Stat330: Homework #4

Due on February 12, 2014 at 3:10pm

Mr. Lanker Section A

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Problem 1

The *discrete uniform* distribution defined on $[a, b]$, where a and b are integers, is the probability distribution where any number between a and b is equally likely to occur. Let X be the discrete uniform variable on $[a, b]$. The pmf of X is:

$$p_X(x) = \begin{cases} \frac{1}{b-a+1} & x \in \{a, a+1, \ldots, b-1, b\} \\ 0 & \text{any other } x \end{cases}$$

Part A

Find $\mathrm{E}[X]$.

Solution

The expected value is the sum of the probabilities multiplied by the x value:

$$\begin{aligned}
\mathrm{E}[X] &= \sum_{x=a}^{b} x \cdot p_X(x) \\
&= \sum_{x=a}^{b} x \cdot \frac{1}{b-a+1} \\
&= \frac{1}{b-a+1} \sum_{x=a}^{b} x \\
&= \frac{1}{b-a+1} \left[\sum_{x=a}^{b} x - \sum_{x=1}^{a-1} x \right] \\
&= \frac{1}{b-a+1} \left[\frac{b}{2}(b+1) - \frac{a-1}{2}(a-1+1) \right] \\
&= \frac{1}{b-a+1} \cdot \frac{1}{2} \cdot \left[b^2 + b - a^2 + a \right] \\
&= \frac{1}{b-a+1} \cdot \frac{1}{2} \cdot \left[(b^2 - a^2) + (b+a) \right] \\
&= \frac{1}{b-a+1} \cdot \frac{1}{2} \cdot [(b+a)(b-a) + (b+a)] \\
&= \frac{1}{b-a+1} \cdot \frac{1}{2} \cdot [(b+a)(b-a+1)] \\
&= \frac{b+a}{2}
\end{aligned}$$

Part B

Suppose c is an integer between a and b. How does $p_X(c)$ change as a and b move further away from each other?

Solution

Since since it is a uniform distribution, the value of $p_X(c)$ will be dependent on the distance of a and b which is given as $\frac{1}{b-a+1}$. Thus as a and b get farther from each other, that is making the denominator of the fraction bigger, thus the probability decreases.

The highest the probability can be is when $a = b$ which would make the fraction be $1/1$. Thus, again, it makes sense that it decreases as a and b grow apart.

Problem 2

Considering the game of *Monopoly*, answer the following questions.

Let X be the outcome of a single roll of the dice, with "success" considered to be rolling doubles and "failure" rolling anything else. Then $X \sim \text{Bernoulli}(p)$.

Part A

What is p?

Solution

The probability of rolling doubles, or $n/|\Omega|$ where the size of $\Omega = 36$, the total possibilities and $n = 6$ because a 1 can only match a 1, and a 2 can only match a 2, and so on. Thus $p = 6/36 = 1/6 \approx .167$.

Part B

What are $\text{E}[X]$ and $\text{Var}[X]$?

Solution

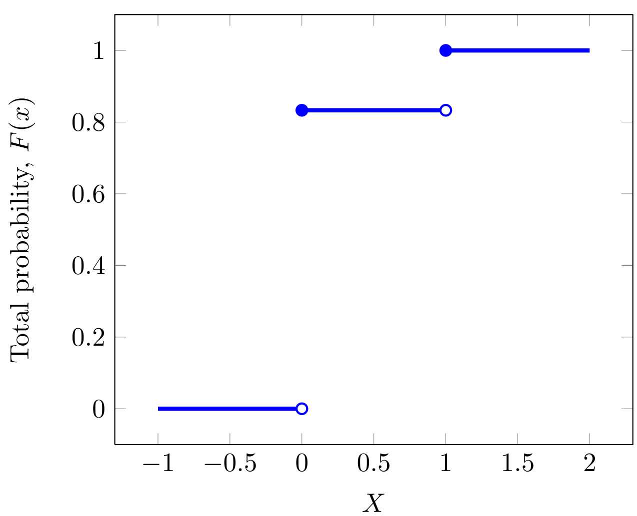
Since our distribution is a Bernoulli distribution, the expectation of such a distribution is $\text{E}[X] = p$, so with probability $p \approx .167$, the expectation is thus $\text{E}[X] = .167$.

Similarly, the variance for a Bernoulli distribution is defined as $\text{Var}[X] = pq$ with p and $q = 1 - p \approx .833$ thus $\text{Var}[X] = pq \approx .139$.

Part C

Construct a graph of the cumulative distribution function $F_X(t)$.

Solution



Now let's take a look at the number of turns that are needed until doubles are rolled. Let Y be the random variable representing the number of rolls of the dice until doubles comes up. Then $Y \sim \text{Geometric}(p)$.

Part D

Problem 2 continued on next page. . .

Using your answers for 2a for p, what is the expected number of turns a player will need to get out of jail?

