0.1 Predator-prey model

Consider the predator-prey model:

$$\frac{dx_1}{dt} = rx_1(1-x_1) - \frac{cx_1x_2}{\alpha + x_1} \quad (1)$$
$$\frac{dx_2}{dt} = -dx_2 + \frac{cx_1x_2}{\alpha + x_1}$$

In this project we will, **(a)** Carry out a complete analysis of the Hopf Bifurcation for this system, **(b)** Phase portraits and time series analysis for solutions near the Hopf Bifurcation value, **(c)** The biological feasibility of the model and **(d)** The existence of the periodic solution when he coexistence equilibria is unstable.

(a) Hopf Bifurcation Analysis :

To simplify this model (1) here we have introduced a new time variable τ such that $dt = (\alpha + x_1)d\tau$. Thus gives a new polynomial system such that it has same orbits as the origin one.

$$\frac{dx_1}{dt} = rx_1(1-x_1)(\alpha + x_1) - cx_1x_2 \quad (2)$$
$$\frac{dx_2}{dt} = -\alpha dx_2 + (c-d)x_1x_2$$

This model (2) has one trivial $E_0(0,0)$ and one non-trivial $E^*(\frac{\alpha d}{c-d}, \frac{r\alpha}{c-d}[1-\frac{\alpha d}{c-d}])$ exists when $c > d$ and $c - d - \alpha d > 0$. The jacobian matrix of the system (2) is:

$$J = \begin{bmatrix} r\alpha + 2rx_1 - 2r\alpha x_1 - 3rx_1^2 - cx_2 & -cx_1 \\ (c-d)x_2 & -\alpha d + (c-d)x_1 \end{bmatrix}$$

The variational matrix of system (2) at trivial equilibrium point $E_0(0,0)$ is

$$J(E_0(0,0)) = \begin{bmatrix} r\alpha & 0 \\ 0 & -\alpha d \end{bmatrix}$$

here we have two eigenvalue $\lambda_1 = r\alpha$ and $\lambda_2 = -\alpha d$. Since the eigenvalues are not complex conjugate with zeor real part, hence Hopf Bifurcation does not occure at the trivial equilibrium point $E_0(0,0)$.

Again, The variational matrix of system (2) at trivial equilibrium point $E^*(\frac{\alpha d}{c-d}, \frac{r\alpha}{c-d}[1-\frac{\alpha d}{c-d}])$ is

$$J(E^*) = \begin{bmatrix} \frac{\alpha r d(c+d)}{(c-d)^2}[\frac{c-d}{c+d} - \alpha] & -\frac{\alpha c d}{c-d} \\ \frac{\alpha r(c-d(1+\alpha))}{c-d} & 0 \end{bmatrix}$$

here the eigenvalues takes the form $\lambda_{1,2} = \mu(\alpha) \pm i\omega_0(\alpha)$ where

$$\mu(\alpha) = \frac{trace}{2} = \frac{\alpha rd(c+d)}{2(c-d)^2}[\frac{c-d}{c+d} - \alpha]$$
$$\omega_0^2(\alpha) = \frac{4det - trace^2}{2} = \frac{1}{2}[\frac{4\alpha rd(c+d)}{2(c-d)^2} - \frac{rd(c+d)}{(c-d)^2}[\frac{c-d}{c+d} - \alpha]^2$$

For Hopf Bifurcation, we consider our bifurcation parameter α_0 such that $\mu(\alpha_0) = 0$. Thus at $\mu(\alpha_0) = 0$ we have two values, either $a_0 = 0$ or $a_0 = \frac{c-d}{c+d}$. Now at $a_0 = 0$ we have $\omega_0^2(\alpha) = 0$, thus Hopf bifurcation does not occurs at parameter $\alpha_0 = 0$. Again at $a_0 = \frac{c-d}{c+d}$ we have $\omega_0^2(\alpha) = \frac{rc^2d(c-d)}{(c+d)^2} > 0$. Since at at $a_0 = \frac{c-d}{c+d}$, our $\omega_0^2(\alpha) > 0$ and the eigenvalues at the non-trivial equilibrium takes the following form $\lambda_{1,2} = \pm\omega_0(\alpha)$ where $\omega_0(\alpha) > 0$, thus Hopf bifurcation occurs at $a_0 = \frac{c-d}{c+d}$.
Now, we need to verify this. Here

$$\mu'(\alpha_0) = \frac{rd(c+d)}{(c-d)}[\frac{c-d}{c+d} - 2\alpha]$$
$$\text{at } a_0 = \frac{c-d}{c+d}, \text{ we have } \mu'(\alpha) = -\frac{rd}{2(c-d)} < 0$$

Since at at $a_0 = \frac{c-d}{c+d}$ our $\mu'(\alpha_0) \neq 0$ thus the genericity conditions also holds.

