Chapter 6 Analog Filter Design

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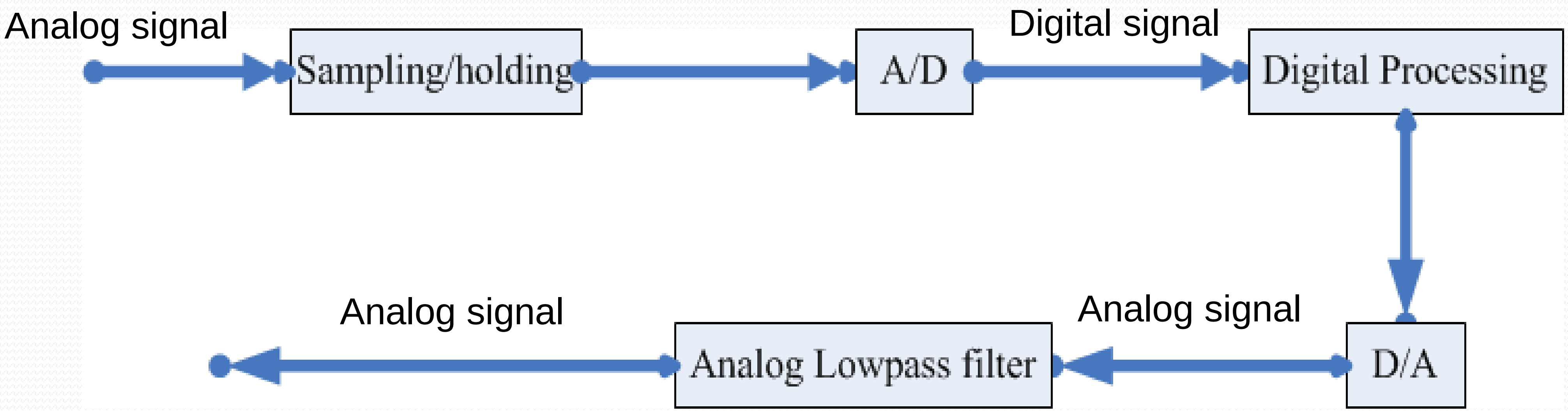
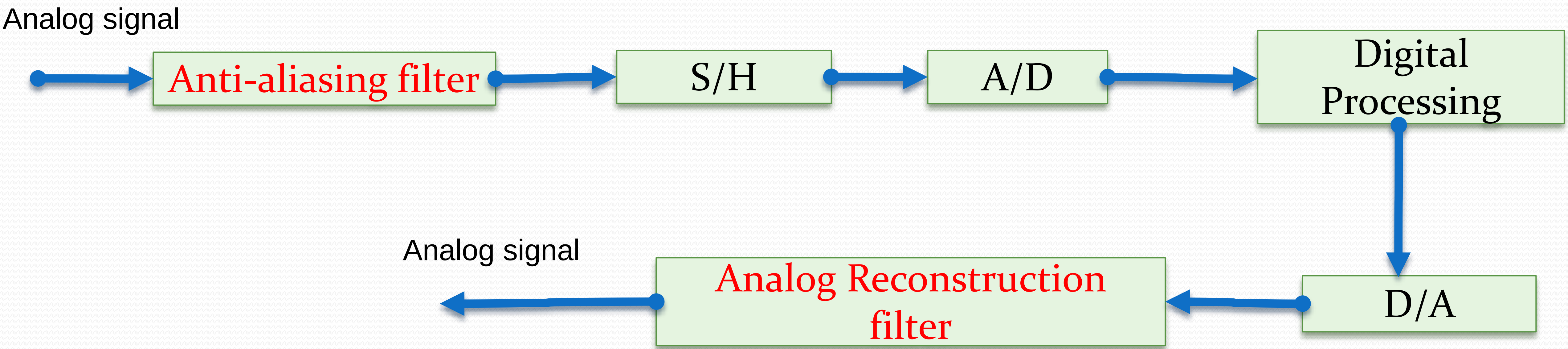
review



Analog signal
Sampling/holding
A/D
Digital signal
Digital Processing
Analog signal
Analog Lowpass filter
Analog signal
D/A



Analog signal
Anti-aliasing filter
S/H
A/D
Digital Processing
Analog signal
Analog Reconstruction filter
D/A

motivation

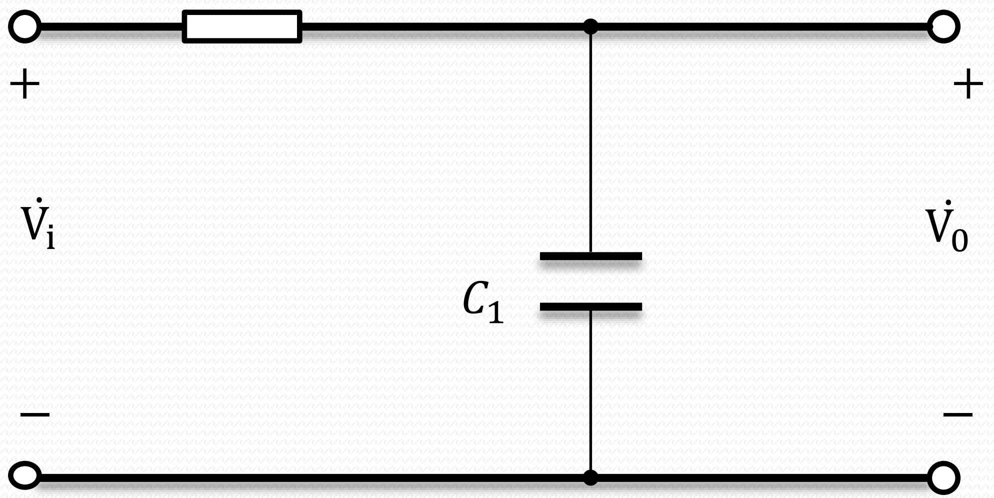
- To prevent aliasing, an anti-aliasing analog filter is often placed before S/H
- To smooth the D/A converter output, a reconstruction(smoothing) analog filter is often placed after D/A
- Ideal filters cannot not be realized since impulse response is not absolutely summable， sometimes digital filter(especially for the design of IIR digital filter) has to be converted from analog filter
- In order to develop stable and realizable system, the ideal frequency response specifications are relaxed by a transition band between the passband and the stopband o permit the magnitude response to decay gradually

Contents

- Anolog filter
  - Butterworth approximation
  - LP analog Filter design
  - Other analog filters design
- Sampling theorem
- Anti-aliasing analog filter
  - determine the sampling frequency T for practical filters
- Reconstruction analog filter
  - to contain all the information of original signal as much as possible
- From analog filter to digital filter
  - for IIR digital filter and FIR digital filter

6.1 Analog filter design

- what the difference between analog filter and digital filter design?
  - analog filter is implemented by analog circuit
  - analog filter is designed by using approximation technique
- How an analog filter comes from
  - analog filter circuit
  - the gain of output
    - $\dot{A}_{VH} = \frac{\dot{V}_o}{\dot{V}_i} = \frac{1}{1+j(f/f_H)}, f_H = \frac{1}{2\pi R_1 C_1}$
    - $A_{VH} = \frac{1}{\sqrt{1+(f/f_H)^2}},$
    - $\varphi_{VH} = -arctg(f/f_H)$



Butterworth approximation

- The magnitude-squared response

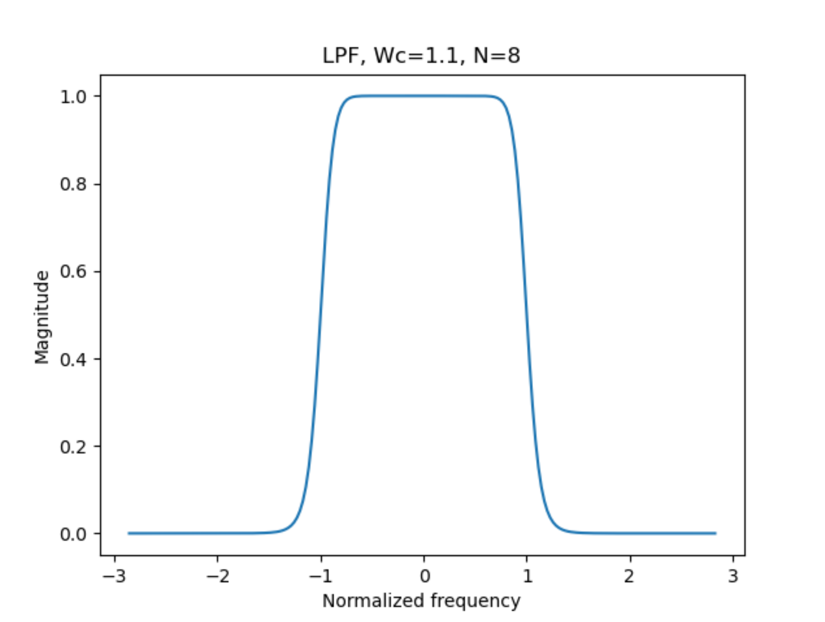
$$|H_a(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}},$$

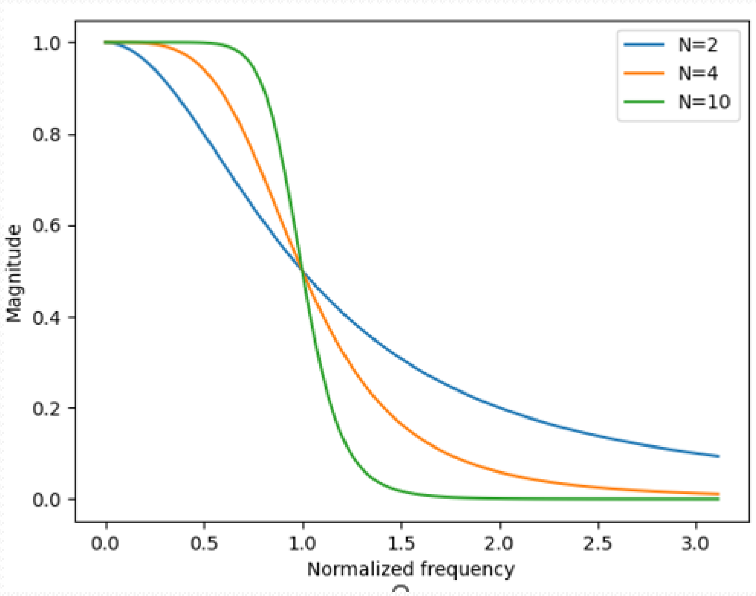
- N—orders
  - as the filter order N increase, the transition band decrease .
  - 2N-1 derivatives at Ω=0 are equal to zero
  - maximally flat magnitude at Ω=0
- Ω_c--cutoff frequency

  $|H_a(j\Omega_c)|^2 = \frac{1}{2}$
- The gain

$$g(\Omega) = 10\log_{10}|H_a(j\Omega)|^2\, dB$$

  - Ω=0, the gain is zero
  - $g(\Omega_c) = 10\log_{10}(\frac{1}{2}) \cong -3dB$
  - Ω= Ωc, Ω >> Ωc, $|H_a(j\Omega)|^2 \approx \frac{1}{\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$





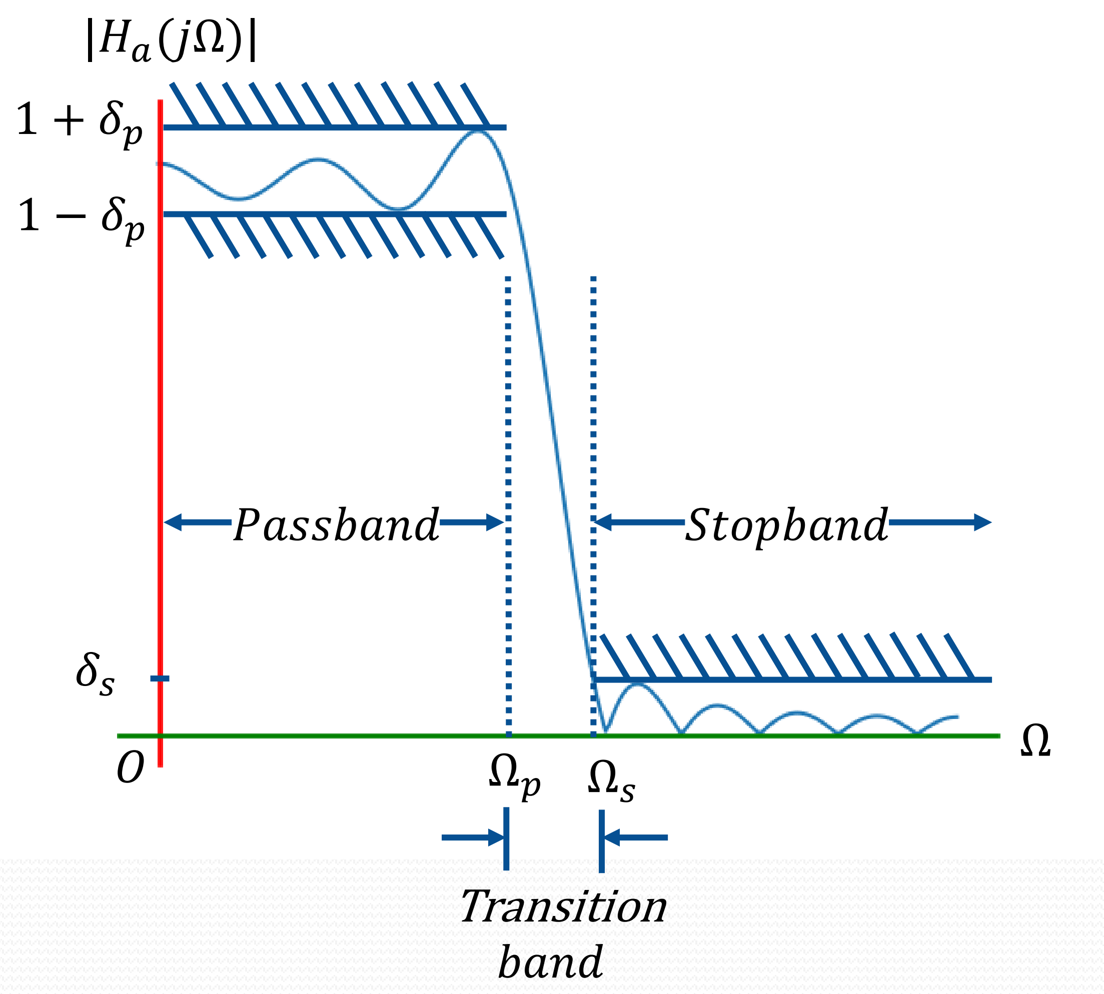
- How to determine a Butterworth analog filter?
  - Based on the specifications of Gibbs phenomenon
- passband error δ_p
- stopband error δ_s
- Passband edge frequency Ω_p

  $1-\delta_p \le |H_a(j\Omega)| \le 1+\delta_p, for\ |\Omega| \le \Omega_p$
- Stopband edge frequency Ω_s

  $|H_a(j\Omega)| \le \delta_s, for\ \Omega_s \le |\Omega| < \infty$
- Transition band $\left|\Omega_p - \Omega_s\right|$
- Peak passband ripple

