

# **Stat330: Homework #2**

Due on January 29, 2014 at 3:10pm

*Mr. Lanker Section A*

**Josh Davis**

## Problem 1

Binomial theorem.

### Part A

What is the coefficient of  $x^5y^3$ ?

### Solution

$$c \cdot x^5y^3 = \binom{8}{3}x^5y^3 = \frac{8!}{5!3!}x^5y^3 = 56x^5y^3$$

### Part B

What is the coefficient of  $x^3y^5$ ?

### Solution

$$c \cdot x^3y^5 = \binom{8}{5}x^3y^5 = \frac{8!}{5!3!}x^3y^5 = 56x^3y^5$$

---

## Problem 2

Simpson's Paradox.

### Part A

A black urn contains 5 red and 6 green balls and a white urn contains 3 red and 4 green balls. You are allowed to choose an urn and then choose a ball at random from the urn. Assume that choosing any ball in the urn is equally likely. If you choose a red ball you get a prize. Which urn should you choose to draw from?

### Solution

For the black urn, the probability of getting a red is:

$$\Pr(\text{black}) = \frac{5 \text{ red}}{5 + 6 \text{ total}} = \frac{5}{11} \approx .455$$

$$\Pr(\text{white}) = \frac{3 \text{ red}}{3 + 4 \text{ total}} = \frac{3}{7} \approx .429$$

Therefore we have a higher probability of getting a red from the black urn and should choose from that one.

### Part B

A second black urn contains 6 red and 3 green balls and a second white urn contains 9 red and 5 green balls.

### Solution

$$\Pr(\text{2nd black}) = \frac{6 \text{ red}}{6 + 3 \text{ total}} = \frac{6}{9} \approx .667$$

$$\Pr(\text{2nd white}) = \frac{9 \text{ red}}{9 + 5 \text{ total}} = \frac{9}{14} \approx .643$$

The 2nd black has a higher probability than the 2nd white so we should choose from that one.

### Part C

The two black urns are combined as well as the two white urns.

### Solution

$$\Pr(\text{combined black}) = \frac{5 + 6 \text{ red}}{11 + 9 \text{ total}} = \frac{11}{20} \approx .55$$

$$\Pr(\text{combined white}) = \frac{3 + 9 \text{ red}}{7 + 14 \text{ total}} = \frac{12}{21} \approx .571$$

The combined white urn has a higher probability of getting a red so we should choose that one.

---

## Problem 3

A group of 4 undergraduates and 5 graduates are available to fill four student government posts.

### Part A

Find the probability that at least three undergraduates will be among the four chosen assuming the selection was random?

### Solution

$$\frac{\text{at least 3 undergrads}}{\text{total possibilities}} = \frac{\text{exactly 3} + \text{4 undergrads}}{\text{total possibilities}} = \frac{\binom{4}{3}\binom{5}{1} + \binom{4}{4}\binom{5}{0}}{\binom{9}{4}} = \frac{4 \cdot 5 + 1 \cdot 1}{126} = \frac{21}{126} \approx .167$$

### Part B

What is the probability of no undergraduates being selected again assuming the selection was random?

### Solution

$$\frac{\text{no undergrads}}{\text{total possibilities}} = \frac{\binom{4}{0}\binom{5}{4}}{\binom{9}{4}} = \frac{1 \cdot 1}{126} \approx .008$$

### Part C

Later we will talk about a  $p$ -value, which represents the probability happening under certain model assumptions. Here our model assumptions are that the selection process was purely random. It is common that if the  $p$ -value is very low, say less than .05, we reject that our model assumptions are true. Our  $p$ -value here is the result from part b. What does this value say about the model statement that the selections were made at random?

### Solution

Since our result from part b was  $\approx .008$  and since that is lower than .05, this says that our assumption that the selection process was purely random was false.

## Problem 4

Suppose that there are two events  $A$  and  $B$  in  $\Omega$  such that  $\Pr(A) > 0$  and  $\Pr(B) > 0$ . Further, suppose that the two events are mutually exclusive. Can they also be independent? Explain.

### Solution

We know that  $\Pr(A) > 0$ ,  $\Pr(B) > 0$ , and that  $A$  and  $B$  are mutually exclusive which means that  $A \cap B = \{\}$ . So that probability of  $\Pr(A \cap B) = 0$  is by definition.

The definition of independent events means that  $\Pr(A \cap B) = \Pr(A) \Pr(B)$ . This cannot be true because we know that  $\Pr(A \cap B) = 0$  and thus one of the probabilities of the events must be zero, which goes against the assumption. So no, they cannot also be independent.

---

## Problem 5

Let  $A$  be the event that processor 1 is in use and  $B$  be the event that processor 2 is in use.

### Part A

Calculate  $\Pr(A \mid B)$ .

### Solution

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{.7}{.8} = .875$$

### Part B

What is  $\Pr(A)$ ?

### Solution

$$\Pr(A) = .75$$

### Part C

Are  $A$  and  $B$  independent?

