affect the probabilities of events in future subintervals. An exponential random variable is the continuous analog of a geometric random variable, and they share a similar lack of memory property.

The exponential distribution is often used in reliability studies as the model for the time until failure of a device. For example, the lifetime of a semiconductor chip might be modeled as an exponential random variable with a mean of 40,000 hours. The lack of memory property of the exponential distribution implies that the device does not wear out. That is, regardless of how long the device has been operating, the probability of a failure in the next 1000 hours is the same as the probability of a failure in the first 1000 hours of operation. The lifetime L of a device with failures caused by random shocks might be appropriately modeled as an exponential random variable. However, the lifetime L of a device that suffers slow mechanical wear, such as bearing wear, is better modeled by a distribution such that $P(L < t + \Delta t \mid L > t)$ increases with t. Distributions such as the Weibull distribution are often used, in practice, to model the failure time of this type of device. The Weibull distribution is presented in a later section.

EXERCISES FOR SECTION 4-8

4-90. Suppose X has an exponential distribution with $\lambda = 2$. Determine the following:
(a) $P(X \le 0)$ (b) $P(X \ge 2)$
(c) $P(X \le 1)$ (d) $P(1 < X < 2)$
(e) Find the value of x such that $P(X < x) = 0.05$.

4-91. Suppose X has an exponential distribution with mean equal to 10. Determine the following:
(a) $P(X > 10)$ (b) $P(X > 20)$
(c) $P(X < 30)$
(d) Find the value of x such that $P(X < x) = 0.95$.

4-92. Suppose X has an exponential distribution with a mean of 10. Determine the following:
(a) $P(X < 5)$
(b) $P(X < 15 \mid X > 10)$
(c) Compare the results in parts (a) and (b) and comment on the role of the lack of memory property.

4-93. Suppose the counts recorded by a Geiger counter follow a Poisson process with an average of two counts per minute.
(a) What is the probability that there are no counts in a 30-second interval?
(b) What is the probability that the first count occurs in less than 10 seconds?
(c) What is the probability that the first count occurs between 1 and 2 minutes after start-up?

4-94. Suppose that the log-ons to a computer network follow a Poisson process with an average of 3 counts per minute.
(a) What is the mean time between counts?
(b) What is the standard deviation of the time between counts?
(c) Determine x such that the probability that at least one count occurs before time x minutes is 0.95.

4-95. The time between calls to a plumbing supply business is exponentially distributed with a mean time between calls of 15 minutes.
(a) What is the probability that there are no calls within a 30-minute interval?
(b) What is the probability that at least one call arrives within a 10-minute interval?
(c) What is the probability that the first call arrives within 5 and 10 minutes after opening?
(d) Determine the length of an interval of time such that the probability of at least one call in the interval is 0.90.

4-96. The life of automobile voltage regulators has an exponential distribution with a mean life of six years. You purchase an automobile that is six years old, with a working voltage regulator, and plan to own it for six years.
(a) What is the probability that the voltage regulator fails during your ownership?
(b) If your regulator fails after you own the automobile three years and it is replaced, what is the mean time until the next failure?

4-97. Suppose that the time to failure (in hours) of fans in a personal computer can be modeled by an exponential distribution with $\lambda = 0.0003$.
(a) What proportion of the fans will last at least 10,000 hours?
(b) What proportion of the fans will last at most 7000 hours?

4-98. The time between the arrival of electronic messages at your computer is exponentially distributed with a mean of two hours.
(a) What is the probability that you do not receive a message during a two-hour period?
(b) If you have not had a message in the last four hours, what is the probability that you do not receive a message in the next two hours?
(c) What is the expected time between your fifth and sixth messages?

4-99. The time between arrivals of taxis at a busy intersection is exponentially distributed with a mean of 10 minutes.

(a) What is the probability that you wait longer than one hour for a taxi?
(b) Suppose you have already been waiting for one hour for a taxi. What is the probability that one arrives within the next 10 minutes?
(c) Determine x such that the probability that you wait more than x minutes is 0.10.
(d) Determine x such that the probability that you wait less than x minutes is 0.90.
(e) Determine x such that the probability that you wait less than x minutes is 0.50.

4-100. The number of stork sightings on a route in South Carolina follows a Poisson process with a mean of 2.3 per year.
(a) What is the mean time between sightings?
(b) What is the probability that there are no sightings within three months (0.25 years)?
(c) What is the probability that the time until the first sighting exceeds six months?
(d) What is the probability of no sighting within three years?

4-101. According to results from the analysis of chocolate bars in Chapter 3, the mean number of insect fragments was 14.4 in 225 grams. Assume the number of fragments follows a Poisson distribution.
(a) What is the mean number of grams of chocolate until a fragment is detected?
(b) What is the probability that there are no fragments in a 28.35-gram (one-ounce) chocolate bar?
(c) Suppose you consume seven one-ounce (28.35-gram) bars this week. What is the probability of no insect fragments?

4-102. The distance between major cracks in a highway follows an exponential distribution with a mean of 5 miles.
(a) What is the probability that there are no major cracks in a 10-mile stretch of the highway?
(b) What is the probability that there are two major cracks in a 10-mile stretch of the highway?
(c) What is the standard deviation of the distance between major cracks?
(d) What is the probability that the first major crack occurs between 12 and 15 miles of the start of inspection?
(e) What is the probability that there are no major cracks in two separate 5-mile stretches of the highway?
(f) Given that there are no cracks in the first 5 miles inspected, what is the probability that there are no major cracks in the next 10 miles inspected?