To compute the first Lyapunov coefficient, we need to fix the parameter α as its critical value α_0. At $\alpha = \alpha_0$, the nontrivial equilibrium E^* has the following coordinates.

$$x_1^* = \frac{d}{c+d} \text{ and } x_2^* = \frac{rc}{(c+d)^2}$$

Now, we translate the origin of the coordinates to this equilibrium by the change of variables

$$x_1 = x_1^* + \eta_1$$
$$x_2 = x_2^* + \eta_2$$

This transforms system (2) into a new system, such that

$$\frac{d\eta_1}{dt} = -\frac{cd}{c+d}\eta_2 - \frac{rd}{c+d}\eta_1^2 - c\eta_1\eta_2 - e\eta_1^3 \tag{3}$$
$$\frac{d\eta_2}{dt} = \frac{rc(c-d)}{(c+d)^2}\eta_1 + (c-d)\eta_1\eta_2$$

This system can be represented as:

$$\frac{d\eta}{dt} = A\eta + F(\eta_1, \eta_2) \text{ and } \eta\epsilon\mathbb{R}^2$$

where $A = A(\alpha_0)$ and $F = F(\eta_1, \eta_2)$ is the nonlinear functions. The matrix $A = A(\alpha_0)$ is in the form

$$A = \begin{bmatrix} 0 & -\frac{cd}{c+d} \\ \frac{\omega_0^2(c+d)}{cd} & 0 \end{bmatrix}$$

Clearly this matrix A have two eigenvalues $\lambda_1 = i\omega_0$ and $\lambda_2 = -i\omega_0$.
Consider V_1 is the eigenvector corresponding to eigenvalue, $\lambda_1 = i\omega_0$. Here

$$V_1 = \begin{bmatrix} t \\ -\frac{i\omega_0(c+d)t}{cd} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{i\omega_0(c+d)}{cd} \end{bmatrix}$$

here $v_1 = t$ is a free variable and further we have considered $t = 1$.
Similarly, Consider V_2 is the eigenvector corresponding to eigenvalue, $\lambda_2 = -i\omega_0$. Here

$$V_2 = \begin{bmatrix} 1 \\ \frac{i\omega_0(c+d)}{cd} \end{bmatrix}$$

To apply the Hopf Bifurcation Implicit theorem(to count the first Lyapunov coefficient) we have considered the following transformation of the system (3). Consider

$$\eta = Cy \text{ where } \eta = (\eta_1, \eta_2)^T, y = (y_1, y_2)^T$$
$$\text{i.e } y' = C^{-1}\eta' = C^{-1}(A\eta + F) = C^{-1}ACy + C^{-1}F$$

Here we consider the matrix C consists of real and complex part of eigenvector V_2.(we can also use eigenvector V_1)

$$C = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\omega_0(c+d)}{cd} \end{bmatrix}$$

and the inverse of matrix C is

$$C^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{cd}{\omega_0(c+d)} \end{bmatrix}$$

Now we have,

$$C^{-1}AC = \begin{bmatrix} 1 & 0 \\ 0 & \frac{cd}{\omega_0(c+d)} \end{bmatrix} \begin{bmatrix} 0 & -\frac{cd}{c+d} \\ \frac{\omega_0^2(c+d)}{cd} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{\omega_0(c+d)}{cd} \end{bmatrix} = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix}$$

And we know,$\eta = Cy$, thus

$$\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\omega_0(c+d)}{cd} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

then we have $\eta_1 = y_1$ and $\eta_2 = \frac{\omega_0(c+d)}{cd} y_2$. Now $F(y_1, y_2)$=

$$\begin{bmatrix} -\frac{rd}{c+d}y_1^2 - \frac{\omega_0(c+d)}{d}y_1y_2 - ry_1^3 \\ \frac{\omega_0(c^2-d^2)}{cd}y_1y_2 \end{bmatrix}$$

Then

$$C^{-1}F = \begin{bmatrix} 1 & 0 \\ 0 & \frac{cd}{\omega_0(c+d)} \end{bmatrix} \begin{bmatrix} -\frac{rd}{c+d}y_1^2 - \frac{\omega_0(c+d)}{d}y_1y_2 - ry_1^3 \\ \frac{\omega_0(c^2-d^2)}{cd}y_1y_2 \end{bmatrix} = \begin{bmatrix} -\frac{rd}{c+d}y_1^2 - \frac{\omega_0(c+d)}{d}y_1y_2 - ry_1^3 \\ (c-d)y_1y_2 \end{bmatrix}$$

