Parallel Coordinates: A Tool for Visualizing Multi-Dimensional Geometry

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PARALLEL COORDINATES : A TOOL FOR VISUALIZING MULTI-DIMENSIONAL GEOMETRY

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ABSTRACT

A methodology for **visualizing** analytic and synthetic geometry in R^N is presented. It is based on a system of **parallel coordinates** which induces a non-projective mapping between N-Dimensional and 2-Dimensional sets. Hypersurfaces are represented by their planar images which have some geometrical properties analogous to the properties of the hypersurface that they represent. A *point* ← → *line* duality when $N = 2$ generalizes to lines and hyperplanes enabling the representation of polyhedra in R^N. The representation of a class of convex and non-convex hypersurfaces is discussed together with an algorithm for constructing and *displaying any interior point.* The display shows some local properties of the hypersurface and provides information on the point's *proximity* to the boundary. Applications to Air Traffic Control, Robotics, Computer Vision, Computational Geometry, Statistics, Instrumentation and other areas are discussed.

Introduction

Why is our space three-dimensional? The physicist Paul Ehrenfest found that planetary orbits in N-dimensional Euclidean space R^N are stable if and only if $N = 3$, precluding other dimensional universes from having a long career [18]. Another *dimensionality* result is that rigid bodies rotating in R^N have an axis of rotation if *N is an odd number* [40] showing that even the parity of the dimension number can have far reaching implications! For scientists and others studying multi-variate relations or data sets, understanding the underlying **geometry** of a problem can provide crucial insights into what is possible and what is not. This need to augment our perception, limited as it is by the experience of our three-dimensional habitation, has attracted considerable attention and various **visualization** methodologies have been proposed (see for example [1], [2], [3], [4], [5], [8], [10], [12], [22], [23], [26], [27], [35], [37], [41], [42], [43], [44] and [45]).

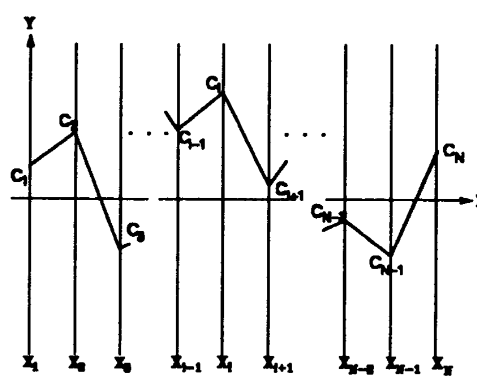

Figure 1: – Parallel axes for R^N.

The polygonal line shown represents the point $C = (c_1, \ldots, c_{i-1}, c_i, c_{i+1}, \ldots, c_N)$.

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Here a methodology based on a multi-dimensional system of *Parallel Coordinates* whose development began in 1978 is described. Preliminary results on some representations and construction algorithms for N-Dimensional Lines and Hyperplanes appeared in 1981 [28]. The results were extended in [29], [33] and [34]. Interest in this method grew in recent applications to Robotics ([14], [20], [21]), Statistics ([6], [13], [26], [24], and [46]), Computational Geometry [32], and other areas (see [16], [17], [25], [31], [38]). In conjunction with the design of the new Air Traffic Control system an algorithm for early conflict detection and resolution[1], [30], was derived using some results on lines in P^N the projective N-space ([33], [34]).

Parallel Coordinates bears a superficial similarity with *Nomography* [9] . It differs from **all** other visualization methodologies in a fundamental way. Parallel Coordinates yield graphical *representations* of Multi-Dimensional **relations rather than just finite point sets**. These representations have a rigorous mathematical structure leading to algorithms for *synthetic* constructions. In short, it is a system for doing and **visualizing** analytic and synthetic multi-dimensional Geometry.

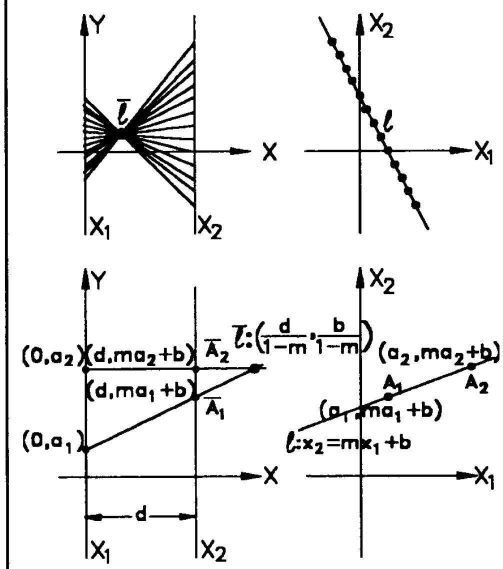

Figure 2: – In the plane parallel coordinates induce duality.

Top part shows the family of segments, representing points on a given line, *intersecting* at a point. Lower part shows the duality line $\ell \leftarrow \rightarrow \bar{\ell}$ point in general.

Definitions and Basic Results in 2-dimensions

On the plane with xy-Cartesian coordinates, and starting on the y-axis, N copies of the real line, labeled x_1 , x_2 , ... , x_N, are placed equidistant and perpendicular to the x-axis. They are the axes of the *parallel coordinate* system for Euclidean N-Dimensional Space R^N all having the same positive orientation as the y-axis --see Figure 1. A point C with coordinates (c_1 , c_2 , ... , c_N) is represented by the polygonal line whose N vertices are at ($i - 1$, c_i) on the x_i-axis for $i = 1$, ... , N. In effect, a 1-1 correspondence between points in R^N and planar polygonal lines with vertices on x_1 , x_2 , ... , x_N is established.

Points are denoted by capitals and lines (or arcs of curves) by lower-case letters and planes by lower case greek letters respectively. In parallel coordinates, the corresponding symbols are shown with a bar superscript (i.e. $\bar{\ell}$ represents the line ℓ, $\bar{P}$ represents the point P etc.).

We have embarked on a program involving the definition via parallel coordinates of a mapping

[1] Patent Pending.

