——○

$$(\epsilon_i, \epsilon_i) = 1 \qquad DA = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

- Root system

$$\tilde{\Phi}^+_{\mathrm{even}} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\} \quad \tilde{\Phi}^+_{\mathrm{iso}} = \{\}$$

- Lemma

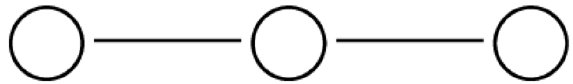
$$\Gamma^{(2|1)} = \Gamma^{(3|2)} = \Gamma^{(23|1)} = \Gamma^{(32|1)} = \Gamma^{(3|12)} = \Gamma^{(3|21)} = \mathcal{R}$$

- Proof (The case $\Gamma^{(23|1)}$)
  - h: ○——○ → ○——○——○ defined by $e_1 \mapsto e_1$, $e_2 \mapsto e_{23}$ is an algebra hom by higher-order relations:

$$e_1^2 e_{23} - (q + q^{-1}) e_1 e_{23} e_1 + e_{23} e_1^2 = e_{23}^2 e_1 - (q + q^{-1}) e_{23} e_1 e_{23} + e_1 e_{23}^2 = 0$$

  - $[m]_{q^{d_{\alpha_2+\alpha_3}}}! = [m]_{q^{d_{\alpha_2}}}!$ and $[m]_{q^{d_{\alpha_1+\alpha_2+\alpha_3}}}! = [m]_{q^{d_{\alpha_1+\alpha_2}}}!$ hold.
  - Then we have

$$e_2^{(a)} e_{12}^{(b)} e_1^{(c)} = \sum_{i,j,k} \mathcal{R}^{a,b,c}_{i,j,k} e_1^{(k)} e_{21}^{(j)} e_2^{(i)} \mapsto e_{23}^{(a)} e_{123}^{(b)} e_1^{(c)} = \sum_{i,j,k} \mathcal{R}^{a,b,c}_{i,j,k} e_1^{(k)} e_{(23)1}^{(j)} e_{23}^{(i)}$$

The case 

- <u>Previous Theorem</u> [Y20]

$$\rho_1 = p(\alpha_1)p(\alpha_3)$$
$$\rho_2 = p(\alpha_1+\alpha_2)p(\alpha_2+\alpha_3)$$
$$\rho_3 = p(\alpha_2)p(\alpha_1+\alpha_2+\alpha_3)$$

$$\begin{aligned}&\sum(-1)^{\rho_1(i_1o_6+x_4)+\rho_2x_2x_5+\rho_3i_3i_4}\\ &\qquad\times \Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3}\Gamma^{(3|12)}{}^{x_1,o_4,o_5}_{i_1,x_4,x_5}\Gamma^{(32|1)}{}^{x_2,x_4,o_6}_{i_2,i_4,x_6}\Gamma^{(2|1)}{}^{x_3,x_5,x_6}_{i_3,i_5,i_6}\\ &=\sum(-1)^{\rho_1(o_1i_6+x_4)+\rho_2x_2x_5+\rho_3o_3o_4}\\ &\qquad\times \Gamma^{(2|1)}{}^{o_3,o_5,o_6}_{x_3,x_5,x_6}\Gamma^{(23|1)}{}^{o_2,o_4,x_6}_{x_2,x_4,i_6}\Gamma^{(3|21)}{}^{o_1,x_4,x_5}_{x_1,i_4,i_5}\Gamma^{(3|2)}{}^{x_1,x_2,x_3}_{i_1,i_2,i_3}\end{aligned}$$

- <u>Previous Lemma</u>

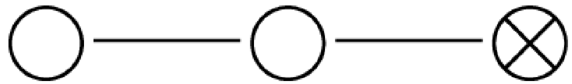
$$\Gamma^{(2|1)}=\Gamma^{(3|2)}=\Gamma^{(23|1)}=\Gamma^{(32|1)}=\Gamma^{(3|12)}=\Gamma^{(3|21)}=\mathcal{R}$$

- The theorem is specialized as follows:

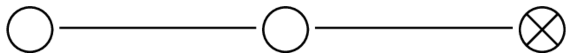
$$\sum\mathcal{R}^{o_1,o_2,o_3}_{x_1,x_2,x_3}\mathcal{R}^{x_1,o_4,o_5}_{i_1,x_4,x_5}\mathcal{R}^{x_2,x_4,o_6}_{i_2,i_4,x_6}\mathcal{R}^{x_3,x_5,x_6}_{i_3,i_5,i_6}=\sum\mathcal{R}^{o_3,o_5,o_6}_{x_3,x_5,x_6}\mathcal{R}^{o_2,o_4,x_6}_{x_2,x_4,i_6}\mathcal{R}^{o_1,x_4,x_5}_{x_1,i_4,i_5}\mathcal{R}^{x_1,x_2,x_3}_{i_1,i_2,i_3}$$

  - This is exactly the tetrahedron equation of [Kapranov-Voevodsky94]:

$$\mathcal{R}_{123}\mathcal{R}_{145}\mathcal{R}_{246}\mathcal{R}_{356}=\mathcal{R}_{356}\mathcal{R}_{246}\mathcal{R}_{145}\mathcal{R}_{123}$$

The case 

$\epsilon_1 - \epsilon_2 \qquad \epsilon_2 - \epsilon_3 \qquad \epsilon_3 - \delta_4$



$$(\epsilon_i, \epsilon_i) = 1 \qquad (\delta_i, \delta_i) = -1$$

$$DA = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

- Root system

$$\tilde{\Phi}^+_{\text{even}} = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} \quad \tilde{\Phi}^+_{\text{iso}} = \{\alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$$

- Lemma

$$\Gamma^{(2|1)} = \mathcal{R} \quad \Gamma^{(3|2)} = \Gamma^{(23|1)} = \Gamma^{(32|1)} = \Gamma^{(3|12)} = \Gamma^{(3|21)} = \mathcal{L}$$

- The theorem is specialized as follows:

$$\sum \mathcal{L}^{o_1,o_2,o_3}_{x_1,x_2,x_3} \mathcal{L}^{x_1,o_4,o_5}_{i_1,x_4,x_5} \mathcal{L}^{x_2,x_4,o_6}_{i_2,i_4,x_6} \mathcal{R}^{x_3,x_5,x_6}_{i_3,i_5,i_6} = \sum \mathcal{R}^{o_3,o_5,o_6}_{x_3,x_5,x_6} \mathcal{L}^{o_2,o_4,x_6}_{x_2,x_4,i_6} \mathcal{L}^{o_1,x_4,x_5}_{x_1,i_4,i_5} \mathcal{L}^{x_1,x_2,x_3}_{i_1,i_2,i_3}$$

  - This is exactly the tetrahedron equation of [Bazhanov-Sergeev06]:

$$\mathcal{L}_{123}\mathcal{L}_{145}\mathcal{L}_{246}\mathcal{R}_{356} = \mathcal{R}_{356}\mathcal{L}_{246}\mathcal{L}_{145}\mathcal{L}_{123}$$

The case $\bigcirc$ — $\bigotimes$ — $\bigotimes$

$$\begin{array}{ccc} \epsilon_1 - \epsilon_2 & \epsilon_2 - \delta_3 & \delta_3 - \epsilon_4 \\ \bigcirc & \text{———} \bigotimes \text{———} & \bigotimes \end{array} \qquad \begin{array}{l} (\epsilon_i, \epsilon_i) = 1 \\ (\delta_i, \delta_i) = -1 \end{array} \qquad DA = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

- Root system

$$\tilde{\Phi}^+_{\mathrm{even}} = \{\alpha_1, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\} \quad \tilde{\Phi}^+_{\mathrm{iso}} = \{\alpha_2, \alpha_3, \alpha_1 + \alpha_2\}$$

- <u>Lemma</u>

$$\Gamma^{(2|1)} = \mathcal{L}, \quad \Gamma^{(3|2)} = \Gamma^{(3|12)} = \Gamma^{(3|21)} = \mathcal{N}(q^{-1}), \quad = \Gamma^{(23|1)} = \Gamma^{(32|1)} = \mathcal{R}$$

