Problem 1 (multiclass logistic regression)

Part 1:

a. Write out the log likelihood L of data $(x_1, y_1), ..., (x_n, y_n)$ using an i.i.d. assumption.

$$L = \prod_{i=1}^{n} \prod_{j=1}^{k} (\frac{e^{x_i^T w_j}}{\sum_{l=1}^{k} e^{x_i^T w_l}})^{\mathbb{1}(y_i=j)}$$

First, we observe that while $p(y|x, w_1, ..., w_n)$ is a product of the likelihood of each w, the indicator effectively cancels out every $w_i, i \neq c$ as $z^{\mathbb{1}(y=c)} = z^0 = 1$, where c is the true class of data point x_i. Furthermore, since we are only considering x_i in the context of the correct class, we can eliminate the indicator function also.

With that in mind, we can cast the joint probability of the data as:

$$L = \prod_{i=1}^{n} (\frac{e^{x_i^T w_i}}{\sum_{j=1}^{k} e^{x_i^T w_j}})$$

Taking the natural log of this, we have:

$$\mathcal{L} = lnL = \sum_{i=1}^{n} ln(\frac{e^{x_i^T w_i}}{\sum_{j=1}^{k} e^{x_i^T w_j}})$$

$$\mathcal{L} = \sum_{i=1}^{n} (x_i^T w_i - ln(\sum_{j=1}^{k} e^{x_i^T w_j}))$$

b. Calculate $\nabla_{w_i} \mathcal{L}$ and $\nabla_{w_i}^2 \mathcal{L}$.

$\nabla_{w_i} \mathcal{L}$

Beginning with the log likelihood:

$$lnL = \sum_{i=1}^{n} (x_i^T w_i - ln(\sum_{j=1}^{k} e^{x_i^T w_j}))$$

$$\nabla_{w_i} \mathcal{L} = \sum_{i=1}^{n} (x_i - \frac{1}{\sum_{j}^{k} e^{x_i^T w_j}} (e^{x_i^T w_i}) x_i)$$

$$\nabla_{w_i} \mathcal{L} = \sum_{i=1}^{n} (x_i)(1 - \frac{e^{x_i^T w_i}}{\sum_{j}^{k} e^{x_i^T w_j}})$$

The numerator $e^{x_i^T w_i}$ makes sense because we take the gradient with respect to w_i. Therefore, the terms of the summation are 0 for every $w_j, w_j \neq w_i$. To interpret the gradient, we see how the greater the probability of x_i, the smaller $\nabla_{w_i} lnL$, or the less of an impact a change in w_i will have on the likelihood.

—

$\nabla^2_{w_i}\mathcal{L}$

To calculate the Hessian, we consider each term $\mathcal{L}_i$ separately, and calculate the second derivative for that term.

This gives us:

$$\nabla_{w_i}\mathcal{L}_i = \sum_{i=1}^{n}(x_i)(1 - \frac{e^{x_i^T w_i}}{\sum_j^k e^{x_i^T w_j}})$$

Using the quotient rule to take the derivative of the rightmost term:

$$\nabla_{w_i}\mathcal{L}_i = \sum_{i=1}^{n}(x_i)(1 - \frac{x_i e^{x_i^T w_i}\sum_j^k e^{x_i^T w_j} - x_i e^{2x_i^T w_i}}{(\sum_j^k e^{x_i^T w_j})^2})$$

Problem 2 (Gaussian kernels)

To begin:

$$K(u,v) = \int \phi_t(u)\phi_t(v)dt$$

Expanding:

$$K(u,v) = \int \frac{1}{(2\pi\beta')^{d/2}} e^{(-\frac{||u-t||^2}{2\beta'})} \frac{1}{(2\pi\beta')^{d/2}} e^{(-\frac{||v-t||^2}{2\beta'})} dt$$

Rearranging:

$$K(u,v) = \frac{1}{(2\pi\beta')^{d/2}} \int \frac{1}{(2\pi\beta')^{d/2}} e^{(-\frac{||u-t||^2+||v-t||^2}{2\beta'})} dt$$

