《微分几何》（彭家贵）习题

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Chapter 1

曲线的局部理论

1.1 习题

题 2.1

求下列曲线的弧长与曲率:
(1) $y = ax^2$;
(2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$;
(3) $\boldsymbol{r}(t) = (a\cosh t, b\sinh t)$;
(4) $\boldsymbol{r}(t) = (t, a\cosh\frac{t}{a})\ (a > 0)$.

解 2.1

(1) 可设 $r(t) = (t, at^2)$, 于是

$$\begin{aligned} s(t) &= \int_0^t |\boldsymbol{r}'(u)| du \\ &= \int \sqrt{1+4a^2u^2} du \\ &= \frac{1}{2} t\sqrt{1+4a^2t^2} + \frac{1}{2a}\text{arcsinh} 2at \end{aligned}$$

另一方面有

$$\frac{ds}{dt} = \sqrt{1+4a^2t^2}$$

于是

$$\boldsymbol{t} = \frac{d\boldsymbol{r}}{dt}\frac{dt}{ds} = (1+4a^2t^2)^{-\frac{1}{2}}(1, 2at)$$

由于 $\boldsymbol{n}$ 是 $\boldsymbol{t}$ 逆时针旋转 $\frac{\pi}{2}$ 得到, 即

$$\boldsymbol{n} = (1+4a^2t^2)^{-\frac{1}{2}}(-2at, 1)$$

而

$$\frac{d\boldsymbol{t}}{ds} = \frac{d\boldsymbol{t}}{dt}\frac{dt}{ds} = (1+4a^2t^2)^{-2}(-4a^2t, 2a)$$

根据 $\kappa(t)\boldsymbol{n} = \frac{d\boldsymbol{t}}{ds}$ 有

$$\kappa(t) = 2a(1+4a^2t^2)^{-\frac{3}{2}}.$$

(2) 取一种参数化表示为 $\boldsymbol{r}(t)=(a\cos t, b\sin t)$, 则

$$\begin{aligned}s(t)&=\int_0^t|\boldsymbol{r}'(u)|du\\&=\int_0^t\sqrt{a^2\sin^2u+b^2\cos^2u}du\end{aligned}$$

此时有

$$\frac{ds}{dt}=\sqrt{a^2\sin^2t+b^2\cos^2t}$$

因此

$$\boldsymbol{t}=\frac{d\boldsymbol{r}}{dt}\frac{dt}{ds}=(a^2\sin^2t+b^2\cos^2t)^{-\frac{1}{2}}(-a\sin t, b\cos t)$$

又因为 $\boldsymbol{n}$ 为 $\boldsymbol{t}$ 逆时针旋转 $\frac{\pi}{2}$, 故

$$\boldsymbol{n}=(a^2\sin^2t+b^2\cos^2t)^{-\frac{1}{2}}(-b\cos t,-a\sin t)$$

而

$$\frac{d\boldsymbol{t}}{ds}=\frac{d\boldsymbol{t}}{dt}\frac{dt}{ds}=(a^2\sin^2t+b^2\cos^2t)^{-2}(-ab^2\cos t,-a^2b\sin t)$$

根据 $\kappa(t)\boldsymbol{n}=\frac{d\boldsymbol{t}}{ds}$, 有

$$\kappa(t)=ab(a^2\sin^2t+b^2\cos^2t)^{-\frac{3}{2}}.$$

(3) 由

$$\begin{aligned}s(t)&=\int_0^t|\boldsymbol{r}'(u)|du\\&=\int_0^t\sqrt{a^2\sinh^2u+b^2\cosh^2u}du\end{aligned}$$

注意到

$$\frac{ds}{dt}=\sqrt{a^2\sinh^2t+b^2\cosh^2t}$$

因此

$$\boldsymbol{t}=\frac{d\boldsymbol{r}}{dt}\frac{dt}{ds}=(a^2\sinh^2t+b^2\cosh^2t)^{-\frac{1}{2}}(a\sinh t, b\cosh t)$$

而由于 $\boldsymbol{n}$ 为 $\boldsymbol{t}$ 的逆时针旋转 $\frac{\pi}{2}$, 故

$$\boldsymbol{n}=(a^2\sinh^2t+b^2\cosh^2t)^{-\frac{1}{2}}(-b\cosh t, a\sinh t)$$

此时

$$\frac{d\boldsymbol{t}}{ds}=\frac{d\boldsymbol{t}}{dt}\frac{dt}{ds}=(a^2\sinh^2t+b^2\cosh^2t)^{-2}(ab^2\cosh t,-a^2b\sinh t)$$

