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Bounded Budget Connection (BBC) games or how to make friends and influence people, on a budget ☆

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ABSTRACT

Motivated by applications in social and peer-to-peer networks, we introduce the *Bounded Budget Connection* (BBC) game and study its pure Nash equilibria. We have a collection of n players, each with a budget for purchasing links. Each link has a cost and a length. Each node has a preference weight for each node, and its objective is to purchase outgoing links within its budget to minimize its sum of preference-weighted distances to the nodes. We show that determining if a BBC game has pure Nash equilibria is NP-hard. We study (n,k)-uniform BBC games, where all link costs, lengths, and preferences are equal and every budget equals k. We show that pure Nash equilibria exist for all (n,k)-uniform BBC games and all equilibria are essentially fair. We construct a family of equilibria spanning the spectrum from minimum to maximum social cost. We also analyze best-response walks and alternative node objectives.

1. Introduction

1.1. Motivation

You are the campaign manager for a Presidential candidate and it is the start of what will be a long and grueling series of primaries and caucuses to determine your party's nominee. You have a limited budget for the campaign, in terms of both money and time. You need to understand the (organized as well as informal) networks of connections and influence that exist within the nation to decide how best to allocate your scarce resources so as to have the optimal impact on voters. Many of the players and political operatives you choose to reach out to are not only being courted by other candidates but also have their own ambitions (maybe at the regional or town levels) and agendas. Your actions affect and, in turn, are affected by the actions of the others, voters and candidates, in the race. You need to understand the calculus of allegiance in the world of ever-shifting political loyalties: what should you do and with whom should you ally so as to effectively counteract and neutralize the strategies of your opponents while maximizing your chances of winning the required votes?

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You are the founder of a social networking website, such as a friend finder site or a site where people trade timeshares on vacation homes. Your income from the site depends on how well people are connected to one another. The more easily users can find others to befriend or trade timeshares with, the more money you make. People have natural bounds on their time and cognitive resources and hence are limited in the number of people they can maintain direct ties with (also known as the Dunbar limit in the sociology literature). They must rely on friends of friends, and friends of friends of friends and so on to reach other people in the network. It is no surprise that you are concerned with understanding how the structure of networks generated by individuals expressing their natural preferences and aversions will affect your ability to monetize the network. Could it be possible that left to their own devices people will generate poorly connected networks?

You are designing the next killer application, the next Napster, the next Kazaa, the next big thing in the world of unstructured peer-to-peer file sharing networks or overlay networks. You know that people will hack the open source reference implementation of the client to create nodes that will behave strategically, selecting their first hop neighbors to selfishly optimize their utility. In unstructured P2P file sharing, nodes employ scoped flooding or multiple parallel random walks to reach other nodes and thus have to adhere to small out-degrees to prevent clogging. For analogous reasons of scalability, overlay networks require constraints on the out-degree of nodes so as to reduce the number of links that need monitoring and to reduce the amount of link state information that needs to be disseminated. You know that the success of your killer application depends critically on the connectedness of the network. Every node will independently attempt to minimize its average latency to the subset of nodes of interest. Will this lead to an operating point that is close to the social optimum, or can it lead to an anarchic situation characterized by an impoverishment of connectivity?

In this paper we define and study a graph-theoretic game called the Bounded Budget Connection (BBC) game that abstracts each of the three situations above where strategic nodes acting under a *cost budget* form connections (friends) with a view to optimizing their proximity (influence) to the nodes of interest. This is a big problem space that allows for a variety of models to capture different situations. In addition to different notions of connection cost and proximity e.g., symmetric and asymmetric, uniform and nonuniform, metric spaces, etc., one can also consider a variety of solution and equilibrium concepts (other than Nash equilibria) as well as the dynamics of the resultant complex systems. There are many earlier works that touch on similar issues as detailed in Section 1.3. We believe the budget constraint is an important real-world restriction and consider our paper to be a preliminary step towards understanding and characterizing the rich and elegant structures that exist in this domain.

1.2. Our results

To capture the above scenarios we posit the following Bounded Budget Connection (BBC) game. We have a collection of n players or nodes each of whom has a budget for purchasing links; each link has a cost as well as a length, and each node has a set of preference weights for each of the remaining nodes. The objective of each node is to use its budget to buy a set of outgoing links from itself so as to minimize its sum of preference-weighted distances to the remaining nodes.

Our goal in this paper is to study the structural and complexity-theoretic properties of pure Nash equilibria. We first present our results on nonuniform BBC games, the most general kind of BBC games. Nonuniform BBC games are best explained by defining their complement. Uniform BBC games are those in which all link costs are equal, all link lengths are equal, all preference weights are equal, and all budgets are equal. Nonuniform BBC games are BBC games which are not uniform BBC games.

- We show that determining the existence of a pure Nash equilibrium in nonuniform BBC games is NP-hard. To be precise, we prove the NP-hardness of determining the existence of a pure Nash equilibrium when link costs, link lengths or preference weights are nonuniform[1]

Next, we present our results on uniform BBC games. We can assume, without loss of generality, that all link weights, link lengths and preference weights are equal to 1 and all budgets are equal to k, thus allowing us to talk of (n,k)-uniform BBC games.

- We show that a pure Nash equilibrium or stable graph exists for all (n,k)-uniform BBC games, and that all stable graphs are essentially fair (i.e., all nodes have similar costs). We provide an explicit construction of a family of stable graphs that spans the spectrum from minimum total social cost to maximum total social cost. To be precise, we show that the price of stability is $\Theta(1)$ and the price of anarchy is $\Omega(\sqrt{\frac{n/k}{\log_k n}})$ and $O(\sqrt{\frac{n}{\log_k n}})$. Observe that our bounds for the price of anarchy are essentially tight when k is a constant.
- Inspired by the existence of stable graphs in the uniform case, we next tackle the question of finding stable graphs that are "regular" in a strong sense (to be defined subsequently). Such graphs, if they exist, would have applications to overlay and P2P networks. Unfortunately we are able to show that even the class of Abelian Cayley graphs (a strict superset of "regular" graphs) does not possess a stable graph. In other words, stability and regularity are mutually

[1] We believe that this question is not just NP-hard but in fact Σ_2-complete. Further, we also conjecture that pure Nash equilibria do exist in all cases where only the budgets are non-uniform.

incompatible. This has the implication that, as the designer of a P2P or overlay network, one has to give up stability in order to get the simplicity and convenience of regularity.

Lastly, we consider the dynamics of best response moves.

- We show that in any (n, k)-uniform BBC game, a (suitably defined and entirely natural) best response walk converges to a strongly connected configuration within n^2 steps.
- We show that uniform BBC games are not (ordinal) potential games by presenting a loop for best response walks. This underscores the importance of our explicit constructions of stable graphs, as it rules out the possibility of demonstrating existence of Nash equilibria through suitably defined potential functions.

We end by showing that there are analogous results for the case where the cost function is the maximum (instead of sum) of the weighted distances.

1.3. Related work

Network formation models. Notions of group and network formation along with concepts of influence have been investigated by a number of different communities, starting with researchers in economics and game theory and followed by work in combinatorial optimization and computer science. A significant difference between the previous models and ours is that earlier models do not place any budget constraints on the links. None of the utility functions adopted in prior work (mostly linear in the link costs) can strictly enforce the budget constraint; hence, they do not capture scenarios in which there are hard constraints on the number or cost of links for individual nodes. The mathematical techniques underlying our proofs are combinatorial and, for the most part, elementary. The techniques used in our work are not significantly different from past work in this area. The particular constructions and the results obtained, however, are different and much more involved in some cases (like the Forest of Willows family of Section 4.1).

We present a brief summary of selected models and results in network creation games. The work of Jackson and Wolinsky [1] models and analyzes the stability of networks when nodes themselves choose to form or sever links; their model is different from ours in that they studied different stylized models that included production and allocation functions under the (relatively weak) concept of pairwise stability, along with side payments. Bala and Goyal study a model of directed network formation where nodes incur costs based on the number of incoming links [2]. Galeotti, Goyal, and Kamphorst [3] give a characterization of equilibrium networks in a model with link costs similar to those in [2], but where the links formed by each node are bidirectional and thus can benefit both endpoints. Goyal and Vega-Redondo study a model in which each node plays a bilateral game with each neighbor and formulate a game in which a node selects its adjacent links as well as the strategy to play in the bilateral games [4]. Page and Wooders introduce and analyze an abstract game that models how farsighted individuals choose their network formation strategies in the presence of coalitional network preferences and coalitional moves [5].

In [6], Fabrikant, Luthra, Maneva, Papadimitriou, and Shenker define and study a network creation game in which nodes select undirected links and optimize a cost which is the sum of the number of edges, scaled by a parameter $\alpha > 0$, and the sum of distances to the rest of the nodes. They present several results on the price of anarchy, which is the ratio of the cost of the worst-case Nash equilibrium to the social optimum cost [7]. Further results in this direction are obtained by Albers, Eilts, Even-Dar, Mansour, and Roditty [8] and Demaine, Hajiaghayi, Mahini, and Zadimoghaddam [9]. Halevi and Mansour extend this model to the case where each node is only interested in connecting to a subset of the other nodes [10]. Even-Dar and Kearns [11] impose a cost for the purchase of a link rather than a fixed budget; unlike [6], they consider a stochastic model and associated small-world effects. In [12], Moscibroda, Schmid, and Wattenhofer study a variant in which the nodes are embedded in a metric space and the distance component of the cost is replaced by the stretch with respect to the metric. They obtain tight bounds on the price of anarchy and show that the problem of deciding the existence of pure Nash equilibria is NP-hard.

Network formation under the requirement for bilateral consent for building links is studied by Corbo and Parkes [13]. Even-Dar, Kearns, and Suri [14] study a similar network creation game restricted to a bipartite graph, with nodes representing buyers and sellers. Our model closely follows the works of Chun, Fonseca, Stoica, and Kubiatowicz [15] and Laoutaris, Smaragdakis, Bestavros, and Byers [16], who present experimental studies of network formation games involving non-unit link lengths.

