Reconstruction of one-punctured elliptic curves in positive characteristic by their geometric fundamental groups

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2019/03/12

Anabelian Geometry

k : a finitely generated extension field of prime fields
U : a scheme $/k$

U is "anabelian" $\Rightarrow$
the geometry of U can be recovered from $\pi_1(U)$

If U is a smooth geometrically connected curve $/k$,

$$U \text{ is "anabelian"} \overset{?}{\Leftrightarrow} U \text{ is hyperbolic} \overset{\text{def}}{\Leftrightarrow} 2-2g-n<0$$

Grothendieck conjecture for (hyperbolic) curves

k : (finitely generated field $/\mathbb{Q}$, $g = 0$) $\rightarrow$ OK (Nakamura)

k : (finite field, $n > 0$) or
(finitely generated field $/\mathbb{Q}$, $n > 0$) $\rightarrow$ OK (Tamagawa)

k : (finite field) or
(sub-p-adic ($k \hookrightarrow \exists L$: fin. gen. $/\mathbb{Q}_p$)) $\rightarrow$ OK (Mochizuki)

k : alg. cl. field of positive characteristic $\rightarrow$ today

(k : alg. cl. field of characteristic 0 $\Rightarrow \pi_1(U) \simeq \Pi_{g,n}$)

Main result

Theorem (Tamagawa)

p, p': prime numbers
$U = (\mathbb{P}^1 \backslash S) \,/\, \overline{\mathbb{F}_p}, \ \ \#S > 0$
U' : a (smooth connected) curve / $\overline{\mathbb{F}_{p'}}$
Then,

$$\pi_1(U) \simeq \pi_1(U') \Rightarrow U \simeq_{sch} U'$$

Theorem (S.)

p : an odd prime number
p': a prime number
$U = (E \backslash S) \,/\, \overline{\mathbb{F}_p}, \ \ \#S = 1$ *($\exists E$: an elliptic curve / $\overline{\mathbb{F}_p}$)*
U' : a (smooth connected) curve / $\overline{\mathbb{F}_{p'}}$
Then,

$$\pi_1(U) \simeq \pi_1(U') \Rightarrow U \simeq_{sch} U'$$

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1. Reconstruction of various invariants (Tamagawa)
2. Linear relations of the images in $\mathbb{P}^1$
3. Combination of two additive structures

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Notation

k : an algebraically closed field of positive characteristic
p : the characteristic of k
U : a smooth connected curve $/k$
X : the smooth compactification of U
$g = g_U$: the genus of X
$S_U = X \backslash U$, $n = n_U = \#(S_U)$
$\pi_1(U)$: the étale fundamental group of U
$\pi_1^{tame}(U)$: the tame fundamental group of U
G^{ab} : the abelianization of a profinite group G
G^p : the maximal pro-p quotient of a profinite group G
$(= \lim_{H \lhd_{op} G, p \nmid [G:H]} G/H)$
$G^{p'}$: the maximal prime-to-p quotient of a profinite group G
$(= \lim_{H \lhd_{op} G, p \nmid [G:H]} G/H)$
$r = r_U$: the p-rank of the Jacobian variety of X
(hence $0 \leq r \leq g$)

$\pi_1(U) \rightsquigarrow p$ (if $(g, n) \neq (0, 0)$)

Let $\epsilon = \begin{cases} 0 \ (n = 0) \\ 1 \ (n > 0) \end{cases}$

Theorem (Corollary of G.A.G.A. theorems)

$$\pi_1^{(-)}(U)^{ab} \simeq \begin{cases} (\hat{\mathbb{Z}}^{p'})^{\oplus 2g+n-\epsilon} \times \mathbb{Z}_p^{\oplus r} & (n = 0 \textit{ or } (-) = \textit{tame}) \\ (\hat{\mathbb{Z}}^{p'})^{\oplus 2g+n-\epsilon} \times \prod_{j \in J} \mathbb{Z}_p & (n > 0 \textit{ and } (-) = \textit{unrestricted}) \end{cases}$$

here, $\#J = \#k$

l : prime number

$p = l \Leftrightarrow \pi_1(U)^{ab,l'}$ is a free $\hat{\mathbb{Z}}^{l'}$-module

$\therefore \pi_1(U) \rightsquigarrow p$

$\pi_1(U) \rightsquigarrow \chi = 2 - 2g - n$

$$\left(\pi_1(U)^{ab} \simeq \begin{cases} (\hat{\mathbb{Z}}^{p'})^{\oplus 2g} \times \mathbb{Z}_p^{\oplus r} & (n=0) \\ (\hat{\mathbb{Z}}^{p'})^{\oplus 2g+n-1} \times \prod_{i \in I} \mathbb{Z}_p, \ \#I = \#k & (n>0) \end{cases} \right)$$

