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ORIGINAL ARTICLE

Dynamic complexities of a modified Leslie–Gower model in deterministic and stochastic environments

Pritam Saha[1] · Akidul Haque[2] · Md. Shahidul Islam[2,3] · Uttam Ghosh[1]

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Abstract
In this paper, we explore the Leslie–Gower type prey–predator model with a Holling-IV functional response, examining both deterministic and stochastic environments. In the deterministic analysis, we establish the positivity and boundedness of solutions, as well as the stability criteria for various equilibria and different bifurcations, including transcritical, saddle-node, and Hopf bifurcations. For the stochastic component, we demonstrate the existence and uniqueness of global positive solutions and identify conditions for persistence in the mean. Additionally, we derive the stationary distribution and probability density function for the stochastic model. We conduct a stochastic sensitivity analysis by approximating the confidence domain, showing that the size of the confidence ellipse is influenced by the level of noise intensity. When the confidence ellipse intersects the separatrix, critical transitions or tipping points may occur. In such instances, the system may not revert to its previous state depending on the intensity of fluctuations. Most theoretical findings are supported by numerical simulations.

Keywords Leslie–Gower model · Regime shift · Probability density function · Stochastic sensitivity · Confidence ellipse

Introduction

The interplay between prey and predator is crucial in preserving the ecosystem biodiversity. Mathematical Ecology is the branch of science in which dynamics of various interacting species with their surroundings is studied using analytical approaches to identify the species persistence as well as diversity (Turchin 2003; Roth 2016). These interactions may be different types likely: between individuals, populations with organisms and their environment. In nature, there are large number of interacting species and study of their dynamics considering them at a time is an difficult task. To overcome this difficulty researchers considers similar types of species in a same framework and formulate their model with two or three dimension in most of the cases.

To connect the dynamics of the competing species/ food chain model (Lotka 1925; Volterra 1931) independently developed mathematical model known as the Lotka–Volterra model in the following form:

$$\frac{dx}{dt} = rx - mxy$$
$$\frac{dy}{dt} = -\delta y + cmxy$$

with non-negative initial conditions, $x(t)$ and $y(t)$ denotes the densities of prey and predator populations respectively, used constants are assumed to be positive have some specific biological meaning will be discussed later.

Several types of prey–predator models used to investigate the dynamics of thee ecological systems, such as the type Lotka-Volterra type (Lotka 1925; Volterra 1931),

✉ Uttam Ghosh
uttam_math@yahoo.co.in

Pritam Saha
pritamsaha1219@gmail.com

Akidul Haque
akidulhaque12@gmail.com

Md. Shahidul Islam
mshahidul11@yahoo.com; mshahid@du.ac.bd

1 Department of Applied Mathematics, University of Calcutta, Kolkata, India

2 Department of Mathematics, University of Dhaka, Dhaka, Bangladesh

3 Bangabandhu Sheikh Mujibur Rahman Science and Technology University (BSMRSTU), Pirojpur, Bangladesh

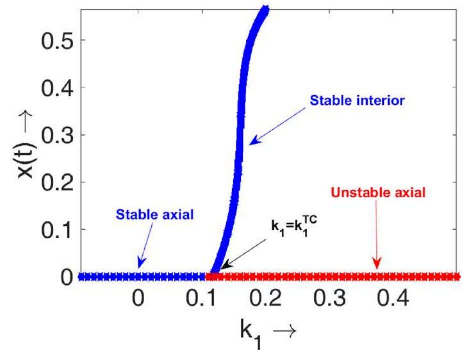


Fig. 1 Schematic bifurcation diagram with with respect to k_1 and other parameters are taken from Table 1

Gauss-type (Gause 1934), Leslie–Gower type (Lesile and Gower 1960) etc., prey–predator model. The generalized Gauss model (Gause 1934; Freedman 1980) for the interaction of the two species is as follows:

$$\begin{aligned}\frac{dx}{dt} &= xg(x) - p(x)y\\ \frac{dy}{dt} &= -\delta y + h(x)y\end{aligned}$$

with suitable initial conditions, where the function $g(x)$ satisfies $g(0) > 0$ and $g'(x) > 0$ for all $x > 0$. The environment where prey has a limited amount of food, they cannot expand indefinitely in the absence of predators, there exists a $k > 0$ such that $g(k) = 0$, and k is referred to as the carrying capacity of the prey. There are some instance where the growth of predator directly related to the abundance of prey (Graham and Lambin 2002). The work of Graham and Lambin (2002) demonstrated that field-vole (*Microtus agrestis*) survival can be affected by reducing least weasel (*Mustela nivalis*) predation, they also showed that weasels abundance decreases in summer and autumn, while the vole (Microtus agrestis) population always declined to low density at these season. To study the dynamics of these types of populations Lesile and Gower (1960) proposed prey–predator system and later many authors (Hsu and Huang 1995; Freedman and Mathsen 1993) and reference therein studied the similar types of model considering different types of biological effects. The general form of Leslie–Gower model is as follows:

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right) - p(x)y\\ \frac{dy}{dt} &= sy\left(1-\frac{y}{hx}\right)\end{aligned} \tag{1.1}$$

with suitable initial conditions, r and k are the intrinsic growth rate and the carrying capacity of the prey. The rate at which prey are captured per predator is denoted by the symbol $p(x)$, is also known as the functional response of a predator and describes the change in prey density per unit time per predator as the prey density changes. In ecology, it is assumed that $p(0) = 0$ and $p'(x) > 0$ for every $x > 0$ for the majority of examples that arise. The classical Gauss-type prey–predator system and its expanded variant are different from system (1.1). In the (1.1), the predator develops logistically at an intrinsic growth rate of s, and the environment's carrying capacity is proportional to the size of the available prey, or hx, where h is constant. The term $\frac{y}{hx}$ is known as the term (Lesile and Gower 1960). System (1.1) with different functional responses $p(x)$ have been studied extensively by the researchers in Ko and Ryu (2006), Zhang et al. (2021), Bashkirtseva et al. (2015) and Huang et al. (2006). Holling-types functional response is widely used to analyze the dynamical behavior of system (1.1) (Holling 1959a). The functional response of Holling type I is linear. The linear growth assumes that the consumer's processing time for food items is minimal. The functional response of type II is frequently approximated by a rectangular hyperbola type or Crytoid, which assumes that prey density and prey consumption are opposing behaviors. Similar to type II, type III functional response occurs at high prey density levels. However, in this case, the graphical relationship between processing food and looking for food is a super linearly growing function of prey devoured by predators at low prey density levels (Holling 1959a, b, 1965).

In this paper, our main concern to study the dynamics of a Predator-prey model using a Holling-type IV functional response, which is in general commonly known as the Monod–Haldane (Collings 1997) functional response, is denoted as $p(x) = \frac{mx}{a+bx+x^2}$. In their investigations on the kinetics of phenol oxidation by washed cells, Sokol and Howell (1987) proposed a simplified Holling type IV function of the form $p(x) = \frac{mx}{n+x^2}$ Sokol and Howell (1987). After introduction of Holling-type IV functional response the system (1.1) becomes:

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right) - \frac{mxy}{n+x^2}\\ \frac{dy}{dt} &= sy\left(1-\frac{y}{hx}\right)\end{aligned} \tag{1.2}$$

In the above prey–predator system (1.1), the predator is a specialist type because it depends on a single kind of prey to thrive and will go extinct without it. In contrast, some predators are generalized predators with a variety of alternative food (another prey species) to choose from and the ability to switch to other food sources to remain active even when their preferred prey species is in short supply (Zhang et al. 2021; Souna et al. 2023) and reference there in. There are many research on predator–prey models that focus on the specialized predator, but relatively few of them take into account the impact of generalist predation.

In traditional Gauss-type models (Freedman 1980), generalist predation can be taken into account in a variety of ways, such as:

1. Taking the alternative resources are constant and directly incorporate density-dependent mortality into the prey's equation;
2. Letting the alternative resources are not constant and the predator's equation takes on a logistic or logistic-like form when the prey is absent;
3. Assuming the alternative resources are not constant and take into account high-dimensional models, such as two prey and one predator species.

According to Aziz-Alaoui and Okiye (2011) we proposed the modified Leslie–Gower model taking into account alternate food sources for predators based on the model in system (1.2). Then the above system reduces to:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\frac{mxy}{n+x^2} \\ \frac{dy}{dt} &= sy\left(1-\frac{y}{hx+p}\right) \end{aligned} \tag{1.3}$$

with the initial conditions $x(0)\geq 0, y(0)\geq 0,\ hx+p$ represents the new carrying capacity for predators, and $p>0$ can be viewed as additional constant carrying capacity coming from all other food sources and model parameters are non-negative and their significance is given in Table 1. Further, when $x=0$, the predator in system (1.3) becomes $\frac{dy}{dt}=sy(1-\frac{y}{p})$, that is a logistic growth equation, It implies that even when a particular prey species is scarce, predators can survive by switching to other food sources. Thus, the predator in system (1.3) can be seen as a generalist predator.

Another important fact in nature is the effect of various environmental noises in biological system (Allen 2007; Yuan et al. 2020). The environmental noises can modify the qualitative behavior of the deterministic model through non-periodic oscillation or regime shift when bi-stability exists in any dynamical system (Bashkirtseva and Ryashko 2011; Bashkirtseva et al. 2015). Das et al. (2020) proposed a mathematical model is proposed which describes the dynamics of the prey–predator (low predator and top predator) to investigate the effect of white noise and diffusion on the population dynamics of the considering prey–predator system. The authors in Zhang et al. (2021) considered stochastic predator prey system with modified Leslie–Gower type model with white noise and they established the condition of global persistence of solutions. Using numerical simulations they showed that the system co-existence steady state solutions of deterministic solution may go to extinction with the increase of intensity noise. Mandal et al. (2020) proposed a fractional-order prey–predator type ecological model with the functional response of Holling type II and the effect of harvesting with the influence of a super-predator on the conventional predator. Han and Jiang (2022) considered the two dimensional predator prey model effect of fear, they found the explicit expression of the probability density function and using numerical simulation they verified the increase of strength of fluctuation implies the extinction one or both the species. The study in Devi and Jana (2020) deals with the dynamics of a stochastically perturbed time-variant prey–predator model of aquatic animals hilsa (prey) and eel (predator). Authors in Mondal et al. (2022) considered three dimensional predator prey model with sexually reproductive top predator and establish that the effect of environmental fluctuation may change the dynamics from from co-existence of all the species to extinction state. The authors in Yuan et al. (2020) and Garain and Mandal (2022) and reference therein used the concept of stochastic sensitivity to study the ecological models and they showed that they may arises the stochastic transition or regime shift when the confidence ellipse touches or crosses the line of separatrix of basin of attractor.