Network formation games have also been studied in the context of Internet inter-domain routing. A coalitional game-theoretic problem modeling BGP is introduced by Papadimitriou in [17] and studied further by Markakis and Saberi [18]. A fractional version is studied by Haxell and Wilfong [19]. Also related is the work on designing strategy-proof mechanisms for BGP by Feigenbaum, Papadimitriou, Sami, and Shenker [20] as well as the recent work on strategic network formation through AS-level contracts by Anshelevich, Shepherd, and Wilfong [21]. Johari, Mannor, and Tsitsiklis [22] consider a contracts-based model of network formation where links do not have predefined costs but are subject to negotiation, and nodes attempt to minimize incoming traffic by obtaining compensation in return.

Optimization in network formation. Combinatorial optimization aspects are explored by Kempe, Kleinberg, and Tardos [23, 24] where the goal is to pick an initial set in a stochastic model with maximal expected influence. This model is extended further by Bharathi, Kempe, and Salek [25] to a competitive setting within the stochastic framework where different players compete (sequentially) to maximize their expected influence.

More recent work. Subsequent to the original publication of this work [26], several new studies on network creation games have been published. Recent work by Alon, Demaine, Hajiaghayi, and Leighton considers network creation games in which each node aims to minimize its maximum diameter or its average distance to other nodes by swapping one incident edge at a time [27]. Bei, Chen, Teng, Zhang, and Zhu [28] study network formation in which the objective of each node is to maximize its *betweenness*, which is a measure of how much information passes through a node among all pairwise interactions. Brautbar and Kearns [29] consider the objective of maximizing the clustering coefficient, a measure of local connectivity widely used in social sciences. Dev introduces a notion of identity and link cost sharing to network formation games [30]. Each node chooses its identity and makes offers for link creation. A link gets formed if the offers made by the two endpoints can pay for the cost of the link, which itself depends on the identities of the endpoints. The work of Demaine, Hajiaghayi, Mahini, and Zadimoghaddam [31] considers network creation games in which the link creation costs are shared between the endpoints, and links are chosen from an underlying host graph (as opposed to the complete graph).

In recent work, Ehsani et al. [32] study a bounded budget connection game very similar to ours with one distinction: the links in our model are directed, while their model has bidirectional links. This difference in the model changes the structure of the networks obtained both qualitatively and quantitatively. For instance, the price of anarchy bounds under the two different models are asymptotically far apart. The model of Ehsani et al. could be viewed as a budgeted version of the model of [3]. Kawald and Lenzner study the dynamics of network creation games under the models of [27] and [32].

2. Problem definition

A *Bounded Budget Connection* game (henceforth, a BBC game) is specified by a tuple $\langle V, w, c, \ell, b\rangle$, where V is a set of nodes, $w : V \times V \to \mathbb{Z}$, $c : V \times V \to \mathbb{Z}$, $\ell : V \times V \to \mathbb{Z}$, and $b : V \to \mathbb{Z}$ are functions. For any $u, v \in V$, $w(u, v)$ indicates u's preference, or affinity, for communicating with v, $c(u, v)$ denotes the cost of directly linking u to v, and $\ell(u, v)$ denotes the length of the link (u, v), if established. For any node $u \in V$, $b(u)$, specifies the budget u has for establishing outgoing directed links: the sum of the costs of the links u establishes should not exceed $b(u)$.

A strategy for node u is a subset S_u of $\{(u, v) : v \in V\}$ such that $\sum_{v:(u,v)\in S_u} c(u, v) \le b(u)$. Let S_u denote a strategy chosen by node u and let $S = \{S_u : u \in V\}$ denote the collection of strategies. The network formed by S is simply the directed graph $G(S) = (V, E)$ where $E = \bigcup_u S_u$. The utility of a node u in $G(S)$ is given by $-\sum_v w(u, v)d(u, v)$, where $d(u, v)$ is the shortest path from u to v in $G(S)$ according to the lengths given by ℓ. For convenience, we assume that if no path exists in $G(S)$ from u to v, then $d(u, v)$ is given by some large integer $M \gg n \max_{u,v} \ell(u, v)$; we refer to M as the *disconnection penalty*.

Following the standard game-theoretic terminology, we say that a strategy selection $S = \{S_u : u \in V\}$ is stable if it is a pure Nash equilibrium for the BBC game; in particular, for each u, S_u is an optimal strategy for u assuming that the strategy for every $v \ne u$ is fixed as in S.

A major focus of our work is on *uniform games*, in which (a) $c(u, v)$ is identical for all u, v; (b) $w(u, v)$ is identical for all u, v; (c) $\ell(u, v)$ is identical for all u, v; and (d) $b(u)$ is identical for all $u \in V$. In a uniform game, we may assume without loss of generality that $c(u, v) = w(u, v) = \ell(u, v) = 1$ for all u, v, and $b(u) = k$, for all $u \in V$, for some integer k. We refer to the preceding uniform game as an (n, k)-uniform game where $n = |V|$. We refer to BBC games that are not uniform as *non-uniform* games.

3. Nonuniform games

In this section we show there exist instances of non-uniform BBC games that do not have a pure Nash equilibrium. Furthermore, we prove that it is NP-hard to determine whether a given instance of a non-uniform BBC game has a pure Nash equilibrium.

The following theorem is subsumed by Theorem 2, but the proof includes a very simple example of a BBC game with no pure Nash equilibrium. We will reuse this example in the proof of Theorem 4.

Theorem 1. *For any $k \ge 1$, $n \ge k^2 + k + 4$ there exists a nonuniform BBC game with no pure Nash equilibrium with n nodes, a budget of k per node, a cost of* 1 *per edge, nonuniform affinities and nonuniform link lengths.*

Proof. We first construct a BBC game G with $n = 4$, $k = 1$, uniform costs, nonuniform lengths, and nonuniform affinities, such that G has no pure Nash equilibrium. We then show how to extend the claim to larger values of n and k.

We will create 4 nodes, as shown in Fig. 1, labeled A, A^*, B, and B^*. A and A^* have positive affinity only for each other, B has positive affinity only for B^*, and B^* has positive affinity only for A^*. The lengths of edges (A, B), (A, B^*), (B, A^*), and (A^*, A) are all 1, the length of edge (B^*, A^*) is 2, and the length of all other edges is 4.

Clearly, A^* will purchase edge (A^*, A) and B^* will purchase edge (B^*, A^*), since any other options for each of these nodes would have length 4 and would therefore make the distance to their goal significantly further. For A, the strategy of choosing edge (A, B^*) (giving utility 3 for A) dominates the strategy of choosing edge (A, A^*) (which gives utility 4).

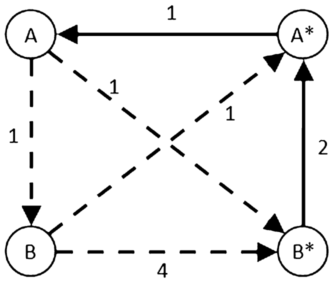


Fig. 1. Simple example of a BBC game with no pure Nash equilibrium. A has affinity only for A^*, A^* only for A, B only for B^*, and B^* only for A^*. All edges have cost 1 and all nodes have budget 1. Some edge lengths are shown in the figure. The rest of the edges have length 4.

Therefore A will either choose (A, B) or (A, B^*). Similarly, for B, the strategy of choosing edge (B, B^*) (which gives a utility of 4 for B) dominates the strategy of choosing edge (B, A) (which gives a utility ≥ 5), so B will choose either (B, B^*) or (B, A^*).

If A purchases (A, B^*), then A pays 3 regardless of what B chooses. In this case B will purchase (B, A^*), and pay 3 using path $B \to A^* \to A \to B^*$. However, if B purchases (B, A^*), then A would do better to purchase (A, B) and pay 2. If A does not purchase edge (A, B^*), then B must purchase edge (B, B^*) to avoid paying the disconnection penalty, which will cause A to switch back to edge (A, B^*). Therefore there is no pure Nash equilibrium.

This example can be extended to higher values of n and k as follows. First, add k sets $S_1, S_2, \dots, S_k$, each consisting of $k+1$ additional nodes. Each node in a set S_i has positive affinity only for the other nodes in S_i. Each edge to or from a node in S_i has length 1 and cost 1. Each of A, A^*, B, and B^* will have positive affinity for one node in each of $S_1, S_2, \dots, S_{k-1}$, in addition to the affinities specified above.

If $n > k^2 + k + 4$, we will add nodes as needed; call this set of additional nodes T. Each of these will have positive affinity for one node in each of the k sets. The length of all edges from these nodes to nodes in any S_i will be 1. All other edges to or from these nodes will have length 4.

Since a node in some S_i has affinity for exactly k other nodes, and has an available length 1 edge to each of these, it will purchase exactly the set of edges to the other nodes in S_i. Similarly, each node in T will connect exactly to the nodes for which it has positive affinity. Now, consider a node from the original game G. It must build at least one of its edges within the original game G, or else it will have to pay the disconnection penalty. Case 1: it builds only one edge within the original graph G. Here, the best option is to build $k-1$ edges directly to the nodes in the sets S_i in which it is interested, paying $(k-1)$ plus the cost of accessing its goal node within G (which is some value between 2 and 4 as analyzed above), or at most $k-1+4 = k-3$. Case 2: it builds $j > 1$ edges within the original graph G. Here, only $k-j$ edges are available to access the $k-1$ goal nodes outside of G. In order to access more than one node with the same purchased edge, it must purchase an edge of length 4 to one of the nodes in T and use this to access at least 2 of the goal nodes. Therefore, it will pay at least $(k-3) + (2 \times (4+1))$ + the cost of accessing its goal node within G, which is at least $k-3+10+2 = k+9 > k-3$. So the node would do better to point only one edge into G, leaving the same non-equilibrium as described above. $\square$

Theorem 2. *For any $k \geq 4$, $n \geq 2k^2 + k + 7$ there exists a BBC game with no pure Nash equilibrium (n nodes, budget k per node) in which only the affinities are nonuniform. All costs and lengths are* 1.

Proof. We first construct a BBC game G with $n = 28$, nonuniform k, uniform costs, uniform lengths, and nonuniform affinities, such that G has no pure Nash equilibrium. We then show how to extend the claim to uniform values of k, and then to larger values of n and k.