Focusing only on the terms in the exponent:

$$||u-t||^2 + ||v-t||^2$$

$$= u^T u + t^T t - 2u^T t + v^T v + t^T t - 2v^T t$$

$$= 2||t||^2 - 2(u+v)^T t + ||u||^2 + ||v||^2$$

$$= ||t||^2 - (u+v)^T t + \frac{||u||^2}{2} + \frac{||v||^2}{2}$$

Then we complete the square:

$$= ||t||^2 - (u+v)^T t + ||\frac{u+v}{2}||^2 - ||\frac{u+v}{2}||^2 + \frac{||u||^2}{2} + \frac{||v||^2}{2}$$

$$= ||t - \frac{u+v}{2}||^2 - ||\frac{u+v}{2}||^2 + \frac{||u||^2}{2} + \frac{||v||^2}{2}$$

Placing this back into context:

$$K(u,v) = \frac{1}{(2\pi\beta')^{d/2}} \int \frac{1}{(2\pi\beta')^{d/2}} e^{-\frac{1}{2\beta'}(||t-\frac{u+v}{2}||^2 - ||\frac{u+v}{2}||^2 + \frac{||u||^2}{2} + \frac{||v||^2}{2})} dt$$

Rearranging, we can form a Gaussian in t:

$$K(u,v) = \frac{1}{(2\pi\beta')^{d/2}} e^{-\frac{1}{2\beta'}(-||\frac{u+v}{2}||^2 + \frac{||u||^2}{2} + \frac{||v||^2}{2})} \int \frac{1}{(2\pi\beta')^{d/2}} e^{-\frac{1}{2\beta'}(||t-\frac{u+v}{2}||^2)} dt$$

The term to be integrated is now a well-formed Gaussian in t. We can integrate this to 1, leaving us with:

$$K(u,v) = \frac{1}{(2\pi\beta')^{d/2}} e^{-\frac{1}{2\beta'}(-||\frac{u+v}{2}||^2 + \frac{||u||^2}{2} + \frac{||v||^2}{2})}$$

Returning to the exponent, we have:

$$-||\frac{u+v}{2}||^2 + \frac{||u||^2}{2} + \frac{||v||^2}{2}$$

$$\frac{||u||^2}{2} - ||\frac{u+v}{2}||^2 + \frac{||v||^2}{2}$$

$$\frac{||u||^2}{2} - (\frac{u+v}{2})^T(\frac{u+v}{2}) + \frac{||v||^2}{2}$$

$$\frac{||u||^2}{2} - \frac{(u+v)^T(u+v)}{4} + \frac{||v||^2}{2}$$

$$\frac{||u||^2}{2} - \frac{||u||^2 + 2u^Tv + ||v||^2}{4} + \frac{||v||^2}{2}$$

$$\frac{2||u||^2 - ||u||^2 - 2u^Tv - ||v||^2 + 2||v||^2}{4}$$

$$\frac{||u||^2 - 2u^Tv + ||v||^2}{4}$$

$$\frac{||u-v||^2}{4}$$

Returning to context:

$$K(u,v) = \frac{1}{(2\pi\beta')^{d/2}} e^{-\frac{1}{2\beta'}(\frac{||u-v||^2}{4})}$$

$$K(u,v) = \frac{1}{(2\pi\beta')^{d/2}} e^{-(\frac{||u-v||^2}{8\beta'})}$$

Now, we define $\alpha = f_1(\beta')$ and $\beta = f_2(\beta')$, to achieve our desired result:

$$K(u,v) = \alpha e^{-(\frac{||u-v||^2}{\beta})}$$

Problem 3 (Classification)