Then, $\epsilon = 0 \Leftrightarrow n = 0 \Leftrightarrow \pi_1(U)^{ab}$ is finitely generated $\hat{\mathbb{Z}}$-module

$\therefore \pi_1(U) \rightsquigarrow \epsilon$

$\chi = 2 - \epsilon - rank_{\hat{\mathbb{Z}}^{p'}}(\pi_1(U)^{ab,p'})$

$\therefore \pi_1(U) \rightsquigarrow \chi$

$\pi_1(U) \rightsquigarrow r$

By Hurwitz's formula,
$ker(\pi_1(U) \to \pi_1^{tame}(U)) \subset H \Leftrightarrow \chi_H = (\pi_1(U) : H)\chi$
$\therefore \pi_1(U) \rightsquigarrow \pi_1^{tame}(U)$

$r = rank_{\mathbb{Z}_p}(\pi_1^{tame}(U)^{ab,p})$
$\therefore \pi_1(U) \rightsquigarrow r$

$(\pi_1^{tame}(U)^{ab} \simeq (\hat{\mathbb{Z}}^{p'})^{\oplus 2g+n-\epsilon} \times \mathbb{Z}_p^{\oplus r})$

$\pi_1(U) \rightsquigarrow (g, n)$

$(\pi_1(U) \rightsquigarrow \epsilon)$
$\underline{n = 0}$
$g = \frac{1}{2}(2 - \chi)$
$\therefore \pi_1(U) \rightsquigarrow (g, n)$

$\pi_1(U) \rightsquigarrow (g, n)$

$\underline{n > 0}$

Theorem (Deuring-Shafarevich formula)

Let $H \lhd_{op} \pi_1(U)$ such that $[\pi_1(U) : H] = p^m$.
Then, $r_H - 1 + n_H = (\pi_1(U) : H)(r - 1 + n)$

Clearly, $n_H \geq n$ holds.
Thus, $n \geq \frac{1}{p-1} max_{H \lhd_{op} \pi_1(U), [\pi_1(U):H]=p}(r_H - 1 - p(r-1))$ holds.

Using Riemann-Roch theorem, we can prove the existence of an étale covering $U_H \to U$ such that $n_H = n$.
Thus, $n = \frac{1}{p-1} max_{H \lhd_{op} \pi_1(U), [\pi_1(U):H]=p}(r_H - 1 - p(r-1))$ holds.
$\therefore \pi_1(U) \rightsquigarrow (g, n)$

$\pi_1(U) \rightsquigarrow \pi_1(X)$

By Hurwitz's formula,
$ker(\pi_1(U) \to \pi_1(X)) \subset H \Leftrightarrow 2g_H - 2 = (\pi_1(U) : H)(2g - 2)$
$\therefore \pi_1(U) \rightsquigarrow \pi_1(X)$

$\pi_1(U) \rightsquigarrow S_U$ (only construction)

K : the function field of U
$\tilde{K}$: the maximal Galois extension of K in K^{sep} that is unr. over U
$\tilde{X}$: the normalization of X in $\tilde{K}$
$\tilde{S_U}$: the inverse image of S_U under $\tilde{X} \to X$
$Sub(G)$: the set of closed subgroups of G
$I_{\tilde{P}} \in Sub(\pi_1(U))$: the inertia subgroup associated to $\tilde{P} \in \tilde{S_U}$

By using the discussion of the tame case and representation theory of finite groups, we can prove that $\tilde{S_U} \to Sub(\pi_1(U))$ $(\tilde{P} \mapsto I_{\tilde{P}})$ is injective and $\pi_1(U) \rightsquigarrow Im(\tilde{S_U} \to Sub(\pi_1(U)))$.

We can identify S_U with $\tilde{S_U}/\pi_1(U)$.

Summary of this section

$$\pi_1(U) \rightsquigarrow p,\ g,\ n,\ \pi_1(X),\ S_U$$

In the situation of the main result, we see that U and U' are defined over $\overline{\mathbb{F}_p}$ and $(g_U, n_U) = (g_{U'}, n_{U'})$.

$$\begin{tikzcd} H \arrow[r, leftrightarrow, "\sim"] \arrow[d, hook, "op"] & H' \arrow[d, hook, "op"] \\ \pi_1(U) \arrow[r, leftrightarrow, "\sim"] & \pi_1(U') \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} U_H \arrow[d, "f\acute{e}t"] & U'_{H'} \arrow[d, "f\acute{e}t"] \\ U & U' \end{tikzcd}$$

$$\Rightarrow S_{U_H} \simeq S_{U'_{H'}}$$

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Notation and assumptions

In this section, we assume that X is a hyperelliptic curve and $p \neq 2$.