We will create 8 nodes, as shown in Fig. 2, labeled A, B, C, D, and E, p_1, p_2, and q. As in the proof of Theorem 1, we will force the edges for C, E, D, p_1, p_2, and q by giving them positive affinities for exactly as many nodes as they can afford to link to directly. C has budget 2 and affinity only for A and D. E has budget 3 and affinity only for B, p_1 and q. D has budget 0. p_1 has budget 1 and affinity only for p_2, p_2 has budget 1 and affinity only for D, and q has budget 1 and affinity only for C. Each of the nodes C, D, E, p_1, p_2, and q will purchase exactly the edges to the nodes for which they have some affinity.

This leaves only A and B with any decision to make. A has budget 1 and affinity for B, C and D. B has budget 1 and affinity for A, E and D. A always has the option to point to E and be able to access all of its preferred nodes. Therefore, A will never point to C or to D, since both of these choices will force it to pay the disconnection penalty, regardless of B's choice. If A points to E, then the best choice for B is to point to C (pointing to D would cause it to pay the disconnection penalty, pointing to A has utility 8, pointing to E has utility 9, and pointing to C has utility 7). If B points to C, then A can improve by pointing to B (paying 6 rather than 9 by pointing to E). If A points to B, then B must point to E to avoid paying the disconnection penalty. If B points to E, then A will do best by also pointing to E (paying 9 rather than 10 by pointing to B), causing B to move back to C. Therefore there is no pure Nash equilibrium.

To make the budgets uniform, we can add $2(k-1)+1$ sets of $k+1$ nodes each, call them $S_1, \dots, S_{k-1}, T_1, \dots, T_{k-1}, R$. Each node in each set will have affinity for each other node in the same set. A will have affinity for one node in each S_i,

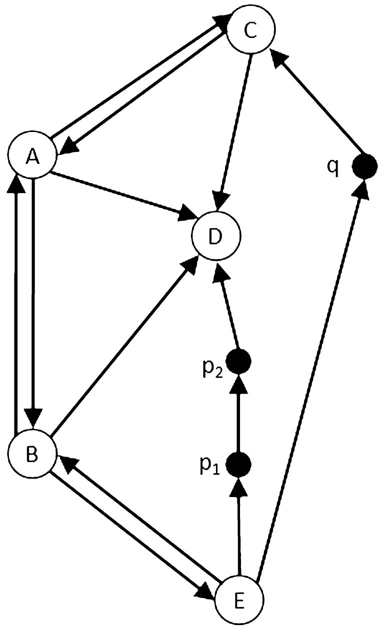


Fig. 2. Example of a BBC game with no pure Nash equilibrium, only budget and affinity are nonuniform. All edges have cost 1 and length 1. D has budget 0, C has budget 2, E has budget 3, and all other nodes have budget 1. The arrows in the figure show affinities $= 1$. The rest of the affinities are 0.

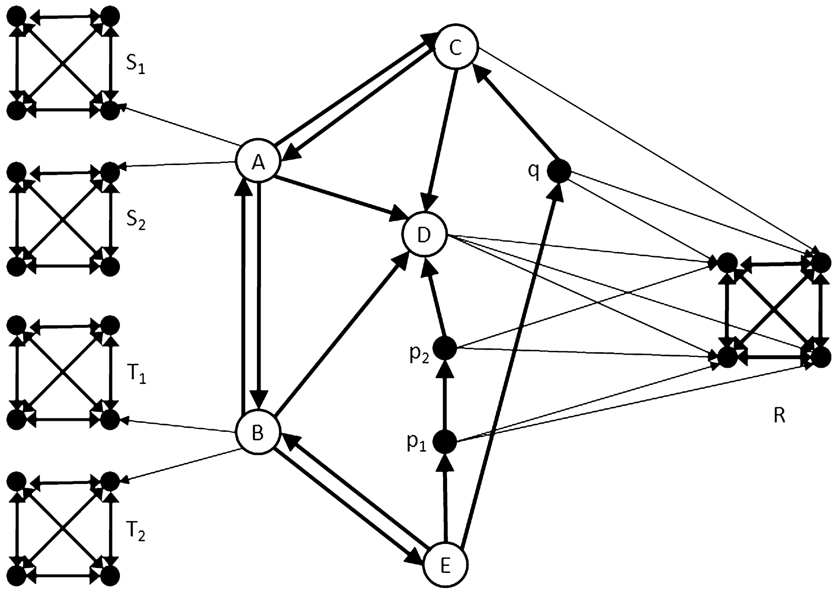


Fig. 3. Example of a BBC game with no pure Nash equilibrium, only affinity is nonuniform. All edges have cost 1 and length 1. All nodes have budget 3. The arrows in the figure show affinities $= 1$. The rest of the affinities are 0.

B will have affinity for one node in each T_i, and each other node will have affinity for exactly enough nodes in R to fill the remainder of its budget. These new affinities force the remaining edges without changing the existing example. See Fig. 3.

In order to extend to higher values of n, simply add additional nodes with affinity only for the nodes in R. $\square$

Theorem 3. *There exists a BBC game for any $n \geq 5$ with no pure Nash equilibrium in which only the costs are nonuniform and all budgets $k = 2$.*

Proof. Consider the example shown in Fig. 4. The displayed edges have cost 1. The rest of the edges have cost 3. Each node has budget 2. All lengths and affinities are uniform. Since the edges missing from the figure are too expensive to ever be

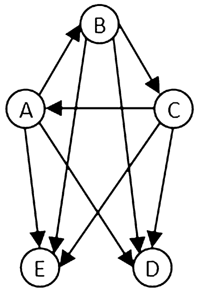


Fig. 4. Example of a BBC game with no pure Nash equilibrium in which only the costs are nonuniform. The displayed edges cost 1. The rest of the edges cost 3. Each node has budget 2. This example can also be used to show a game with no pure Nash equilibrium in which only the lengths are nonuniform. For this case, the displayed edges have length 1, and the rest of the edges have length 4.

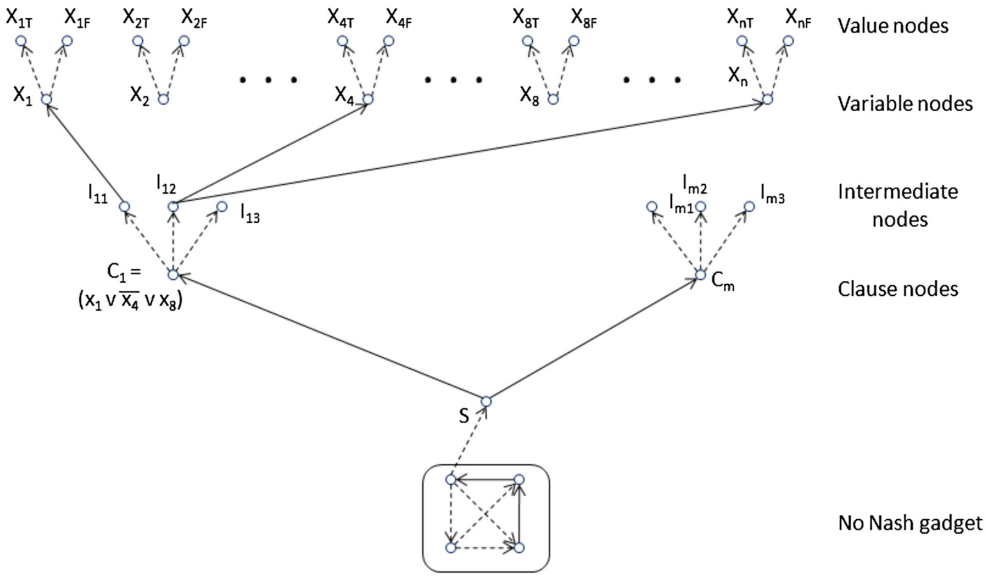


Fig. 5. Construction to prove NP-hardness of pure Nash equilibrium detection.

included, A, B or C cannot remove the edges from the cycle $A \to B \to C \to A$ without paying the disconnection penalty. D and E cannot afford to purchase any edges.

A, B, and C can each afford one other edge, and the affordable options for each are the edge to D and the edge to E. At least two of A, B, and C must point to the same node (either E or F). Without loss of generality, assume two of the nodes point to E. If both A and B point to E, then A's utility is at least 7. A can switch its edge from E to F and get utility 6. If both B and C point to E, then B's utility is at least 7, but B can switch from E to F and get utility 6. If both A and C point to E, then C's utility is at least 7, but C can switch from E to F and get utility 6. Therefore, for any set of legally purchased edges, some node can improve by changing an edge, so there is no equilibrium.

This can be trivially extended to larger n by adding additional nodes with only cost 3 edges into and out of these nodes. $\square$

Theorem 4. *It is NP-hard to determine whether a given instance of the non-uniform BBC game has a pure Nash equilibrium.*

Proof. The proof is by a reduction from 3SAT. Let ϕ be a 3SAT formula with n variables, $x_1, \ldots, x_n$, and m clauses, $c_1, \ldots, c_m$. We create a non-uniform BBC instance as follows. For each variable x_i in ϕ, we introduce 3 nodes: a *variable node* X_i and two *truth nodes* X_{iT} and X_{iF}. For each clause c_j, we introduce a clause node C_j and three intermediate nodes I_{j1}, I_{j2}, and I_{j3}, one for each of the three literals in the clause. We also have an additional node S and a gadget G consisting of the nodes illustrated in Fig. 1. Our construction is depicted in Fig. 5.

For all u, v in V, we set $c(u,v)$ to be 1. The length of every link shown in Fig. 5 is 1 and the length of every other link is a large number L greater than the number of nodes; we set the disconnection penalty M to be nL. The budget for each truth node is 0, the budget for S is m, and the budget for every other node is 1.

We now define the affinities. Let V denote the set of all nodes. Affinities for the truth nodes are irrelevant, since they have budget 0. For node X_i, we set $w(X_i, v)$ to be 1 for $v \in \{X_{iT}, X_{iF}\}$ and 0 for all other v; thus, X_i equally prefers to communicate with X_{iT} and X_{iF} and with no other node.

