Horn Clauses in Hybrid-Dynamic First-Order Logic

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Reconfiguration paradigm

- The present work is a part of a larger project: logical foundations of reconfiguration paradigm
- In many cases, the applications with reconfigurable features involve safety-critical areas. For example, the new generation of software-driven medical devices such as imaging machines, pill cameras, artificial pacemakers, the insulin infusion pump, etc.
- The safety requirements can be fulfilled only with formal methods
- The safety requirements can be fulfilled only by applying formal methods.
- *"One of the main issues is that there is no real formal method of implementing the reconfiguration of an application*" [Szepesi and Ciocarlie, Theory Appl. Math. Comput. Sci. 2011].

Hybrid-dynamic logics

Safe **reconfigurable systems**	**Medical devices**	**Driver applications**	**Intelligent transportation systems**	**Cloud systems**	
	They interact with users and the environment in which they coexist				

⇑

All developments above are powered by **formal methods**	**Formal methods** are mathematically-based techniques for the specification and verification of software systems, and the only way to ensure safety of applications

⇑

All developments above are powered by **math/logic**	The design of a formal method relies on	1. **model-theoretic studies** 2. **proof-theoretic support**

Reconfigurable systems can be regarded as transition systems in the following way:

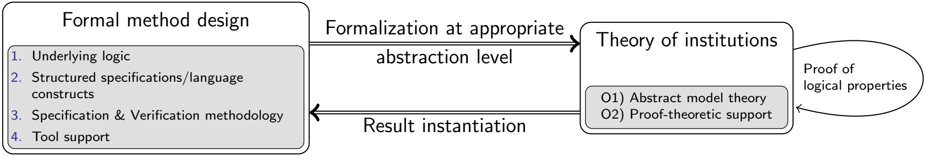
- the configurations are states, and
- switching from one configuration to another is a transition.

Hybrid dynamic logics are modal logics that can describe transitions systems and express the dynamics of (re)configurations:

- the configurations of the software in use today may be modeled with first-order logic, higher-order logic, rewriting logic, etc; these are considered **base logics**;
- the construction of a hybrid logic on top of a base logic is called **hybridization** [Diaconescu and Madeira, Math. Struct. Comput. Sci. 2016].

Formal method design

- **Institution** category-based formalization of the intuitive definition of logical system [Goguen and Burstall, Journal of the ACM 1992]
- **Hybrid institution** formalizes the notion of hybrid logic by supplementing the definition of **institution** with an additional structure to extract (a) nominals and modalities from signatures, and (b) frames from models.



- the design of a formal method consists of several steps depicted in the left node of the figure above
- each step relies on some model-theoretic or proof-theoretic properties which should be defined at an appropriate abstraction level in the framework of hybrid institutions
- the proof of model-theoretic and proof-theoretic properties should be performed within the framework of hybrid institutions as depicted in the right side of the figure above
- once proved the abstract results are instantiated to concrete logical systems

Preparation status

- Definition of hybrid institution and foundation of logic programming in hybrid institutions [Gaina, Theor. Comput. Sci. 2017]
- Abstract proof calculus for hybrid institutions whose sentences are Horn clauses [Gaina, Formal Asp. Comput. 2017]
- Proof calculus for the Horn clauses of hybrid-dynamic first-order logic [Gaina and Tutu, TABLEAUX 2019]
- Forcing in hybrid institutions [Gaina, Journal of the ACM accepted]

The following figure shows the advancements for this project and how this work will be completed in the present project.