$x : X \to \mathbb{P}^1$: a finite morphism of degree 2
with ramified points $\lambda_0, \lambda_\infty, \lambda_1, \cdots, \lambda_{2g}$

We also assume that $x^{-1}(x(S_U)) = S_U$, $\lambda_0, \lambda_\infty, \lambda_1, \cdots, \lambda_{2g} \in S_U$ and $\{\lambda_0, \lambda_\infty, \lambda_1, \cdots, \lambda_{2g}\} \neq S_U$.

$\varphi : \pi_1(U) \to \pi_1(\mathbb{P}^1 \backslash x(S_U))$
$\psi : \pi_1(\mathbb{P}^1 \backslash x(S_U)) \to \pi_1(\mathbb{P}^1 \backslash x(S_U))^{ab,p'}$
$L_U = ker(\psi \circ \varphi)$

$(\pi_1(U), L_U) \rightsquigarrow x(S_U)$

For each $\mu \in S_U$ and $P \in x(S_U)$, we fix $\tilde{\mu} \in \tilde{S_U}$ above μ and $\tilde{P} \in \tilde{S_U}$ above P respectively.
($\tilde{X}$= the normalization of $\mathbb{P}^1$ in $\tilde{K}$)
By G.A.G.A. theorems, if $x(\mu) = P$,

$$(\psi \circ \varphi)(I_{\tilde{\mu}}) = \begin{cases} \psi(I_{\tilde{P}}) & (x \text{ is unramified at } \lambda) \\ 2\psi(I_{\tilde{P}}) & (x \text{ is ramified at } \lambda) \end{cases}$$

Thus, for any μ and $\nu \in S_U$,

$$\mu \sim \nu \overset{\text{def}}{\Leftrightarrow} x(\mu) = x(\nu) \Leftrightarrow$$
$$(I_{\tilde{\mu}} L_U)/L_U = (\psi \circ \varphi)(I_{\tilde{\mu}}) = (\psi \circ \varphi)(I_{\tilde{\nu}}) = (I_{\tilde{\nu}} L_U)/L_U$$

We can identify $x(S_U)$ with $S_U/\sim$.
$\therefore (\pi_1(U), L_U) \rightsquigarrow x(S_U)$

Additive structure on $\mathbb{P}^1(k)\backslash\{P_\infty\}$ ass. to P_0 and P_∞

Fix P_0 and $P_\infty \in \mathbb{P}^1(k)$ s.t. $P_0 \neq P_\infty$. Let $\phi : \mathbb{P}^1 \simeq \mathbb{P}^1$ be a k-isomorphism such that $\phi(P_0) = 0$ and $\phi(P_\infty) = \infty$.
Then the bijection $\mathbb{P}^1(k)\backslash\{P_\infty\} \simeq \mathbb{P}^1(k)\backslash\{\infty\} = k$ does not depend on the choice of ϕ up to scalar multiplication.
Then the additive str. on k induces one on $\mathbb{P}^1(k)\backslash\{P_\infty\}$

Thus, we can define a linear relation of $x(S_U)\backslash\{x(\lambda_\infty)\}$ ass. to $x(\lambda_0)$ and $x(\lambda_\infty)$

$$\sum_{P \in x(S_U)\backslash\{x(\lambda_\infty)\}, a_P \in \mathbb{F}_p} a_P P = 0$$

$(\pi_1(U), L_U) \rightsquigarrow \sum_{P \in x(S_U)\setminus\{x(\lambda_\infty)\}, a_P \in \mathbb{F}_p} a_P P = 0$ or not (sketch)

Step 1(construct a suitable covering)
Let $\tilde{a_P} \in \{0, 1, \cdots, p-1\} \subset \mathbb{Z}$ s.t. $\tilde{a_P}$ mod $p = a_P$, $s = \sum_P \tilde{a_P}$
and $H \lhd_{op} \pi_1(U)$ the open normal subgroup of $\pi_1(U)$
corresponding to the Kummer covering defined by
$y^{p-1} = (x - P_0)^{s-1} \prod_{P \in x(S_U)\setminus\{P_0, P_\infty\}} (x - P)^{-\tilde{a_P}}$

exponent of poly. $\leftrightarrow$ ramification index $\leftrightarrow$ index of inertia subgp.