Part A: KNN

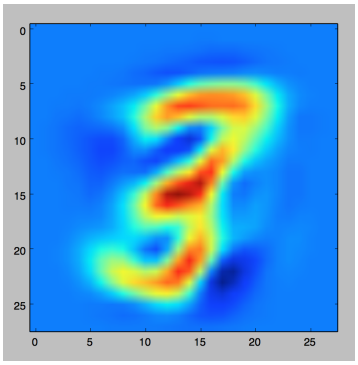
K	Acc
1	.948
2	.93
3	.918
4	.902
5	.894

Table 1: Prediction accuracy for KNN

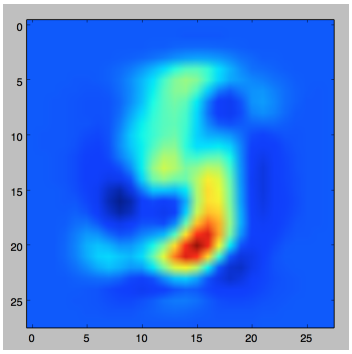
Misclassified examples:

K = 1:

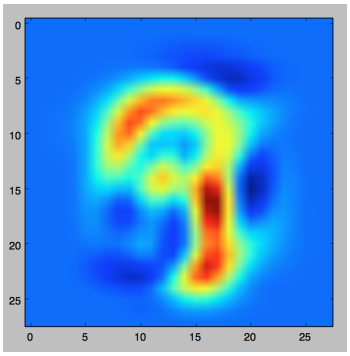
Class: 3, Prediction: 8



Class: 5, Prediction: 4

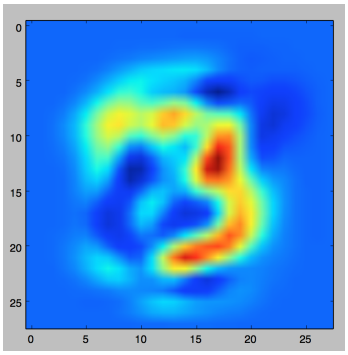


Class: 8, Prediction: 9

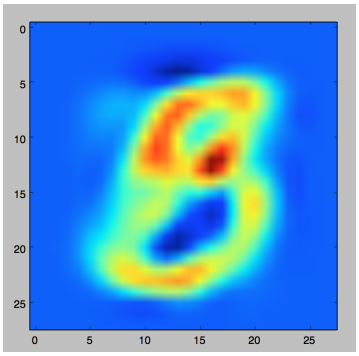


K = 3:

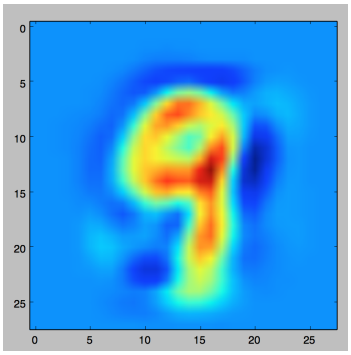
Class: 3, Prediction: 7



Class: 8, Prediction: 3

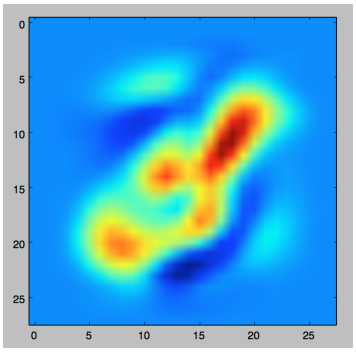


Class: 9, Prediction: 4

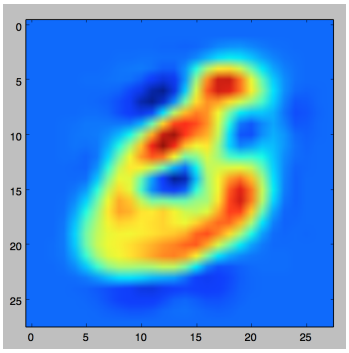


Class: 0, Prediction: 5

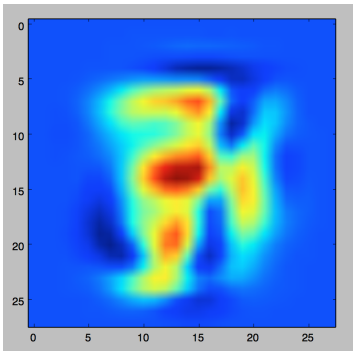
K = 5:



Class: 5, Prediction: 6



Class: 7, Prediction: 2



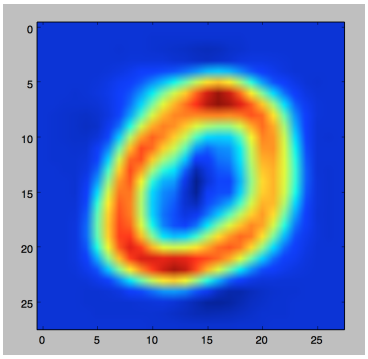
Part B: Bayes

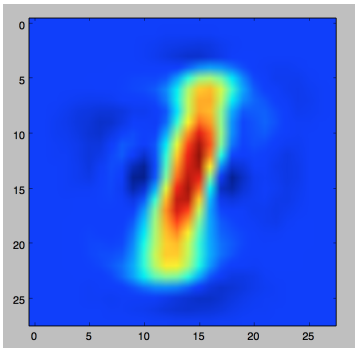
	0	1	2	3	4	5	6	7	8	9
0	48	0	0	1	0	1	0	0	0	0
1	0	49	0	0	0	0	0	0	1	0
2	0	0	48	0	1	0	1	0	0	0
3	0	0	1	47	0	0	0	0	2	0
4	0	0	0	0	48	0	0	0	1	1
5	0	0	0	1	0	45	2	0	1	1
6	0	0	0	0	1	5	43	0	0	1
7	0	0	2	0	2	0	0	46	0	0
8	0	0	1	0	0	1	0	0	47	1
9	1	0	0	0	2	0	0	0	0	47

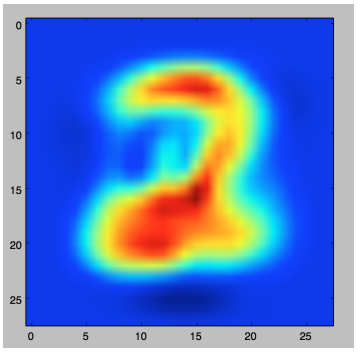
Table 2: Confusion Matrix for Bayes Classifier

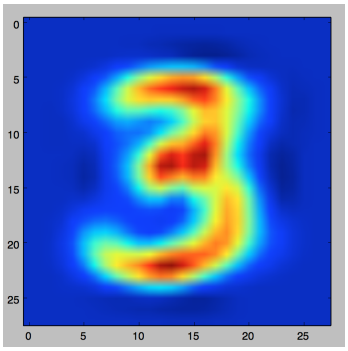
Prediction accuracy: 0.93600000000000005

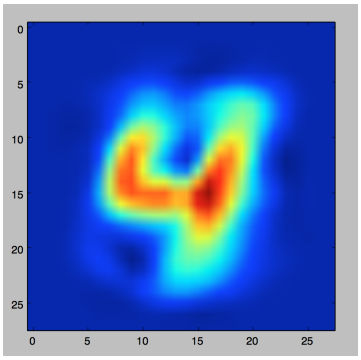
Image means:

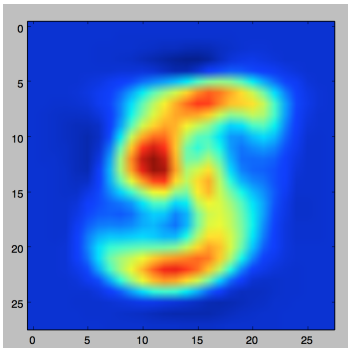


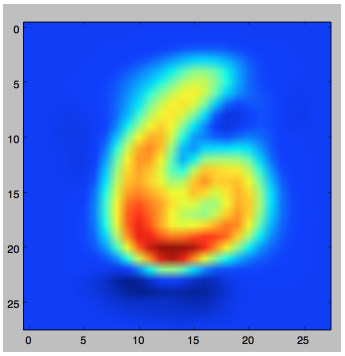


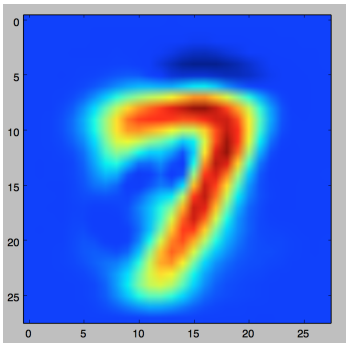


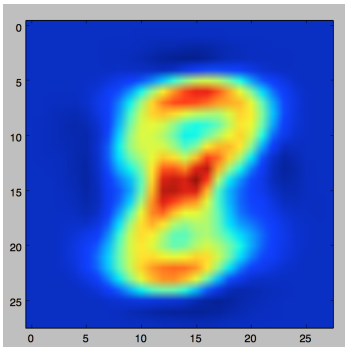

0
5
10
15
20
25
0 5 10 15 20 25

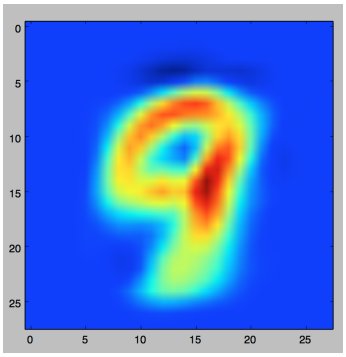

0
5
10
15
20
25
0 5 10 15 20 25


0
5
10
15
20
25
0 5 10 15 20 25


0
5
10
15
20
25
0 5 10 15 20 25





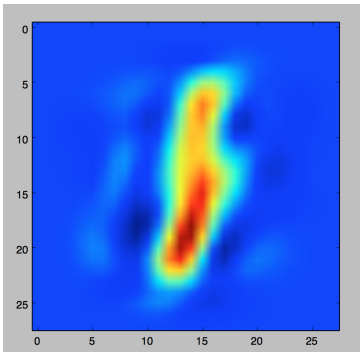


Misclassified examples:

X 84

Class: 1

Prediction: 8



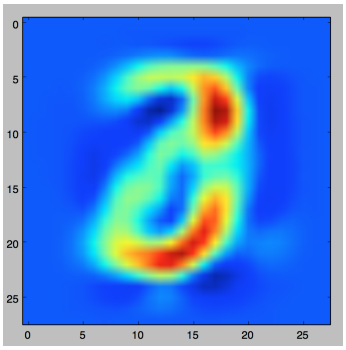
X 189

Class: 3

Class	Likelihood
0	1.61427396527e-30
1	0.000190440181997
2	0.000184578956757
3	1.57209260318e-08
4	0.0013882095176
5	2.43343185745e-13
6	2.49257721585e-10
7	5.52356463049e-09
8	8.23010497024
9	4.24372709689e-12

Table 3: Probability distributions

Prediction: 2



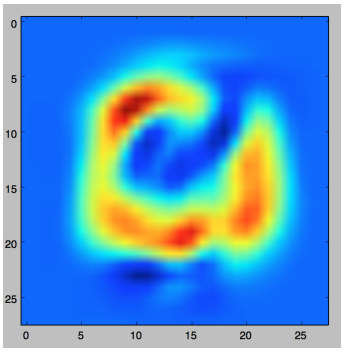
Class	Likelihood
0	7.85200802045e-05
1	5.53960717342e-132
2	0.26868012579
3	0.207323802348
4	1.37871629701e-37
5	7.0913139412e-13
6	2.10121631612e-18
7	8.64158255829e-51
8	9.39609382955e-07
9	3.50286382359e-24