Table 1 Model parameters, their biological significance and the empirical values

Parameters	Biological significance	Dimensionless parameters	Values of dimensionless parameters
r	Prey intrinsic growth rate	...	...
k	Environmental carrying capacity	...	...
s	Predator intrinsic growth rate	$c=\frac{s}{r}$	0.05
m	Consumption rate	...	...
$\frac{1}{h}$	$\frac{1}{\text{Measure of the quality}}$	$b=\frac{rk}{hm}$	5.2
p	Amount of additional food	$k_2=\frac{p}{hk}$	0.61
n	Environmental protection	$k_1=\frac{n}{k^2}$	0.12

In Mao et al. (2002), Mao et al. mentioned that it can be perturbed system parameter(s) stochastically in order to formulate the environmentally fluctuating system i.e., the system with stochastic differential equation(SDE). Here we give perturbation to the intrinsic growth rate of both the populations such that the growth rate rx is modified as $rx + \sigma_1 x \frac{dB_1}{dt}$ and sy is modified as $sy + \sigma_2 y \frac{dB_2}{dt}$ where σ_1, σ_2 denote intensities of white noises, B_1, B_2 denote independent standard Brownian motion. Using these perturbation system (1.3) takes the following form:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1 - \frac{x}{k}\right) - \frac{mxy}{n + x^2} + r\sigma_1 x dB_1 \\ \frac{dy}{dt} &= sy\left(1 - \frac{y}{hx + p}\right) + sy\sigma_2 dB_2 \end{aligned} \tag{1.4}$$

with suitable initial conditions. Using the scaling $x = k\bar{x}$, $y = \frac{rk^2}{m}\bar{y}$, $t = \frac{\tau}{r}$ and dropping bar, system (1.3) and (1.4) respectively, reduces to:

$$\begin{aligned} \frac{dx}{dt} &= x(1 - x) - \frac{xy}{x^2 + k_1} \\ \frac{dy}{dt} &= cy\left(1 - \frac{by}{x + k_2}\right) \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} \frac{dx}{dt} &= x(1 - x) - \frac{xy}{x^2 + k_1} + \sigma_1 x dB_1 \\ \frac{dy}{dt} &= cy\left(1 - \frac{by}{x + k_2}\right) + cy\sigma_2 dB_2 \end{aligned} \tag{1.6}$$

with suitable initial conditions and $k_1 = \frac{n}{k^2}$, $b = \frac{rk}{hm}$, $c = \frac{s}{r}$, $k_2 = \frac{p}{hk}$.

The rest part of the manuscript is organized in the following manner: in "Study of deterministic model" section we have studied the positivity and boundedness of solutions, stability and bifurcation of different equilibrium points. "Study of stochastic model" section is dedicated to study the stochastic model through the investigation of existence and uniqueness of positive solutions, stationary distribution, stochastic persistence, generating the probability density function and the study of stochastic sensitivity analysis for stochastic transition. Lastly, some concluding remarks are given in "Conclusion" section.

Study of deterministic model

Positivity and boundedness of the solutions

Here we shall establish the positivity and boundedness of the solutions of the proposed system. Biologically it is important as the first one gives the existence of feasible solutions and second gives the finite value of the solutions of the proposed system.

Theorem 1 *All the solutions of the system (1.5) are positive for any positive initial conditions.*

Proof

$$\frac{dx}{x} = \left[(1 - x) - \frac{y}{x^2 + k_1}\right]dt = \gamma(x, y)dt \text{ (say)}$$

Integrating, from 0 to t, we get

$$x(t) = x_0 \exp \int_0^t \gamma(x(t), y(t))dt.$$

So, $x(t) \geq 0$ whenever $x_0 \geq 0$.

Similarly using second equation, we can obtain $y(t) \geq 0$ whenever $y_0 \geq 0$. Therefore, any solutions of the system (1.5) initiating from the first quadrant of $(x - y)$ plane will stay in that quadrant. □

Theorem 2 *All the solutions of the system (1.5) in ${\mathbb{R}_+}^2$, are uniformly bounded and lie in $\Omega = \{(x, y) \in {\mathbb{R}_+}^2 : 0 \leq x + \frac{1}{a}y \leq \frac{Q}{\lambda}\}$ where Q and λ are defined in the proof.*

Proof It is clear from the first equation of (1.5) that $\frac{dx}{dt} \leq x(1 - x)$ and hence $x(t) \leq 1$.

Let us consider $W = x + \frac{1}{a}y$. Then

$$\frac{dW}{dt} = x(1 - x) - \frac{xy}{x^2 + k_1} + \frac{1}{a}\left[cy\left(1 - \frac{by}{x + k_2}\right)\right]$$

$$\begin{aligned} i.e.\ \frac{dW}{dt} + \lambda W &= x(1 - x) - \frac{xy}{x^2 + k_1} \\ &\quad + \frac{1}{a}\left[cy\left(1 - \frac{by}{x + k_2}\right)\right] + \lambda x + \frac{\lambda}{a}y \end{aligned}$$

$$i.e.\ \frac{dW}{dt} + \lambda W \leq x[(1 - x) + \lambda] + \frac{1}{a}\left(c + \lambda - \frac{by}{1 + k_2}\right)y \leq Q,$$

where λ is positive constant and $Q = Max\left\{x[(1 - x) + \lambda] + \frac{1}{a}(c + \lambda - \frac{by}{1+k_2})y\right\}$. Solving this, we get

$$W(x(t), y(t)) < \frac{Q}{\lambda} + e^{-\lambda t}W(x(0), y(0)),$$

$$i.e. W(x(t), y(t)) < \frac{Q}{\lambda} \text{ as } t \to \infty.$$

Hence, all the solutions of the system (1.5) that start in ${\mathbb{R}_+}^2$ are confined to the positively invariant region Ω. □

Existence of equilibria and their local stability

The Jacobian matrix of system (1.3) at any equilibrium $E(x^*, y^*)$ takes the following form:

$$J_E = \begin{bmatrix} 1 - 2x^* - \dfrac{k_1 y^* - x^{*^2} y^*}{(x^{*^2} + k_1)^2} & -\dfrac{x^*}{x^{*^2} + k_1} \\ \dfrac{bcy^{*^2}}{(x^* + k_2)^2} & c - \dfrac{2bcy^*}{x^* + k_2}. \end{bmatrix}. \quad (2.1)$$

Therefore

$$Det(J_E) = \left(1 - 2x^* - \frac{k_1 y^* - x^{*^2} y^*}{(x^{*^2} + k_1)^2}\right)\left(c - \frac{2bcy^*}{x^* + k_2}\right) + \frac{bcx^* y^{*2}}{(x^{*^2} + k_1)(x^* + k_2)^2},$$

$$Tr(J_E) = 1 + c - 2x^* - \frac{k_1 y^* - x^{*^2} y^*}{(x^{*^2} + k_1)^2} - \frac{2bcy^*}{x^* + k_2}.$$

The characteristic equation of (2.1) is given by

$$\lambda^2 - T\lambda + D = 0 \quad (2.2)$$

where $T = Trace(J_E))$ and $D = Det(J_E)$.

The system (1.3) has three boundary equilibrium points, namely $E_0 = (0, 0)$, $E_1 = (1, 0)$ and $E_2 = \left(0, \frac{k_2}{b}\right)$.

1. The Jacobian matrix of system (1.3) at $E_0(0, 0)$ is $J(E_0) = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$ and clearly eigenvalues are 1 and $c > 0$. Hence $E_0(0, 0)$ is always a unstable node.
2. The Jacobian matrix of system (1.3) at $E_1(1, 0)$ is $J(E_1) = \begin{bmatrix} -1 & -\dfrac{1}{1 + k_1} \\ 0 & c \end{bmatrix}$ So, the eigenvalues are $-1(<0)$ and $c(>0)$ and $E_1(1, 0)$ is always a saddle.
3. The Jacobian matrix of system (1.3) at $E_2\left(0, \dfrac{k_2}{b}\right)$ is

$$J(E_2) = \begin{bmatrix} 1 - \dfrac{k_2}{bk_1} & 0 \\ \dfrac{c}{b} & -c \end{bmatrix}$$

So, the eigenvalues are $1 - \dfrac{k_2}{bk_1}$ and $-c(<0)$. Depending on sign of $1 - \dfrac{k_2}{bk_1}$, the equilibrium point $E_2\left(0, \dfrac{k_2}{b}\right)$ may show the following three types behaviors -

   (a) E_2 is hyperbolic stable if $k_2 < bk_1$.
   (b) E_2 is a hyperbolic saddle if $k_2 > bk_1$.
   (c) E_2 is the degenerate equilibrium point $k_2 = bk_1$.

4. Co-existence equilibrium $E_3(x^*, y^*)$ is root of the equations

$$(1 - x^*) - \frac{y^*}{x^{*2} + k_1} = 0, c\left(1 - \frac{by^*}{x^* + k_2}\right) = 0.$$

Solving we get, $y^* = \frac{1}{b}(x^* + k_2)(>0)$, x^* is a root of the equation:

$$bx^3 - bx^2 + (bk_1 + 1)x + k_2 - bk_1 = 0. \quad (2.3)$$

The Eq. (2.3) may have one or three equilibrium points. Jacobian of the system (1.3) at an positive equilibrium $(E_3(x^*, y^*))$ is:

$$J_{E_3} = \begin{bmatrix} 1 - 2x^* - \dfrac{k_1 y^* - x^{*2} y^*}{R^2} & -\dfrac{x^*}{R} \\ \dfrac{bcy^{*2}}{S^2} & c - \dfrac{2bcy^*}{S} \end{bmatrix}$$

where $R = x^{*2} + k_1$ and $S = x^* + k_2$. The corresponding characteristic equation is given in (2.3) and hence $E_3(x^*, y^*)$ is locally asymptotically stable if $T(J_{E_3}) < 0$ and $D(J_{E_3}) > 0$ (Mondal et al. 2022).

Bifurcation analysis

In this part of the paper we shall study the qualitative behaviour of the solution through the study the differnt kinds of bifurcation. Fist we shall study the transcritical bifurcation in which the dynamical behaviour likely exchange of stability will be studied. The saddle bifurcation investigate the creation or destroy pair of equilibrium, for the considered problem it represents the generation of co-existence equilibrium points i.e. persistence of both the species. At last the hopf bifurcation will be studied which indicative of generation of periodic solution about the co-existence equilibrium point.

Transcritical bifurcation

Theorem 3 *The system (1.3) experiences transcritical bifurcation at* $E_2\left(0, \dfrac{k_2}{b}\right)$ *about* $k_1 = k_1^{TC} = \dfrac{k_2}{b}$.

Proof Considering $F(X, k_1) = \begin{bmatrix} f(X, k_1) \\ g(X, k_1) \end{bmatrix}$ where $f(X, k_1) = x(1 - x) - \dfrac{xy}{x^2 + k_1}$ and $g(X, k_1) = cy - \dfrac{bcy^2}{x + k_2}$, $X = (x, y)^T$.

Therefore, $F_{k_1}(E_2, k_1^{TC}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Also $DF(E_2, k_1^{TC}) = \begin{bmatrix} 0 & 0 \\ 0 & -c \end{bmatrix}$. Eigenvector corresponding to zero eigenvalue of $DF(E_2, k_1^{TC})$ and it's transpose are V and W, respectively then $V = \begin{bmatrix} b \\ 1 \end{bmatrix}$ and $W = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Again we have,

$$DF_{k_1}(E_2, k_1^{TC}) = \begin{bmatrix} -\dfrac{2k_2}{bk_1^3} & 0 \\ 0 & 0 \end{bmatrix}$$

and $D^2F(E_2, k_1^{TC})V = \begin{bmatrix} -b\left(2b + \dfrac{1}{k_1}\right) \\ 0 \end{bmatrix}$.

