Stat 330: Homework #8

Due on April 2, 2014 at 3:00pm

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Problem 1

Let X represent the outcome of a roll of a 20-sided die. The values X between $a = 1$ to $b = 20$ follows a discrete uniform probability distribution, with $\mathrm{E}[X] = \frac{a+b}{2}$ and a variance of $\frac{b^2-1}{12}$. Thus $\mathrm{E}[X] = 10.5$, $\mathrm{Var}[X] = 33.25$ and $\sigma = 5.766$.

Suppose you roll this die 33 times, $n = 33$. Let S be the sum of these 33 rolled values and let Y be the random variable that represents the mean of these 33 values minus the overall mean of 10.5:

$$Y = \frac{S}{33} - 10.5$$

Part A

Determine the $\Pr\{Y < -0.5\}$:

Solution

$$\begin{aligned}
\Pr\{Y < -0.5\} &= \Pr\{Y < -0.5\} \\
&= \Pr\left\{\frac{Y-\mu}{\sqrt{\sigma^2/n^2}}\right\} \\
&= \Pr\left\{\frac{-0.5-10.5}{\sqrt{33^2/5.766^2}}\right\} = \Pr\{z < -0.5\} = \Phi(-0.5) \\
&= 0.3085
\end{aligned}$$

Part B

Determine the $\Pr\{Y < -2.5\}$:

Solution

$$\begin{aligned}
\Pr\{Y < -2.5\} &= \Pr\{Y < -2.5\} \\
&= \Pr\left\{\frac{Y-\mu}{\sqrt{\sigma^2/n^2}}\right\} \\
&= \Pr\left\{\frac{-2.5-10.5}{\sqrt{33^2/5.766^2}}\right\} = \Pr\{z < -2.5\} = \Phi(-2.5) \\
&= 0.0062
\end{aligned}$$

Part C

Let Z be a standard normal random variable, determine the probability that:

(c) Z is less than -0.5: $\Phi(-0.5) = 0.3085$

(d) Z is less than -2.5: $\Phi(-2.5) = 0.0062$

Simulate 1000 values for Y (by drawing 33 random variables from an appropriate discrete uniform distribution) and calculate the proportion of these 1000 values that are:

(e) less than -0.5: 0.3095

(f) less than -2.5: 0.0056

Problem 2

In Happyland, the weather each day is characterized by these three state: let state 1=a sunny day, state 2=a cloudy day, and state 3=a rainy day.

The weather from one day to the next has the following transition probabilities:

1. If today is sunny, the probability that it is sunny tomorrow is 0.4 and the probability that it is rainy tomorrow is 0.2.
2. If today is cloudy, the probability that it is sunny tomorrow is 0.5 and equal probability that it is cloudy or rainy tomorrow.
3. If today is rainy, the probability that it is rainy again tomorrow is 0.25, with no chance of it being sunny tomorrow.

Use the concept of steady-state probabilities to get the probability of a rainy day this time next year in Happyland. Your answer should be in the form of a fraction.

You may check your work using a simluation, but it is not required.

Solution

Using the information, we can get the following transition probability matrix:

$$P = \begin{pmatrix} 0.4 & 0.4 & 0.2 \\ 0.5 & 0.25 & 0.25 \\ 0 & 0.75 & 0.25 \end{pmatrix}$$

Using the fact that $\pi P = \pi$, we want to do the following:

$$(\pi_1, \pi_2, \pi_3) = (\pi_1, \pi_2, \pi_3) \begin{pmatrix} 0.4 & 0.4 & 0.2 \\ 0.5 & 0.25 & 0.25 \\ 0 & 0.75 & 0.25 \end{pmatrix} = (0.4\pi_1 + 0.5\pi_2 + 0\pi_3, 0.4\pi_1 + 0.25\pi_2 + 0.75\pi_3, 0.2\pi_1 + 0.25\pi_2 + 0.25\pi_3)$$

This gives us the equations:

$$\begin{cases} 0.5\pi_2 = 0.6\pi_1 \\ 0.4\pi_1 + 0.75\pi_3 = 0.75\pi_2 \\ 0.2\pi_1 + 0.25\pi_2 = 0.75\pi_3 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

Using algebra to solve the equations, this gives us the steady state probabilities of:

$$\pi_1 = 0.3488, \quad \pi_2 = 0.4186, \quad \pi_3 = 0.2326$$

Thus the probability of a rainy day (state 3) this time next year in happy land is 0.2326.