	Hybrid institutions	Hybrid-dynamic institutions
Proof calculi (for all sentences)	[Gaina, Journal of the ACM accepted]	future work
Birkhoff proof calculi (for Horn clauses only)	[Gaina, Formal Asp. Comput. 2017]	[Gaina and Tutu, TABLEAUX 2019] + future work
Structured specifications (initiality + imports)	[Gaina, Theor. Comput. Sci. 2017]	[Gaina and Tutu, TABLEAUX 2019] + future work

Semantics first: Kripke models

- mathematical structures used to model the behaviour of a system
- underlie the semantics of modal, hybrid, dynamic and temporal logics

- typically, a Kripke structure consists in a (hyper)graph where:
  - each node (possible world) represents a state of the system
  - each edge (part of an accessibility relation) represents a transition

- the original definition has been extended in various ways:
  - complex algebraic structures as labels of the states
  - structured actions as labels of transitions
  - model constraints

Hybrid-Dynamic First-Order Logic with user-defined sharing

A logic for specifying and reasoning about Kripke structures

... obtained by enriching first-order logic
with features that are characteristic to
hybrid logics (nominals and local-satisfaction operators) and
dynamic logics (structured actions over modalities)

... and with a number of distinctive attributes:

- a first-order structure on possible worlds
- supports sharing between possible worlds / rigidity constraints
- hybrid terms (annotated with nominals)

Signatures

Definition. An HDFOL-signature is a tuple $\Delta = (\Sigma^n, \Sigma^r \subseteq \Sigma)$, where:

- $\Sigma^n = (F^n, P^n)$ is a single-sorted first-order signature of nominals, where F^n_i is a set of nominal operations of arity $i \in \mathbb{N}$
- $\Sigma^r = (S^r, F^r, P^r)$ is a many-sorted signature of rigid symbols, where $F^r_{ar \to s}$ is a set of operations of arity $ar \in S^{r*}$ and sort $s \in S^r$
- $\Sigma = (S, F, P)$ is a many-sorted first-order signature of both rigid and flexible symbols

An HDFOL-signature morphism $\varphi\colon \Delta \to \Delta'$ consists of a pair of first-order signature morphisms $\varphi^n\colon \Sigma^n \to \Sigma'^n$ and $\varphi\colon \Sigma \to \Sigma'$ such that $\varphi(\Sigma^r) \subseteq \Sigma'^r$.

Semantics

Definition. A Kripke model of $\Delta = (\Sigma^n, \Sigma^r \subseteq \Sigma)$ is a pair $\langle W, M \rangle$:

- W is a Σ^n-model, whose carrier set we denote by $|W|$
- $M = (M_w)_{w \in |W|}$ is a family of Σ-models, indexed by worlds, such that $M_{w_1,\varsigma} = M_{w_2,\varsigma}$ for all $w_1, w_2 \in |W|$ and symbols ς in Σ^r

(rigid symbols have the same interpretation across possible worlds)

A homomorphism $h \colon \langle V, N \rangle \to \langle W, M \rangle$ is also a pair, consisting of first-order homomorphisms $h \colon V \to W$ and $h_v \colon N_v \to M_{h(v)}$, for every world $v \in |V|$, such that $h_{v_1,s} = h_{v_2,s}$ for all $v_1, v_2 \in |V|$ and $s \in S^r$.

Syntax

The set actions over Δ is defined in an inductive fashion, according to the following grammar:

$$\mathfrak{a} ::= \lambda \in P_2^n \mid \mathfrak{a} \,\mathrm{;}\, \mathfrak{a} \mid \mathfrak{a} \cup \mathfrak{a} \mid \mathfrak{a}^*$$

The atomic sentences defined over a signature Δ are given by:

$$\rho ::= \overbrace{k_1 = k_2 \mid \lambda(k')}^{\text{nominal equations and relations}} \mid \underbrace{t_1 =_{k,s} t_2 \mid \varpi(t) \mid \pi(k;t)}_{\text{hybrid equations and relations}}$$

Full sentences over Δ are built from atomic sentences according to the following grammar:

$$\gamma ::= \rho \mid \mathfrak{a}(k_1, k_2) \mid @_k\, \gamma \mid \neg\gamma \mid \bigwedge \Gamma \mid {\downarrow} z \cdot \gamma' \mid \forall X \cdot \gamma''$$

where z is a nominal variable, X is a set of variables, and γ', γ'' are sentences over the extended signatures $\Delta[z]$ and $\Delta[X]$, respectively.

