Stat330: Homework #2

Due on January 29, 2014at 3:10pm

Mr. Lanker Section A

Josh Davis

Problem 1

Binomial theorem.

Part A

What is the coefficient of x^5y^3?

Solution

$$c \cdot x^5y^3 = \binom{8}{3}x^5y^3 = \frac{8!}{5!3!}x^5y^3 = 56x^5y^3$$

Part B

What is the coefficient of x^3y^5?

Solution

$$c \cdot x^3y^5 = \binom{8}{5}x^3y^5 = \frac{8!}{5!3!}x^3y^5 = 56x^3y^5$$

Problem 2

Simpson's Paradox.

Part A

A black urn contains 5 red and 6 green balls and a white urn contains 3 red and 4 green balls. You are allowed to choose an urn and then choose a ball at random from the urn. Assume that choosing any ball in the urn is equally likely. If you choose a red ball you ge a prize. Which urn should you choose to draw from?

Solution

For the black urn, the probability of getting a red is:

$$\Pr(\text{black}) = \frac{5 \text{ red}}{5 + 6 \text{ total}} = \frac{5}{11} \approx .455$$

$$\Pr(\text{white}) = \frac{3 \text{ red}}{3 + 4 \text{ total}} = \frac{3}{7} \approx .429$$

Therefore we have a higher probability of getting a red from the black urn and should choose from that one.

Part B

A second black urn contains 6 red and 3 green balls and a second white urn contains 9 red and 5 green balls.

Solution

$$\Pr(\text{2nd black}) = \frac{6 \text{ red}}{6 + 3 \text{ total}} = \frac{6}{9} \approx .667$$

$$\Pr(\text{2nd white}) = \frac{9 \text{ red}}{9 + 5 \text{ total}} = \frac{9}{14} \approx .643$$

The 2nd black has a higher probability than the 2nd white so we should choose from that one.

Part C

The two black urns are combined as well as the two white urns.

Solution

$$\Pr(\text{combined black}) = \frac{5 + 6 \text{ red}}{11 + 9 \text{ total}} = \frac{11}{20} \approx .55$$

$$\Pr(\text{combined white}) = \frac{3 + 9 \text{ red}}{7 + 14 \text{ total}} = \frac{12}{21} \approx .571$$

The combined white urn has a higher probability of getting a red so we should choose that one.

Problem 3

A group of 4 undergraduates and 5 graduates are available to fill four student government posts.

Part A

Find the probability that at least three undergraduates will be among the four chosen assuming the selection was random?

Solution

$$\frac{\text{at least 3 undergrads}}{\text{total possibilities}} = \frac{\text{exactly 3 } + \text{ 4 undergrads}}{\text{total possibilities}} = \frac{\binom{4}{3}\binom{5}{1} + \binom{4}{4}\binom{5}{0}}{\binom{9}{5}} = \frac{4 \cdot 5 + 1 \cdot 1}{126} = \frac{21}{126} \approx .167$$

Part B

What is the probability of no undergraduates being selected again assuming the selection was random?

Solution

$$\frac{\text{no undergrads}}{\text{total possibilities}} = \frac{\binom{4}{0}\binom{5}{4}}{\binom{9}{5}} = \frac{1 \cdot 5}{126} \approx .040$$

Part C

Later we will talk about a p-value, which represents the probability happening under certain model assumptions. Here our model assumptions are that the selection process was purely random. It is common that if the p-value is very low, say less than .05, we reject that our model assumptions are true. Our p-value here is the result from part b. What does this value say about the model statement that the selections were made at random?

Solution

Since our result from part b was $\approx .040$ and since that is lower than .05, this says that our assumption that the selection process was purely random was false.

Problem 4

Suppose that there are two events A and B in Ω such that $\Pr(A) > 0$ and $\Pr(B) > 0$. Further, suppose that the two events are mutually exclusive. Can they also be independent? Explain.

Solution

We know that $\Pr(A) > 0$, $\Pr(B) > 0$, and that A and B are mutually exclusive which means that $A \cap B = \{\}$. So that probability of $\Pr(A \cap B) = 0$ is by definition.

The definition of independent events means that $\Pr(A \cap B) = \Pr(A)\Pr(B)$. This cannot be true because we know that $\Pr(A \cap B) = 0$ and thus one of the probabilities of the events must be zero, which goes against the assumption. So no, they cannot also be independent.

Problem 5

Let A be the event that processor 1 is in use and B be the event that processor 2 is in use.

Part A

Calculate $\Pr(A \mid B)$.

Solution

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{.7}{.8} = .875$$

Part B

What is $\Pr(A)$?

Solution

$$\Pr(A) = .75$$

Part C

Are A and B independent?