Now the system (3) takes the following form

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} -\frac{rd}{c+d}y_1^2 - \frac{\omega_0(c+d)}{d}y_1y_2 - ry_1^3 \\ (c-d)y_1y_2 \end{bmatrix}$$

here the transformed system follows the standard form of Hopf Bifurcation implicit theorem. So we can use this theorem to find the first Lyapunov coefficient. We know

$$a = \frac{1}{16}[f_{y_1y_1y_1} + f_{y_1y_2y_2} + g_{y_1y_1y_2} + g_{y_1y_1y_1}] + \frac{1}{16\omega_0}[f_{y_1y_2}(f_{y_1y_1} + f_{y_2y_2}) - g_{y_1y_2}(g_{y_1y_1} + g_{y_2y_2}) - f_{y_1y_1}g_{y_1y_1} + f_{y_2y_2}g_{y_2y_2}]$$

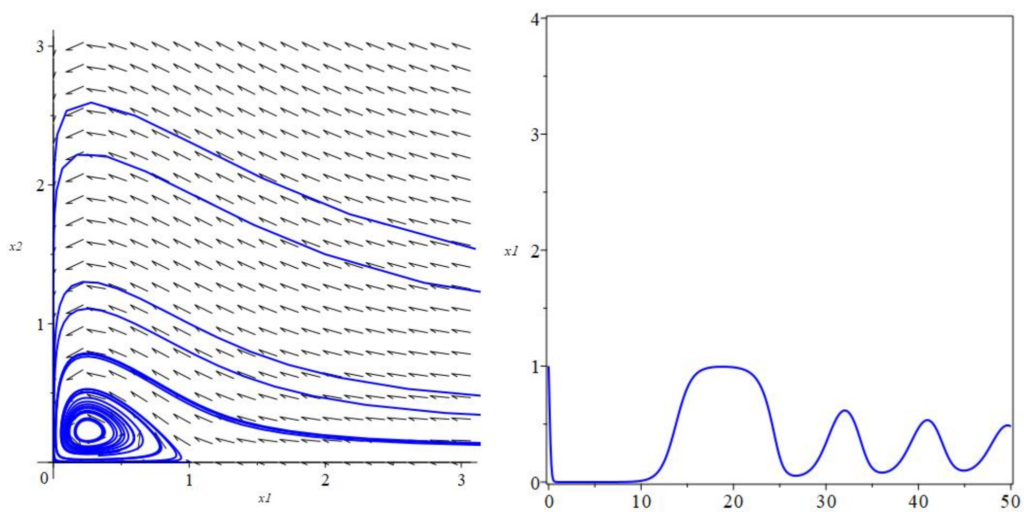
here $f(y_1, y_2) = -\frac{rd}{c+d}y_1^2 - \frac{\omega_0(c+d)}{d}y_1y_2 - ry_1^3$ and $g(y_1, y_2) = (c-d)y_1y_2$. Now taking partial derivative at $(0, 0)$ we get, $f_{y_1} = 0$, $f_{y_1y_1} = -\frac{2rd}{c+d}$, $f_{y_1y_1y_1} = -6r$, $f_{y_1y_2} = -\frac{\omega_0(c+d)}{d}$, $f_{y_2} = 0$, $f_{y_2y_2} = 0$, $f_{y_1y_2y_2} = 0$, $f_{y_2y_2y_2} = 0$, $g_{y_1} = 0$,$g_{y_1y_1} = 0$, $g_{y_1y_1y_1} = 0$, $g_{y_1y_2} = (c-d)$, $g_{y_2} = 0$, $g_{y_2y_2} = 0$, $g_{y_1y_1y_2} = 0$. Thus

$$\begin{aligned} a &= \frac{1}{16}(-6r) + \frac{1}{16\omega_0}[-\frac{\omega_0(c+d)}{d}(-\frac{2rd}{c+d})] \\ &= -\frac{r}{4} < 0 \end{aligned}$$

Since $a < 0$ then according to the Hopf Bifurcation explicit theorem, there is a stable limit cycle bifurcating at nontrivial equilibrium point $E^*(\frac{\alpha d}{c-d}, \frac{r\alpha}{c-d}[1 - \frac{\alpha d}{c-d}])$ when the bifurcation parameter α passes through $\alpha_0 = \frac{c-d}{c+d}$.