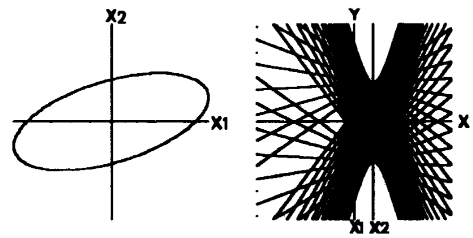

Figure 3: – A (point) curve is mapped into a (line) curve formed by the tangents at each one of its points.

Here an ellipse is mapped into a hyperbola. In general, conics are always mapped to conics.

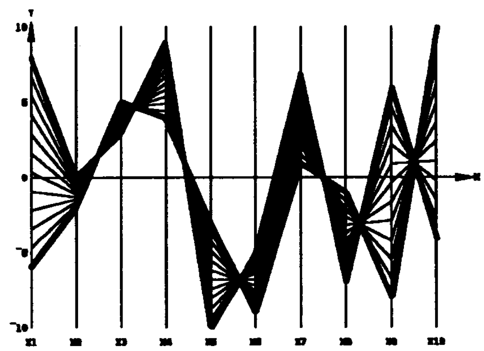

Figure 4: – Interval on a line in P^{10}.

$$I : 2^{P^N} \rightarrow 2^{P^2}.$$

That is, a relation among N-variables (a subset of P^N) is mapped into a subset of P^2, providing a 2-dimensional image whose properties facilitate the visualization and study of the corresponding N-dimensional hypersurface. For a line ℓ in P^N the representation (i.e. $I(\ell)$) is constructed as the *envelope* (see [7]) of the family of polygonal lines in P^2 representing the points of ℓ. In general, a smooth convex hypersurface $S \subset P^N$ is be represented by $I(S)$; the envelope of its tangent hyperplanes at each point which in turn is based on the representation of hyperplanes in P^N.

Points on the plane are represented by segments between the x_1 and x_2-axis and, in fact, by the *line* containing the segment. In Figure 2, the distance between the x_1 and x_2 axes is "d". The line

(1) $$\ell : x_2 = mx_1 + b \, , \; m < \infty$$

is a collection of points A. They are represented by the collection of lines $\bar{A}$ on the xy-plane which when $m \neq 1$ intersect at the *point*:

(2) $$\bar{\ell} : (\frac{d}{1 - m} \, , \, \frac{b}{1 - m}) \, ,$$

given with respect to the xy-Cartesian coordinates. For lines with $m = 1$, we consider xy and x_1x_2 as two copies of the *Projective Plane* [15] so that the line ℓ corresponds to the *ideal point* $\bar{\ell}$ with tangent direction (i.e. slope) b/d. Conversely, in the x_1x_2-projective plane the *ideal point* with slope m is mapped into the *vertical line* at $x = d/(1 - m)$ of the xy-projective plane. Hence, we have a *duality* between points and lines of the

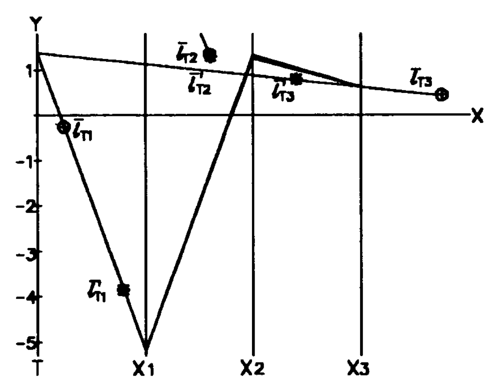

Figure 5: – Nearly Intersecting Lines

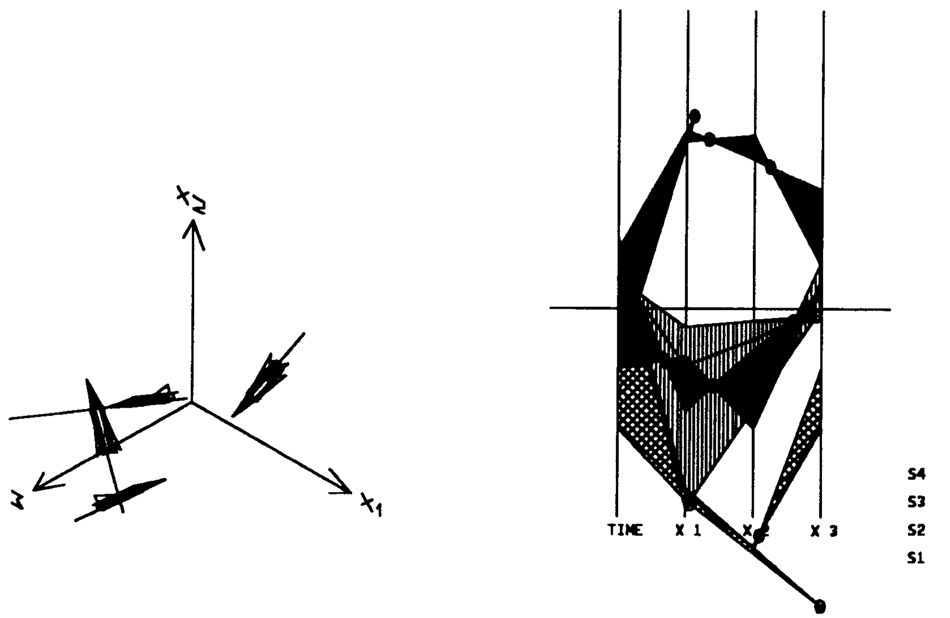

Figure 6: – Aircraft flying straight line paths for specified time periods i.e segments in 4-D (3-D space and time).

Here 3 paths (in 3-D space) intersect yet two aircraft actually collide. That is only two aircraft go through the same point **at the same time**. Such a situation is impossible to discern in 3-D though it is easily found (both algorithmically and visually) in parallel coordinates.

projective plane. This duality as expressed by means of homogeneous coordinates is a linear transformation a--*correlation*--between the line coordinates $[m, -1, b]$ of ℓ and the point coordinates $(d, b, 1-m)$ of $\bar{\ell}$. By means of the correlation C_A the collection of points on a curve is mapped into a collection of lines which can be considered as tangents to another curve (*"line curve"*--see Figure 3). On the plane conics map into conics but this property is more general. Sections of a double cone whose base is *any* bounded convex, let's call them *gconics*, map into other such sections corresponding, to bounded and unbounded convex sets as well *hstars* (generalized hyperbolas). This yields a new duality between bounded and unbounded convex sets and hstars as well as a duality between Convex Merge (Convex Union) and Intersection. Based on these results efficient new algorithms for Convex Hull construction, and the Convex Merge and Intersection of Convex sets were derived (see [32]). For non-convex curves there is a duality between cusps and inflection points.