Now,

- $W^T F_{k_1}(E_2, k_1^{TC}) = 0$,
- $W^T[DF_{k_1}(E_2, k_1^{TC})V] = -\dfrac{k_2}{k_1^3} \neq 0$,
- $W^T[D^2F_{k_1}(E_2, k_1^{TC})(V, V)] = -b\left(2b + \dfrac{1}{k_1}\right) \neq 0$.

Thus all the conditions of transcritical bifurcation (Perko 2000) hold at equilibrium point $E_2\left(0, \dfrac{k_2}{b}\right)$ when k_1 crosses the critical value k_1^{TC}. □

Saddle-node bifurcation

Suppose at $b = b_{SN}$ any two roots of the characteristic equation coincide and suppose that coincident interior equilibrium is $E^*(x^*, y^*)$. Using Sotomayar (Perko 2000) we shall establish here that the system experiences saddle-node bifurcation when $b = b_{SN}$.

Theorem 4 *The system (1.3) goes through saddle-node bifurcation at coincident interior equilibrium E^* when b crosses the critical value $b = b_{SN}$.*

Proof The Jacobian matrix at E^* is $J_{E^*} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ where, $a_{11} = 1 - 2x^* - \dfrac{k_1y^* - x^{*2}y^*}{(x^{*2} + k_1)^2}$, $a_{12} = -\dfrac{x^*}{R}$, $a_{21} = \dfrac{bcy^{*2}}{S^2}$, $a_{22} = c - \dfrac{2bcy^*}{x^* + k_2}$. It can be easily shown that at b_{SN} one of the eigenvalues of J_{E^*} is 0. Let V and W be the eigenvectors corresponding to the eigenvalue 0 for the matrix J_{E*} and J_{E*}^T respectively, then $V = \begin{bmatrix} -a_{12} \\ a_{11} \end{bmatrix}$ and $W = \begin{bmatrix} -a_{21} \\ a_{11} \end{bmatrix}$. Now, $F_b(E^*(x^*, y^*)) = \begin{bmatrix} 0 \\ -\dfrac{cy^{*2}}{x^* + k_2} \end{bmatrix}$ and

$$D^2F(E^*(x^*, y^*))(V, V) = \begin{bmatrix} f_{xx}a_{12}^2 - 2f_{xy}a_{12}a_{11} + f_{yy}a_{11}^2 \\ g_{xx}a_{12}^2 - 2g_{xy}a_{12}a_{11} + g_{yy}a_{11}^2 \end{bmatrix}.$$

Thus, $W^T F_b(E^*(x^*, y^*)) = -\dfrac{a_{11}cy^{*2}}{x^* + k_2} \neq 0$,

and

$$\begin{aligned} &W^T D^2F(E^*(x^*, y^*))(V, V) \\ &= -a_{21}(f_{xx}a_{12}^2 - 2f_{xy}a_{12}a_{11} + f_{yy}a_{11}^2) \\ &\quad + a_{11}(g_{xx}a_{12}^2 - 2g_{xy}a_{12}a_{11} + g_{yy}a_{11}^2) \neq 0 \end{aligned}$$

.

Therefore the conditions of Sotomayor theorem hold at the interior equilibrium $E^*(x^*, y^*)$ about parameter $b = b_{SN}$ (Perko 2000). □

The presence of saddle-node bifurcation is biologically significant because, on one side of the bifurcation curve, the system lacks any interior equilibrium points, while on the other side, it features two interior equilibrium points: one stable and one saddle.

Hopf bifurcation

The Hopf bifurcation occurs when an equilibrium changes its stability through the creation of a closed orbit around it. Suppose we consider b as the bifurcation parameter and determine its critical value b_{HB}, for which $T = 0$, $D > 0$.

Theorem 5 *The system (1.3) possesses a Hopf bifurcation about the positive equilibrium $E^*(x^*, y^*)$ if b passes through the critical value $b = b_{HB}$. Moreover there exists a threshold quantity θ^* (which is defined in the proof) such that $\theta^* < 0$, the periodic solution of the system (1.3) about $E^*(x^*, y^*)$ will be stable (supercritical Hopf bifurcation), while for $\theta^* > 0$, the solution is unstable (subcritical Hopf bifurcation).*

Proof The system (1.3) will experience Hopf bifurcation at the equilibrium point $E^*(x^*, y^*)$, if both roots of the Eq. (2.3) are purely imaginary, which is possible if $T = 0$, $D > 0$ and the transversality condition is also satisfied (Wiggins 2003). At $b = b_{HB}$, $T = 0$, $D > 0$ and the condition of transversality, $\dfrac{\partial}{\partial b}(Re\lambda)|_{b=b_{HB}} = -\dfrac{c}{S}y_1^* \neq 0$ is satisfied. Hence the proposed system (1.3) satisfied the conditions of hopf bifurcation at $b = b_{HB}$.

To determine the direction of Hopf bifurcation at $E^*(x^*, y^*)$ of system (1.3) we shall apply center manifold theorem (Wiggins 2003). Using the transformation $x = x' + x^*$ and $y = y' + y^*$ in system (1.3),we get (neglecting the dashes),

$$\begin{aligned} \frac{dx}{dt} &= (x + x^*)(1 - x - x^*) - \frac{(x + x^*)(y + y^*)}{(x + x^*)^2 + k_1} \\ \frac{dy}{dt} &= c(y + y^*)(1 - \frac{b(y + y^*)}{x + x^* + k_2}) \end{aligned} \tag{2.4}$$

Expanding the above equation about (x^*, y^*) by using Taylor?s theorem, we have (neglecting higher order terms)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} \tag{2.5}$$

where,

$$\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} a_{11}x^2 + a_{12}xy + a_{13}y^2 + a_{14}x^3 + a_{15}x^2y + a_{16}xy^2 + a_{17}y^3 \\ a_{21}x^2 + a_{22}xy + a_{23}y^2 + a_{24}x^3 + a_{25}x^2y + a_{26}xy^2 + a_{27}y^3, \end{pmatrix}$$

the coefficients a_{ij}'s are given in Appendix 1. The Eq. (2.3) is the characteristic equation of the corresponding linear part of (2.5). The Eq. (2.5) has a pair of purely imaginary roots if the condition of hopf bifurcation is satisfied. The roots of the characteristic equation are as follows:

$$\lambda_{x,y} = \frac{a_{11} + a_{22}}{2} \pm i\frac{\sqrt{4(a_{11}a_{22} - a_{12}a_{21}) - (a_{11} + a_{22})^2}}{2} = p(b) + iq(b). \tag{2.6}$$

Above roots are purely imaginary when $b = b_{HB}$. Considering the transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = p\begin{pmatrix} u \\ v \end{pmatrix}, \tag{2.7}$$

where $p = \begin{pmatrix} \pi & 0 \\ w_{21} & w_{22} \end{pmatrix}$, $w_{21} = \frac{a_{11}\pi}{-a_{12}}$ and $w_{22} = \frac{q\pi}{a_{12}}$. Then the system (2.4) reduce to-

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & q \\ \frac{a_{11}^2}{q} + \frac{a_{12}a_{21}}{q} & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix}, \tag{2.8}$$

where $f(u,v) = d_{11}u^2 + d_{12}uv + d_{13}v^2 + d_{14}u^3 + d_{15}u^2v + d_{16uv^2} + d_{17}v^3$ and $g(u,v) = e_{11}u^2 + e_{12}uv + e_{13}v^2 + e_{14}u^3 + e_{15}u^2v + e_{16uv^2} + e_{17}v^3$ (neglecting the higher order terms for both d_{ij}'s and e_{ij}'s are given in Appendix 2).

Now to determine the stability of the periodic solution, we compute the first Lyapunov coefficient (θ^*) sign of which will give the stability or instability of the periodic solution.

$$\begin{aligned}\theta^* &= \frac{1}{16}(f_{uuu} + f_{uvv} + g_{uuv} + g_{vvv}) \\ &\quad + \frac{1}{16q}(f_{uu}(f_{uu} + f_{vv}) - g_{uv}(g_{uu} + g_{vv}) - f_{uu}g_{uu} + f_{vv}g_{vv})_{(0,0)} \\ &= \frac{1}{16}(6d_{14} + 2e_{15}) + \frac{1}{16q}[4d_{11}^2 - e_{12}(2e_{11} + 2_{e13}) - 4d_{11}e_{11}].\end{aligned}$$

Based on the sign of θ^*, the nature of hopf bifurcation may be supercritical or subcritical (Perko 2000; Wiggins 2003). □

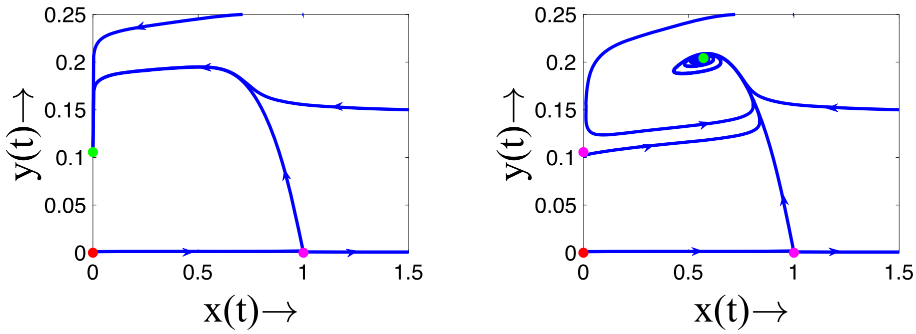
Numerical simulations

Transcritical bifurcation

The bifurcation diagram in Fig. 1 illustrates that the stability of the axial (prey-free) equilibrium shifts as one interior equilibrium emerges. For $k_1 < k_1^{TC}$, the axial equilibrium is stable and for $k_1 > k_1^{TC}$ it is unstable whereas interior equilibrium is stable. In Fig. 2, we have given the phase portrait for two cases, namely $k_1 < k_1^{TC}$ and $k_1 > k_1^{TC}$. Therefore at $k_1 = k_1^{TC}$, stability shifts, leading to the creation of an interior equilibrium, indicating a transcritical bifurcation. This finding is biologically significant because it establishes a lower limit for the environmental protection coefficient; if this limit is not met, the predator population risks extinction.