$\therefore (\pi_1(U), L_U) \rightsquigarrow H$

$(\pi_1(U), L_U) \rightsquigarrow \sum_{P\in x(S_U)\setminus\{x(\lambda_\infty)\}, a_P\in\mathbb{F}_p} a_P P = 0$ or not (sketch)

Step 2

By Artin-Schreier theory,

$$Hom(\pi_1(X_H)^{ab}/p, \mathbb{F}_p)) = Hom_{conti}(\pi_1(X_H), \mathbb{F}_p)) = H^1_{et}(X_H, \mathbb{F}_p)$$
$$= H^1(X_H, \mathcal{O}_{X_H})[F-1]$$

Thus, $(\pi_1(U), L_U) \rightsquigarrow (H^1(X_H, \mathcal{O}_{X_H})[F-1] = 0$ or not$)$

By calculating the Frobenius map F and using the defining equation of X_H, we see that the vanishing of (a part of) $H^1(X_H, \mathcal{O}_{X_H})[F-1]$ is equivalent to the linear relation.

$\therefore (\pi_1(U), L_U) \rightsquigarrow \sum_{P\in x(S_U)\setminus\{x(\lambda_\infty)\}, a_P\in\mathbb{F}_p} a_P P = 0$ or not

Summary of this section

$$(\pi_1(U), L_U) \rightsquigarrow x(S_U), \quad \sum_{P \in x(S_U)\setminus\{x(\lambda_\infty)\}, a_P \in \mathbb{F}_p} a_P P = 0$$

If we have the following diagram.

$$\begin{tikzcd}
L_{U_H} \arrow[r, leftrightarrow, "\sim"] \arrow[d, hook] & L_{U'_{H'}} \arrow[d, hook] \\
H \arrow[r, leftrightarrow, "\sim"] \arrow[d, hook, "op"] & H' \arrow[d, hook, "op"] \\
\pi_1(U) \arrow[r, leftrightarrow, "\sim"] & \pi_1(U')
\end{tikzcd}
\quad \longleftrightarrow \quad
\begin{tikzcd}
\mathbb{P}^1 & & \mathbb{P}^1 & \\
& U_H \arrow[ul, "x"'] \arrow[d, "f\,\acute{e}t"] & & U'_{H'} \arrow[ul, "x'"'] \arrow[d, "f\,\acute{e}t"] \\
& U & & U'
\end{tikzcd}$$

We obtain $\sigma : x(S_{U_H}) \simeq x'(S_{U'_{H'}})$ and see that

$$\sum_{P \in x(S_U)\setminus\{x(\lambda_\infty)\}, a_P \in \mathbb{F}_p} a_P P = 0$$
$$\Leftrightarrow \quad \sum_{P \in x(S_U)\setminus\{x(\lambda_\infty)\}, a_P \in \mathbb{F}_p} a_P \sigma(P) = 0$$

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Notation and assumptions

In this section, we asuume that $k \simeq \overline{\mathbb{F}_p}$, $g = 1$ and $\#(X \backslash U) = 1$. Let $\{\mathcal{O}\} = X \backslash U$.

$\pi_1(X\backslash\{\mathcal{O}\}) \rightsquigarrow \pi_1(X\backslash X[m])$

$$\pi_1(X\backslash X[m]) \simeq ker(\pi_1(X\backslash\{\mathcal{O}\}) \to \pi_1(X) \to \pi_1(X)/m)$$
$$\therefore \pi_1(X\backslash\{\mathcal{O}\}) \rightsquigarrow \pi_1(X\backslash X[m])$$

$\pi_1(X\backslash\{\mathcal{O}\}) \rightsquigarrow X[m]$ with a group structure

We already know $\pi_1(X\backslash\{\mathcal{O}\}) \rightsquigarrow \pi_1(X\backslash X[m]) \rightsquigarrow X[m]$

Fix $\mathcal{P} \in X[m]$
The action of $\pi_1(X\backslash\{\mathcal{O}\})/\pi_1(X\backslash X[m])$ ($\simeq X[m]$) on $X[m]$ defines the group structure on $X[m]$ with identity $\mathcal{P}$

$\therefore \pi_1(X\backslash\{\mathcal{O}\}) \rightsquigarrow X[m]$ with a group structure

