Stat 330: Homework #7

Due on March 14, 2014 at 3:00pm

Mr. Lanker, Section A

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Problem 1

Problem 6.3 from Baron.

An offspring of a black dog is black with probability 0.6 and brown with probability 0.4. An offspring of a brown dog is black with probability 0.2 and brown with probability 0.8.

Part A

Write the transition probability matrix of this Markov chain.

Solution

The transition probability matrix, P, is:

$$P = \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix}$$

Part B

Rex is a brown dog. Compute the probability that his grandchild is black.

Solution

We know that Rex is P^0, his child is P^1, then his grandchild is P^2. This gives us:

$$P^2 = \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix}^2 = \begin{pmatrix} 0.44 & 0.56 \\ 0.28 & 0.72 \end{pmatrix}$$

We want to know the probability that Rex's grandchild is black which is $P^2_{i,j}$ where i and j indicate the location in the matrix. We know Rex is brown, which is the second row, $i = 2$, and we want the probability that the pup is black, which is the first column, $j = 1$. Thus this gives us $P^2_{2,1}$ which is 0.28.

Part C

Using the R code, **hw7code.R**, use it to do the rest.

Compute the probability that a dog of the 4th generation (great-great-grand-pup?) is black.

Solution

```
## [1] "The state probabilities (1=black, 2=brown) after 4 generations are:"
##        [,1]   [,2]
## [1,] 0.3248 0.6752
```

This gives the probability that the 4th generation is black as 0.3248.

Part D

Compute the probability that a dog of the 20th generation is black.

Solution

```
## [1] "The state probabilities (1=black, 2=brown) after 20 generations are:"
##        [,1]   [,2]
## [1,] 0.3333 0.6667
```

This gives the probability that the 20th generation is black as 0.3333.

Problem 1 continued on next page. . .

Part E

Compute the probability that a dog of the 100th generation is black.

Solution

```
## [1] "The state probabilities (1=black, 2=brown) after 100 generations are:"
##        [,1]   [,2]
## [1,] 0.3333 0.6667
```

This gives the probability that the 100th generation is black as 0.3333.

Part F

Based on your last answers, what do you think π is?

Solution

Based off of this, I think that $\pi = \left(\frac{1}{3}, \frac{2}{3}\right)$.

Part G

Based on (f), give P^∞.

Solution

$$P^\infty = \begin{pmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

Problem 2

Problem 6.7 from Baron.

Part A

Fill in the blanks.

Solution

The filled in matrix looks like:

$$P = \begin{pmatrix} 0.3 & 0.7 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Part B

Show that this is a regular Markov chain

Solution

To show that this Markov chain is regular, there must be a point when $P_{i,j}^{(h)}$ for all i, j and some h where the probability at point (i, j) is greater than 0.

Let's continue the Markov chain to see if we can reach a transition matrix that satisfies the above:

$$P = \begin{pmatrix} 0.3 & 0.7 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad P^2 = \begin{pmatrix} 0.09 & 0.21 & 0.7 \\ 1 & 0 & 0 \\ 0.3 & 0.7 & 0 \end{pmatrix}, \quad P^3 = \begin{pmatrix} 0.727 & 0.063 & 0.21 \\ 0.3 & 0.7 & 0 \\ 0.09 & 0.21 & 0.7 \end{pmatrix}, \quad P^4 = \begin{pmatrix} 0.4281 & 0.5089 & 0.063 \\ 0.09 & 0.21 & 0.7 \\ 0.727 & 0.063 & 0.21 \end{pmatrix}$$

At P^4, we can see that all the probabilities are non-zero thus showing that our Markov chain is regular.

Part C

Compute the steady-state probabilities using a system of equations.

Solution

Using the fact that $\pi P = \pi$, we want to do the following:

$$(\pi_1, \pi_2, \pi_3) = (\pi_1, \pi_2, \pi_3) \begin{pmatrix} 0.3 & 0.7 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = (0.3\pi_1 + 0\pi_2 + 1\pi_3, 0.7\pi_1 + 0\pi_2 + 0\pi_3, 0\pi_1 + 1\pi_2 + 0\pi_3)$$

This gives us the equations:

$$\begin{cases} 0.3\pi_1 + \pi_3 = \pi_1 \\ 0.7\pi_1 = \pi_2 \\ \pi_2 = \pi_3 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

Using algebra to solve the equations, this gives us the steady state probabilities of:

$$\pi_1 = 0.4167, \quad \pi_2 = 0.2917, \quad \pi_3 = 0.2917$$

Problem 3

Problem 6.1 from Baron.

