Problem 1 (multiclass logistic regression)

a. Write out the log likelihood L of data $(x_1, y_1), ..., (x_n, y_n)$ using an i.i.d. assumption.

$$L = \prod_{i=1}^{n} \prod_{j=1}^{k} (\frac{e^{x_i^T w_j}}{\sum_{l=1}^{k} e^{x_i^T w_l}})^{\mathbb{1}(y_i=j)}$$

First, we observe that while $p(y|x, w_1, ..., w_n)$ is a product of the likelihood of each w, the indicator effectively cancels out every $w_i, i \neq c$ as $z^{\mathbb{1}(y=c)} = z^0 = 1$, where c is the true class of data point x_i. Furthermore, since we are only considering x_i in the context of the correct class, we can eliminate the indicator function also.

With that in mind, we can cast the joint probability of the data as:

$$L = \prod_{i=1}^{n} (\frac{e^{x_i^T w_i}}{\sum_{j=1}^{k} e^{x_i^T w_j}})$$

Taking the natural log of this, we have:

$$\mathcal{L} = lnL = \sum_{i=1}^{n} ln(\frac{e^{x_i^T w_i}}{\sum_{j=1}^{k} e^{x_i^T w_j}})$$

$$\mathcal{L} = \sum_{i=1}^{n} (x_i^T w_i - ln(\sum_{j=1}^{k} e^{x_i^T w_j}))$$

b. Calculate $\nabla_{w_i} \mathcal{L}$ and $\nabla^2_{w_i} \mathcal{L}$.

$\nabla_{w_i} \mathcal{L}$

Beginning with the log likelihood:

$$lnL = \sum_{i=1}^{n} (x_i^T w_i - ln(\sum_{j=1}^{k} e^{x_i^T w_j}))$$

$$\nabla_{w_i} \mathcal{L} = \sum_{i=1}^{n} (x_i - \frac{1}{\sum_{j}^{k} e^{x_i^T w_j}} (e^{x_i^T w_i}) x_i)$$

$$\nabla_{w_i} \mathcal{L} = \sum_{i=1}^{n} (x_i)(1 - \frac{e^{x_i^T w_i}}{\sum_{j}^{k} e^{x_i^T w_j}})$$

The numerator $e^{x_i^T w_i}$ makes sense because we take the gradient with respect to w_i. Therefore, the terms of the summation are 0 for every $w_j, w_j \neq w_i$. To interpret the gradient, we see how the greater the probability of x_i, the smaller $\nabla_{w_i} lnL$, or the less of an impact a change in w_i will have on the likelihood.

$\nabla^2_{w_i}\mathcal{L}$

To calculate the Hessian, we consider each term $\mathcal{L}_l$ separately, and calculate the second derivative for that term.

This gives us:

$$\nabla_{w_i}\mathcal{L}_l = \sum_{i=1}^{n}(x_{il})(1 - \frac{e^{x_i^T w_i}}{\sum_j^k e^{x_i^T w_j}})$$

Using the quotient rule to take the derivative of the rightmost term:

$$\nabla^2_{w_i}\mathcal{L}_l = \sum_{i=1}^{n}(x_{il})(1 - \frac{x_i e^{x_i^T w_i}\sum_j^k e^{x_i^T w_j} - x_i e^{2x_i^T w_i}}{(\sum_j^k e^{x_i^T w_j})^2})$$

$$\nabla^2_{w_i}\mathcal{L}_l = \sum_{i=1}^{n}(x_{il})(1 - \frac{x_i(e^{x_i^T w_i}\sum_j^k e^{x_i^T w_j} - e^{2x_i^T w_i})}{(\sum_j^k e^{x_i^T w_j})^2})$$

If we set $c = e^{x_i^T w_i}\sum_j^k e^{x_i^T w_j} - e^{2x_i^T w_i}$ and $d = (\sum_j^k e^{x_i^T w_j})^2$, both scalars, we have:

$$\nabla^2_{w_i}\mathcal{L}_l = \sum_{i=1}^{n}(x_{il})(1 - \frac{x_i c}{e})$$

As x_{il} is also a scalar (element l of gradient vector $\mathcal{L}$), and x_i is a vector, this expression evaluates to a vector. Thus, evaluating this expression for every $\mathcal{L}_l$ in $(1, 2, ...d)$ where d is the number of elements (dimensions) in $\mathcal{L}$, (and summing across every x_i in X), we generate the Hessian matrix:

$$\nabla^2_{w_i}\mathcal{L} = [\nabla^2_{w_i}\mathcal{L}_1, \nabla^2_{w_i}\mathcal{L}_2, ..., \nabla^2_{w_i}\mathcal{L}_d]^T$$

Problem 2 (Gaussian kernels)

