Chapter 5 z-transform

授课教师：蔡珣
山东大学 软件学院

Motivation

- $H(e^{j\omega})$ is difficult to manipulate it for the realization of a digital filter.
- Transfer function (z-transform) is polynomial function with real coefficients and is more amenable(易控) for synthesis
- z-transform can be easily used to implement a stable and causal LTI system in discrete time domain

Contents

- Definition of z-transform and Inverse z-transform
- Computations of z-transform
- Properties of z-transform
- Computations of Inverse z-transform
- Transfer function of LTI system
- Simple FIR and IIR Filter design by z-transform
- Filter design without distortion

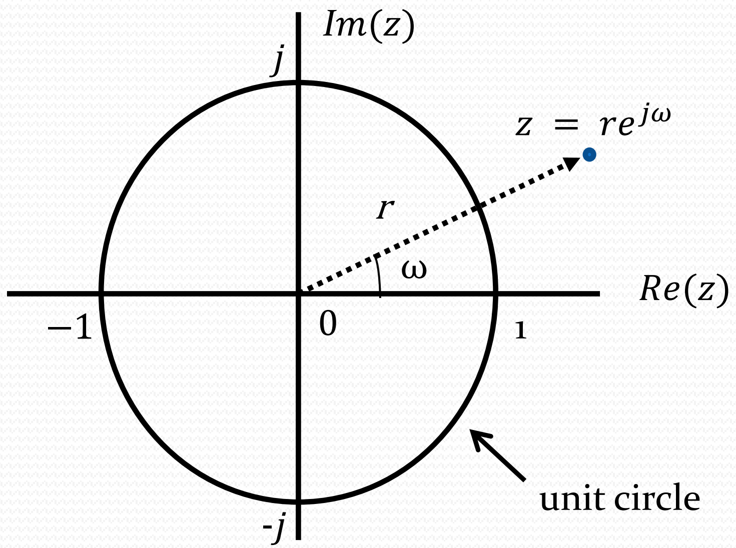
5.1 **Definitions** for z-transform

- z-transform X(z) is a generalization of DTFT $X(e^{j\omega})$ so can be used to design digital filters.

$$X(e^{j\omega}) \rightarrow X(re^{j\omega})$$

$$\sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} \xrightarrow{z=re^{j\omega}} \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z)$$

$$z = Re(z) + j\, Im(z)$$



- In power series form
- In rational (有理) form
  - In cascade form of z-transform
    - Zeros and Poles
  - In Parallel form of z-transform
- ROC(The region of convergence)
  - If and only if within the ROC , z-transform existed uniquely
  - If and only if the ROC include unit circle then DTFT existed
  - DTFT existed doesn‘t mean the existence of z-transform
    - Ex. Ideal LP filter h[n]

$$z\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n},$$

$$x[n] = \left(\frac{1}{2\pi}j\right)\oint X(z)z^{n-1}dz$$

$$X(z) = \frac{\sum_n b[n]z^{-n}}{\sum_m a[m]z^{-m}} = \frac{K\prod_n (z-z_1)(z-z_2)\cdots(z-z_n)}{\prod_m (z-p_1)(z-p_2)\cdots(z-p_m)}$$

$$z_i\ - the\ zero, p_i\ - the\ pole$$

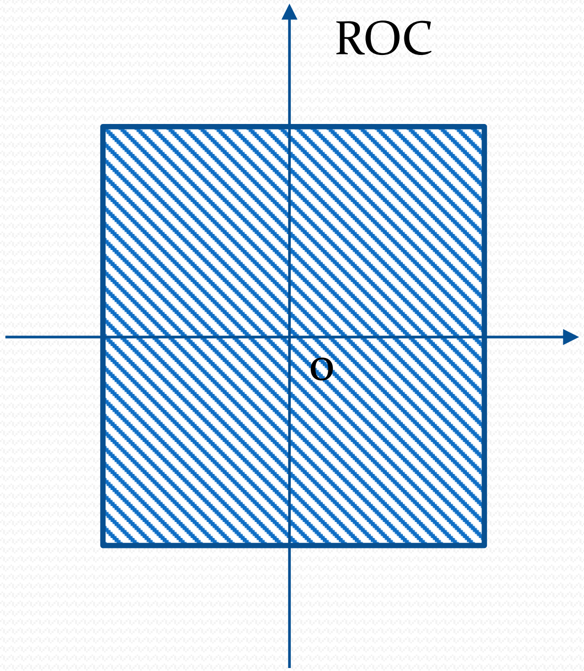
$$X(z) = \sum_k \frac{h_k + i_k z^{-1}}{1 + f_z z^{-1} + g_k z^{-2}}$$