Multi-Dimensional Lines

A line ℓ in P^N can, after possibly some manipulation, be described in terms of $N-1$ linearly independent equations of the form:

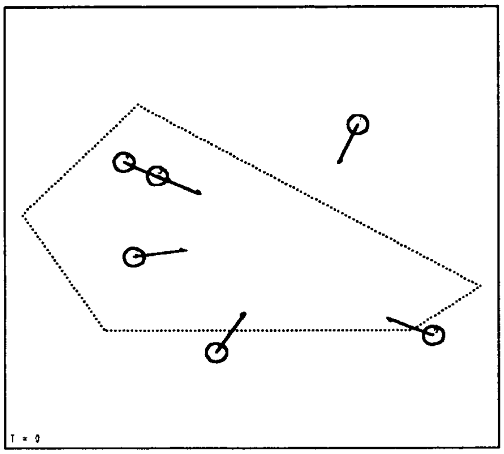

Figure 7: – Six aircraft flying at the same altitude.

Initial positions (T = 0 sec.) and circles centered at each aircraft with radius 2.5 nm (separation standard) are shown to scale. Arrows represent the velocity vectors.

(3) $$\ell_{ij} : x_i = m_{ij}x_j + b_{ij} \ .$$

There is no loss of generality in assuming that $i < j$. The gist is that ℓ is represented in parallel coordinates by $N - 1$ points of the $xy - plane$, whose homogeneous coordinates are:

(4) $$\bar{\ell}_{ij} : ((i - 1)(1 - m_{ij}) + i - j, b_{ij}, 1 - m_{ij}) \ ,$$

indexed by two subscripts so that the point $\bar{\ell}_{ij}$ represents the linear relation ℓ_{ij}. When $N = 2$ this reduces to the previous *point* $\leftarrow \rightarrow$ *line* duality.

In Figure 4 the points $\bar{\ell}$ correspond to the *adjacent variables* parametrization where $j = i + 1$. Another common parametrization involves a *base variable* such as $i = 1$ and $j \in \{2, 3, ..., N\}$. A polygonal line $\bar{P}$ represents a point $P \in \ell$ if and only if the $\bar{\ell}_{ij} \in \bar{P}_{ij}$, where $\bar{P}_{ij}$ is the ij portion (i.e. line) representing the projection of P on the $x_i x_j$-plane of $\bar{P}$. A word of caution on the construction of points of ℓ based on this representation; all equations in (3) where $m_{ij} = 0$ need to be grouped together and the corresponding portion of $\bar{P}$ constructed first or last. From (4) any other point say $\bar{\ell}_{rs}$, representing the linear relationship between x_r and x_s can be obtained by a simple construction. In fact, if $P, Q \in \ell$ $\bar{\ell}_{rs} = \bar{P}^{rs} \cap \bar{Q}^{rs}$ where P^{rs} is the projection of P on the 2-plane $x_r x_s$. Also it turns out that the points $\bar{\ell}_{ij}$, $\bar{\ell}_{jk}$ and $\bar{\ell}_{ki}$ are always collinear.

There is an easy construction algorithm to find the intersection of two lines if it exists. The same construction can be used to discover *proximity* when the two lines are nearly intersecting or better yet when their minimum distance is less than some specified bound. This can be computed very efficiently *without need for computing the actual distance*. The construction is shown in Figure 5.

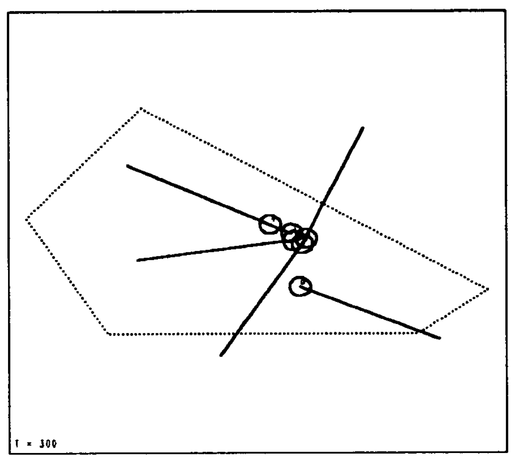

Figure 8: – Conflicts among the six aircraft.

A conflict occurs when the separation between any two aircraft is less than 5 nm (i.e. two circles intersect). Here several conflicts occur within the first 5 minutes. The time elapsed in seconds is indicated in the lower left hand corner.

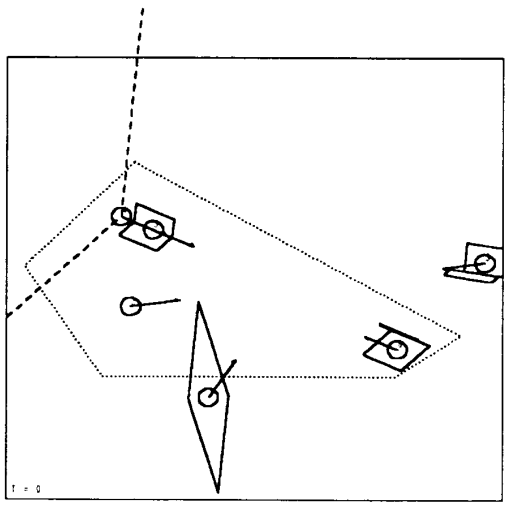

Figure 9: – Conflict Parallelograms

Here potential conflicts between aircraft 4 (where the two dashed lines intersect) and remaining aircraft are analyzed. An aircraft with the same velocity as 4 in order to avoid entering a circle must necessarily avoid the parallelogram(s) associated with that circle.