Phase Portrait and Time Series Analysis

Here we have considered the parameter values $r = 1.2$, $c = 3$, $d = 1$. This implies $\alpha_0 = \frac{3-1}{3+1} = 0.5$(bifurcation value)



(a) Phase portrait (b) Time series

Figure 1: when $\alpha = \alpha_0 = 0.5$

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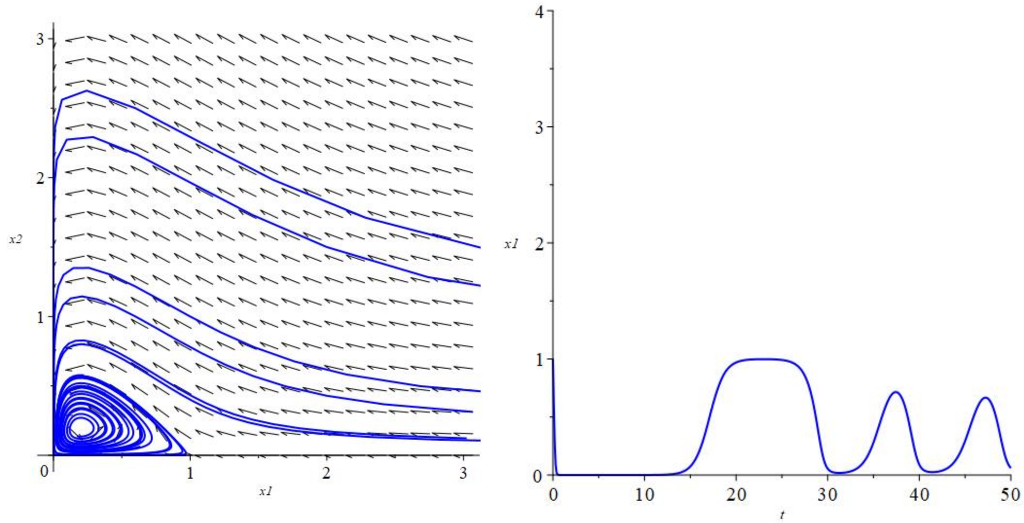
Model Feasibility

In this section, we analyzed the biological significance of the model. In theory, the boundedness of a system implies that the solutions trajectories of the system is biologically valid



(a) Phase portrait (b) Time series

Figure 2: when $\alpha = 0.4 (\alpha < \alpha_0)$

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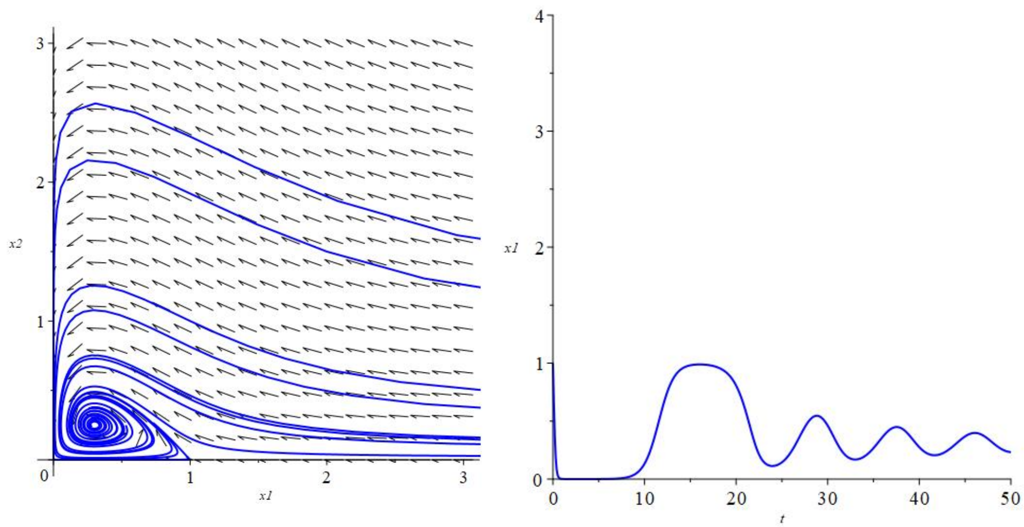

(a) Phase portrait (b) Time series

Figure 3: when $\alpha = 0.6 (\alpha > \alpha_0)$

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and well behaved. Moreover, It is also necessary to demonstrate that all solutions of the system, starting from positive initial conditions, remain positive over time $t > 0$.