Saddle-node and Hopf bifurcation

To demonstrate the presence of saddle-node and Hopf bifurcations, we will create a one parameter bifurcation diagram that varies with the parameter b, while keeping the other parameters constant as in Table 1 (see Fig. 3). The figure clearly illustrates the presence of both saddle-node and Hopf bifurcations. System (3.2) has two saddle-node bifurcation points where the stable and unstable branches intersect. Additionally, there are two Hopf regions highlighted in magenta. We also present the phase portrait for five distinct branches of the one-parameter bifurcation diagram (see Fig. 4). First for $b = 5.78$ (lower blue region), there exists one stable equilibrium with two axial and one trivial equilibrium (see Fig. 4a). For $b = 6.11$, system contains three interior equilibria, namely one stable spiral, one saddle and one unstable spiral (lower magenta region). stable limit cycle (green) arises around the unstable spiral equilibrium and an unstable limit cycle (red) arises around stable spiral equilibrium (see Fig. 4b). Then for $b = 6.13$ (red region), two unstable spiral interior exists and one saddle interior. In this region a stable limit cycle enclosing both unstable spiral equilibrium (see Fig. 4c). Next, for $b = 6.22$, among three interior one is stable spiral, one unstable spiral and other saddle (higher magenta region) (see Fig. 4d). Finally, for $b = 6.49$ (upper blue region), one stable interior arises with two saddle axial and unstable trivial point(see Fig. 4e). In Fig. 3, crossing from the lower blue region to the red region results in a change in



(a) $k_1 = 0.09, b = 5.78$ (b) $k_1 = 0.15, b = 5.78$

Fig. 2 Complete phase portrait either size of $k_1 = k_1^{TC}$ as shown in Fig. 1: **a** For $k_1 = 0.09, b = 5.78$, **b** For $k_1 = 0.15, b = 5.78$ and others are as in Table 1

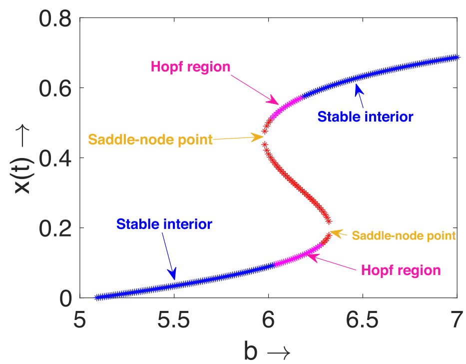
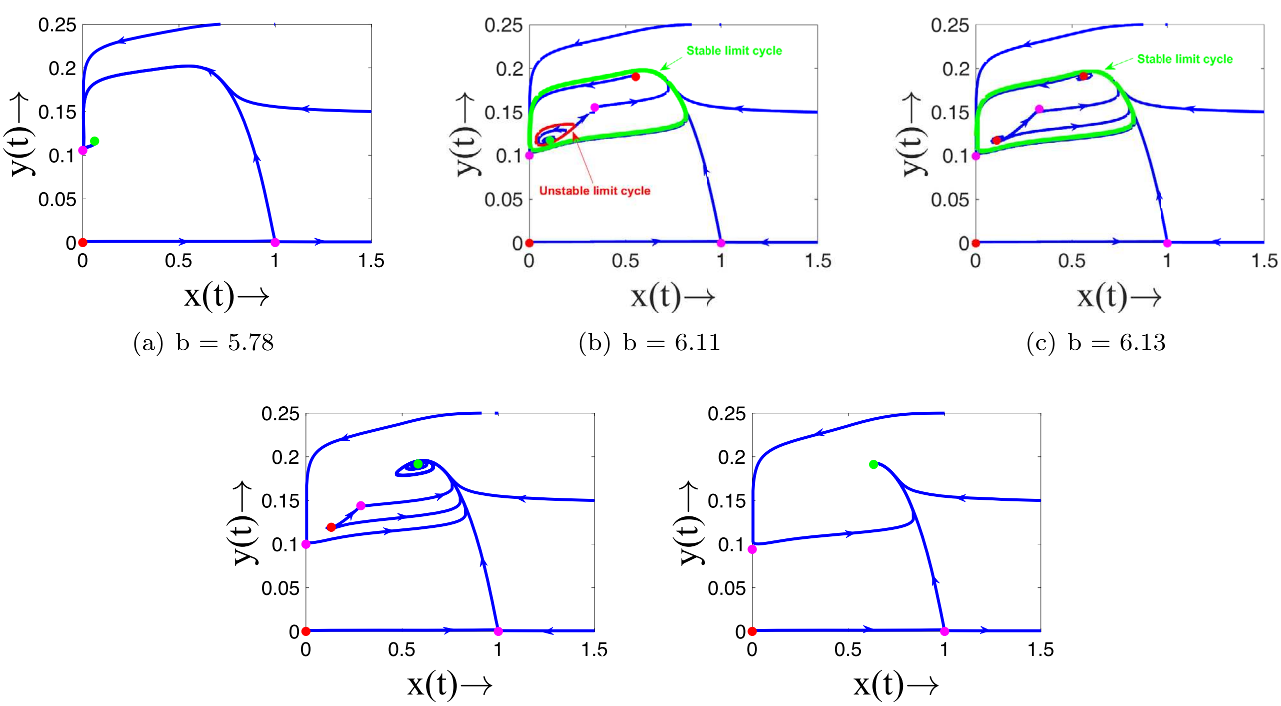


Fig. 3 Schematic diagram with saddle-node and hopf bifurcation

the number of equilibrium points from 1 to 3. Conversely, moving from the red region to the upper blue region shifts the equilibrium points from 3 back to 1. This indicates the occurrence of a saddle-node bifurcation. Additionally, crossing the lower magenta line destabilizes the stable spiral while both the spiral and limit cycles remain stable. On the other hand, crossing the upper magenta line stabilizes the previously unstable spiral and generates a stable limit cycle around it. These transitions signify a Hopf bifurcation. From a biological perspective, this parameter value is crucial, as it represents a domain of non-dimensional food quality where the system exhibits complex dynamics through the generation of periodic solutions.



(a) b = 5.78 (b) b = 6.11 (c) b = 6.13

(d) b = 6.22 (e) b = 6.49

Fig. 4 Phase portrait for **a** $b = 5.78$, **b** $b = 6.11$, **c** $b = 6.13$, **d** $b = 6.22$, **e** $b = 6.49$ and others are as in Table 1

Study of stochastic model

Existence and uniqueness of positive global solution

For existence of unique global solution of SDE, system has to satisfy linear growth condition and also local Lipschitz condition (Allen 2007). As the system (1.6) does not satisfy linear growth condition, the solution may be exploded at a finite time. First we show existence of unique positive local solution of (1.6) using change of variable (Ji et al. 2009) and then we prove this solution is global using Lyapunov analysis method (Dalal et al. 2008).

Theorem 6 *The SDE* (1.6) *has unique positive global solution in* $Int(\mathbb{R}^3_+)$ *for any initial* $(x(0), y(0)) \in Int(\mathbb{R}^3_+)$.

Proof By the transformation $u(t) = log(x(t))$, $v(t) = log(y(t))$ and using *Itô* formula, we get

$$\begin{aligned} du(t) &= \left(1 - e^{u(t)} - \frac{e^{v(t)}}{e^{2u(t)} + k_1}\right)dt + \sigma_1 dB_1(t) \\ dv(t) &= c\left(1 - \frac{be^{v(t)}}{e^{u(t)+k_2}}\right) + \sigma_2 dB_2(t) \end{aligned} \tag{3.1}$$

with $u(0) = log(x(0))$, $v(0) = log(y(0))$.

This equations satisfy local Lipschitz condition, hence system (3.1) has unique local solution $(u(t), v(t))$ for $t \in [0, \tau_e)$, τ_e is explosion time. Therefore, $x(t) = e^{u(t)}$, $y(t) = e^{v(t)}$ is unique positive local solution of (1.6) for positive initial condition. Now we show this solution is global i.e. we have to show $\tau_e = \infty$ a.s.

We can choose a large p_0 such that $x_0, y_0 \in \left[\frac{1}{p_0}, p_0\right]$. Then, define stopping time,

$$\tau_p = \inf\left\{t \in [0, \tau_e) : x \notin \left(\frac{1}{p}, p\right) \text{or } y \notin \left(\frac{1}{p}, p\right)\right\} \quad \text{for}$$

each $p \geq p_0$.

Since, τ_p is increasing with p and $\tau_\infty \leq \tau_p$ where $\tau_\infty = \lim_{p\to\infty} \tau_p$. To show $\tau_e = \infty$, we have to prove $\tau_\infty = \infty$. We prove this result by method of contradiction. If it does not hold, then there exists $T > 0$ and $\epsilon \in (0, 1)$ satisfying $P(\tau_\infty \leq T) > \epsilon$. Therefore, we can get a $p_1 \geq p_0$ such that $P(\tau_p \leq T) \geq \epsilon$ for all $p \geq p_1$. Let us consider a C^2- function: $Int(\mathbb{R}^2_+) \to Int((\mathbb{R}^2_+)$ by $V(x, y) = (x - lnx + 1) + (y - lny + 1)$. Using *Itô* formula, we get

$$\begin{aligned} dV &= (x-1)\left[1 - x - \frac{y}{x^2 + k_1} + \frac{\sigma_1^2}{2}\right]dt + (y-1) \\ &\quad \left[c\left(1 - \frac{by}{x + k_2}\right) + \frac{\sigma_2^2}{2}\right]dt + (x-1)\sigma_1 dB_1(t) \\ &\quad + (y-1)c\sigma_2 dB_2(t), \\ &= \left[-(x-1)^2 - \frac{y(x-1)}{x^2 + k_1} + \frac{\sigma_1^2}{2}\right]dt \\ &\quad + \left[c(y-1) - \frac{bcy(y-1)}{x + k_2} + \frac{\sigma_1^2}{2}\right]dt + (x-1)\sigma_1 dB_1(t) \\ &\quad + (y-1)c\sigma_2 dB_2(t), \\ &\leq \left[2x + \left(\frac{1}{k_1} + c + \frac{bc}{k_2}\right)y\right]dt \\ &\quad + \frac{\sigma_1^2 + \sigma_2^2}{2}dt + (x-1)\sigma_1 dB_1(t) + (y-1)c\sigma_2 dB_2(t). \end{aligned}$$

Using $x \leq 2(x + 1 - lnx)$, we get

$$\begin{aligned} dV &\leq \left[4(x + 1 - lnx) + 2\left(\frac{1}{k_1} + c + \frac{bc}{k_2}\right)(y + 1 - lny)\right]dt \\ &\quad + \frac{\sigma_1^2 + \sigma_2^2}{2}dt + (x-1)\sigma_1 dB_1(t) \\ &\quad + (y-1)c\sigma_2 dB_2(t). \end{aligned}$$

Let, $\frac{\sigma_1^2 + \sigma_2^2}{2} = p_1$ and $p_2 = \max\left\{4, 2\left(\frac{1}{k_1} + c + \frac{bc}{k_2}\right), p_1\right\}$, then

$$dV \leq p_2(V + 1)dt + (x-1)\sigma_1 dB_1(t) + (y-1)c\sigma_2 dB_2(t).$$

Integrating from 0 to $\tau_p \wedge T$ (τ_p is stopping time), we get

$$\begin{aligned} \int_0^{\tau_p\wedge T} &\leq p_2 \int_0^{\tau_p\wedge T} (V + 1)dt + \sigma_1 \int_0^{\tau_p\wedge T} (x-1)dB_1(t) \\ &\quad + c\sigma_2 \int_0^{\tau_p\wedge T} (y-1)dB_2(t). \end{aligned}$$

By *Itô* formula, we get

$$\begin{aligned} V(x(\tau_p \wedge T), y(\tau_p \wedge T)) &\leq V(x(0), y(0)) \\ &\quad + p_2 \int_0^{\tau_p\wedge T} (V(x, y) + 1)dt. \end{aligned}$$

Taking expectation and by Fubini's theorem, we have

$$\begin{aligned} EV(x(\tau_p \wedge T), y(\tau_p \wedge T)) \le V(x(0), y(0)) \\ + p_2 E \int_0^{\tau_p \wedge T} (V(x,y)+1)dt, \\ = V(x(0), y(0)) + p_2 T \\ + p_2 \int_0^{\tau_p \wedge T} EV(x,y)dt. \end{aligned}$$