To begin:

$$K(u, v) = \int \phi_t(u)\phi_t(v)dt$$

Expanding:

$$K(u, v) = \int \frac{1}{(2\pi\beta')^{d/2}} e^{(-\frac{||u-t||^2}{2\beta'})} \frac{1}{(2\pi\beta')^{d/2}} e^{(-\frac{||v-t||^2}{2\beta'})} dt$$

Rearranging:

$$K(u, v) = \frac{1}{(2\pi\beta')^{d/2}} \int \frac{1}{(2\pi\beta')^{d/2}} e^{(-\frac{||u-t||^2+||v-t||^2}{2\beta'})} dt$$

Focusing only on the terms in the exponent:

$$||u-t||^2 + ||v-t||^2$$

$$= u^Tu + t^Tt - 2u^Tt + v^Tv + t^Tt - 2v^Tt$$

$$= 2||t||^2 - 2(u+v)^Tt + ||u||^2 + ||v||^2$$

$$= ||t||^2 - (u+v)^Tt + \frac{||u||^2}{2} + \frac{||v||^2}{2}$$

Then we complete the square:

$$= ||t||^2 - (u+v)^Tt + ||\frac{u+v}{2}||^2 - ||\frac{u+v}{2}||^2 + \frac{||u||^2}{2} + \frac{||v||^2}{2}$$

$$= ||t - \frac{u+v}{2}||^2 - ||\frac{u+v}{2}||^2 + \frac{||u||^2}{2} + \frac{||v||^2}{2}$$

Placing this back into context:

$$K(u,v) = \frac{1}{(2\pi\beta')^{d/2}} \int \frac{1}{(2\pi\beta')^{d/2}} e^{-\frac{1}{2\beta'}(||t-\frac{u+v}{2}||^2 - ||\frac{u+v}{2}||^2 + \frac{||u||^2}{2} + \frac{||v||^2}{2})} dt$$

Rearranging, we can form a Gaussian in t:

$$K(u,v) = \frac{1}{(2\pi\beta')^{d/2}} e^{-\frac{1}{2\beta'}(-||\frac{u+v}{2}||^2 + \frac{||u||^2}{2} + \frac{||v||^2}{2})} \int \frac{1}{(2\pi\beta')^{d/2}} e^{-\frac{1}{2\beta'}(||t-\frac{u+v}{2}||^2)} dt$$

The term to be integrated is now a well-formed Gaussian in t. We can integrate this to 1, leaving us with:

$$K(u,v) = \frac{1}{(2\pi\beta')^{d/2}} e^{-\frac{1}{2\beta'}(-||\frac{u+v}{2}||^2 + \frac{||u||^2}{2} + \frac{||v||^2}{2})}$$

Returning to the exponent, we have:

$$-||\frac{u+v}{2}||^2 + \frac{||u||^2}{2} + \frac{||v||^2}{2}$$

$$\frac{||u||^2}{2} - ||\frac{u+v}{2}||^2 + \frac{||v||^2}{2}$$

$$\frac{||u||^2}{2} - (\frac{u+v}{2})^T(\frac{u+v}{2}) + \frac{||v||^2}{2}$$

$$\frac{||u||^2}{2} - \frac{(u+v)^T(u+v)}{4} + \frac{||v||^2}{2}$$

$$\frac{||u||^2}{2} - \frac{||u||^2 + 2u^Tv + ||v||^2}{4} + \frac{||v||^2}{2}$$

$$\frac{2||u||^2 - ||u||^2 - 2u^Tv - ||v||^2 + 2||v||^2}{4}$$

$$\frac{||u||^2 - 2u^Tv + ||v||^2}{4}$$

$$\frac{||u-v||^2}{4}$$

Returning to context:

$$K(u,v) = \frac{1}{(2\pi\beta')^{d/2}} e^{-\frac{1}{2\beta'}(\frac{||u-v||^2}{4})}$$

$$K(u,v) = \frac{1}{(2\pi\beta')^{d/2}} e^{-(\frac{||u-v||^2}{8\beta'})}$$

Now, we define $\alpha = f_1(\beta')$ and $\beta = f_2(\beta')$, to achieve our desired result:

$$K(u,v) = \alpha e^{-(\frac{||u-v||^2}{\beta})}$$

Problem 3 (Classification)

K	Acc
1	.948
2	.93
3	.918
4	.902
5	.894

Table 1: Prediction accuracy for KNN

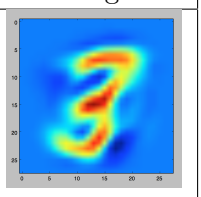
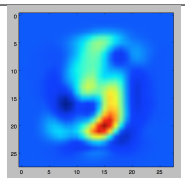
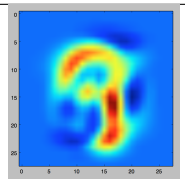
Class	Prediction	Image
3	8	
5	4	
8	9	