A small computer lab has 2 terminals. The number of students working in this lab is recoreded at the end of every hour. A computer assistant notices the following pattern:

1. If there are 0 or 1 students in a lab, then the number of students in 1 hour has a 50% chance to increase by 1 or remain unchanged.
2. If there are 2 students in a lab, then the number of students in 1 hour has a 50% chance to decrease by 1 or reamin unchanged.

Part A

Write the transition probability matrix for this Markov chain.

Solution

The transition probability matrix, P, is:

$$P = \begin{pmatrix} 0.50 & 0.50 & 0 \\ 0 & 0.50 & 0.50 \\ 0 & 0.50 & 0.50 \end{pmatrix}$$

Part B

Is this a regular Markov chain? Justify your answer.

Solution

No, it is not a regular Markov chain. By looking at how the matrix multiplication works out, we can see that for $P_{2,1}$ and $P_{3,1}$, they will always be zero. This is because of the following:

$$P_{2,1} = 0(0.5) + (0.5)0 + (0.5)0 \quad P_{3,1} = 0(0.5) + (0.5)0 + (0.5)0$$

Thus we can see that since we are only multiplying by P, we will never satisfy the regular Markov Chain requirement because $P_{2,1}$ and $P_{3,1}$ will always be 0.

Part C

Suppose there is nobody in the lab at 7am. What is the probability of nobody working in the lab at 10am?

Solution

We need to calculate 3 iterations of our P probability transition matrix. This gives us the following where P^{k+1} indicates the matrix after k hours:

$$P = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{pmatrix}, \quad P^2 = \begin{pmatrix} 0.25 & 0.5 & 0.25 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{pmatrix}, \quad P^3 = \begin{pmatrix} 0.125 & 0.5 & 0.375 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{pmatrix}, \quad P^4 = \begin{pmatrix} 0.0625 & 0.5 & 0.4375 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{pmatrix},$$

We want the starting probabilty that nobody is in the lab, which is the first row, or $(1, 0, 0)P^4$. Thus the probabilities are:

$$0 \text{ students} = 0.0625, \quad 1 \text{ student} = 0.5, \quad 2 \text{ students} = 0.4375$$

Part D

List the absorbing zone for this chain.

Solution

The absorbing zone for this chain is the zone that has state 2 and state 3, because there is no way to move back to state 1.

Problem 4

Use Problem 4 from the **hw7code.R** for the following parts.

Part A

Write the transition probability matrix.

Solution

The probability matrix will look like:

$$P = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \end{pmatrix}$$

Part B

Is this a regular Markov chain? Justify your answer.

Solution

Let's continue the Markov chain to see if we can reach a transition matrix that satisfies the above:

$$P = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \end{pmatrix}, \quad P^2 = \begin{pmatrix} 0.5 & 0.25 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{pmatrix},$$

At P^2, we can see that all the probabilities are non-zero thus showing that our Markov chain is regular.

Part C

Suppose there is nobody in the lab at 7am. Give the probabilities that there are 0, 1, or 2 students working in the lab at 10am.

Solution

```
## [1] "The state probabilities (1=no students, 2=one student, 3=two students"
## [1] "after 3 generations are:"
##       [,1]   [,2]   [,3]
## [1,] 0.375 0.3125 0.3125
```

Thus this gives the probabilities of 0 students as 0.375 as 1 student 0.3125 as 2 student 0.3125.

Part D

Suppose there is nobody in the lab at 7am. Give the probabilities that there are 0, 1, or 2 students working in the lab at 7pm.

Solution

```
## [1] "The state probabilities (1=no students, 2=one student, 3=two students"
## [1] "after 12 generations are:"
##        [,1]   [,2]   [,3]
## [1,] 0.3334 0.3334 0.3333
```

Thus this gives the probabilities of 0 students as 0.3334 as 1 student 0.3334 as 2 student 0.3333.

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Part E

Suppose there is nobody in the lab at 7am. Give the probabilities that there are 0, 1, or 2 students working in the lab at 7am the next morning.

Solution

```
## [1] "The state probabilities (1=no students, 2=one student, 3=two students"
## [1] "after 24 generations are:"
##        [,1]   [,2]   [,3]
## [1,] 0.3333 0.3333 0.3333
```

Thus this gives the probabilities of 0 students as 0.3333 as 1 student 0.3333 as 2 student 0.3333.

Part F

Based on your last answers, give the steady-state probabilities π.