### Solution

No,  $A$  and  $B$  are not independent because  $\Pr(A \mid B) \neq \Pr(A)$  as our previous parts show.

### Part D

Calculate  $\Pr(B \mid A)$ .

### Solution

$$\Pr(B \mid A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{.7}{.75} \approx .933$$

### Part E

Show that  $\Pr(A \mid B) \Pr(B) = \Pr(B \mid A) \Pr(A)$ .

### Solution

*Proof.*

$$\begin{aligned} \Pr(A \mid B) \Pr(B) &= \frac{\Pr(A \cap B)}{\Pr(B)} \Pr(B) \\ &= \Pr(A \cap B) \\ &= \frac{\Pr(A \cap B)}{\Pr(A)} \Pr(A) \\ &= \Pr(B \mid A) \Pr(A) \end{aligned}$$

□

Using our values:

$$\begin{aligned} \Pr(A \mid B) \Pr(B) &= (.875)(.8) \approx .700 \\ \Pr(B \mid A) \Pr(A) &= (.933)(.75) \approx .700 \end{aligned}$$

---

## Problem 6

Suppose that  $A$  and  $B$  are independent events with  $\Pr(A) > 0$  and  $\Pr(B) > 0$ . Show that  $\bar{A}$  and  $\bar{B}$  are also independent.

### Solution

*Proof.* Suppose that  $A$  and  $B$  are events with probability  $\Pr(A) > 0$  and  $\Pr(B) > 0$ . And that the events are also independent, which means that  $\Pr(A \cap B) = \Pr(A) \Pr(B)$ .

We want to show that  $\bar{A}$  and  $B$  are independent or in other words, we want to show that  $\Pr(\bar{A} \cap B) = \Pr(\bar{A}) \Pr(B)$ .

Since  $B = (A \cap B) \cup (\bar{A} \cap B)$ , then  $\Pr(B) = \Pr(A \cap B) + \Pr(\bar{A} \cap B)$ . Which gives us:

$$\begin{aligned} \Pr(\bar{A} \cap B) &= \Pr(B) - \Pr(A \cap B) \\ &= \Pr(B) - \Pr(A) \Pr(B) \text{ because of independence} \\ &= \Pr(B)(1 - \Pr(A)) \\ &= \Pr(B) \Pr(\bar{A}) \end{aligned}$$

Thus by assuming that  $A$  and  $B$  are independent, we have shown that  $\bar{A}$  and  $B$  are also independent, or  $\Pr(\bar{A} \cap B) = \Pr(\bar{A}) \Pr(B)$ . Thus the proof is complete.  $\square$

## Problem 7

Firing based on positive drug use. Was she fired unjustly?

Let  $A$  be the event that the person is a drug user and let  $B$  be the event that the person tested positive for drugs. We have the following equations:

$$\Pr(A) = .02 \quad \Pr(\bar{A}) = .08 \quad \Pr(B | A) = .99 \quad \Pr(B | \bar{A}) = .01$$

### Solution

We want to determine what the probability that she was not a drug user given that she tested positive, or  $\Pr(\bar{A} | B)$ . By using Bayes Rule and the law of total probability we get:

$$\begin{aligned} \Pr(\bar{A} | B) &= \frac{\Pr(B | \bar{A}) \Pr(\bar{A})}{\Pr(B)} \\ &= \frac{\Pr(B | \bar{A}) \Pr(\bar{A})}{\Pr(B | \bar{A}) \Pr(\bar{A}) + \Pr(B | A) \Pr(A)} \\ &= \frac{(.01)(.08)}{(.01)(.08) + (.99)(.02)} \\ &\approx 4\% \end{aligned}$$

Considering the possibility that the test is accurate is 99% and the possibility that she is not a drug user given she tests positive (a false positive) is 4%, I'd say she doesn't have a legitimate claim.

## Problem 8

In a bag there are two standard dice. One of the dice is fair and the other is loaded so that the die rolls a 6 exactly half of the time. You reach into the bag, randomly select a dice, and roll a 6. What is the probability that you selected the loaded die?

### Solution

Let  $A$  be the event that we pick the loaded die and let  $B$  be the event that we roll a 6.

Since there are two dice in the bag,  $\Pr(A) = .5$  because we are drawing at random. We also know that  $\Pr(B | A) = .5$  and  $\Pr(B | \bar{A}) = .167$  based on the problem description.

We want to find the probability that we picked the loaded die given that we rolled a six which is  $\Pr(A | B)$ .

Using Bayes Rule and the law of total probability:

$$\begin{aligned}\Pr(A | B) &= \frac{\Pr(B | A) \Pr(A)}{\Pr(B)} \\ &= \frac{\Pr(B | A) \Pr(A)}{\Pr(B | A) \Pr(A) + \Pr(B | \bar{A}) \Pr(\bar{A})} \\ &= \frac{(.5)(.5)}{(.5)(.5) + (.167)(1 - .5)} \\ &\approx .750\end{aligned}$$