Solution

Since it is Geometric, the expected value for a geometric distribution is $\mathrm{E}[Y] = 1/p$, plugging in our values gives us $\mathrm{E}[Y] = 1/.167 \approx 5.99$ turns.

Part E

What is the probability that a player will need four or more rolls to get out of jail?

Solution

By needing four or more rolls to get out of jail will mean the complement of not getting doubles in 4 rolls. Thus:

$$\begin{aligned}
P[Y \geq 4] &= 1 - P[Y < 4] \\
&= 1 - P(3) - P(2) - P(1) \\
&\approx 1 - .116 - .139 - .167 \quad \text{using the pdf function, } P(x) \\
&\approx .578
\end{aligned}$$

Problem 3

A student is taking a 10 question multiple choice exam, where each question has four possible answers and the correct answer is chosen randomly and independent of all other answers. If the student guesses "C" on every question, then X, the number of questions that the student answers correctly, follows a $\sim$ Binomial$(10, 0.25)$ distribution.

Part A

Find E$[X]$ and Var$[X]$.

Solution

Since the distribution is a Binomial one, the expected value of it is E$[X] = np$ where n is the number of trials and p is the probability of a success. Thus E$[X] = (10)(.25) = 2.5$.

For the variance of a Binomial distribution, it is defined as Var$[X] = npq$ where n and p are the same as previously, and $q = 1 - p = .75$. Thus Var$[X] = (10)(.25)(.75) = 1.875$.

Part B

Does E$[X]$ represent a possible outcome? Does this matter? Why or why not?

Solution

No, it isn't possible to get 2.5 questions right on the exam. This doesn't matter because the expectation is how many someone would expect to get right on 10 questions for guessing C over and over again for a very long time. It isn't the number he **will** get right, but rather the number he **should** get right. Thus it makes more sense that over a 20 question exam, the guesser would be expected to get 5 right.

Part C

What is the probability that the student answers no questions correctly?

Solution

In this case, $n = 10$ questions and given our values of $p = .25$ and $q = .75$. We want to see the probability of him getting 0 questions right. Thus the value we are looking for is when $x = 0$ because x. Thus given our pmf, $p(x) = \binom{n}{x} p^x q^{n-x}$. Inserting our values gives us:

$$\begin{aligned} P[X = 0] = p(0) &= \binom{10}{0} p^0 q^{10-0} \\ &= \frac{10!}{0!10!} 1(.75)^1 0 \\ &= 1 \cdot 1 \cdot .75^1 0 \\ &\approx .056 \end{aligned}$$

Part D

The student will pass the exam if he answers 6 or more questions correctly. What is the probability that the student passes the exam?

Solution

Using the above equation, we want to see the probability of $P[X \geq 6]$. Which gives us the following:

Problem 3 continued on next page. . .

$$\begin{aligned}
P[X \geq 6] &= p(6) + p(7) + p(8) + p(9) + p(10) \\
&= \sum_{x=6}^{10} \binom{10}{x} p^x q^{10-x} \\
&= 1 - P[X < 6] \\
&= 1 - P[X \leq 5] \quad \text{because its discrete} \\
&\approx 1 - .980 \quad \text{using the cdf function, } F(x) \\
&\approx .020
\end{aligned}$$

Problem 4

In some city, the probability of rain on any day is 0.60, determined using historical climate records. It is known that on rainy days the number of traffics has a $\sim$ Poisson(4) distribution.

An accident occurred years ago and there are no precise weather records available, but we do know that on the day of the accident in question there were 8 accidents in the city. Use Baye's Rule to determine the probability that it was in fact a rainy day.

Solution

Let there be two events, event R that it rained that day and E that there were 8 accidents that day. We know that $\Pr(R) = .6$ and $\Pr(\overline{R}) = .4$.

We want to know what the probability that given there were 8 accidents that it was a rainy day, or $\Pr(R \mid E)$.

We also know that the probability that there were 8 accidents given that it rained follows a Poisson distribution with $\lambda = 4$. In other words, $\Pr(E \mid R) \sim \text{Poisson}(4)$. Thus this gives us:

$$\Pr(E \mid R) \sim \text{Poisson}(4) = p(8) = e^{-4} \frac{4^8}{8!} \approx .030$$

We know that Baye's Rule is that $\Pr(E \mid R) \Pr(R) = \Pr(R \mid E) \Pr(E)$. Since we are looking for $\Pr(R \mid E)$, we get the following:

$$\Pr(R \mid E) = \frac{\Pr(E \mid R) \Pr(R)}{\Pr(E)}$$

Using the law of total probability

$$\begin{aligned}
\Pr(R \mid E) &= \frac{\Pr(E \mid R) \Pr(R)}{\Pr(E \mid R) \Pr(R) + \Pr(E \mid \overline{R}) \Pr(\overline{R})} \\
&= \frac{(.030)(.6)}{(.030)(.6) + (1 - .030)(.4)} \\
&= \frac{.018}{.406} \\
&\approx .044
\end{aligned}$$

Problem 5

Fred Hoiberg had a free throw shooting percentage of 85.4%. Suppose that in pratice one day he shoots free throws until his first miss. Let X be the number of shots he takes.