5.2 Computation of z-transform

- Finite length sequence

$$g[n], N_1 \leq n \leq N_2, and\ |g[n]| < \infty.$$

$$G(z) = \sum_{n=N_1}^{N_2} g[n]z^{-n}$$



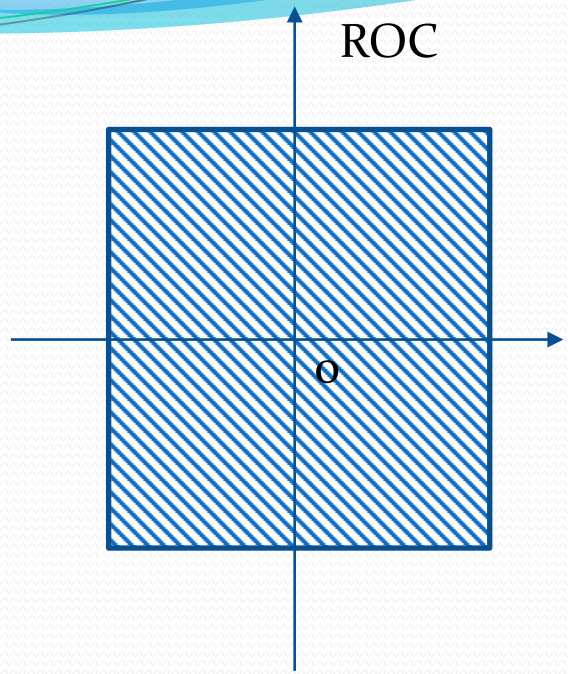
  - case1:N_1=$N_2 = 0, G(z) = g[0], ROC\ for\ entired\ z - plane$
  - case2:$N_2 > 0, poles\ at\ z = 0, so\ ROC\ for\ z \neq 0$
  - case3:$N_1 < 0, poles\ at\ z = \infty, so\ ROC\ for\ z \neq \infty$
  - in general, z-transform of a finite-length bounded sequence
  - convergence everywhere in the z-plane except possibly at z =0 and/or z =∞

- Ex. $x[n] = \begin{cases} \alpha^n, M \le n \le N-1. \\ 0, otherwise. \end{cases}$
  - solution:

$$X(z) = \sum_{n=M}^{N-1} \alpha^n z^{-n} = z^{-M} \sum_{n=0}^{N-M-1} \alpha^n z^{-n}$$

$$= z^{-M} \left(\frac{1 - \alpha^{N-M} z^{-(N-M)}}{1 - \alpha z^{-1}} \right).$$

  - For $\mathrm{N} > \mathrm{M} > 0$, ROC is the entire z-plane excluding the $\mathrm{z} = 0$.
  - For $\mathrm{M} < 0$ and $\mathrm{N} > 0$, the ROC is the entire z-plane excluding $\mathrm{z} = 0$ and $\mathrm{z} = \infty$.
  - For $\mathrm{M} > \mathrm{N} > 0$, the ROC is entire z-plane excluding $\mathrm{z} = 0$ and $\mathrm{z} = \infty$.
  - For $0 > \mathrm{M} > \mathrm{N}$, the ROC is entire z-plane excluding $\mathrm{z} = \infty$.