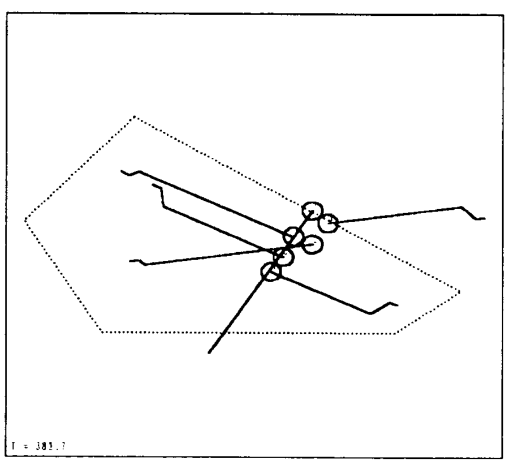

Figure 10: – Three Pairs of Tangent circles

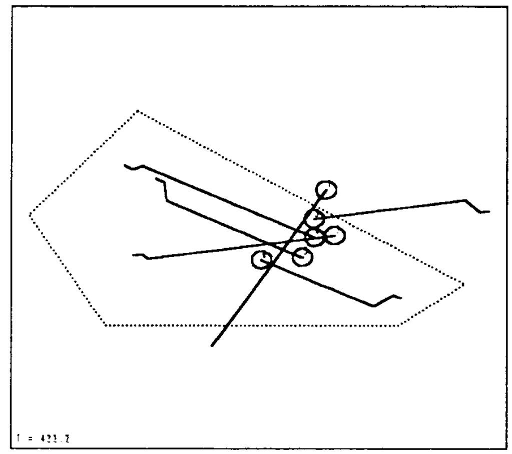

Figure 11: – Triple Tangency

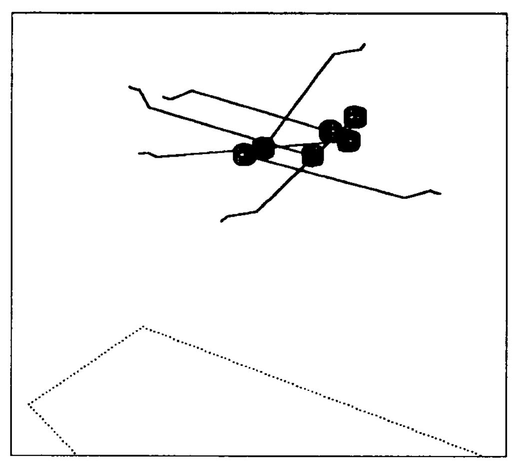

Figure 12: – Resolution in 3 dimensions

Aircraft are at different altitudes and protected space is cylindrical

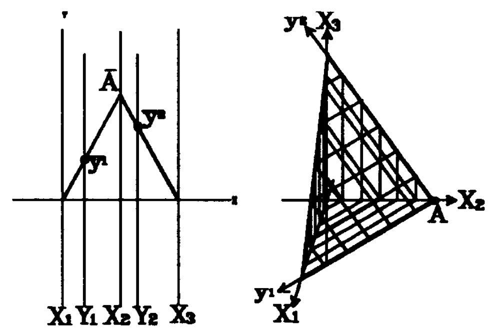

Figure 13: – In P^3 planes are represented by two vertical lines and a polygonal line $\bar{A}$

This generalizes to N-dimensions where hyperplanes are represented by N-1 parallel lines and a polygonal line representing one of its points.

There is an important application to *Air Traffic Control* based on this representation. The trajectory of an aircraft is a function of four variables, time T and three space variables say x_1, x_2, x_3. A straight line trajectory for constant vector velocity can be represented by three stationary points say T:1, 1:2, 2:3. The time axis can be thought of as a "clock". At any given time T, the position of the aircraft is found by selecting the value of T on the T-axis. Such a representation also clarifies the problem of understanding collision as the intersection in space **and** time (i.e. both aircraft must be at the same position at the same time; a 4-D intersection) as contrasted to passing through the same position but at different times. An example is shown in Figure 6.

Based on this methodology an algorithm for **Conflict Detection and Resolution** was derived and tested on some complex scenarios provided by the FAA in conjunction with the design of the new Air Traffic Control system [36]. By way of illustration a resolution not involving altitude changes (i.e. in 2 dimensions) for one of these scenarios (scenario 8) is presented. The initial position and constant velocity of the six aircraft involved is shown in Figure 7. As can be seen from Figure 8 several conflicts arise. An expert Air Traffic controller resolved the conflicts with four (4) aircraft remaining at the same altitude and changing the altitude of two. It is usually more desirable to avoid altitude changes. Our algorithm was able to resolve all conflicts without altitude changes. There are two key steps.

The first involves the computation of certain parallelograms which enclose the circular safety zones centered on each aircraft. For example in Figure 9 the parallelograms with respect to aircraft 4 and

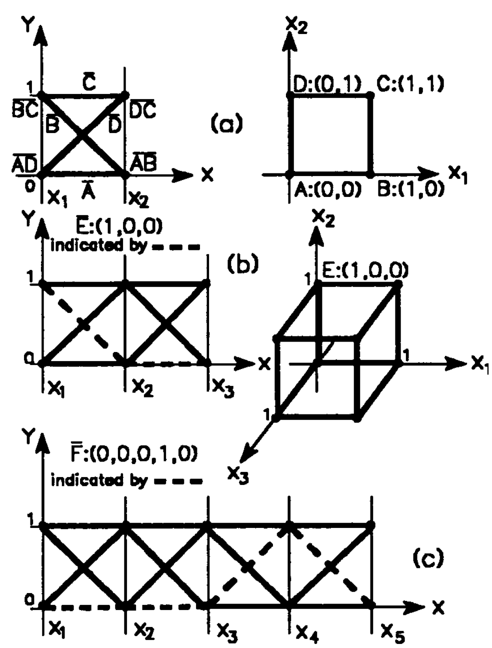

Figure 14: – Hypercube Representation in Parallel Coordinates.

Graph of square (a), cube in 3-D (b) and Cube in 5-D (c) all having unit side.