  $\alpha_p = -20\log_{10}(1-\delta_p)\text{dB}$
- Minimum stopband attenuation

  $\alpha_s = -20\log_{10}\delta_s\text{dB}$



- Ex. If a desired peak passband ripple of a lowpass filter α_p is 0.01dB, and the minimum attenuation in the stopband α_s is 70dB. Determine $\delta_p$and δ_s.
  - Solution:

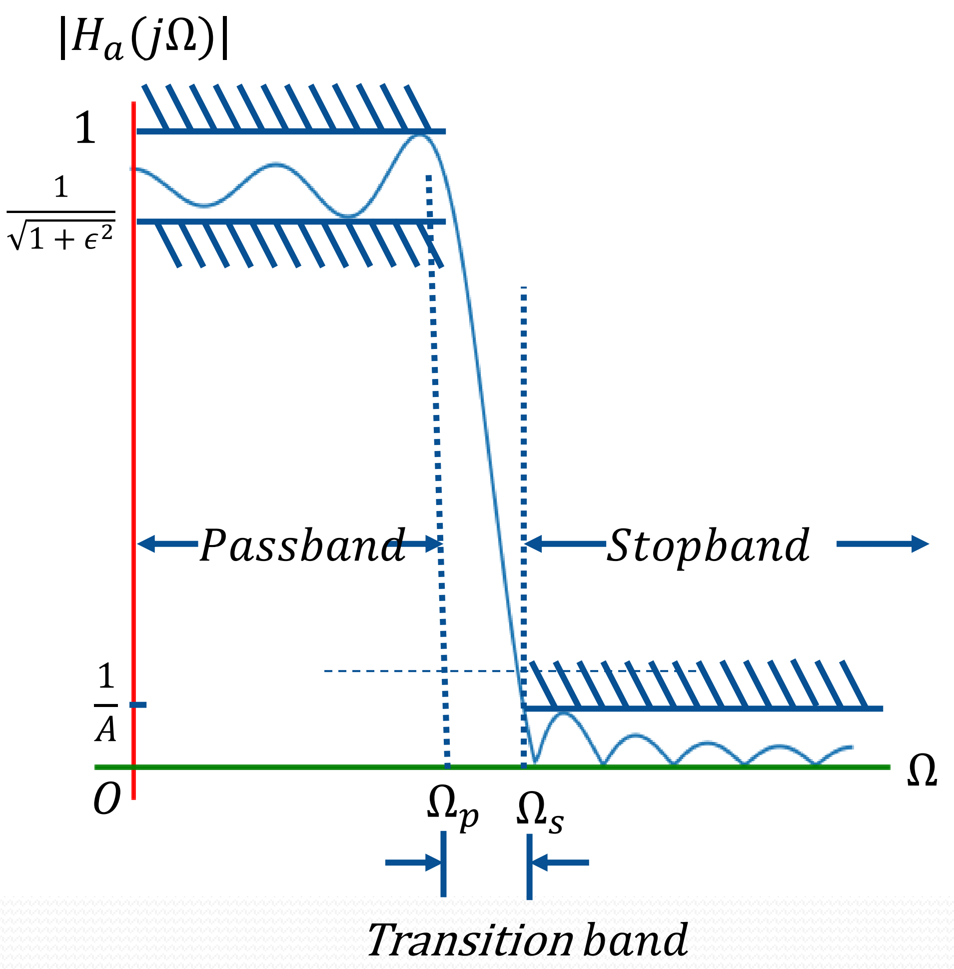
$$\delta_p = 1 - 10^{-\alpha_p/20} = 0.00115$$
$$\delta_s = 10^{-\alpha_s/20} = 0.0003162$$

- Normalized form
  - maximum value of the magnitude in the passband is 1
  - minimum passband ripple

$$|H_a(j\Omega)|_{\min\_passband} = \frac{1}{\sqrt{1+\varepsilon^2}}$$

- Maximum stopband ripple $\frac{1}{A}$

$$\alpha_s = -20\log_{10}\frac{1}{A}\,\mathrm{dB}$$



- Given passband Ω_p and the minimum passband magnitude $|H_a(\Omega_p)|^2$, the stopband edge Ω_s and the maximum stopband ripple $|H_a(\Omega_s)|^2$, based on the following formulations, we can determine Ω_c and N

$$|H_a(\Omega_p)|^2 = \frac{1}{1+(\frac{\Omega_p}{\Omega_c})^{2N}} = \frac{1}{1+\varepsilon^2}, \quad (1)$$

$$|H_a(\Omega_s)|^2 = \frac{1}{1+(\frac{\Omega_s}{\Omega_c})^{2N}} = \frac{1}{A^2}, \quad (2)$$

$$N = \frac{1}{2}\frac{\log_{10}\left[\frac{(A^2-1)}{\varepsilon^2}\right]}{\log_{10}(\frac{\Omega_s}{\Omega_p})} = \frac{\log_{10}\left(\frac{1}{k_1}\right)}{\log_{10}\left(\frac{1}{k}\right)}$$

Ω_c

- Ex. A analog lowpass filter $H_a(j\Omega)$ having a maximally flat lowpass characteristic with a 1-dB at passband frequency Ω_p which is $1kH_z$ and a minimum attenuation of 40 dB at stopband frequency Ω_s which is $5kH_z$. Determine N of Butterworth filter
- Solution:

$$\left|H_a(j\Omega_p)\right|^2 = \frac{1}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}} = \frac{1}{1+\varepsilon^2}$$

$$\Rightarrow \log_{10}\left(\frac{1}{1+\varepsilon^2}\right) = -1 \Rightarrow \varepsilon^2 = 0.25895$$

$$|H_a(j\Omega_s)|^2 = \frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} = \frac{1}{A^2}$$

$$\Rightarrow \log_{10}\left(\frac{1}{A^2}\right) = -40 \Rightarrow A^2 = 10{,}000$$

$$\frac{1}{k_1} = \frac{\sqrt{A^2-1}}{\varepsilon} = 196.51334, \qquad \frac{1}{k} = \frac{\Omega_p}{\Omega_s} = \frac{5000}{1000} = 5$$

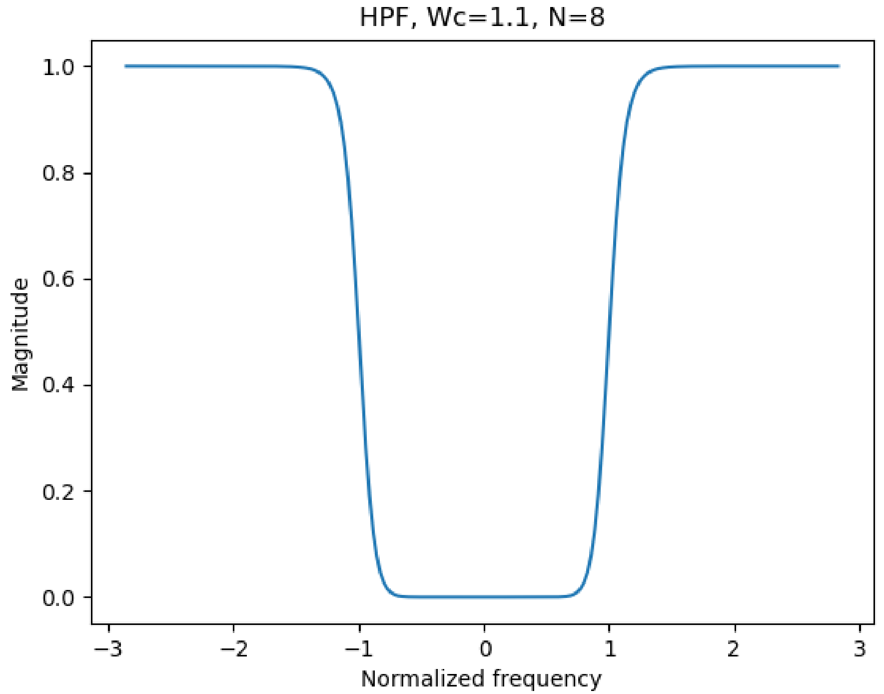
$$N = \left\lceil \frac{\frac{1}{2}\log_{10}[\frac{(A^2-1)}{\varepsilon^2}]}{\log_{10}\left(\frac{\Omega_s}{\Omega_p}\right)} \right\rceil = \left\lceil \frac{\log_{10}\left(\frac{1}{k_1}\right)}{\log_{10}\left(\frac{1}{k}\right)} \right\rceil = \left\lceil \frac{\log_{10}(196.51334)}{\log_{10}(5)} \right\rceil \approx \lceil 3.2811022 \rceil = 4$$

The other analog filters

- Transform analog lowpass filter into the desired analog filter
  - Determine the specifications of lowpass filter from the desired specifications of highpass filter
  - Transform lowpass filter to the other kinds of filter
- Analog highpass filter design

$$\Omega = -\frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}}$$

  - $\hat{\Omega}$--highpass filter frequency,
  - $\hat{\Omega}_p$--highpass passband frequency
  - $\left|H_a(j\Omega_p)\right|^2 : \quad |\Omega| < \Omega_p \leftrightarrow \left|\hat{\Omega}\right| \geq \hat{\Omega}_p$
  - $|H_a(j\Omega_s)|^2 : |\Omega| \geq \Omega_s \leftrightarrow 0 \leq \left|\hat{\Omega}\right| \leq \hat{\Omega}_s$



- Ex. Design an analog Butterworth highpass filter with the following specifications: passband edge at $4kH_z$, stopband edge at $1kH_z$, passband ripple of 0.1dB, and minimum stopband attenuation of 40dB.
  - Solution:

(1) chose the normalized passband edge Ω_p of lowpass filter to be 1 r/s, then normalized stopband edge of the lowpass filter is given by

$$\Omega_s = \frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}_s} = \frac{1 \times 4000}{1000} = 4$$

$$\log_{10}\left(\frac{1}{1+\varepsilon^2}\right) = -0.1dB, \quad \log_{10}\left(\frac{1}{A^2}\right) = -40dB$$

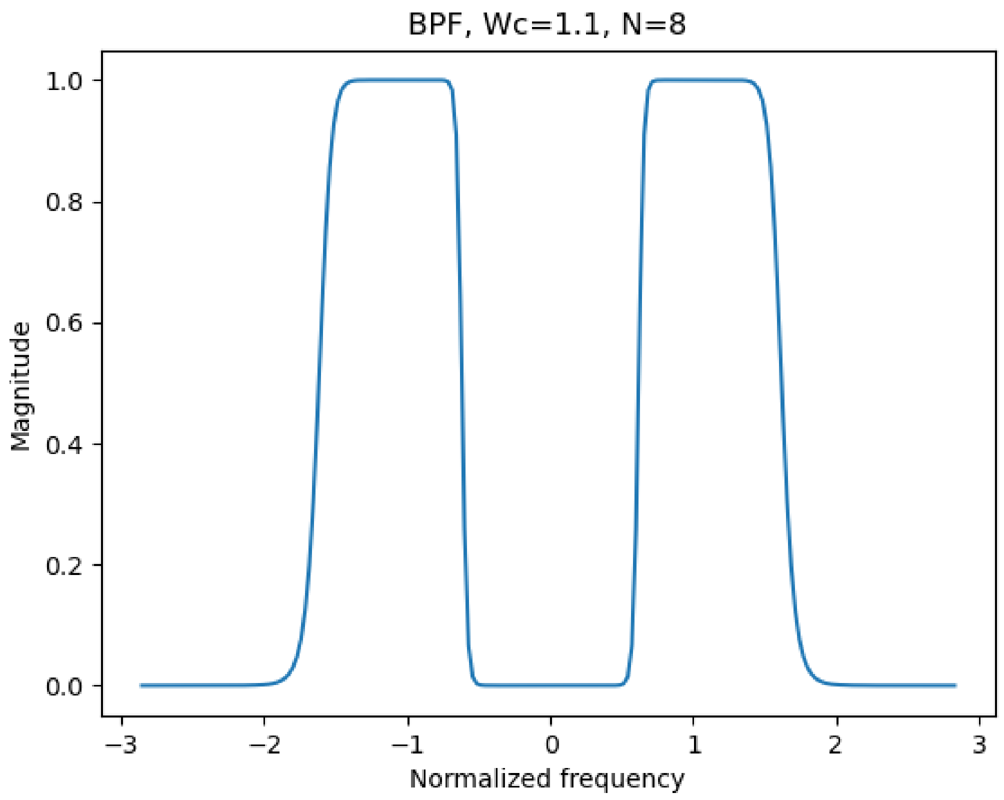
$$N = \left\lceil \frac{1}{2} \frac{\log_{10}[\frac{(A^2-1)}{\varepsilon^2}]}{\log_{10}\left(\frac{\Omega_s}{\Omega_p}\right)} \right\rceil$$, and Ω_c is -3dB cutoff frequency of analog lowpass filter

(2) Transform the lowpass filter into highpass filter by replace $\Omega = \frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}}$

- Analog bandpass filter

$$\Omega = -\Omega_p \frac{\hat{\Omega}_o^2 - \hat{\Omega}^2}{\hat{\Omega} B_W}, B_w = (\hat{\Omega}_{p2} - \hat{\Omega}_{p1})$$

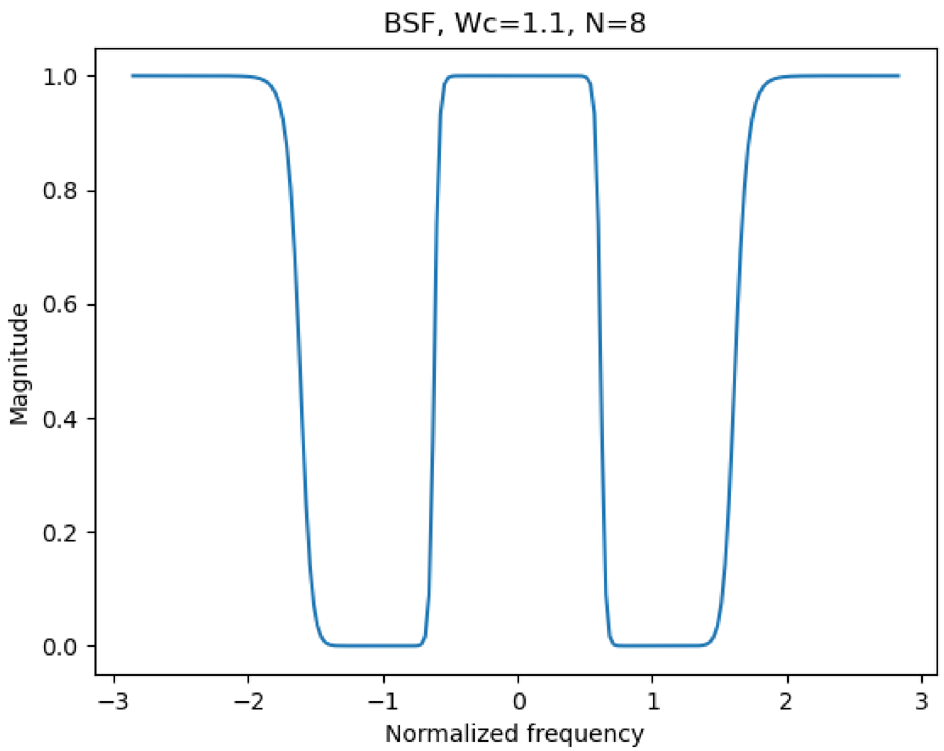
  - $\hat{\Omega}_o^2$ --Passband center frequency
  - $\Omega = 0 \leftrightarrow \hat{\Omega} = \hat{\Omega}_0$
  - $\Omega_p \leftrightarrow \hat{\Omega}_{p2}\ and\ (-\hat{\Omega}_{p1})$,
  - $\Omega_s \leftrightarrow \hat{\Omega}_{s2}\ and\ (-\hat{\Omega}_{s1})$,
  - $|\Omega| \leq \Omega_p \leftrightarrow \hat{\Omega}_{p1} \leq |\hat{\Omega}| \leq \hat{\Omega}_{p2}$
  - $\hat{\Omega}_{p1}\hat{\Omega}_{p2} = \hat{\Omega}_{s2}\hat{\Omega}_{s1} = \hat{\Omega}^2{}_o$