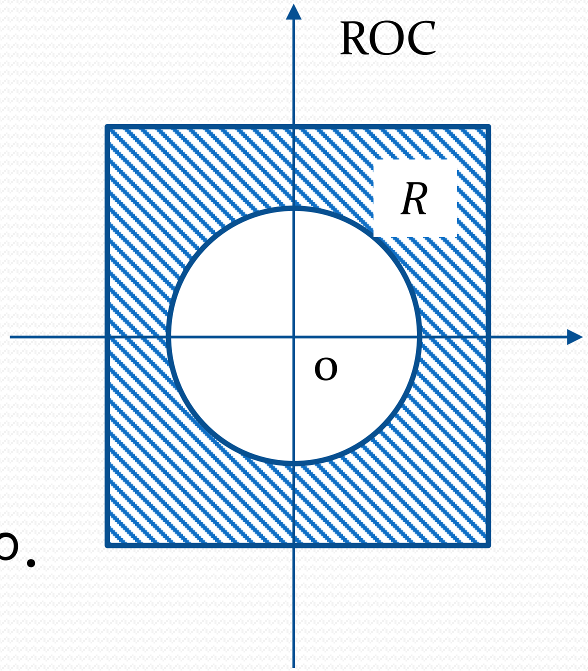


z-transform for Infinite sequence

- Right-side infinite-length sequence
  - Case1: $u_1[n], n \geq 0.$

$$U_1(z) = \sum_{n=0}^{\infty} u_1[n] z^{-n}$$

R.O.C $|z| > R_1$, including the point $z = \infty$.



  - Case2: $u_2[n], n \geq M, M < 0.$

$$U_2(z) = \sum_{n=M}^{0} u_2[n] z^{-n} + \sum_{n=0}^{\infty} u_2[n] z^{-n}$$

R.O.C $|z| > R_2$, excluding the point $z = \infty$.

- $Ex.\, h[n] = (-0.6)^n \mu[n]$,
  - solution :

$$H(z) = \sum_{n=0}^{\infty} (-0.6)^n z^{-n} = \sum_{n=0}^{\infty} (-0.6z^{-1})^n$$

$$= \frac{1}{1 + 0.6z^{-1}} = \frac{z}{z + 0.6},$$

$$|0.6z^{-1}| < 1 \rightarrow ROC = |z| > 0.6$$

- $Ex. w[n] = \left((-0.5)^{n-2} + (0.2)^{n-1}\right)\mu[n]$
  - Solution: rewrite w$[n]$ as

$$w[n] = (4(-0.5)^n + 2(0.2)^n)\mu[n].$$

using table , we have

$$(-0.5)^n\mu[n] \overset{z}{\leftrightarrow} \frac{1}{1+0.5z^{-1}}, ROC: |z| > 0.5,$$

$$(0.2)^n\mu[n] \overset{z}{\leftrightarrow} \frac{1}{1-0.2z^{-1}}, ROC: \ |z| > 0.2$$

$$W(z) = 4\left(\frac{1}{1+0.5z^{-1}}\right) + 2\left(\frac{1}{1-0.2z^{-1}}\right) = \frac{6+0.2z^{-1}}{(1+0.5z^{-1})(1-0.2z^{-1})},$$

$$ROC: |z| > 0.5$$

- Left-side infinite-length sequence
  - Case1: $v_1[n], n < 0$, which is called anti-causal sequence.

$$V_1(z) = \sum_{n=-\infty}^{0} v_1[n] z^{-n},$$

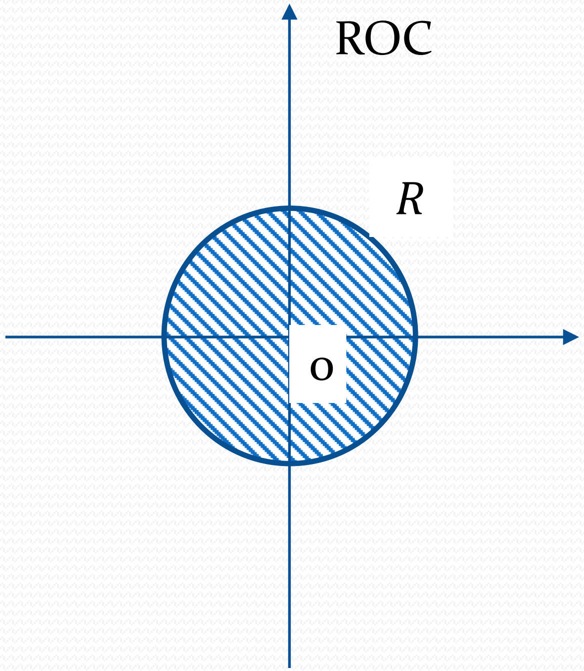
ROC $|z| < R_3$, including the point $z = 0$.