Local-satisfaction relation

Definition. Given a Kripke model $\langle W, M\rangle$ of a signature Δ and a possible world $w \in |W|$, we have, for atomic sentences:

- $\langle W, M\rangle \models^w k_1 = k_2$ iff $W_{k_1} = W_{k_2}$
- $\langle W, M\rangle \models^w \lambda(k)$ iff $W_k \in W_\lambda$
- $\langle W, M\rangle \models^w t_1 =_k t_2$ iff $\langle W, M\rangle_{t_1} = \langle W, M\rangle_{t_2}$
- $\langle W, M\rangle \models^w \varpi(t)$ iff $\langle W, M\rangle_t \in M_{w,\varpi}$
- $\langle W, M\rangle \models^w \pi(k; t)$ iff $\langle W, M\rangle_t \in M_{w',\pi}$, where $w' = W_k$

Local-satisfaction relation

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- $\langle W, M\rangle \models^w \mathfrak{a}(k_1, k_2)$ iff $(W_{k_1}, W_{k_2}) \in W_{\mathfrak{a}}$
- $\langle W, M\rangle \models^w @_k\,\gamma$ iff $\langle W, M\rangle \models^{w'} \gamma$, where $w' = W_k$
- $\langle W, M\rangle \models^w \neg\gamma$ iff $\langle W, M\rangle \not\models^w \gamma$
- $\langle W, M\rangle \models^w \bigwedge\Gamma$ iff $\langle W, M\rangle \models^w \gamma$ for all $\gamma \in \Gamma$
- $\langle W, M\rangle \models^w {\downarrow} z \cdot \gamma$ iff $\langle W, M\rangle^{z\leftarrow w} \models^w \gamma$
- $\langle W, M\rangle \models^w \forall X \cdot \gamma$ iff $(W', M') \models^w \gamma$ for all $\Delta[X]$-expansions (W', M') of $\langle W, M\rangle$

Expressivity and relationship to other modal logics

Support for conventional modal operators

- $[\mathfrak{a}]\gamma \triangleq {\downarrow} z \cdot \forall z' \cdot \mathfrak{a}(z, z') \Rightarrow @_{z'}\, \gamma$
- $\langle\mathfrak{a}\rangle\gamma \triangleq {\downarrow} z \cdot \exists z' \cdot \mathfrak{a}(z, z') \wedge @_{z'}\, \gamma$

Support for (linear) temporal operators

- $\bigcirc\, \gamma \triangleq {\downarrow} z \cdot @_{\mathsf{next}(z)}\, \gamma$
- $\rho \,\mathsf{Until}\, \gamma \triangleq \exists z \cdot \Diamond(z \wedge \gamma) \wedge \Box(\Diamond z \Rightarrow \rho)$

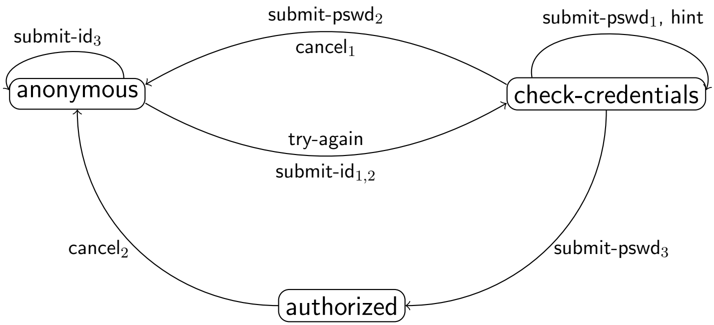
Hybrid (annotated) terms vs ordinary terms

- $c_1(k) = \sigma(k; c_2(k))$ is equivalent to $@_k\, c_1 = \sigma(c_2)$
- $c_3(k) = \sigma(k; c_4(k_0))$ is equivalent to $\exists x \cdot @_{k_0}\, x = c_4 \wedge @_k\, c_3 = \sigma(x)$

Implicit vs explicit dependence on possible worlds

- the nominal sentence k is equivalent to ${\downarrow} z \cdot z = k$
- the store sentence ${\downarrow} z \cdot \gamma$ is equivalent to $\exists z \cdot (z \wedge \gamma)$