由 $\kappa(t)\boldsymbol{n}=\frac{d\boldsymbol{t}}{ds}$, 有曲率

$$\kappa(t)=-ab(a^2\sinh^2t+b^2\cosh^2t)^{-\frac{3}{2}}.$$

(4) 由

$$\begin{aligned}s(t)&=\int_0^t|\boldsymbol{r}'(u)|du\\&=\int_0^t\sqrt{1+\sinh^2\frac{u}{a}}du\\&=a\sinh\frac{t}{a}\end{aligned}$$

注意到

$$\frac{ds}{dt}=\sqrt{1+\sinh^2\frac{t}{a}}$$

因此

$$\boldsymbol{t}=\frac{d\boldsymbol{t}}{dt}\frac{dt}{ds}=(1+\sinh^2\frac{t}{a})^{-\frac{1}{2}}(1,\sinh\frac{t}{a}) \tag{1.1}$$

因 $\boldsymbol{n}$ 为 $\boldsymbol{t}$ 逆时针旋转 $\frac{\pi}{2}$, 故

$$\boldsymbol{n}=(1+\sin^2\frac{t}{a})^{-\frac{1}{2}}(-\sinh\frac{t}{a},1)$$

而

$$\frac{d\boldsymbol{t}}{ds}=\frac{d\boldsymbol{t}}{dt}\frac{dt}{ds}=(1+\sinh^2\frac{t}{a})^{-2}(-\frac{1}{a}\sinh\frac{t}{a}\cosh\frac{t}{a},\frac{1}{a}\cosh\frac{t}{a})$$

由于 $\kappa(t)\boldsymbol{n}=\frac{d\boldsymbol{t}}{ds}$, 则曲率

$$\kappa(t)=\frac{1}{a}\cosh\frac{t}{a}(1+\sinh^2\frac{t}{a})^{-\frac{3}{2}}.$$

题 2.2

设曲线 $\boldsymbol{r}(t)=(x(t),y(t))$, 证明它的曲率为

$$\kappa(t)=\frac{x'(t)y''(t)-x''(t)y'(t)}{((x')^2+(y')^2)^{-\frac{3}{2}}}$$

证明 2.2

由

$$\frac{ds}{dt}=((x'(t))^2+(y'(t))^2)^{-\frac{1}{2}}$$

则

$$\boldsymbol{t}=\frac{d\boldsymbol{r}}{dt}\frac{dt}{ds}=((x'(t))^2+(y'(t))^2)^{-\frac{1}{2}}(x'(t),y'(t))$$

由于 $\boldsymbol{n}$ 是 $\boldsymbol{t}$ 逆时针旋转 $\frac{\pi}{2}$ 得到, 故

$$\boldsymbol{n}=((x'(t))^2+(y'(t))^2)^{-\frac{1}{2}}(-y'(t),x'(t))$$

注意到

$$(x'(t)((x'(t))^2+(y'(t))^2)^{-\frac{1}{2}})_t=x'(t)(x'(t)y''(t)-x''(t)y'(t))((x'(t))^2+(y'(t))^2)^{-\frac{3}{2}}$$

$$(y'(t)((x'(t))^2+(y'(t))^2)^{-\frac{1}{2}})_t=y'(t)(y'(t)x''(t)-y''(t)x'(t))((x'(t))^2+(y'(t))^2)^{-\frac{3}{2}}$$

于是

$$\frac{d\boldsymbol{t}}{ds}=\frac{d\boldsymbol{t}}{dt}\frac{dt}{ds}=((x'(t))^2+(y'(t))^2)^{-2}(x'(t)y''(t)-x''(t)y'(t))(-y'(t),x'(t))$$

根据 $\kappa(t)\boldsymbol{n}=\frac{d\boldsymbol{t}}{ds}$, 我们有

$$\kappa(t)=(x'(t)y''(t)-x''(t)y'(t))((x'(t))^2+(y'(t))^2)^{-\frac{3}{2}}.$$

题 2.3

设曲线 C 在极坐标 (r,θ) 上的表示为 $r=f(\theta)$, 证明曲线 C 的曲率表达式为

$$\kappa(\theta)=\frac{f^2(\theta)+2\left(\frac{df}{d\theta}\right)^2-f(\theta)\frac{d^2f}{d\theta^2}}{\left(f^2(\theta)+\left(\frac{df}{d\theta}\right)^2\right)^{\frac{3}{2}}}$$

证明 2.3

令

$$\begin{aligned}x(\theta)&=f(\theta)\cos\theta\\ y(\theta)&=f(\theta)\sin\theta\end{aligned}$$