Now consider a clause c_j. For each intermediate node I_{jk}, we set $w(I_{jk}, v)$ to be 1 if $v = X_i$ and 0 otherwise. If the kth literal of c_j is x_i, then for the intermediate node I_{jk}, we set $w(I_{jk}, v)$ to 1 for $v = X_{iT}$ and 0 for all other v; else, we set $w(I_{jk}, v)$ to 1 for $v = X_{iF}$ and 0 for all other v. For the clause node C_j, we set the affinities as follows. If x_i is in clause C_j, then $w(C_j, X_{iT})$ is 1; if $\overline{x_i}$ is in clause K_j, then $w(C_j, X_{iF})$ is 1; for all other v, $w(C_j, v)$ is 0.

We next consider the gadget G. The affinities of B, A^*, and B^* remain the same (see the proof of Theorem 1). For node A, we set $w(A, A^*) = 2m - 1$, $w(A, v) = 2$ for all intermediate nodes v, and $w(A, v) = 0$ for all other nodes v.

For node S, we set $w(S, v) = 1$ if v is a clause node and 0 otherwise. Node S has a budget of m and will therefore be able to reach all desired nodes each with distance 1.

We now show that ϕ is satisfiable if and only if the above BBC game has a pure Nash equilibrium. Suppose ϕ is satisfiable. Consider a satisfying assignment for ϕ. If x_i is true, we set the link from X_i to X_{iT}; otherwise, we set the link from X_i to X_{iF}; in either case, X_i has attained its highest utility possible. The intermediate nodes just link to their respective variable nodes and attain their best utility. For each clause c_j, there exists a literal in the clause, say the kth literal, which is satisfied. If the literal equals variable x_i, then the intermediate node I_{jk} has a path to X_{iT} through X_i. So we set the link from the clause node C_j to I_{jk}. A clause node prefers to communicate with three of the truth nodes but can communicate with at most one in any stable network owing to budget constraints. Furthermore, the three-hop path achieved from the clause node to a truth node is the shortest possible, so each clause node has also attained its best utility. We finally consider the nodes in the gadget G. A^* and B^* will still obtain their best utility by pointing to A and A^*, respectively. If A links to S, it's utility will be

$$\begin{aligned}
&(2m-1)\cdot\big(\text{distance to } A^*\big) + \sum_{j=1}^{m}\sum_{d=1}^{3} 2\cdot(\text{distance to } I_{jd}) \\
&= (2m-1)\cdot M + \sum_{j=1}^{m}\big((2\cdot 3) + (2\cdot 2\cdot M)\big) \\
&= 6mM - M + 6m
\end{aligned}$$

If A were to link within G, its utility will be at least

$$\begin{aligned}
&(2m-1)\cdot\big(\text{distance to } A^*\big) + \sum_{j=1}^{m}\sum_{d=1}^{3} 2\cdot(\text{distance to } I_{jd}) \\
&\geq \big((2m-1)\cdot 2\big) + \sum_{j=1}^{m}\sum_{d=1}^{3} 2M \\
&= 4m - 2 + 6mM
\end{aligned}$$

Since $M > 2m + 2$, A will link to S. Since A is not linking to B^*, B must link to B^* to avoid paying M. Thus, the constructed network is stable.

If the BBC game has a pure Nash equilibrium, then A has to link to S since the gadget by itself does not have a pure Nash equilibrium, by the proof of Theorem 1. Since A links to S, it must be true that the cost to A by pointing to S is at most the cost to A if it were to point within G. Assuming s clause nodes are pointing directly to truth nodes (because they cannot access their truth nodes via intermediate nodes), the utility for A pointing to S is

$$\begin{aligned}
&(2m-1)\cdot\big(\text{distance to } A^*\big) + \sum_{j=1}^{m}\sum_{d=1}^{3} 2\cdot(\text{distance to } I_{jd}) \\
&= (2m-1)\cdot M + (s+2m)2M + (m-s)6 \\
&= 2mM - M + 2sM + 4mM + 6m - 6s \\
&= 6mM - M + 2sM + 6m - 6s
\end{aligned}$$

The utility for A pointing within G is at most

$$(2m-1)\cdot\big(\text{distance to } A^*\big) + \sum_{j=1}^{m}\sum_{d=1}^{3} 2\cdot(\text{distance to } I_{jd})$$

$$\geq \big((2m-1)\cdot 3\big) + \sum_{j=1}^{m}\sum_{d=1}^{3} 2M$$
$$= 6m - 3 + 6mM$$

Solving for s so that $6mM - M + 2sM + 6m - 6s \leq 6m - 3 + 6mM$ gives $s \leq \frac{1}{2}$, so s must be 0. This means that each clause node must be pointing to some intermediate node. A clause node links to an intermediate node only if the intermediate node has a path either to a node X_{iT}, where x_i is in the clause, or to a node X_{iF}, where $\overline{x_i}$ is in the clause. This is because if no intermediate node for the clause has such a path, then the clause node would link to S. This yields the following satisfying assignment for ϕ: set x_i to true if X_i has a link to X_{iT}, and false otherwise.

In the above reduction, the budget function is nonuniform. By using additional nodes as in the extension to higher k in the proof to Theorem 1, the reduction can be easily adapted to work where the budget of each node is k, for $k \geq m$. □

4. Uniform games

Although non-uniform games lack stability, the simplest version of the framework has many interesting properties. We define a uniform (n,k)-BBC game as a game in which all preferences, costs, and lengths are 1, and each node has a budget of k links. In other words, in this graph, all the nodes are equally interested in communicating with all other nodes, any connection can be established for the same cost, and the utility function is calculated using hop counts.

We show that a Nash equilibrium, or stable graph, exists for the uniform (n,k)-BBC game with any values of n and k and that all stable graphs are essentially fair (all nodes in a stable graphs have similar cost). We also establish nearly tight bounds on the price of anarchy and price of stability. Although we describe a class of stable graphs, showing that there are multiple Nash equilibria, we show that no regular graph – a graph in which all nodes imitate the same configuration of links – can ever be stable. We finally provide some initial results about the dynamics of non-stable uniform graphs, as individual nodes keep changing their links to improve their cost.

4.1. Nash equilibria

The main result of this section is the following.

Theorem 5. *For any $n \geq 2$ and any positive integer k, uniform stable (n,k)-graphs exist, and in any stable graph the cost of any node is $\Theta(1)$ times the cost of any other node. The price of anarchy is $\Omega(\frac{\sqrt{(n/k)}}{\log_k n})$, $O(\sqrt{\frac{n}{\log_k n}})$ (for $k \geq 2$). The price of stability is $\Theta(1)$.*

To prove Theorem 5, we first show fairness. Then we describe a class of stable graphs for any k and prove that they are stable. The graphs in this class have total cost ranging from $O(n^2 \log_k n)$ to $\Omega(n^2\sqrt{\frac{n}{k}})$. This gives a lower bound on the price of anarchy and the price of stability. Then, we give an upper bound on the diameter of any stable graph and use this to obtain an upper bound on the price of anarchy.

Lemma 1 *(Fairness). In any stable graph for the (n,k)-uniform game, the cost of any node is at most $n + n\lfloor \log_k n\rfloor$ more than, and at most $2 + 1/k + o(1)$ times, the cost of any other node.*

Proof. Let G be a stable graph for the (n,k)-uniform game and let r be a node in G that has the smallest cost C^*. Consider the shortest path tree T rooted at r. Let v be any other node. Within $\lfloor \log_k n\rfloor$ hops from v, there exists a node u that has at least one edge not in T. Since G is stable, node u has cost at most $C^* + n$, since it can achieve this cost by attaching one of its links not in T to r. Therefore, the cost of v is at most $C^* + n + n\log_k n$, since the distance from v to any node w is at most $\log_k n$ more than that of u to w. Noting that C^* is at least $\sum_{0\leq i<\log_k n} ik^i \geq (n - n/k)\lfloor \log_k n\rfloor$ completes the proof of the lemma. □

In order to give an upper bound on the price of anarchy and the price of stability, we define a class of graphs that is stable. We call this class the “Forest of Willows” graphs (see Fig. 6).

Definition 1 *(Forest of Willows graphs).* There are k directed, complete, k-ary trees of height h (rooted at nodes $r_1, r_2, \ldots, r_k$). Each of these trees has k^h leaves. Beneath each leaf, there is a tail of length l (l nodes not including the leaf). Let R_i be the nodes in the tree rooted at r_i plus the tails beneath this tree. The last node in each tail has an edge to the root of each of the k trees. The second to last node of a tail in R_i has an edge to each r_j, $j \neq i$. If a tail node in R_i does not have an edge to r_i, the node above it has an edge to r_i and any $k-2$ other roots. If a tail node in R_i does have an edge to r_i, the node above it has an edge to each r_j, $j \neq i$. We call this the *initial configuration*. This graph has n nodes, where $n = k(2^{h+1} - 1 + 2^h l)$. This can be extended to other values of n by adding additional leaves as evenly as possible across the trees. However, for the sake of simplicity, the following proof of stability assumes that n is of the above form.

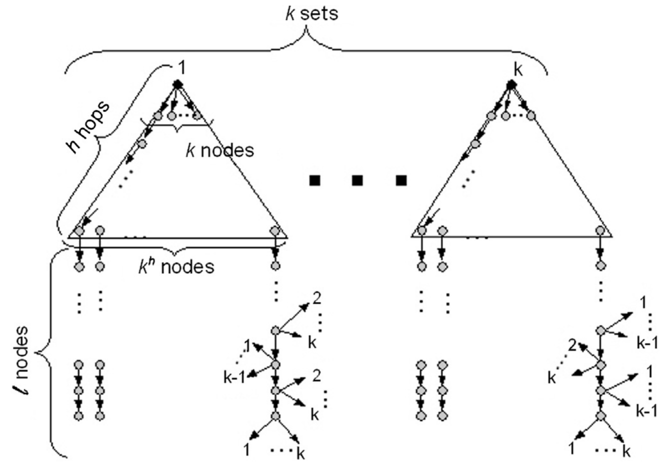


Fig. 6. The "Forest of Willows" stable graphs: k sections, each has a complete k-ary tree of height h. Under each leaf, there is a tail of length l. The last node in each tail has an edge to the root of each tree. The second to last node of a tail has an edge to the root of each tree other than its own. The rest of the tail nodes alternate between pointing to all the roots except their own or all the roots except one (arbitrary, but not its own).