- Analog bandstop filter

$$\Omega = \Omega_s \frac{\hat{\Omega} B_W}{\hat{\Omega}_o^2 - \hat{\Omega}^2}, B_w = \left(\hat{\Omega}_{s2} - \hat{\Omega}_{s1}\right)$$

- $\hat{\Omega}_o^2$ – stopband center frequency
- $\Omega = \pm\infty \leftrightarrow \pm\hat{\Omega}_0,$
- $\Omega_s \leftrightarrow \hat{\Omega}_{s1}\ and\ \left(-\hat{\Omega}_{s2}\right),$
- $-\Omega_s \rightarrow -\hat{\Omega}_{s1}\ and\ \hat{\Omega}_{s2},$
- $|\Omega| \leq \Omega_p \leftrightarrow -\hat{\Omega}_{p1} \leq \hat{\Omega} \leq \hat{\Omega}_{p1}, -\infty \leq \hat{\Omega} \leq -\hat{\Omega}_{p2}, and - \hat{\Omega}_{p2} \leq \hat{\Omega} \leq \infty$
- $\hat{\Omega}_{p1}\hat{\Omega}_{p2} = \hat{\Omega}_{s2}\hat{\Omega}_{s1} \ = \hat{\Omega}^2{}_o$

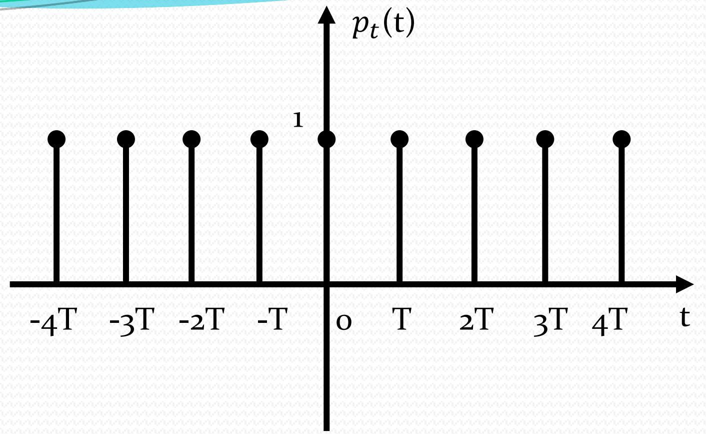


6.2 Sample theorem

- FT of the Impulse signal

$$p_t(t) = \sum_n \delta(t - nT)$$

$$F(p_t(t)) = F\left(\sum_n \delta(t - nT)\right) = \frac{2\pi}{T}\sum_n \delta(\Omega - n\Omega_T))$$



- Proof
  - Since $p_t(t)$ is periodic, so

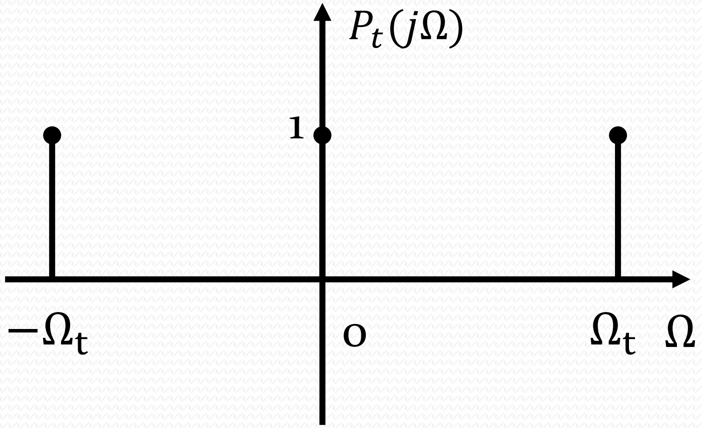
$$p_t(t) = \sum_n P(\mathrm{n}\Omega_T)e^{jn\Omega_T t}, \text{ where } P(n\Omega_T) \text{ is Fourier series}$$

$$P(n\Omega_T) = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jn\Omega_T t}dt = \frac{1}{T}, \text{ where } \Omega_T = \frac{2\pi}{T}$$

$$p_t(t) = \sum_n P(n\Omega_T)e^{jn\Omega_T t} = \frac{1}{T}\sum_n e^{jn\Omega_T t}, \text{ where } \Omega_T = \frac{2\pi}{T}$$

$$P(j\Omega) = F\big(p_t(t)\big) = \int_{-\infty}^{\infty} p_t(t)e^{-j\Omega t}dt = \int_{-\infty}^{\infty} \frac{1}{T}\sum_n e^{jn\Omega_T t}\, e^{-j\Omega t}dt$$

$$= \frac{1}{T}\sum_n \int_{-\infty}^{\infty} e^{j(n\Omega_T - \Omega)t}dt = \frac{2\pi}{T}\sum_n \delta(\Omega - n\Omega_T))$$



- From continuous time FT to DTFT

$$X_s(j\Omega) = \frac{1}{T}\sum_n X_a\left(j\left(\Omega - \frac{2\pi}{T}n\right)\right) = \sum_n X_a(j(\Omega - \Omega_T n))$$

- Proof
  - $X_a(j\Omega) = F(x_a(t))$
  - $x[n] = x_a(nT) = x_a(t)|_{t=T} = x_a(t)p_t(t)$

    $$= x_a(t)\sum_n \delta(t - nT)$$
  - $X_s(j\Omega) = F\big(x_a(nT)\big)$------ $\Omega = \frac{\omega}{T}$

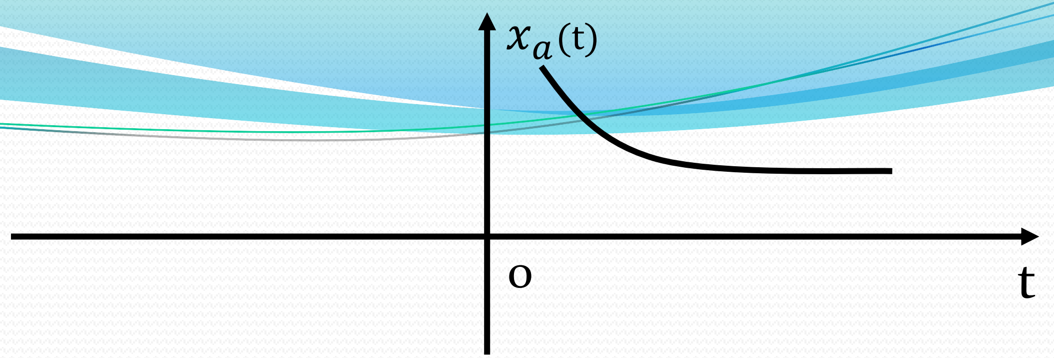
    $$= F\left(x_a(t)\sum_n \delta(t - nT)\right)$$

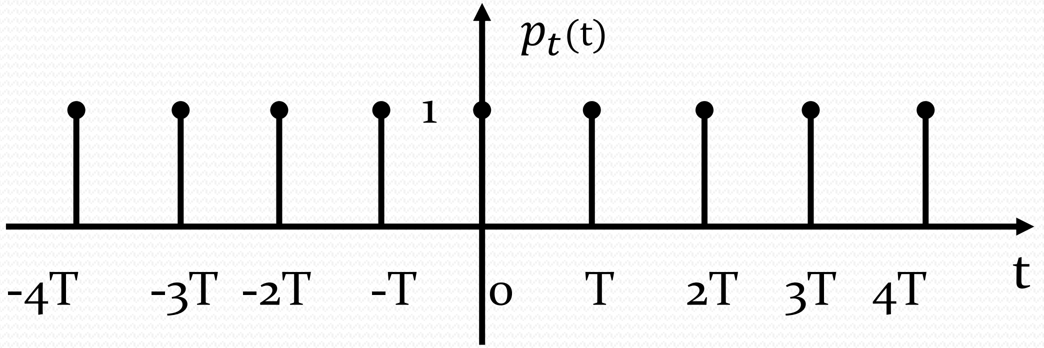
    $$= F\big(x_a(t)\big) * F\left(\sum_n \delta(t - nT)\right)$$

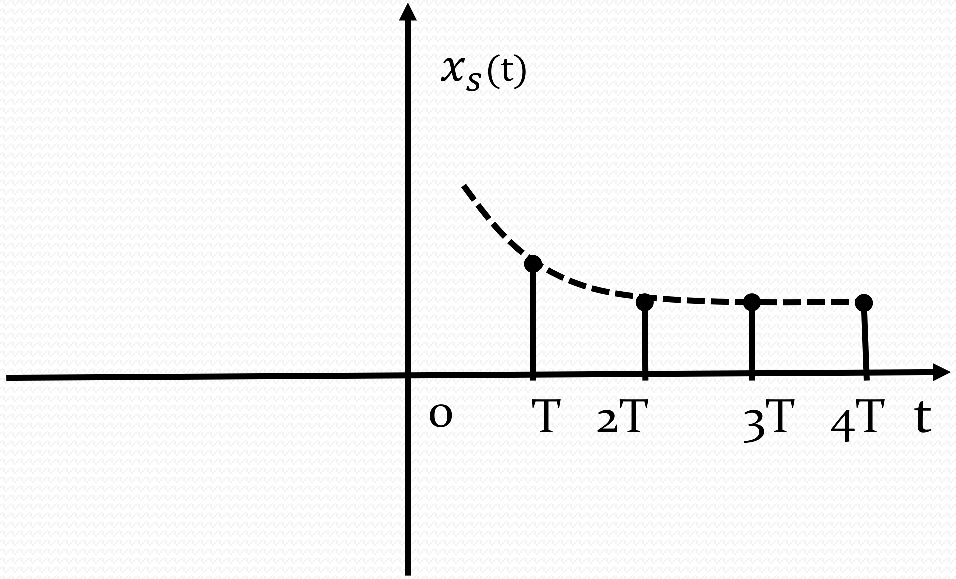
    $$= X_a(\mathrm{j}\Omega) * \frac{2\pi}{T}\sum_n \delta(\Omega - n\Omega_T))$$

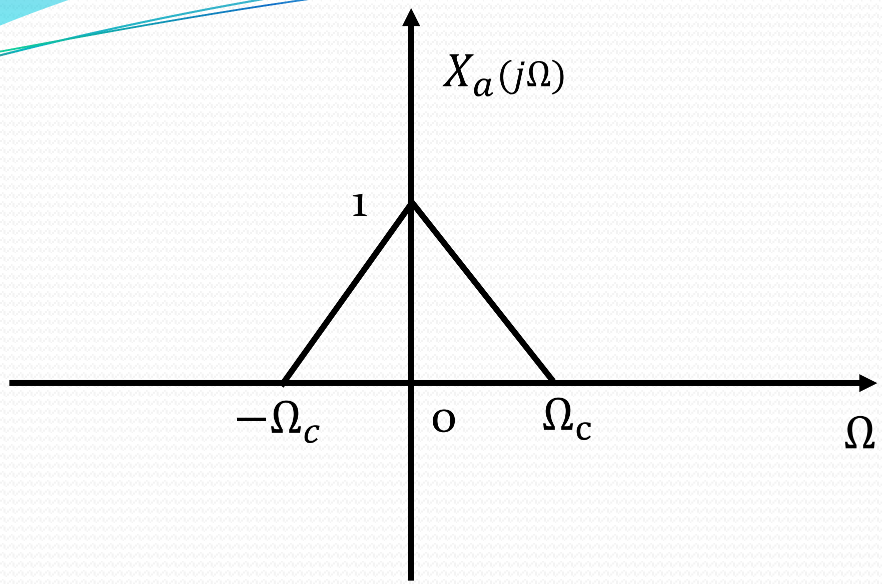
    $$= \frac{2\pi}{T}\frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(\mathrm{j}\lambda)\sum_n \delta(\Omega - n\Omega_T - \lambda)\, d\lambda$$

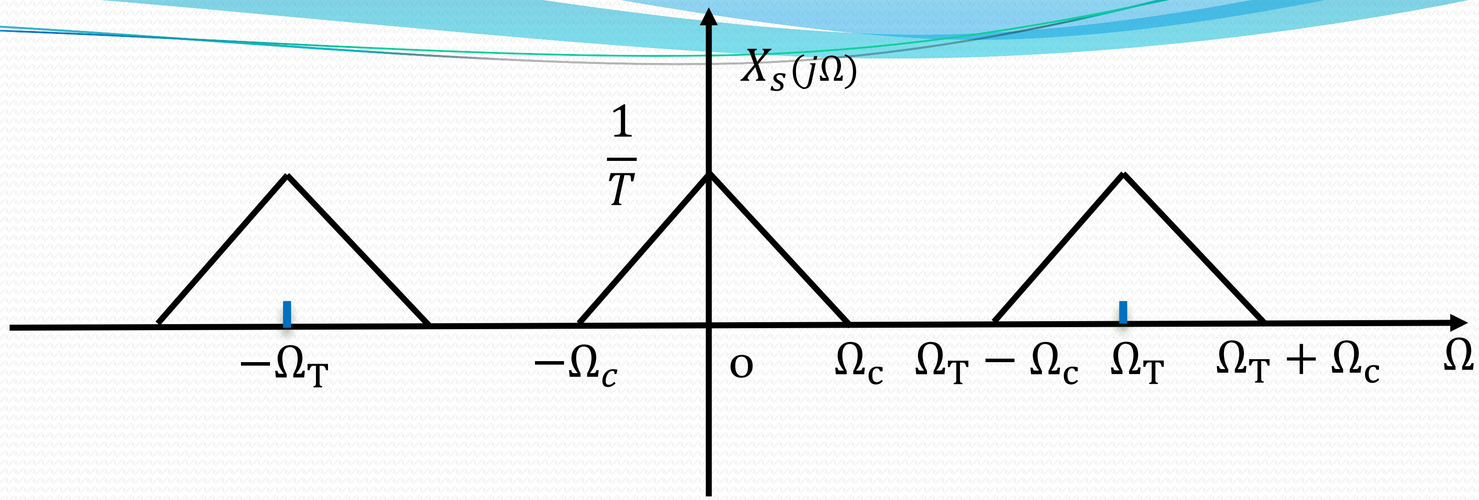
    $$= \frac{1}{T}\sum_n X_a(j\Omega - jn\Omega_T)$$