Solution

No, A and B are not independent because $\Pr(A \mid B) \neq \Pr(A)$ as our previous parts show.

Part D

Calculate $\Pr(B \mid A)$.

Solution

$$\Pr(B \mid A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{.7}{.75} \approx .933$$

Part E

Show that $\Pr(A \mid B)\Pr(B) = \Pr(B \mid A)\Pr(A)$.

Solution

Proof.

$$\begin{aligned}
\Pr(A \mid B)\Pr(B) &= \frac{\Pr(A \cap B)}{\Pr(B)}\Pr(B) \\
&= \Pr(A \cap B) \\
&= \frac{\Pr(A \cap B)}{\Pr(A)}\Pr(A) \\
&= \Pr(B \mid A)\Pr(A)
\end{aligned}$$

$\square$

Using our values:

$$\begin{aligned}
\Pr(A \mid B)\Pr(B) &= (.875)(.8) \approx .700 \\
\Pr(B \mid A)\Pr(A) &= (.933)(.75) \approx .700
\end{aligned}$$

Problem 6

Suppose that A and B are independent events with $\Pr(A) > 0$ and $\Pr(B) > 0$. Show that $\overline{A}$ and $\overline{B}$ are also independent.

Solution

Proof. Suppose that A and B are events with probability $\Pr(A) > 0$ and $\Pr(B) > 0$. And that the events are also independent, which means that $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

We want to show that $\overline{A}$ and B are independent or in other words, we want to show that $\Pr(\overline{A} \cap B) = \Pr(\overline{A})\Pr(B)$.

Since $B = (A \cap B) \cup (\overline{A} \cap B)$, then $\Pr(B) = \Pr(A \cap B) + \Pr(\overline{A} \cap B)$. Which gives us:

$$
\begin{aligned}
\Pr(\overline{A} \cap B) &= \Pr(B) - \Pr(A \cap B) \\
&= \Pr(B) - \Pr(A)\Pr(B) \text{ because of independence} \\
&= \Pr(B)(1 - \Pr(A)) \\
&= \Pr(B)\Pr(\overline{A})
\end{aligned}
$$

Thus by assuming that A and B are independent, we have shown that $\overline{A}$ and B are also independent, or $\Pr(\overline{A} \cap B) = \Pr(\overline{A})\Pr(B)$. Thus the proof is complete. □

Problem 7

Firing based on positive drug use. Was she fired unjustly?

Let A be the event that the person is a drug user and let B be the event that the person tested positive for drugs. We have the following equations:

$$\begin{aligned}\Pr(A) &= .02\\ \Pr(\overline{A}) &= .08\\ \Pr(B \mid A) &= .99\\ \Pr(B \mid \overline{A}) &= .01\end{aligned}$$

Solution

We want to determine what the probability that she was not a drug user given that she tested positive, or $\Pr(\overline{A} \mid B)$. By using Bayes Rule and the law of total probability we get:

$$\begin{aligned}\Pr(\overline{A} \mid B) &=\\ &= \frac{\Pr(B \mid \overline{A})\Pr(\overline{A})}{\Pr(B)}\\ &= \frac{\Pr(B \mid \overline{A})\Pr(\overline{A})}{\Pr(B \mid \overline{A})\Pr(\overline{A}) + \Pr(B \mid A)\Pr(A)}\\ &= \frac{(.01)(.08)}{(.01)(.08) + (.99)(.02)}\\ &\approx 4\%\end{aligned}$$

Considering the possibility that the test is accurate is 99% and the possibility that she is not a drug user given she tests positive (a false positive) is 4%, I'd say she doesn't have a legitimate claim.

Problem 8

In a bag there are two standard dice. One of the dice is fair and the other is loaded so that the die rolls a 6 exactly half of the time. You reach into the bag, randomly select a dice, and roll a 6. What is the probability that you selected the loaded die?

Solution

Let A be the event that we pick the loaded die and let B be the event that we roll a 6.

Since there are two dice in the bag, $\Pr(A) = .5$ because we are drawing at random. We also know that $\Pr(B \mid A) = .5$ and $\Pr(B \mid \overline{A}) = .167$ based on the problem description.

We want to find the probability that we picked the loaded die given that we rolled a six which is $\Pr(A \mid B)$.

Using Bayes Rule and the law of total probability:

$$\begin{aligned}
\Pr(A \mid B) &= \\
&= \frac{\Pr(B \mid A)\Pr(A)}{\Pr(B)} \\
&= \frac{\Pr(B \mid A)\Pr(A)}{\Pr(B \mid A)\Pr(A) + \Pr(B \mid \overline{A})\Pr(\overline{A})} \\
&= \frac{(.5)(.5)}{(.5)(.5) + (.167)(1 - .5)} \\
&\approx .750
\end{aligned}$$