We restrict h and l by requiring:

$$\frac{(h+l)^2}{4} + h + 2l + 1 < \frac{n}{k}$$

By definition of the graph structure, $h \in O(\log_k n)$. Any l $(0 \le l < 2\sqrt{\frac{n}{k}})$ obey the requirements. Notice that $l < 2\sqrt{\frac{n}{k}}$ implies $h > \frac{\log_2 n}{2} - \frac{\log_2 k}{2} - 1$. Also notice that the diameter of this graph is $\Theta(h+l)$, so as k approaches $\frac{n}{\log^2 n}$, this class converges to a single graph: a collection of k complete k-ary trees with edges from the leaves to the roots.

For ease of notation, we use *descendants of* x for a node $x \in R_i$ to refer to x plus all nodes $y \in R_i$ such that x is on the unique shortest path from r_i to y. We use D_x to refer to the number of descendants of x. *Ancestors of* x for $x \in R_i$ refer to all the nodes in the shortest path from r_i to x (not including x). We use δ_x to refer to the number of ancestors of x (which is the same as the number of hops from r_i to x). When x is clear from the context, we use D and δ instead of D_x and δ_x.

Since any node that is δ hops below some r_i is symmetric to any other node δ hops below any r_j, we only need to consider whether nodes in a single R_i (say R_1) would move any edges. None of the edges that make up the trees or the tails will be moved, or else the graph would become disconnected. So we only need to consider edges from leaf nodes or tail nodes to roots (call these *non-essential edges*).

With this symmetry in mind, we must verify that no node in R_1 will move any of its links. First, we show for that any node u in R_j, the number of hops from r_j to u times the number of descendants of u is smaller than the number of nodes in R_j that are *not* descendants of u. Intuitively, this is like isolating a single potential link end point: if a node were to move one of its links from r_j to u, the decrease to its cost would be smaller than the increase to its cost, even if the distance to each node only increased by one hop. Next, we show that a node would never move its links to one of its own ancestors or descendants, and a node would never place multiple links that have an ancestor/descendant relationship to each other. Once we've eliminated the possibility of related links, it is a relatively small step using our initial lemma to show that no node would ever place its links on non-root nodes. Finally, we show that the nodes would not move their links between roots, completing the proof that Forest of Willows graphs are stable.

The following lemma is used throughout this proof.

Lemma 2. *Let u be a given node in R_1. If $\delta_u > 1$, then $\frac{n}{k} - D_u - l \ge D_u\delta_u$. If $\delta_u = 1$, then $\frac{n}{k} - D_u \ge D_u$.*

Proof. Case 1: u is a tree node (so $1 \le \delta \le h$). Here, regardless of the values of h and l:

if $\delta > 1$:

$$D = \frac{n}{k^{\delta+1}} - \sum_{i=1}^{\delta} \frac{1}{k^i} < \frac{n}{k^{\delta+1}}$$

$$\frac{n}{k} - D(\delta+1) - l > n\left(\frac{1}{k} - \frac{(\delta+1)}{k^{\delta+1}}\right) - l \ge \frac{n}{k^2} - l \quad \text{if } k \ge 3, \text{ since } \delta \ge 1$$

$$> 0 \quad \text{since there are } k \text{ sections with at least } k \text{ tails per section}$$

and

$$\frac{n}{k} - D(\delta+1) - l > n\left(\frac{1}{k} - \frac{(\delta+1)}{k^{\delta+1}}\right) - l \geq \frac{n}{8} - l > 0 \quad \text{since } h \geq 3$$

if $\delta = 1$:

$$D = \frac{n}{k^2} - \frac{1}{k}$$

$$\frac{n}{k} - 2D = \frac{n}{k} - \frac{2n}{k^2} + \frac{2}{k} = \frac{n(1-\frac{2}{k})+2}{k} > 0$$

Case 2: u is a tail node.

$$D = h + l - \delta + 1$$

$$\frac{n}{k} - D(\delta+1) - l = \frac{n}{k} + \delta^2 - (h+l)(\delta+1) - l - 1$$

The second derivative with respect to δ is positive, so we only need to check this at the point where $\frac{d}{d\delta} = 0$ (a minima).

$$\delta = \frac{h}{2} + \frac{l}{2} \qquad D = \frac{h}{2} + \frac{l}{2} + 1$$

$$\frac{n}{k} - D(\delta+1) - l = \frac{n}{k} - \frac{(h+l)^2}{4} - h - 2l - 1$$

$$> 0 \quad \text{by our restrictions on } h \text{ and } l \qquad \square$$

Lemma 3. *If node $x \in R_1$ benefits by moving any of its non-essential edges to one of its descendants, and if u_1 is the closest such descendant, then x will also benefit by moving this edge to another node (distinct from u_1) that is δ_{u_1} hops from a root.*

Proof. Suppose x placed at least one of its non-essential edges at node u_1, a descendant of x. Suppose the $k-2$ other non-essential edges were placed at nodes $u_2, u_3, \dots, u_{k-1}$, and if any other u_j is also a descendant of x, then $\delta_{u_j} > \delta_{u_1}$.

The total decrease in hop count by moving the edges from our original placement is at most $\sum_{j=1}^{k-2}(D_{u_j}\delta_{u_j}) - D_{u_1}\delta_x$ (since the sum counts all of the descendants of u_1 as having a decrease of δ_{u_1}, but they actually only decreased by $\delta_{u_1} - \delta_x$).

The total increase in hop count is at least $\frac{(k-1)n}{k} - \sum_{j=2}^{k-1}(D_{u_j}) - D_x$ (since each of these non-essential edges used to point to a root, and the distance to all the descendants of these roots that are not also descendants of x or one of the u_j will now increase by at least one hop).

By moving to another node δ_{u_1} hops below a root (that is not an ancestor or descendant of x or of any of the other u_j), the total decrease in hop count will increase by at least $D_{u_1}\delta_x$. Meanwhile, the increase in hop count can only get lower. Therefore, if x would make the previous move, x would also make the new move. $\square$

Lemma 4. *If node $x \in R_1$ benefits by moving any of its non-essential edges to one of its ancestors, u_1 ($u_1 \neq r_1$), then x will also benefit by moving this edge to another node δ_{u_1} hops from a root.*

Proof. Suppose x placed at least one of its non-essential edges at node u_1, an ancestor of x. Suppose the $t-1$ other non-essential edges were placed at nodes $u_2, u_3, \dots, u_t$ (t may be $k-1$ or k, depending on the location of x). By Lemma 3, we can assume none of $u_2, \dots, u_t$ is a descendant of x.

The total decrease in hop count by moving the edges from our original placement is at most $\sum_{j=1}^{t}(D_{u_j}\delta_{u_j}) - D_x\delta_{u_1}$ (since the sum counts all of the descendants of u_1 as having a decrease of δ_{u_1}, but actually the descendants of x did not decrease at all).

The total increase in hop count is at least $\frac{tn}{k} - \sum_{j=1}^{t}(D_{u_j})$ (since each of these non-essential edges used to point to a root, and the distance to all descendants of these roots that are not also descendants of one of the u_j will now increase by at least one hop).

By moving to another node δ_{u_1} hops below a root (that is not an ancestor or descendant of x or of any of the other u_j), the total decrease in hop count will increase by at least $D_x\delta_{u_1}$. Meanwhile, the increase in hop count can only get lower. Therefore, if x would make the previous move, x would also make the new move. $\square$

Lemma 5. *If x will benefit by moving any two of its non-essential edges to nodes, $\{u_1, u_2\} \in R_1$, such that u_1 is an ancestor of u_2, then it will also benefit by moving to a node δ_{u_1} hops below r_1 and a node δ_{u_2} hops below r_1 (neither of which is an ancestor or descendant of x or of any other u_j).*

Notice there will always exist two such nodes because there are k branches of each tree and at most k non-essential edges, and u_2 must be at least 2 hops below a root, where there are k^2 branches (so we can always avoid an ancestor or descendant of x as well).

Proof. Suppose x placed two of its non-essential edges at nodes $\{u_1, u_2\}$ such that u_1 is an ancestor of u_2. Suppose the other non-essential edges were placed at nodes $u_3, u_4, \ldots, u_t$ (none of which is an ancestor or descendant of x). Also assume there is no u_j on the shortest path from u_1 to u_2.

The total decrease in hop count by moving the edges from our original placement is at most $\sum_{j=1}^{t}(D_{u_j}\delta_{u_j}) - D_{u_2}\delta_{u_1}$ (since the sum counts all of the descendants of u_2 as having a decrease of δ_{u_2}, but actually the decrease was only $\delta_{u_2} - \delta_{u_1}$).

The total increase in hop count is at least $\frac{tn}{k} - \sum_{j=1}^{t}(D_{u_j}) + D_{u_2} - D_x$ (since each of these non-essential edges used to point to a root, and the distance to all nodes that are descendants of these roots but not of x or one of the u_j will now increase by at least one hop).

By changing the move as suggested in this lemma, the total decrease in hop count will increase by at least $D_{u_2}\delta_{u_1}$. Meanwhile, the increase in hop count can only get lower. Therefore, if x would make the previous move, x would also make the new move. □

Lemma 6. *Forest of Willows graphs are stable.*

Proof. Consider any possible selections of non-essential edges for a node $x \in R_1$. Suppose t of these, $\{u_1, u_2, \ldots, u_t\}$, are moved away from the roots they point to in the initial configuration (to nodes at least one hop below a root). Also assume that no u_i is an ancestor or descendant of x or of any other u_j (we can make this assumption because of Lemmas 3, 4, and 5). Then, some nodes in each of t trees will get at least one hop further away from x. D_{u_i} nodes will get δ_{u_i} hops closer (for all u_i). $D_x (\le l)$ nodes will stay the same distance. The change in total hop count is at least the total increase minus the total decrease.

$$\begin{aligned}
\text{change in total hop count} &\ge \frac{nt}{k} - \sum_{i=1}^{t} D_{u_i} - l - \sum_{i=1}^{t} D_{u_i}\delta_{u_i} \\
&= \frac{nt}{k} - \sum_{i=1}^{t} D_{u_i}(\delta_{u_i} + 1) - l \\
&= \sum_{i=1}^{t}\left(\frac{n}{k} - D_{u_i}(\delta_{u_i} + 1)\right) - l \\
&\ge 0 \quad \text{if } \exists i \text{ such that } \delta_{u_i} > 1, \text{ by Lemma 2}
\end{aligned}$$

When $\delta_{u_i} = 1$ for all i, we must consider two cases.