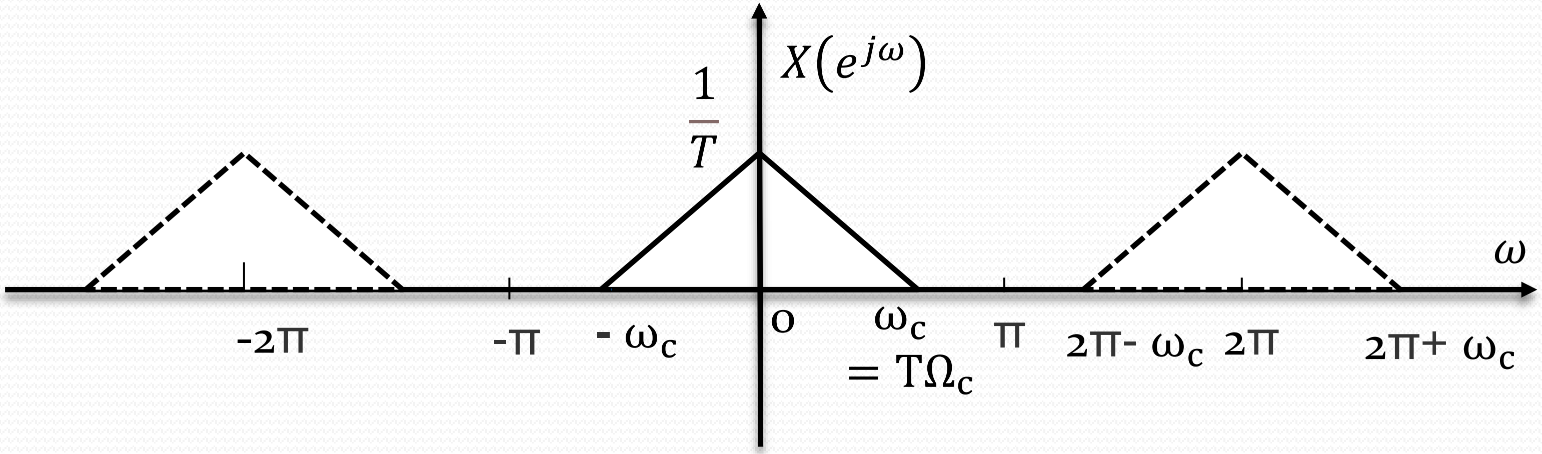


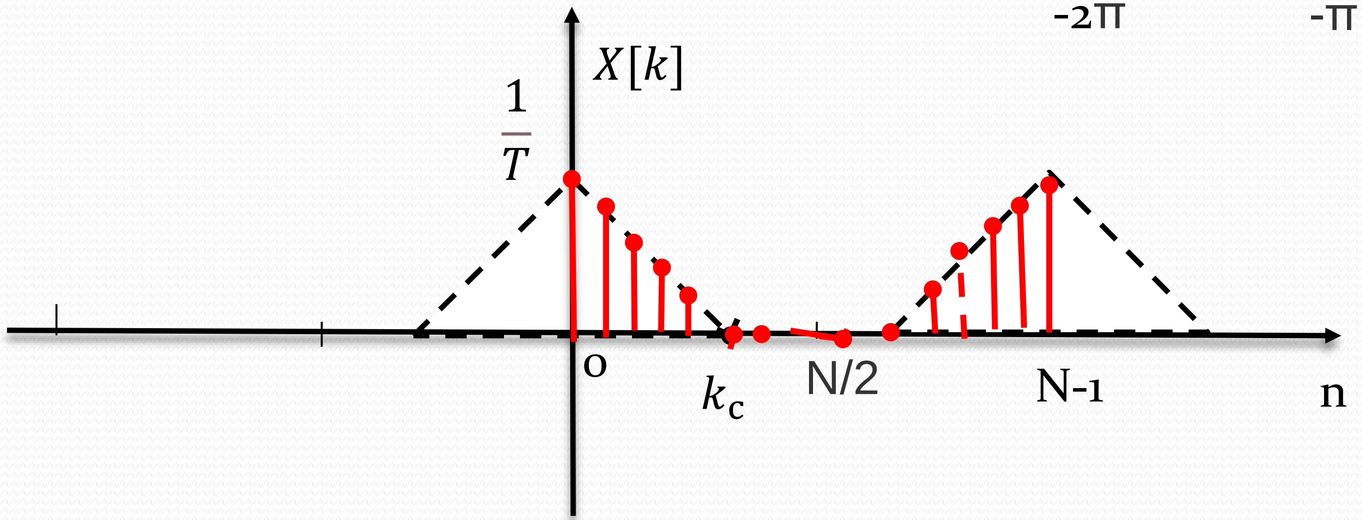




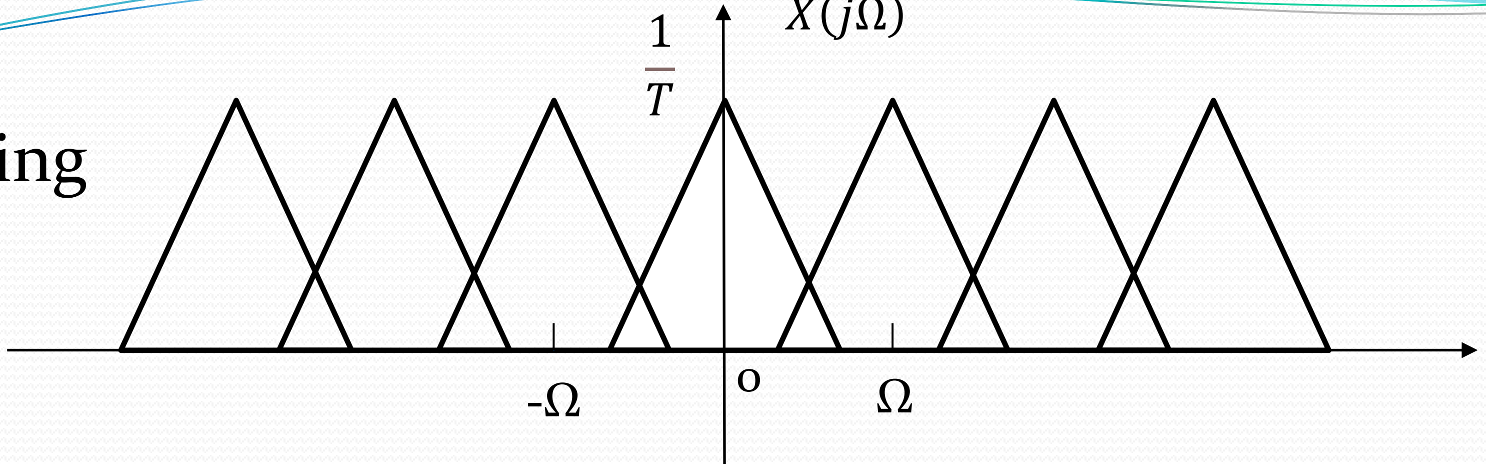

$X_a(j\Omega)$
1
$-\Omega_c$
0
Ω_c
Ω

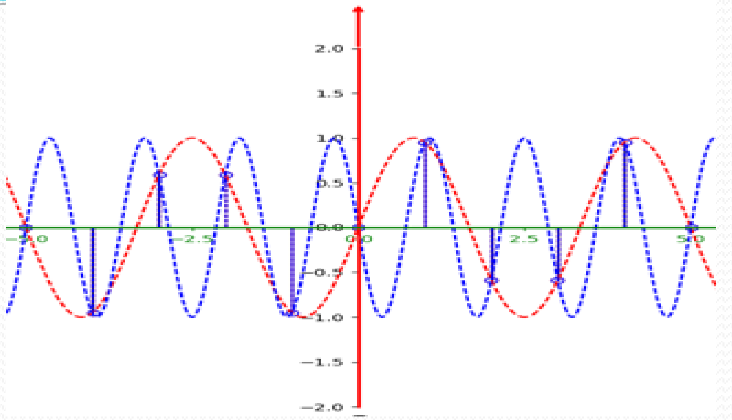

$X_s(j\Omega)$
$\frac{1}{T}$
$-\Omega_T$
$-\Omega_c$
0
Ω_c
$\Omega_T - \Omega_c$
Ω_T
$\Omega_T + \Omega_c$
Ω


$X(e^{j\omega})$
$\frac{1}{T}$
ω
-2π
-π
$-\omega_c$
0
$\omega_c = T\Omega_c$
π
$2\pi - \omega_c$
2π
$2\pi + \omega_c$


$X[k]$
$\frac{1}{T}$
0
k_c
N/2
N-1
n

- Aliasing





- Nyquist sampling theorem

$$\Omega_T = \frac{2\pi}{T} \geq 2\Omega_{\max},$$

  - Ω_T−−−−angular sampling frequency(rad/s)
  - $\Omega_T/2$---Nyquist frequency (rad/s)
  - $-\Omega_T/2 \leq \Omega \leq \Omega_T/2$−−−base band or Nyquist band, also band of $X_a(j\Omega)$
  - Ω_{max}--the highest frequency of the signal
  - T– sampling period (s) ,$\frac{1}{T} = f_x$ ---sampling frequency (Hz)

The relation between DTFT and FT

- $X_s(j\Omega) = F\{x_s(t)\} = \sum_n x[nT]F\{\delta(t - nT)\}$

$$= \sum_n x[n]e^{-jnT\Omega}$$

$\omega = T\Omega \Longrightarrow$

$$X\left(e^{j\omega n}\right) = \sum_n x[n]e^{-j\omega n}$$

- $X_s(j\Omega) = \sum_n X_a\left(j(\Omega - \Omega_T n)\right)$

$\omega = T\Omega \Longrightarrow$

$$X\left(e^{j\omega n}\right) = \frac{1}{T}\sum_n X_a(j\omega/T - jn2\pi/T)$$

- $\Omega_T T = \frac{2\pi}{T}T = 2\pi$

Examples of Sampling theorem

- Ex1: $x_a(t) = \cos(\Omega_0 t)$ is sampled and get $x[n] = \cos(\frac{\pi}{4} n)$, T = $\frac{1}{1000}$ $seconds$, find possible value of Ω_0 if no aliasing.
  - Solution:

$$x[n] = x_a(nT) = \cos(\Omega_0 nT) = \cos(\frac{\pi}{4} n)$$

$$\Omega_0 nT = \Omega_0 n \frac{1}{1000} = \frac{\pi}{4} n + 2k\pi n$$

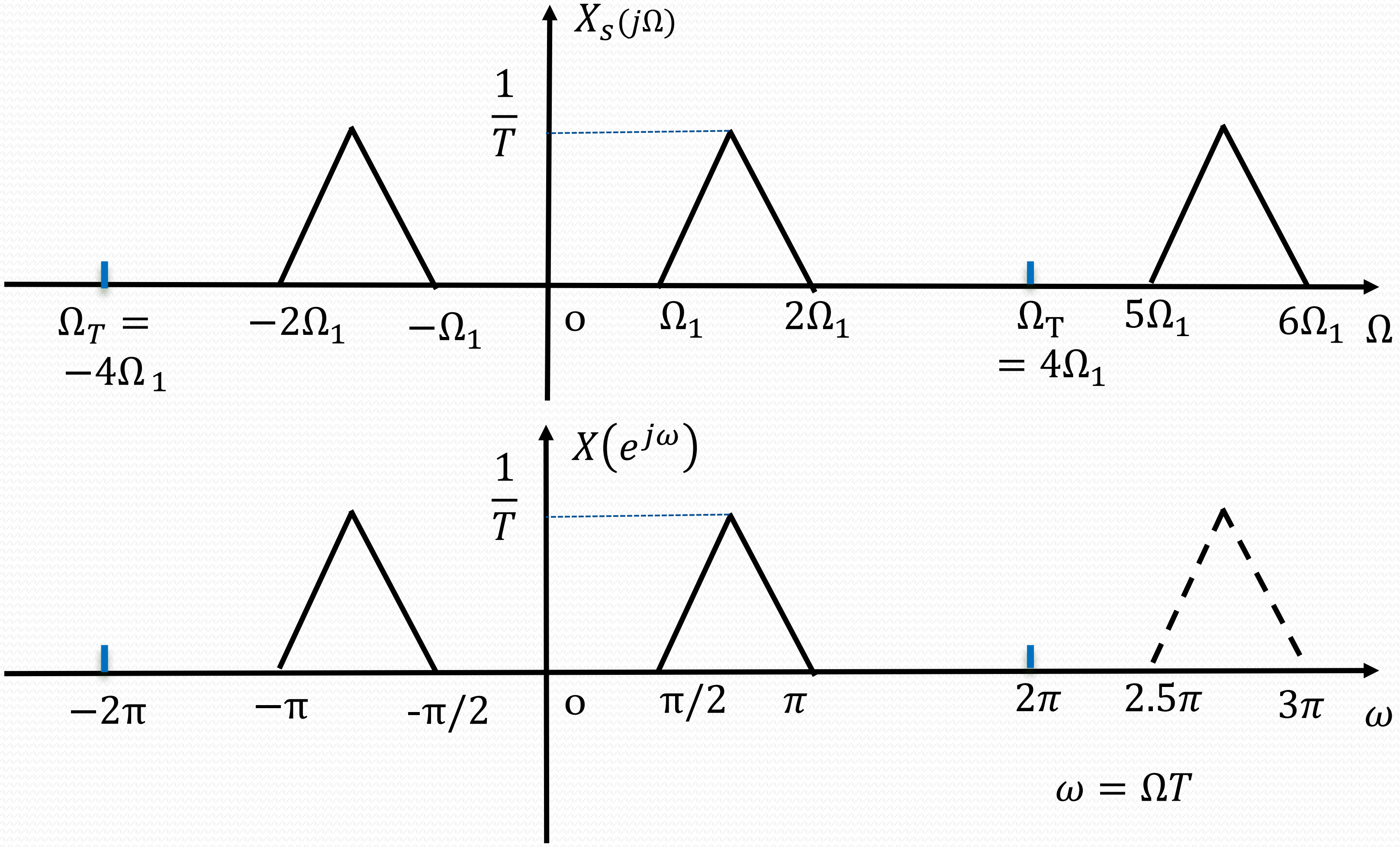
$$\Omega_0 = 1000 \times \left(\frac{\pi}{4} + 2k\pi\right)$$