Event-based transition system



Events:

- submit-id, submit-pswd, cancel
- try-again, hint

Attributes:

- $\underbrace{\text{status, uid, attpts,}}_{\text{observations}}$
- $\underbrace{\text{type-id, type-pswd}}_{\text{random values}}$

Setting the stage for Birkhoff completeness

Goal: syntactic characterization of the satisfiability relations $\Gamma \models_\Delta \gamma$

$$\langle W, M\rangle \models \gamma \text{ for all } \Delta\text{-models } \langle W, M\rangle \text{ such that } \langle W, M\rangle \models \Gamma$$

- both Γ and γ belong to the Horn-clause fragment of HDFOL

Definition. By Horn clause, we mean a sentence obtained from atomic sentences by repeated applications of the following sentence-building operators, in any order:

- *retrieve*
- *implication* (hypothesis: only atoms or action relations)
- *store*
- *universal quantification*
- *necessity*, *next*

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A layered approach

- develop progressively a series of syntactic entailment relations

$$\Gamma \vdash \gamma$$

- where each layer builds on the previous one to allow for more general antecedents or consequents
- three major steps / entailment relations

  Atomic completeness: both Γ and γ are atomic
  Quasi-completeness: Γ is arbitrary, but γ is atomic
  Horn-clause completeness: both Γ and γ are arbitrary

- all are sound and complete; only the first two are also compact
- atomic completeness is much more difficult to establish than it may seem at first sight...

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Atomic completeness

Proof rules

Lemma. For every set Γ of nominal equations over a signature Δ, there exists a reachable initial model (W^Γ, M^Γ) such that

$$\Gamma \models \rho \quad \text{iff} \quad (W^\Gamma, M^\Gamma) \models \rho \quad \text{iff} \quad \Gamma \vdash \rho$$

for all nominal or hybrid equations ρ over Δ.

Theorem (Atomic completeness). Every set Γ of atomic sentences over a signature Δ has a reachable initial model (W^Γ, M^Γ) such that

$$\Gamma \models \rho \quad \text{iff} \quad (W^\Gamma, M^\Gamma) \models \rho \quad \text{iff} \quad \Gamma \vdash \rho$$

for all atomic sentences ρ over Δ.

Quasi-completeness

Proof rules

Theorem (Quasi-completeness). Let:

- Γ be a set of clauses over a signature Δ,
- $\Gamma_0 = \{\rho \in \mathrm{Sen}^{\mathsf{HDCLS}}(\Delta) \mid \Gamma \vdash \rho \;\&\; \rho$ is atomic$\}$,
- $(W^{\Gamma_0}, M^{\Gamma_0})$ a reachable initial model of Γ_0 as before.

Then the following statements are equivalent:

1. $\Gamma \models \rho$
2. $(W^{\Gamma_0}, M^{\Gamma_0}) \models \rho$
3. $\Gamma \vdash \rho$

Horn-clause completeness

Proof rules

Theorem (Birkhoff completeness). The entailment relation generated by the rules presented thus far is sound and complete.

$$\Gamma \models \gamma \qquad \text{if and only if} \qquad \Gamma \vdash \gamma$$

Moreover, in the absence of $(\mathsf{Star}_\mathsf{I})$, it is also compact.

Proposition (Lack of compactness). HDCLS does not admit an entailment relation that is sound, complete, and also compact.

Conclusions

To sum up

- we have introduced a hybrid-dynamic first-order logic
- seen how it relates to modal/temporal/hybrid logics
- presented a sound and complete calculus for its Horn fragment
- touched upon the limits of compactness

Ongoing work

- decidability properties and support for executable specifications
- case studies & a prototype implementation

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Thank you!