Applying Gronwall's inequality (Mao 1997), we get

$$EV(x(\tau_p \wedge T), y(\tau_p \wedge T)) \le (V(x(0), y(0)) + p_2 T) e^{p_2 T}$$

$$i.e. V(x(0), y(0)) + p_2 T) e^{p_2 T}$$

$$\ge \epsilon \left[(p + 1 - lnp) \wedge \left(\frac{1}{p} + 1 + lnp \right) \right].$$

As $p \to \infty$, $\infty > V(x(0), y(0)) + p_2 T) e^{p_2 T} = \infty$ which leads to a contradiction. Hence, $\tau_\infty = \infty$ which completes the proof. □

Stochastic persistence

First we define $\langle x(t) \rangle = \frac{1}{t} \int_0^t x(\omega) d\omega$ and $\langle x(t) \rangle_* = \lim_{t \to \infty} inf \frac{1}{t} \int_0^t x(\omega) d\omega.$

Theorem 7 *The solutions of SDE (1.6) are strongly persistent in mean if* $\frac{\sigma_1^2}{2} + \frac{1}{k_1} < 1$ *and* $\frac{\sigma_2^2}{2} < c$ *holds.*

Proof Let, $V(x(t)) = ln(x(t))$ for $x(t) \in (0, \infty)$. Using *Itô* formula, we have

$$d(lnx) = \left[1 - x - \frac{y}{x^2 + k_1} - \frac{\sigma_1^2}{2} \right] dt + \sigma_1 dB_1(t).$$

Integrating from 0 to t, we get

$$\begin{aligned} \ln\left(\frac{x(t)}{x(0)}\right) &= t(1 - \langle x(t) \rangle) - t \left\langle \frac{y(t)}{x^2(t) + k_1} \right\rangle \\ &\quad - \frac{\sigma_1^2}{2} t + \sigma_1 B_1(t) \\ &\le t(1 - \langle x(t) \rangle) - \frac{1}{k_1} t - \frac{\sigma_1^2}{2} t + \sigma_1 B_1(t) \end{aligned}$$

$$\text{i.e.} \frac{1}{t} \ln\left(\frac{x(t)}{x(0)}\right) \le \left(1 - \frac{\sigma_1^2}{2} - \frac{1}{k_1}\right) - \frac{1}{t} \int_0^t x(t) dt + \frac{\sigma_1 B_1(t)}{t}.$$

By lemma 4.1 in Mandal and Banerjee (2012), we get $\langle x(t) \rangle_* \ge 1 - \frac{\sigma_1^2}{2} - \frac{1}{k_1} > 0$ if $\frac{\sigma_1^2}{2} + \frac{1}{k_1} < 1.$

Let $V(y(t)) = \ln(y(t))$ for $y(t) \in (0, \infty)$. By *Itô* formula,

$$d(\ln y) = \left[c \left(1 - \frac{by}{x + k_2} \right) - \frac{\sigma_2^2}{2} \right] dt + c\sigma_2 dB_2(t).$$

Integrating from 0 to t, we get

$$\frac{1}{t} \ln \frac{y(t)}{y(0)} = c - \frac{\sigma_2^2}{2} - \frac{bc}{k_2} \langle y(t) \rangle + \frac{c\sigma_2 B_2(t)}{t}.$$

By lemma 4.1 in Mandal and Banerjee (2012), we get $\langle y(t) \rangle_* \ge \frac{k_2}{bc} \left(c - \frac{\sigma_2^2}{2} \right) > 0$ if $\frac{\sigma_2^2}{2} < c$. This completes the proof. □

Stationary distribution

Theorem 8 *The SDE (1.6) has a stationary distribution* $\overline{q}(.)$ *for* $k_2 \le k_1$,

$$\delta_1 < \min\left\{ \left(M_2 - \frac{(x^{*2} + k_1 + 1)}{2} \right) \left(x^* + \frac{M_1}{2\left(M_2 - \frac{(x^{*2} + k_1 + 1)}{2} \right)} \right), \left(1 - \frac{x^{*2} + k_1 + 1}{2} \right) y^{*2} \right\}$$

where (x^*, y^*) *is an positive equilibrium point of (1.5),*

$$\delta_1 = M_1 x^* + \frac{M_1^2}{4\left(M_2 - \frac{(x^{*2} + k_1 + 1)}{2} \right)} + kM_1,$$

$M_1 = \frac{(x^{*2} + k_1) x^*}{2} \sigma_1^2 + \frac{y^*}{2bc} \sigma_2^2,$ $M_2 = \left((x^{*2} + k_1) - \frac{y^* x^*}{k_1} \right) k_2$ *with* $M_2 - \frac{a(x^{*2} + k_1 + 1)}{2} > 0$ *and* $1 - \frac{x^{*2} + k_1 + 1}{2} > 0$ *simultaneously.*

Proof Since (x^*, y^*) is a positive equilibrium of (1.5), therefore

$$1 = x^* + \frac{y^*}{x^{*2} + k_1}, 1 = \frac{by^*}{x^* + k_2}$$

Let us define a positive definite function $V : Int(\mathbb{R}_+^2 \to Int(\mathbb{R}_+)$ by

$$\begin{aligned} V(x,y) &= (x^{*2} + k_1)\left(x - x^* - x^* \ln \frac{x}{x^*} \right) \\ &\quad + \frac{1}{bc} \left(y - y^* - y^* \ln \frac{y}{y^*} \right) \end{aligned}$$

$= \quad V_1 + V_2 \quad \text{where} \quad V_1 = (x^{*2} + k_1)\left(x - x^* - x^* \ln \frac{x}{x^*}\right),$

$V_2 = \frac{1}{bc}\left(y - y^* - y^* \ln \frac{y}{y^*}\right).$

Using *Itô* formula, we have

$$
\begin{aligned}
dV_1 &= (x^{*2} + k_1)\left(1 - \frac{x^*}{x}\right)dx + \frac{(x^{*2} + k_1)x^*}{2} \times \frac{\sigma_1^2 x^2}{x^2},\\
&= \Bigg[(x^{*2} + k_1)(x - x^*)\left((1 - x) - \frac{y}{x^2 + k_1}\right)\\
&\quad + \frac{(x^{*2} + k_1)}{2} x^* \sigma_1^2\Bigg]dt + \sigma_1(x^{*2} + k_1)dB_1,\\
&= \Bigg[(x^{*2} + k_1)(x - x^*)\left(x^* + \frac{y^*}{x^{*2} + k_1} - x - \frac{y}{x^2 + k_1}\right)\\
&\quad + \frac{(x^{*2} + k_1)}{2} x^* \sigma_1^2\Bigg]dt + \sigma_1(x^{*2} + k_1)dB_1,\\
&= \Bigg[-\left(x^{*2} + k_1 - \frac{y^*(x + x^*)}{x^2 + k_1}\right)(x - x^*)^2\\
&\quad - \frac{(x - x^*)(y - y^*)(x^{*2} + k_1)}{x^2 + k_1} + \frac{(x^{*2} + k_1)x^* \sigma_1^2}{2}\Bigg]dt\\
&\quad + \sigma_1(x^{*2} + k_1)dB_1 \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
dV_2 &= \frac{1}{2}\left(1 - \frac{y^*}{y}\right)dy + \frac{y^*}{2c} \times \frac{\sigma_2^2 y^2}{y^2},\\
&= \left[(y - y^*)\left(1 - \frac{by}{x + k_2}\right) + \frac{y^*}{2c}\sigma_2^2\right]dt\\
&\quad + \frac{1}{c}\sigma_2(y - y^*)dB_2,\\
&= \left[(y - y^*)\left(\frac{by^*}{x^* + k_2} - \frac{by}{x + k_2}\right) + \frac{y^*}{2c}\sigma_2^2\right]dt\\
&\quad + \frac{1}{c}\sigma_2(y - y^*)dB_2,\\
&= \left[-\frac{(y - y^*)^2}{x + k_2} + \frac{b(x - x^*)(y - y^*)}{x + k_2} + \frac{y^*}{2c}\sigma_2^2\right]dt\\
&\quad + \frac{1}{c}\sigma_2(y - y^*)dB_2.
\end{aligned}
$$

Now

$$
\begin{aligned}
dV &= dV_1 + dV_2\\
&= LVdt + \sigma_1(x^{*2} + k_1)dB_1 + \frac{1}{bc}\sigma_2(y - y^*)dB_2.
\end{aligned}
$$

where

$$
\begin{aligned}
LV &= -(x^{*2} + k_1)(x - x^*)^2 - \frac{(x^{*2} + k_1)(x - x^*)(y - y^*)}{x^2 + k_1}\\
&\quad + \frac{y^*(x + x^*)(x - x^*)^2}{x^2 + k_1} - \frac{(y - y^*)^2}{x + k_2}\\
&\quad + \frac{a(x - x^*)(y - y^*)}{x + k_2} + M_1\\
&\leq -(x^{*2} + k_1)(x - x^*)^2 + \frac{(x^{*2} + k_1)|x - x^*||y - y^*|}{x^2 + k_1}\\
&\quad + \frac{y^*(x + x^*)(x - x^{*2})}{x^2 + k_1} - \frac{(y - y^*)^2}{x + k_2}\\
&\quad + \frac{|x - x^*||y - y^*|}{x + k_2} + M_1\\
&\leq -(x^{*2} + k_1)(x - x^*)^2 + \frac{y^*(x + x^*)(x - x^*)^2}{k_1}\\
&\quad + \frac{|x - x^*||y - y^*|(x^{*2} + k_1 + 1)}{x + k_2} - \frac{(y - y^*)^2}{x + k_2} + M_1\\
&= \frac{-\left((x^{*2} + k_1) - \frac{y^*(x + x^*)}{k_1}\right)(x + k_2)(x - x^*)^2 + (x^{*2} + k_1 + 1)|x - x^*||y - y^*| - (y - y^*)^2}{x + k_2} + M_1\\
&\leq \frac{-\left((x^{*2} + k_1) - \frac{y^* x^*}{k_1}\right)k_2(x - x^*)^2 + (x^{*2} + k_1 + 1)|x - x^*||y - y^*| - (y - y^*)^2}{x + k_2} + M_1.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(x+k_2)LV \le & -M_2(x-x^*)^2 + (x^{*2}+k_1+1) \\
& |x-x^*||y-y^*| - (y-y^*)^2 + M_1(x+k_2) \\
& \le -M_2(x-x^*)^2 + (x^{*2}+k_1+1) \\
& \left(\frac{(x-x^*)^2+(y-y^*)^2}{2}\right) - (y-y^*)^2 \\
& + M_1(k_2+x) \\
& = -\left(M_2 - \frac{(x^{*2}+k_1+1)}{2}\right)(x-x^*)^2 \\
& - \left(1-\frac{x^{*2}+k_1+1}{2}\right)(y-y^*)^2 + M_1(k_2+x) \\
& = -\left(M_2 - \frac{(x^{*2}+k_1+1)}{2}\right) \\
& \left[x - \left(x^* + \frac{M_1}{2\left(M_2 - \frac{(x^{*2}+k_1+1)}{2}\right)}\right)\right]^2 \\
& - \left(1-\frac{x^{*2}+k_1+1}{2}\right)(y-y^*)^2 + \delta_1.
\end{aligned}
$$

If $\delta_1 < \min \left\{\left(M_2 - \frac{(x^{*2}+k_1+1)}{2}\right)(x^* + \frac{M_1}{2\left(M_2 - \frac{(x^{*2}+k_1+1)}{2}\right)})^2, \left(1-\frac{x^{*2}+k_1+1}{2}\right)y^{*2}\right\}$, then the ellipsoid

$$
\begin{aligned}
& \left(M_2 - \frac{(x^{*2}+k_1+1)}{2}\right) \\
& \left[x - \left(x^* + \frac{M_1}{2\left(M_2 - \frac{(x^{*2}+k_1+1)}{2}\right)}\right)\right]^2 \\
& + \left(1-\frac{x^{*2}+k_1+1}{2}\right)(y-y^*)^2 = \delta
\end{aligned}
$$

totally lies in $\mathbb{R}_+^2$. So, there exists a neighbourhood O such that $\overline{O} \subset U_2 = Int(\mathbb{R}_+^2)$ where $\overline{O}$ is compact. Therefore for $x \in O\ U_2, LV < 0$ which implies second condition for lemma 5.1 (Mandal and Banerjee 2012) is satisfied.