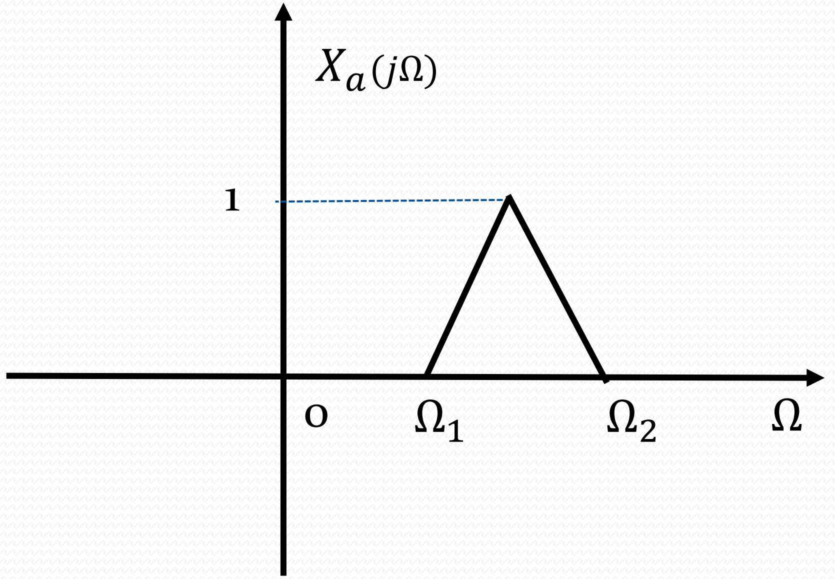
based on the sample theorem $\Omega_s \geq 2\Omega_0$,

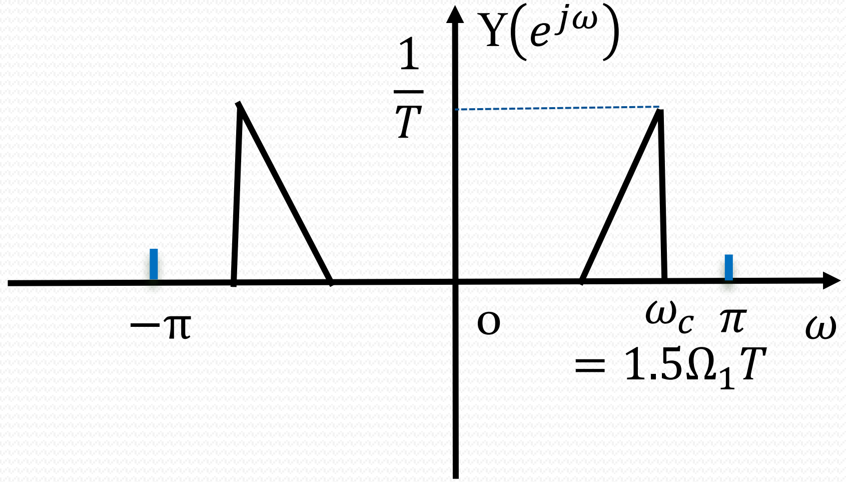
$$\Omega_T = \frac{2\pi}{T} = 2000\pi, \Omega_0 \leq \frac{\Omega_T}{2} = 1000\pi$$

So k=0, and $\Omega_0 = \frac{1000\pi}{4}$

- Ex: $X_a(j\Omega)$ is known, with $\Omega_1 \le \Omega \le \Omega_2$, $x_a(t)$ is sampled by $T = \frac{\pi}{\Omega_2}$, which is $\Omega_T = 2\Omega_2$; (a)if $\Omega_2 = 2\Omega_1$, sketch $X_s(j\Omega)$ and $X(e^{j\omega})$(b) if we want to filter out Y(jΩ) with 0≤Ω_ 1.5Ω_1, sketch the ideal digital filter and $Y(e^{j\omega})$







- Ex4: If $X_c(j\Omega)$, has $|\Omega_c| = 2000\pi$, find T so that $y_c(t) = {x_c}^2(t)$ without aliasing.
- Solution:
  - based on properties of DTFT

$$y_c(t) = {x_c}^2(t) \rightarrow Y_c(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * X_c(j\Omega)$$

$$X_c(j\Omega) \text{ has } |\Omega_c| = 4000\pi(r/s),$$

$$Y_c(j\Omega) \text{ has } |\Omega_c| = 8000\pi(r/s),$$

  - So for aliasing,

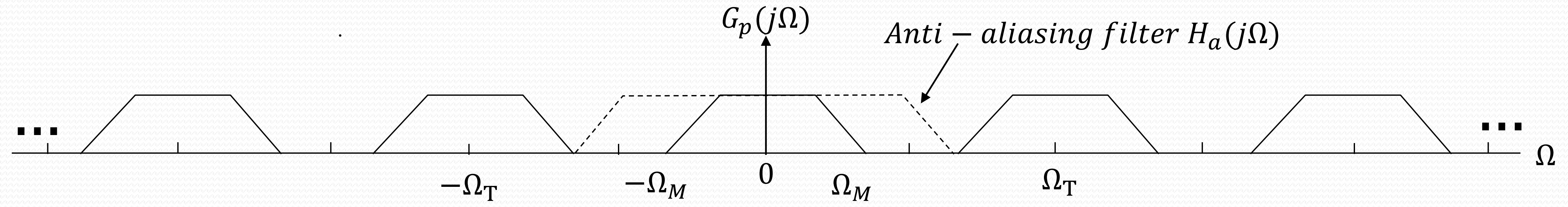
$$\Omega_T = \frac{2\pi}{\mathrm{T}} \geq 2 * 8000\pi(r/s),$$

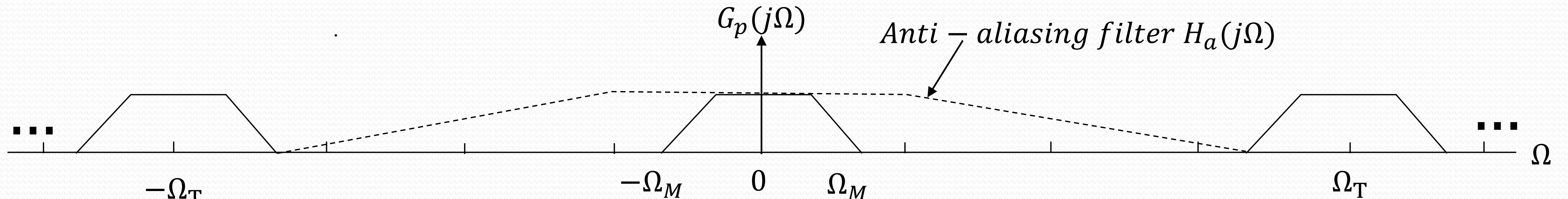
$$\mathrm{T} \leq \frac{1}{8} \times 10^{-3}(\mathrm{s})$$

6.2 Anti-aliasing filter design

- Why use anti-aliasing filter
  - To enforce the continuous signal to satisfy the condition of sampling prior to sampling
  - limited sampling frequency which circuit can get
- The ideal anti-aliasing filter

$$H_a(j\Omega) = \begin{cases} 1, & |\Omega| < \frac{\Omega_T}{2}, \\ 0, & |\Omega| \geq \frac{\Omega_T}{2}. \end{cases} \qquad \Omega_p < \Omega_s < \frac{\Omega_T}{2}$$

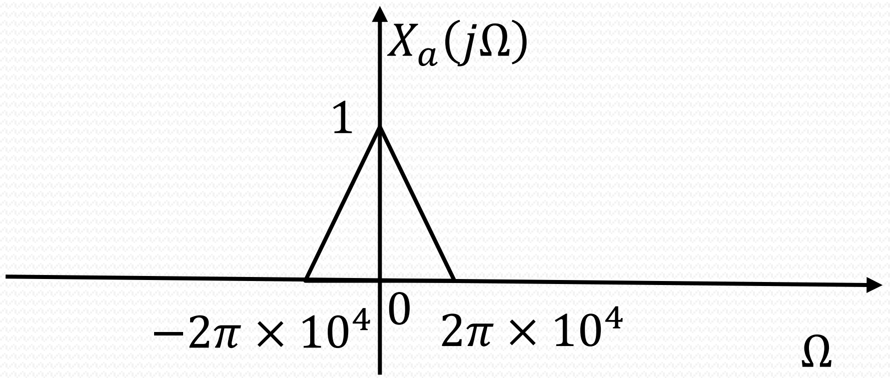




- Ex. For a continuous time signal x(t) with frequency spectrum X(jΩ) with $-2\pi\times10^4 r/s \le \Omega_x \le 2\pi\times10^4 r/s$,If sampling period T=$2\times10^{-4}$s, to prevent aliasing, write out the anti-aliasing filter with largest cut-off frequency before sampling X(jΩ).
  - Solution:

$\Omega_T = \pi\times10^4$

$$H_a(j\Omega)=\begin{cases}1, & |\Omega|<\dfrac{\Omega_T}{2}=0.5\pi\times10^4,\\ 0, & |\Omega|\ge\dfrac{\Omega_T}{2}=0.5\pi\times10^4\end{cases}$$



- The practical anti-aliasing filter design

$$\Omega_p < \Omega_{stop} \le \frac{\Omega_T}{2}$$

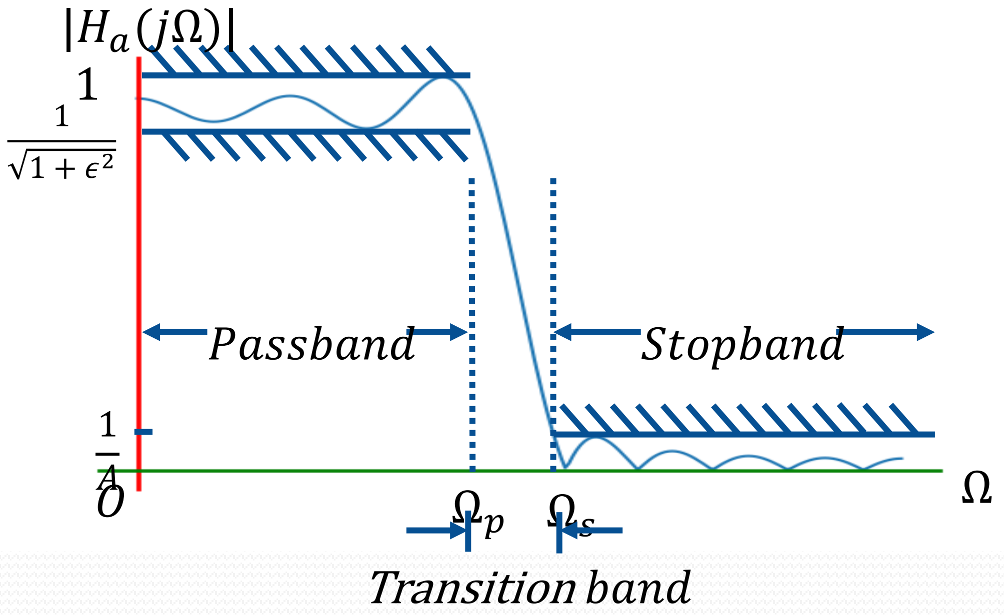
  - The maximum distortion will be in the replicas of input spectrum adjacent the baseband
  - the maximum aliasing frequency is at

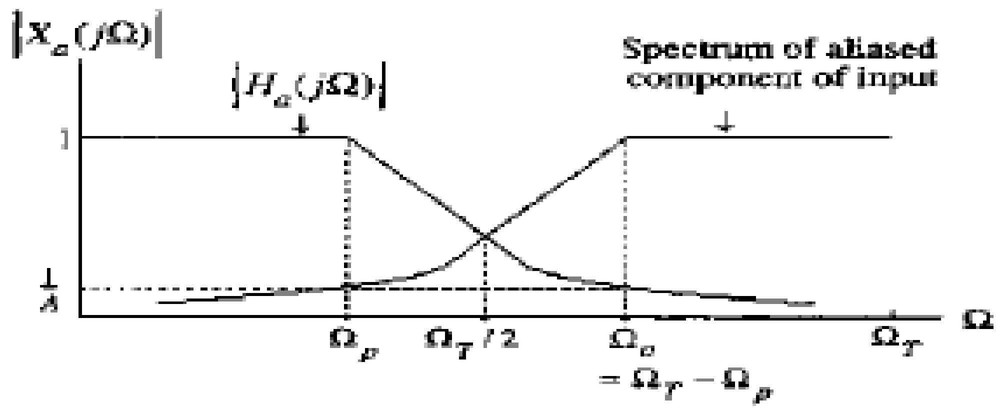
    $\Omega_o = \Omega_T - \Omega_p$
  - Minimum attenuation for anti-aliasing filter at Ω_o should less than $1/A$

Attenuation difference(level)

$$= 10\log_{10}\left[\frac{1+(\frac{\Omega_o}{\Omega_c})^{2N}}{1+(\frac{\Omega_p}{\Omega_c})^{2N}}\right] \cong 10\log_{10}\left(\frac{\Omega_o}{\Omega_p}\right)^{2N}$$

  - $N = \dfrac{attenuation\ in\ the\ table}{corresponding\ factor\ in\ the\ table}$





Ω_0	$2\Omega_p$	$3\Omega_p$	$4\Omega_p$
Attenuation in dB	6.02N	9.54N	12.04N
Ω_T	$3\Omega_p$	$4\Omega_p$	$5\Omega_p$

- Ex. Consider an anti-aliasing filter with a Butterworth lowpass filter, if the minimum stopband attenuation $at\ \Omega_0$ is 60dB and $\Omega_T = 3\Omega_p$ then from table , determination of the order N of the anti-aliasing filter
  - solution:

if $\Omega_T = 3\Omega_p$,from table we can get an attenuation level which is 6.02N,

since attenuation $at\ \Omega_0$ is 60dB , from table,

$$6.02N=60$$

$$N=\lceil 60/6.02 \rceil = 10$$

6.3 Analog reconstruction filter design

- Why use analog reconstruction filter ?
  - To recover the original continuous time signal from discrete time signal
  - To eliminate all the replicas of the spectrum outside the baseband.
  - To sample the required signal
- The ideal reconstruction filter

$$H_r(j\Omega) = \begin{cases} T, |\Omega| \leq \Omega_c \\ 0, |\Omega| > \Omega_c \end{cases},$$

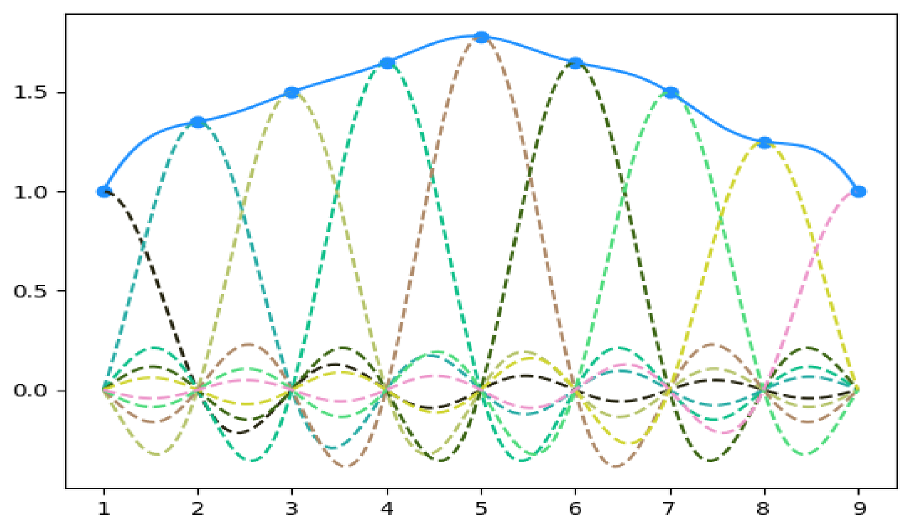
Ω_c--the highest frequency of the signal to be preserved

$$h_r(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_r(j\Omega)e^{j\Omega t}d\Omega = \frac{\sin(\Omega_c t)}{\Omega_T t/2}, \quad -\infty < t < \infty$$