Problem 2 continued on next page. . .

Simulation

Let's verify this with a simluation function in R below:

```r
val <- c(0.4, 0.5, 0, 0.4, 0.25, 0.75, 0.2, 0.25, 0.25)

happyland <- function(K) {
    P <- matrix(val, nrow = 3, ncol = 3)

    print(P)

    # 2. enter starting state probabilities
    pr <- c(0.4, 0.4, 0.2)

    for (k in 1:K) {
        pr <- pr %*% P

        if (k == K) {
            print("The state probabilities (1=sunny day, 2=cloudy day, 3=rainy day")
            print(sprintf("after %i days are:", k))
            print(pr)
            return(pr)
        }
    }
}
```

We want a year from today, so this gives:

```r
# 365 days later
happyland(356)

##      [,1] [,2] [,3]
## [1,]  0.4 0.40 0.20
## [2,]  0.5 0.25 0.25
## [3,]  0.0 0.75 0.25
## [1] "The state probabilities (1=sunny day, 2=cloudy day, 3=rainy day"
## [1] "after 356 days are:"
##        [,1]   [,2]   [,3]
## [1,] 0.3488 0.4186 0.2326
##        [,1]   [,2]   [,3]
## [1,] 0.3488 0.4186 0.2326
```

Everything checks out!

Problem 3

Suppose that the average number of telephone calls arriving at the switchboard of a small corporation is 30 calls per hour or $\lambda = 30$ hour^{-1} or $\lambda = 0.5$ min^{-1}. Assume that the arriving calls follows a Poisson process.

Part A

What is the probability that no calls will arrive in a 3-minute period?

Solution

Let T = time until the first occurrence. We know that $T \sim \text{Exp}(\lambda)$ and want to the probability that no calls arrive in a 3-minute period or when $\Pr(T \geq 3)$. This gives us $T \sim \text{Exp}(0.5)$ where the cdf is $F(x) = 1 - e^{-\lambda x}$.

$$\Pr(T \geq 3) = 1 - \Pr(T < 3) = 1 - (1 - e^{-0.5 \cdot 3}) = e^{-1.5} \approx 0.2231$$

Part B

What is the probability that four or more calls will arrive in a 5-minute interval?

Solution

Let X = # occurrences in time t. We know that $X \sim \text{Poisson}(\lambda t)$ and $t = 5$. We want to the probability that four or more calls arrive in a 5-minute period or when $\Pr(X > 5)$. This gives us $X \sim \text{Poisson}(0.5 \cdot 5)$ where the pdf is $\Pr(x) = e^{-\lambda} \frac{\lambda^x}{x!}$ this gives us:

$$\begin{aligned}
\Pr[X \geq 4] &= 1 - \Pr[X \leq 3] \\
&= 1 - \Pr[X = 0] - \Pr[X = 1] - \Pr[X = 2] - \Pr[X = 3] \\
&\approx 1 - 0.082 - 0.205 - 0.257 - 0.213 \quad \text{using the above formula} \\
&\approx 0.243
\end{aligned}$$

Part C

What is the probability that one minute will pass before the next incoming call?

Solution

This is an Exponential distribution. Thus if we let T = time until first call, then $T \sim \text{Exp}(0.5$. We want $\Pr[T > 1]$ thus:

$$\Pr[T > 1] = 1 - \Pr[T < 1] \approx 1 - 1 - e^{-0.5} \approx 0.607$$

Part D

If you need to go refill your coffee, which will take two minutes to do, what is the probability you will have missed at most two calls during that trip to the coffeemaker? (*Hint*: find the probability that the third incoming call occurs after two minutes.)

Solution

This will take the form of a Gamma distribution where $\alpha = 3$ for the 3 missed calls and $\lambda = 0.5$. This gives $T \sim \text{Gamma}(\alpha = 3, \lambda = 0.5)$. We are looking for the probability that you will have missed at most two calls during the trip. Or when the 3rd call comes after 2 minutes, $\Pr[T > 2]$. Using R to calculate, this gives us:

```
pgamma(t, shape = 3, rate = 0.5)
```

Thus $\Pr[T > 2] = 1 - \Pr[T \leq 2] = 1 - 0.0803 = 0.9197$.

Problem 4

A university barbershop. The campus barbershop has only one barber but unlimited chairs waiting for customers. No matter how many customers are in the shop, arriving customers always choose to wait.