Atomic completeness

Rules for nominal terms

$$(\mathsf{R^n}) \quad \frac{}{\Gamma \vdash k = k} \qquad\qquad (\mathsf{S^n}) \quad \frac{\Gamma \vdash k_1 = k_2}{\Gamma \vdash k_2 = k_1}$$

$$(\mathsf{T^n}) \quad \frac{\Gamma \vdash k_1 = k_2 \qquad \Gamma \vdash k_2 = k_3}{\Gamma \vdash k_1 = k_3}$$

$$(\mathsf{F^n}) \quad \frac{\Gamma \vdash k_1 = k_2}{\Gamma \vdash o(k_1) = o(k_2)} \qquad\qquad (\mathsf{P^n})$$

$$\frac{\Gamma \vdash \lambda(k_1) \qquad \Gamma \vdash k_1 = k_2}{\Gamma \vdash \lambda(k_2)}$$

Atomic completeness

Rules for sharing

$$(\mathsf{W}^{\mathsf{h}}) \quad \frac{\Gamma \vdash k = k'}{\Gamma \vdash t =_{k,s} \delta_{k'/k}(t)} \quad \text{where } s \in S^r$$

$$(\mathsf{W}^{\mathsf{r}}) \quad \frac{\Gamma \vdash t_1 =_{k_1,s} t_2}{\Gamma \vdash t_1 =_{k_2,s} t_2} \quad \text{where } s \in S^r$$

$$(\mathsf{W}^{\mathsf{f}}) \quad \frac{\Gamma \vdash k = k' \qquad \Gamma \vdash t_1 =_k t_2}{\Gamma \vdash \delta_{k/k'}(t_1) =_{k'} \delta_{k/k'}(t_2)}$$

Atomic completeness

Rules for hybrid terms

$$(\mathsf{F}^{\mathsf{r}})\quad \frac{\Gamma \vdash t_1 =_{k,ar} t_2}{\Gamma \vdash \sigma(t_1) =_{k,s} \sigma(t_2)} \qquad (\mathsf{F}^{\mathsf{f}})\quad \frac{\Gamma \vdash t_1 =_{k,ar} t_2}{\Gamma \vdash \sigma(k;t_1) =_{k,s} \sigma(k;t_2)}$$

$$(\mathsf{P}^{\mathsf{r}})\quad \frac{\Gamma \vdash t_1 =_k t_2 \quad \Gamma \vdash \pi(t_1)}{\Gamma \vdash \pi(t_2)} \qquad (\mathsf{P}^{\mathsf{f}})\quad \frac{\Gamma \vdash t_1 =_k t_2 \qquad \Gamma \vdash \pi(k;t_1)}{\Gamma \vdash \pi(k;t_2)}$$

$$(\mathsf{P}^{\mathsf{h}})\quad \frac{\Gamma \vdash k_1 = k_2 \quad \Gamma \vdash \pi(k_1;t_1)}{\Gamma \vdash \pi(k_2;\delta_{k_1/k_2}(t_1))} \qquad (\mathsf{Ret_0})\quad \frac{\Gamma \vdash @_k\, \rho}{\Gamma \vdash \rho}$$

Quasi-completeness

Rules for action relations

$$(\mathrm{Comp})\quad \frac{\Gamma \vdash \mathfrak{a}_1(k_1,k_2) \qquad \Gamma \vdash \mathfrak{a}_2(k_2,k_3)}{\Gamma \vdash (\mathfrak{a}_1 \mathbin{;} \mathfrak{a}_2)(k_1,k_3)}$$

$$(\mathrm{Union})\quad \frac{\Gamma \vdash \mathfrak{a}_i(k_1,k_2)}{\Gamma \vdash (\mathfrak{a}_1 \cup \mathfrak{a}_2)(k_1,k_2)}$$

$$(\mathrm{Refl})\quad \frac{\Gamma \vdash k_1 = k_2}{\Gamma \vdash \mathfrak{a}^*(k_1,k_2)} \qquad (\mathrm{Star})\quad \frac{\Gamma \vdash \mathfrak{a}(k_i,k_{i+1}) \text{ for } 0 \leq i < n}{\Gamma \vdash \mathfrak{a}^*(k_0,k_n)}$$

