Lecture 19

AVL Tree Algorithms

Balanced Trees (Review)

worst case $h = N-1$
$\rightarrow O(N) = h$

BST Operations take $O(h)$ steps

- in the worst case, h can grow linearly in N

Strategy: Keep the tree balanced

- similar number of keys in left / right subtrees
- want $h = O(\log N)$

AVL Trees

- balanced by height
- height invariant: heights of left & right subtrees differ by at most 1
- base case: The empty tree is balanced

Minimum Tree Density — AVL Trees

Let $N(h)$ represent the number of nodes

- Q. What's the minimum value of $N(h)$?

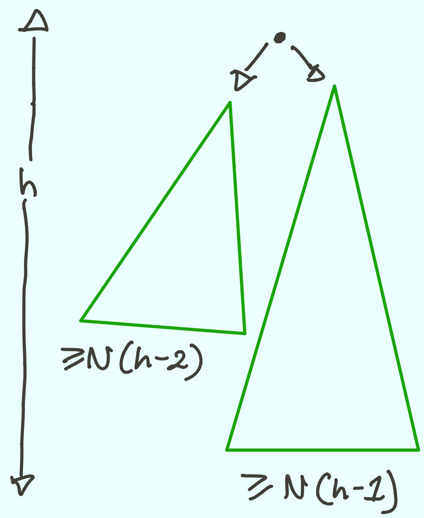
$N(h) \geq N(h-1) + N(h-2) + 1$

$N(-1) = 0$, $N(0) = 1$

Pattern?

- Fibonacci sequence

Claim: $N(h) \geq F_{h+3} - 1$



h	min $N(h)$		
−1	0	1	(F_2)
0	1	2	(F_3)
1	2	3	(F_4)
2	4	5	(F_5)
3	7	8	(F_6)
4	12	13	(F_7)

Minimum Tree Density — AVL Trees Cont'd

An AVL Tree holds at least $N(h) \geq F_{h+3} - 1$ nodes.

<u>Proof (by induction on *h*):</u>

Strategy:

- Base case? Both $h=-1$ & $h=0$ satisfy the claim
- Inductive case? Consider an AVL tree of height $k \geq 1$. Then:

$$N(k) \geq N(k-1) + N(k-2) + 1$$
$$\geq (F_{k+2} - 1) + (F_{k+1} - 1) + 1$$
$$= F_{k+2} + F_{k+1} - 1$$
$$= F_{k+3} - 1 \quad \checkmark$$

F_N grows exponentially:

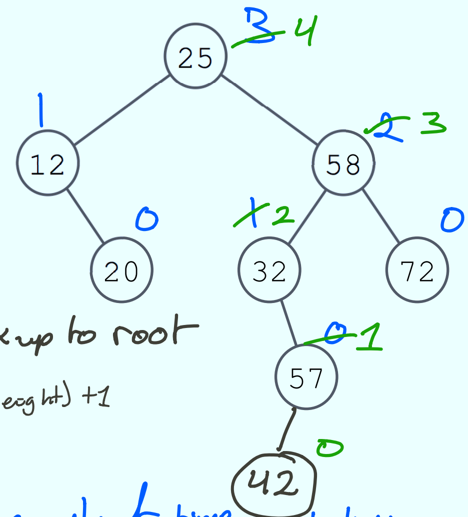
$F_N \approx \varphi^n / \sqrt{5} \Rightarrow N \geq \varphi^{h+3} / \sqrt{5} - 1$

$\varphi = \frac{\sqrt{5}+1}{2}$ ↳ golden ratio

$h \leq c \log_{\varphi} N$

$h = O(\log N)$

Tree Maintenance



Add `.height` property to each node

For `.insert(x)`, augment tree in the usual manner, and:

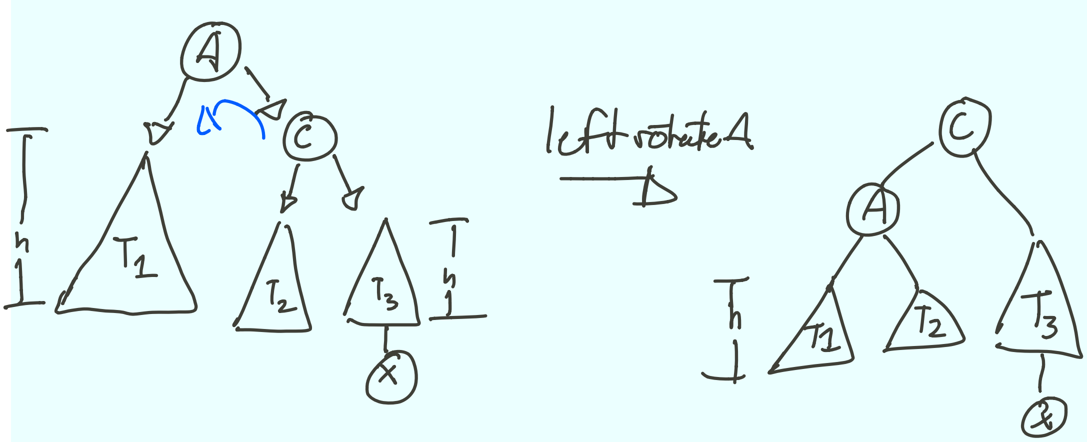
- update height fields bottom up (from x up to root)
    - `t->height` = max (t->left->height, t->right->height) + 1
- E.g., `.insert(42)`
- if node unbalanced, then perform a constant time rotation
    - rotate unbalanced ancestor closest to x