1. **Positivity :** Consider the initial values $x_1(0), x_2(0) > 0$. We want to prove that the

solution $x_1(t)$ and $x_2(t)$ with these initial conditions satisfy $x_1(t) > 0$ and $x_2(t) > 0$ for all $t > 0$.
At $x_2 = 0$, from system(1) we have,

$$\dot{x}_1 = rx_1(1 - x_1)$$

We have two fixed point 0 and 1 in the x axis. $\dot{x}_1 > 0$ is increasing in the interval $(0, 1)$ i.e when $x_1 < 1$ and $\dot{x}_1 < 0$ is decreasing in the interval $(1, \infty)$ i.e when $x_1 > 1$.
Similarly, at $x_1 = 0$, from system(1) we have,

$$\dot{x}_2 = -dx_2 < 0$$

which is always decreasing. Thus, from the isocline analysis, we can say that if we start from a point on the boundaries of the first quadrant, it will always stay there and cannot leave the boundaries. Hence, every solution of the system is positive.

2. **Boundedness :** Here we want to show that every solution is bounded. From system(2) we have $\dot{x}_1 = rx_1(1 - x_1)(\alpha + x_1) - cx_1x_2$; we know that $\dot{x}_1 < rx_1(1 - x_1)$ then their exist $M > 0$ such that for all $t \geq M$, and we consider $x(t) \leq 2$.
Consider $s = x_1 + \delta x_2$. Then $\dot{s} = \dot{x}_1 + \delta \dot{x}_2$. Now we get

$$\dot{s} = rx_1(1 - x_1) - \tfrac{cx_1x_2}{\alpha+x_1} - d\delta x_2 + \tfrac{c\delta x_1 x_2}{\alpha+s_2}$$

Consider $\delta c = c$ implies

$$\begin{aligned}
\dot{s} &= rx_1(1 - x_1) - d\delta x_2 \\
\dot{s} &= rx_1 - r{x_1}^2 - d\delta x_2 \\
\dot{s} &= (r + d)x_1 - r{x_1}^2 - ds \\
\dot{s} &< 2(r + d) - ds \text{ for all } t \geq M.
\end{aligned}$$

Thus $\dot{s} < 0$ if $s > \frac{2(r+d)}{d}$. Then their exist $M_1 \geq M$ such that for all $t \geq M$ we have $s < \frac{2(r+d)}{d} + 1$. Hence the solution is bounded.

Periodic Solution

To determine if the model always has a periodic solution when the coexistence equilibrium is unstable we can use the Poincare-Bendixon theorem. Since the solutions of this system(1) is positive and bounded so we can apply Poincare-Bendixon Theorem to show that this system(1) have a periodic solution. We know that coexistence equilibrium $E^*(\frac{\alpha d}{c-d}, \frac{r\alpha}{c-d}[1 - \frac{\alpha d}{c-d}])$ exists when $c > d$ and $c - d - \alpha d > 0$.

For the isoclines, $\dot{x}_1 = 0$ implies that $x_2 = r(\alpha + x_1)(1 - x_1)$, which represent a parabola of this pattern $X^2 = 4aY$ and $\dot{x}_2 = 0$ implies $x_1 = \frac{\alpha d}{c-d}$, where $c > d$, which represent a line parallel to y axis. The x coordinate of vertex of the parabola $\dot{x}_1 = 0$ is located at $x_1^{(0)} = \frac{1-\alpha}{2} = \frac{d}{c+d}$ at $\alpha = \frac{c-d}{c+d}$. For the stability of E^*, we need to check the determinate and trace of the Jacobian matrix at E^* and show that $\det(J) > 0$ and $\text{trace}(J) < 0$ at E^*. We have

$$\text{Trace}(J)= \frac{\alpha r d(c+d)}{(c-d)^2}[\frac{c-d}{c+d} - \alpha] < 0 \text{ if } \alpha > \frac{c-d}{c+d}$$

and

$$\text{Det}(J)= \frac{\alpha^2 rcd}{(c-d)(1+\alpha)(c-d)^2} > 0 \text{ (always positive).}$$

Thus, E^* is stable if $\alpha > \frac{c-d}{c+d} = \alpha_0$ and unstable if $\alpha < \frac{c-d}{c+d} = \alpha_0$. Hence, if the line $\dot{x}_2 = 0$ lies to the right of the maximum, the point is stable; conversely, if this line is to the left, the point is unstable. Due to its repellor character, positive half-orbits of the system cannot approach the nontrivial equilibrium. Consequently, the Poincare-Bendixson Theorem implies the existence of at least one periodic orbit in the positive quadrant. Given that any such orbit must encircle an equilibrium, we deduce the existence of at least one periodic orbit around the nontrivial equilibrium in the system.