Part A

Define the probability distribution of X, carefully determining the appropriate parameter(s).

Solution

The probability distribution that this problem will take is the Geometric distribution on the random variable X where X is the number of shots until he misses. The parameters just need to be x, but counterintuitively, our "success" isn't really a success, but rather a miss. Thus our p will be $p = 1 - .854 = .146$.

The equation for the probability distribution is then $P(x) = (1-p)^{x-1}p$.

Part B

What is the probability that Hoiberg makes his first two shots but misses his third short?

Solution

We are looking for the probability that the first miss will happen after three shots, or or $P[X = 3]$. This gives:

$$P[X = 3] = P(3) = (1-p)^{3-1}p = (.854)^2(.146) \approx .106$$

Part C

What is the probability that Hoiberg makes more than 15 shots?

Solution

We want the probability that he makes 15 shots in a row until his first miss, or $P[X = 16]$. This gives:

$$P[X = 16] = P(16) = (1-p)^{16-1}p = (.854)^{15}(.146) \approx .014$$

Part D

What is the probability that Hoiberg takes between two and four (inclusive) shots?

Solution

We want the probability that he can make between 2 and four shots. This is $P[2 \leq X \leq 4]$. Giving us:

$$\begin{aligned}
P[2 \leq X \leq 4] &= P(2) + P(3) + P(4) \\
&= (1-p)^{2-1}p + (1-p)^{3-1}p + (1-p)^{4-1}p \\
&= (.854)^1(.146) + (.854)^2(.146) + (.854)^3(.146) \\
&= .125 + .106 + .091 \\
&\approx .322
\end{aligned}$$

Problem 6

On average, a certain region in Alaska experiences on average 12 severe earthquakes every 10 years. Assume that the timings of sever earthquakes are independent. If you have a three year assignment in this region of Alaska:

Part A

What is the probability that there are no severe earthquakes during your assignment?

Solution

This problem takes the form of the Poisson distribution. We have a rate and a value that has one of the units of the rate. Thus we can calculate our lambda, $\lambda = (12/10)(3) = 3.6$ occurences (earthquakes).

We know the pmf of a Poisson distribution is: $P(x) = e^{-\lambda}\frac{\lambda^x}{x!}$. We want to find the probability that there are no severe earthquakes or $P[X = 0]$. Using the above equation, $P[X = 0] = .027$.

Part B

What is the probability that there are three or more severe earthquakes during your assignment?

Solution

Using the above information, we want to know $P[X \geq 3]$ so we can calculate it as follows:

$$\begin{aligned}
P[X \geq 3] &= 1 - P[X < 3] \\
&= 1 - P[X = 0] - P[X = 1] - P[X = 2] \\
&\approx 1 - .027 - .098 - .177 \quad \text{using the above formula} \\
&\approx .698
\end{aligned}$$

Problem 7

A type of photo paper has on average 1.5 blemishes per square foot. You can reasonably assume that blemishes are randomly distributed. If the photograph you are printing is .4 square feet, what is the probability that the randomly selected photo paper for your photograph will have at least one blemish?

Solution

This problem takes the form of the Poisson distribution. We have a rate and a value that has one of the units of the rate. Thus we can calculate our lambda, $\lambda = (1.5)(0.4) = .6$ occurences (blemishes).

We know the pmf of a Poisson distribution is: $P(x) = e^{-\lambda}\frac{\lambda^x}{x!}$. We want to find the probability that we get at least one blemish which is $P[X \geq 1]$. Then:

$$\begin{aligned}
P[X \geq 1] &= 1 - P[X < 1] \\
&= 1 - P[X = 0] \\
&\approx 1 - .549 \quad \text{by using the equation above} \\
&\approx .451
\end{aligned}$$

Problem 8

The probability that you attend lecture is .95. If there are 30 lectures during the semester, what is the probability that you miss 2 or more lectures throughout the semester? There is no attendance requirement, and the probability of missing any lecture is independent of other lectures.

Solution

If we think about attend as a success and missing lecture as a failure, then this takes the form of a Binomial distribution.

This gives us the equation of $p(x) = \binom{30}{x} p^x q^{30-x}$ where $p = .95$ and thus $q = .05$.

The probability of missing 2 or more lectures is the same as attending less than 29 lectures. Thus we can write it as $P[X \leq 28]$. This can also be written as: $1 - P[X > 28]$:

$$\begin{aligned} P[X \leq 28] &= 1 - P[X > 28] \\ &= 1 - p(29) - p(30) \\ &\approx 1 - .339 - .215 \quad \text{by using the equation above} \\ &\approx .446 \end{aligned}$$