Chapter **10**

Tests of Hypotheses

Introduction

Every statistical investigation aims at collecting information about some aggregate or collection of individuals or of their attributes, rather than the individuals themselves. In statistical language, such a collection is called *a population* or *universe*. For example, we have the population of products turned out by a machine, of lives of electric bulbs manufactured by a company etc. A population is finite or infinite, according as the number of elements is finite or infinite. In most situations, the population may be considered infinitely large. A finite subset of a population is called *a sample* and the process of selection of such samples is called *sampling*. The basic objective of the theory of sampling is to draw inference about the population using the information of the sample.

Parameters and Statistics

Generally, in statistical investigations, our ultimate interest will lie in one or more characteristics possessed by the members of the population. If there is only one characteristic of importance, it can be assumed to be a variable x. If x_i be the value of x for the ith member of the sample, then $(x_1, x_2, \ldots., x_n)$ are referred to as sample observations. Our primary interest will be to know the values of different statistical measures such as mean and variance of the population distribution of x. Statistical measures, calculated on the basis of population values of x are called *parameters*. Corresponding measures computed on the basis of sample observations are called *statistics*.

Sampling Distribution

If a number of samples, each of size n, (i.e. each containing n elements) are drawn from the same population and if for each sample the value of some statistic, say, mean is calculated, a set of values of the statistic will be obtained.

Note *The values of the statistic will normally vary from one sample to another, as the values of the population members included in different samples, though drawn from the same population, may be different. These differences in the values of a statistic are said to be due to sampling fluctuations.*

If the number of samples is large, the values of the statistic may be classified in the form of a frequency table. The probability distribution of the statistic that would be obtained if the number of samples, each of same size were infinitely large is called the *sampling distribution* of the statistic. If we adopt random sampling technique that is the most popular and frequently used method of sampling [the discussion of which is beyond the scope of this book], the nature of the sampling distribution of a statistic can be obtained theoretically, using the theory of probability, provided the nature of the population distribution is known.

Like any other distribution, a sampling distribution will have its mean, standard deviation and moments of higher order. The standard deviation of the sampling distribution of a statistic is of particular importance in tests of hypotheses and is called *the standard error* of the statistic.

Estimation and Testing of Hypotheses

In sampling theory, we are primarily concerned with two types of problems which are given below:

(i) Some characteristic or feature of the population in which we are interested may be completely unknown to us and we may like to make a guess about this characteristic entirely on the basis of a random sample drawn from the population. This type of problem is known as the problem of *estimation*.

(ii) Some information regarding the characteristic or feature of the population may be available to us and we may like to know whether the information is tenable (or can be accepted) in the light of the random sample drawn from the population and if it can be accepted, with what degree of confidence it can be accepted. This type of problem is known as the problem of *testing of hypotheses*.

Tests of Hypotheses and Tests of Significance

When we attempt to make decisions about the population on the basis of sample information, we have to make assumptions or guesses about the nature of the population involved or about the value of some parameter of the population. Such assumptions, which may or may not be true, are called *statistical hypotheses*. Very often, we set up a hypothesis which assumes that there is no significant difference between the sample statistic and the corresponding population parameter or between two sample statistics. Such a hypothesis of no difference is called a *null hypothesis* and is denoted by H_0. A hypothesis that is different from (or complementary to) the null hypothesis is called an *alternative hypothesis* and is denoted by H_1. A procedure for deciding whether to accept or to reject a null hypothesis (and hence to reject or to accept the alternative hypothesis respectively) is called *the test of hypothesis.*

If θ_0 is a parameter of the population and θ is the corresponding sample statistic, usually there will be some difference between θ_0 and θ since θ is based on sample observations and is different for different samples. Such a difference which is caused due to sampling fluctuations is called *insignificant difference*. The difference that arises due to the reason that either the sampling procedure is not purely random or that the sample has not been drawn from the given population is known as *significant difference*. This procedure of testing whether the difference between θ_0 and θ is significant or not is called *the test of significance.*

Critical Region and Level of Significance

If we are prepared to reject a null hypothesis when it is true or if we are prepared to accept that the difference between a sample statistic and the corresponding parameter is significant, when the sample statistic lies in a certain region or interval, then that region is called the *critical region* or *region of rejection*. The region complementary to the critical region is called *the region of acceptance.*

In the case of large samples, the sampling distributions of many statistics tend to become normal distributions. If 't' is a statistic in large samples, then t follows a normal distribution with mean $E(t)$, which is the corresponding population parameter, and S.D. equal to S.E. (t). Hence, $Z = \dfrac{t - E(t)}{\text{S}\cdot\text{E}\cdot(t)}$ is a standard normal variate, i.e., Z (called *the test statistic*) follows a normal distribution with mean zero and S.D unity.

It is known from the study of normal distribution, that the area under the standard normal curve between $t = -1.96$ and $t = +1.96$ is 0.95. Equivalently the area under the general normal curve of 't' between $[E(t) - 1.96 \text{ S.E. } (t)]$ and $[E(t) + 1.96 \text{ S.E. } (t)]$ is 0.95. In other words, 95 per cent of the values of t will lie between $[E(t) \mp 1.96 \text{ S.E. } (t)]$ or only 5 per cent of values of t will lie outside this interval.

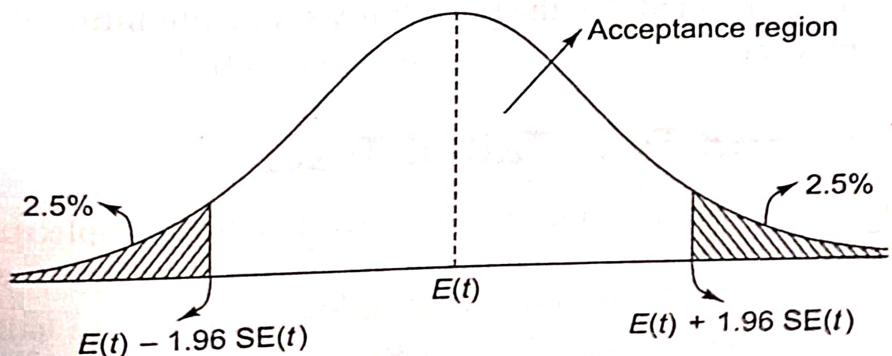

Fig. 10.1

If we are prepared to accept that the difference between t and $E(t)$ is significant when t lies in either of the regions $[-\infty, E(t) - 1.96\text{ S.E. }(t)]$ and $[E(t) + 1.96\text{ S.E. }(t), \infty]$ then these two regions constitute the critical region of 't'.

The probability 'α' that a random value of the statistic lies in the critical region is called *the level of significance* and is usually expressed as a percentage.

From the study of normal distributions, it is known that

$$P\{E(t) - 1.96\text{ S.E. }(t) < t < E(t) + 1.96\text{ S.E. }(t)\} = 0.95$$

i.e. $$P\left\{\left|\frac{t - E(t)}{\text{S}\cdot\text{E}\cdot(t)}\right| < 1.96\right\} = 0.95$$

i.e. $$P\{|Z| > 1.96\} = 0.05 \quad \text{or} \quad 5\%$$

Thus, when t lies in either of the two regions constituting the critical region given above, the level of significance is 5 per cent.

Note *The level of significance can also be defined as the maximum probability with which we are prepared to reject H_0 when it is true. In other words, the total area of the region of rejection expressed as a percentage is called the level of significance.*

(The specification of critical region and the choice of level of significance will depend upon the nature of the problem and is a matter of judgement for those who carry out the investigation. Usually, the levels of significance are taken as 5%, 2% or 1%.)

Errors in Hypotheses Testing

The level of significance is fixed by the investigator and as such it may be fixed at a higher level by his wrong judgement. Due to this, the region of rejection becomes larger and the probability of rejecting a null hypothesis, when it is true, becomes greater. The error committed in rejecting H_0, when it is really true, is called *Type I error*. This is similar to a good product being rejected by the consumer and hence Type I error is also known as *producer's risk*. The error committed in accepting H_0, when it is false, is called *Type II error*. As this error is similar to that of accepting a product of inferior quality, it is also known as *consumer's risk*.

The probabilities of committing Type I and II errors are denoted by α and β respectively. It is to be noted that the probability α of committing Type I error is the level of significance.

One-Tailed and Two-Tailed Tests

If θ_0 is a population parameter and θ is the corresponding sample statistic and if we set up the null hypothesis $H_0 : \theta = \theta_0$, then the alternative hypothesis which is complementary to H_0 can be any one of the following:

(i) $H_1 : \theta \neq \theta_0$, i.e. $\theta > \theta_0$ or $\theta < \theta_0$

(ii) $H_1 : \theta > \theta_0$

(iii) $H_1 : \theta < \theta_0$.

H_1 given in (i) is called a two tailed alternative hypothesis, whereas H_1 given in (ii) is called a right-tailed alternative hypothesis and H_1 given in (iii) is called a left-tailed alternative hypothesis.

When H_0 is tested while H_1 is a one-tailed alternative (right or left), the test of hypothesis is called *a one-tailed test*.

When H_0 is tested while H_1 is two-tailed alternative, the test of hypothesis is called *a two-tailed test*.

The application of one-tailed or two-tailed test depends upon the nature of the alternative hypothesis. The choice of the appropriate alternative hypothesis depends on the situation and the nature of the problem concerned.

Critical Values or Significant Values

The value of the test statistic z for which the critical region and acceptance region are separated is called *the critical value* or *the significant value* of z and denoted by z_α, when α is the level of significance. It is clear that the value of z_α depends not only on α but also on the nature of alternative hypothesis.

When $$z = \frac{t - E(t)}{S \cdot E \cdot (t)},$$ we have seen that

$$P(|z| < 1.96) = 95 \text{ per cent and } \quad P(|z| > 1.96) = 5 \text{ per cent}.$$

Thus, $z = \pm 1.96$ separate the critical region and the acceptance region at 5% level of significance for a two-tailed test. That is the critical values of z in this case are ± 1.96.

In general, the critical value z_α for the level of significance α is a given by the equation $P(|z| > z_\alpha) = \alpha$ for a two-tailed test, by the equation $P(z > z_\alpha) = \alpha$ for the right-tailed test and by the equation

$P(z < -z_\alpha) = \alpha$ for the left-tailed test. These equations are solved by using the normal tables.

Note *If z_α is the critical value of z corresponding to the level of significance α in the right-tailed test, then $P(z > z_\alpha) = \alpha$.*

By symmetry of the standard normal distribution followed by z, $P(z < -z_\alpha) = \alpha$.

$$\therefore \quad P(|z| > z_\alpha) = P\{(z > z_\alpha) + (z < -z_\alpha)\}$$
$$= P\{z > z_\alpha) + P(z < -z_\alpha)$$
$$= 2\alpha$$

That is, z_α is the critical value of z corresponding to the LOS (level of significance) 2α.

Thus, the critical value of z for a single tailed test (right or left) at LOS 'α' is the same as that for a two-tailed test of LOS '2α'].

The critical values for some standard LOS's are given in the following table both for two-tailed and one-tailed tests.

Table 10.1

Nature of test \ LOS	1% (.01)	2% (.02)	5% (.05)	10% (.1)
Two-tailed	$\|z_\alpha\| = 2.58$	$\|z_\alpha\| = 2.33$	$\|z_\alpha\| = 1.96$	$\|z_\alpha\| = 1.645$
Right-tailed	$z_\alpha = 2.33$	$z_\alpha = 2.055$	$z_\alpha = 1.645$	$z_\alpha = 1.28$
Left-tailed	$z_\alpha = -2.33$	$z_\alpha = -2.055$	$z_\alpha = -1.645$	$z_\alpha = -1.28$

Procedure for Testing of Hypothesis

1. Null hypothesis H_0 is defined.
2. Alternative hypothesis H_1 is also defined after a careful study of the problem and also the nature of the test (whether one-tailed or two tailed) is decided.
3. LOS 'α' is fixed or taken from the problem if specified and z_α is noted.
4. The test-statistic $z = \dfrac{t - E(t)}{\text{S}\cdot\text{E}\cdot(t)}$ is computed.
5. Comparison is made between $|z|$ and z_α. If $|z| < z_\alpha$, H_0 is accepted or H_1 is rejected, i.e., it is concluded that the difference between t and $E(t)$ is not significant at α% L.O.S.

On the other hand, if $|z| > z_\alpha$, H_0 is rejected or H_1 is accepted, i.e. it is concluded that the difference between t and E (t) is significant at α% L.O.S.

Interval Estimation of Population Parameters

It was pointed out that the objective of the theory of sampling is to estimate population parameters with the help of the corresponding sample statistics. Estimation of a parameter by single value is referred to as *point estimation*, the study of which is beyond the scope of this book. However, an alternative procedure is to give an interval within which the parameter may be supposed to lie. This is called *interval estimation*. The interval within which the parameter is expected to lie is called *the confidence interval* for that parameter. The end points of the confidence interval are called *confidence limits* or *fiducial limits*.

We have already seen that

$$P\{|z| \le 1.96\} = 0.95$$

i.e. $$P\left\{\left|\frac{t - E(t)}{\text{S}\cdot\text{E}\cdot(t)}\right| \le 1.96\right\} = 0.95$$

i.e. $$P\{t - 1.96 \text{ S.E. } (t) \le E(t) \le t + 1.96 \text{ S.E. } (t)\} = 0.95$$

This means that we can assert, with 95% confidence, that the parameter E (t) will lie between t – 1.96 S.E. (t) and t + 1.96 S.E. (t). Thus, $\{t - 1.96$ S.E. (t), $t + 1.96$ S.E. (t) are the 95% confidence limits for E (t).

Similarly, $\{t - 2.58$ S.E. (t), $t + 2.58$ S.E. $(t)\}$ is the 99% confidence interval for E (t).

Tests of Significance for Large Samples

It is generally agreed that, if the size of the sample exceeds 30, it should be regarded as a large sample. The tests of significance used for large samples are different from the ones used for small samples for the reason that the following assumptions made for large samples do not hold for small samples:

1. The sampling distribution of a statistic is approximately normal, irrespective of whether the distribution of the population is normal or not.
2. Sample statistics are sufficiently close to the corresponding population parameters and hence may be used to calculate the standard error of the sampling distribution.

Test 1 *Test of significance of the difference between sample proportion and population proportion.*

Let X be the number of successes in n independent Bernoulli trials in which the probability of success for each trial is a constant $= P$ (say). Then it is known that X follows a binomial distribution with mean $E(X) = nP$ and variance $V(X) = nPQ$.

When n is large, X follows $N(nP, \sqrt{nPQ})$, i.e. a normal distribution with mean nP and S.D. $\sqrt{nPQ}$, where $Q = 1 - P$.

$$\therefore \quad \frac{X}{n} \text{ follows } N\left\{\frac{nP}{n}, \sqrt{\frac{nPQ}{n^2}}\right\}$$

Now, $\frac{X}{n}$ is the proportion of successes in the sample consisting of n trials, that is denoted by p. Thus, the sample proportion p follows $N\left(P, \sqrt{\frac{PQ}{n}}\right)$.

Therefore, test statistic $z = \dfrac{p - P}{\sqrt{\dfrac{PQ}{n}}}$.

If $|z| \le z_\alpha$, the difference between the sample proportion p and the population proportion P is not significant at $\alpha\%$ L.O.S.

Note *1. If P is not known, we assume that p is nearly equal to P and hence S.E. (p) is taken as* $\sqrt{\frac{pq}{n}}$. *Thus* $z = \dfrac{p - P}{\sqrt{\dfrac{pq}{n}}}$.

2. 95 per cent confidence limits for P are then given by $\dfrac{|P - p|}{\sqrt{\dfrac{pq}{n}}} \le 1.96$, *i.e. they are* $\left(p - 1.96\sqrt{\frac{pq}{n}},\ p + 1.96\sqrt{\frac{pq}{n}}\right)$.

Test 2 *Test of significance of the difference between two sample proportions.*

Let p_1 and p_2 be the proportions of successes in two large samples of size n_1 and n_2 respectively drawn from the same population or from two population with the same proportion P.

Then, p_1 follows $N\left(P, \sqrt{\dfrac{PQ}{n_1}}\right)$ and p_2 follows $N\left(P, \sqrt{\dfrac{PQ}{n_2}}\right)$.

Therefore, $p_1 - p_2$, which is a linear combination of two normal variables, also follows a normal distribution.

Now $$E(p_1 - p_2) = E(p_1) - E(p_2) = P - P = 0$$

$$V(p_1 - p_2) = V(p_1) + V(p_2) \qquad (\because \text{ the two samples are independent})$$

$$= PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

$$\therefore \quad (p_1 - p_2) \text{ follows } N\left\{0, \sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}\right\}$$

$$\therefore \quad \text{the test statistic } z = \frac{p_1 - p_2}{\sqrt{PQ\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}.$$

If P is not known, an unbiased estimate of P based on both samples, given by

$$\hat{P} = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}, \text{ is used in the place of } P.$$

As before, if $|z| \le z_\alpha$, the difference between the two sample proportions p_1 and p_2 is not significant at α per cent L.O.S.

Note *A sample statistic θ is said to be an unbiased estimate of the parameter θ_0, if $E(\theta) = \theta_0$. In the present case,*

$$E\left\{\frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}\right\} = \frac{1}{n_1 + n_2}\{n_1 E(p_1) + n_2 E(p_2)\}$$

$$= \frac{1}{n_1 + n_2}(n_1 P + n_2 P) = P.$$

$$\therefore \quad \text{an unbiased estimate of } P \text{ is } \left(\frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}\right).$$

Test 3 *Test of significance of the difference between sample mean and population mean.*

Let $X_1, X_2, \ldots, X_n$ be the sample observations in a sample of size n, drawn from a population that is $N(\mu, \sigma)$.

Then each X_i follows $N(\mu, \sigma)$.

It is known that if X_i $(i = 1, 2, ..., n)$ are independent normal variates with mean μ_i and variance σ_i^2, then $\Sigma\, c_i\, x_i$ is a normal variate with mean $\mu = \Sigma\, c_i\, \mu_i$ and variance $\sigma^2 = \Sigma\, c_i^2\, \sigma_i^2$.

Now, putting $\quad c_i = \dfrac{1}{n}$, $\mu_i = \mu$ and $\sigma_i = \sigma$, we get

$$\Sigma\, c_i\, x_i = \frac{1}{n}\,\Sigma x_i = \bar{X}\,, \; \Sigma c_i\, \mu_i = \frac{1}{n}\mu + \frac{1}{n}\mu + ... + \frac{1}{n}\mu \;(n \text{ terms}) = \mu$$

and
$$\Sigma\, c_i^2\, \sigma_i^2 = \frac{1}{n^2}\sigma^2 + \frac{1}{n^2}\sigma^2 + ... + \frac{1}{n^2}\sigma^2 \;(n \text{ terms})$$
$$= \frac{\sigma^2}{n}.$$

Thus, if X_i are n independent normal variates with the same mean μ and same variance σ^2, then their mean $\bar{X}$ follows a $N\left(\mu, \dfrac{\sigma}{\sqrt{n}}\right)$. Even if the population, from which the sample is drawn, is non-normal, it is known (from central limit theorem) that the above result holds good, provided n is large.