$\pi_1(X\backslash\{\mathcal{O}\}) \rightsquigarrow L_{X\backslash X[m]}$

$\pi_1(X\backslash\{\mathcal{O}\}) \rightsquigarrow L_{X\backslash X[m]} \Leftrightarrow$
$\pi_1(X\backslash\{\mathcal{O}\}) \rightsquigarrow M \overset{\text{def}}{=} ker(\pi_1(X\backslash X[m])^{ab,p'} \to \pi_1(\mathbb{P}^1\backslash x(X[m]))^{ab,p'})$

Let $W \overset{\text{def}}{=}$ the sum of all inertia subgroups in $\pi_1(X\backslash X[m])^{ab,p'}$
$W^- \overset{\text{def}}{=} W \cap M$
By observing the action of $X[m]$ on W and $\pi_1(X\backslash X[m])^{ab,p'}$, we can prove

$\pi_1(X\backslash\{\mathcal{O}\}) \rightsquigarrow (W^-, (\pi_1(X\backslash X[m])^{ab,p'})^{X[m]})$

$\pi_1(X\backslash\{\mathcal{O}\}) \rightsquigarrow L_{X\backslash X[m]}$

Fact

- $\pi_1(X\backslash X[m])^{ab,p'}/M$ is torsion free
- $(\pi_1(X\backslash X[m])^{ab,p'})^{X[m]} \subset M$
- $\#(M/(\pi_1(X\backslash X[m])^{ab,p'})^{X[m]} + W^-)) < \infty$

$\therefore \pi_1(X\backslash\{\mathcal{O}\}) \rightsquigarrow L_{X\backslash X[m]}$

Reconstruction of λ invariants

Assume X (resp. X') is defined by $y^2 = x(x-1)(x-\lambda)$ (resp. $y^2 = x(x-1)(x-\lambda')$), $\mathcal{O} = \infty$ (resp. $\mathcal{O}' = \infty$) and $\pi_1(X \backslash \{\mathcal{O}\}) \simeq \pi_1(X' \backslash \{\mathcal{O}'\})$.
Let f (resp. f') $\in \mathbb{F}_p[T]$ be the minimal polynomial of λ (resp. λ').
By taking suitable m, we can assume that $(1, *_1), (\lambda, *_\lambda), (\lambda^2, *_{\lambda^2}), \cdots, (\lambda^{deg(f)}, *_{\lambda^{deg(f)}}) \in X[m]$ (here, $(*_\nu)^2 = \nu(\nu-1)(\nu-\lambda)$)

Reconstruction of λ invariants

$$\pi_1(X\backslash\{\mathcal{O}\}) \rightsquigarrow \begin{cases} \sum_{P\in x(X[m])\backslash\{x(\infty)\}, a_P\in\mathbb{F}_p} a_P P = 0 \\ \text{group structure of } X[m] \end{cases}$$

By the addition law of elliptic curves,

- $x((\lambda^i, *_{\lambda^i}) + (\lambda^i + 1, *_{\lambda^i+1})) + x((-\lambda^i, *_{-\lambda^i}) - \cdots$
  $= -8\lambda^{2i+1} + 4\lambda^{2i} + 4\lambda$
- $x((\lambda^i, *_{\lambda^i}) + (\lambda^i + 1, *_{\lambda^i+1})) + x((\lambda^i, *_{\lambda^i}) + (\lambda^i - 1, *_{\lambda^i-1})) - \cdots$
  $= 12\lambda^{2i} - 8\lambda^{i+1} - 8\lambda^i + 4\lambda$

$$\begin{array}{ccc} X[m] & \longleftrightarrow & X'[m] \\ 1 & \longleftrightarrow & 1 \\ \lambda & \longleftrightarrow & \lambda' \\ \lambda^2 & \longleftrightarrow & \lambda'^2 \\ & \vdots & \end{array}$$

Reconstruction of λ invariants

We can regard $f(\lambda)$ as a linear relation of $1, \lambda, \lambda^2, \cdots, \lambda^{deg(f)}$ / $\mathbb{F}_p$
$\therefore f(\lambda) = 0 \Leftrightarrow f(\lambda') = 0$
$\therefore f = f'$
There is an isom $\alpha : \overline{\mathbb{F}_p} \simeq \overline{\mathbb{F}_p}$ s.t. $\alpha(\lambda) = \lambda'$
$\therefore X \backslash \{\mathcal{O}\} \simeq (X \backslash \{\mathcal{O}\}) \times_{\overline{\mathbb{F}_p}, \alpha} \overline{\mathbb{F}_p} = X' \backslash \{\mathcal{O}'\}$

$\blacksquare$

Thank you for your attention!