The reconstruction filter is also called smoothing filter

- the recovered continous time signal in time-domain

$$\hat{Y}_a(j\Omega) = H_r(j\Omega)Y_s(j\Omega) = T \cdot u(\Omega - |\Omega_C|) \cdot Y_s(j\Omega)$$



$$Y_s(j\Omega) = \sum y[n]\, e^{-j\Omega nT}$$

$$\hat{g}_a(t) = \frac{T}{2\pi}\int_{\frac{-\pi}{T}}^{\frac{\pi}{T}} \sum y[n]\, e^{-j\Omega nT}\, e^{j\Omega t} d\Omega$$

$$= \sum y[n] \int_{-\Omega_c}^{\Omega_c} e^{j\Omega(t-nT)} \frac{T}{2\pi} d\Omega$$

$$= \sum y[n] \operatorname{sinc}\left(\frac{\pi}{T}(t - nT)\right)$$

$$\int_{-\Omega_c}^{\Omega_c} e^{j\Omega(t-nT)} * \frac{T}{2\pi} d\Omega$$

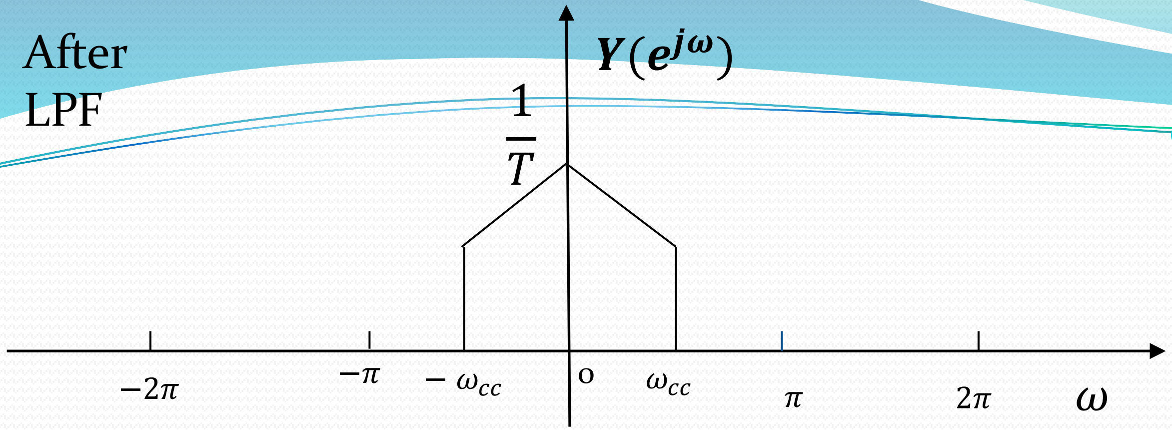
$$= \frac{e^{j\frac{\pi}{T}(t-nT)} - e^{-j\frac{\pi}{T}(t-nT)}}{\pi(2j)(t-nT)\frac{1}{T}}$$

When $\Omega_c = \frac{\Omega_s T}{2} = \frac{\pi}{T}$

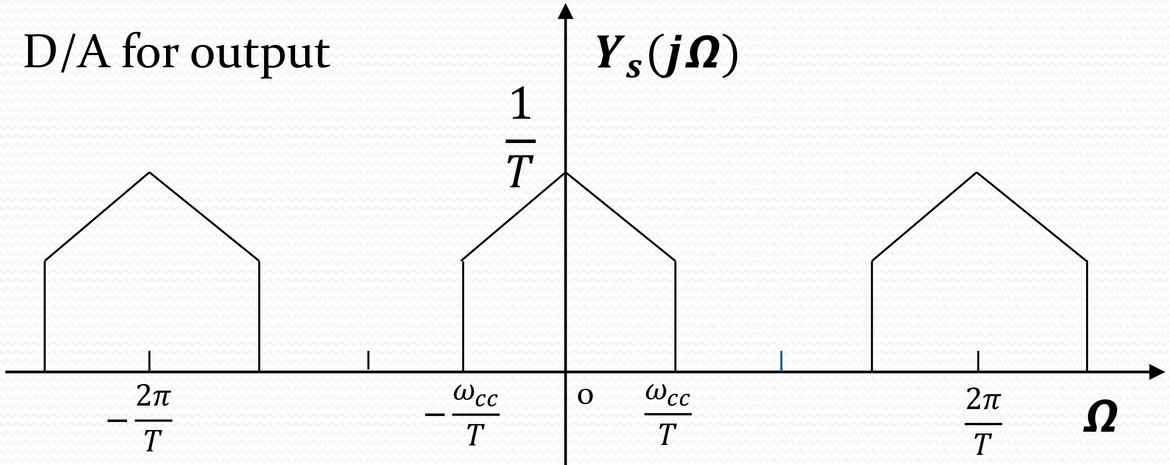
$$= \frac{\sin\left(\Omega_c(t-kT)\right)}{\Omega_c(t-kT)}$$

- the process of recovering in frequency domain

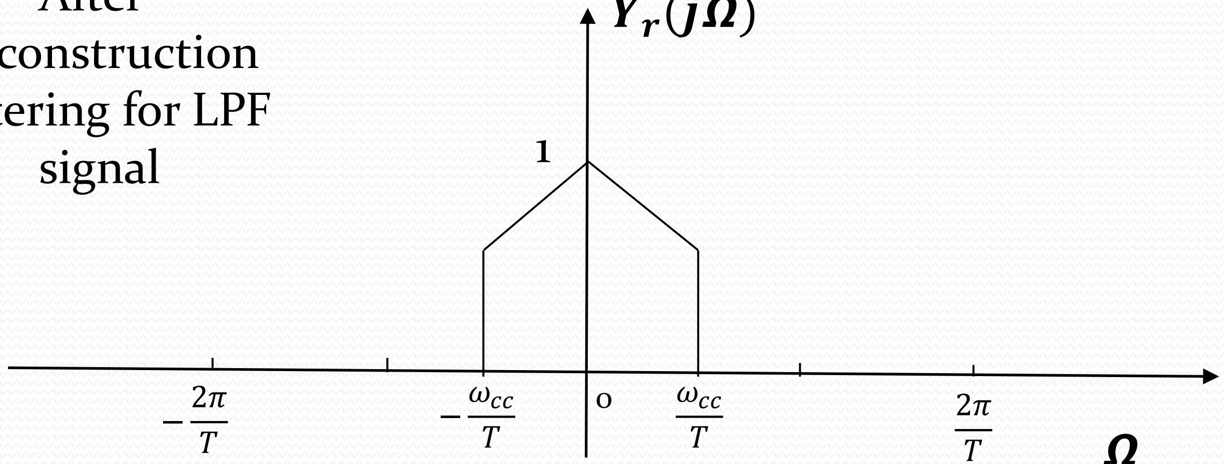
After LPF



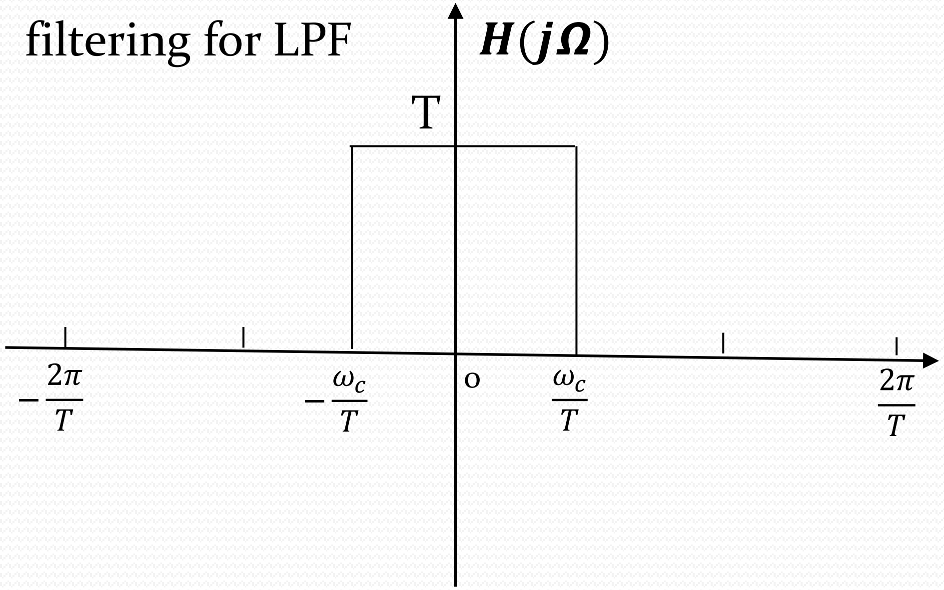
D/A for output



After reconstruction filtering for LPF signal

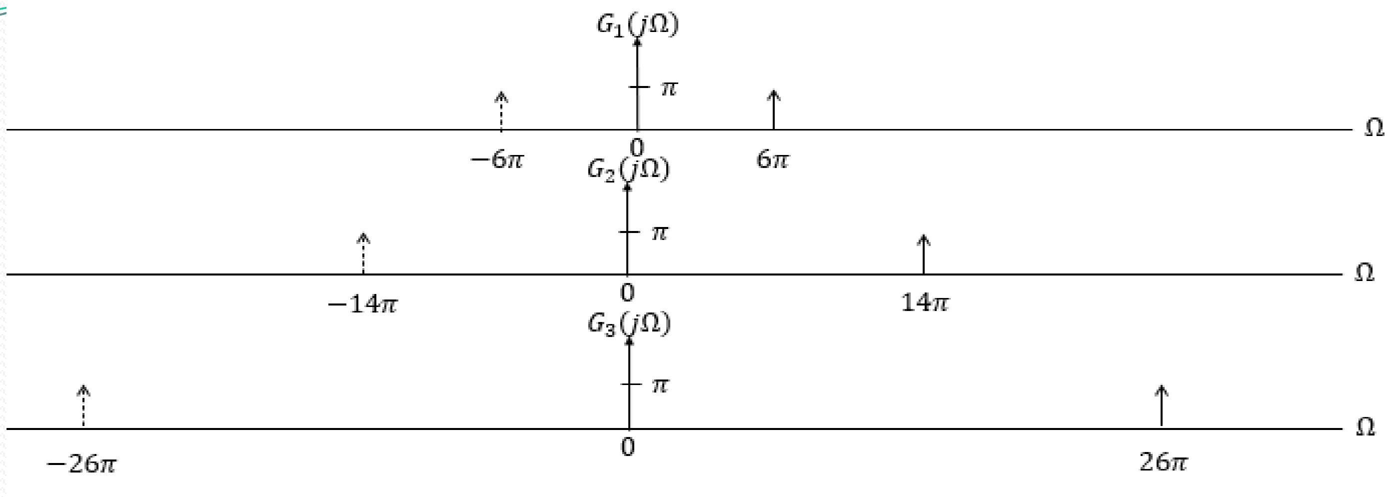


Reconstruction filtering for LPF



normally, $\omega_c = \omega_{cc}$

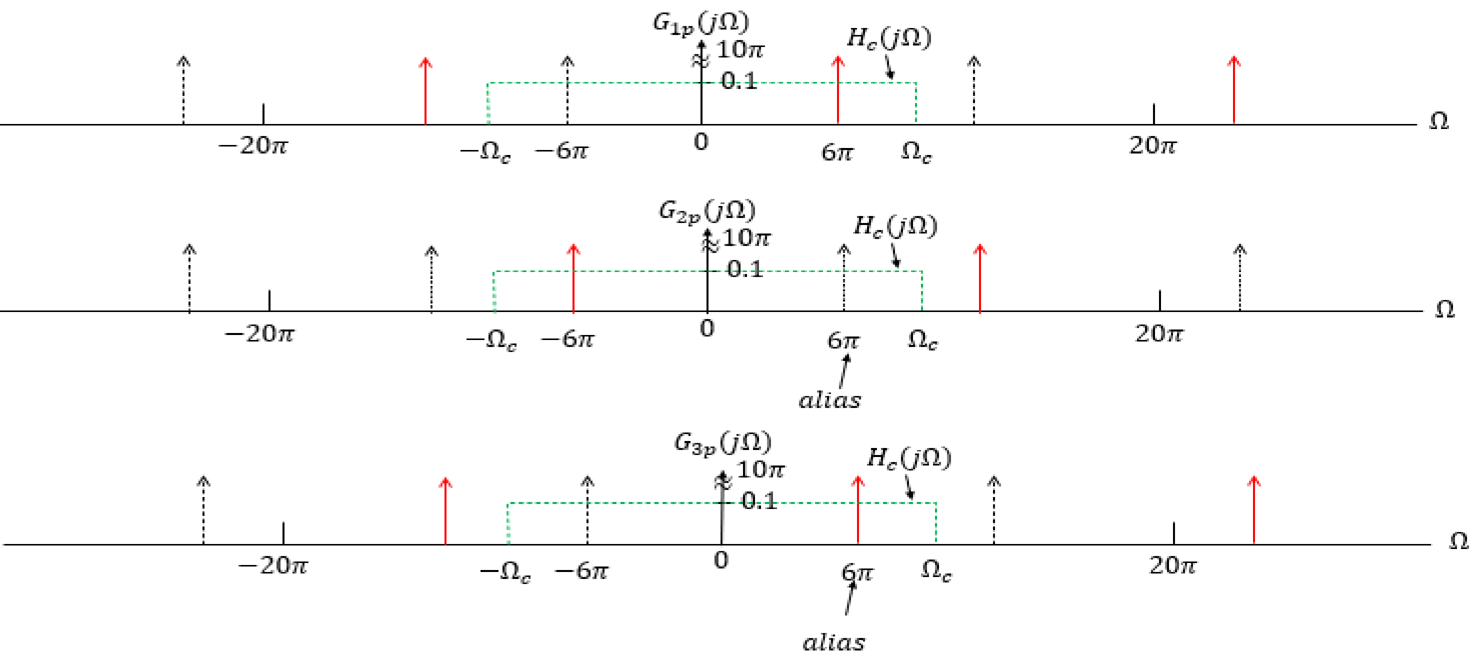
Aliasing of reconstruction



Spectrum of $\cos(6\pi t)$

Spectrum of $\cos(14\pi t)$

Spectrum of $\cos(26\pi t)$



Spectrum of sampled $\cos(6\pi t)$ with
$\Omega_T = 20\pi > 2 \times 6\pi$

Spectrum of sampled $\cos(14\pi t)$ with
$\Omega_T = 20\pi < 2 \times 14\pi$

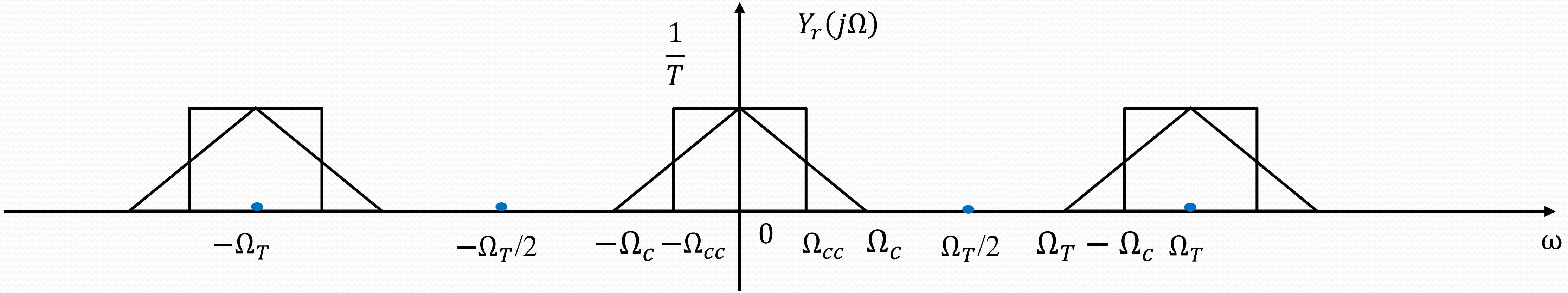
Spectrum of sampled $\cos(26\pi t)$ with
$\Omega_T = 20\pi < 2 \times 26\pi$