求导有

$$\begin{aligned}x'(\theta)&=f'\cos\theta-f\sin\theta\\ y'(\theta)&=f'\sin\theta+f\cos\theta\end{aligned}$$

$$\begin{aligned}x''(\theta)&=f''\cos\theta-2f'\sin\theta-f\cos\theta\\ y''(\theta)&=f''\sin\theta+2f'\cos\theta-f\sin\theta\end{aligned}$$

带入曲率的计算公式中有

$$\begin{aligned}\kappa(\theta)&=\frac{x'(\theta)y''(\theta)-x''(\theta)y'(\theta)}{((x'(\theta))^2+(y'(\theta))^2)^{\frac{3}{2}}}\\ &=\frac{(f'\cos\theta-f\sin\theta)(f''\sin\theta+2f'\cos\theta-f\sin\theta)-(f''\cos\theta-2f'\sin\theta-f\cos\theta)(f'\sin\theta+f\cos\theta)}{((f'\cos\theta-f\sin\theta)^2+(f'\sin\theta+f\cos\theta)^2)^{\frac{3}{2}}}\\ &=\frac{f^2(\theta)+2\left(\frac{df}{d\theta}\right)^2-f(\theta)\frac{d^2f}{d\theta^2}}{\left(f^2(\theta)+\left(\frac{df}{d\theta}\right)^2\right)^{\frac{3}{2}}}\end{aligned}$$

题 2.4

求下列曲线的曲率和挠率

(1) $\boldsymbol{r}(t)=(a\cosh t,a\sinh t,bt)$ $(a>0)$
(2) $\boldsymbol{r}(t)=(3t-t^2,3t^2,3t+t^2)$
(3) $\boldsymbol{r}(t)=(a(1-\sin t),a(1-\cos t),bt)$ $(a>0)$
(4) $\boldsymbol{r}(t)=(at,\sqrt{2}a\ln t,\frac{a}{t})$ $(a>0)$

解 2.4

(1) 由

$$\frac{ds}{dt}=\sqrt{a^2\sinh^2t+a^2\cosh^2t+b^2}=\sqrt{2a^2\cosh^2t-a^2+b^2}$$

知

$$\boldsymbol{t}=\frac{d\boldsymbol{r}}{dt}\frac{dt}{ds}=(2a^2\cosh^2t-a^2+b^2)^{-\frac{1}{2}}(a\sinh t,a\cosh t,b)$$

故

$$\frac{d\boldsymbol{t}}{ds}=\frac{d\boldsymbol{t}}{dt}\frac{dt}{ds}=(2a^2\cosh^2 t-a^2+b^2)^{-2}(a(a^2+b^2)\cosh t,-a(a^2-b^2)\sinh t,-2a^2b\sinh t\cosh t)$$

则曲率为

$$\kappa(t)=|\frac{d\boldsymbol{t}}{ds}|=a(2a^2\cosh^2 t-a^2+b^2)^{-\frac{3}{2}}(2b^2\cosh^2 t+a^2-b^2)^{\frac{1}{2}}$$

于是

$$\begin{aligned}\boldsymbol{n}&=\frac{1}{\kappa(t)}\frac{d\boldsymbol{t}}{ds}\\&=(2a^2\cosh^2 t-a^2+b^2)^{-\frac{1}{2}}(2b^2\cosh^2 t+a^2-b^2)^{-\frac{1}{2}}((a^2+b^2)\cosh t,(b^2-a^2)\sinh t,-2ab\sinh t\cosh t)\end{aligned}$$

则

$$\boldsymbol{b}(s)=\boldsymbol{t}\wedge\boldsymbol{n}=(2b^2\cosh^2 t+a^2-b^2)^{-\frac{1}{2}}(-b\sinh t,b\cosh t,-a)$$

注意到

$$\begin{aligned}\dot{\boldsymbol{b}}(s)&=\frac{d\boldsymbol{b}(s)}{dt}\frac{dt}{ds}\\&=(2b^2\cosh^2 t+a^2-b^2)^{-\frac{3}{2}}(2a^2\cosh^t-a^2+b^2)^{-\frac{1}{2}}(-b(a^2+b^2)\cosh t,b(a^2-b^2)\sinh t,2ab^2\sinh t\cosh t)\end{aligned}$$

于是根据 $\dot{\boldsymbol{b}}=-\tau(t)\boldsymbol{n}$ 有

$$\tau(t)=\frac{b}{2b^2\cosh^2 t+a^2-b^2}$$

(2) 由

$$\begin{aligned}\frac{ds}{dt}&=\sqrt{(3-2t)^2+(6t)^2+(3+2t)^2}\\&=\sqrt{44t^2+18}\end{aligned}$$