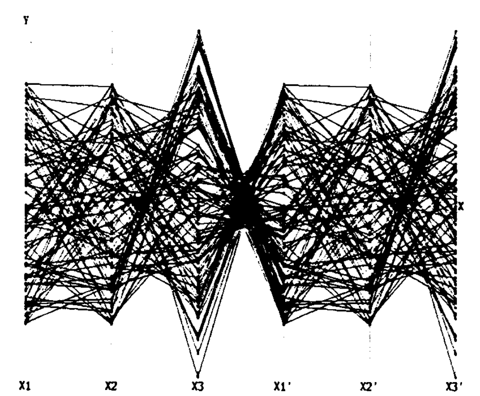

Figure 15: – On the first 3 axes a set of coplanar points is shown

The points are not equally spaced with respect to any of the variables.

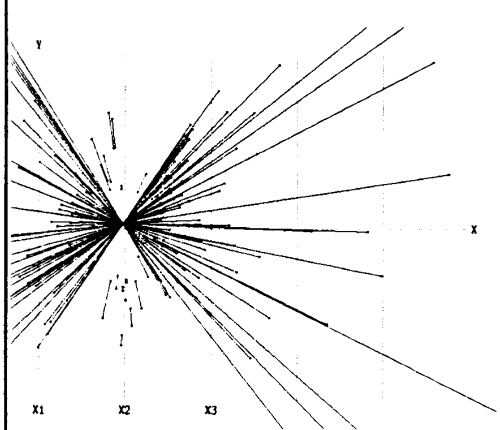

Figure 16: – Coplanarity

The two planar points representing lines on 3-dimensional plane form a pencil of lines.

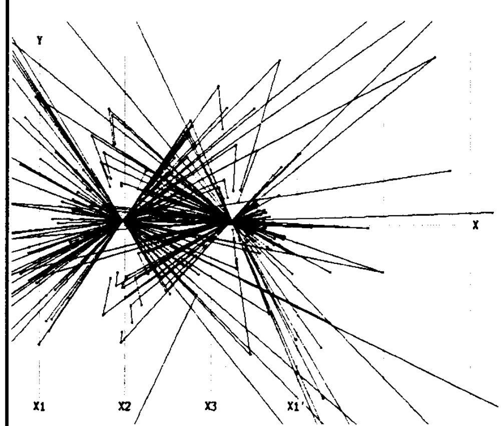

Figure 17: – A plane in 3-dimensions is represented by 2 points.

the remaining aircraft are shown. There may be none, one or two adjacent parallelograms with each circle and they are computed from the relative velocity of that aircraft with respect to aircraft 4. Any aircraft flying with the same velocity as 4 in order to avoid the circles **must avoid entering the parallelograms**. The second step involves the representation of lines in parallel coordinates by points so that the trajectory information for each circle is represented by intervals on the same line. Aircraft 4, in this case, is in conflict with another aircraft if and only if the points representing the trajectory of 4 are interior to the intervals associated with that aircraft. The interval associated with aircraft k consists of the points representing all paths parallel to the path of 4 which enter the parallelogram of k. The closest conflict-free path for 4, parallel to the original trajectory, is represented by one of the end-points of the union of these intervals. If that end-point lies on yet another interval I_j the union is enlarged by including I_j until eventually a free end-point is found. Unfortunately, it is not possible to provide a short exposition on this interesting application and a proper one would take us too far afield. The

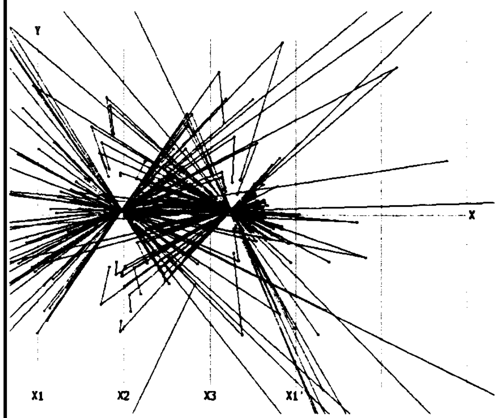

Figure 18: – Near coplanarity in "noisy" data that is points on a slab rather than a plane.

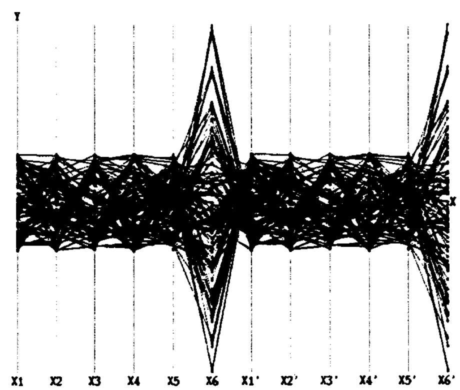

Figure 19: – Points (0-flats) on a "noisy" hyperplane (i.e. slab) in 6-dimensions

interested reader is referred to two papers currently in preparation [11] and [30].

Let's see how this works out. To resolve the conflicts, at $T = 0$ five aircraft start maneuvers which eventually lead each of them to a new course parallel, and within 5nm of the original one, where they continue flying at their previous constant velocity with the circles exquisitely missing each other. This is shown in Figure 10. There and also in Figure 11 are seen instances of tangent circles. This is because the algorithm minimally disturbs the aircraft from their original paths. In Figure 12 the resolution is applied in 3-D to cylindrical protected airspaces. The generalization of this approach to more general Motion Planning in Robotics is currently being investigated [20], [21].