  - Case2: $v_2[n], n \leq N, N > 0$.

$$V_2(z) = \sum_{n=-\infty}^{0} v_2[n] z^{-n} + \sum_{n=0}^{N} v_2[n] z^{-n},$$

it has N poles at $z = 0$.

ROC $|z| < R_4$, excluding the point $z = 0$.



- Ex. z-transform of an anti-causal exponential sequence

$$x[n] = -\alpha^n \mu[-n-1]$$

  - solution:

$$X(z) = -\sum_{n=-\infty}^{-1} \alpha^n z^{-n} = -\sum_{n=1}^{\infty} \alpha^{-n} z^n = -\alpha^{-1} z \sum_{n=0}^{\infty} \alpha^{-n} z^n$$

$$= \frac{-\alpha^{-1} z}{1-\alpha^{-1} z},$$

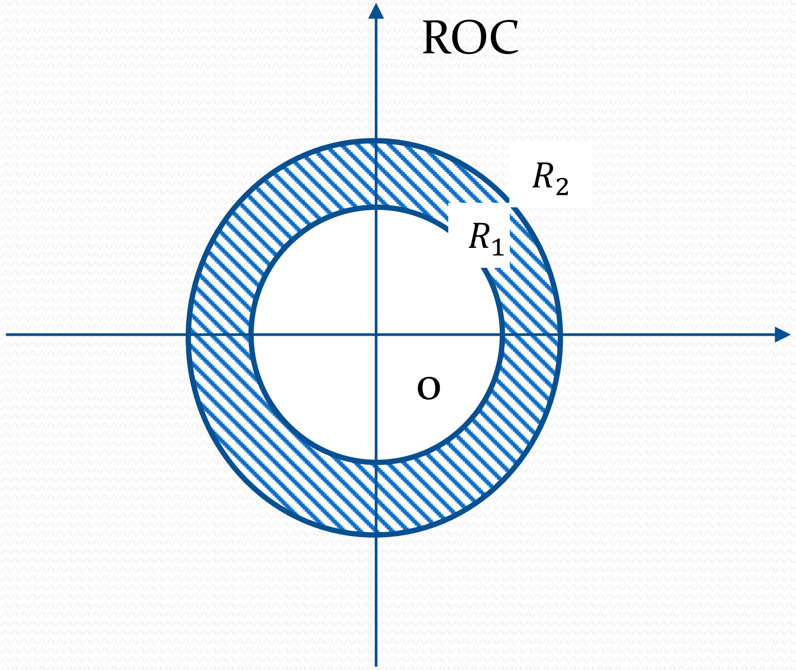
$$for\ |a^{-1} z| < 1,\ \ ROC\ is\ |z| < |\alpha|.$$

- Two-side infinite-Length Sequence

$$w[n],\ -\infty < n < \infty.$$

$$W(z) = \sum_{n=-\infty}^{\infty} w[n]z^{-n}$$

$$= \sum_{n=0}^{\infty} w_1[n]z^{-n} + \sum_{n=-\infty}^{-1} w_2[n]z^{-n},$$



- The first can be interpreted as the transform of a right-sided sequence, the second is left-sided.
- If the first converge exterior to the circle $|z_1| = R_1$ and the second converge interior to the circle $|z_2|=R_2$, then if $R_1<R_2$, there is overlapping R.O.C. $R_1<|z|<R_2$.
- If $R_1>R_2$, the z-transform doesn't exist.