Table 4: Probability distributions

X 456

Class: 9

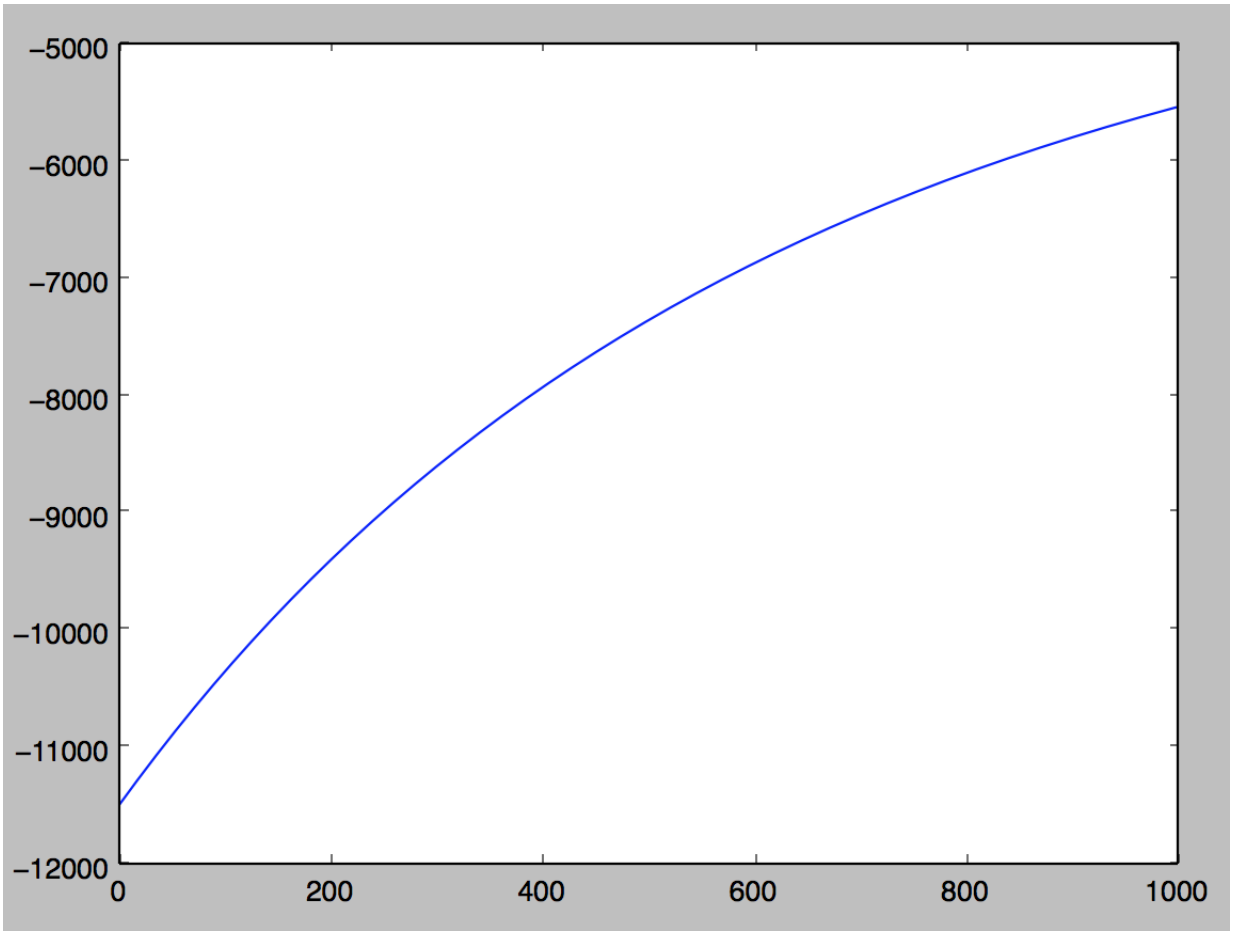
Prediction: 0



Class	Likelihood
0	1.01943498002e-09
1	0.0
2	4.77168981732e-13
3	1.84914477161e-37
4	2.09951907729e-33
5	2.61861276808e-50
6	3.49193403358e-23
7	2.28328889615e-105
8	3.81858520041e-39
9	5.89572887373e-43

Table 5: Probability distributions

Part C: Logistic Regression



Plot of log likelihood error by number of iterations.

	0	1	2	3	4	5	6	7	8	9
0	42	0	1	1	0	2	3	0	1	0
1	0	35	0	0	0	0	0	0	15	0
2	1	0	36	3	0	0	3	0	7	0
3	1	0	1	37	0	1	0	0	10	0
4	0	0	1	0	33	1	0	0	5	10
5	0	0	0	11	2	25	1	0	8	3
6	0	0	1	0	6	2	37	0	4	0
7	0	0	1	0	1	0	0	34	7	7
8	0	0	0	0	0	2	0	0	47	1
9	0	0	0	0	2	0	1	0	2	45

Table 6: Confusion Matrix for Logistic Regression Classifier

Prediction accuracy: 0.74199999999999999