Also, $M = \min\{\sigma_1^2 x^2, c^2\sigma_2^2 y^2, (x,y) \in \overline{O}\} > 0$ such that $\sum_{i,j=1}^2 a_{ij}\xi_i\xi_j = \sigma_1^2 x^2 \xi_1^2 + c^2\sigma_2^2 y^2 \xi_2^2 \ge M|\xi|^2$ for all $(x,y) \in \overline{O}, \xi \in \mathbb{R}_+^2$, which implies first condition of lemma 5.1 (Mandal and Banerjee 2012) is also satisfied. Therefore system (1.6) has a stationary distribution $\overline{q}(.)$. □

Probability density function

Now we find the approximation expression of the probability density function of the stationary distribution $\overline{q}(.)$. First we define a quasi-positive equilibrium $\overline{E}^* = (\overline{x}^*, \overline{y}^*)$ which satisfies

$$
\begin{aligned}
& (1-\overline{x}^*) - \frac{\overline{y}^*}{\overline{x}^{*^2}+k_1} - \frac{\sigma_1^2}{2} = 0, \\
& c\left(1 - \frac{b\overline{y}^*}{\overline{x}^*+k_2}\right) - \frac{c^2\sigma_2^2}{2} = 0.
\end{aligned} \tag{3.2}
$$

If the value of environmental noise is zero, then $\overline{E}^* = (\overline{x}^*, \overline{y}^*)$ coincides with positive equilibrium of the deterministic system (1.3). Hence $\overline{E}^* = (\overline{x}^*, \overline{y}^*)$ is biologically valid assumption for the stochasticity.

Theorem 9 *For $(\overline{x}^*, \overline{y}^*) \in {\mathbb{R}_+}^2$, the stationary distribution $\overline{q}(.)$ about $\overline{E}^*$ has a log-normal probability density function $\phi(x,y)$ which is*

$$
\phi(x,y) = \frac{q_1 q_2}{\pi\sqrt{rxy}} e^{-\frac{1}{2}\left(\ln\frac{x}{\overline{x}^*}, \ln\frac{y}{\overline{y}^*}\right)\Pi^{-1}\left(\ln\frac{x}{\overline{x}^*}, \ln\frac{y}{\overline{y}^*}\right)^T}
$$

if $q_1 > 0$ and $q_2 > 0$ where r, q_1, q_2 are defined in the proof.

Proof We divide the proof into three steps. The first two steps are introduced for necessary transformations of (1.5) and the last step is to solve Fokker-Planck equation of density function.

Step 1: (Logarithmic transformation) Let $(u_1, u_2)^T = (\ln x, \ln y)^T$. Using *Itô* formula, we get

$$
\begin{aligned}
du_1 &= \left((1-e^{u_1}) - \frac{e^{u_2}}{e^{2u_1}+k_1} - \frac{\sigma_1^2}{2}\right)dt + \sigma_1 dB_1(t), \\
du_2 &= \left(1 - \frac{be^{u_2}}{e^{u_1}+k_2} - \frac{c^2\sigma_2^2}{2}\right)dt + c\sigma_2 dB_2(t).
\end{aligned} \tag{3.3}
$$

Now we consider the following equations:

$$
\begin{aligned}
& (1-e^{u_1^*}) - \frac{e^{uu_2^*}}{e^{2uu_1^*}+k_1} - \frac{\sigma_1^2}{2} = 0, \\
& \left(1 - \frac{be^{uu_2^*}}{e^{uu_1^*}+k_2}\right) - \frac{c\sigma_2^2}{2} = 0.
\end{aligned} \tag{3.4}
$$

If $(\overline{x}^*, \overline{y}^*) \in {\mathbb{R}_+}^2$, then system (3.4) has a solution $u_1^*, u_2^* = (\ln\overline{x}^*, \ln\overline{y}^*)$.

Step 2: (Linearized transformation) Let $B(t) = (B_1(t), B_2(t))^T$ and $A(t) = (u_1, u_2)^T$ satisfying $v_k = u_k - u_k^*, k = 1, 2$. Then (3.3) around $(u_1^*, u_2^*)^T$ has the form:

$$dA(t) = \begin{pmatrix} -p_{11} & -p_{12} \\ p_{21} & -p_{22} \end{pmatrix} A(t)dt + \begin{pmatrix} \sigma_1 & 0 \\ 0 & c\sigma_2 \end{pmatrix} dB(t) \\ = E_0 A(t)dt + \Gamma dB(t) \tag{3.5}$$

where $p_{11} = x^* - \frac{2y^*}{(x^{*2}+k_1)^2}$, $p_{12} = \frac{y^*}{x^{*2}+k_1}$, $p_{21} = \frac{bcx^*y^*}{(x^{*2}+k_1)^2}$, $p_{22} = \frac{bcy^*}{x^{*2}+k_1}$, $\Gamma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & c\sigma_2 \end{pmatrix}$, $E_0 = \begin{pmatrix} -p_{11} & -p_{12} \\ p_{21} & -p_{22} \end{pmatrix}$.

Step 3: Using Han and Jiang (2022), system (3.5) has an invariant density function $\phi(A(t))$ which satisfies Fokker-Planck equation:

$$\frac{\sigma_1^2}{2}\frac{\partial^2\phi}{\partial v_1^2} + \frac{c^2\sigma_2^2}{2}\frac{\partial^2\phi}{\partial v_2^2} - \frac{\partial}{\partial v_1}[(-a_{11}v_1 - a_{12}v_2)\phi] \\ - \frac{\partial}{\partial v_2}[(-a_{21}v_1 - a_{22}v_2)\phi] = 0 \\ i.e. \frac{\partial}{\partial v_1}\left[\frac{\sigma_1^2}{2}\frac{\partial\phi}{\partial v_1} + (a_{11}v_1 + a_{12}v_2)\phi\right] \\ + \frac{\partial}{\partial v_2}\left[\frac{\sigma_2^2}{2}\frac{\partial\phi}{\partial v_2} + (a_{22}v_1 - a_{21}v_2)\phi\right] = 0. \tag{3.6}$$

According to Han and Jiang (2022) and Roozen (1989), $\phi(A(t))$ has the form $\phi(A(t)) = c_1 e^{-\frac{1}{2}(v_1,v_2)Q(v_1,v_2)^T}$, $Q = (q_{ij})_{2\times 2}$ is real symmetric matrix and c_1 is determined by $\int_{\mathbb{R}^2} \phi(v_1, v_2)dv_1 d_2 = 1$ and Q satisfies

$$Q\Gamma^2 Q + E_0^T Q + QE_0 = 0. \tag{3.7}$$

Letting $\Pi = Q^{-1}$, we have

$$\Gamma^2 + E_0\Pi + \Pi A_0^T = 0. \tag{3.8}$$

Therefore, Π is the solution of (3.7) is unique and positive definite, then $Q = \Pi^{-1}$. then $\phi(A(t))$ is a normal density function and $c_1 = \frac{|\Pi|^{-\frac{1}{2}}}{2\pi}$.

Lemma 2.6 (Han and Jiang 2022) implies

$$\Pi = \frac{1}{2q_1q_2}\begin{pmatrix} \sigma_1^2(q_2 + p_{22}^2 + \sigma_2 p_{12}^2) & \sigma_1^2 p_{21}a_{22} - \sigma_2^2 p_{11}p_{12} \\ \sigma_1^2 p_{21}p_{22} - \sigma_2 p_{12}^2 p_{11}p_{12} & \sigma_1^2 p_{21}^2 + \sigma_2^2(q_2 + p_{11}^2) \end{pmatrix}$$

where $q_1 = p_{11} + p_{22}$, $q_2 = p_{11}p_{22} + p_{12}p_{21}$.

Therefore $c_1 = (2\pi)^{-1}|\Pi|^{-\frac{1}{2}} = \frac{2q_1q_2}{2\pi\sqrt{r}} > 0$, where

$$r = q_2(\sigma_1^4 p_{21}^2 + \sigma_2^4 p_{12}^2) + \sigma_1^2\sigma_2^2(q_2^2 + (p_{11}^2 + p_{22}^2)q_2 \\ + (p_{11}p_{22} + p_{12}p_{21})^2).$$

From (2.8), (3.5) and the transformation $A(t) = \left(\ln\frac{x(t)}{\overline{x}^*}, \ln\frac{y(t)}{\overline{y}^*}\right)$, it follows that the stationary distribution $\overline{q}(.)$ about $\overline{E}^*$ has a log-normal distribution. Hence the stationary distribution $\overline{q}(.)$ about $\overline{E}^*$ approximately a log-normal probability density function $\phi(x, y)$ which implies $\phi(x,y) = \frac{q_1q_2}{\pi\sqrt{rxy}} e^{-\frac{1}{2}\left(\ln\frac{x}{\overline{x}^*}, \ln\frac{y}{\overline{y}^*}\right)} \Pi^{-1}\left(\ln\frac{x}{\overline{x}^*}, \ln\frac{y}{\overline{y}^*}\right)^T$. □

Stochastic sensitivity and confidence ellipse

Let us consider the following nonlinear stochastic system

$$\dot{X} = f(X) + \sigma\rho(X)\dot{B}, \tag{3.9}$$

where X is a n-vector, $\rho(X)$ represents $n \times n$ matrix for the disturbances with intensity σ, $B(t)$ represents standard Wiener process.

Now, random trajectories of (3.9) leave a deterministic attractor and also form a corresponding stochastic attractor for stationary probability distribution $\psi(X, \sigma)$, which satisfies corresponding Fokker-Planck equation. But this type of solution is very hard to get. In such cases, asymptotic based quasipotential $V(X) = -\lim_{\sigma\to 0}\sigma^2\log(\psi(X,\sigma))$ are being used (Bashkirtseva and Ryashko 2009, 2004) and the approximation of $\psi(X, \sigma)$ is given by $\psi(X,\sigma) \approx \exp\left(\frac{-V(X)}{\sigma^2}\right)$.