Hyperplanes

Since a line in 2-dimensions is represented by a point whose x-coordinate is $1 / (1 - m)$ the set of parallel lines is represented by a vertical line at $x = 1 / (1 - m)$ in parallel coordinates. On a hyperplane in P^N a coordinate system with $N - 1$ axes (and the N-1 sets of parallel lines to each axis) is represented in parallel coordinates

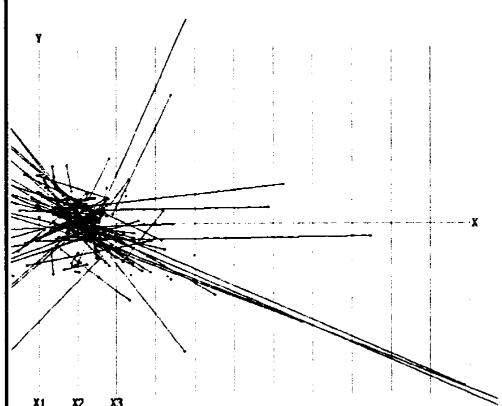

Figure 20: – Portions of Lines (1-flats) formed from the previous points. No structure is apparent

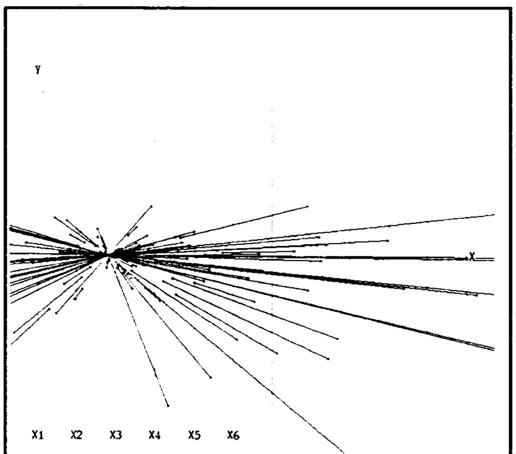

Figure 21: – Portions of 4-Flats formed from the 3-flats. The points are very near to a hyperplane (5-Flat) in 6-dimensions.

by $N-1$ vertical lines. Hence, a hyperplane in P^N is represented by $N-1$ vertical lines and a polygonal line $\bar{P}$ representing one of its points Figure 13. A feel for the power of the representation can be gained from Figure 14 from which, with a bit of practice, the vertices, edges and faces, and their interrelationship, of the hypercube can be recognized. A large set of equally spaced points on a hyperplane will show intersections along the vertical lines. There is an alternate elegant hyperplane representation due to Eickemeyer [19]. To motivate it let us look a set of 3-dimensional points (consider at first only the portion in the first 3 || axes) shown in Figure 15 which happen to be on a plane π and are not equally spaced. So the vertical lines revealing co-planarity of the previous representation are not seen. In fact, no discernible "structure" seems evident. Now let us form *lines* on the plane π from pairs of these points. These lines are represented by a pair of planar points indexed with two subscripts (i.e. two out of the three *collinear* points $\bar{\ell}_{12}$, $\bar{\ell}_{23}$ $\bar{\ell}_{13}$). The collection of lines through the $\bar{\ell}$ points for every line ℓ on any plane turns out to form a pencil of lines at a point $\bar{\pi}_{123}$ as shown on Figure

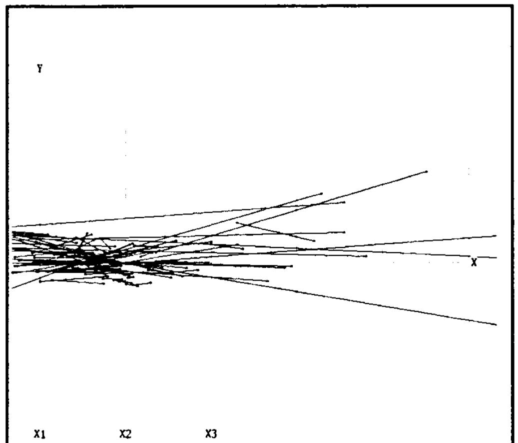

Figure 22: – Portion of lines formed from a point set.

16. In fact this turns out to be a necessary and sufficient condition for coplanarity! Since a planar point has two independent coordinates more information is needed to identify the *specific* plane.

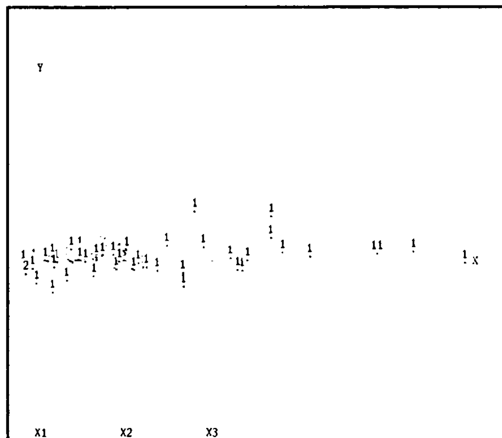

Figure 23: – Histogram showing number of intersections per position

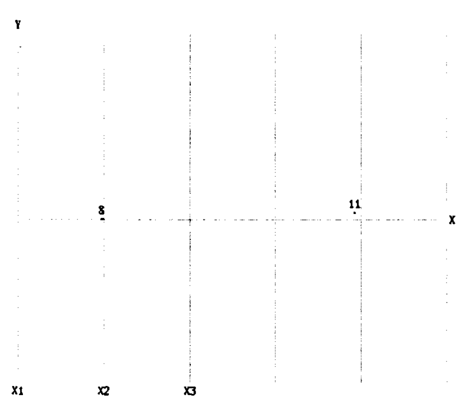

Figure 24: – Only two points have more than one intersection.

This shows that the original points belonged to two different planes. To completely identify the planes this process may be repeated with another axes permutation to obtain two more points.

This may be done by taking another copy of the 3 parallel coordinate axes as shown on the right-hand-side of Figure 15 (hence the six axes seen) and represent the original points on π with respect to these axes which effectively have a different horizontal distance between them than the original set, repeating the process and getting another pencil of lines as shown Figure 17 at another point say $\bar{\pi}_{231'}$ which turns out to have the same y-coordinate as $\bar{\pi}_{123}$ so the coordinates of the two points have enough information between them to identify the plane π that they represent. It is rather remarkable that a plane in 3-dimensions, a two dimensional object technically called **2-flat**, can be represented in terms of two pencils of lines which in turn represent its lines (1-dimensional objects called **1-flats**) in terms of **two** planar pencil of lines. Recall, that a 1-flat (i.e. in 2-dimensions) is represented by **one** pencil of lines. To distinguish the situation where points represent lines we label them with two indices and points representing