Anti-aliasing reconstruction filter design

- Ω_c--cut-off angular frequency of reconstruction filter
- Ω_{cc}---cut-off angular frequency of the filtered analog signal which is required to be reconstructed

$$\begin{cases} \Omega_{\mathrm{cc}} \le \Omega_T - \Omega_c \\ \Omega_{\mathrm{cc}} \le \Omega_c \end{cases}$$

Ideal reconstruction filter



- Ex: If $X_a(j\Omega)$ is known with $\Omega_c = 2\pi \times 10^4$, if reconstruction filter has no aliasing, Determine range of T
  - solution :

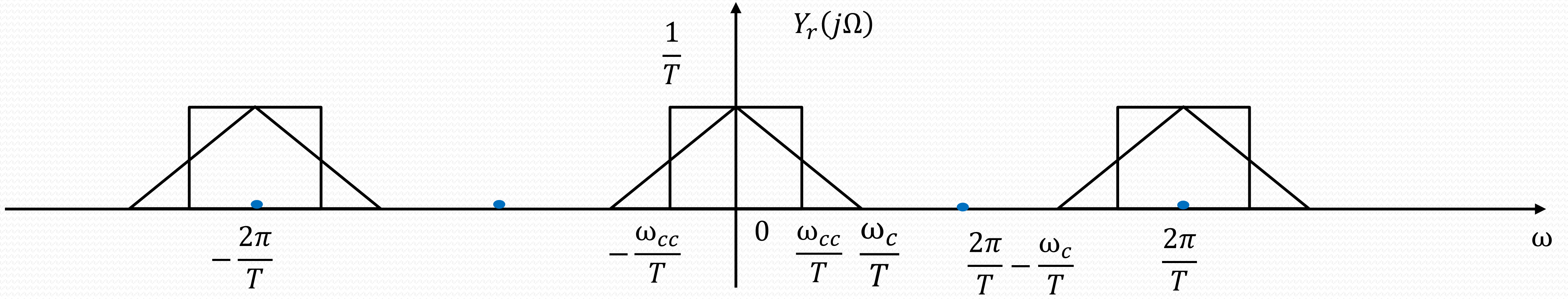
$$\Omega_c = 2\pi \times 10^4$$

$$\text{based on} \begin{cases} \Omega_{cc} \le \Omega_T - \Omega_c \\ \Omega_{cc} \le \Omega_c \end{cases} \rightarrow \begin{cases} \frac{\omega_{cc}}{T} \le \frac{2\pi}{T} - 2\pi \times 10^4 \\ \frac{\omega_{cc}}{T} \le 2\pi \times 10^4 \end{cases},$$

$$\begin{cases} 2\pi \times 10^4 \le \frac{2\pi - \omega_{cc}}{T} \\ \frac{\omega_{cc}}{T} \le 2\pi \times 10^4 \end{cases} \rightarrow \begin{cases} T \le \frac{2\pi - \omega_{cc}}{2\pi \times 10^4} \\ \frac{\omega_{cc}}{2\pi \times 10^4} \le T \end{cases},$$

$$\frac{\omega_{cc}}{2\pi \times 10^4} \le T \le \frac{2\pi - \omega_{cc}}{2\pi \times 10^4}$$

Ideal reconstruction filter



reconstruct the bandpass signals

- if Ω_{max} is very large, but signal band is $\Delta\Omega$ is small, then $\Omega_T \geq 2\Omega_{max}$ will be very large spectral gap, which is not be practical.

Assume

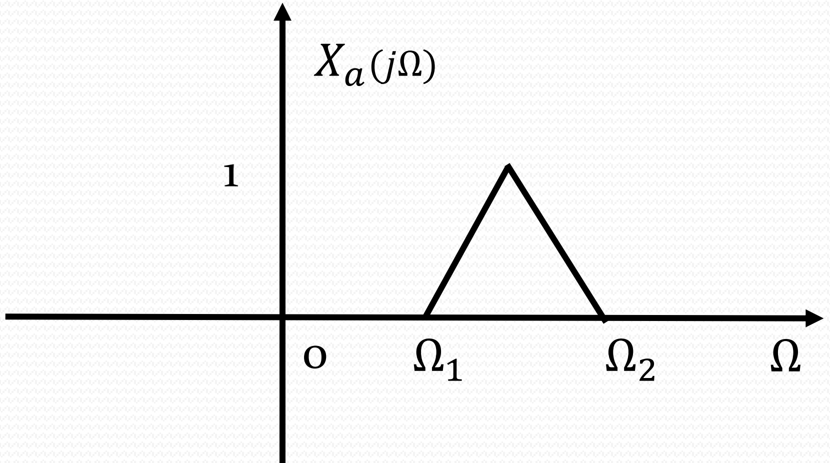
$$\Omega_{max} = \mathrm{M}(\Delta\Omega)$$

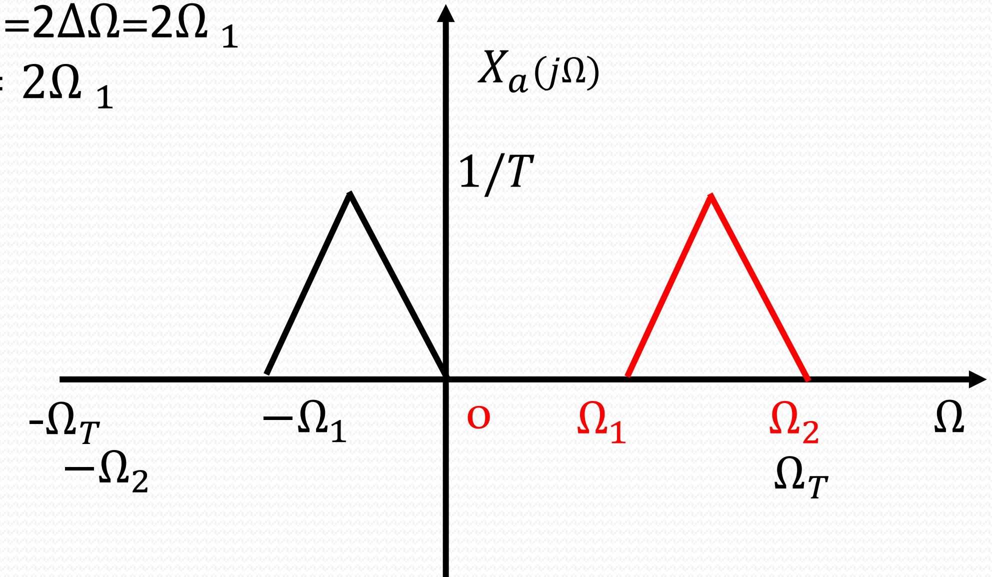
  - Choose

$$\Omega_T \geq 2(\Delta\Omega) = \frac{2\Omega_{max}}{M} < \Omega_{max}$$

- Ex2: $X_a(j\Omega)$ is known, with $\Omega_1 \leq \Omega \leq \Omega_2$ which is $\Omega_2 = 2\Omega_1$; find the lowest sampling rate Ω_T, if $x_a(t)$ can be exactly reconstructed.
  - Solution:

$$\Omega_{max} = \Omega_2 = 2(\Omega_2 - \Omega_1) = 2\Delta\Omega = 2\Omega_1$$
$$\text{Then } \Omega_T \geq 2\Delta\Omega = 2\Omega_1$$





From analog filter to digital filter

- FIR digital from analog filter
  - Directly based on $\Omega = \frac{\omega}{T}$, determine the cut-off frequency of FIR filter
  - Determine the order N(the length) of FIR digital filter based on Gibbs specifications
  - Choose Windows function
- IIR digital filter design from analog filter
  - Bilinear transformation method
  - Impulse Invariance method

FIR digital design from analog filter

- Determine ω_c of ideal filter based on $\omega_c = \Omega T$

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases} \Rightarrow \quad h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, -\infty < n \le \infty$$

- Windowing the ideal filter (from IIR filter to FIR filter)

$$\hat{h}_{LP}[n] = \begin{cases} \dfrac{\sin \omega_c n}{\pi n}, 0 \le n \le N-1 \\ 0, \text{otherwise} \end{cases} \Rightarrow \quad \hat{h}_{LP}[n] = h_{LP}[n] \cdot w[n]$$

- Determine the order of FIR filter
  - Kaiser's Formula

$$N \cong \frac{-20\log_{10}\left(\sqrt{\delta_p \delta_s}\right) - 13}{14.6\,(\omega_s - \omega_p)/2\pi}$$

IIR filter design from analog filter

- Impulse Invariance Method
  - Given the analog filters impulse response, sample it to get the digital filter unit impulse response

$$h_s(t) = T \sum_{k=-\infty}^{\infty} h_a(kt)\delta(t - kT)$$

  - Frequency response relation is found as

$$H_s(j\Omega) = T \cdot \left(\frac{1}{T}\right) \sum_{k=-\infty}^{\infty} H_a\left(j\left(\Omega - \left(\frac{2\pi}{T}\right)k\right)\right)$$

$$H_s(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_a\left(j\left(\frac{\omega}{T} - 2\pi k\right)\right)$$

  - It will be effected by aliasing.
    - For LP and BP it can sometimes work or can't work due to aliasing;
    - For HP and BR filter, it can't work due to aliasing

- Bilinear transformation

$$j\Omega = \frac{1 - z^{-1}}{1 + z^{-1}}$$

$when\ z = e^{j\omega},$

$$j\Omega = \frac{\left(\frac{2j}{T}\right)\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)/2j}{\left(e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}\right)/2}$$

$$= \frac{2j\,\sin(\frac{\omega}{2})}{T\,\cos(\frac{\omega}{2})}$$

when $\Omega = \frac{2}{T}\tan(\frac{\omega}{2})$, will not cause aliasing

- **Ex. An analog lowpass filters has the following characteristics:**

$$\left||H_c(j\Omega)| - 1\right| \le \delta_{\mathrm{p}}, \text{for} |\Omega| \le \Omega_p$$
$$|H_c(j\Omega)| \le \delta_s, \text{ for } |\Omega| \ge \Omega_s$$

(a)For constant Ω_p, find T_p such that $\omega_p = \pi/2$

Solution : $\Omega_p = \frac{2}{T_p}\tan(\omega_p/2) = (2/T_p)\tan(\pi/4)$

$$T_p = (2/\Omega_p)\tan(\pi/4)$$

(b) With Ω_p fixed, sketch ω_p, as a function of T_p

Solution : $\omega_p = 2\tan^{-1}(\Omega_p \cdot T_p/2)$

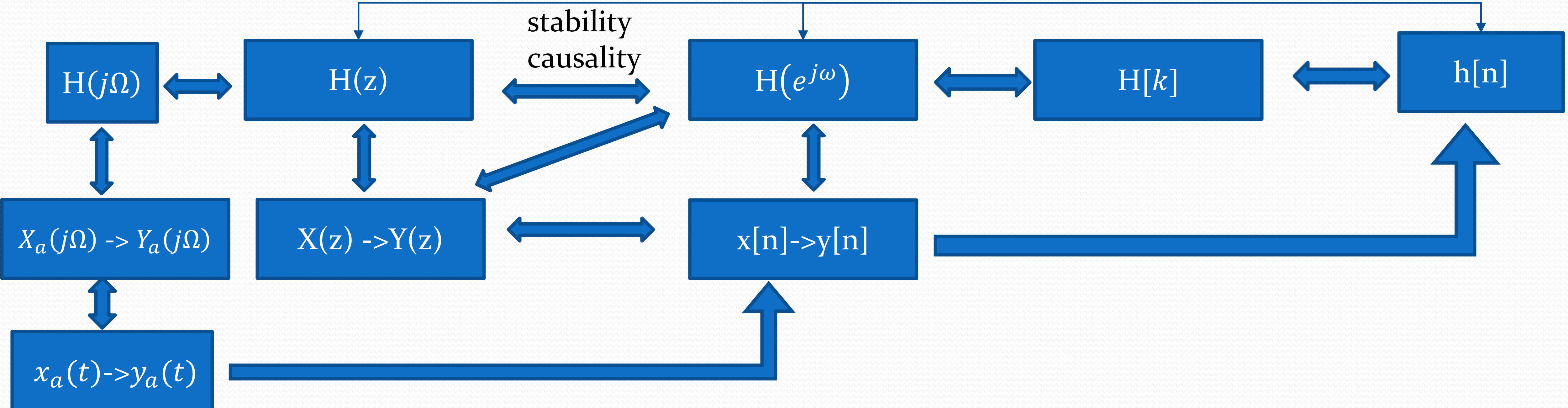
(c)With Ω_p and Ω_s fixed, sketch $\Delta\omega = \omega_s -$ as a function of T_p

Solution :$\Delta\omega = \omega_s - \omega_p = 2\tan^{-1}\left(\Omega_s \cdot \frac{T_p}{2}\right) - 2\tan^{-1}\left(\Omega_p \cdot \frac{T_p}{2}\right)$

$\approx T_p(\Omega_s - \Omega_p) \qquad \because \tan^{-1}(x) \approx x, for\ \ x \to 0$

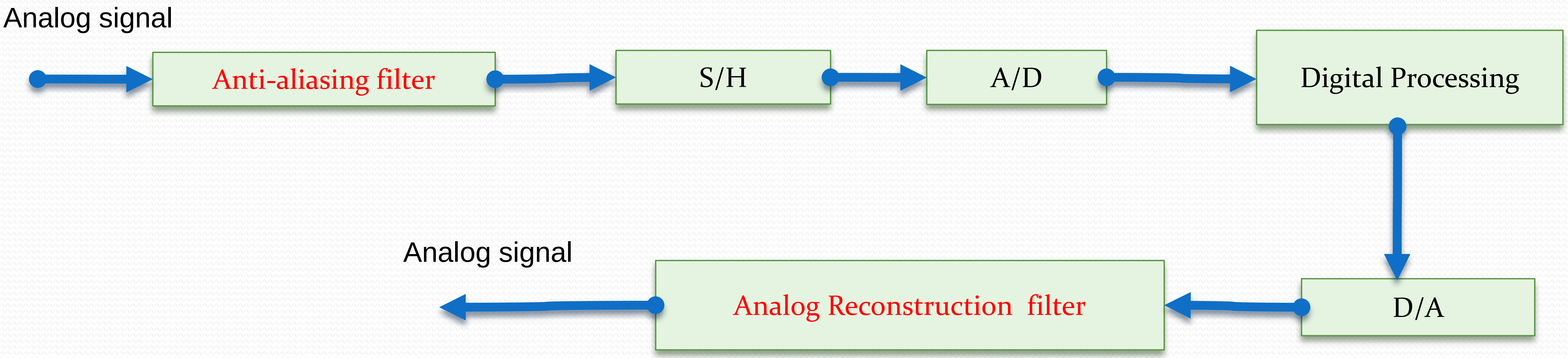
Procedure of practical digital filter design