Next we consider stochastically forced equilibrium. For exponentially stable equilibrium $\overline{X}$, when stochastic states of (3.9) concentrate about $\overline{X}$, the quadratic approximation of quasipotential $V(X)$ is as follows: $V(X) \approx \frac{1}{2}(X - \overline{X}, W^{-1}(X - \overline{X}))$ where (,) denotes scalar product. Now we can get an asymptotic of the stationary distribution of (3.9) in Guassian form:

$$\psi(X,\sigma) \approx k\exp\left(\frac{-(X-\overline{X}), W^{-1}(X-\overline{X})}{\sigma^2}\right)$$

with $\sigma^2 W$ as covariance matrix. The stochastic sensitivity matrix W (unique) satisfies the equation

$$EW + WE^T = -Q, E = \frac{\partial f}{\partial X}(\overline{X}), Q = FF^T, F = \sigma(\overline{X}). \tag{3.10}$$

Let the explicit form of W is given by $W = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$ and

$$E = \frac{\partial f}{\partial t}(\overline{X})|_{(\overline{x},\overline{y})} = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}.$$

Therefore the relation (3.10) takes the form (using the

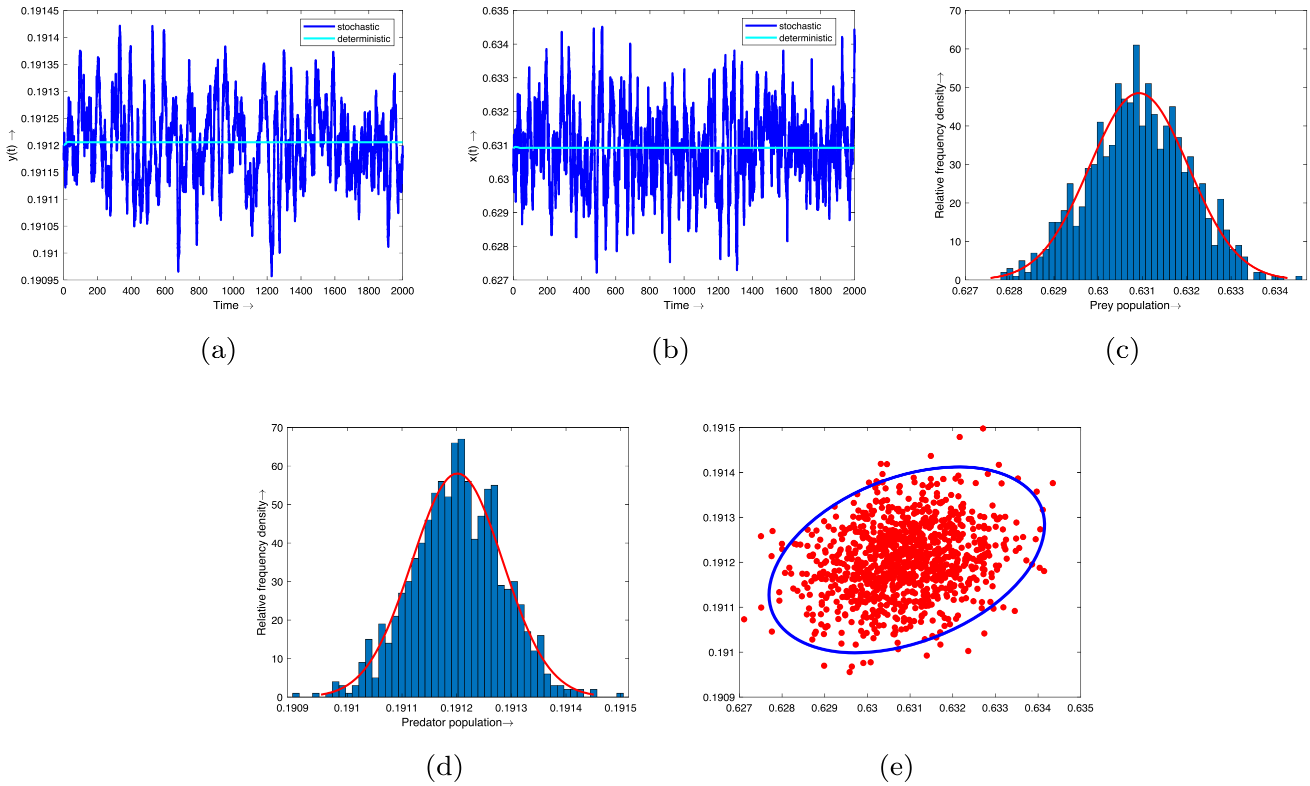


Fig. 5 Time series of the populations for $\sigma_1 = 0.0007$, $\sigma_2 = 0.0001$ and other parameter are as in Table 1 with $b = 6.49$ **a** Prey population, **b** Predator population, **c** Stationary distribution of prey population, **d** Stationary distribution of predator population, **e** scatter diagram with confidence ellipse

methodology in Yuan et al. 2020; Bashkirtseva and Ryashko 2011)

$$\begin{aligned} &2e_{11}w_{11} + e_{12}w_{12} + e_{12}w_{21} = -x^2 \\ &e_{21}w_{11} + (e_{11} + e_{22})w_{12} + e_{12}w_{22} = 0 \\ &e_{21}w_{11} + (e_{11} + e_{22})w_{21} + e_{12}w_{22} = 0 \\ &2e_{22}w_{22} + e_{21}w_{12} + e_{21}w_{21} = -c^2y^2. \end{aligned} \tag{3.11}$$

In two dimension, the equation of confidence ellipse is as follows:

$$(X - \overline{X}), W^{-1}(X - \overline{X}) = -2\sigma^2 \ln(1 - p) \tag{3.12}$$

where p is fiducial probability.

Taking the parameters as $c = 0.45, k_1 = 0.03, k_2 = 0.065, b = 4.6$, the proposed deterministic system contains two stable interior equilibrium those are $E_1(0.10171, 0.03624)$ and $E_3(0.66293, 0.15824)$. The confidence ellipse for interior equilibrium E_1 is

$$\begin{aligned} &9.94666(x - 0.10171)^2 - 2 \\ &\times 44.41101(x - 0.10171)(y - 0.03624) \\ &+ 297.65937(y - 0.03624)^2 = 2\sigma^2 \ln\frac{1}{(1-p)}, \end{aligned}$$

and the confidence ellipse for interior equilibrium E_3 is

$$\begin{aligned} &1.58383(x - .66293)^2 - 2 \\ &\times 6.07533(x - .66293)(y - .15824) \\ &+ 52.78786(y - .15824)^2 = 2\sigma^2 \ln\frac{1}{(1-p)}. \end{aligned}$$

Numerical simulations

Persistence and stationary distribution

To validate the theoretical findings regarding stationary distribution and persistence in mean, we set $b = 6.49$ and use the parameters listed in Table 1, with initial populations $x(0) = 0.6309$, $y(0) = 0.1912$. The noise levels are $\sigma_1 = 0.0007$ and $\sigma_2 = 0.0001$. In Fig. 5a, b, we present the time series for both prey and predator in deterministic and stochastic systems. Since the conditions of Theorems 7 and 8 are met, system (1.6) exhibits a stationary distribution

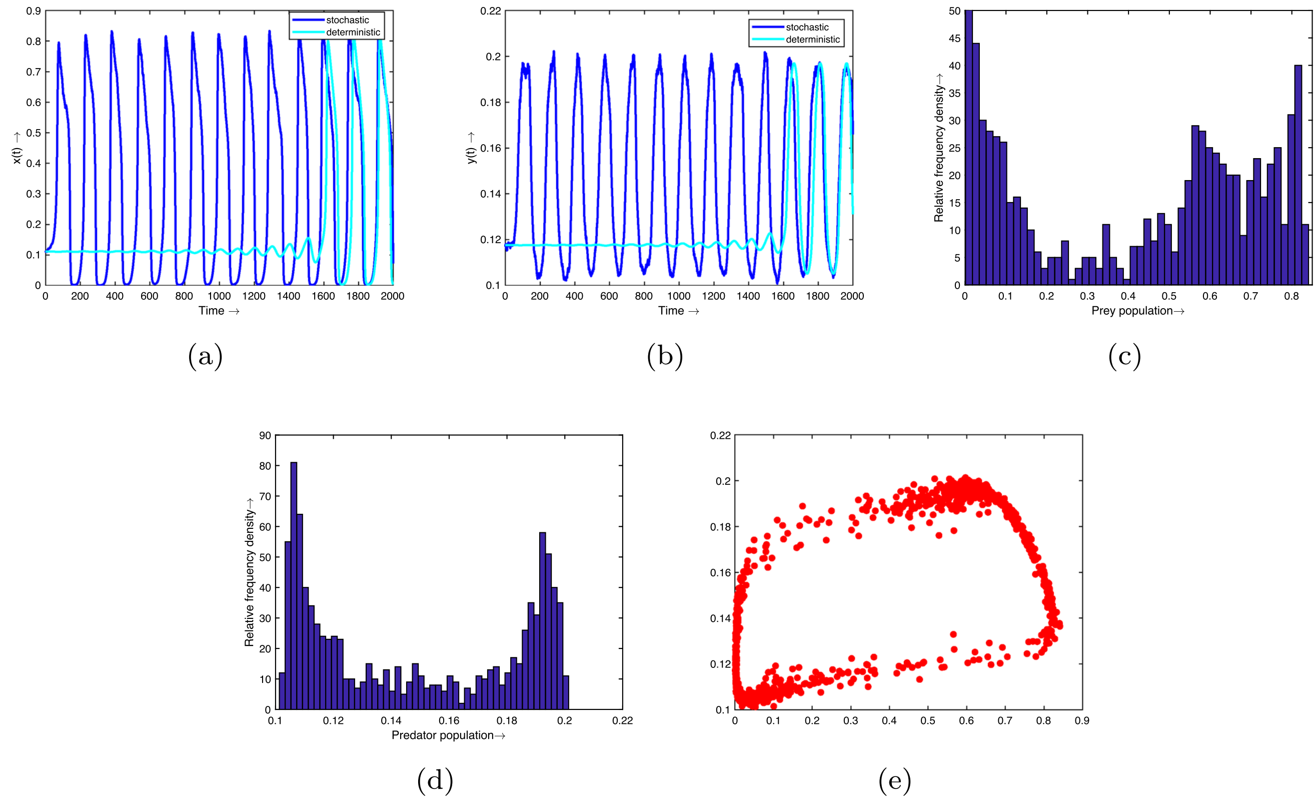


Fig. 6 Time series of the populations for $\sigma_1 = 0.005$, $\sigma_2 = 0.005$ and other parameter are as in Table 1 with $b = 6.13$ **a** Prey population, **b** Predator population, **c** stationary distribution of prey population, **d** stationary distribution of predator population, **e** Scatter diagram

and a solution that is persistent in mean. The solution demonstrates fluctuations around the equilibrium point due to the presence of white noise. As shown in Fig. 5c, d, the trajectories for prey and predator fluctuate around means of 0.1 and 0.036, respectively. Additionally, we include a confidence ellipse with a fiducial probability of 0.99, and the scatter plots illustrate that most data points lie within this ellipse (see Fig. 5e).

Next, we examine the stationary distribution for a periodic initial condition. We start with the initial populations set to $x(0) = 0.1112$, $y(0) = 0.1176$, using the parameters listed in Table 1, where $b = 6.13$ and the noise levels are $\sigma_1 = 0.005$, $\sigma_2 = 0.005$. Figure 6a, b present the time series for both prey and predator populations in these systems. As shown in Fig. 6c, d, the solution trajectories for both species fluctuate around their mean values. Additionally, Fig. 6e displays the scatter plot for this scenario.