Case 1: $x \in R_1$ does not have an edge to r_1 in the initial configuration (or does not move this edge). In this case, the total increase is at least $\frac{nt}{k} - \sum_{i=1}^{t} D_{u_i}$. (The $-l$ is not there, because x is not located under a root that increases.) This gives a change in total hop count $\ge \frac{nt}{k} - \sum_{i=1}^{t} D_{u_i}(\delta_{u_i} + 1) > 0$ (by the $\delta = 1$ condition in Lemma 2).

Case 2: $x \in R_1$ has an edge to r_1 in the initial configuration and moves this edge. All of the nodes u_i are 1 hop below roots, and none is an ancestor of x. There is a single node, u_1, that is 1 hop from r_1 that is an ancestor of x: the distance to each of the descendants of u_1 that are not also descendants of x (at least $D_{u_1} + 1 - l$ nodes) will increase by at least 2 hops (x cannot be the second to last node in a tail because it had an edge to r_1. If x is the last node of a tail, then the new distance to r_1 is at least $h - 1$. If x is at least 2 hops from the end of a tail, then there are at least 2 hops to the closest node pointing to r_1).

Therefore, the total increase in trees other than R_1 is at least $\frac{n(t-1)}{k} - \sum_{i=1}^{t-1} D_{u_i}$, and the increase in R_1 is at least $2(D_{u_1} + 1 - l)$. This gives the following total change in hop count.

$$\begin{aligned}
\text{change in total hop count} &\ge \frac{n(t-1)}{k} - \sum_{i=1}^{t-1} D_{u_i}(\delta_{u_i} + 1) + (2 - \delta_{u_1})D_{u_1} + 2 - 2l \\
&= \sum_{i=1}^{t-1}\left(\frac{n}{k} - 2D_{u_i}\right) + D_{u_1} + 2 - 2l \\
&\ge 0 \quad \text{by Lemma 2 and the fact that } D_{u_1} \text{ includes at} \\
&\qquad \text{least 3 tails (when } \delta_{u_1} = 1) \text{ as long as } k > 1 \text{ and } h \ge 3.
\end{aligned}$$

Therefore, x does not have incentive to move any of its non-essential edges to nodes other than roots.

Finally, we must verify that x has no incentive to move an edge from one root to another.

Case 1: $x \in R_1$ has edges to all roots except r_1 in the initial configuration.

In this case, consider what would happen if x moved an edge from some root r_j to r_1. The distance to descendants of r_1 but not of x would decrease by at most 1 hop, since the node beneath x in the tail already has an edge to r_1. This is a decrease of at most $\frac{n}{k} - 2$ (x has only $k-1$ non-essential edges, so at least x and one node below it keep the same distance). Meanwhile, the distance to all the descendants of r_j will increase by at least one hop. This gives an increase of at least $\frac{n}{k}$. Since the increase is always larger than the decrease, there is no incentive for this move.

Case 2: $x \in R_1$ has edges to all roots except some r_j ($j \neq 1$) in the initial configuration.

In this case, first consider what would happen if x moved an edge from r_1 to r_j. The distance to descendants of r_1 but not of x (at least $\frac{n}{k} - l$ nodes) would increase by at least 2, since it is 2 hops to another node with an edge to r_1. So there is an increase of at least $\frac{2n}{k} - 2l$. The distance to the $\frac{n}{k}$ descendants of r_j would decrease by 1 hop, since the node beneath x already points to r_j. So the decrease is at most $\frac{n}{k}$. The increase is always larger than the decrease, so there is no incentive for this move.

Next consider what would happen if x moved an edge from some $r_g \neq r_1$ to r_j. The distance to the $\frac{n}{k}$ descendants of r_g would increase by 1 hop, while the distance to the $\frac{n}{k}$ descendants of r_j would decrease by 1 hop. Therefore, this move does not make any difference to x, so x has no incentive to move. $\square$

Lemma 7. *The diameter of any uniform stable (n,k)-graph ($k \geq 2$) is $O(\sqrt{n \log_k n})$, and there is at least one node whose distance to any other node is $O(\sqrt{n})$.*

Proof. Let G be a stable graph for the (n,k)-uniform game, and let Δ denote the diameter of G, given by a path from a node r to a node v. Consider a shortest path tree from r; so the depth of this tree is Δ and v is a leaf of T. Let P denote the set of nodes on the path from r to v in T, not counting r; so $|P| = \Delta$. Let C be the sum of distances from r to the $n - \Delta$ nodes not in P. The sum of distances from r to the Δ nodes in P is exactly $\Delta(\Delta+1)/2$. So the cost of r is $C + \Delta(\Delta+1)/2$.

The cost of v is at most $C + n - \Delta/2 + \Delta(\Delta/2+1)/4 + \Delta(\Delta/2+1)/4$ since v can use one of its at least two edges to connect to r and the other to connect to a node halfway along the path from r to v. Simplifying, we obtain that the cost of v is at most $C + n + \Delta^2/4$. By Lemma 1, the cost of v is at least $C + \Delta(\Delta+1)/2 - n - n\log_k n$. We thus obtain the inequality:

$$C + n + \Delta^2/4 \geq C + \Delta(\Delta+1)/2 - n - n\log_k n,$$

yielding $\Delta = O(\sqrt{n \log_k n + 2n})$.

Using the fact that the cost of v is at least C (in place of the reference to Lemma 1) in the above proof gives the second part of the lemma. $\square$

Proof of Theorem 5. The first claim directly follows from Lemma 1. In any graph with max degree k, every node must have cost at least $\Omega(n \log_k n)$. Forest of Willows graphs with $l = 0$ have total cost per node $O(n \log_k n)$. Therefore, the price of stability is $\Theta(1)$.

If $l = 0$, a Forest of Willows graph has total cost per node $= O(n \log_k n)$. Therefore, the social utility has total cost (over all nodes) $O(n^2 \log_k n)$. If $l = \Omega(\sqrt{\frac{n}{k}})$, the total cost (over all nodes) is $\Omega(n^2 \sqrt{\frac{n}{k}})$. Therefore, the price of anarchy is $\Omega(\frac{\sqrt{(n/k)}}{\log_k n})$.

Finally, Lemma 7 implies that the total cost of any node in the worst Nash equilibrium cannot be higher than $O(\sqrt{n \log_k n})$, so the total cost is $O(n\sqrt{n \log_k n})$. We already know that the cost of the social optimum is $\Omega(n \log_k n)$. Therefore, the price of anarchy is $O(\sqrt{\frac{n}{\log_k n}})$. $\square$

4.2. Stability of regular graphs

In the context of overlay or peer-to-peer networks, a natural degree-k graph to consider is to map the nodes to $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ and have the k edges for all nodes be defined by k offsets a_i, $0 \leq i < k$: the ith edge from node x goes to $x + a_i \bmod n$. We refer to such graphs as *regular graphs*. For a suitable choice of the offsets, these graphs have diameter $O(n^{1/k})$. In this section, we study Abelian Cayley graphs, a more general class that includes regular graphs. We show that these graphs are not stable for $k \geq 2$. Cayley graphs have been widely studied by mathematicians and computer scientists, and arise in several applications including expanders and interconnection networks (e.g., see [33–35]).

A Cayley graph $G(H,S)$ is defined by a group H and a subset S of k elements of H. The elements of H form the nodes in G, and we have an edge (u,v) in G if and only if there exists an element a in S such that $u \cdot a = v$, where $\cdot$ is the group operation. A Cayley graph $G(H,S)$ is referred to as an Abelian Cayley graph if H is Abelian (that is, the operation $\cdot$ is commutative). The regular graph described in the preceding paragraph is exactly the Cayley graph with the group H being the Abelian additive group $\mathbb{Z}_n$ and $S = \{a_i \bmod n : 0 \leq i \leq k\}$.

We prove that no pure Nash equilibria exist in Abelian Cayley graphs, using a particular embedding of these graphs into k-dimensional grids. Let $G(H,S)$ be a given Abelian Cayley graph and let the k elements of S be a_i, $0 \leq i < k$. We assume without loss of generality that S does not contain the identity of H since these edges only form self-loops, which clearly cannot belong to any stable graph. Each edge of the graph G can be labeled by the index of the element of S that creates it; that is, if $v = u \cdot a_i$, then we call the edge (u, v) an *i-edge*. The edge labels naturally induce labels on paths as follows. If a path contains x_i i-edges, then we label the path by the vector $\vec{x} = (x_1, \ldots, x_i, \ldots, x_k)$. Note that the length of a path with label $\vec{x}$ is simply $\sum_{1\le i\le k} x_i$. Furthermore, the commutativity of the underlying group operator implies that for all nodes v and all path labels $\vec{x}$, every path that starts from v and has label $\vec{x}$ ends at the same node.

We say that node v has label $\vec{x}$ if there exists a shortest path from r to v that has label $\vec{x}$. For any node v, while two shortest paths from r to v share the same sum of label-coordinates, the actual path labels may be different; therefore, a node may have multiple labels. However, a particular label is assigned to at most one node.

We next prove that for $k \ge 2$ no Abelian Cayley graph is stable. For $k = 1$, it is trivial to see that the simple directed cycle is an Abelian Cayley graph and is stable.