$\therefore$ the test statistic $z = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}}$.

As usual, if $|z| \le z_\alpha$, the difference between the sample mean $\bar{X}$ and the population mean μ is not significant at $\alpha\%$ L.O.S.

Note *1. If σ is not known, the sample S.D. 's' can be used in its place, as s is nearly equal to σ when n is large.*

2. 95% confidence limits for μ are given by $\dfrac{|\mu - \bar{X}|}{\sigma/\sqrt{n}} \le 1.96$, i.e. $\left(\bar{X} - 1.96\dfrac{\sigma}{\sqrt{n}}, \bar{X} + 1.96\dfrac{\sigma}{\sqrt{n}}\right)$, if σ is known. If σ is not known, then the 95% confidence interval is $\left(\bar{X} - \dfrac{1.96\, s}{\sqrt{n}}, \bar{X} + \dfrac{1.96\, s}{\sqrt{n}}\right)$.

Test 4 *Test of significance of the difference between the means of two samples.*

Let $\bar{X}_1$ and $\bar{X}_2$ be the means of two large samples of sizes n_1 and n_2 drawn from two populations (normal or non-normal) with the same mean μ and variances σ_1^2 and σ_2^2 respectively.

Then $\bar{X}_1$ follows a $N\left(\mu, \dfrac{\sigma_1}{\sqrt{n_1}}\right)$ and $\bar{X}_2$ follows a $N\left(\mu, \dfrac{\sigma_2}{\sqrt{n_1}}\right)$ either exactly or approximately.

$\therefore \bar{X}_1 - \bar{X}_2$ also follows a normal distribution.

$$E(\bar{X}_1 - \bar{X}_2) = E(\bar{X}_1) - E(\bar{X}_2) = \mu - \mu = 0.$$

$$V(\bar{X}_1 - \bar{X}_2) = V(\bar{X}_1) + V(\bar{X}_2)$$

($\because$ $\bar{X}_1$ and $\bar{X}_2$ are independent, as the samples are independent)

$$= \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

Thus, $(\bar{X}_1 - \bar{X}_2)$ follows a $N\left\{0, \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right\}$

$\therefore$ the test statistic
$$z = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \tag{1}$$

If $|z| \le z_\alpha$, the difference between $(\bar{X}_1 - \bar{X}_2)$ and 0 or the difference between $\bar{X}_1$ and $\bar{X}_2$ is not significant at α per cent L.O.S.

Note *If the samples are drawn from the same population, i.e., if $\sigma_1 = \sigma_2 = \sigma$ then*

$$z = \frac{\bar{X}_1 - \bar{X}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \tag{2}$$

2. *If σ_1 and σ_2 are not known and $\sigma_1 \ne \sigma_2$, σ_1 and σ_2 can be approximated by the sample S.D.'s σ_1 and σ_2. Hence, in such a situation,*

$$z = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \qquad \text{(3) [from (1)]}$$

3. *If σ_1 and σ_2 are equal and not known, then $\sigma_1 = \sigma_2 = \sigma$ is approximated by $\sigma^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}$. Hence, in such a situation,*

$$z = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\left(\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}\right)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}, \quad \textit{from (2)}$$

i.e.
$$z = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{s_1^2}{n_2} + \frac{s_2^2}{n_1}}} \tag{4}$$

4. *The difference in the denominators of the values of z given in (3) and (4) may be noted.*

Test 5 *Test of significance of the difference between sample S.D. and population S.D.*

Let 's' be the S.D. of a large sample of size n drawn from a normal population with S.D. σ. Then it is known that s follows a $N\left(\sigma, \dfrac{\sigma}{\sqrt{2n}}\right)$ approximately.

$$\therefore \quad \text{the test statistic } z = \frac{s-\sigma}{\sigma/\sqrt{2n}}$$

As before, the significance of the difference between s and σ is tested.

Test 6 *Test of significance of the difference between S.D.'s of two large samples.*

Let s_1 and s_2 be the S.D.'s of two large samples of sizes n_1 and n_2 drawn from a normal population with S.D. σ.

$$s_1 \text{ follows a } N\left(\sigma, \frac{\sigma}{\sqrt{2n_1}}\right) \text{ and } s_2 \text{ follows a } N\left(\sigma, \frac{\sigma}{\sqrt{2n_2}}\right).$$

$$\therefore \quad (s_1 - s_2) \text{ follows a } N\left\{0,\ \sigma\sqrt{\frac{1}{2n_1}+\frac{1}{2n_2}}\right\}.$$

$$\therefore \quad \text{the test statistic } z = \frac{s_1 - s_2}{\sigma\sqrt{\dfrac{1}{2n_1}+\dfrac{1}{2n_2}}}.$$

As usual, the significance of the difference between s_1 and s_2 is tested.

Note *If σ is not known, it is approximated by $\sqrt{\left(\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}\right)}$, when n_1 and n_2 are large. In this situation,*

$$z = \frac{s_1 - s_2}{\sqrt{\left(\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}\right)\left(\dfrac{1}{2n_1}+\dfrac{1}{2n_2}\right)}}$$

i.e.

$$z = \frac{s_1 - s_2}{\sqrt{\dfrac{s_1^2}{2n_2}+\dfrac{s_2^2}{2n_1}}}$$

Worked Example 10(A)

Example 1

Experience has shown that 20 per cent of a manufactured product is of top quality. In one day's production of 400 articles, only 50 are of top quality. Show that either the production of the day chosen was not a representative sample or the hypothesis of 20 per cent was wrong. Based on the particular day's production, find also the 95 per cent confidence limits for the percentage of top quality product.

$H_0 : P = \frac{1}{5}$, i.e., 20 per cent of the products manufactured is of top quality.

$H_1 : P \neq \frac{1}{5}$.

p = proportion of top quality products in the sample

$$= \frac{50}{400} = \frac{1}{8}$$

From the alternative hypothesis H_1, we note that two-tailed test is to be used. Let us assume that LOS (level of significance)

$$= 5\%. \quad \therefore z_\alpha = 1.96$$

$$z = \frac{p-P}{\sqrt{\frac{PQ}{n}}} = \frac{\frac{1}{8}-\frac{1}{5}}{\sqrt{\frac{1}{5}\times\frac{4}{5}\times\frac{1}{400}}}$$, since the size of the sample = 400.

$$= -\frac{3}{40} \times 50 = -3.75$$

Now, $|z| = 3.75 > 1.96$.

The difference between p and P is significant at 5 per cent level.

Also, H_0 is rejected. Hence H_0 is wrong or the production of the particular day chosen is not a representative sample.

95 per cent confidence limits for P are given by

$$\frac{|p-P|}{\sqrt{\frac{pq}{n}}} \leq 1.96$$

Note *We have taken $\sqrt{\frac{pq}{n}}$ in the denominator, because P is assumed to be unknown, for which we are trying to find the confidence limits and P is nearly equal to p.*

$$p - \sqrt{\frac{pq}{n}} \times 1.96 \le P \le p + \sqrt{\frac{pq}{n}} \times 1.96$$

i.e. $$0.125 - \sqrt{\frac{1}{8} \times \frac{7}{8} \times \frac{1}{400}} \times 1.96 \le P \le 0.125 + \sqrt{\frac{1}{8} \times \frac{7}{8} \times \frac{1}{400}} \times 1.96$$

i.e. $$0.093 \le P \le 0.157$$

∴ 95 per cent confidence limits for the percentage of top quality product are 9.3 and 15.7.

Example 2

The fatality rate of typhoid patients is believed to be 17.26 per cent. In a certain year 640 patients suffering from typhoid were treated in a metropolitan hospital and only 63 patients died. Can you consider the hospital efficient?

$H_0 : p = P$, i.e. the hospital is not efficient. $H_1 : p < P$.

One-tailed (left-tailed) test is to be used

Let us assume that LOS = 1%. ∴ $z_\alpha = -2.33$

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$$z = \frac{p - P}{\sqrt{\frac{PQ}{n}}}, \text{ where } p = \frac{63}{640} = 0.0984 \text{ and}$$

$$P = 0.1726 \quad \text{and hence} \quad Q = 0.8274.$$

$$z = \frac{0.0984 - 0.1726}{\sqrt{\frac{0.1726 \times 0.8274}{640}}} = -4.96$$

∴ $$|z| > |z_\alpha|$$

∴ the difference between p and P is significant, i.e., H_0 is rejected and H_1 is accepted.

i.e. the hospital is efficient in bringing down the fatality rate of typhoid patients.

Example 3

A salesman in a departmental store claims that at most 60 percent of the shoppers entering the store leaves without making a purchase. A random sample of 50 shoppers showed that 35 of them left without making a purchase. Are these sample results consistent with the claim of the salesman? Use a level of significance of 0.05.

Let P and p denote the population and sample proportions of shoppers not making a purchase.

$$H_0 : p = P$$

$$H_1 : p > P, \text{ since } p = 0.7 \quad \text{and} \quad P = 0.6$$

One-tailed (right-tailed) test is to be used.

$$\text{LOS} = 5\% \quad \therefore z_\alpha = 1.645$$

$$z = \frac{p - P}{\sqrt{\dfrac{PQ}{n}}} = \frac{0.7 - 0.6}{\sqrt{\dfrac{0.6 \times 0.4}{50}}} = 1.443$$

$\therefore \quad |z| < z_\alpha$

$\therefore$ the difference between p and P is not significant at 5 percent level.

i.e. H_0 is accepted and H_1 is rejected.

i.e. the sample results are consistent with the claim of the salesman.

Example 4

Show that for a random sample of size 100, drawn with replacement, the standard error of sample proportion cannot exceed 0.05.

The items of the sample are drawn one after another with replacement.

$\therefore$ the proportion (probability) of success in the population, i.e. P remains a constant.

We know that the sample proportion p follows a $N\left(P, \sqrt{\dfrac{PQ}{n}}\right)$

$$\text{i.e. standard error of } p = \sqrt{\frac{PQ}{n}} = \frac{1}{10}\sqrt{PQ} \quad (\because n = 100) \tag{1}$$

Now $\quad (\sqrt{P} - \sqrt{Q})^2 \geq 0$

i.e. $\quad P + Q - 2\sqrt{PQ} \geq 0$

i.e. $\quad 1 - 2\sqrt{PQ} \geq 0 \quad \text{or} \quad \sqrt{PQ} \leq \dfrac{1}{2} \qquad (2)$

Using (2) in (1), we get,

S.E. of $p \leq \dfrac{1}{20}$. i.e. S.E. of p cannot exceed 0.05.

Example 5

A cubical die is thrown 9000 times and a throw of three or four is observed 3240 times. Show that the die cannot be regarded as an unbiased one and find the extreme limits between which the probability of a throw of three or four lies.

H_0 : the die is unbiased, i.e. $P = \dfrac{1}{3}$ (= the probability of getting 3 or 4)

$H_1 : P \neq \dfrac{1}{3}$

Two-tailed test is to be used.

Let $\quad \text{LOS} = 5\% \quad \therefore z_\alpha = 1.96$

Though we may test the significance of the difference between the sample and population proportions, we shall test the significance of the difference between the number X of successes in the sample and that in the population.

When n is large, X follows a $N\,(nP, \sqrt{nPQ})$ [Refer to Test 1].

$$\therefore \quad z = \frac{X - nP}{\sqrt{nPQ}} = \frac{3240 - \left(9000 \times \frac{1}{3}\right)}{\sqrt{9000 \times \frac{1}{3} \times \frac{2}{3}}} = 5.37$$

$$\therefore \quad |z| > z_\alpha$$

$\therefore$ The difference between X and nP is significant. i.e., H_0 is rejected.

i.e., the die cannot be regarded as unbiased.

If X follows a $N\,(\mu, \sigma)$, then the reader can easily verify that $P\,(\mu - 3\sigma \le X \le \mu + 3\sigma) = .9974$.

The limits $\mu \pm 3\sigma$ are considered as the extreme (confidence) limits within which X lies.

Accordingly, the extreme limits for P are given by

$$\frac{|P - p|}{\sqrt{\frac{pq}{n}}} \le 3 \qquad \text{[Refer to Example (1)]}$$

i.e., $$p - 3\sqrt{\frac{pq}{n}} \le P \le p + 3\sqrt{\frac{pq}{n}}$$

i.e., $$0.36 - 3\sqrt{\frac{0.36 \times 0.64}{9000}} \le P \le 0.36 + 3\sqrt{\frac{0.36 \times 0.64}{9000}}$$

i.e., $$0.345 \le P \le 0.375.$$

Example 6

In a large city A, 20 per cent of a random sample of 900 school boys had a slight physical defect. In another large city B, 18.5 percent of a random sample of 1600 school boys had the same defect. Is the difference between the proportions significant?

$$p_1 = 0.2, \quad p_2 = 0.185, \quad n_1 = 900 \quad \text{and} \quad n_2 = 1600$$

$$H_0 : p_1 = p_2$$

$$H_1 : p_1 \ne p_2$$

Two-tailed test is to be used.

Let L.O.S. be 5% $\quad \therefore z_\alpha = 1.96$

$$z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \qquad (1)$$

Since P, the population proportion, is not given, we estimate it as $\hat{P} = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{180 + 296}{900 + 1600} = 0.1904.$

Using in (1), we have

$$z = \frac{0.2 - 0.185}{\sqrt{0.1904 \times 0.8096 \times \left(\frac{1}{900} + \frac{1}{1600}\right)}} = 0.92$$

$|z| \le z_{\alpha}$. Therefore, the difference between p_1 and p_2 is not significant at 5 per cent level.

Example 7

Before an increase in excise duty on tea, 800 people out of a sample of 1000 were consumers of tea. After the increase in duty, 800 people were consumers of tea in a sample of 1200 persons. Find whether there is significant decrease in the consumption of tea after the increase in duty.

Let p_1 and p_2 be the proportions of the consumers before and after the increase in duty respectively.

Then $\quad p_1 = \frac{800}{1000} = \frac{4}{5}$ and $p_2 = \frac{800}{1200} = \frac{2}{3}$.

$$H_0 : p_1 = p_2$$
$$H_1 : p_1 > p_2$$

One-tailed (right-tailed) test is to be used. Let LOS be 1%. $\therefore z_{\alpha} = 2.33$.

$$z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}, \text{ where } p \simeq \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$$

$$= \frac{800 + 800}{2200} = 0.7273$$

$$= \frac{0.8 - 0.67}{\sqrt{0.7273 \times 0.2727 \times \left(\frac{1}{1000} + \frac{1}{1200}\right)}}$$

$$= \frac{0.13 \times \sqrt{1000 \times 1200}}{\sqrt{0.7273 \times 0.2727 \times 2200}} = 6.82$$

Now, $\quad |z| > z_{\alpha}$

$\therefore$ the difference between p_1 and p_2 is significant at 1% level.

i.e., H_0 is rejected and H_1 is accepted.

i.e., there is significant decrease in the consumption of tea after the increase in duty.

Example 8

15.5 per cent of a random sample of 1600 undergraduates were smokers, whereas 20% of a random sample of 900 postgraduates were smokers in a state. Can we conclude that less number of undergraduates are smokers than the postgraduates?

$$p_1 = 0.155 \quad \text{and} \quad p_2 = 0.2; \quad n_1 = 1600 \quad \text{and} \quad n_2 = 900$$

$H_0 : p_1 = p_2$

$H_1 : p_1 < p_2$

One-tailed (left-tailed) test is to be used. Let LOS be 5%. $\therefore z_\alpha = -1.645$.

$$z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}, \text{ where } P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = 0.1712$$

$$= \frac{0.155 - 0.2}{\sqrt{0.1712 \times 0.8288 \times \left(\frac{1}{1600} + \frac{1}{900}\right)}} = \frac{-0.045 \times 1200}{\sqrt{0.1712 \times 0.8288 \times 2500}}$$

$$= -2.87$$

Now, $\quad |z| > |z_\alpha|$

$\therefore$ the difference between p_1 and p_2 is significant.

i.e. H_0 is rejected and H_1 is accepted.

i.e. The habit of smoking is less among the undergraduates than among the postgraduates.

Example 9

Diff b/w mean.

A sample of 100 students is taken from a large population. The mean height of the students in this sample is 160 cm. Can it be reasonably regarded that, in the population, the mean height is 165 cm, and the S.D. is 10 cm?

$$\bar{x} = 160, \quad n = 100, \quad \mu = 165 \quad \text{and} \quad \sigma = 10.$$

$H_0 : \bar{x} = \mu$ (i.e. the difference between $\bar{x}$ and μ is not significant)

$H_1 : \bar{x} \neq \mu$.

Two-tailed test is to be used.

Let LOS be 1% $\therefore z_\alpha = 2.58$

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{160 - 165}{10 / \sqrt{100}} = -5$$

Now, $\quad |z| > z_\alpha$.

$\therefore$ the difference between $\bar{x}$ and μ is significant at 1% level.

i.e. H_0 is rejected.

i.e. it is not statistically correct to assume that $\mu = 165$.

Example 10

The mean breaking strength of the cables supplied by a manufacturer is 1800 with a S.D. of 100. By a new technique in the manufacturing process, it is claimed that the breaking strength of the cable has increased. In order to test this claim, a sample of 50 cables is tested and it is found that the mean breaking strength is 1850. Can we support the claim at 1 per cent level of significance?

$$\bar{x} = 1850, \quad n = 50, \quad \mu = 1800 \quad \text{and} \quad \sigma = 100$$

$$H_0 : \bar{x} = \mu$$

$$H_1 : \bar{x} > \mu$$

One-tailed (right-tailed) test is to be used.

$$\text{LOS} = 1\% \quad \therefore z_\alpha = 2.33$$

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{1850 - 1800}{100 / \sqrt{50}} = 3.54$$

Now, $|z| > z_\alpha$.

$\therefore$ the difference between $\bar{x}$ and μ is significant at 1 per cent level.

i.e. H_0 is rejected and H_1 is accepted.

i.e. based on the sample data, we may support the claim of increase in breaking strength.

Example 11

The mean value of a random sample of 60 items was found to be 145 with a S.D. of 40. Find the 95% confidence limits for the population mean. What size of the sample is required to estimate the population mean within five of its actual value with 95% or more confidence, using the sample mean?