Stochastic sensitivity and confidence ellipse

The changes in the system described by Eq. (1.6) are illustrated in Fig. 7 for various noise intensities. We start by plotting confidence ellipses around the equilibria E_1 and E_3. The size of these ellipses is influenced by the noise intensity and the chosen fiducial probability. At lower noise intensities ($\sigma_1 = 0.01$, $\sigma_2 = 0.01$, the confidence ellipses do not intersect the separatrix. As a result, trajectories originating from different initial conditions converge to distinct equilibria in their respective regions. Specifically, solutions approach E_1 within the separatrix and E_3 outside of it. The corresponding phase portrait is depicted in Fig. 7d.

We now increase the noise intensity to $\sigma_2 = 0.1$, while keeping σ_1 constant at $\sigma_1 = 0.01$. As a result, the confidence ellipse surrounding E_3 expands and touches the separatrix. The corresponding phase portrait is shown in Fig. 7h. At this point, a critical transition occurs as one trajectory jumps to another.

Next, we raise the noise intensity to $\sigma_1 = 0.08$, while keeping σ_2 constant at 0.01. At these levels, the confidence ellipse around E_1 expands and comes into contact with the separatrix. This situation also triggers a critical transition, where the trajectory of one equilibrium shifts to another (see Fig. 7l).

Finally, we raise both intensities to $\sigma_1 = 0.25$ and $\sigma_2 = 0.15$. As a result, the confidence ellipses around E_1 and E_3 expand and intersect. In this scenario, the solution trajectories exhibit chaotic behavior, as illustrated in

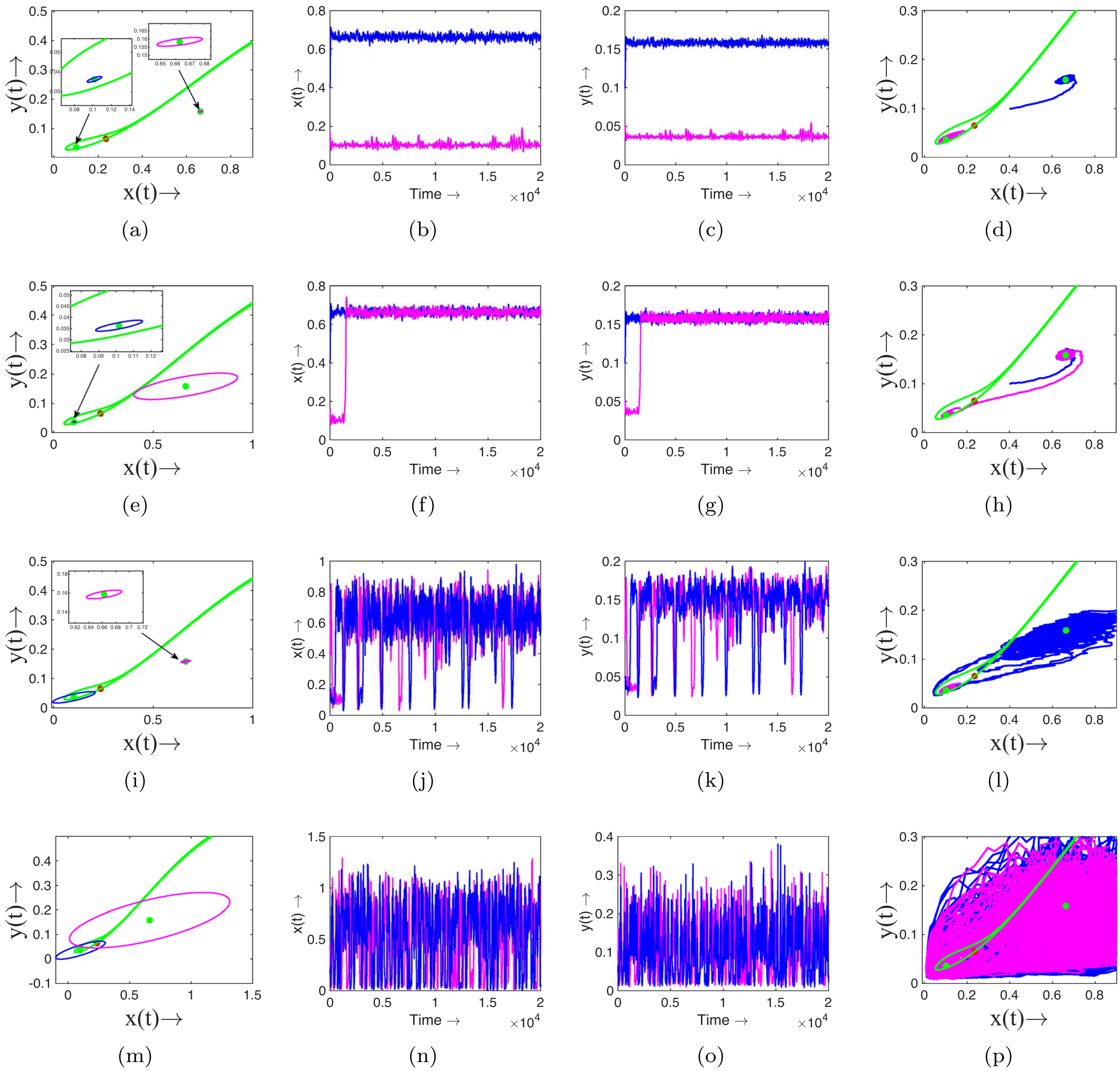


Fig. 7 Basin of attractor of deterministic system with confidence ellipse, Time series of prey population, Time series of predator population and phase portrait: **a–d** for $\sigma = (0.01, 0.01)$, **e–h** for $\sigma = (0.1, 0.01)$, **i–l** for $\sigma = (0.08, 0.01)$, **m–p** for $\sigma = (0.25, 0.15)$

Fig. 7p. This indicates that the noise intensity levels play a crucial role in controlling the dynamics of the proposed biological system, facilitating transitions between equilibrium points.

Conclusion

This paper examines how prey and predator populations interact using both predictable (deterministic) and unpredictable (stochastic) approaches. The study employs a Leslie–Gower type model, which typically describes how populations grow over time, combined with a Holling-IV functional response, a mathematical way to express how predators consume prey at varying prey densities. Deterministic analysis focuses on how the populations behave under idealized conditions without random fluctuations. It looks at stable states, oscillations, and other predictable dynamics. Stochastic examination incorporates randomness, examining how unpredictable factors (like environmental changes or random birth/death events) affect the interactions between prey and predators. Overall, the paper aims to provide a comprehensive understanding of these ecological dynamics under different conditions, contributing to the field of population ecology.

In the deterministic section, we demonstrate the positivity and boundedness of solutions, along with a stability analysis of equilibrium points. The model exhibits various bifurcations, including transcritical, saddle-node, and Hopf bifurcations. Notably, in the case of Hopf bifurcation, a stable limit cycle emerges around an unstable spiral equilibrium. Under certain parameter values, the system can have up to three interior equilibria, with two of them potentially being stable, leading to bistability. In summary, the deterministic analysis of the model reveals complex behaviors including stable and unstable equilibria, oscillations, and the potential for multiple stable states, all of which can change as parameters are varied.

In the stochastic section, we incorporate independent Brownian noise. We first establish the existence and uniqueness of global solutions, followed by conditions for persistence in mean. We also assess the criteria for stationary distributions and their corresponding probability density functions. Our analysis reveals the transition dynamics of the stochastic model near tipping points, highlighting noise-induced transitions from coexistence to extinction zones. Using stochastic simulation frameworks (SSF), we illustrate confidence domains around equilibria. At lower densities, stochastic trajectories tend to cluster around stable equilibria, while increased noise levels allow these trajectories to cross the separatrix between stable equilibria and concentrate near alternative equilibria. The threshold for stochastic transitions among stable equilibria is determined through the construction of confidence ellipses using the SSF method. At higher noise levels, the phase portrait indicates chaotic dynamics within the stochastic model. We believe our findings provide valuable insights into understanding critical transitions in a two-dimensional prey–predator system.

Appendix 1: Coefficients of a_{ij} i = 1, 2; j = 1, ..., 9

$a_{11} = 1 - 2x^* + \frac{y^*}{x^{*2}+k_1} - \frac{2x^{*2}y^*}{(x^{*2}+k_1)^2}$, $a_{12} = \frac{x^*}{x^{*2}+k_1}$, $a_{13} = -1 - \frac{3x^*y^*}{(x^{*2}+k_1)^2}$, $a_{14} = \frac{1}{x^{*2}+k_1} - \frac{2x^{*2}}{(x^{*2}+k_1)^2}$, $a_{15} = 0$, $a_{16} = \frac{-y^*}{(x^{*2}+k_1)^2}$, $a_{17} = \frac{-3x^*}{(x^{*2}+k_1)^2}$, $a_{18} = 0$, $a_{19} = 0$, $a_{21} = \frac{bcy^{*2}}{(x^{*2}+k_2)^2}$, $a_{22} = bc - \frac{2bcy^*}{x^*+k_2}$, $a_{23} = \frac{-bcy^{*2}}{(x^{*2}+k_2)^3}$, $a_{24} = \frac{2bcy^*}{(x^{*2}+k_2)^2}$, $a_{25} = \frac{-bc}{x^*+k_2}$, $a_{26} = \frac{bcy^*}{(x^{*2}+k_2)^4}$, $a_{27} = \frac{-2bcy^*}{(x^{*2}+k_2)^3}$, $a_{28} = \frac{bc}{(x^{*2}+k_2)^2}$, $a_{29} = 0$.

Appendix 2: Coefficients of A_{ij}, d_{ij}, e_{ij} i = 1, 2; j = 1, ..., 7

$A_{13} = \frac{a_{11}}{q\pi}a_{13} + \frac{a_{12}}{q\pi}a_{23}$, $A_{14} = \frac{a_{11}}{q\pi}a_{14} + \frac{a_{12}}{q\pi}a_{24}$, $A_{15} = \frac{a_{12}}{q\pi}a_{25}$, $A_{16} = \frac{a_{11}}{q\pi}a_{16} + \frac{a_{12}}{q\pi}a_{26}$, $A_{17} = \frac{a_{11}}{q\pi}a_{17} + \frac{a_{12}}{q\pi}a_{27}$, $A_{18} = \frac{a_{12}}{q\pi}a_{28}$, $A_{19} = 0$, $d_{11} = \frac{1}{\pi}a_{13} + \frac{1}{\pi}a_{14}p_{21}$, $d_{12} = \frac{1}{\pi}a_{14}p_{22}$, $d_{13} = 0$, $d_{14} = \frac{1}{\pi}a_{16} + \frac{1}{\pi}a_{17}p_{21}$, $d_{15} = \frac{1}{\pi}a_{17}p_{22}$, $d_{16} = 0$, $d_{17} = 0$, $e_{11} = A_{13} + A_{14}p_{21} + A_{15}p_{21}^2$, $e_{12} = A_{14}p_{21} + 2A_{15}p_{21}p_{22}$, $e_{13} = A_{15}p_{22}^2$, $e_{14} = A_{16} + A_{17}p_{21} + A_{18}p_{21}^2$, $e_{15} = A_{17}p_{22} + 2A_{18}p_{21}p_{22}$, $e_{16} = A_{18}p_{22}^2$, $e_{17} = 0$.

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Data availability Data of this study will be made available from the corresponding author on reasonable request.

Declarations

Conflict of interest Authors declare that they have no Conflict of interest.

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