Table 2: Misclassified examples for K=1

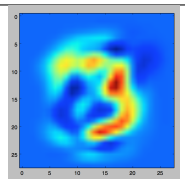
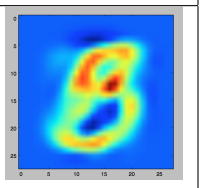
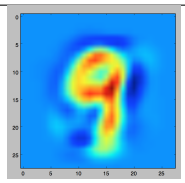
Class	Prediction	Image
3	7	
8	3	
9	4	

Table 3: Misclassified examples for K=3

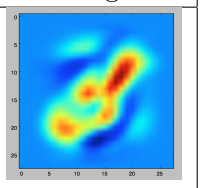
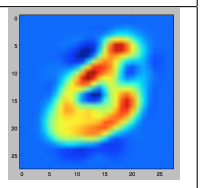
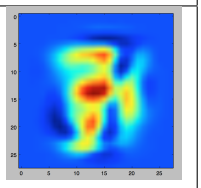
Class	Prediction	Image
0	5	
5	6	
7	2	

Table 4: Misclassified examples for K=5

	0	1	2	3	4	5	6	7	8	9
0	48	0	0	1	0	1	0	0	0	0
1	0	49	0	0	0	0	0	0	1	0
2	0	0	48	0	1	0	1	0	0	0
3	0	0	1	47	0	0	0	0	2	0
4	0	0	0	0	48	0	0	0	1	1
5	0	0	0	1	0	45	2	0	1	1
6	0	0	0	0	1	5	43	0	0	1
7	0	0	2	0	2	0	0	46	0	0
8	0	0	1	0	0	1	0	0	47	1
9	1	0	0	0	2	0	0	0	0	47

Table 5: Confusion Matrix for Bayes Classifier (Prediction accuracy = .936)

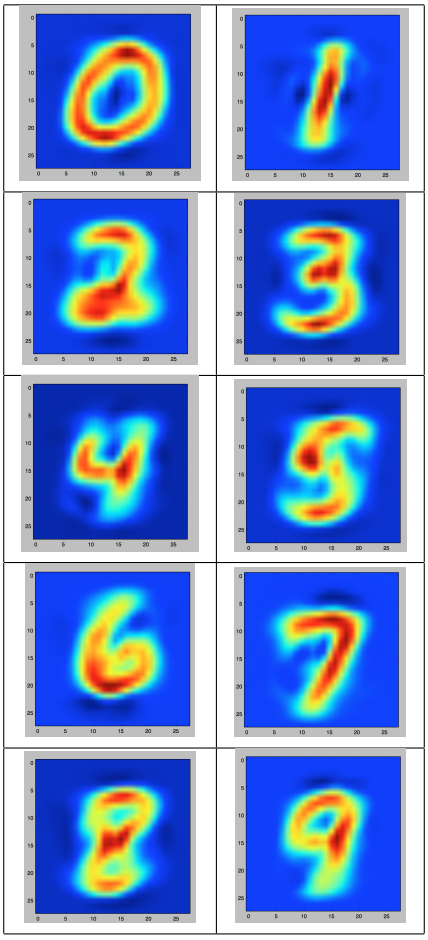


Table 6: Image means for Bayes classifier

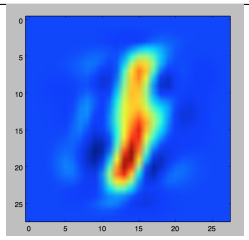
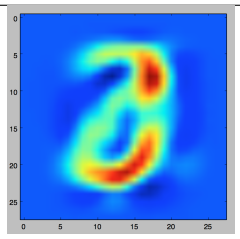
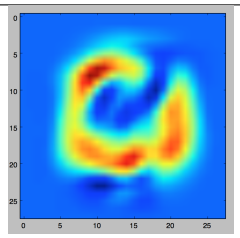
i	Class	Prediction	Image
84	1	8	
189	3	2	
456	9	0	

Table 7: Misclassified examples with Bayesian classifier

Class	Likelihood
0	1.61427396527e-30
1	0.000190440181997
2	0.000184578956757
3	1.57209260318e-08
4	0.0013882095176
5	2.43343185745e-13
6	2.49257721585e-10
7	5.52356463049e-09
8	8.23010497024
9	4.24372709689e-12