95% confidence limits for μ are given by

$$\frac{|\mu - \bar{x}|}{\sigma / \sqrt{n}} \le 1.96$$

Since the population S.D. σ too is not given, we can approximate it by the sample S.D.s. therefore 95% confidence limits for μ are given by $\frac{|\mu - \bar{x}|}{s/\sqrt{n}} \le 1.96$

i.e., $$\bar{x} - 1.96\frac{s}{\sqrt{n}} \le \mu \le \bar{x} + 1.96\frac{s}{\sqrt{n}}.$$

i.e., $$145 - \frac{1.96 \times 40}{\sqrt{60}} \le \mu \le 145 + \frac{1.96 \times 40}{\sqrt{60}}$$

i.e., $$134.9 \le \mu \le 155.1$$

We have to find the value of n such that

$$P\{\bar{x} - 5 \le \mu \le \bar{x} + 5\} \ge \cdot 95$$

$$P\{-5 \le \mu - \bar{x} \le 5\} \ge \cdot 95$$

i.e., $$P\{|\mu - \bar{x}| \le 5\} \ge \cdot 95 \quad \text{or}$$

i.e., $$P\{|\bar{x} - \mu| \le 5\} \ge \cdot 95$$

∴ $$P\left\{\frac{|\bar{x} - \mu|}{\sigma/\sqrt{n}} \le \frac{5}{\sigma/\sqrt{n}}\right\} \ge \cdot 95$$

i.e., $$P\left\{|z| \le \frac{5\sqrt{n}}{\sigma}\right\} \ge \cdot 95, \text{ where } z \text{ is the standard normal variate} \quad (1)$$

We know that $P\{|z| \le 1.96\} = \cdot 95$

∴ the least value of $n = n_L$ that will satisfy (1) is given by $\dfrac{5\sqrt{n_L}}{\sigma} = 1.96$

i.e., $$\sqrt{n_L} = \frac{1.96\, s}{5} \quad (\because \quad \sigma \simeq s)$$

i.e., $$n_L = \left(\frac{1.96 \times 40}{5}\right)^2$$

i.e., $$n_L = 245.86$$

∴ the least size of the sample = 246.

Example 12

A normal population has a mean of 0.1 and S.D. of 2.1. Find the probability that the mean of a sample of size 900 drawn from this population will be negative.

Since $\bar{x}$ follows a $N\left(\mu, \dfrac{\sigma}{\sqrt{n}}\right)$, $z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}}$ is the standard normal variate.

Now, $$P(\bar{x} < 0) = P\{\bar{x} - 0.1 < -0.1\}$$

$$= P\left\{\frac{\bar{x} - 0.1}{(2.1)/\sqrt{900}} < \frac{-0.1}{(2.1)/\sqrt{900}}\right\}$$

$$= P\{z < -1.43\}$$

$$= P\{z > 1.43\},$$

by symmetry of the standard normal distribution.

$$= 0.5 - P\{0 < z < 1.43\}$$

$$= 0.5 - 0.4236 \text{ (from the normal tables)}$$

$$= 0.0764.$$

Example 13

In a random sample of size 500, the mean is found to be 20. In another independent sample of size 400, the mean is 15. Could the samples have been drawn from the same population with S.D. 4?

$$\bar{x}_1 = 20, \quad n_1 = 500; \quad \bar{x}_2 = 15, \quad n_2 = 400; \quad \sigma = 4$$

$H_0 : \bar{x}_1 = \bar{x}_2$, i.e. the samples have been drawn from the same population.

$H_1 : \bar{x}_1 \neq \bar{x}_2$.

Two-tailed test is to be used.

Let LOS be 1% $\quad \therefore z_\alpha = 2.58$

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$ (Refer to Note 1 under Test 4)

$$= \frac{20 - 15}{4\sqrt{\frac{1}{500} + \frac{1}{400}}} = 18.6$$

Now, $\quad |z| > z_\alpha$

$\therefore$ the difference between $\bar{x}_1$ and $\bar{x}_2$ is significant at 1% level.

i.e. H_0 is rejected

i.e. the samples could not have been drawn from the same population.

Example 14

A simple sample of heights of 6400 Englishmen has a mean of 170 cm and a S.D. of 6.4 cm, while a simple sample of heights of 1600 Americans has a mean of 172 cm and a S.D. of 6.3 cm. Do the data indicate that Americans are, on the average, taller than Englishmen?

$$n_1 = 6400, \; \bar{x}_1 = 170 \quad \text{and} \quad s_1 = 6.4$$

$$n_2 = 1600, \; \bar{x}_2 = 172 \quad \text{and} \quad s_2 = 6.3$$

$H_0 : \mu_1 = \mu_2$ or $\bar{x}_1 = \bar{x}_2$,

i.e., the samples have been drawn from two different populations with the same mean.

$H_1 : \bar{x}_1 < \bar{x}_2$ or $\mu_1 < \mu_2$.

Left-tailed test is to be used.

Let LOS be 1%. $\quad \therefore z_\alpha = -2.33$

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

[$\because \sigma_1 \simeq s_1$ and $\sigma_2 \simeq s_2$. Refer to Note 2 under Test 4]

$$= \frac{170-172}{\sqrt{\frac{(6.4)^2}{6400}+\frac{(6.3)^2}{1600}}} = -11.32$$

Now, $\quad |z| > |z_\alpha|$

$\therefore$ the difference between $\overline{x}_1$ and $\overline{x}_2$ (or μ_1 and μ_2) is significant at 1% level.

i.e. H_0 is rejected and H_1 is accepted.

i.e. Americans are, on the average, taller than Englishmen.

Example 15

Test the significance of the difference between the means of the samples, drawn from two normal populations with the same S.D. from the following data:

Table 10.2

	Size	*Mean*	*S.D.*
Sample 1	100	61	4
Sample 2	200	63	6

$$H_0 : \overline{x}_1 = \overline{x}_2 \quad \text{or} \quad \mu_1 = \mu_2$$

$$H_1 : \overline{x}_1 \neq \overline{x}_2 \quad \text{or} \quad \mu_1 \neq \mu_2$$

2

Two-tailed test is to be used.

Let LOS be 5% $\therefore z_\alpha = 1.96$

$$z = \frac{\overline{x}_1 - \overline{x}_2}{\sqrt{\frac{s_1^2}{n_2}+\frac{s_2^2}{n_1}}}$$

[Refer to Note 3 under Test 4; The populations have the same S.D.]

$$= \frac{61-63}{\sqrt{\frac{4^2}{200}+\frac{6^2}{100}}} = -3.02$$

Now, $\quad |z| > z_\alpha$

$\therefore$ the difference between $\overline{x}_1$ and $\overline{x}_2$ (or μ_1 and μ_2) is significant at 5% level.

i.e. H_0 is rejected and H_1 is accepted.

i.e. The two normal populations, from which the samples are drawn, may not have the same mean, though they may have the same S.D.

Example 16

The average marks scored by 32 boys is 72 with a S.D. of 8, while that for 36 girls is 70 with a S.D. of 6. Test at 1% level of significance whether the boys perform better than girls.

$$H_0 : \bar{x}_1 = \bar{x}_2 \quad (\text{or } \mu_1 = \mu_2)$$

$$H_1 : \bar{x}_1 > \bar{x}_2$$

Right-tailed test is to be used.

$$\text{LOS} = 1\% \quad \therefore z_\alpha = 2.33$$

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$$

(The two populations are assumed to have S.D.'s $\sigma_1 \simeq s_1$ and $\sigma_2 \simeq s_2$)

$$= \frac{72 - 70}{\sqrt{\dfrac{8^2}{32} + \dfrac{6^2}{36}}} = 1.15$$

$|z| < z_\alpha$.

$\therefore$ the difference between $\bar{x}_1$ and $\bar{x}_2$ (μ_1 and μ_2) is not significant at 1% level.

i.e., H_0 is accepted and H_1 is rejected.

i.e., statistically, we cannot conclude that boys perform better than girls.

Example 17

The heights of men in a city are normally distributed with a mean of 171 cm and S.D. of 7 cm, while the corresponding values for women in the same city are 165 cm and 6 cm respectively. If a man and a woman are chosen at random from this city, find the probability that the woman is taller than the man.

Let $\bar{x}_1$ and $\bar{x}_2$ denote the mean heights of men and women respectively.

Then $\bar{x}_1$ follows a $N(171, 7)$ and $\bar{x}_2$ follows a $N(165, 6)$

$\therefore$ $\bar{x}_1 - \bar{x}_2$ also a follows a normal distribution.

$$E(\bar{x}_1 - \bar{x}_2) = E(x_1) - E(\bar{x}_2) = 171 - 165 = 6$$

$$V(\bar{x}_1 - \bar{x}_2) = V(\bar{x}_1) + V(\bar{x}_2) = 49 + 36 = 85 \text{ [Refer to Test 4]}$$

$\therefore$ S.D. of $(\bar{x}_1 - \bar{x}_2) = \sqrt{85} = 9.22$.

$\therefore$ $\bar{x}_1 - \bar{x}_2$ follows a $N(6, 9.22)$

Now, $P(\bar{x}_2 > \bar{x}_1) = P(\bar{x}_1 - \bar{x}_2 < 0)$

$$= P\left\{\frac{(\bar{x}_1 - \bar{x}_2) - 6}{9.22} < \frac{-6}{9.22}\right\}$$

$= P\{z < -0.65\}$, where z is the standard normal variate.

$= P\{z > 0.65\}$, by symmetry.

$= 0.5 - P(0 < z < 0.65)$

$= 0.5 - 0.2422 = 0.2578$.

Example 18

Two populations have the same mean, but the S.D. of one is twice that of the other. Show that in samples, each of size 500, drawn under simple random conditions, the difference of the means will, in all probability, not exceed 0.3 σ, where σ is the smaller S.D.

Let $\bar{x}_1$ and $\bar{x}_1$ be the means of the samples of size 500 each. Let their S.D.'s be σ and 2σ respectively.

$$\bar{x}_1 \text{ follows a } N\left(\mu, \frac{\sigma}{\sqrt{500}}\right) \text{ and}$$

$$\bar{x}_2 \text{ follows a } N\left(\mu, \frac{2\sigma}{\sqrt{500}}\right)$$

$\therefore$ $\bar{x}_1 - \bar{x}_2$ also follows a normal distribution

$$E(\bar{x}_1 - \bar{x}_2) = E(\bar{x}_1) - E(\bar{x}_2) = \mu - \mu = 0$$

$$V(\bar{x}_1 - \bar{x}_2) = V(\bar{x}_1) + V(\bar{x}_2)$$

$$= \frac{\sigma^2}{500} + \frac{4\sigma^2}{500} = \frac{\sigma^2}{100}$$

$\therefore$ S.D. of $(\bar{x}_1 - \bar{x}_2) = \dfrac{\sigma}{10}$

Thus, $(\bar{x}_1 - \bar{x}_2)$ follows a $N\left(0, \dfrac{\sigma}{10}\right)$.

$$\therefore \quad P\{|\bar{x}_1 - \bar{x}_2| \le 0.3\,\sigma\}$$

$$= P\left\{\frac{|(\bar{x}_1 - \bar{x}_2) - 0|}{\sigma/10} \le \frac{0.3\,\sigma}{\sigma/10}\right\}$$

$= P\{|z| \le 3\}$, where z is the standard normal variate

$$= 0.9974 \simeq 1.$$

$\therefore$ $|\bar{x}_1 - \bar{x}_2|$ will not exceed 0.3 σ almost certainly.

Example 19

A manufacturer of electric bulbs, according to a certain process, finds the S.D. of the life of lamps to be 100 hours. He wants to change the process, if the new process results in a smaller variation in the life of lamps. In adopting a new process, a sample of 150 bulbs gave an S.D. of 95 hours. Is the manufacturer justified in changing the process?

$$\sigma = 100, \quad n = 150 \quad \text{and} \quad s = 95$$

$$H_0 : s = \sigma$$
$$H_1 : s < \sigma$$

Left-tailed test is to be used.

Let LOS be 5%. $\therefore z_\alpha = -1.645$

$$z = \frac{s-\sigma}{\sigma/\sqrt{2n}} = \frac{95-100}{100/\sqrt{300}} = -0.866$$

Now, $|z| < |z_\alpha|$

$\therefore$ the difference between s and σ is not significant at 5% level.

i.e. H_0 is accepted and H_1 is rejected.

i.e. The manufacturer is not justified in changing the process.

Example 20

The S.D. of a random sample of 1000 is found to be 2.6 and the S.D. of another random sample of 500 is 2.7. Assuming the samples to be independent, find whether the two samples could have come from populations with the same S.D.

$$n_1 = 1000, \quad s_1 = 2.6\,; \quad n_2 = 500, \quad s_2 = 2.7$$

$$H_0 : s_1 = s_2 \quad (\text{or } \sigma_1 = \sigma_2)$$
$$H_1 : s_1 \neq s_2$$

Two-tailed test is to be used.

Let LOS be 5%. $\therefore z_\alpha = 1.96$

$$z = \frac{s_1 - s_2}{\sqrt{\dfrac{s_1^2}{2n_2} + \dfrac{s_2^2}{2n_1}}}, \text{ since } \sigma \text{ is not known.}$$

$$= \frac{2.6-2.7}{\sqrt{\dfrac{(2.6)^2}{1000} + \dfrac{(2.7)^2}{2000}}} = -0.98$$

Now, $|z| < z_\alpha$

$\therefore$ the difference between s_1 and s_2 (and hence between σ_1 and σ_2) is not significant at 5% level,

i.e., H_0 is accepted.

i.e., the two samples could have come from populations with the same S.D.

Exercise 10(A)

Part-A (Short-answer Questions)

1. What is the difference between population and sample?
2. Distinguish between parameter and statistic.

3. What do you mean by sampling distribution?
4. What is meant by standard error?
5. What do you mean by estimation?
6. What is meant by hypothesis testing?
7. Define null hypothesis and alternative hypothesis.
8. What is meant by test of significance?
9. What do you mean by critical region and acceptance region?
10. Define level of significance.
11. Give the general form of a test statistic.
12. Define type I and type II errors.
13. Define producer's risk and consumer's risk.
14. What is the relation between type I error and level of significance?
15. Define one-tailed and two-tailed tests.
16. Define critical value of a test statistic.
17. What is the relation between the critical value and level of significance?
18. What is the relation between the critical values for a single tailed test and a two-tailed test?
19. Write down the 1% and 5% critical values for right-tailed and two-tailed tests.
20. What do you mean by interval estimation and confidence limits?
21. Write down the general form of 95% confidence limits of a population parameter in terms of the corresponding sample statistic.
22. What is the standard error of the sample proportion, when the population proportion is (i) known, and (ii) not known?
23. What is the standard error of the difference between two sample proportions when the population proportion is (i) known, and (ii) not known?
24. What do you mean by unbiased estimate? Give an example.
25. Write down the form of the 98% confidence interval for the population mean in terms of (i) population S.D., and (ii) Sample S.D.
26. What is the standard error of the difference between the means of two large samples drawn from different populations with (i) known S.D.'s, and (ii) unknown S.D.'s?
27. What is the standard error of the difference between the means of two large samples drawn from the same population with (i) known S.D., and (ii) unknown S.D.?
28. What is the standard error of the difference between the S.D.'s of two large samples drawn from the same population with (i) known S.D., and (ii) unknown S.D.?

Part-B

29. Out of 200 individuals, 40 per cent show a certain trait and the number expected on a certain theory is 50 per cent. Find whether the number observed differs significantly from expectation.

30. A coin is thrown 400 times and is found to result in 'Head' 245 times. Test whether the coin is a fair one.
31. A manufacturer of light bulbs claims that on the average 2 per cent of the bulbs manufactured by his firm are defective. A random sample of 400 bulbs contained 13 defective bulbs. On the basis of the this sample, can you support the manufacturer's claim at 5% level of significance?
32. 100 people were affected by cholera and out of them only 90 survived. Would you reject the hypothesis that the survival rate, if affected by cholera, is 85 per cent in favour of the hypothesis that it is more at 5 per cent level of significance?
33. A random sample of 400 mangoes was taken from a big consignment and 40 were found to be bad. Prove that the percentage of bad mangoes in the consignment will, in all probability, lie between 5.5 and 14.5.
34. A random sample of 64 articles produced by a machine contained 14 defectives. Is it reasonable to assume that only 10 per cent of the articles produced by the machine are defective? If not, find the 99 per cent confidence limits for the percentage of defective articles produced by the machine.
35. Certain crosses of the pea gave 5321 yellow and 1804 green seeds. The expectation is 25 per cent of green seeds based on a certain theory. Is this divergence significant or due to sampling fluctuations?
36. During a countrywide investigation, the incidence of T.B. was found to be 1 per cent . In a college with 400 students, 5 are reported to be affected whereas in another with 1200 students, 10 are found to be affected. Does this indicate any significant difference?
37. A random sample of 600 men chosen from a certain city contained 400 smokers. In another sample of 900 men chosen from another city, there were 450 smokers. Do the data indicate that (i) the cities are significantly different with respect to smoking habit among men? (ii) the first city contains more smokers than the second?
38. A sample of 300 spare parts produced by a machine contained 48 defectives. Another sample of 100 spare parts produced by another machine contained 24 defectives. Can you conclude that the first machine is better than the second?
39. In two large populations, there are 30 per cent and 25 per cent respectively of fair haired people. Is this difference likely to be hidden in samples of sizes 1200 and 900 respectively drawn from the two populations?

$$\left[\textbf{Hint: } H_0 : P_1 - P_2 = 0, \text{ or } P_1 = P_2 \text{ and } H_1 : P_1 \neq P_2 \text{ and } z = \frac{p_1 - p_2}{\sqrt{\dfrac{P_1 Q_1}{n_1} + \dfrac{P_2 Q_2}{n_2}}}\right]$$

40. A machine produces 16 defective bolts in a batch of 500 bolts. After the machine is overhauled, it produces three defective bolts in a batch of 100 bolts. Has the machine improved?
41. There were 956 births in a year in town A, of which 52.5 per cent were males, while in towns A and B combined together this proportion in a total of 1406 births was 0.496. Is there any significant difference in the proportion of male births in the two towns?
42. A cigarette-manufacturing company claims that its brand A cigarettes outsells its brand B by 8 per cent. It is found that 42 out of a sample of 200 smokers prefer brand A and 18 out of another sample of 100 smokers prefer brand B. Test at 5 per cent L.O.S. whether the 8 per cent difference is a valid claim.

