Equivalent characterizations of the adjoint

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Abstract

We explore equivalent definitions for categorical adjunctions. A baseline understanding of category theory is assumed.

Contents

1 Introduction

Adjunctions are important. I should probably write more words here about the fabled history of the adjunction or the usefulness thereof, but I can't be bothered—further, this is just a silly bit that I can send to my friends; hi!! :D

Note that some of this PDF may be incorrect, and a couple properties are not verified in section 3. Throughout, let $\mathcal{C}, \mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors.

2 Definitions

We begin by defining some terminology.[1]

2.1 Universal morphisms

Let $X \in \mathcal{D}$ be an object. A **universal morphism** from X to F is an object $A \in \mathcal{C}$ and a morphism $u : X \to F(A)$ such that for all $A' \in \mathcal{C}$, letting $K = F(A')$ there is a unique morphism $\varphi : K \to L$ such that following triangle commutes:

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[dr, "u"'] & K \arrow[d, dashed, "\varphi"] \\ & L. \end{tikzcd}$$

[1] For all prerequisites, see Dummit and Foote, *Abstract Algebra*, 3rd ed., Appendix II.

2.2 Comma categories

Consider the diagram $\mathcal{A} \xrightarrow{S} \mathcal{C} \xleftarrow{T} \mathcal{B}$ of categories. The **comma category** $(S \downarrow T)$ has objects triples (A, B, f) where $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $f : S(A) \to T(B)$ and morphisms pairs $(a : A \to A', b : B \to B')$ which make the following diagram commute:

$$\begin{tikzcd} S(A) \arrow[r, "f"] \arrow[d, "S(a)"] & T(B) \arrow[d, "T(b)"] \\ S(A') \arrow[r, "f'"] & T(B') \end{tikzcd}$$

If $\mathcal{A} = A$ is a single-object category, we write $(S \downarrow T)$ as $(S(A) \downarrow T)$. We do the same thing if B has one object.

We may now rephrase the definition of a universal morphism. A **universal morphism** from Y to F is an initial object of the category $(Y \downarrow F)$. We may also now easily dualize this concept: a **universal morphism** from F to Y is a terminal object in the category $(F \downarrow Y)$.

3 Adjunctions

We now present equivalent formulations of adjunctions.

3.1 Universal morphisms and comma categories

Say F is a **left adjoint** if, for each $Y \in \mathcal{D}$, there is a universal morphism from F to Y. If the universal morphism is $u : F(X_Y) \to Y$ and $G(Y) = X_Y$, we say that F is **left adjoint to** G and write $F \dashv G$.

An immediate corollary to our above definition is the following equivalent one: a functor $F : \mathcal{C} \to \mathcal{D}$ is a **left adjoint** if the comma category $(F \downarrow Y)$ has an initial object for all objects $Y \in \mathcal{D}$

3.2 Unit–counit

Given a natural transformation η, one may easily check (as any good girl will) that the $(F\eta)_X = F(\eta_X)$ form a natural transformation $F\eta$. Then, we say F is **left adjoint to** G if there are natural transformations $\eta : id_{\mathcal{C}} \to GF$ and $\varepsilon : FG \to id_{\mathcal{D}}$ such that the following triangles commute:

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "\mathrm{id}_F"'] & FGF \arrow[d, "\varepsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "\mathrm{id}_G"'] & GFG \arrow[d, "G\varepsilon"] \\ & G \end{tikzcd}$$

3.3 Hom-set bijection

We say F is left adjoint to G if, for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, there exists a bijection $\varphi : \mathrm{Hom}(F(X), Y) \to \mathrm{Hom}(X, F(Y))$ natural in X and Y. The naturality condition just means that, given any $f : X' \to X$ and $g : Y \to Y'$, the following diagrams commute:

$$\begin{tikzcd} \mathrm{hom}(F(X), Y) \arrow[r, "\varphi"] \arrow[d, "F(f)^*"] & \mathrm{Hom}(X, G(Y)) \arrow[d, "f^*"] \\ \mathrm{Hom}(F(X'), Y) \arrow[r, "\varphi"] & \mathrm{Hom}(X', G(Y)) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{hom}(F(X), Y) \arrow[r, "\varphi"] \arrow[d, "g_*"] & \mathrm{Hom}(X, G(Y)) \arrow[d, "G(g)_*"] \\ \mathrm{Hom}(F(X), Y') \arrow[r, "\varphi"] & \mathrm{Hom}(X, G(Y')). \end{tikzcd}$$

4 Proof of equivalence

We summarize the above definitions in a theorem.
Theorem. Let $\mathcal{C}, \mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. The following are equivalent:

1. For each object $Y \in \mathcal{D}$, there is a universal morphism $u : F(X_Y) \to Y$ where $X_Y = G(Y)$.
2. For all objects $Y \in \mathcal{D}$, there exists an initial object $(G(Y), Y, u)$ in $(F \downarrow Y)$.
3. There are natural transformations $\eta : id_{\mathcal{C}} \to GF$ and $\varepsilon : FG \to id_{\mathcal{D}}$ such that $\varepsilon F \circ F\eta = id_F$ and $G\varepsilon \circ \eta G = id_G$
4. For all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, there exists a bijection $\varphi : \mathrm{Hom}(F(X), Y) \to \mathrm{Hom}(X, G(Y))$ natural in X and Y.

Proof. We have already seen why (1) $\Leftrightarrow$ (2).
(1) $\Rightarrow$ (3): First, we construct G. If $F(X) \to Y$ is a universal morphism, then put $G(Y) = X$. Let $FG(Y_i) \xrightarrow{u_i} Y_i$ be universal morphisms (for $i = 1, 2$) and let $f : Y_1 \to Y_2$. Since u_2 is a universal morphism, there is exactly one map φ such that the following commutative diagram commutes:

$$\begin{tikzcd}
FG(Y_1) \arrow[r, "u_1"] \arrow[d, dashed, "F(\varphi)"] & Y_1 \arrow[d, "f"] \\
FG(Y_2) \arrow[r, "u_2"] & Y_2
\end{tikzcd}$$

Then, put $G(f) = \varphi$. It is easy to show that G is a functor. Letting $\varepsilon_{Y_i} = u_i$, the above square shows that ε is a natural transformation. Let η_X be the unique map factoring $id_{F(X)}$ through the universal morphism associated with X. Pictorially, set η_X so that the diagram

$$\begin{tikzcd}
F(X) \arrow[d, dashed, "F(\eta_X)"'] \arrow[dr, "id_{F(X)}"] & \\
FGF(X) \arrow[r, "\varepsilon F_X"'] & F(X)
\end{tikzcd}$$

commutes. We put $\eta_X = \varphi$. To see that η is natural, let $f : X_1 \to X_2$ and consider the diagram

$$\begin{tikzcd}
F(X_1) \arrow[rrr, "F(f)"] \arrow[dr, "id_{F(X_1)}"] \arrow[dd, dashed, "F\eta_{X_1}"'] & & & F(X_2) \arrow[dl, "id_{F(X_2)}"'] \arrow[dd, dashed, "F\eta_{X_2}"] \\
& F(X_1) \arrow[r, "F(f)"] & F(X_2) & \\
FGF(X_1) \arrow[rrr, "FGF(f)"] \arrow[ur, "\varepsilon F_{X_1}"] & & & FGF(X_2). \arrow[ul, "\varepsilon F_{X_2}"']
\end{tikzcd}$$

The left and right triangles commute by definition, and the bottom trapezoid commutes by naturality of ε. Hence, if $P = FGF(f) \circ F\eta_{X_1}$ is the composition of the blue arrows and $Q = F\eta_{X_2} \circ F(f)$ is the composition of the red arrows, then $\varepsilon F_{X_2} \circ P = \varepsilon F_{X_2} \circ Q$. Let φ be this map. By the universality, there is exactly one arrow in $\mathcal{C}$ which (after applying F) factors φ through εF_{X_2}. Since both P and Q give such a factoring, we conclude that they are equal. This shows naturality of $F\eta$. Naturality of η is also easily seen as there is only one map in $\mathcal{C}$ (namely, the preimage of $P = Q$) which factors φ after applying F.

We also see that $\varepsilon F_X \circ F\eta_X = id_{F(X)}$ by the above diagram. By definition, we have the commutative diagram:

$$\begin{tikzcd}
FG(Y) \arrow[r, dashed, "F(\eta_{G(Y)})"] \arrow[dr, "id_{FG(Y)}"'] & FGFG(Y) \arrow[d, "\varepsilon F_Y"] \\
& FG(Y).
\end{tikzcd}$$

Our aim is to show that

$$\begin{tikzcd}
G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "id_{G(Y)}"'] & GFG(Y) \arrow[d, "G\varepsilon_Y"] \\
& G(Y)
\end{tikzcd}$$

commutes, and by universality of εF_Y, it suffices to show that $\varepsilon F_Y = G\varepsilon_Y$.