1. The customers arrive according to a Poisson process. The average time between arrivals is 30 minutes, or $\lambda = \frac{1}{2}$.
2. The lone barber completes servering a customer at a rate of x per hour, where x is the number of customers in the barbershop.

Assume that inter-arrival times and service times are independent exponential random variables.

Part A

Draw a state diagram with possible states and corresponding birth/death rates. Since there are an infinite number of states, show enough states to show the pattern or arrival and service rates.

Solution

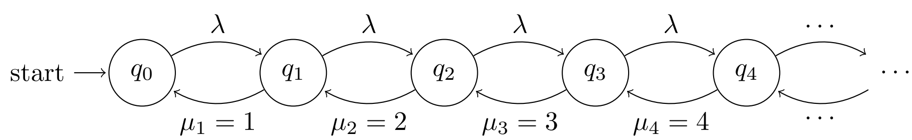

Figure 1: State Diagram for a Barbershop

Part B

What is the (large t) probability that the shop is empty? (Note: $e^x = \sum\limits_{k=0}^{\infty} \frac{x^k}{k!}$)

Solution

We want the "large t" probability which is P_0. This gives us:

$$P_0 = (S)^{-1} = \left(1 + \frac{\lambda_0}{\mu_1} + \cdots\right)^{-1} = \left(1 + \frac{2}{1} + \frac{2}{1}\frac{2}{2} + \frac{2}{1}\frac{2}{2}\frac{2}{3} + \cdots\right)^{-1} = \left(\sum_{k=0}^{\infty} \frac{2^k}{k!}\right)^{-1} = (e^2)^{-1} \approx 0.135$$

Part C

Let X be the number of customers in the store. Determine the "large t" probability function for the different states of X. What probability distribution is this, and what is the value of the parameter in that distribution?

Solution

(a) $p_0 = \frac{1}{S} = 0.135$

(b) $p_1 = p_0 \frac{\lambda_0}{\mu_1} = p_0 \frac{2}{1} = 0.270$

(c) $p_2 = p_0 \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} = p_0 \frac{2^2}{1 \cdot 2} = 0.270$

(d) $p_3 = p_0 \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3}$

This gives us the pmf that is the same as $X \sim \text{Poisson}(2)$.

Part D

What is the "large t" probability that a customer arrives when the barbershop has more than two persons

Problem 4 continued on next page...

waiting?

Solution

We want the probability that a customer arrives when the barbershop has more than two persons waiting, this is when $P[X > 2]$, this gives us:

$$\begin{aligned} P[X > 3] &= 1 - P[X \leq 3] \\ &= 0.143 \end{aligned}$$

Part E

What is the expected number of customers in the barbershop at any time (for "large t" time)? *Hint:* Use the properties of the known probability distribution.

Solution

We want the expected value of our distribution, since we know that our distribution is Poisson, the expected value of a Poisson distribution is just $\mathrm{E}[X] = \lambda$ where our $\lambda = 2$.

Problem 5

Problem 7.14 from Baron.

Trucks arrive at a weigh station according to a Poisson process with the average rate of 1 truck every 10 minutes. Inspection time is Exponential with the average of 3 minutes. When a truck is on a scale, the other arrived trucks stay in line waiting for their turn.

Instead of answering the book questions, fill in the table below theoretical values versus simluation values for these quantities:

Use the provided R code, **hw8code.R** to simulate the performance of this queuing system. Your simulation should have a sample of 25,000 time values, as currently set up in the code (drawing 25% of the 100,000 arrivals). You will need to change the rate parameters at the beginning of the code to match this queuing system.

Solution

We know that the $\lambda_A = 1/10\text{min}^{-1}$ and then $\lambda_S = 1/3\text{min}^{-1}$.

	r	π_0	π_1	π_2	$\sum_{k=3}^{\infty} \pi_k$	$\mathrm{E}[X]$
Theor.	$\frac{\lambda_A}{\lambda_S} = 0.3$	$1 - r = 0.7$	$0.7r^1 = 0.21$	$0.7r^2 = 0.063$	$1 - \sum_{k=0}^{\infty} \pi_k = 0.027$	$\frac{r}{1-r} = 0.4286$
Sim	0.3005	0.6995	0.2132	0.0616	0.0258	0.4257

	$\mathrm{Var}[X]$	$\mathrm{E}[X_s]$	$\mathrm{E}[X_w]$	$\mathrm{E}[R]$	$\mathrm{E}[S]$	$\mathrm{E}[W]$
Theor.	$\frac{r}{(1-r)^2} = 0.6122$	$\frac{r-r^2}{(1-r)} = 0.3$	$\frac{r^2}{(1-r)} = 0.1286$	$\frac{1}{\lambda_S(1-r)} = 4.286$	$\frac{1-r}{\lambda_S(1-r)} = 3$	$\frac{r}{\lambda_S(1-r)} = 1.286$
Sim	0.6047	0.3005	0.1252	4.2606	2.9794	1.2811

Problem 6

Problem 7.23 from Baron.