[**Hint:** $H_0 : P_1 - P_2 = .08;\; H_1 : P_1 - P_2 \neq .08$ and

$$Z = \frac{(p_1 - p_2) - (P_1 - P_2)}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}, \quad \text{where } P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$$]

43. A sample of 900 items is found to have a mean of 3.47 cm. Can it be reasonably regarded as a simple sample from a population with mean 3.23 cm and S.D. 2.31 cm?
44. A sample of 400 observations has mean 95 and S.D. 12. Could it be a random sample from a population with mean 98? What should be the maximum value of the population mean so that the sample can be regarded as one drawn from it almost certainly?
45. A manufacturer claims that, the mean breaking strength of safety belts for air passengers produced in his factory is 1275 kgs. A sample of 100 belts was tested and the mean breaking strength and S.D. were found to be 1258 kg and 90 kg respectively. Test the manufacturer's claim at 5 per cent level of significance.
46. An I.Q. test was given to a large group of boys in the age group of 18 to 20 years, who scored an average of 62.5 marks. The same test was given to a fresh group of 100 boys of the same age group. They scored an average of 64.5 marks with a S.D. 12.5 marks. Can we conclude that the fresh group of boys have better I.Q.?
47. The guaranteed average life of a certain brand of electric bulb is 1000 hours with a S.D. of 125 hours. It is decided to sample the output so as to ensure that 90 per cent of the bulbs do not fall short of the guaranteed average by more than 2.5 per cent. What should be the minimum sample size?

48. A random sample of 100 students gave a mean weight of 58 kg with a S.D. of 4 kg. Find the 95 per cent and 99 per cent confidence limits of the mean of the population.
49. The means of two simple samples of 1000 and 2000 items are 170 cm and 169 cm. Can the samples be regarded as drawn from the same population with S.D. 10, at 5 per cent level of significance?
50. The mean and S.D. of sample sizes of 400 are 250 and 40 respectively. Those of another sample of size 400 are 220 and 55. Test at 1% level of significance whether the means of the two populations from which the samples have been drawn are equal.
51. Intelligence tests were given to two groups of boys and girls of the same age group chosen from the same college and the following results were got:

Table 10.3

	Size	*Mean*	*S.D.*
Boys	100	73	10
Girls	60	75	8

Examine if the difference between the means is significant.
52. A sample of 100 bulbs of brand *A* gave a mean lifetime of 1200 hours with a S.D. of 70 hours, while another sample of 120 bulbs of brand *B* gave a mean lifetime of 1150 hours with a S.D. of 85 hours. Can we conclude that brand *A* bulbs are superior to brand *B* bulbs?
53. In a college, 60 junior students are found to have a mean height of 171.5 cm and 50 senior students are found to have a mean height of 173.8 cm. Can we conclude, based on this data, that the juniors are shorter than seniors at (i) 5% level of significance, and (ii) 1% level of significance, assuming that the S.D. of students of that college is 6.2 cm?
54. Two samples drawn from two different populations gave the following results:

Table 10.4

	Size	*Mean*	*S.D.*
Sample I	400	124	14
Sample II	250	120	12

Find the 95% confidence limits for the difference of the population means.

$$\left[\textbf{Hint:} (\bar{x}_1 - \bar{x}_2) - 1.96\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \le \mu_1 - \mu_2 \le (\bar{x}_1 - \bar{x}_2) + 1.96\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}\right]$$

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55. Two samples drawn from two different populations gave the following results:

Table 10.5

	Size	Mean	S.D.
Sample I	100	582	24
Sample II	100	540	28

Test the hypothesis, at 5% level of significance, that the difference of the means of the populations is 35.

$$\left[\textbf{Hint: } z = \frac{(\bar{x}_1 - \bar{x}_2) - 35}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}\right]$$

56. Two populations have their means equal, but the S.D. of one is twice the other. Show that, in the samples of size 2000 drawn one from each, the difference of the means will in all probability, not exceed 0.15 σ, where σ is the smaller S.D.
57. In a certain random sample of 72 items, the S.D. is found to be 8. Is it reasonable to suppose that it has been drawn from a population with S.D.7?
58. In a random sample of 200 items, drawn from a population with S.D. 0.8, the sample S.D. is 0.7. Can we conclude that the sample S.D. is less than the population S.D. at 1% level of significance?
59. The S.D. of a random sample of 900 members is 4.6 and that of another independent sample of 1600 members is 4.8. Examine if the two samples could have been drawn from a population with S.D. 4.0?
60. Examine whether the two samples for which the data are given in Table 10.6 could have been drawn from populations with the same S.D.:

Table 10.6

	Size	*S.D.*
Sample I	100	5
Sample II	200	7

Tests of Significance for Small Samples

The tests of significance discussed in the previous section hold good only for large samples, i.e., only when the size of the sample $n \geq 30$. When the sample is small, i.e., $n < 30$, the sampling distributions of many statistics are not normal, even though the parent populations may be normal. Moreover, the assumption of near equality of population parameters and the corresponding sample statistics will not be justified for small samples. Consequently, we have to develop entirely different tests of significance that are applicable to small samples.

Student's *t*-Distribution

A random variable T is said to follow student's t-distribution or simply t-distribution, if its probability density function is given by

$$f(t) = \frac{1}{\sqrt{\nu}\,\beta\left(\frac{\nu}{2}, \frac{1}{2}\right)} \cdot \left(1 + \frac{t^2}{\nu}\right)^{-(\nu+1)/2}, \quad -\infty < t < \infty.$$

ν is called the number of degrees of freedom of the t-distribution.

Note *t-distribution was defined by the mathematician W.S.D. Gosset whose pen name is Student.*

Properties of t-Distribution

1. The probability curve of the t-distribution is similar to the standard normal curve and is symmetric about $t = 0$, bell-shaped and asymptotic to the t-axis as shown in Fig. 10.2.
2. For sufficiently large value of n, the t-distribution tends to the standard normal distribution.
3. The mean of the t-distribution is zero.
4. The variance of the t-distribution is $\frac{\nu}{\nu - 2}$, if $n > 2$ and is greater than 1, but it tends to 1 as $\nu \to \infty$.

Uses of t-Distribution

The t-distribution is used to test the significance of the difference between

1. The mean of a small sample and the mean of the population.
2. The means of two small samples and
3. The coefficient of correlation in the small sample and that in the population, assumed zero.

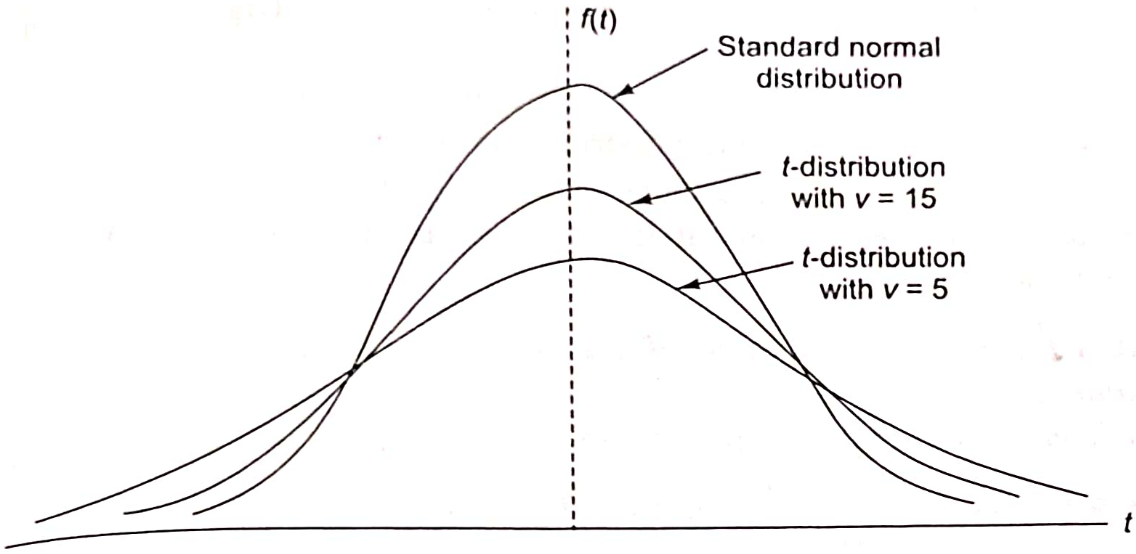

Fig. 10.2

Note on Degree of Freedom

The number of degrees of freedom, usually denoted by the Greek alphabet ν, can be interpreted as the number of useful bits of information generated by a sample of given size for estimating a population parameter. Suppose we wish to find the mean of a sample with observations $x_1, x_2, \ldots, x_n$. We have to use all the 'n' values taken by the variable with full freedom (i.e., without putting any constraint or restriction on them) for computing $\bar{x}$. Hence, $\bar{x}$ is said to have n degrees of freedom.

Suppose we wish to further compute the S.D. 's' of this sample using the formula $s^2 = \frac{1}{n}\sum(x_i - \bar{x})^2$. Though we use the n values $x_1 - \bar{x}, x_2 - \bar{x}, \ldots, x_n - \bar{x}$ for this computation, they do not have 'n' degrees of freedom, as they depend on $\bar{x}$ which has been already calculated and fixed. Since there is one restriction regarding the value of $\bar{x}$, 's' is said to have $(n - 1)$ degrees of freedom.

If we compute another statistic of the sample based on $\bar{x}$ and s, then that statistic will be assumed to have $(n - 2)$ degrees of freedom, and so on.

Thus, the number of independent variates used to compute the test statistic is known as the number of degrees of freedom of that statistic. In general, the number of degrees of freedom is given by $\nu = n - k$, where n is the number of observations in the sample and k is the number of constraints imposed on them or k is the number of values that have been found out and specified by prior calculations.

Critical Values of t and the t-Table

The critical value of t at level of significance α and degrees of freedom ν is given by $P\{|t| > t_\nu(\alpha)\} = \alpha$ for two-tailed test, as in the case of normal distribution and large samples and by $P\{t > t_\nu(\alpha)\} = \alpha$ for the right-tailed test also, as in the case of normal distribution. The critical value of t for a single (right or left) tailed test at LOS 'α' corresponding to ν degrees of freedom is the same as that for a two-tailed test at LOS '2α' corresponding to the same degrees of freedom.

Critical values $t_\nu(\alpha)$ of the t-distribution for two-tailed tests corresponding to a few important levels of significance and a range of values of ν have been published by Prof. R.A. Fisher in the form of a table, called the t-table, which is given in the Appendix.

Test 1 *Test of significance of the difference between sample mean and population mean.*

If $\bar{x}$ is the mean of a sample of size n, drawn from a population $N(\mu, \sigma)$, we have seen that $z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}}$ follows a $N(0, 1)$.

If σ, the S.D. of the population is not known, we have to estimate it using the sample S.D.'s'. From the theory of estimation, it is known that $s\sqrt{\dfrac{n}{n-1}}$ is an unbiased estimate of σ with $(n-1)$ degrees of freedom. When n is large, $\dfrac{n}{n-1} \simeq 1$ and, hence, s was taken as a satisfactory estimate of σ and hence $z = \dfrac{\bar{x} - \mu}{s/\sqrt{n}}$ was assumed to follow a $N(0, 1)$. But when n is small, we cannot use s as an estimate of σ, since

$$\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{\bar{x} - \mu}{s\sqrt{\dfrac{n}{n-1}} \cdot \dfrac{1}{\sqrt{n}}} = \frac{\bar{x} - \mu}{s/\sqrt{n-1}}$$

Now, $\dfrac{\bar{x} - \mu}{s/\sqrt{n-1}}$ does not follow a normal distribution, but follows a t-distribution with number of degrees of freedom $\nu = n - 1$. Hence, $\dfrac{\bar{x} - \mu}{s/\sqrt{n-1}}$ is denoted by t and is taken as the test-statistic.

Sometimes $t = \dfrac{\bar{x} - \mu}{s/\sqrt{n-1}}$ is also taken as $t = \dfrac{\bar{x} - \mu}{S/\sqrt{n}}$,

where $S^2 = \dfrac{1}{n-1}\sum_{r=1}^{n}(x_r - \bar{x})^2$ and is called students 't'.

We shall use only $t = \dfrac{\bar{x} - \mu}{s/\sqrt{n-1}}$, where s is the sample S.D.

We get the value of $t_v(\alpha)$ for the L.O.S. α and $v = n - 1$ from the t-table.

If the calculated value of t satisfies $|t| < t_v(\alpha)$, the null hypothesis H_0 is accepted at L.O.S.'α' otherwise, H_0 is rejected at L.O.S.'α'.

Note *95% confidence interval of m is given by*

$$\left|\frac{\bar{x}-\mu}{s/\sqrt{n-1}}\right| \le t_{0.05}, \quad \text{since } P\left\{\left|\frac{\bar{x}-\mu}{s/\sqrt{n-1}}\right| \le t_{0.05}\right\} = 0.95$$

i.e. by $\bar{x} - t_{0.05}\dfrac{s}{\sqrt{n-1}} \le m \le \bar{x} + t_{0.05} \times \dfrac{s}{\sqrt{n-1}}$, *where $t_{0.05}$ is the 5 per cent critical value of t for n $(= n - 1)$ degrees of freedom for a two-tailed test.*

Test 2 *Test of significance of the difference between means of two small samples drawn from the same normal population.*

In Test (4) for large samples, the test statistic used to test the significance of the difference between the means of two samples from the same normal population was taken as

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}, \text{ which follows a } N(0, 1) \tag{1}$$

If σ is not known, we may assume that $\sigma = \sqrt{\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}}$, when n_1 and n_2 are large, where s_1 and s_2 are the sample S.D.'s. This assumption no longer holds good when n_1 and n_2 are small.

In fact, it is known from the theory of estimation, that an estimate of σ is $\sqrt{\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}}$ with $(n_1 + n_2 - 2)$ degrees of freedom, when n_1 and n_2 are small.

Using this value of σ in (1), the test statistic becomes

$$\frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}\right)\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}},$$

which does not follow a N (0, 1), but follows a t-distribution with $v = (n_1 + n_2 - 2)$ degrees of freedom. Hence, the t-test is applied in this case.

Note *1. If $n_1 = n_2 = n$ and if the samples are independent i.e., the observations in the two samples are not at all related, then the test statistic is given by*

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2 + s_2^2}{n-1}}} \quad \text{with} \quad \nu = 2n - 2 \tag{2}$$

2. If $n_1 = n_2 = n$ and if the pairs of values of x_1 and x_2 are associated in some way (or correlated), the formula (2) for t in Note (1) should not be used. In this case, we shall assume that $H_0 : \bar{d} (= \bar{x} - \bar{y}) = 0$ and test the significance of the difference between $\bar{d}$ and 0, using the test statistic $t = \dfrac{\bar{d}}{s/\sqrt{n-1}}$ with $\nu = n - 1$, where $d_i = x_i - y_i$ $(i = 1, 2, \ldots\ldots, n)$,

$$\bar{d} = \bar{x} - \bar{y}; \text{ and } \quad s = \text{S.D. of } d\text{'s} = \frac{1}{n}\sum_{i=1}^{n}(d_i - \bar{d})^2 .$$

Snedecor's *F*-Distribution

A random variable *F* is said to follow snedecor's *F*-distribution or simply *F*-distribution, if its probability density function is given by

$$f(F) = \frac{(\nu_1/\nu_2)^{\nu_1/2}}{\beta\left(\dfrac{\nu_1}{2}, \dfrac{\nu_2}{2}\right)} \cdot \frac{F^{\frac{\nu_1}{2}-1}}{\left(1 + \dfrac{\nu_1 F}{\nu_2}\right)^{(\nu_1+\nu_2)/2}}, \quad F > 0.$$

Note *(The mathematical variable corresponding to the random variable F is also taken as F.) ν_1 and ν_2 used in $f(F)$ are the degrees of freedom associated with the F-distribution.*

Properties of the F-Distribution

1. The probability curve of the *F*-distribution is roughly sketched in Fig. 10.3.

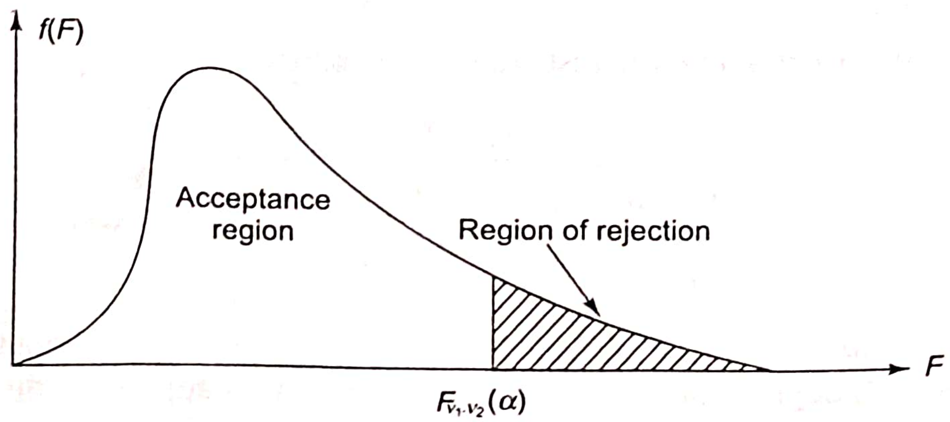

Fig. 10.3

2. The square of the t-variate with n degrees of freedom follows a F-distribution with 1 and n degrees of freedom.

3. The mean of the F-distribution is $\dfrac{v_2}{v_2 - 2}$ $(v_2 > 2)$.

4. The variance of the F-distribution is $\dfrac{2v_2^2\,(v_1 + v_2 - 2)}{v_1\,(v_2 - 2)^2\,(v_2 - 4)}$ $(v_2 > 4)$.

Use of F-Distribution

F-distribution is used to test the equality of the variance of the populations from which two small samples have been drawn.

F-test of significance of the difference between population variances and F-table.

To test the significance of the difference between population variances, we shall first find their estimates, $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ based on the sample variances s_1^2 and s_2^2 and then test their equality. It is known that $\hat{\sigma}_1^2 = \dfrac{n_1 s_1^2}{n_1 - 1}$ with the number of degree of freedom $v_1 = n_1 - 1$ and $\hat{\sigma}_2^2 = \dfrac{n_2 s_2^2}{n_2 - 1}$ with the number of degrees of freedom $v_2 = n_2 - 1$.

It is also known that $F = \dfrac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2}$ follows a F-distribution with v_1 and v_2 degrees of freedom. If $\hat{\sigma}_1^2 = \hat{\sigma}_2^2$, then $F = 1$. Hence, our aim is to find how far any observed value of F can differ from unity due to fluctuations of sampling.

Snedecor has prepared tables that give, for different values of v_1 and v_2, the 5 per cent and 1 per cent critical values of F. An extract from these tables is given in the Appendix. If F denotes the observed (calculated) value and $F_{v_1, v_2}(\alpha)$ denotes the critical (tabulated) value of F at LOS α, then $P\{F > F_{v_1, v_2}(\alpha)\} = \alpha$.