Quasi-completeness

Rules for Horn clauses

$$(\mathsf{Ret}_@)\quad \frac{\Gamma \vdash @_{k_1} @_{k_2}\, \gamma}{\Gamma \vdash @_{k_2}\, \gamma} \qquad\qquad (\mathsf{Ret}_\mathsf{I})\quad \frac{\Gamma \vdash \gamma}{\Gamma \vdash @_k\, \gamma}$$

$$(\mathsf{Imp}_\mathsf{E})\quad \frac{\Gamma \vdash @_k\, (\bigwedge H \Rightarrow \gamma)}{\Gamma \cup H \vdash @_k\, \gamma}$$

$$(\mathsf{Store}_\mathsf{E})\quad \frac{\Gamma \vdash @_k \downarrow z \cdot \gamma}{\Gamma \vdash @_k\, \theta_{z \leftarrow k}(\gamma)} \qquad\qquad (\mathsf{Subst}_\mathsf{q})\quad \frac{\Gamma \vdash @_k\, \forall X \cdot \gamma}{\Gamma \vdash @_k\, \theta(\gamma)}$$

Horn-clause completeness

Additional rules for Horn clauses

$$(\mathsf{Ret_E})\quad \frac{\Gamma \vdash_{\Delta[z]} @_z\, \gamma}{\Gamma \vdash_{\Delta} \gamma} \qquad\qquad (\mathsf{Imp_I})\quad \frac{\Gamma \cup H \vdash @_k\, \gamma}{\Gamma \vdash @_k\, (\bigwedge H \Rightarrow \gamma)}$$

$$(\mathsf{Store_I})\quad \frac{\Gamma \vdash @_k\, \theta_{z\leftarrow k}(\gamma)}{@_k \downarrow z \cdot \gamma} \qquad\qquad (\mathsf{Quant_I})\quad \frac{\Gamma \vdash_{\Delta[X]} @_k\, \gamma}{\Gamma \vdash_{\Delta} @_k\, \forall X \cdot \gamma}$$

Horn-clause completeness

Additional rules for action relations

$$(\mathsf{Comp_I})\quad \frac{E \cup \{\mathfrak{a}_1(k_1, z), \mathfrak{a}_2(z, k_2)\} \vdash_{\Delta[z]} e}{E \cup \{(\mathfrak{a}_1 \fatsemi \mathfrak{a}_2)(k_1, k_2)\} \vdash_{\Delta} e}$$

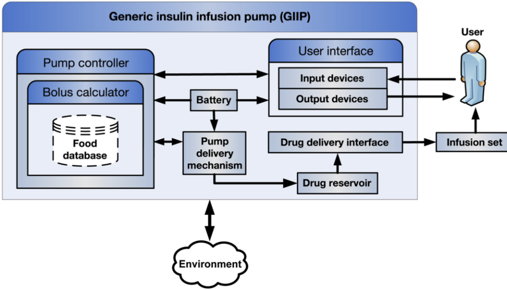
$$(\mathsf{Union_I})\quad \frac{E \cup \{\mathfrak{a}_i(k_1, k_2)\} \vdash e \text{ for } i \in \{1, 2\}}{E \cup \{(\mathfrak{a}_1 \cup \mathfrak{a}_2)(k_1, k_2)\} \vdash e}$$

$$(\mathsf{Star_I})\quad \frac{E \cup \{\mathfrak{a}^n(k_1, k_2)\} \vdash e \text{ for all } n \in \mathbb{N}}{E \cup \{\mathfrak{a}^*(k_1, k_2)\} \vdash e}$$

Generic insulin infusion pump

Pump controller - an abstract representation of generic insulin pump software

- Main functionality: command the pump delivery mechanism to propel insulin stored in the drug reservoir to the patient through the drug delivery interface and the infusion set
- Overall responsibility: ensure correct operation of the model



More about pump controller functionality:

- interacts with the patient through a user interface
- it alerts the patient when abnormal conditions arise
- recommends appropriate bolus dosages with the help of a bolus calculator and a food database
- manages and checks parameters and programs related to insulin administration;
- logs important data and events during pump use to facilitate clinical use analysis and problem diagnosis