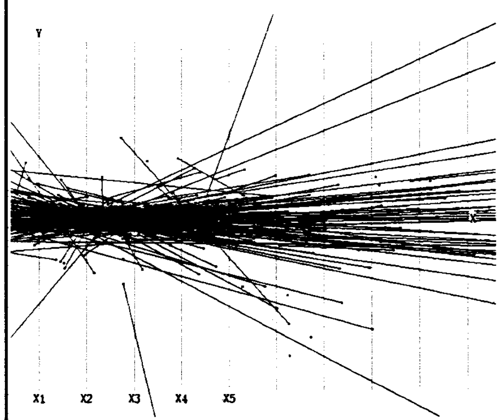

Figure 25: – Portions of 4-flats formed from a set a points in 5-D

planes are labeled with **three** indices. Altogether then in order to discover coplanarity of *points* it is recommended that *lines* be formed by appropriately sampling the points by pairs. Then coplanarity can be found *independently of the co-*

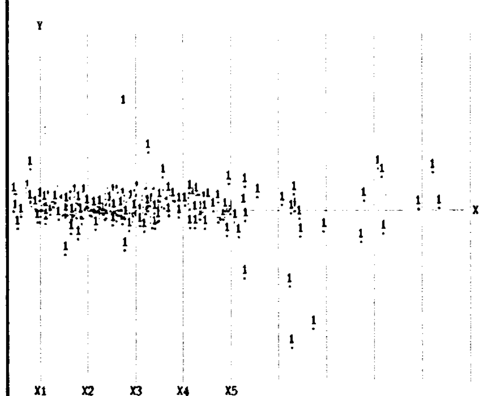

Figure 26: – Number of intersections per position

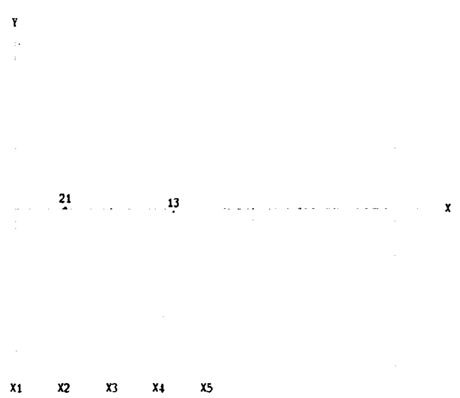

Figure 27: – Two "hits" with more than 1 intersection. Points are on two hyperplanes.

ordinate system and VIEWPOINT of the observer which does not enter in this representation.

It is not difficult to think of situations where discovering coplanarity is of interest, i.e. Computer Vision, Statistics, CAD/CAM etc.. In such cases one would expect the data to have errors so it is important to see if this representation is numerically stable with "noisy" data. So a set of points which is within "20% of coplanarity" in the sense that given:

$$(5) \qquad \pi,\ \pi_n : c_1x_1 + c_2x_2 + c_3x_3 = c + c_n$$

points on the nominal plane π are found with c_n while points on the "noisy" plane π_n are computed by allowing the variation $|c_n| = \pm .2|c|$ from the nominal. Repeating the process by forming lines again two pencils of lines are clearly seen, Figure 18, but now with some variation. It would be interesting to study the density of the clusters of intersecting lines picking the two points or better the two positions where the highest number of lines intersect as the *"best planar fit"* to the data in some reasonable sense.

This representation, as well as the numerical stability, turns out to be completely general. As an example consider the 6-dimensional points with the same kind of variation from some nominal hyperplane shown in the first six axes in Figure 19 Repeating the process as is shown in Figure 20, for the 1-flats by joining successive pairs of points representing linear objects of increasing dimensionality the near coplanarity is discovered from the pencil of lines formed by the points representing the 4-dimensional hyperplanes (4-flats) contained in the hyperplane as shown in Figure 21. Repeating this process for the successive axes permutations 234561', 34561'2', 4561'2'3', 561'2'3'4' yields 4 more such points. The position of these five points, labeled by the six indices corresponding to their axes permutation completely identify the nominal hyperplane. In general, in N-dimensions an (N-1)-flat is represented by N-1 pencils of lines (i.e. points) representing (N-2)-flats each point labeled with N indices.

Points belonging to several planes may also be "separated" in this way. Starting with 100 points in 3-dimensions 100 lines are formed by sampling pairs of points as shown in Figure 22. The number of intersecting lines per pixel is recorded, Figure 23, with only two positions having more that 1 "hits", Figure 24 indicating that the original point set was on *two* different planes. This separation method works also in higher dimensions, see Figure 25, where 4-flats are formed from a set of 500 points in 5-dimensions, the number of intersections per position is shown on Figure 26 with two positions having more than one Figure 27 showing that the points are on two separate hyperplanes.

In general a p -- *flat* in N-space, specified by the (N-p) linearly independent equations,

$$(6) \qquad \begin{gathered} \pi_{i_1 \dots i_{(p+1)}} : \sum_{k=1}^{p+1} c_{i_k} x_{i_k} = c_o \ , \\ 1 \le i_k \le N\ ,\ i_j \ne i_r,\ j \ne r \end{gathered}$$

is represented by the $(N-p)p$ points, given in homogeneous coordinates, by

$$(7)\qquad \bar{\pi}_{i_1^* \ldots i_{(p+1)}^*} : \left(\sum_{k=1}^{p+1} d_{i_k^*} c_{i_k},\ c_o,\ \sum_{k=1}^{p+1} c_{i_k}\right) ,$$

where the axes $x_1, \ldots, x_N$ are followed by the axes $x_1', \ldots, x_{N'}$, d_i is distance from the y-axis to the x_i-axis and

$$(8)\qquad i_k^* = \begin{cases} i_k & j \le k \\ i'_k & j > k \end{cases} .$$

This is a very recent result and its ramifications are just being explored. The identification of several hyperplanes and in fact the subject of Geometric Modeling of polyhedra in N-dimensions with the very exciting prospect of visualizing polyhedra in any dimensions in terms of connected indexed points are now open to investigation.