Note *F-test is not a two-tailed test and is always a right-tailed test, since F cannot be negative. Thus, if $F > F_{v_1, v_2}(\alpha)$, then the difference between F and 1, i.e., the difference between $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ is significant at LOS 'α'. In other words, the samples may not be regarded as drawn from the same population or from populations with the same variance. If $F < F_{v_1, v_2}(\alpha)$, the difference is not significant at LOS α.*

1. *We should always make $F > 1$. This is done by taking the larger of the two estimates of σ^2 as σ_1^2 and by assuming that the corresponding degree of freedom as v_1.*
2. *To test if two small samples have been drawn from the same normal population, it is not enough to test if their means differ significantly or not, because in this test we assumed that the two samples came from the same population or from populations with equal variance. So, before applying the t-test for the significance of the difference of two sample means, we should satisfy ourselves about the equality of the population variances by F-test.*

Worked Example 10(B)

Example 1

Tests made on the breaking strength of 10 pieces of a metal wire gave the results: 578, 572, 570, 568, 572, 570, 570, 572, 596 and 584 kg. Test if the mean breaking strength of the wire can be assumed as 577 kg.

Let us first compute sample mean $\bar{x}$ and sample S.D.'s' and then test if $\bar{x}$ differs significantly from the population mean $\mu = 577$.

We take the assumed mean $A = \dfrac{568 + 596}{2} = 582$

$$di = x_i - A$$

$$\therefore \quad x_i = d_i + A$$

$$\therefore \quad \bar{x} = \frac{1}{n}\sum x_i = \frac{1}{n}\sum d_i + A$$

$$= \frac{1}{10} \times (-68) + 582 = 575.2 \text{ (see Table 10.7 given below)}$$

Table 10.7

x_i	$d_i = x_i - A$	d_i^2
578	– 4	16
572	– 10	100
570	– 12	144
568	– 14	196
572	– 10	100
570	– 12	144
570	– 12	144
572	– 10	100
596	14	196
584	2	4
Total	– 68	1144

$$s^2 = \frac{1}{n}\sum d_i^2 - \left(\frac{1}{n}\sum d_i\right)^2$$

$$= \frac{1}{10} \times 1144 - \left(\frac{1}{10} \times -68\right)^2 = 68.16$$

$$\therefore \quad s = 8.26$$

Now,
$$t = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} = \frac{575 \cdot 2 - 577}{8 \cdot 26/\sqrt{9}} = -0.65$$

and
$$\nu = n - 1 = 9.$$
$$H_0 : \bar{x} = \mu \quad \text{and} \quad H_1 : \bar{x} \neq \mu.$$

Let LOS be 5%. Two-tailed test is to be used.

From the t-table, for $\nu = 9$, $t_{5\%} = 2.26$. Since $|t| < t_{5\%}$, the difference between $\bar{x}$ and μ is not significant or H_0 is accepted. ∴ the mean breaking strength of the wire can be assumed as 577 kg at 5% LOS

Example 2

A machinist is expected to make engine parts with axle diameter of 1.75 cm. A random sample of 10 parts shows a mean diameter 1.85 cm. with a S.D. of 0.1 cm. On the basis of this sample, would you say that the work of the machinist is inferior?

$$\bar{x} = 1.85, \quad s = 0.1, \quad n = 10 \quad \text{and} \quad \mu = 1.75.$$
$$H_0 : \bar{x} = \mu; \quad H_1 : \bar{x} \neq \mu$$

Two-tailed test is to be used. Let L.O.S. be 5%

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} = \frac{0 \cdot 10}{0 \cdot 1/\sqrt{9}} = 3 \quad \text{and} \quad \nu = n - 1 = 9.$$

From the t-table, for $\nu = 9$, $t_{0.05} = 2.26$ and $t_{0.01} = 3.25$.

$$\therefore \quad |t| > t_{0.05} \text{ and } |t| < t_{0.01}$$

∴ H_0 is rejected and H_1 is accepted at 5% level, but H_0 is accepted and H_1 is rejected at 1% level. That is, at 5% LOS, the work of the machinist can be assumed to be inferior, but at 1% LOS, the work cannot be assumed to be inferior.

Example 3

A certain injection administered to each of 12 patients resulted in the following increases of blood pressure:

$$5, 2, 8, -1, 3, 0, 6, -2, 1, 5, 0, 4.$$

Can it be concluded that the injection will be, in general, accompanied by an increase in B.P.?

The mean of the sample is given by $\bar{x} = \frac{1}{n}\sum x = \frac{31}{12} = 2.58$

The S.D. 's' of the sample is given by

$$s^2 = \frac{1}{n}\sum x^2 - \left(\frac{1}{n}\sum x\right)^2 = \frac{1}{12} \times 185 - (2.58)^2 = 8.76$$

$$\therefore \quad s = 2.96$$

$$H_0 : \bar{x} = \mu,$$

where $\mu = 0$, i.e. the injection will not result in increase in B.P.

$$H_1 : \bar{x} > \mu$$

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Right-tailed test is to be used. Let L.O.S. be 5%. Now, $t_{5\%}$ for one-tailed test for ($\nu = 11$) = $t_{10\%}$ for two-tailed test for ($\nu = 11$) = 1.80 (from t-table)

Now $$t = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} = \frac{2 \cdot 58 - 0}{2 \cdot 96/\sqrt{11}} = 2.89$$

We see that $|t| > t_{10\%}\ (\nu = 11)$

$\therefore$ H_0 is rejected and H_1 is accepted.

i.e. we may conclude that the injection is accompanied by an increase in B.P.

Example 4

The mean lifetime of a sample of 25 bulbs is found as 1550 hours with an S.D. of 120 hours. The company manufacturing the bulbs claims that the average life of their bulbs is 1600 hours. Is the claim acceptable at 5% level of significance?

$$\bar{x} = 1550, \quad s = 120, \quad n = 25 \quad \text{and} \quad \mu = 1600.$$

$$H_0 : \bar{x} = \mu \quad \text{and} \quad H_1 : \bar{x} < \mu.$$

Left-tailed test is to be used. LOS = 5%

Now, $$t = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} = \frac{-50\sqrt{24}}{120} = -2.04 \text{ and } \quad \nu = 24$$

$t_{5\%}$ for one-tailed test for ($\nu = 24$) = $t_{10\%}$ for two-tailed test for ($\nu = 24$) = 1.71.

We see that $|t| > |t_{10\%}|$

$\therefore$ H_0 is rejected and H_1 is accepted at 5% LOS,

i.e., the claim of the company cannot be accepted at 5% LOS

Example 5

The heights of ten males of a given locality are found to be 175, 168, 155, 170, 152, 170, 175, 160, 160 and 165 cms. Based on this sample, find the 95% confidence limits for the height of males in that locality.

We shall first find the mean $\bar{x}$ and S.D. 's' of the sample, by taking the assumed mean $A = 165$ (Table 10.8).

$$d_i = x_i - A$$

$$\therefore \quad \bar{x} = A + \bar{d}$$

$$= 165 + \frac{1}{10} \times 0 = 165.$$

$$s^2 = \frac{1}{n}\sum d_i^2 - \left(\frac{1}{n}\sum d_i\right)^2$$

$$= \frac{1}{10} \times 578 = 57.8$$

$$\therefore \quad s = 7.6$$

From the t-table,

$$t_{5\%}\ (\nu = 9) = 2.26.$$

Galaxy F34 5G

The 95% confidence limits for μ are

$$\left(\bar{x} - 2 \cdot 26 \frac{s}{\sqrt{n-1}},\ \bar{x} + 2 \cdot 26 \frac{s}{\sqrt{n-1}}\right)$$

i.e.

$$\left(165 - \frac{2 \cdot 26 \times 7 \cdot 6}{\sqrt{9}},\ 165 + \frac{2 \cdot 26 \times 7 \cdot 6}{\sqrt{9}}\right)$$

i.e. (159.3, 170.7)

i.e. the heights of males in the locality are likely to lie within 159.3 cm and 170.7 cm.

Table 10.8

x_i	$d_i = x_i - A$	d_i^2
175	10	100
168	3	9
155	– 10	100
170	5	25
152	– 13	169
170	5	25
175	10	100
160	– 5	25
160	– 5	25
165	0	0
Total	0	578

Example 6

Two independent samples of sizes 8 and 7 contained the following values:

Sample I : 19, 17, 15, 21, 16, 18, 16, 14

Sample II : 15, 14, 15, 19, 15, 18, 16

Is the difference between the sample means significant?

Table 10.9

Sample I			Sample II		
x_1	$d_1 = x_1 - 18$	d_1^2	x_2	$d_2 = x_2 - 16$	d_2^2
19	1	1	15	– 1	1
17	–1	1	14	– 2	4
15	– 3	9	15	– 1	1
21	3	9	19	3	9
16	– 2	4	15	– 1	1
18	0	0	18	2	4
16	– 2	4	16	0	0
14	– 4	16			
Total	– 8	44	Total	0	20

For sample I. $\bar{x}_1 = 18 + \bar{d}_1 = 18 + \dfrac{1}{8}\sum d_1$

$$= 18 + \frac{1}{8} \times (-8) = 17.$$

$$s_1^2 = \frac{1}{n_1}\sum d_1^2 - \left(\frac{1}{n_1}\sum d_1\right)^2$$

$$= \frac{1}{8} \times 44 - \left(\frac{1}{8} \times -8\right)^2 = 4.5$$

$\therefore$ $s_1 = 2.12.$

For sample II, $\bar{x}_2 = 16 + \bar{d}_2 = 16 + \dfrac{1}{7}\sum d_2 = 16.$

$$s_2^2 = \frac{1}{n_2}\sum d_2^2 - \left(\frac{1}{n_2}\sum d_2\right)^2$$

$$= \frac{1}{7} \times 20 - \left(\frac{1}{7} \times 0\right)^2 = 2.857$$

10+10-2
20-2
18

$\therefore$ $s_2 = 1.69$

$H_0 : \bar{x}_1 = \bar{x}_2$ and $H_1 : \bar{x}_1 \neq \bar{x}_2$

Two-tailed test is to be used. Let LOS be 5 %

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}\right)\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}} = \frac{17 - 16}{\sqrt{\left(\dfrac{8 \times 4{\cdot}5 + 7 \times 2{\cdot}857}{13}\right)\left(\dfrac{1}{8} + \dfrac{1}{7}\right)}}$$

$= 0.93$

Also, $\nu = n_1 + n_2 - 2 = 13.$

From the t-table, $t_{5\%}\ (\nu = 13) = 2.16$

Since $|t| < t_{5\%}$, H_0 is accepted and H_1 is rejected.

i.e. the two sample means do not differ significantly at 5% LOS

Example 7

Table 10.10 gives the biological values of protein from cow's milk and buffalo's milk at a certain level. Examine if the average values of protein in the two samples significantly differ.

Table 10.10

Cow's milk (x_1):	1.82	2.02	1.88	1.61	1.81	1.54
Buffalo's milk (x_2):	2.00	1.83	1.86	2.03	2.19	1.88

$$n = 6$$

$$\bar{x}_1 = \frac{1}{6} \times 10.68 = 1.78$$

$$s_1^2 = \frac{1}{6} \times \Sigma x^2{}_1 - (\bar{x}_1)^2 = \frac{1}{6} \times 19.167 - (1.78)^2 = 0.0261$$

$$\bar{x}_2 = \frac{1}{6} \times 11.79 = 1.965$$

$$s_2^2 = \frac{1}{6} \times \Sigma x^2{}_2 - (\bar{x}_2)^2 = \frac{1}{6} \times 23.2599 - (1.965)^2 = 0.0154$$

As the two samples are independent, the test statistic is given by

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2 + s_2^2}{n-1}}}$$

with $v = 2n - 2$ [Refer to Note (2) under Test (2)]

$$t = \frac{1{\cdot}78 - 1{\cdot}965}{\sqrt{\dfrac{0{\cdot}0261 + 0{\cdot}0154}{5}}} = \frac{-0{\cdot}185}{\sqrt{{\cdot}0083}} = -2.03 \text{ and } \quad v = 10.$$

$$H_0 : \bar{x}_1 = \bar{x}_2 \quad \text{and} \quad H_1 : \bar{x}_1 \neq \bar{x}_2.$$

Two-tailed test is to be used. Let LOS be 5%

From t-table, $t_{5\%}\ (v = 10) = 2.23$.

Since $|t| < t_{5\%}\ (v = 10)$, H_0 is accepted.

i.e., the difference between the mean protein values of the two varieties of milk is not significant at 5% level.

Example 8

Samples of two types of electric bulbs were tested for length of life and the following data were obtained.

	Size	Mean	S.D.
Sample I	8	1234 hours	36 hours
Sample II	7	1036 hours	40 hours

Is the difference in the means sufficient to warrant that type I bulbs are superior to type II bulbs?

$$\bar{x}_1 = 1234, \quad s_1 = 36, \quad n_1 = 8; \ \bar{x}_2 = 1036, \quad s_2 = 40, \quad n_2 = 7$$

$$H_0 : \bar{x}_1 = \bar{x}_2\,; \quad H_1 : \bar{x}_1 > \bar{x}_2.$$

Right-tailed test is to be used. Let LOS be 5%.

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}\right)\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}} = \frac{198}{\sqrt{\left(\dfrac{21568}{13}\right)\left(\dfrac{1}{8} + \dfrac{1}{7}\right)}} = \frac{198}{21{\cdot}0807}$$

$= 9.39$

$\nu = n_1 + n_2 - 2 = 13$

$t_{5\%}$ ($\nu = 13$) for one-tailed test $= t_{10\%}$ ($\nu = 13$) for two tailed test $= 1.77$ (from t-table)

Now, $t > t_{10\%}$ ($\nu = 13$)

$\therefore$ H_0 is rejected and H_1 is accepted.

i.e. Type I bulbs may be regarded superior to type II bulbs at 5% LOS.

Example 9

The mean height and the S.D. height of eight randomly chosen soliders are 166.9 cm. and 8.29 cm. respectively. The corresponding values of six randomly chosen sailors are 170.3 cm and 8.50 cm. respectively. Based on this data, can we conclude that soldicers are, in general, shorter than sailors?

$\bar{x}_1 = 166.9, \quad s_1 = 8.29, \quad n_1 = 8; \quad \bar{x}_2 = 170.3, \quad s_2 = 8.50, n_2 = 6.$

$$H_0 : \bar{x}_1 = \bar{x}_2 ; \quad H_1 : \bar{x}_1 < \bar{x}_2.$$

Left-tailed test is to be used. Let LOS be 5%.

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}\right)\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}} = \frac{-3\cdot 4}{\sqrt{\left(\dfrac{983\cdot 29}{12}\right)\left(\dfrac{1}{8} + \dfrac{1}{6}\right)}}$$

$$= -0.695$$

$$\nu = n_1 + n_2 - 2 = 12$$

$t_{5\%}$ ($\nu = 12$) for one-tailed test $= t_{10\%}$ ($\nu = 12$) for two tailed test $= 1.78$ (from t-table)

Now, $|t| < t_{10\%}$ ($\nu = 12$)

$\therefore$ H_0 is accepted and H_1 is rejected.

i.e. based on the given data, we cannot conclude that soldicers are in general, shorter than sailors.

Example 10

The following data relate to the marks obtained by 11 students in two tests, one held at the beginning of a year and the other at the end of the year after intensive coaching. Do the data indicate that the students have benefited by coaching?

Test 1:	19,	23,	16,	24,	17,	18,	20,	18,	21,	19,	20
Test 2:	17,	24,	20,	24,	20,	22,	20,	20,	18,	22,	19

The given data relate to the marks obtained in two tests by the same set of students. Hence, the marks in the two tests can be regarded as correlated and so the t-test for paired values should be used.

Let $\qquad d = x_1 - x_2,$

where x_1, x_2 denote the marks in the two tests.

Thus, the values of d are 2, – 1 – 4, 0, – 3, – 4, 0, – 2, 3, – 3, 1.

$$\Sigma d = -11 \quad \text{and} \quad \Sigma d^2 = 69$$

$$\therefore \quad \bar{d} = \frac{1}{n}\Sigma d = \frac{1}{11} \times -11 = -1$$

$$s^2 = s_d^2 = \frac{1}{n}\Sigma d^2 - (\bar{d})^2 = \frac{1}{11} \times 69 - (-1)^2 = 5.27$$

$$\therefore \quad s = 2.296$$

$H_0 : \bar{d} = 0$, i.e. the students have not benefited by coaching; $H_1 : \bar{d} < 0$ (i.e. $\bar{x}_1 < \bar{x}_2$).

One-tailed test is to be used. Let LOS be 5%

$$t = \frac{\bar{d}}{s/\sqrt{n-1}} = \frac{-1}{2 \cdot 296/\sqrt{10}} = -1.38 \quad \text{and} \quad \nu = 10$$

$t_{5\%}$ ($\nu = 10$) for one-tailed test = $t_{10\%}$ ($\nu = 10$) for two-tailed test = 1.81 (from t-table).

Now, $|t| < t_{10\%}$ ($\nu = 10$)

$\therefore$ H_0 is accepted and H_1 is rejected.

i.e. there is no significant difference between the two sets of marks.

i.e. the students have not benefitted by coaching.

Example 11

A sample of size 13 gave an estimated population variance of 3.0, while another sample of size 15 gave an estimate of 2.5. Could both samples be from populations with the same variance?

$$n_1 = 13, \quad \hat{\sigma}_1^2 = 3.0 \quad \text{and} \quad \nu_1 = 12$$

$$n_2 = 15, \quad \hat{\sigma}_2^2 = 2.5 \quad \text{and} \quad \nu_2 = 14.$$

$H_0 : \hat{\sigma}_1^2 = \hat{\sigma}_2^2$, i.e. The two samples have been drawn from populations with the same variance.

$H_1 : \hat{\sigma}_1^2 \neq \hat{\sigma}_2^2$. Let L.O.S. be 5%

$$F = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \frac{3 \cdot 0}{2 \cdot 5} = 1.2$$

$$\nu_1 = 12 \quad \text{and} \quad \nu_2 = 14.$$

$F_{5\%}$ ($\nu_1 = 12$, $\nu_2 = 14$) = 2.53, from the F-table.

$F < F_{5\%}$. $\therefore$ H_0 is accepted

i.e. the two samples could have come from two normal populations with the same variance.

Example 12

Two samples of sizes nine and eight gave the sums of squares of deviations from their respective means equal to 160 and 91 respectively. Can they be regarded as drawn from the same normal population?

$$n_1 = 9, \quad \Sigma(x_i - \bar{x})^2 = 160, \quad \text{i.e. } n_1 s_1^2 = 160$$

$$n_2 = 8, \quad \Sigma(y_i - \bar{y})^2 = 91, \quad \text{i.e. } n_2 s_2^2 = 91$$

$$\hat{\sigma}_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{1}{8} \times 160 = 20; \quad \hat{\sigma}_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{1}{7} \times 91 = 13$$

Since $\hat{\sigma}_1^2 > \hat{\sigma}_2^2$, $\nu_1 = n_1 - 1 = 8$ and $\nu_2 = n_2 - 1 = 7$

$$H_0 : \hat{\sigma}_1^2 = \hat{\sigma}_2^2 \quad \text{and} \quad H_1 : \hat{\sigma}_1^2 \neq \hat{\sigma}_2^2.$$

Let the LOS be 5%.

$$F = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \frac{20}{13} = 1.54$$

$F_{5\%}$ $(\nu_1 = 8, \nu_2 = 7) = 3.73$, from the F-table.