Theorem 6. *For any $k \ge 2$, no Abelian Cayley graph with degree k and n nodes is stable, for $n \ge c2^k$, for a suitably large constant c.*

Proof. We now consider the impact of replacing the i-edge from root r to $r_i = r \cdot a_i$ by the edge from r to $r_i' = r \cdot a_i \cdot a_i$. The node r equals $(0, 0, \ldots, 0)$, while the node r_i equals $(0, 0, \ldots, 1, \ldots, 0)$ with a 1 in the ith coordinate. (We note that r and r_i are distinct since a_i is not identity.) For every node v that has a label $\vec{v}$ such that $v_i \ge 2$, the distance decreases by 1. Let $S_i = \{v : v \text{ has a label } \vec{v} \text{ with } v_i \ge 2\}$ be the set of such nodes. On the other hand, the only node whose distance from r increases is the node r_i; this is because any path in the original graph starting from r, having exactly one i-edge (r, r_i) and having length at least two, can be substituted by another path of the same length with an i-edge as its second edge.

We bound the increase in the distance from r to r_i in terms of the diameter Δ of the graph. Let $w = r \cdot a_j^{-1} \ne r_i$ denote a node that has an edge to r in G. Since the shortest path to any vertex other than r_i has not increased, the distance from r to r_i is at most $\Delta + 2$, given by a shortest path from r to w, followed by an i-edge and then by a j-edge. Thus, when the edge (r, r_i) is replaced by the edge (r, r_i'), the total utility for node r decreases by at least $|S_i| - (\Delta + 2)$. By the definition of S_i, this is precisely the set

$$G \setminus \bigcup_{0\le i<k} S_i = \{\vec{v} : 0 \le v_i \le 1 \text{ for all } i\}.$$

Therefore, there exists i, $0 \le i < k$, such that $|S_i| \ge (n - 2^k)/k$, and the graph G is not stable if $(n - 2^k)/k$ exceeds $\Delta + 1$. By Lemma 7, for G to be stable we must have $\Delta = O(\sqrt{n \log_k n})$. Using this upper bound on Δ, if $n \ge c2^k$ (for an appropriately large constant c), then $(n - 2^k)/k$ exceeds $\Delta + 1$, implying that G is not stable. □

Corollary 1. *For any $k > 4$, the 2^k-node hypercube is not stable for the $(2^k, k)$-uniform game.*

If the degree k is more than nearly half the size of the graph, then any degree-k n-node Abelian Cayley graph is stable.

Lemma 8. *For all $k > \frac{n-2}{2}$ any degree-k n-node Abelian Cayley graph is stable.*

4.3. Dynamics of best response walks

Given the existence of pure Nash equilibria for (n, k)-uniform games, it is natural to ask whether an equilibrium can be obtained by a sequence of local links changes. In particular, we consider a specific type of best response walk: in each step, a node tests for its stability and, if it is not stable, moves its links to the set of nodes that optimize its cost. We assume for convenience that only one node attempts to change its links in any step of the best response walk.

We first show that, starting from any initial state, the best response walk converges to a strongly connected graph in $O(n^2)$ steps, as long as every node is allowed to execute a best response step once every n steps. Furthermore, there exists an initial state such that a best response walk takes $\Omega(n^2)$ steps to converge to strong connectivity. We next study convergence to stability and show that there exists an initial state from which a particular best response walk does not converge to a stable graph. This means that the (n, k)-uniform game is *not an ordinal potential game*, a characteristic which justifies our use of a constructive proof for the existence of Nash equilibria.

Convergence to a strongly connected graph. For a given node u, we define the *reach* of u to be the number of nodes to which it has paths. Since the cost of disconnection is assumed to be $M > n$, when we execute a best-response step for a node u, the reach of u cannot decrease.

Lemma 9. *Suppose the graph G is not strongly connected, and a node u changes its edges according to a best response step. Then, after the step, the reach of any node other than u either remains the same or is at least the new reach of u.*

Proof. If a node v has a path to u, then the reach of v is at least the reach of u after the best response step. Otherwise, the reach of v does not change. □

The above lemma indicates that whenever a best response step causes a change, the vector that consists of all the reach values in increasing order becomes lexicographically larger. In order to show convergence, we need to argue progress. We will do so by showing that whenever the graph is not strongly connected, there exists a node that can improve its reach. In fact, we use a stronger property that allows us to bound the convergence time.

Consider best response walks that operate in a round-robin manner. In each round, each node (one at a time in an arbitrary order) executes a best response step. The order may vary from round to round. Let G_r refer to the graph before round r.

Lemma 10. *If G_r is not strongly connected at the start of round r, then the minimum reach increases by at least one during the round.*

Proof. Consider the strongly connected components of the given graph G_r. Consider the component graph CG in which we have a vertex for each strongly connected component and edge between two components whenever there is an edge from a vertex in one component to the other. This graph is a dag. Let m denote the minimum reach in G_r. By Lemma 9, nodes with reach greater than m will continue to have reach greater than m. So we only need to consider nodes with reach m. All of these nodes lie in sink components.

Consider any sink component C. We first argue that there exists a node in C that can improve its reach by executing a best response step. Consider a vertex u in C that has an edge from a vertex v in another component. Let w be a vertex in the sink component that has an edge to u. All of u, v, and w exist by definition of strongly connected components (and our assumption that the out-degree of every vertex is at least 1). If w replaces the edge (w,u) with (w,v), it can reach all vertices in the sink component as well as the component containing v. The latter set is clear; for the former set, note that all we have done is "replace" the direct edge (w,u) by the two-hop path $w \to u \to v$.

For any sink component C, let v be the first node in C in the round order that improves its reach through a best response step. Note that v exists by the argument of the preceding paragraph. Furthermore, in the step prior to v's best response, the reach of every node in C is m. After v's best response, the reach of v increases to at least $m+1$, as does that of *every* node in C, since they each have a path to v. By Lemma 9, after every subsequent step, the reach of any node in C is at least $m+1$. Therefore, it follows that at the end of the round, the reach of every node in a sink component of CG increases; hence, the minimum reach increases, completing the proof of the lemma. □

Theorem 7. *The best response walk converges to a strongly connected graph in n^2 steps.*

Proof. By Lemma 10, the minimum reach increases by at least one. Since the initial reach is 1 and the maximum reach is n, the number of steps for the best response walk to converge to a strongly connected graph is at most n^2. □

The above theorem is essentially tight. In the following scenario (with $k=1$), a best response walk may take $\Omega(n^2)$ steps to converge to a strongly connected graph. Consider a graph G of $n=r+p$ nodes that is a directed ring over $r \geq n/2$ nodes together with a directed path of $p=n-r$ nodes that ends at one of the nodes in the ring. Suppose a round begins at the tail T of the directed path, which can reach all nodes, proceeds along the path and then along the ring in the direction of the ring. The p nodes on the path cannot improve their reach. Furthermore, the first $r-p$ nodes on the ring (in round-robin order) also cannot improve their reach in a best response step. The $(r-p+1)$st node can improve its reach by connecting to T, yielding a new graph G' that is a directed ring over $r+1$ nodes and a directed path of $n-r$ nodes. If we repeat this process, the number of steps to converge is $\Omega(n^2)$.

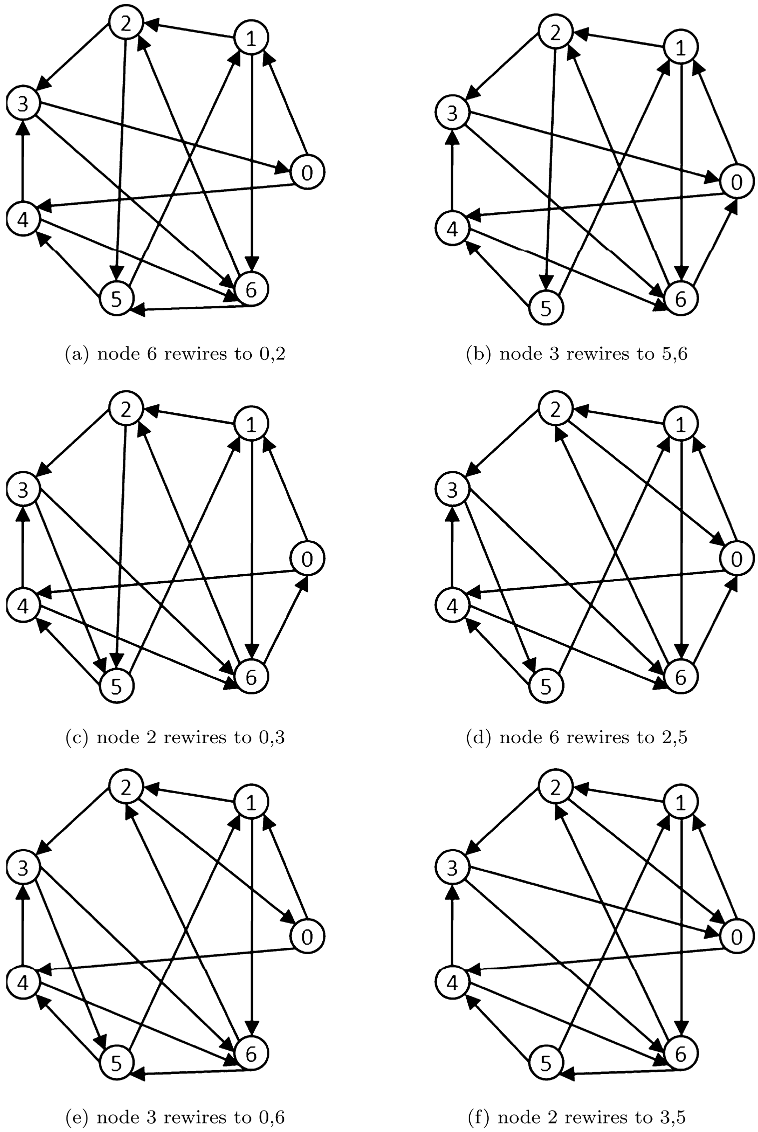
Cycles in best response walks. Unlike strong connectivity, convergence to a pure Nash equilibrium is not guaranteed. In the following simple example, a round-robin best-response walk contains loops. This simple example is a $(7,2)$-uniform game that starts from the top-left configuration of Fig. 7. The nodes take turns in round-robin order, starting with node 6 then nodes $0, 1, 2$, and so on. Tracing the example, one can verify that after 6 deviations (nodes $6, 3, 2, 6, 3, 2$ re-linking in this order, implying that missing nodes are stable), the graph returns to the initial configuration thus completing a loop.