- $Ex.\ \ v[n] = \alpha^n \mu[n] - \beta^n \mu[-n-1].$
  - Solution :

$$Denote\ x_1[n] = \alpha^n \mu[n], x_2[n] = -\beta^n \mu[-n-1].$$

due to linearity property of the z-transform,

$$V(z) = X_1(z) + X_2(z).$$

$$X_1(z) = \frac{1}{1-\alpha z^{-1}}, for\ |z| > |\alpha|,$$

$$X_2(z) = \frac{-\beta^{-1}z}{1-\beta^{-1}z}, for\ |z| < |\beta|.$$

$$Therefore, V(z) = \frac{1}{1-\alpha z^{-1}} + \frac{-\beta^{-1}z}{1-\beta^{-1}z}, for\ R.O.C = |\alpha| < |z| < |\beta|$$

$$if\ |\beta| > |\alpha|, it\ exists, else\ doesn't\ exist.$$

- $Ex. \quad v[n] = \alpha^n \mu[n] - \beta^n \mu[-n-1].$
  - Solution : $Denote\ x_1[n] = \alpha^n \mu[n], x_2[n] = -\beta^n \mu[-n-1].$

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$$for\ R.O.C = |\alpha| < |z| < |\beta|\ , if\ |\beta| > |\alpha|, it\ exists, else\ doesn't\ exist.$$

- Ex. Nonexistence of ROC of the z-transform of a Two-sided infinite-Length Sequence

$$u[n] = \alpha^n, -\infty \le n \le \infty.$$

  - solution:

$$U(z) = \sum_{n=-\infty}^{-1} \alpha^n z^{-n} + \sum_{n=0}^{\infty} \alpha^n z^{-n},$$

$$the\ first\ term\ converges\ for\ |z| < |\alpha|,$$
$$the\ second\ term\ converges\ for\ |z| > |\alpha|,$$
$$hence, there\ is\ no\ overlap\ of\ the\ two\ ROCs.$$

- **Summary of ROC of a z-transform**
  - ROC of the z-transform of an finite-length sequence, $M<=n<=N$
    - It is the entire of z-plane except possibly $z=0$ and/or $z=\infty$
  - ROC of z-transform of a right-sided infinite-length sequence , $M<=n< \infty$
    - It is exterior to a circle in the z-plane passing through the pole furthest from the origin $z=0$
  - ROC of the z-transform of a left-Sided infinite-Length sequence, $\infty<n<=N$
    - It is interior to a circle in the z-plane passing through the pole nearest to the $z=0$
  - ROC of the z-transform of a Two-sided infinite-Length sequence
    - If existence, it is a ring bounded by two circles in the z-plane passing through two poles with no poles inside the ring
    - Nonexistence of the z-transform of a Two-sided infinite-Length Sequence

Some commonly used z-transform pairs

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$\mu[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$\alpha^n\mu[n]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$n\alpha^n\mu[n]$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
$(n+1)\alpha^n\mu[n]$	$\dfrac{1}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
$(r^n \cos\omega_0 n)\mu[n]$	$\dfrac{1-(r\cos\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$
$(r^n \sin\omega_0 n)\mu[n]$	$\dfrac{(r\sin\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$

5.3 Properties of z-transform(similar to DTFT)

Property	Sequence	z-Transform	ROC
	$g[n]$ $h[n]$	$\mathrm{G}(z)$ $\mathrm{H}(z)$	R_g R_h
Conjugation	$g^*[n]$	$G^*(z^*)$	R_g
Time-reversal	$g[-n]$	$G(1/z)$	$1/R_g$
Linearity	$\alpha g[n]+\beta h[n]$	$\alpha\mathrm{G}(z)+\beta\mathrm{H}(z)$	Includes $R_g \cap R_h$
Time-shifting	$g[n-n_0]$	$z^{-n_0}\mathrm{G}(z)$	R_g, except possibly the point $z=0$ or ∞
Multiplication by an sequence	$\alpha^n g[n]$	$G(z/\alpha)$	$\lvert\alpha\rvert R_g$
Differentiation of $G(z)$	$ng[n]$	$-z\dfrac{d\mathrm{G}(z)}{dz}$	R_g, except possibly the point $z=0$ or ∞
Convolution	$g[n] \odot h[n]$	$\mathrm{G}(z)\mathrm{H}(z)$	Includes $R_g \cap R_h$
Modulation	$g[n]h[n]$	$\dfrac{1}{2\pi j}\oint_C \mathrm{G}(v)\mathrm{H}(z/v)v^{-1}\,dv$	Includes $R_g R_h$
Parseval's relation	$\sum_{n=-\infty}^{\infty} g[n]\,h^*[n]=\dfrac{1}{2\pi j}\oint_C \mathrm{G}(v)H^*(1/v^*)v^{-1}\,dv$		