Since $F < F_{5\%}$, H_0 is accepted.

i.e. the two samples could have come from two normal populations with the same variance.

We cannot say that the samples have come from the same population, as we are unable to test if the means of the samples differ significantly or not.

Example 13

Two independent samples of eight and seven items respectively had the following values of the variable.

Sample 1 :	9,	11,	13,	11,	15,	9,	12,	14
Sample 2 :	10,	12,	10,	14,	9,	8,	10	

Do the two estimates of population variance differ significantly at 5% level of significance?

For the first sample, $\Sigma x_1 = 94$ and $\Sigma x^2{}_1 = 1138$

$$\therefore \quad s_1^2 = \frac{1}{n_1} \sum x_1^2 - \left(\frac{1}{n_1} \sum x_1\right)^2$$

$$= \frac{1}{8} \times 1138 - \left(\frac{1}{8} \times 94\right)^2 = 4.19$$

For the second sample, $\Sigma x_2 = 73$ and $\Sigma x_2{}^2 = 785$

$$\therefore \quad s_2^2 = \frac{1}{n_2} \sum x_2^2 - \left(\frac{1}{n_2} \sum x_2\right)^2$$

$$= \frac{1}{7} \times 785 - \left(\frac{1}{7} \times 73\right)^2 = 3.39$$

$$\hat{\sigma}_1^2 = \frac{n_1}{n_1 - 1} s_1^2 = 4.79 \text{ and } \hat{\sigma}_2^2 = \frac{n_2}{n_2 - 1} s_2^2 = 3.96$$

Since $\hat{\sigma}_1^2 > \hat{\sigma}_2^2$, $\nu_1 = 7$ and $\nu_2 = 6$

$$H_0 : \hat{\sigma}_1^2 = \hat{\sigma}_2^2 \text{ and } H_1 : \hat{\sigma}_1^2 \neq \hat{\sigma}_2^2$$

$$F = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \frac{4.79}{3.96} = 1.21$$

$F_{5\%}$ $(\nu_1 = 7, \nu_2 = 6) = 4.21$, from the F-table. Since $F < F_{5\%}$, H_0 is accepted.

i.e. $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ do not differ significantly at 5% level of significance.

Example 14

Two random samples gave the following data:

	Size	Mean	Variance
Sample I	8	9.6	1.2
Sample II	11	16.5	2.5

Can we conclude that the two samples have been drawn from the same normal population?

Refer to Note (2) under F-test. To conclude that the two samples have been drawn from the same population, we have to check first that the variances of the populations do not differ significantly and then check that the sample means (and hence the population means) do not differ significantly.

$$\hat{\sigma}_1^2 = \frac{8 \times 1{\cdot}2}{7} = 1.37 ; \quad \hat{\sigma}_2^2 = \frac{11 \times 2.5}{10} = 2.75$$

$$F = \frac{\hat{\sigma}_2^2}{\hat{\sigma}_1^2} = \frac{2.75}{1.37} = 2.007 \text{ with degrees of freedom 10 and 7.}$$

From the F-table, $F_{5\%}$ $(10, 7) = 3.64$

If $H_0 : \hat{\sigma}_1^2 = \hat{\sigma}_2^2$ and $H_1 : \hat{\sigma}_1^2 \neq \hat{\sigma}_2^2$,

H_0 is accepted, since $F < F_{5\%}$

i.e. the variances of the populations from which samples are drawn may be regarded as equal.

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}\right)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{-6{\cdot}9}{\sqrt{\left(\frac{9{\cdot}6 + 27{\cdot}5}{17}\right)\left(\frac{1}{8} + \frac{1}{11}\right)}}$$

$$= -\frac{6{\cdot}9}{0{\cdot}6864} = -10.05$$

and $\qquad \nu = n_1 + n_2 - 2 = 17.$

$t_{5\%}\ (\nu = 17) = 2.11$, from the t-table.

If $\quad H_0 : \bar{x}_1 = \bar{x}_2$ and $H_1 : \bar{x}_1 \neq \bar{x}_2$, H_0 is rejected, since $|t| > t_{5\%}$.

i.e. the means of two samples (and so the populations) differ significantly.

$\therefore$ the two samples could not have been drawn from the same normal population.

Example 15

The nicotine contents in two random samples of tobacco are given below.

Sample I :	21	24	25	26	27	
Sample II :	22	27	28	30	31	36.

Can you say that the two samples came from the same population?

$$\bar{x}_1 = \text{Mean of sample I} = \frac{123}{5} = 24.6$$

$$\bar{x}_2 = \text{Mean of sample II} = \frac{174}{6} = 29.0$$

$$s_1^{\,2} = \text{Variance of sample I} = \frac{1}{5}\ \Sigma(x_i - 24.6)^2 = 4.24$$

$$s^2_{\,2} = \text{Variance of sample II} = \frac{1}{6}\ \Sigma(x_i - 29.0)^2 = 18.0$$

$$\hat{\sigma}_1^2 = \frac{5}{4} \times 4.24 = 5.30 \text{ and } \nu = 4;\ \hat{\sigma}_2^2 = \frac{6}{5} \times 18.0 = 21.60 \text{ and } \nu = 5$$

$$H_0 : \hat{\sigma}_1^2 = \hat{\sigma}_2^2;\ H_1 : \hat{\sigma}_1^2 \neq \hat{\sigma}_2^2$$

$$F = \frac{\hat{\sigma}_2^2}{\hat{\sigma}_1^2} = \frac{21{\cdot}60}{5{\cdot}30} = 4.07$$

$F_{5\%}\ (5, 4) = 6.26.$

Since $\quad F < F_{5\%}$, H_0 is accepted.

$\therefore$ the variances of the two populations can be regarded as equal.

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}\right)\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}} = \frac{-4{\cdot}4}{\sqrt{\left(\dfrac{21{\cdot}2 + 108{\cdot}0}{9}\right)\left(\dfrac{1}{5} + \dfrac{1}{6}\right)}}$$

$$= \frac{-4{\cdot}4}{2{\cdot}2943} = -1.92$$

and $\qquad \nu = 9.$

From t-table, $F_{5\%}\ (\nu = 9) = 2.26.$

If $\quad H_0 : \bar{x}_1 = \bar{x}_2$ and $H_1 : \bar{x}_1 \neq \bar{x}_2$, H_0 is accepted

since $\quad |t| < F_{5\%}.$

That is, the means of two samples (and hence the populations) do not differ significantly. Therefore, the two samples could have been drawn from the same normal population.

Exercise 10(B)

Part-A (Short-answer Questions)

1. Write down the probability density of student's t-distribution.
2. State the important properties of the t-distribution.
3. Give any two uses of t-distribution.
4. What do you mean by degrees of freedom?
5. How will you get the critical value of t for a single-tailed test at level of significance a?
6. What is the test statistic used to test the significance of the difference between small sample mean and population mean?
7. Give the 95% confidence interval of the population mean in terms of the mean and S.D. of a small sample.
8. What is the test statistic used to test the significance of the difference between the means of two small samples?
9. Give an estimate of the population variance in terms of variances of two small samples. What is the associated number of degrees of freedom?
10. What is the test statistic used to test the significance of the difference between the means of two small samples of the same size? What is the associated number of degree of freedom?
11. What is the test statistic used to test the significance of the difference between the means of two small samples of the same size, when the sample items are correlated?
12. Write down the probability density function of the F-distribution.
13. State the important properties of the F-distribution.
14. What is the use of F-distribution?
15. Why is the F-distribution associated with two numbers of degrees of freedom?

Part-B

16. A random sample of ten boys had the following I.Q.'s: 70, 120, 110, 101, 88, 83, 95, 98, 107, 100. Does the data support the assumption of a population mean I.Q. of 100? Find a reasonable range in which most of the mean I.Q. values of samples of ten boys lie.
17. A random sample of 16 values from a normal population showed a mean of 103.75 cm. and the sum of the squares of deviations from this mean is equal to 843.75 square cms. Show that the assumption of a mean of 108.75 cm for the population is not reasonable. Obtain 95% and 99% fiducial limits for the same.

18. The mean weekly sales of soap bars in departmental stores is 145 bars per store. After an advertising campaign, the mean weekly sales in 17 stores for a typical week increased to 155 and showed a S.D. of 16. Was the advertising campaign successful?
19. The annual rainfall at a certain place is normally distributed with mean 30. If the rainfalls during the past eight years are 31.1, 30.7, 24.3, 28.1, 27.9, 32.2, 25.4 and 29.1, can we conclude that average rainfall during the past eight years is less than the normal rainfall
20. A machine is expected to produce nails of 7 cm length. A random sample of 10 nails were found to measure : 7.2, 7.3, 7.1, 6.9, 6.8, 6.5, 6.9, 6.8, 7.1 and 7.2 cm. On the basis of this sample, what can be said about the reliability of the machine?
21. A random sample of eight envelopes is taken from the letterbox of a post office and their weights in grams are found to be: 12.2, 11.9, 12.5, 12.3, 11.6, 11.7, 12.2 and 12.4. Find the 95% and 99% confidence limits for the mean weight of the envelopes in the letter box.
22. The average production of 16 workers in a factory was 107 with a S.D. of 9, while 12 workers in another comparable factory had an average production of 111 with a S.D. of 10. Can we say that the production rate of workers in the latter factory is more than that in the former factory?
23. Two different types of drugs A and B were tried on certain patients for increasing weight. 5 persons were given drug A and 7 persons were given drug B. The increase in weight (in kg.) is given below

Drug A :	3.6	5.5	5.9	4.1	1.4		
Drug B :	4.5	3.6	5.5	6.8	2.7	3.6	5.0

Do the two drugs differ significantly with regard to their effect in increasing weight?
24. Samples of 12 foremen in one division and 10 foremen in another division of a factory were selected at random and the following data were obtained (Table 10.11).

Table 10.11

	Division I	*Division II*
Sample size	12	10
Average monthly salary of foremen (Rs.)	5250	4900
S.D. of salary (Rs.)	152	165

Can you conclude that foremen in Division I get more salary than foremen in Division II?
25. Two independent groups of 10 children were tested to find how many digits they could repeat from memory after hearing them. The results are as follows.

Group A:	8	6	5	7	6	8	7	4	5	6
Group B:	10	6	7	8	6	9	7	6	7	7

Is the difference between mean scores of the two groups significant?

26. Table 10.12 gives the marks obtained by 12 students in two tests, one held before coaching and the other after coaching. Does the data indicate that the coaching was effective in improving the performance of students?

Table 10.12

Test I:	55,	60,	65,	75,	49,	25,	18,	30,	35,	54,	61,	72
Test II:	63,	70,	70,	81,	54,	29,	21,	38,	32,	50,	70,	80

27. In one sample of 8 items, the sum of the squares of deviations of the sample values from the sample mean was 84.4 and in another sample of 10 observations it was 102.6. Test whether this difference is significant at 5% level.

28. Two random samples drawn from two normal populations gave the following observations.
Sample I: 20, 16, 26, 27, 23, 22, 18, 24, 25, 19
Sample II: 17, 23, 32, 25, 22, 24, 28, 18, 31, 33, 20, 27
Test whether the two populations have the same variance.

29. Two random samples gave the following results. (Table 10.13)

Table 10.13

Sample No.	*Size*	*Mean*	*Variance*
I	16	440	40
II	25	460	42

Test whether the samples have been drawn from the same normal population.

30. Two random samples gave the following results. (Table 10.14)

Table 10.14

Sample No.	*Size*	*Sum of the values*	*Sum of the squares of values*
I	10	150	2340
II	12	168	2460

Test whether the samples have been drawn from the same normal population.

Chi-Square Distribution

If $X_1, X_2, \ldots, X_n$ are normally distributed independent random variables, then it is known that $(X^2_1 + X_2^2 + \ldots + X^2_n)$ follows a probability distribution, called chi-square (χ^2-distribution) distribution with n degrees of freedom.

The probability density function of the χ^2-distribution is given by

$$f(\chi^2) = \frac{1}{2^{\nu/2}\sqrt{\left(\frac{\nu}{2}\right)}} \cdot (\chi^2)^{\nu/2-1} e^{-\chi^2/2}$$

$0 < \chi^2 < \infty$, where ν is the number of degrees of freedom.

Properties of χ^2-Distribution

1. A rough sketch of the probability curve of the χ^2-distribution for $\nu = 3$ and $\nu = 6$ is given in Fig. 10.4.

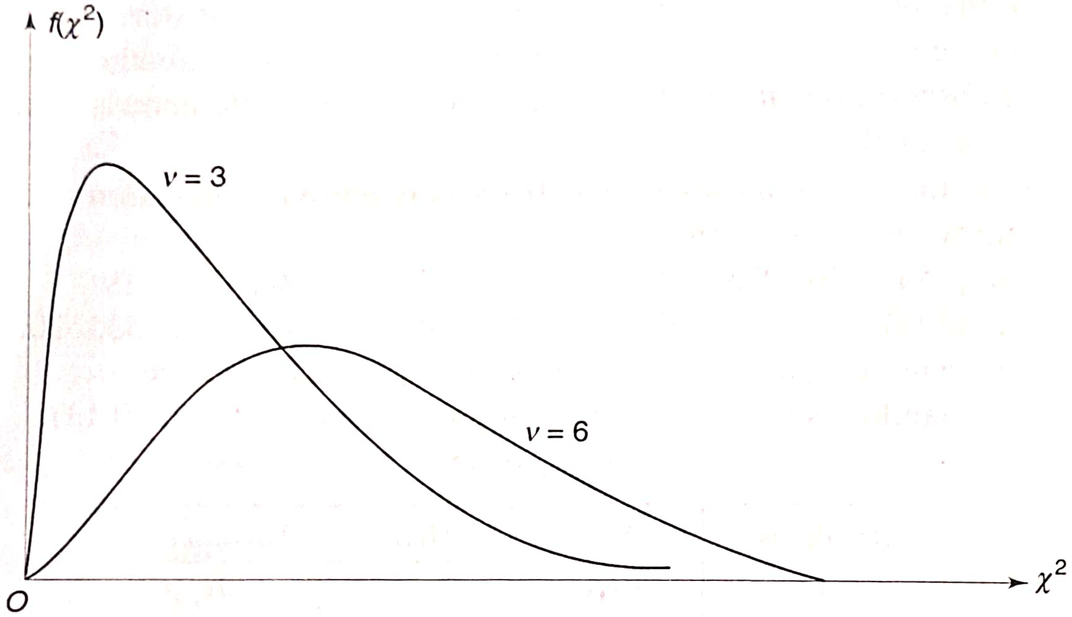

Fig. 10.4

2. As ν becomes smaller and smaller, the curve is skewed more and more to the right. As ν increases, the curve becomes more and more symmetrical.
3. The mean and variance of the χ^2-distribution are ν and 2ν respectively.
4. As n tends to ∞, the χ^2-distribution becomes a normal distribution.

Uses of χ^2-Distribution

1. χ^2-distribution is used to test the goodness of fit. i.e., it is used to judge whether a given sample may be reasonably regarded as *a* simple sample from a certain hypothetical population.
2. It is used to test the independence of attributes. i.e., if a population is known to have two attributes (or traits), then χ^2-distribution is used to test whether the two attributes are associated or independent, based on a sample drawn from the population.

χ^2-Test of Goodness of Fit

On the basis of the hypothesis assumed about the population, we find the expected frequencies $E_i (i = 1, 2, ..., n)$, corresponding to the observed frequencies

$O_i (i = 1, 2, ..., n)$ such that $\sum E_i = \sum O_i$. It is known that $\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$ follows approximately a χ^2-distribution with degrees of freedom equal to the number of independent frequencies. In order to test the goodness of fit, we have to determine how far the differences between O_i and E_i can be attributed to fluctuations of sampling and when we can assert that the differences are large enough to conclude that the sample is not a simple sample from the hypothetical population. In other words, we have to determine how large a value of χ^2 we can get so as to assume that the sample is a simple sample from the hypothetical population.

The critical value of χ^2 for v degrees of freedom at α level of significance, denoted by $\chi_v^2(\alpha)$ is given by

$$P[\chi^2 > \chi^2_v(\alpha)] = \alpha.$$

Critical values of the χ^2-distribution corresponding to a few important levels of significance and a range of values of v are available in the form of a table called χ^2-table, which is given in the Appendix.

If the calculated $\chi^2 < \chi^2_v(\alpha)$, we will accept the null hypothesis H_0 which assumes that the given sample is one drawn from the hypothetical population, i.e. we will conclude that the difference between the observed and expected frequencies is not significant at α % LOS If $\chi^2 > \chi^2_v(\alpha)$, we will reject H_0 and conclude that the difference is significant.

Conditions for the Validity of χ^2-Test

1. The number of observations N in the sample must be reasonably large, say ≥ 50.
2. Individual frequencies must not be too small, i.e. $O_i \geq 10$. In case $O_i < 10$, it is combined with the neighbouring frequencies, so that the combined frequency is ≥ 10.
3. The number of classes n must be neither too small nor too large i.e., $4 \leq n \leq 16$.

χ^2-Test of Independence of Attributes

If the population is known to have two major attributes A and B, then A can be divided into m categories $A_1, A_2, ..., A_m$ and B can be divided into n categories $B_1, B_2, ..., B_n$. Accordingly the members of the population and hence those of the sample can be divided into mn classes. In this case, the sample data may be presented in the form of a matrix containing m rows and n columns and hence mn cells and showing the observed frequencies O_{ij}, in the various cells, where $i = 1, 2, ..., m$ and $j = 1, 2, ... n$. O_{ij} means the number of observed frequencies possessing the attributes A_i and B_j. The matrix or tabular form of the sample data, called an $(m \times n)$ *contingency table* is given below:

Table 10.15

A\B	B_1	B_2	-	B_j	-	B_n	Row Total
A_1	O_{11}	O_{12}	-	O_{1j}	-	O_{1n}	O_{1*}
A_2	O_{21}	O_{22}	-	O_{2j}	-	O_{2n}	O_{2*}
:	-	-	-	-	-	-	-
A_i	O_{i1}	O_{i2}	-	O_{ij}	-	O_{in}	O_{i*}
:	-	-	-	-	-	-	-
A_m	O_{m1}	O_{m2}	O_{mj}	-	-	O_{mn}	O_{m*}
Column Total	O_{*1}	O_{*2}	-	O_{*j}	-	O_{*n}	N

Now, based on the null hypothesis H_0, i.e., the assumption that the two attributes A and B are independent, we compute the expected frequencies E_{ij} for various cells, using the following formula $E_{ij} = \dfrac{O_{i*} \cdot O_{*j}}{N}$, $i = 1, 2, \ldots. m$; and $j = 1, 2, \ldots. n$

i.e. $$E_{ij} = \left\{ \frac{\text{(Total of observed frequencies in the } i^{\text{th}} \text{ row)} \times \text{(total of observed frequencies in the } j^{\text{th}} \text{ column)}}{\text{Total of all cell frequecies}} \right\}$$

Then we compute $\chi^2 = \sum_{i=1}^{m} \sum_{J=1}^{n} \left\{ \dfrac{(O_{ij} - E_{ij})^2}{E_{ij}} \right\}$

The number of degrees of freedom for this χ^2 computed from the $(m \times n)$ contingency table is $v = (m - 1)(n - 1)$.