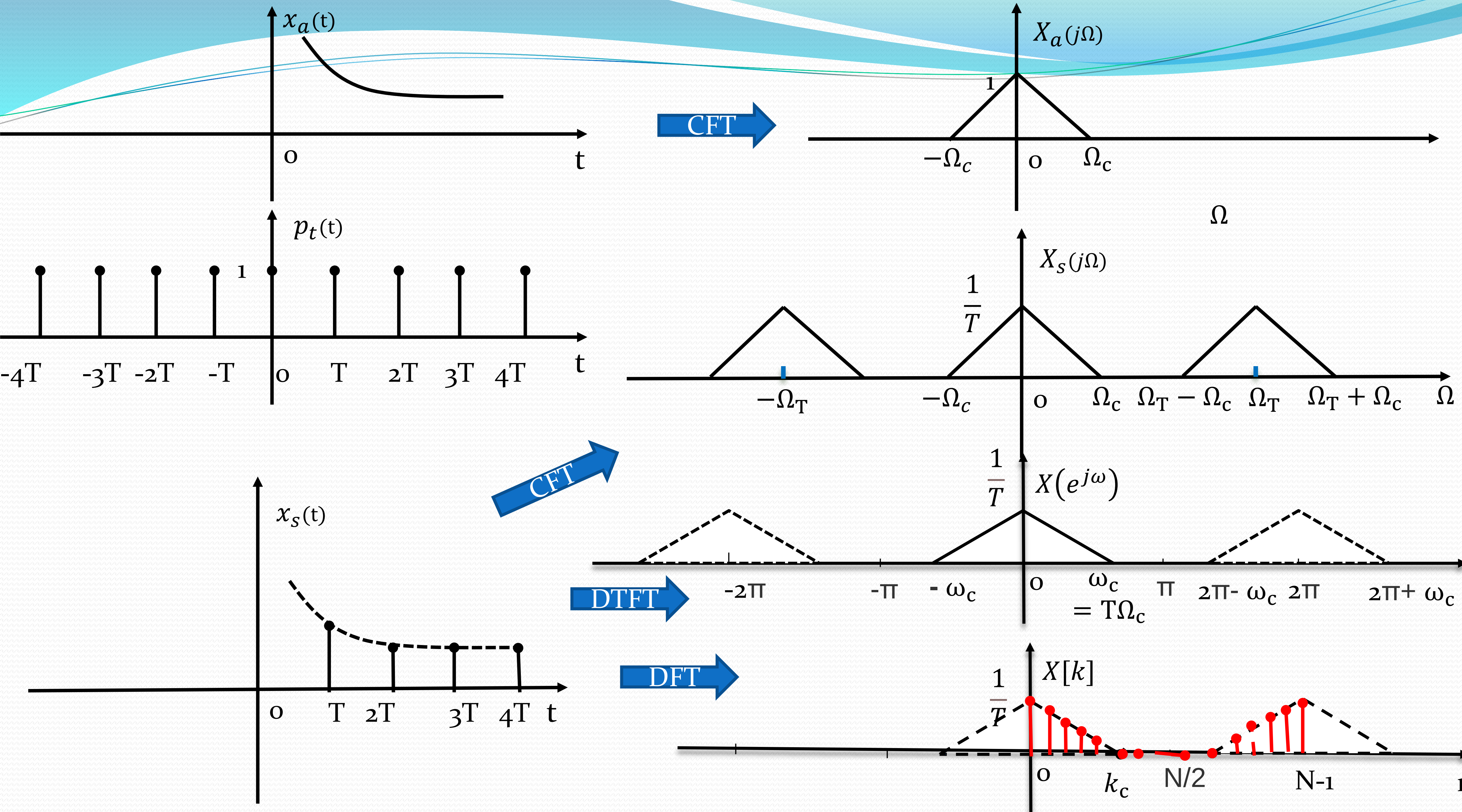
- Step1: Design analog filters for analog signal $H(j\Omega)$ based on specifications of analog filters$H(j\Omega)$
- Step 2 : Mapping analog filter to transfer function $H(z)$
- Step 3: based on stability and causality requirements of LTI system to design $H(z)$
- Step 4: From $H(z)$ to get $H\left(e^{j\omega}\right)$
- Step 5: DFT frequency response to get $H[k]$
- Step 6: IDFT to get $h[n]$



Procedure of practical digital signal processing

- Step1: Based on the sampling theorem to determine the T of $X_a(j\Omega)$, after anti-aliasing filtering $x_a(t)$, sampling $x_a(t)$ to $x[n]$
- Step2 : Design practical digital filter h[n] based on the previous procedure
- Step3 : Do convolution of $x[n]$ with h[n] to get y[n]
- Step4: Design anti-aliasing reconstruction filter
- Step 5: reconstruct $y_a(t)$ from y[n]

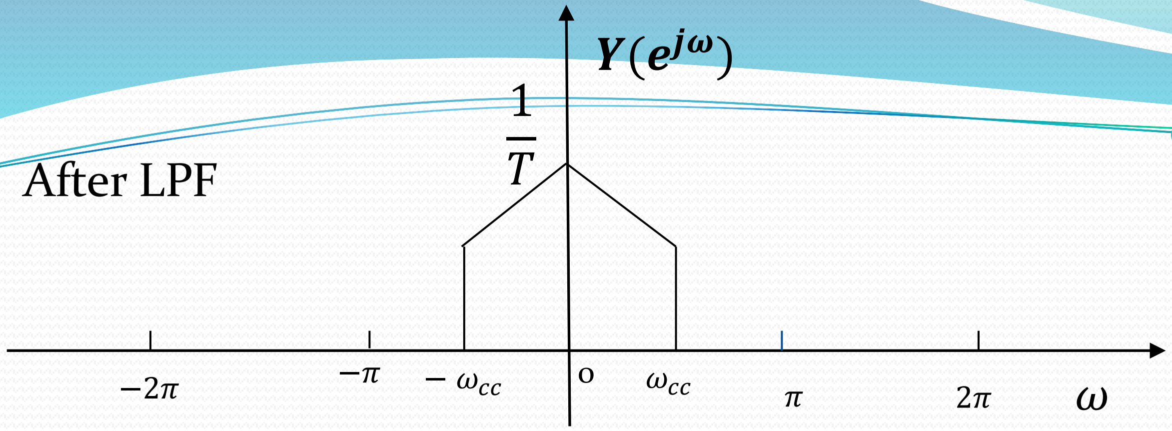




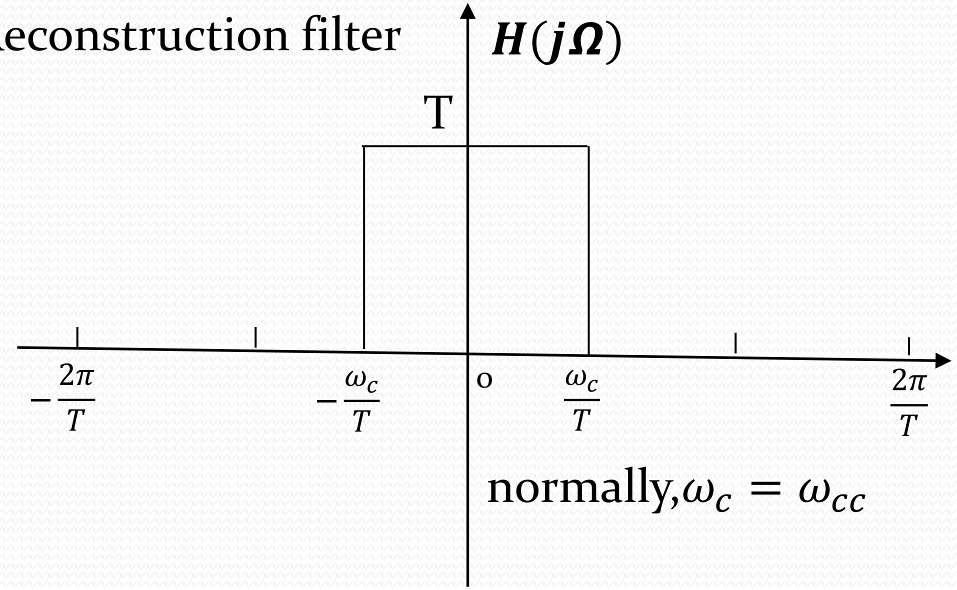
$x_a(t)$
0
t
CFT
$X_a(j\Omega)$
1
$-\Omega_c$
0
Ω_c
Ω
$p_t(t)$
1
-4T
-3T
-2T
-T
0
T
2T
3T
4T
t
$X_s(j\Omega)$
$\frac{1}{T}$
$-\Omega_T$
$-\Omega_c$
0
Ω_c
$\Omega_T - \Omega_c$
Ω_T
$\Omega_T + \Omega_c$
Ω
CFT
$x_s(t)$
0
T
2T
3T
4T
t
$\frac{1}{T}$
$X(e^{j\omega})$
DTFT
-2π
-π
- ω_c
0
ω_c
$= T\Omega_c$
π
2π- ω_c
2π
2π+ ω_c
DFT
$\frac{1}{T}$
$X[k]$
0
k_c
N/2
N-1

- Process of recovering in frequency domain

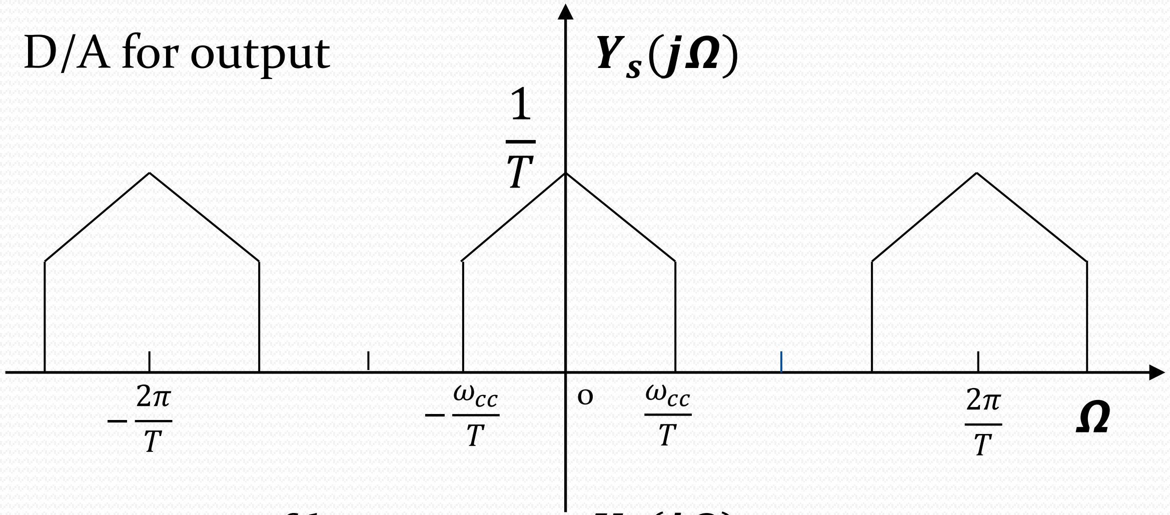
After LPF



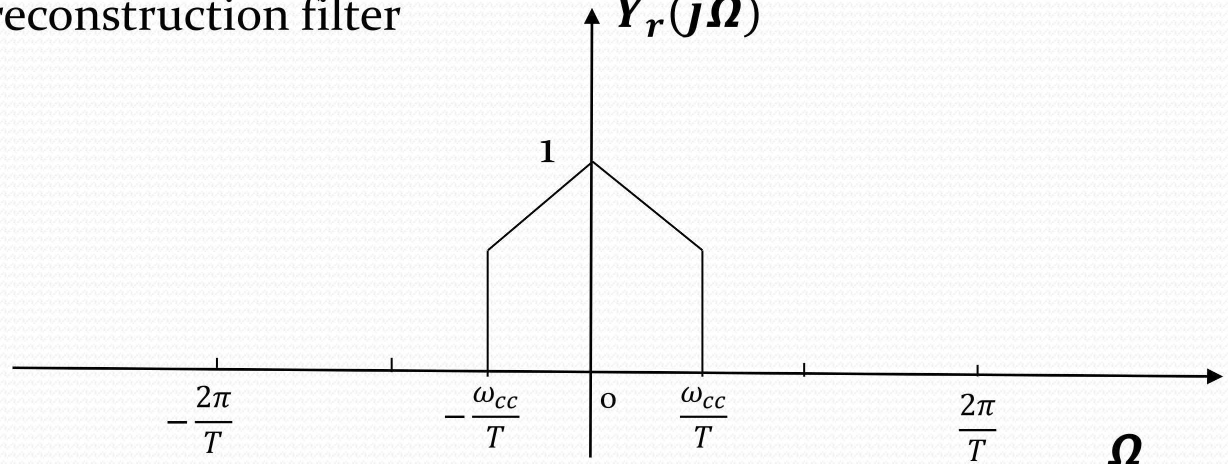
Reconstruction filter



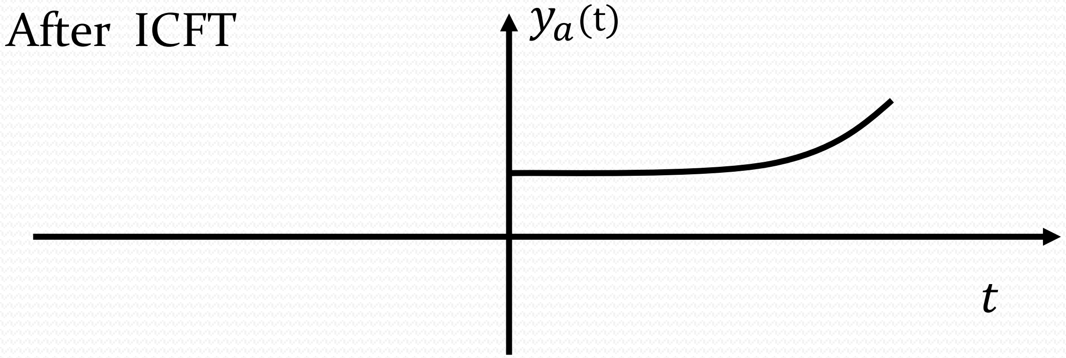
D/A for output



After reconstruction filter



After ICFT



课程结束！