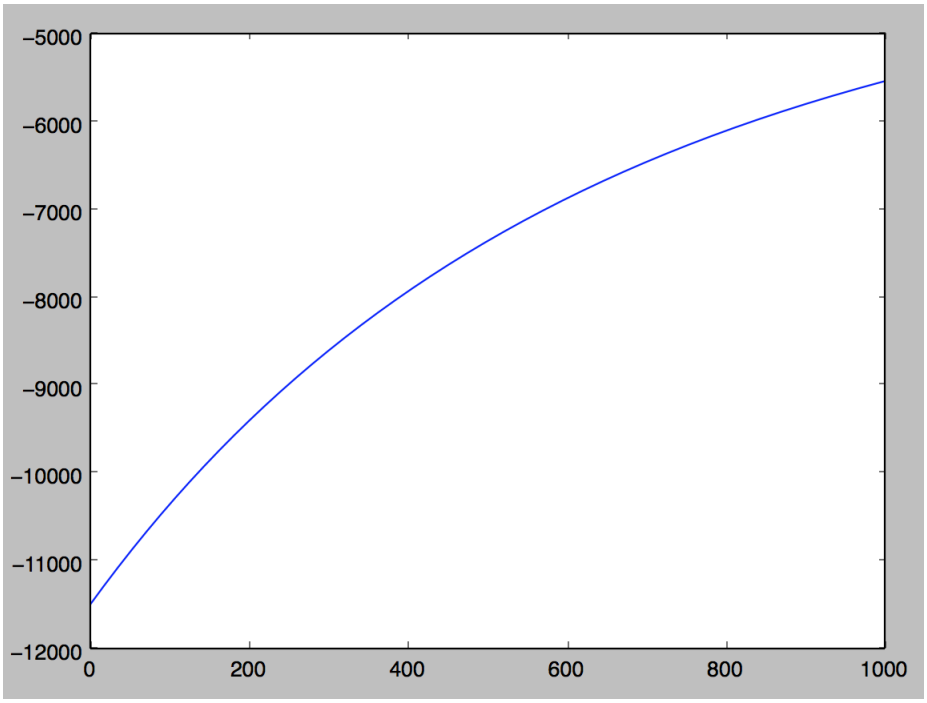
Table 8: Probability distributions for i=84

Class	Likelihood
0	7.85200802045e-05
1	5.53960717342e-132
2	0.26868012579
3	0.207323802348
4	1.37871629701e-37
5	7.0913139412e-13
6	2.10121631612e-18
7	8.64158255829e-51
8	9.39609382955e-07
9	3.50286382359e-24

Table 9: Probability distributions for i=189

Class	Likelihood
0	1.01943498002e-09
1	0.0
2	4.77168981732e-13
3	1.84914477161e-37
4	2.09951907729e-33
5	2.61861276808e-50
6	3.49193403358e-23
7	2.28328889615e-105
8	3.81858520041e-39
9	5.89572887373e-43

Table 10: Probability distributions for i=456



Plot of log likelihood error by number of iterations for Logistic Regression classifier.

	0	1	2	3	4	5	6	7	8	9
0	42	0	1	1	0	2	3	0	1	0
1	0	35	0	0	0	0	0	0	15	0
2	1	0	36	3	0	0	3	0	7	0
3	1	0	1	37	0	1	0	0	10	0
4	0	0	1	0	33	1	0	0	5	10
5	0	0	0	11	2	25	1	0	8	3
6	0	0	1	0	6	2	37	0	4	0
7	0	0	1	0	1	0	0	34	7	7
8	0	0	0	0	0	2	0	0	47	1
9	0	0	0	0	2	0	1	0	2	45

Table 11: Confusion Matrix for Logistic Regression (Prediction accuracy = 0.7419)

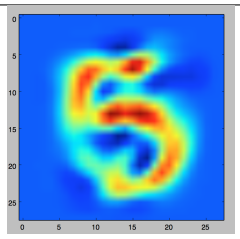
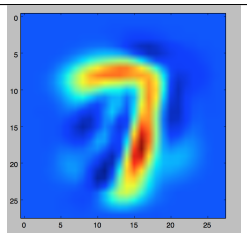
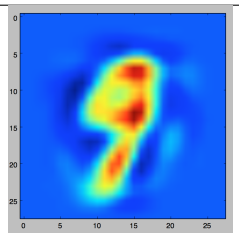
i	Class	Prediction	Image
275	5	3	
373	7	9	
477	9	8	

Table 12: Misclassified examples with Logistic Regression classifier

Class	Likelihood
0	0.056978
1	0.005312
2	0.054814
3	0.279018
4	0.054411
5	0.170424
6	0.064826
7	0.024758
8	0.178350
9	0.111109

Table 13: Softmax probability distributions for i=275

Class	Likelihood
0	0.011427
1	0.004381
2	0.043161
3	0.096474
4	0.061084
5	0.061646
6	0.017383
7	0.309109
8	0.081986
9	0.313349

Table 14: Softmax probability distributions for i=373

Class	Likelihood
0	0.009955
1	0.128215
2	0.039013
3	0.112215
4	0.039321
5	0.150457
6	0.037912
7	0.068410
8	0.256989
9	0.157514

Table 15: Softmax probability distributions for i=477