If $\chi^2 < \chi^2_v(\alpha)$, H_0 is accepted at α % LOS, i.e., the attributes A and B are independent.

If $\chi^2 > \chi^2_v(\alpha)$, H_0 is rejected at α % LOS, i.e., A and B are not independent.

Worked Example 10(C)

Example 1

The following table shows the distribution of digits in the numbers chosen at random from a telephone directory:

Table 10.16

Digit:	0	1	2	3	4	5	6	7	8	9	Total
Frequency:	1026,	1107,	997,	966,	1075,	933,	1107,	972,	964,	853	10,000

Test whether the digits may be taken to occur equally frequently in the directory.

H_0 : The digits occur equally frequently, i.e., they follow a uniform distribution.

Based on H_0, we compute the expected frequencies.

The total number of digits = 10,000.

If the digits occur uniformly, then each digit will occur

$$\frac{10,000}{10} = 1000 \text{ times}$$

$$O_i : 1026, 1107, \ldots, 853$$

$$E_i : 1000, 1000, \ldots, 1000$$

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$$

$$= \frac{1}{1000} \{(26)^2 + (107)^2 + (-3)^2 + (-34)^2 + (75)^2 + (-67)^2 + (107)^2 + (-28)^2 + (-36)^2 + (-147)^2\}$$

$$= 58.542$$

Since ΣE_i was taken equal to ΣO_i (i.e., an information from the sample), $v = n - 1 = 10 - 1 = 9$. From the χ^2–table,

$$\chi^2_{5\%} (n = 9) = 16.919$$

Since $\chi^2 > \chi^2_{5\%}$, H_0 is rejected, i.e., the digits do not occur uniformly in the directory.

Example 2

Table 10.17 gives the number of air-craft accidents that occurred during the various days of a week. Test whether the accidents are uniformly distributed over the week.

Table 10.17

Day:	Mon	Tues	Wed	Thu	Fri	Sat
No. of accidents:	15	19	13	12	16	15

H_0 : Accidents occur uniformly over the week.

Total number of accidents = 90

Based on H_0, the expected number of accidents on any day $= \frac{90}{6} = 15$.

O_i	:	15	19	13	12	16	15
E_i	:	15	15	15	15	15	15

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{1}{15}(0 + 16 + 4 + 9 + 1 + 0) = 2.$$

Since $\quad \sum E_i = \sum O_i, \quad v = 6 - 1 = 5$

From the χ^2-table, $\chi^2_{5\%}$ $(v = 5) = 11.07$.

Since $\chi^2 < \chi^2_{5\%}$, H_0 is accepted.

i.e. accidents may be regarded to occur uniformly over the week.

Example 3

Table 10.18 shows defective articles produced by four machines:

Table 10.18

Machine:	*A*	*B*	*C*	*D*
Production time:	1 hour	1 hour	2 hours	3 hours
No. of detectives:	12	30	63	98

Do the figures indicate a significant difference in the performance of the machines?

H_0 : Production rates of the machines are the same.

Total number of detectives = 203.

Based on H_0, the expected numbers of defectives produced by the machines are

$$E_i: \quad \frac{1}{7} \times 203, \quad \frac{1}{7} \times 203, \quad \frac{2}{7} \times 203, \quad \frac{3}{7} \times 203$$

i.e. $\quad E_i: \quad 29, \quad 29, \quad 58, \quad 87$

$\quad\quad O_i: \quad 12, \quad 30, \quad 63, \quad 98$

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{17^2}{29} + \frac{1^2}{29} + \frac{5^2}{58} + \frac{11^2}{87} = 11.82$$

Since $\quad \sum E_i = \sum O_i, \; v = 4 - 1 = 3$

From the χ^2-table, $\chi^2_{5\%}$ $(v = 3) = 7.815$

Since $\chi^2 > \chi^2_{5\%}$, H_0 is rejected.

i.e., there is significant difference in the performance of machines.

Example 4

The following data represents the monthly sales (in Rs) of a certain retail stores in a leap year. Examine if there is any seasonality in the sales.

6100, 5600, 6350, 6050, 6250, 6200, 6300, 6250, 5800, 6000, 6150 and 6150.

H_0 : There is no seasonability in the sales, i.e. the daily sales are uniform throughout the year or the daily sales follow a uniform distribution.

Based on H_0, we compute the expected frequencies.

The total sales in the year = Rs. 73,200.

If the daily sales are uniform, then the sales on each day

$$= \frac{73,200}{366} = \text{Rs } 200$$

O_i : 6100, 5600, 6350, 6050, 6250, 6200, 6300, 6250, 5800, 6000, 6150, 6150.

Assuming that the months are taken in the usual calendar order, namely, January, February etc. the expected monthly sales are

E_i : 6200, 5800, 6200, 6000, 6200, 6000, 6200, 6200, 6000, 6200, 6000, 6200

Then
$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$$

$$= \frac{(-100)^2}{6200} + \frac{(-200)^2}{5800} + \ldots + \frac{(-50)^2}{6200} = 38.913$$

Since $\sum E_i$ was found as $\sum O_i$ from the sample, $\nu = n - 1 = 12 - 1 = 11$.

From the χ^2-table, $\chi^2_{5\%}$ ($\nu = 11$) = 19.675.

Since $\chi^2 > \chi^2_{5\%}$, H_0 is rejected, i.e., the daily sales are not uniform throughout the year.

Example 5

Theory predicts that the proportion of beans in four groups A, B, C, D should be $9:3:3:1$. In an experiment among 1600 beans, the numbers in the four groups were 882, 313, 287 and 118. Does the experiment support the theory?

H_0 : The experiment supports the theory, i.e., the numbers of beans in the four groups are in the ratio 9 : 3 : 3 : 1

Based on H_0, the expected numbers of beans in the four groups are as follows:

E_i : $\frac{9}{16} \times 1600$, $\frac{3}{16} \times 1600$, $\frac{3}{16} \times 1600$, $\frac{1}{16} \times 1600$

i.e. E_i : 900, 300, 300, 100

O_i : 882, 313, 287, 118

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{18^2}{900} + \frac{13^2}{300} + \frac{13^2}{300} + \frac{18^2}{100} = 4.73$$

Since $\sum E_i = \sum O_i$, $\nu = 4 - 1 = 3$

From the χ^2-table, $\chi^2_{5\%}(\nu = 3) = 7.82$

Since $\chi^2 < \chi^2_{5\%}$, H_0 is accepted.

i.e., the experimental data support the theory.

Example 6

A survey of 320 families with five children each revealed the following distribution:

Table 10.19

No. of boys:	0	1	2	3	4	5
No. of girls:	5	4	3	2	1	0
No. of families:	12	40	88	110	56	14

Is this result consistent with the hypothesis that male and female births are equally probable?

H_0 : Male and female births are equally probable, i.e., P (male birth) $= p = 1/2$ and P (female birth) $= q = 1/2$.

Based on H_0, the probability that a family of 5 children has r male children

$= 5C_r \left(\frac{1}{2}\right)^5$ (by binomial law)

$\therefore$ Expected number of families having r male children $= 320 \times 5\,C_r \times \frac{1}{2^5}$

$= 10 \times 5\,C_r.$

Thus E_i :	10	50	100	100	50	10
and O_i :	12	40	88	110	56	14

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{2^2}{10} + \frac{10^2}{50} + \frac{12^2}{100} + \frac{10^2}{100} + \frac{6^2}{50} + \frac{4^2}{10}$$

$$= 7.16$$

We have used the sample data to get $\sum E_i$ only. The values of p and q have not been found by using the sample data.

$\therefore$ $\quad v = n - 1 = 6 - 1 = 5$ and $\chi^2_{5\%}\ (v = 5) = 11.07$

Since $\quad \chi^2 < \chi^2_{5\%}$, H_0 is accepted.

i.e. male and female births are equally probable.

Example 7

Twelve dice were thrown 4096 times and a throw of six was considered a success. The observed frequencies were as given below.

No. of successes:	0	1	2	3	4	5	6	7 and over
Frequency:	447	1145	1180	796	380	115	25	8

Test whether the dice were unbiased.

H_0 : All the dice were unbiased, i.e., P (getting 6) $= p = \frac{1}{6} \therefore q = \frac{5}{6}$.

Based on H_0, the probability of getting exactly 'r' successes $= 12\,C_r\,p^r\,q^{12-r}$ ($r = 0, 1, 2, \ldots, 12$)

$\therefore$ Expected number of times in which 'r' successes are obtained

$$= 4096 \times 12\,C_r \left(\frac{1}{6}\right)^r \cdot \left(\frac{5}{6}\right)^{12-r}$$

$$= 4096 \times 12\,C_r \times \frac{5^{12-r}}{6^{12}} \quad (r = 0, 1, 2, \ldots, 12)$$

i.e.

$$E_0 = N\,(0 \text{ success}) = N\,(r = 0) = 459.39$$
$$E_1 = N\,(r = 1) = 1102.54$$
$$E_2 = N\,(r = 2) = 1212.80$$
$$E_3 = N\,(r = 3) = 808.53$$
$$E_4 = N\,(r = 4) = 363.84$$
$$E_5 = N\,(r = 5) = 116.43$$
$$E_6 = N\,(r = 6) = 27.17$$
$$E_7 = N\,(r \geq 7) = 5.30$$

Converting E_i's into whole numbers subject to the condition that $\sum E_i = 4096$, we get

E_i:	459,	1103,	1213,	809,	364,	116,	27,	5
O_i:	447,	1145,	1180,	796,	380,	115,	25,	8,

Since E and O corresponding to the last class, i.e., 5 and 8 are less than 10, we combine the last two classes and consider as a single class.

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{12^2}{459} + \frac{42^2}{1103} + \frac{33^2}{1213} + \frac{13^2}{809} + \frac{16^2}{364} + \frac{1^2}{116} + \frac{1^2}{32}$$

$$= 3.76$$

$\nu = n - 1$, since only $\sum E_i$ has been found using the sample data.

$= 7 - 1$ [n must be taken as the number of classes after combination of end classes, if any]

$= 6$

and $\chi^2_{5\%}\ (\nu = 6) = 12.59$, from the χ^2–table. Since $\chi^2 < \chi^2_{5\%}$, H_0 is accepted, i.e. the dice were unbiased.

Example 8

Fit a binomial distribution for the following data and also test the goodness of fit.

x:	0	1	2	3	4	5	6	Total
f:	5	18	28	12	7	6	4	80

To find the binomial distribution $N\,(q + p)^n$, which fits the given data, we require p.

We know that the mean of the binomial distribution is np, from which we can find p. Now the mean of the given distribution is found out and is equated to np.

x:	0	1	2	3	4	5	6	Total
f:	5	18	28	12	7	6	4	80
fx:	0	18	56	36	28	30	24	192

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{192}{80} = 2.4$$

i.e. $np = 2.4$ or $6p = 2.4$, since the maximum value taken by x is n.

$\therefore$ $p = 0.4$ and hence $q = 0.6$

$\therefore$ the expected frequencies are given by the successive terms in the expansion of $80\,(0.6 + 0.4)^6$.

Thus, E_i : 3.73, 14.93, 24.88, 22.12, 11.06, 2.95, 0.33

Converting the E_i's into whole number such that $\sum E_i = \sum O_i = 80$, we get

E_i :	4	15	25	22	11	3	0

Let us now proceed to test the goodness of binomial fit.

O_i :	5	18	28	12	7	6	4

The first class is combined with the second and the last two classes are combined with the last but second class in order to make the expected frequency in each class greater than or equal to 10. Thus, after regrouping, we have,

E_i :	19	25	22	14
O_i :	23	28	12	17

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{4^2}{19} + \frac{3^2}{25} + \frac{10^2}{22} + \frac{3^2}{14} = 6.39$$

We have used the given sample to find

$\sum E_i \, (= \sum O_i)$ and p through its mean.

Hence, $\nu = n - \text{k}$

$= 4 - 2 = 2$

$\chi^2_{5\%}\,(\nu = 2) = 5.99$, from the χ^2-table.

Since $\chi^2 > \chi^2_{5\%}$, H_0, which assumes that the given distribution is approximately a binomial distribution, is rejected, i.e., the binomial fit for the given distribution is not satisfactory.

Example 9

Fit a Poisson distribution for the following distribution and also test the goodness of fit.

x :	0	1	2	3	4	5	Total
f :	142	156	69	27	5	1	400

To find the Poisson distribution whose probability law is

$$P\,(X = r) = \frac{e^{-\lambda}\,\lambda^r}{r!},\ r = 0, 1, 2, \ldots,$$

we require λ, which is the mean of the Poisson distribution.

We find the mean of the given distribution and assume it as λ.

x:	0	1	2	3	4	5	Total
f:	142	156	69	27	5	1	400
fx:	0	156	138	81	20	5	400

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{400}{400} = 1 = \lambda$$

The expected frequencies are given by

$$\frac{N \cdot e^{-\lambda} \lambda^r}{r!} \text{ or } \frac{400 \times e^{-1}}{r!}, \; r = 0, 1, 2, \ldots\infty$$

Thus,

E_i: 147.15, 147.15, 73.58, 24.53, 6.13, 1.23

The values of E_i are very small for $i = 6, 7, \ldots$ and hence neglected.

Converting the values of E_i's into whole numbers such that $\sum E_i = 400$, we get

E_i: 147, 147, 74, 25, 6, 1

Let us now proceed to test the goodness of Poisson fit.

O_i: 142, 156, 69, 27, 5, 1

The last three classes are combined into one, so that the expected frequency in that class may be ≥ 10. Thus, after regrouping, we have

O_i:	142	156	69	33
E_i:	147	147	74	32

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{5^2}{147} + \frac{9^2}{147} + \frac{5^2}{74} + \frac{1^2}{32}$$

$$= 1.09$$

We have used the sample data to find $\sum E_i$ and λ. Hence,

$$n = n - k = 4 - 2 = 2$$

From the χ^2-table, $\chi^2_{5\%}$ $(v = 2) = 5.99$.

Since $\chi^2 < \chi^2_{5\%}$, H_0, which assumes that the given distribution is nearly Poisson, is accepted,

i.e., the Poisson fit for the given distribution is satisfactory.

Example 10

Test the normality of the following distribution by using χ^2-test of goodness of fit

x:	125,	135,	145,	155,	165,	175,	185,	195,	205	Total
f:	1,	1,	14,	22,	25,	19,	13,	3,	2	100

Let us first fit a normal distribution to the given data and then test the goodness of fit.

To fit a normal distribution and hence find the expected frequencies, we require the density function of the normal distribution which involves the mean and S.D. Let us now compute the mean $\bar{x}$ and S.D. 's' of the sample distribution and assume them as μ and σ.

Table 10.20

x	f	$d = \frac{x-165}{10}$	fd	fd^2
125	1	– 4	– 4	16
135	1	– 3	– 3	9
145	14	– 2	– 28	56
155	22	– 1	– 22	22
165	25	0	0	0
175	19	1	19	19
185	13	2	26	52
195	3	3	9	27
205	2	4	8	32
Total	100	–	5	233

$$\bar{x} = A + \frac{c}{N}\sum fd = 165 + \frac{10}{100} \times 5 = 165.5$$

$$s^2 = c^2\left\{\frac{1}{N}\sum fd^2 - \left(\frac{1}{N}\sum fd\right)^2\right\}$$

$$= 10^2\,(2.33 - 0.0025) = 232.75$$

$\therefore$ $$s = 15.26$$

$\therefore$ the density function of the normal distribution which fits the given distribution is $f(x) = \dfrac{1}{15{\cdot}26\sqrt{2\pi}}\, e^{-(x-165{\cdot}5)^2/465{\cdot}5}$.

To find the expected frequency corresponding to a given x, we find $y = f(x)$ and multiply y by the class-width and then by the total frequency N.

Note $y = \dfrac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/2\sigma^2}$. *If we put* $\dfrac{x-\mu}{\sigma} = z$ *then* $y = \dfrac{1}{\sigma}\left\{\dfrac{1}{\sqrt{2\pi}}\, e^{-z^2/2}\right\} = \dfrac{\phi(z)}{\sigma}$, *where* $\phi(z)$ *is density function of the standard normal distribution. Values of* $f(z)$ *are got from the normal table given in the Appendix.*

Table 10.21

x	$y = \dfrac{x - 165\cdot5}{15\cdot26}$	$\phi(z)$	$\dfrac{c\cdot\phi(z)}{\sigma} = \dfrac{10\phi(z)}{15\cdot26}$	*Expected frequency* $= N\,\phi(z)/\sigma$
125	– 2.65	.0119	.0078	0.78
135	– 2.00	.0540	.0354	3.54
145	– 1.34	.1626	.1066	10.66
155	– 0.69	.3144	.2060	20.60
165	– 0.03	.3988	.2613	26.13
175	0.62	.3292	.2157	21.57
185	1.28	.1758	.1152	11.52
195	1.93	.0620	.0406	4.06
205	2.59	.0139	.0091	0.91

Converting the expected frequencies as whole numbers such that $\sum E_i = 100$, we get

E_i: 1, 3, 11, 21, 26, 22, 11, 4, 1

Let us now proceed to test the goodness of normal fit.

Combining the end classes so as to make the individual frequencies greater than 10,

E_i: 15, 21, 26, 22, 16

O_i: 16, 22, 25, 19, 18

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{1^2}{15} + \frac{1^2}{21} + \frac{1^2}{26} + \frac{3^2}{22} + \frac{2^2}{16}$$

$$= 0.82$$

We have used the sample data to find $\sum E_i$, μ and σ. Hence $v = n - k = 5 - 3 = 2$.

From the χ^2-table, $\chi^2_{5\%}\,(v = 2) = 5.99$.

Since $\chi^2 < \chi^2_{5\%}$, H_0, which assumes that the given distribution is nearly normal, is accepted.

i.e. the normal fit for the given distribution is satisfactory.