The above example of a loop in the best response walk shows that the uniform-(n,k)-game is not an ordinal potential game. However, the loop does not rule out the possibility that either (a) a well-chosen best response walk converges from any initial state, or (b) certain best response walks do converge to stability if started from simple initial configurations such as the empty graph.

We have observed experimentally that best response walks in which a node with the maximum cost always makes the next best response step does not always converge to a stable graph. However, based on our experimental data, this best response walk starting from an empty graph does seem to converge to a stable graph. Our experiments also suggest that there may be some exponentially long best-response paths that start in some non-empty initial configuration and end at a stable graph.

5. Max distance utility function

In the BBC games we have studied thus far, the utility of a node u in $G(S)$ given by $-\sum_v w(u,v)d(u,v)$, where $d(u,v)$ is the shortest path from u to v in $G(S)$ according to the lengths given by ℓ. We have also considered a natural variant of



(a) node 6 rewires to 0,2
(b) node 3 rewires to 5,6
(c) node 2 rewires to 0,3
(d) node 6 rewires to 2,5
(e) node 3 rewires to 0,6
(f) node 2 rewires to 3,5

Fig. 7. An example in which a round-robin best-response walk loops. Starting from the top left configuration and following a round-robin best-response walk $6 \to 0 \to 1 \to \ldots \to 6 \to 1 \ldots$ we get back to the initial configuration after 6 deviations (nodes $6, 3, 2, 6, 3, 2$). Turns that are not illustrated imply stable nodes. Next to each node we indicate its cost under the current configuration.

the utility function: the utility of u is $-\max_v w(u,v)d(u,v)$. In order to make it clear that we are using a different cost function, we will call the max distance version *BBC-max games*.

As with the previous cost function, we show there exist instances of the general BBC-max game that have no Nash equilibrium. If we restrict ourselves to the uniform version (uniform (n,k)-BBC-max game), there is a stable graph for any n and $k < n$. It turns out that ratio between the total utility achieved in a Nash equilibrium in a uniform BBC-max game and the social optimum could be much worse than in BBC games. In particular, we establish a lower bound of $\Omega(\frac{n}{k \log_k n})$ on the price of anarchy in BBC-max games.

Theorem 8. *For any $k \geq 4$, $n \geq 2k^2 + k + 7$, there exists a nonuniform max-BBC game with no pure Nash equilibrium (n nodes, budget k per node) in which only the affinities are nonuniform. All costs and lengths are 1.*

Proof. Consider the same example as shown above in Fig. 3 for the proof of Theorem 2. All costs and lengths are 1, all budgets are 3. The arrows show the non-zero affinities. Those nodes with positive affinity for only 3 other nodes will point to those nodes and have a max distance of 1. This leaves only nodes A and B with variable edges. A must point to S_1 and S_2 or else pay the disconnection penalty, and B must point to T_1 and T_2. This leaves one edge each to spare.

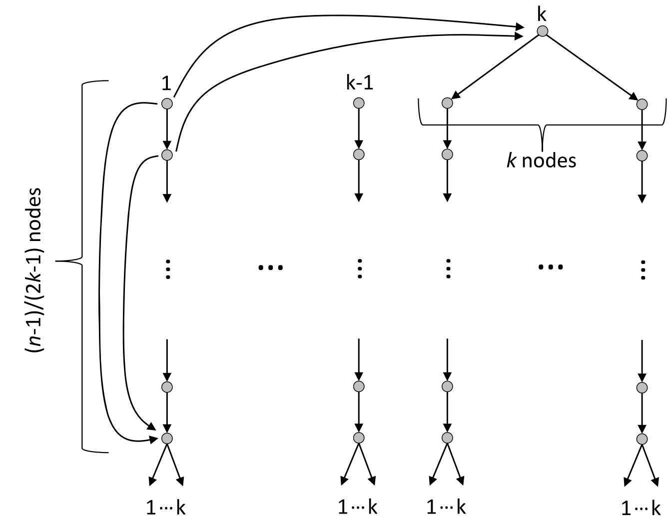


Fig. 8. A high cost Nash equilibrium for the max distance cost function: $2k-1$ paths, one node points to k of them. Each node at the end of a path points to the start of the first $k-1$ paths and the extra node. Each node in a path points to the node at the end of the path and to the extra node. The rest of the edge don't matter.

A always has the option to point to B^* and be able to access all of its preferred nodes. Therefore, A will never point to A^* or to D, since both of these choices will force it to pay the disconnection penalty, regardless of B's choice. If A points to B^*, then the options for B are: (a) point to A, and have max distance 5 to D, (b) point to A^* and have max distance 3 to B^*, (c) point to B^* and have max distance 4 to A, or (d) point to q (the node between B^* and A^*) and have max distance 4 to B^*. All other choices will force it to pay the disconnection penalty. Therefore, B will point to A^*. In this case, A has max distance 4 to D, but A can improve by pointing to B, getting max distance 2. However, if A points to B, then B can only access all of its nodes by pointing to B^*, so B will move its edge. Now, A can improve by moving back to B^*. Therefore there is no pure Nash equilibrium.

As in the proof to Theorem 2, we can extend this to higher values of n by adding additional nodes with affinity only for k of the nodes in R. $\square$

Theorem 9. *The Price of Anarchy for uniform (n,k) BBC-max games is $\Omega(\frac{n}{k\log_k n})$.*

Proof. Consider the following graph for $k>2$. There are $2k-1$ tails, $\{t_1,t_2,\dots,t_{2k-1}\}$, each of length $l=\frac{n-1}{2k-1}$. There is also one "root" node r with edges to the top node in $t_1,t_2,\dots,t_k$. For ease of notation, we will define segments $S_1=\{r,t_1,t_2,\dots,t_k\}$, $S_2=\{t_{k+1}\}$, $S_2=\{t_{k+2}\},\dots,S_k=\{t_{2k-1}\}$, with the head of each segment $S_i(i>1)=$ the first node of the tail, and the head of $S_1=r$. The last node of each tail points to the head of each segment. The rest of the nodes in each tail point to r and to the last node of a tail. The location of the rest of the edges don't matter. See Fig. 8. We will show that this graph is a Nash equilibrium.

First consider whether a node at the end of a tail would benefit by moving any of its edges. Its current max distance $=2+l$ (to a node at the end of t_1, t_2, or t_3). If it does not have one edge to each segment, then the max distance is at least $2+l$ (since it takes at least one hop to get to a node that will point to the segment, and all other nodes that point to the segment point to the head). If the one edge pointing to some segment does not point to the head of the segment, the max distance is 1 (to get to the segment) + the distance to the end of the tail + 1 (to get to the head) + the distance back to where it started $=2+l$.

Next, consider whether a node in the middle of a tail would benefit by moving any of its edges. Its current max distance $=2+l$ (the same distance to the end of any tail other than the tail it lives in). In order to get closer to the end of *every* other tail (since all are currently the same max distance), it would need to point closer to the middle of each tail. For segments $S_2,\dots,S_k$ (other than its own segment), this would be possible by pointing an edge to the head of each segment (or anywhere within the segment). In order to shorten all of these distances, at least $k-2$ edges must be used. However, the only way to reduce the distance to nodes in S_1 would be to point an edge to each of the k tails within the segment (or to $k-1$ edges if this node lives in S_1). There are not enough edges to improve distances to S_1 and to all other tails. Therefore, this node cannot improve its utility.

This example can be extended to the case where $k=2$ with a small adjustment. In this case, there are 3 paths plus one node that points to the head of two of the paths. The nodes at the end of each path point to the root of the single path and the extra node. The second to last nodes in the other paths point to the extra node. The rest of the nodes in the other paths point to the end of a tail.

In the Forest of Willows graphs described in Section 4.1, when $l = 0$ the sum of the max distances $= O(n \log_k n)$. Therefore, the social optimum cost is at most $O(n \log_k n)$. We have just shown that there is a graph with the sum of the max distances $= \Omega(\frac{n^2}{k})$. Therefore, the Price of Anarchy is $\Omega(\frac{n}{k \log_k n})$. □

Theorem 10. *The Price of Stability for uniform (n, k) BBC-max games is $\Theta(1)$.*

Proof. We first argue that the Forest of Willows graphs with $l = 0$ (described in Section 4.1) are also stable under the max cost function. When $l = 0$, then all the leaves of the k-ary trees directly connect to the k roots. No node v that is either a root or an internal node of a k-ary tree has an incentive to change its k links; since the k nodes at the heads of these links have no other incoming links, removal of any of these links results in an infinite cost to v. Given the choices of the k links for all the non-leaf nodes of the k-ary trees, the best response for each leaf node is to connect to the k roots. Thus, the Forest of Willows graphs with $l = 0$ are stable under the max cost function. The total utility in an n-node Forest of Willows graph is $\Theta(n \log_k n)$ Obviously, no node can have max distance less than $\log_k n$ in any graph of degree k. Therefore, the social optimum is at most $O(n \log_k n)$, so the Price of Stability is $\Theta(1)$. □

6. Concluding remarks

In this paper, we introduced a new variant of the network creation game in which there is a budget on the number of links each node can create. Our game is applicable to social networks in which users have limited time and resources to spend creating connections, and to overlay or peer-to-peer networks where each node has only enough bandwidth to connect to a limited number of other nodes. We showed that when different users or nodes have different preferences, or when the distances or costs between users vary, there may be no stable state. However, when all users are identical, there is an equilibrium for any number of users and any number of links per user. In the uniform version, we have given essentially tight bounds on the price of anarchy and the price of stability. We have also shown that no regular graph – a graph in which all nodes imitate the same configuration of links – can ever be stable. This has implications for overlay networks; although it would be natural to consider a symmetric configuration, selfish nodes would not be satisfied.

Our understanding of best response dynamics is currently limited. While we have shown that certain best response walks cycle, we have not been able to analyze best response (or better response) dynamics that start from some specified configuration or are otherwise coordinated. Another direction for future work is to consider variants of our game, such as a fractional version of BBC studied in [36], in which nodes may fractionally divide their budget of links across multiple other nodes. Here, a node's cost is the sum, over all other nodes, of the cost of a unit minimum cost flow to that other node. Unlike (integral) BBC games, every fractional BBC game instance possesses a pure Nash equilibrium [36].

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