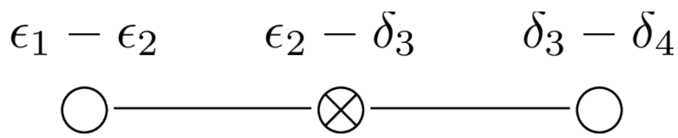
- The theorem is specialized as follows:

$$\sum \mathcal{N}(q^{-1})^{o_1,o_2,o_3}_{x_1,x_2,x_3} \mathcal{N}(q^{-1})^{x_1,o_4,o_5}_{i_1,x_4,x_5} \mathcal{R}^{x_2,x_4,o_6}_{i_2,i_4,x_6} \mathcal{L}^{x_3,x_5,x_6}_{i_3,i_5,i_6}$$
$$= \sum \mathcal{L}^{o_3,o_5,o_6}_{x_3,x_5,x_6} \mathcal{R}^{o_2,o_4,x_6}_{x_2,x_4,i_6} \mathcal{N}(q^{-1})^{o_1,x_4,x_5}_{x_1,i_4,i_5} \mathcal{N}(q^{-1})^{x_1,x_2,x_3}_{i_1,i_2,i_3}$$

  - This gives a new solution to the tetrahedron equation:

$$\mathcal{N}(q^{-1})_{123} \mathcal{N}(q^{-1})_{145} \mathcal{R}_{246} \mathcal{L}_{356} = \mathcal{L}_{356} \mathcal{R}_{246} \mathcal{N}(q^{-1})_{145} \mathcal{N}(q^{-1})_{123}$$

The case $\bigcirc\!-\!\!-\!\!-\!\bigotimes\!-\!\!-\!\!-\!\bigcirc$



$(\epsilon_i, \epsilon_i) = 1$

$(\delta_i, \delta_i) = -1$

$$DA = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

- Root system

$$\tilde{\Phi}^+_{\text{even}} = \{\alpha_1, \alpha_3\} \quad \tilde{\Phi}^+_{\text{iso}} = \{\alpha_2, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$$

- <u>Lemma</u>

$$\Gamma^{(2|1)} = \Gamma^{(23|1)} = \Gamma^{(32|1)} = \mathcal{L}, \quad \Gamma^{(3|2)} = \Gamma^{(3|12)} = \Gamma^{(3|21)} = \mathcal{M}(q^{-1})$$

The case $\bigcirc\!\!-\!\!\!-\!\!\!-\!\!\bigotimes\!\!-\!\!\!-\!\!\!-\!\!\bigcirc$

$$\rho_1 = p(\alpha_1)p(\alpha_3)$$
$$\rho_2 = p(\alpha_1+\alpha_2)p(\alpha_2+\alpha_3)$$
$$\rho_3 = p(\alpha_2)p(\alpha_1+\alpha_2+\alpha_3)$$

- Previous Theorem [Y20]

$$\begin{aligned}&\sum(-1)^{\rho_1(i_1o_6+x_4)+\rho_2x_2x_5+\rho_3i_3i_4}\\ &\qquad\times\Gamma^{(3|2)}{}^{o_1,o_2,o_3}_{x_1,x_2,x_3}\Gamma^{(3|12)}{}^{x_1,o_4,o_5}_{i_1,x_4,x_5}\Gamma^{(32|1)}{}^{x_2,x_4,o_6}_{i_2,i_4,x_6}\Gamma^{(2|1)}{}^{x_3,x_5,x_6}_{i_3,i_5,i_6}\\ &=\sum(-1)^{\rho_1(o_1i_6+x_4)+\rho_2x_2x_5+\rho_3o_3o_4}\\ &\qquad\times\Gamma^{(2|1)}{}^{o_3,o_5,o_6}_{x_3,x_5,x_6}\Gamma^{(23|1)}{}^{o_2,o_4,x_6}_{x_2,x_4,i_6}\Gamma^{(3|21)}{}^{o_1,x_4,x_5}_{x_1,i_4,i_5}\Gamma^{(3|2)}{}^{x_1,x_2,x_3}_{i_1,i_2,i_3}\end{aligned}$$

- Previous Lemma

$$\Gamma^{(2|1)}=\Gamma^{(23|1)}=\Gamma^{(32|1)}=\mathcal{L},\quad \Gamma^{(3|2)}=\Gamma^{(3|12)}=\Gamma^{(3|21)}=\mathcal{M}(q^{-1})$$