Same idea for .delete(x)

Rotations

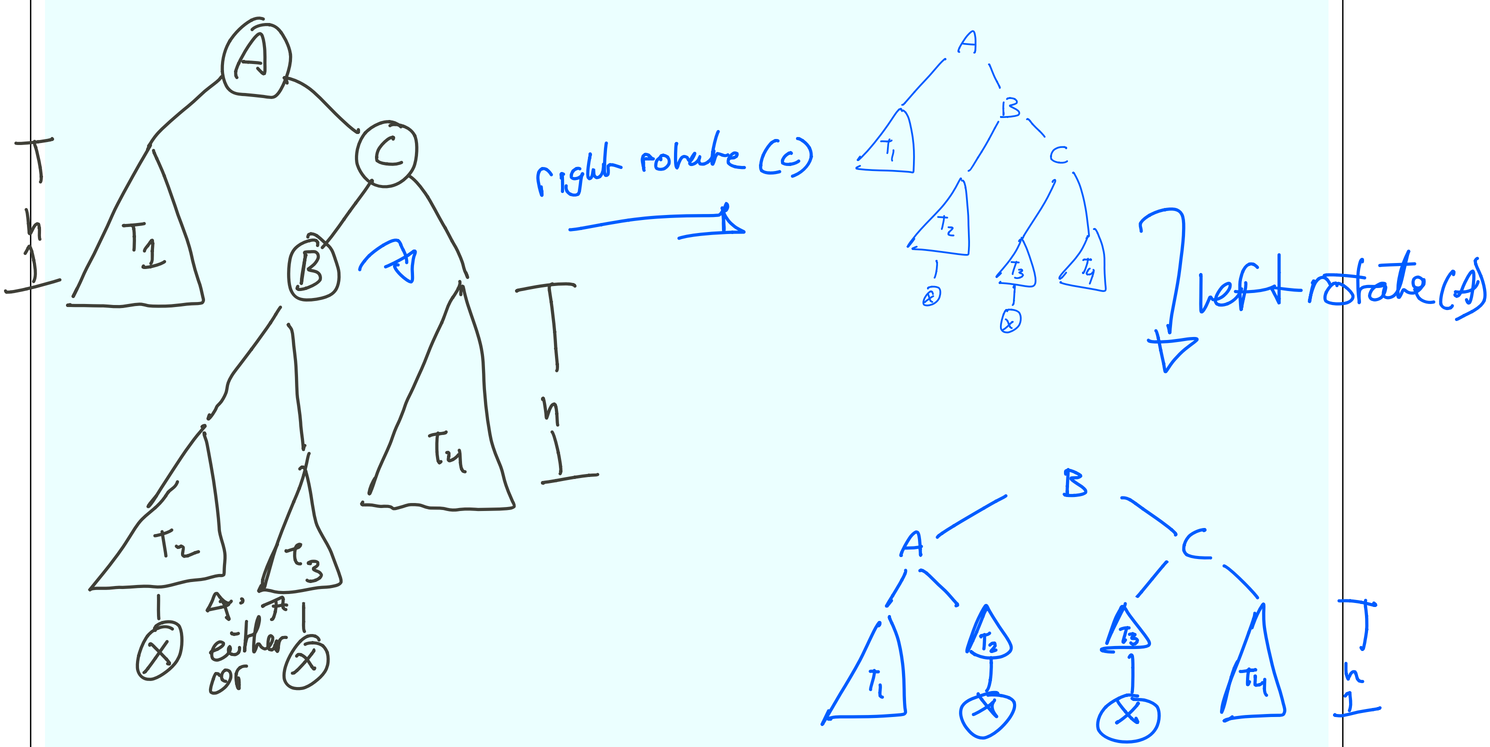
Without loss of generality, assume left child has height *h* and *x* is inserted on the right subtree

<u>Case 1:</u> *x* is on the right / right grandchild



double rotation

Case 2: x is on the right / left grandchild



AVL Rebalancing Example