则

$$\boldsymbol{t}=\frac{d\boldsymbol{t}}{dt}\frac{dt}{ds}=(44t^2+18)^{-\frac{1}{2}}(3-2t,6t,3+2t)$$

故

$$\frac{d\boldsymbol{t}}{ds}=\frac{d\boldsymbol{t}}{dt}\frac{dt}{ds}=(22t^2+9)^{-2}(-3(11t+3),27,-3(11t+3))$$

则曲率为

$$\kappa(t)=|\frac{d\boldsymbol{t}}{ds}|=\frac{3\sqrt{11}}{(22t^2+9)^{\frac{3}{2}}}$$

于是法向量

$$\boldsymbol{n}=\frac{1}{\kappa(t)}\frac{d\boldsymbol{t}}{ds}=\frac{1}{\sqrt{11}}(22t^2+9)^{-\frac{1}{2}}(-(11t+3),9,-(11t-3))$$

则副法向量为

$$\boldsymbol{b}(s)=\boldsymbol{t}\wedge\boldsymbol{n}=\frac{1}{\sqrt{11}}(-\frac{3}{2},-1,\frac{3}{2})$$

而

$$\dot{\boldsymbol{b}}=\frac{d\boldsymbol{b}}{dt}\frac{dt}{ds}=0$$

于是根据 $\dot{\boldsymbol{b}}(s)=-\tau(s)\boldsymbol{n}(s)$, 可知挠率

$$\tau(s)=0.$$

(3) 由

$$\begin{aligned}\frac{ds}{dt} &= \sqrt{(-a\cos t)^2+(a\sin t)^2+b^2}\\ &= \sqrt{a^2\cos^2 t+a^2\sin^2 t+b^2}\\ &= \sqrt{a^2+b^2}\end{aligned}$$

记 $c=\sqrt{a^2+b^2}$, 于是

$$\boldsymbol{t}=\frac{d\boldsymbol{r}}{dt}\frac{dt}{ds}=(-\frac{a}{c}\cos t,\frac{a}{c}\sin t,\frac{b}{c})$$

而由

$$\frac{d\boldsymbol{t}}{ds}=\frac{d\boldsymbol{t}}{dt}\frac{dt}{ds}=(\frac{a}{c^2}\sin t,\frac{a}{c^2}\cos t,0)$$

故曲率

$$\kappa(t)=|\frac{d\boldsymbol{t}}{ds}|=\frac{a}{c^2}$$

则法向量

$$\boldsymbol{n}=\frac{1}{\kappa}\frac{d\boldsymbol{t}}{ds}=(\sin t,\cos t,0)$$

那么副法向量

$$\boldsymbol{b}=\boldsymbol{t}\wedge\boldsymbol{n}=(-\frac{b}{c}\cos t)$$

(4) 由

$$\begin{aligned}\frac{ds}{dt} &= \sqrt{a^2+\left(\sqrt{2}a\frac{1}{t}\right)^2+\left(-\frac{a}{t}\right)^2}\\ &= \sqrt{a^2+\frac{2a^2}{t^2}+\frac{a^2}{t^4}}\\ &= a\sqrt{1+\frac{2}{t^2}+\frac{1}{t^4}}\\ &= a(1+\frac{1}{t})\end{aligned}$$

于是

$$\dot{\boldsymbol{t}}=\frac{d\boldsymbol{r}}{dt}\frac{dt}{ds}=\frac{t}{t+1}(1,\frac{\sqrt{2}}{t},-\frac{1}{t^2})$$

而由

$$\frac{d\boldsymbol{t}}{ds}=\frac{d\boldsymbol{t}}{dt}\frac{dt}{ds}=\frac{t}{a(t+1)}(\frac{1}{(t+1)^2},-\frac{\sqrt{2}}{(t+1)^2},\frac{2t+1}{t^2(t+1)^2})$$

知

$$\kappa(t)=|\frac{d\boldsymbol{t}}{ds}|=\frac{\sqrt{3t^2+(2+\frac{1}{t})^2}}{a(t+1)^3}$$

于是法向量

$$\boldsymbol{n}=\frac{1}{\kappa(t)}\frac{d\boldsymbol{t}}{ds}=\frac{1}{\sqrt{3t^2+(2+\frac{1}{t})^2}}(t,-\sqrt{2}t,2+\frac{1}{t})$$

则副法向量

$$\boldsymbol{b}=\boldsymbol{t}\wedge\boldsymbol{n}=\frac{1}{\sqrt{3t^2+(2+\frac{1}{t})^2}}(\sqrt{2},-2,-\sqrt{2}t)$$