Example 11

The following data are collected on two characters (Table 10.22).

Table 10.22

	Smokers	*Non-smokers*
Literates :	83	57
Illiterates :	45	68

Based on this, can you say that there is no relation between smoking and literacy?

H_0 : Literacy and smoking habit are independent.

Table 10.23

	Smokers	*Non-smokers*	*Total*
Literates	83	57	140
Illiterates	45	68	113
Total	128	125	253

Table 10.24

O	E	E *(rounded)*	$(O-E)^2/E$
83	$\frac{128 \times 140}{253} = 70.83$	71	$122/71 = 2.03$
57	$\frac{125 \times 140}{253} = 69.17$	69	$122/69 = 2.09$
45	$\frac{128 \times 113}{253} = 57.17$	57	$122/57 = 2.53$
68	$\frac{125 \times 113}{253} = 55.83$	56	$122/56 = 2.57$
			$\chi^2 = 9.22$

$$v = (m-1)(n-1)$$
$$= (2 - 1 (2 - 1) = 1.$$

From the χ^2-table, $\chi^2_{5\%}$ $(v = 1) = 3.84$

Since $\chi^2 > \chi^2_{5\%}$, H_0 is rejected.

i.e. there is some association between literacy and smoking.

Example 12

Prove that the value of χ^2 for the 2×2 contingency table

a	b
c	d

is given by

$$\chi^2 = \frac{N(ad - bc)^2}{(a+b)(c+d)(a+c)(b+d)}, \text{ where } N = a + b + c + d.$$

Hence, compute χ^2 for the 2×2 contingency table given in Example 11.

The value of E corresponding to the cell in which $O = a$ is given by $E = \frac{(a+b)(a+c)}{(a+b+c+d)}$.

$\therefore$ the value of χ^2 corresponding to this cell is given by

$$\chi^2 = \left\{a - \frac{(a+b)(a+c)}{a+b+c+d}\right\}^2 \div \frac{(a+b)(a+c)}{(a+b+c+d)}$$

$$= \frac{\{a(a+b+c+d) - (a+b)(a+c)\}^2}{N(a+b)(a+c)}$$

$$= \frac{(ad-bc)^2}{N(a+b)(a+c)}$$

Similarly, the value of χ^2 are found out for the other three cells.

$\therefore \chi^2$ for the table

$$= \frac{(ad-bc)^2}{N}\left\{\frac{1}{(a+b)(a+c)} + \frac{1}{(a+b)(b+d)} + \frac{1}{(a+c)(c+d)} + \frac{1}{(b+d)(c+d)}\right\}$$

$$= \frac{(ad-bc)^2}{N(a+b)(c+d)(a+c)(b+d)}\{(b+d)(c+d) + (a+c)(c+d) + (a+b)(b+d) + (a+b)(a+c)\}$$

$$= \frac{(ad-bc)^2\left[\sum a^2 + 2\sum ab\right]}{N(a+b)(c+d)(a+c)(b+d)} = \frac{(ad-bc)^2(a+b+c+d)^2}{N(a+b)(c+d)(a+c)(b+d)}$$

$$= \frac{N(ad-bc)^2}{(a+b)(c+d)(a+c)(b+d)} \qquad (1)$$

Using (1) for the contingency table in Example 11,

we get $$\chi^2 = \frac{253(83 \times 68 - 45 \times 57)^2}{140 \times 113 \times 128 \times 125} = 9.48.$$

Example 13

Two batches each of 12 animals are taken for test of inoculation. One batch was inoculated and the other batch was not inoculated. The numbers of dead and surviving animals are given in Table 10.25 in both cases. Can the inoculation be regarded as effective against the disease? Make Yate's correction for continuity of χ^2.

Table 10.25

	Dead	*Survived*	*Total*
Inoculated	2	10	12
Not inoculated	8	4	12
Total	10	14	24

Note on Yate's correction for continuity of χ^2.

The χ^2-table was prepared using the theoretical χ^2-distribution which is continuous, whereas the approximate values of χ^2 that we are using are discrete. To rectify this defect, Yates has shown that, when

$$\chi^2 = \Sigma\left[\frac{\left\{|O_i - E_i| - \frac{1}{2}\right\}^2}{E_i}\right]$$

is used, the χ^2-approximation is improved. Yate's correction is used only when $\nu = 1$ and, hence, for a 2×2 contingency table. It is used only when some cell frequency is small, i.e., less than 5.

In the present problem, two cell frequencies are less than 5 each. Hence, we apply Yate's correction (Table 10.26).

Table 10.26

O	E	$\lvert O - E\rvert - 0.5$	$\{\lvert O - E\rvert - 0.5\}^2/E$
2	$\frac{12 \times 10}{24} = 5$	2.5	$6.25/5 = 1.25$
10	$\frac{12 \times 14}{24} = 7$	2.5	$6.25/7 = 0.89$
8	$\frac{12 \times 10}{24} = 5$	2.5	$6.25/5 = 1.25$
4	$\frac{12 \times 14}{24} = 7$	2.5	$6.25/7 = 0.89$
	$\nu = (2 - 1)(2 - 1) = 1$		$\chi^2 = 4.28$

From the χ^2-table, $\chi^2_{5\%}$ $(\nu = 1) = 3.84$

If H_0 : Inoculation and effect on the diseases are independent, then H_0 is rejected as $\chi^2 > \chi^2_{5\%}$ i.e. Inoculation can be regarded as effective against the disease.

Note *Even if Yate's correction is not made, we would have arrived at the same conclusion.*

Example 14

A total number of 3759 individuals were interviewed in a public opinion survey on a political proposal. Of them, 1872 were men and the rest women. 2257 individuals were in favour of the proposal and 917 were opposed to it. 243 men were undecided and 442 women were opposed to the proposal. Do you justify or contradict the hypothesis that there is no association between sex and attitude?

A careful analysis of the problem results in the following contingency (Table 10.27).

Table 10.27

	Favoured	*Opposed*	*Undecided*	*Total*
Men	1154	475	243	1872
Women	1103	442	342	1887
Total	2257	917	585	3759

H_0 : Sex and attitude are independent, i.e., there is no association between sex and attitude.

Table 10.28

O	E (rounded E)	$(O-E)^2/E$
1154	$\frac{1872 \times 2257}{3759} \simeq 1124$	$302/1124 = 0.80$
475	$\frac{1872 \times 917}{3759} \simeq 457$	$182/457 = 0.71$
243	$\frac{1872 \times 585}{3759} \simeq 291$	$482/291 = 7.92$
1103	$\frac{1887 \times 2257}{3759} \simeq 1133$	$302/1133 = 0.79$
442	$\frac{1887 \times 917}{3759} \simeq 460$	$182/460 = 0.70$
342	$\frac{1887 \times 585}{3759} \simeq 294$	$482/294 = 7.84$
	$\nu = (3-1)(2-1) = 2$	$\chi^2 = 18.76$

From the χ^2-table, $\chi^2_{5\%}\ (\nu = 2) = 5.99$

Since $\chi^2 > \chi^2_{5\%}$, H_0 is rejected.

That is, sex and attitude are not independent i.e. there is some association between sex and attitude.

Example 15

The following table gives for a sample of married women, the level of education and the marriage adjustment score:

Table 10.29

Level of education		*Marriage adjustment*				
		Very low	*low*	*high*	*very high*	*Total*
	College	24	97	62	58	241
	High school	22	28	30	41	121
	Middle school	32	10	11	20	73
	Total	78	135	103	119	435

Can you conclude from the above data that the higher the level of education, the greater is the degree of adjustment in marriage?

H_0 : There is no relation between the level of education and adjustment in marriage.

$$\nu = (4-1)(3-1) = 6$$

$$\chi^2_{5\%}\ (\nu = 6) = 12.59$$

Table 10.30

O	E (rounded)	$(O-E)^2/E$
24	43	$19^2/43 = 8.40$
97	75	$22^2/75 = 6.45$
62	57	$5^2/57 = 0.44$
58	66	$8^2/66 = 0.97$
22	22	$0^2/22 = 0.00$
28	37	$9^2/37 = 2.19$
30	29	$1^2/29 = 0.03$
41	33	$8^2/33 = 1.94$
32	13	$19^2/13 = 27.77$
10	23	$13^2/23 = 7.35$
11	17	$6^2/17 = 2.12$
20	20	$0^2/20 = 0.00$
$\chi^2_{5\%}\ (\nu = 6) = 12.59$		$\chi^2 = 57.66$

Since $\chi^2 > \chi^2_{5\%}$, H_0 is rejected.

That is, the level of education and adjustment in marriage are associated.

Thus, we may conclude that the higher the level of education, the greater is the degree of adjustment in marriage.

Exercise 10(C)

Part-A (Short-answer Questions)

1. Define Chi-square distribution.
2. Write down the probability density function of the χ^2-distribution.
3. State the important properties of χ^2-distribution.
4. Give two uses of χ^2-distribution.
5. What is χ^2-test of goodness of fit?
6. State the conditions under which χ^2-test of goodness of fit is valid.
7. What is χ^2-test of independence of attributes?
8. What is contingency table?
9. Write down the value of χ^2 for a 2×2 contingency table with cell frequencies a, b, c and d.

10. What is Yate's correction for continuity of χ^2?

Part-B

11. In 250 digits from the lottery numbers, the frequencies of the digits were as follows:

Digit:	0	1	2	3	4	5	6	7	8	9
Frequency:	23	25	20	23	23	22	29	25	33	27

Test the hypothesis that the digits were randomly drawn.

12. The following table gives the number of fatal road accidents that occurred during the seven days of the week. Find whether the accidents are uniformly distributed over the week.

Day :	Sun	Mon	Tues	Wed	Thu	Fri	Sat
Number :	8	14	16	12	11	14	9

13. In 120 throws of a single dice, the following distribution of faces are obtained:

Face :	1	2	3	4	5	6
Frequency :	30	25	18	10	22	15

Do these results support the equal probability hypothesis?

14. The number of demands for a particular spare part in a shop was found to vary from day to day. In a sample study, the following information was obtained:

Day :	Mon	Tues	Wed	Thu	Fri	Sat
No. of demands :	124	125	110	120	126	115

Test the hypothesis that the number of parts demanded does not depend on the day of the week.

15. According to genetic theory, children having one parent of blood type *M* and the other of blood type *N* will always be one of the three types-*M*, *MN* and *N* and the average proportions of these types will be 1 : 2 : 1. Out of 300 children, having one *M* parent and one N parent, 30 per cent were found to be of type *M*, 45 per cent of type *MN* and the remaining of type *N*. Test the genetic theory by χ^2-test.

16. 5 coins are tossed 256 times. The number of heads observed is given below. Examine if the coins are true.

No. of heads :	0	1	2	3	4	5
Frequency :	5	35	75	84	45	12

17. 5 dice were thrown 243 times and the numbers of times 1 or 2 was thrown (x) are given below:

x :	0	1	2	3	4	5
Frequency :	30	75	76	47	13	2

Examine if the dice were unbiased.

18. Fit a binomial distribution for the following data and also test the goodness of fit.

x : 0 1 2 3 4

19. Fit a binomial distribution for the following data and also test the goodness of fit

x:	0	1	2	3	4	5	6	7	8	9
f:	3	8	11	15	16	14	12	11	9	1

20. Fit a Poisson distribution for the following distribution and also test the goodness of fit.

x:	0	1	2	3	4	5	6	7
f:	314	335	204	86	29	9	3	0

21. Fit a Poisson distribution for the following distribution and also test the goodness of fit.

x:	0	1	2	3	4
f:	123	59	14	3	1

22. The figures given below are (i) the observed frequencies of a distribution, and (ii) the expected frequencies of the normal distribution having the same mean, S.D. and total frequency as in (i).
(i) 1, 12, 66, 220, 495, 792, 924, 792, 495, 220, 66, 12, 1
(ii) 2, 15, 66, 210, 484, 799, 943, 799, 484, 210, 66, 15, 2
Do you think that the normal distribution provides a good fit to the data?

23. Fit a normal distribution to the following data and find also the goodness of fit.

x:	4	6	8	10	12	14	16	18	20	22	24
f:	1	7	15	22	35	43	38	20	13	5	1

24. In an epidemic of certain disease, 92 children contacted the disease. Of these 41 received no treatment and of these 10 showed after effects. Of the remainder who did receive the treatment, 17 showed after effects. Test the hypothesis that the treatment was not effective.

25. Out of 1660 candidates who appeared for a competitive examination, 422 were successful. Out of these, 256 had attended a coaching class and 150 of them came out successful. Examine whether coaching was effective as regards the success in the examination.

26. In a pre-poll survey, out of 1000 rural voters, 620 favoured A and the rest B. Out of 1000 urban voters, 450 favoured B and the rest A. Examine if the nature of the area is related to voting preference.

27. The following information was obtained in a sample of 40 small general shops:

Table 10.31

	Shops in urban areas	*Shops in rural areas*
Owned by men	17	18
Owned by women	3	12

Can it be said that there are more women owners in rural areas than in urban areas? Use Yate's correction for continuity.

28. A certain drug is claimed to be effective in curing cold. In an experiment on 500 persons with cold, half of them were given the drug and half of them were given the sugar pills. The patients' reaction to the treatment are recorded in the following table.

Table 10.32

	Helped	*Harmed*	*No effect*
Drug	150	30	70
Sugar pills	130	40	80

On the basis of this data, can it be concluded that the drug and sugar pills differ significantly in curing cold?

29. A survey of radio listeners' preference for two types of music under various age groups gave the following information.

Table 10.33

Type of music	*Age group*		
	19-25	*26-35*	*Above 36*
Carnatic music :	80	60	90
Film music :	210	325	44
Indifferent :	16	45	132

Is preference for type of music influenced by age?

30. The table given below shows the results of a survey in which 250 respondents were classified according to levels of education and attitude towards students' agitation in a certain town. Test whether the two criteria of classification are independent.

Table 10.34

Education	*Attitude*		
	Against	*Neutral*	*For*
Middle school:	40	25	5
High school:	40	20	5
College:	30	15	30
Postgraduate:	15	15	10

ANSWERS

Exercise 10(A)

29. $z = 2.83$; significant 30. $z = 4.5$; the coin is not fair
31. $z = 1.79$; claim cannot be supported.

32. No, since z (= 1.40) < z_α (= 1.645).
34. No ; (16.7, 27.0)
35. difference due to sampling fluctuations
36. z = 0.725 ; not significant
37. z = 6.49 ; Yes; Yes.
38. Yes, since z (= 1.80) > $z_{.05}$ (= 1.645)
39. z = 2.56 ; the difference cannot be hidden
40. z (= .104) < $z_{.05}$ (= 1.645) ; the machine has improved
41. z (= 3.17) > $z_{.05}$ (= 1.96) ; difference significant.
42. $|z|$ = 1.02 ; the claim is valid
43. z = 3.12 ; No
44. No, since z = 5 ; 96.8
45. Claim cannot be true as z = 1.89 and $z_{5\%}$ = 1.645
46. Yes, since z (= 1.6) < $z_{5\%}$ (= 1.645)
47. 41
48. (57.2, 58.8) and (57.0, 59.0)
49. No, since z = 2.58
50. No, since z = 8.82
51. No, since z = 1.32
52. Yes, since z (= 4.78) > $z_{1\%}$ (= 2.33)
53. Yes, at 5% level, since $|z|$ (= 1.937) > $z_{5\%}$ (= 1.645) and No, at 1% level, since $|z|$ (= 1.937) < $|z_\alpha|$ (= 2.33)
54. (1.98, 6.02)
55. $H_0 : \mu_1 - \mu_2 = 35$ accepted, as z = 1.90
57. Yes, as z = 1.71
58. Yes, as $|z|$ (= 2.5) > $z_{1\%}$ (= 2.33)
59. Yes, as z = 1.70
60. No, as z = 3.61.

Exercise 10(B)

16. $|t|$ = 0.62 ; Yes ; $83.66 < \mu < 110.74$
17. $|t|$ = 2.67; (99.76, 107.74) and (98.22, 109.29)
18. t = 2.5 ; the campaign was successful
19. $|t|$ = 1.44; $t_{10\%}$ = 1.90 ; $\overline{x}$ is not less than μ.
20. $|t|$ = 0.26; machine is reliable
21. (11.82, 12.38); (11.69, 12.51)
22. t = – 1.067 ; No
23. $|t|$ = 0.424 ; No
24. t = 4.33 ; Yes
25. $|t|$ = 1.85 ; Not significant
26. t = 4.0 ; coaching was effective
27. F = 1.057 ; Not significant
28. The populations have the same variance

29. No, though the difference between variances is not significant, the difference between the mean is significant.
30. Yes, as the differences between the means and between the variance are not significant.

Exercise 10(C)

11. $\chi^2 = 5.2$; $\nu = 9$; digits randomly drawn.
12. $\chi^2 = 4.17$; $\nu = 6$; accidents occur uniformly.
13. $\chi^2 = 12.9$; $\nu = 5$; equal probability hypothesis is refuted.
14. $\chi^2 = 1.68$; $\nu = 5$; the demand does not depend on the day of the week
15. $\chi^2 = 4.5$; $\nu = 2$; genetic theory may be correct
16. $\chi^2 = 3.54$; $\nu = 3$; coins are true.
17. $\chi^2 = 2.76$; $\nu = 4$; dice are unbiased
18. E_i : 7, 26, 37, 24, 6 ; $\chi^2 = 0.06$; $\nu = 1$; binomial fit is good. (E_i: 1, 5, 11, 18, 21, 19, 13, 8, 3, 1)
19. $\chi^2 = 11.30$; $\nu = 4$; binomial fit is not satisfactory;
20. E_i : 301, 362, 217, 87, 26, 6, 1; $\chi^2 = 5.40$; $\nu = 4$; Poisson fit is good.
21. E_i : 121, 61, 15, 3, 0 ; $\chi^2 = 0.99$; $\nu = 1$; Poisson fit is good.
22. $x^2 = 3.84$; $\nu = 8$; Normal fit is good.
23. E_i : 2, 5, 13, 25, 37, 42, 36, 23, 12, 4, 1, ; $\chi^2 = 1.68$; $\nu = 4$; Normal fit is good.
24. $\chi^2 = 0.85$; $\nu = 1$; No association between treatment and after-effect.
25. $\chi^2 = 176.12$; $\nu = 1$; coaching was effective.
26. $\chi^2 = 10.09$; $\nu = 1$; some relation between area and voting preference
27. $\chi^2 = 2.48$; $\nu = 1$; No, as there is no relation between area and sex of ownership
28. $\chi^2 = 3.52$; $\nu = 2$; do not differ significantly
29. $\chi^2 = 373.40$; $\nu = 4$; preference for type of music influenced by age.
30. $\chi^2 = 35.42$; $\nu = 6$; the two criteria are not independent.