4-103. The lifetime of a mechanical assembly in a vibration test is exponentially distributed with a mean of 400 hours.
(a) What is the probability that an assembly on test fails in less than 100 hours?
(b) What is the probability that an assembly operates for more than 500 hours before failure?
(c) If an assembly has been on test for 400 hours without a failure, what is the probability of a failure in the next 100 hours?
(d) If 10 assemblies are tested, what is the probability that at least one fails in less than 100 hours? Assume that the assemblies fail independently.
(e) If 10 assemblies are tested, what is the probability that all have failed by 800 hours? Assume the assemblies fail independently.

4-104. The time between arrivals of small aircraft at a county airport is exponentially distributed with a mean of one hour.
(a) What is the probability that more than three aircraft arrive within an hour?
(b) If 30 separate one-hour intervals are chosen, what is the probability that no interval contains more than three arrivals?
(c) Determine the length of an interval of time (in hours) such that the probability that no arrivals occur during the interval is 0.10.

4-105. The time between calls to a corporate office is exponentially distributed with a mean of 10 minutes.
(a) What is the probability that there are more than three calls in one-half hour?
(b) What is the probability that there are no calls within one-half hour?
(c) Determine x such that the probability that there are no calls within x hours is 0.01.
(d) What is the probability that there are no calls within a two-hour interval?
(e) If four nonoverlapping one-half-hour intervals are selected, what is the probability that none of these intervals contains any call?
(f) Explain the relationship between the results in part (a) and (b).

4-106. Assume that the flaws along a magnetic tape follow a Poisson distribution with a mean of 0.2 flaw per meter. Let X denote the distance between two successive flaws.
(a) What is the mean of X?
(b) What is the probability that there are no flaws in 10 consecutive meters of tape?
(c) Does your answer to part (b) change if the 10 meters are not consecutive?
(d) How many meters of tape need to be inspected so that the probability that at least one flaw is found is 90%?
(e) What is the probability that the first time the distance between two flaws exceeds 8 meters is at the fifth flaw?
(f) What is the mean number of flaws before a distance between two flaws exceeds 8 meters?

4-107. If the random variable X has an exponential distribution with mean θ, determine the following:
(a) $P(X > \theta)$ (b) $P(X > 2\theta)$
(c) $P(X > 3\theta)$
(d) How do the results depend on θ?

4-108. Derive the formula for the mean and variance of an exponential random variable.

4-109. Web crawlers need to estimate the frequency of changes to Web sites to maintain a current index for Web searches. Assume that the changes to a Web site follow a Poisson process with a mean of 3.5 days.

(a) What is the probability that the next change occurs in less than two days?

(b) What is the probability that the next change occurs in greater than seven days?

(c) What is the time of the next change that is exceeded with probability 90%?

(d) What is the probability that the next change occurs in less than 10 days, given that it has not yet occurred after three days?

4-110. The length of stay at a specific emergency department in Phoenix, Arizona, had a mean of 4.6 hours. Assume that the length of stay is exponentially distributed.

(a) What is the standard deviation of the length of stay?

(b) What is the probability of a length of stay greater than 10 hours?

(c) What length of stay is exceeded by 25% of the visits?

4-9 ERLANG AND GAMMA DISTRIBUTIONS

An exponential random variable describes the length until the first count is obtained in a Poisson process. A generalization of the exponential distribution is the length until r counts occur in a Poisson process. Consider the following example.

EXAMPLE 4-23 Processor Failure

The failures of the central processor units of large computer systems are often modeled as a Poisson process. Typically, failures are not caused by components wearing out, but by more random failures of the large number of semiconductor circuits in the units. Assume that the units that fail are immediately repaired, and assume that the mean number of failures per hour is 0.0001. Let X denote the time until four failures occur in a system. Determine the probability that X exceeds 40,000 hours.

Let the random variable N denote the number of failures in 40,000 hours of operation. The time until four failures occur exceeds 40,000 hours if and only if the number of failures in 40,000 hours is three or less. Therefore,

$$P(X > 40{,}000) = P(N \le 3)$$

The assumption that the failures follow a Poisson process implies that N has a Poisson distribution with

$$E(N) = 40{,}000(0.0001) = 4 \text{ failures per } 40{,}000 \text{ hours}$$

Therefore,

$$P(X > 40{,}000) = P(N \le 3) = \sum_{k=0}^{3} \frac{e^{-4}4^k}{k!} = 0.433$$

The previous example can be generalized to show that if X is the time until the rth event in a Poisson process, then

$$P(X > x) = \sum_{k=0}^{r-1} \frac{e^{-\lambda x}(\lambda x)^k}{k!}$$

Because $P(X > x) = 1 - F(x)$, the probability density function of X equals the negative of the derivative of the right-hand side of the previous equation. After extensive algebraic simplification, the probability density function of X can be shown to equal

$$f(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{(r-1)!} \quad \text{for } x > 0 \text{ and } r = 1, 2, \ldots .$$

This probability density function defines an **Erlang distribution.** Clearly, an Erlang random variable with $r = 1$ is an exponential random variable.

It is convenient to generalize the Erlang distribution to allow r to assume any nonnegative value. Then the Erlang and some other common distributions become special cases of this generalized distribution. To accomplish this step, the factorial function $(r - 1)!$ has to be generalized to apply to any nonnegative value of r, but the generalized function should still equal ()! when r is a positive integer.