(3) $\Rightarrow$ (1): Our goal is to find a functor G and for each $Y \in \mathcal{D}$ a morphism $u : FG(Y) \to Y$ so that, for all $f : F(X) \to Y$, there exists a unique $\varphi : X \to G(Y)$ such that $u \circ F(\varphi) = f$. Visually, we want to find a commutative diagram

$$\begin{tikzcd}
F(X) \arrow[d, dashed, "F(\varphi)"] \arrow[dr, "f"] & \\
FG(Y) \arrow[r, "u"] & Y.
\end{tikzcd}$$

Let $u = \varepsilon_Y$. We then have the commutative diagram

$$\begin{tikzcd}
FG(Y') \arrow[r, "\varepsilon_{Y'}"] \arrow[d, "FG(\alpha)"] & Y' \arrow[d, "\alpha"] \\
FG(Y) \arrow[r, "u"] & Y.
\end{tikzcd}$$

Relabeling Y' with $F(X)$ and α with f and appending the morphism $F(X) \xrightarrow{(F\eta)_X} FGF(X)$ and $f : F(X) \to Y$, our diagram then becomes

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[drr, bend right, "f"'] & FGF(X) \arrow[r, "\varepsilon_{F(X)}"] \arrow[d, "FG(f)"] & F(X) \arrow[d, "f"] \\
& FG(Y) \arrow[r, "u"] & Y.
\end{tikzcd}$$

The diagram commutes because $\varepsilon F \circ F\eta = id_F$. By commutativity of the above diagram, letting $\varphi = G(f) \circ \eta_X$ we see $f = u \circ F(\varphi)$.
We now show that φ is unique. Suppose $\psi : X \to G(Y)$ is a morphism such that $f = u \circ F(\psi)$. Applying G to and composing with η_X afterwards, we have the diagram

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] & GF(X) \arrow[r, "GF\eta_X"] \arrow[dr, "GF(\psi)"'] \arrow[ddr, bend right, "G(f)"'] & GFGF(X) \arrow[d, "GFG(f)"] \\
& & GFG(Y) \arrow[d, "G\varepsilon_Y"] \\
& & G(Y).
\end{tikzcd}$$

Hence, $G(f) \circ \eta_X = G\varepsilon_Y \circ GF(\psi) \circ \eta_X$. By the naturality square

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "\psi"] & GF(X) \arrow[d, "GF(\psi)"] \\ G(Y) \arrow[r, "\eta_{G(Y)}"] & GFG(Y), \end{tikzcd}$$

we deduce that $\varphi = G(f) \circ \eta_X = G\varepsilon_Y \circ \eta G_Y \circ \psi = \psi$ and φ is unique.

(4) ⇒ (3): Let φ be the natural bijection $\hom(FX, Y) \to \hom(X, GY)$ for $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. Put $\eta_X = \varphi(id_{F(X)})$. We will show that the naturality square

$$\begin{tikzcd} X \arrow[r, "\varphi(id_{F(X)})"] \arrow[d, "f"'] & GF(X) \arrow[d, "GF(f)"] \\ X' \arrow[r, "\varphi(id_{F(X')})"'] & GF(X') \end{tikzcd}$$

commutes. Since φ is natural in both components, the diagram

$$\begin{tikzcd} \hom(F(X), F(X)) \arrow[r, "\varphi"] \arrow[d, "Ff_*"] & \mathrm{Hom}(X, GF(X)) \arrow[d, "GF(f)_*"] \\ \mathrm{Hom}(F(X), F(X')) \arrow[r, "\varphi"] & \mathrm{Hom}(X, GF(X')) \end{tikzcd}$$

commutes. Following where $id_{F(X)} \in \mathrm{Hom}(FX, FX)$ is sent, commutativity yields the equality $GF(f) \circ \varphi(id_{F(X)}) = \varphi(F(f))$. Similarly, we also have the commutative square

$$\begin{tikzcd} \hom(F(X'), F(X')) \arrow[r, "\varphi"] \arrow[d, "Ff^*"] & \mathrm{Hom}(X', GF(X')) \arrow[d, "GF(f)^*"] \\ \mathrm{Hom}(F(X), F(X')) \arrow[r, "\varphi"] & \mathrm{Hom}(X, GF(X')), \end{tikzcd}$$

and following $id_{F(X')}$, we see that $\varphi(id_{F(X')}) \circ GF(f) = \varphi(F(f))$. Combining this with our earlier equality, we conclude that $GF(f) \circ \varphi(id_{F(X)}) = \varphi(id_{F(X')}) \circ GF(f)$ and so $\eta_- = \varphi(id_{F(-)})$ is a natural transformation. That $\varepsilon_- = \varphi^{-1}(id_{G(Y)})$ is natural and that the triangle inequalities hold are left as an exercise.

(1) ⇒ (4): Exercise. □