- By using $\rho_1=0, \rho_2=\rho_3=1$, the theorem is specialized as follows:

$$\begin{aligned}&\sum(-1)^{x_2x_5+i_3i_4}\mathcal{M}(q^{-1})^{o_1,o_2,o_3}_{x_1,x_2,x_3}\mathcal{M}(q^{-1})^{x_1,o_4,o_5}_{i_1,x_4,x_5}\mathcal{L}^{x_2,x_4,o_6}_{i_2,i_4,x_6}\mathcal{L}^{x_3,x_5,x_6}_{i_3,i_5,i_6}\\ &=\sum(-1)^{x_2x_5+o_3o_4}\mathcal{L}^{o_3,o_5,o_6}_{x_3,x_5,x_6}\mathcal{L}^{o_2,o_4,x_6}_{x_2,x_4,i_6}\mathcal{M}(q^{-1})^{o_1,x_4,x_5}_{x_1,i_4,i_5}\mathcal{M}(q^{-1})^{x_1,x_2,x_3}_{i_1,i_2,i_3},\end{aligned}$$

  - There are ``nonlocal" sign factors which can not be eliminated at present.

Concluding remarks: Crystal limit

- Consider non-super cases
  - $\boldsymbol{B}$: canonical basis
  - $\mathbf{i}$: indices of reduced expression of the longest element of Weyl group
  - Lusztig's parametrization: a bijection $b_{\mathbf{i}}: \mathbb{Z}_{\geq 0}^{l} \to \boldsymbol{B}$ associated with $\mathbf{i}$
  - Transition map: $R_{\mathbf{i}}^{\mathbf{i}'} = (b_{\mathbf{i}'})^{-1} \circ b_{\mathbf{i}} : \mathbb{Z}_{\geq 0}^{l} \to \mathbb{Z}_{\geq 0}^{l}$
- Transition maps are obtained by transition matrices with $q \to 0$:

$$\lim_{q\to 0} \mathcal{R}(q)_{i,j,k}^{a,b,c} = \delta_{a,i+j-\min(i,k)}\delta_{b,\min(i,k)}\delta_{c,j+k-\min(i,k)}$$

- A super analog of transition maps is obtained in a similar way.
- Prop [Y20]
  - $\mathcal{L}_{i,j,k}^{a,b,c} = \lim_{q\to 0} \mathcal{L}(q)_{i,j,k}^{a,b,c}$ gives a non-trivial bijection on $\{0,1\}^2 \times \mathbb{Z}_{\geq 0}$.
  - Non-zero elements are given by

$$\mathcal{L}_{0,0,k}^{0,0,c} = \mathcal{L}_{1,1,k}^{1,1,c} = \delta_{k,c}, \quad \mathcal{L}_{0,1,k}^{1,0,c} = \delta_{k+1,c}, \quad \mathcal{L}_{1,0,0}^{1,0,0} = 1, \quad \mathcal{L}_{1,0,k}^{0,1,c} = \delta_{k-1,c}$$

  - $\mathcal{N}_{i,j,k}^{a,b,c} = \lim_{q\to 0} \left(\frac{[b]_q!}{[j]_q!} \mathcal{N}(q)_{i,j,k}^{a,b,c} \right)$ also gives a non-trivial bijection.

Concluding remarks

- Remark
  1. Type B cases give new solutions to the 3D reflection equations.
  2. The crystal limit for $\bigcirc\!\Longrightarrow\!\bullet$ and $\bigotimes\!\Longrightarrow\!\bullet$ take values $0, \pm 1$.

- Summary
  1. The 3D L is characterized as the transition matrix for $\bigcirc\!\!-\!\!\!-\!\!\!-\!\!\bigotimes$.
  2. A new solution to the tetrahedron equation the 3D N is obtained by considering the transition matrix for $\bigotimes\!\!-\!\!\!-\!\!\!-\!\!\bigotimes$.
  3. Several solutions to the tetrahedron equations are obtained without using any result for quantum coordinate rings (c.f. KOY theorem).

- Outlook
  1. Eliminating nonlocal sign factors for $\bigcirc\!\!-\!\!\!-\!\!\!-\!\!\bigotimes\!\!-\!\!\!-\!\!\!-\!\!\bigcirc$
  2. A super analog of KOY theorem
  3. Geometric lifting of a super analog of transition maps