Solution

Based of these answers, I think the steady-state probabilities are:

$$\pi_1 = 0.3333, \quad \pi_2 = 0.3333, \quad \pi_3 = 0.3333$$

Problem 5

Confirm in R that for $Y \sim \text{Exponential}(\lambda = \frac{1}{2})$ that $P(Y \leq 1) = 0.3935$:

```r
pexp(1, rate = 0.5)

## [1] 0.3935
```

Let $X \sim \text{Gamma}(\alpha = 5, \lambda = \frac{1}{2})$, that is time to 5th occurrence with rate of 1 every 2 time units. Calculate $\Pr(X \leq x)$ in R using:

```r
pgamma(x, shape = 5, rate = 0.5)
```

(a) $P(X \leq 1) = 1.7212 \times 10^{-4}$

(b) $P(X \leq 2.5) = 0.0091$

(c) $P(X \leq 5) = 0.1088$

(d) $P(X > 12.5) = 1 - P(X \leq 12.5) = 1 - 0.747 = 0.253$

Problem 6

Problem 4.12 from Baron.

A computer processes tasks in the order they are received. Each task takes an Exponential amount of time with the average of 2 minutes. Compute the probability that a package of 5 tasks is processed in less than

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8 minutes.

Solution

Given the information, we know that $\lambda = 0.5$ minute^{-1}. We want the probability that a package of 5 tasks is processed in less than 8 minutes. This gives us $k = 5$, we know that there is an Exponential distribution between each task. This means that if we let $T =$ time until kth task processed, then $T \sim \text{Gamma}(k, \lambda)$ and we want $\Pr(X < 8)$ which is:

```
pgamma(8, shape = 5, rate = 0.5)
```

Thus this gives us the probability of 0.3712.

Problem 7

Problem 6.20 from Baron.

Power outages are unexpected rare events occurring according to a Poisson process with the average rate of 3 outages per month. Compute the probability of more than 5 power outages during three summer months.

Solution

Given the information, we know that $\lambda = 3$ month^{-1}. We want to calculate the probabilty that 5 outages occur in 3 months. This gives us the value of $t = 3$, so $\lambda t = 9$ where $X = \#$ outages in a month and we want $\Pr(X > 5)$ where $X \sim \text{Poisson}(9)$ which is:

$$\Pr(X > 5) = 1 - \Pr(X \le 5) = 1 - e^{-9} \sum_{i=0}^{5} \frac{9^i}{i!} = 1 - e^{-9} \left(\frac{9^0}{0!} + \frac{9^1}{1!} + \frac{9^2}{2!} + \frac{9^3}{3!} + \frac{9^4}{4!} + \frac{9^5}{5!} \right) \approx 0.8843$$

Problem 8

Problem 6.21 from Baron.

Telephone calls a customer service center according to a Poisson process with the rate of 1 call every 3 minutes. Compute the probability of receiving more than 5 calls during the 12 minutes.

Solution

Given the information, we know that $\lambda = \frac{1}{3}$ minute^{-1}. We want the probability of receiving more than 5 calls during 12 minutes. This gives us $t = 12$ so $\lambda t = 4$ where $X = \#$ of calls during a minute and $\Pr(X > 5)$ where $X \sim \text{Poisson}(4)$. Like before, this gives us:

$$\Pr(X > 5) = 1 - \Pr(X \le 5) = 1 - e^{-4} \sum_{i=0}^{5} \frac{4^i}{i!} = 1 - e^{-4} \left(\frac{4^0}{0!} + \frac{4^1}{1!} + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!} \right) \approx 0.2149$$

Problem 9

Superfan Todd OConnor (played by the late Chris Farley) has a heart attack on average once every 4 years. Todd has his first ever physical exam due to new insurance regulations. His doctor is horrified at his low beef fat-to-pork fat ratio, and his doctor tells him that he must change his diet. Todd resists, but agrees to change his diet after three heart attacks. What is the probability that Todd will not have to change his diet for at least 10 years? Assume a Poisson process is the correct model for Todds heart attacks.

Solution

Given the information, we know that $\lambda = 0.25$ year^{-1}. Let T = time in years until kth heart attack. We know that this means that $T \sim \text{Gamma}(k, \lambda)$ where $k = 3$. Thus we want the probability that $\Pr(T = 10)$. Using R, we know that this is:

```r
pgamma(10, shape = 3, rate = 0.25)
```

Thus this gives us the probability that Todd won't have to change his diet as 0.4562. Poor Todd...