Internet users vist a certain website according to a queuing system with the arrival rate, $\lambda_A = 2\text{min}^{-1}$ and an expected time of 5 minutes on the site, $\lambda_S = 0.2\text{min}^{-1}$, giving $r = \lambda_A/\lambda_S = 10$

Part A

Find the expected number of visitors of the website at any time.

Solution

If we let $X = \#$ of visitors on the site, then we want $\mathrm{E}[X]$. For M/M/∞, the equation for expectation is $\mathrm{E}[X] = r$, thus $\mathrm{E}[X] = 10$.

Part B

Find the fraction of time when nobody is browsing the website.

Problem 6 continued on next page. . .

Solution

We want the probability that $\Pr[X = 0]$, this is P_0 which is the same as $P_0 = \frac{1}{S}$. This gives:

$$P_0 = (S)^{-1} = \left(1 + r^1/1! + r^2/2! + \cdots\right)^{-1} = e^{-r} = e^{-10} \approx 0.0000454$$

Part C

Run the simulated code for #6 and report the estimates for π_0, mean number of visitors, number of visitors, and mean visit time.

Solution

After running the code, we get the following estimates: $\pi_0 = 0$, mean number of visitors $= 9.9403$, variance of number of visitors $= 9.9056$, and the mean visit time $= 4.9657$.

Part D

Comment on what distribution of visitors the website looks like based on the histogram.

Solution

Looking at the graph below, it looks like a Poisson distribution.

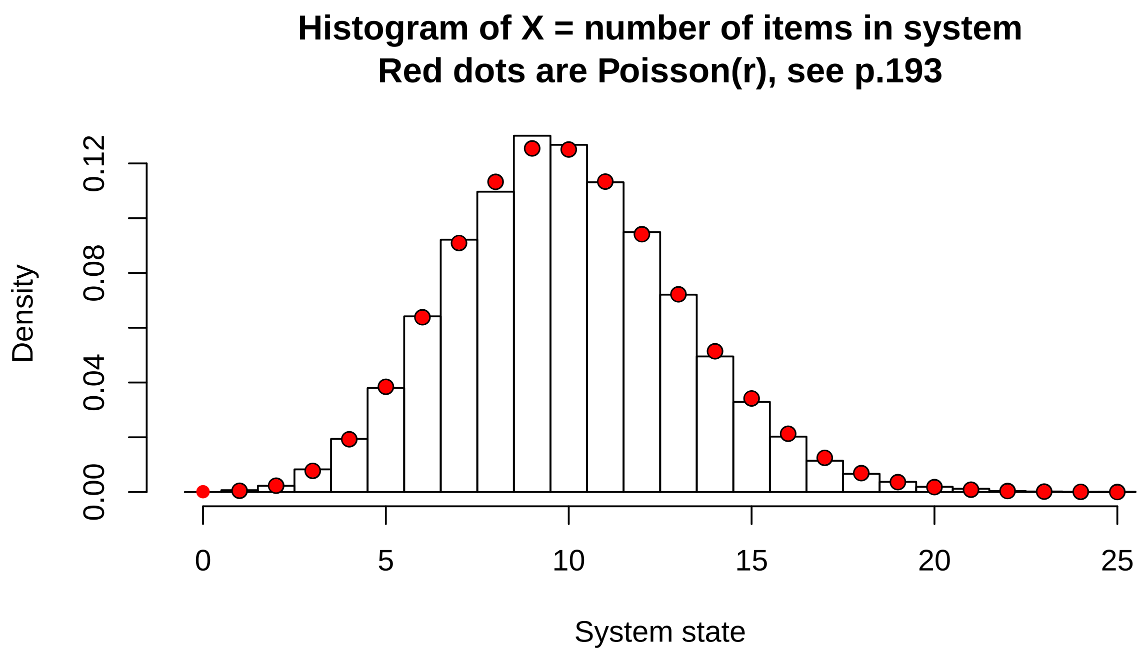

Figure 2: Distribution of visitors