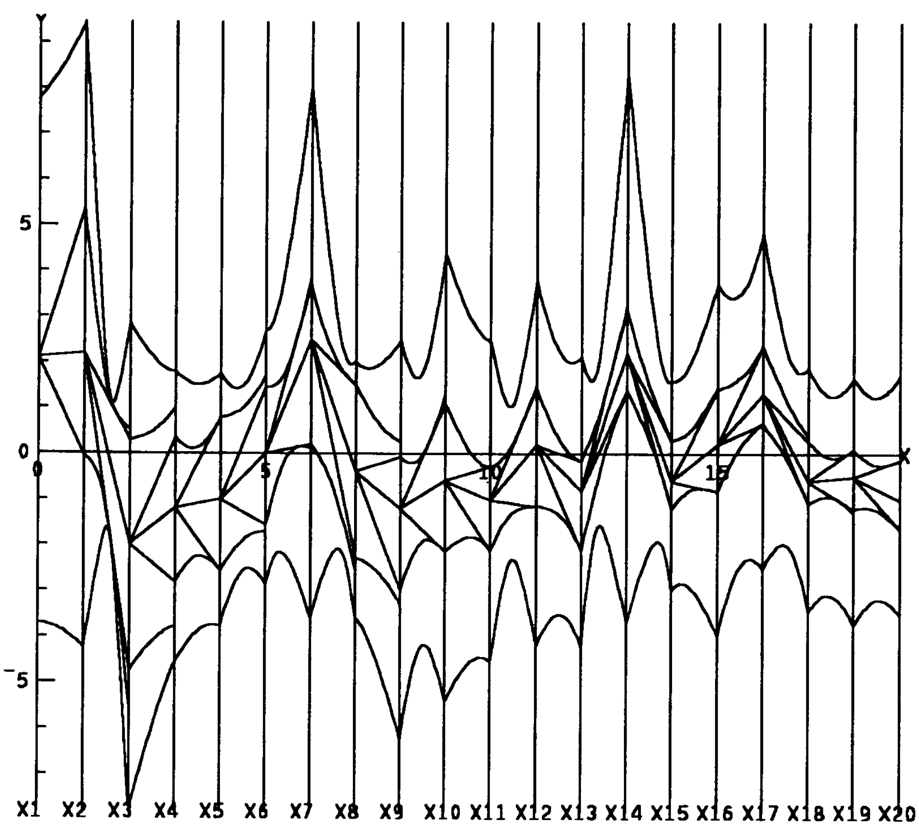

Figure 28: – Finding a Feasible Point for a Process Represented by a Hypersurface

Hypersurfaces, Process Control & Instrumentation

Ideally one would like to have a display where *any pair of variables would be adjacent* and since there are $N(N-1)/2$ pairs it seems like $O(N^2)$ displays with the variables differently ordered are needed. Fortunately this is not the case. Wegman's Theorem [6] states that for $N = 2n$ exactly n "well chosen" displays suffice while for $N = 2n + 1$, $n + 1$ displays suffice. A complete graph of N vertices can be obtained as a union of that many "properly chosen" spanning trees each representing a particular display. Let the ith-node stand for x_i and the edge ij for the pair $x_i x_j$. Each of the aforementioned spanning trees represents a particular display containing all the variables appearing in the order indicated by the edges. Altogether then $O(N)$ displays contain all possible pairs of variables adjacent to each other since all the corresponding spanning trees yield the complete graph. For example, for $N = 6$ the permutations 126354, 231465, 342516 contain every possible pair (independent of the order) of *adjacent* subscripts from 1 to 6.

It was mentioned earlier that a convex hypersurface $S \subset P^N$ is represented by ${}^{4}(S)$ the envelope of is tangent hypersurfaces. This needs to be done stagewise one dimension at a time as for the representation of N-flats in the previous section. This is currently being investigated. In the meantime, below a simplified hypersurface representation and its possible usefulness is illustrated.

Here a general convex hypersurface is represented by computing the *envelope* of the collection of polygonal lines representing its points. The representation of a convex hypersurface in P^{20} is shown in Figure 28. It corresponds to a particular (nonlinear) relation which for our purpose describes a *process* involving 20 parameters labeled $x_1, \ldots, x_{20}$. An N-tuple is a *feasible point for the process* if and only if it is *interior to the hypersurface*. This is the *geometrical equivalent* of the the statement that a particular combination of parameter values satisfies all the constraints imposed on the process. There is an algorithm for finding interior points to such surfaces which is illustrated in the figure.

The interval on the x_i-axis between the upper and lower portions of the envelope indicates the range of parameter x_i. When a value for $x_1 = a_1$ is chosen in the available range of x_1 the number of variables in the relation is reduced by one (i.e. from 20 to 19 variables in our example). Again with reference to Figure 28 from the fixed value of x_1 *tangents* to the upper and lower portions of the envelope are drawn (here the points of tangency are not seen since they lie outside the strip between the x_1 and x_2-axes). The envelope of the 19-dimensional hypersurface is then obtained from the (description) of the original hypersurface with the value $x_1 = a_1$. The *restricted available range* of x_2, due to the constraint of fixing the value of one of the variables, is the interval on the x_2-axis between the upper and lower tangents. A polygonal line is shown which always lies between the intermediary envelopes. It represents an interior point to the hypersurface and *all* interior points can be found in this way.

When the value of a variable is fixed, the *complete* envelope of the 19-dimensional surface may be drawn showing at a glance not only the current value of any of the parameters (in this case x_1) but, unlike a conventional "instrument panel", the *current available range of all the parameters*; this is due to the interrelationship among them. There are other salient features of the display. Note that for the parameters x_{13}, x_{14}, x_{15} the available ranges are the narrowest. This indicates that the feasible point is *closest to the boundary* with respect to these *critical* parameters. Hence, where there is need for rapid decisions in view of a lot of information (e.g. as is piloting very fast aircraft), significant information reduction can be achieved by controlling the critical parameters whose *priority is decided by their proximity to the boundary*. This also brings out the possibility of displaying or highlighting only the information desired (possibly specific subsystems) at any given time. When a point is not feasible the display shows the corresponding polygonal line crossing some of the intermediate envelopes. It is then evident *which parameters must be changed and by how much* in order to bring the polygonal line inside the envelope and the process in control.

Though necessarily brief it is hoped that we have conveyed the basics of a new geometrical tool for representing and visualizing multivariate relations.

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