Note: If R_g denotes the region $R_{g^-}<\lvert z\rvert<R_{g^+}$ and R_h denotes the region $R_{h^-}<\lvert z\rvert<R_{h^+}$, the $1/R_g$ denotes the region$1/R_{g^+}<\lvert z\rvert<1/R_{g^-}$ and $R_g R_h$ denotes the region $R_{g^-}R_{h^-}<\lvert z\rvert<R_{g^+}R_{h^+}$

Comments for properties

- $\alpha^n g(n)$<-> $X(a^{-1}z)$: used for change the stability of system
  - Ex. $\left[\left(\frac{1}{2}\right)^n u(n) + (3)^n u(n)\right]$ is not stable , but when multiply $\left(\frac{1}{4}\right)^n$ it will become stable
- g(n)*h(n)<->G(z)H(z): used for filter design in time domain
  - Ex. $(3z+1+z^{-1})\ (z+3z^{-1}+z^{-2}) = 6\text{z} + 11 + 8z^{-1}+4z^{-2}+z^{-3}$

    => $6\delta(n+1) + 11\delta(n) + 8\delta(n-1)+4\delta(n-2)+\delta(n-3)$

- Ex. Solve z-transform of v[n] by using a first-order time shift equation

$$d_0\mathrm{v}[n]+d_1\mathrm{v}[n-1]=p_0\delta[n]+p_1\delta[n-1],\ |d_0/d_1|<1.$$

  - Solution: from previous table , the z-transform of $\delta[n]$ is 1.

due to time-shifting property of the z-transform

$$d_0V[z]+d_1z^{-1}V[z]=p_0+p_1z^{-1},$$

$$V[z]=\frac{p_0+p_1z^{-1}}{d_0+d_1z^{-1}}.$$

Note, the pole is $z=-d_0/d_1$.

Hence, if v[n] is right-side sequence, the R.O.C is $|z|>d_0/d_1$.

If v[n] is left-sided sequence, the R.O.C is $|z|<d_0/d_1$.

- z-transform determination using differential property
  - Ex. Verify the $Y(z)$ and its ROC of the sequence $y[n] = (n+1)\,\alpha^n \mu[n]$
  - Solution: Let $x[n] = \alpha^n \mu[n]$, then we can write $y[n] = nx[n] + x[n]$.

$$X(z) = \frac{1}{1-\alpha z^{-1}},\ \text{ROC: } |z| > |\alpha|.$$

based on differential property, the z-transform of $nx[n]$ is

$$-z\frac{dX(z)}{dz} = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2},\text{ROC: } |z| > |\alpha|.$$

using the linear property of the z-transform, we obtain

$$Y(z) = \frac{1}{1-\alpha z^{-1}} + \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2} = \frac{1}{(1-\alpha z^{-1})^2},\ \text{ROC: } |z| > |\alpha|.$$

- Enlarge or narrow ROC by Pole-Zero Cancellation
  - Ex. consider $g[n]$ and $h[n]$, with $G(z)$ and $H(z)$ given by：

$$G(z) = \frac{2+1.2z^{-1}}{1-0.2z^{-1}}, |z| > 0.2,$$

$$H(z) = \frac{3}{1+0.6z^{-1}}, |z| > 0.6.$$

The intersection of the two R.O.Cs is $|z| > 0.6$.

The product of the above two z-transform is

$$G(z)H(z) = \left(\frac{2+1.2z^{-1}}{1-0.2z^{-1}}\right)\left(\frac{3}{1+0.6z^{-1}}\right) = \frac{6}{1-0.2z^{-1}},$$

whose ROC is given by $|z| > 0.2$.

5.4 computation of Inverse z-transform

- By using Residual theorems(Cauchy integral theorem)
  - Leads to contour clockwise in region convergence
- Power series by definition
  - Ex. $H(z) = 0.2z + 1 + 0.3z^{-1}$

$$h[n] = 0.2\delta[n+1] + \delta[n] + 0.3\delta[n-1]$$

- Rational form by using Partial-fraction expansion

$$G(z) = \sum_{l=1}^{N} \frac{\rho_l}{1 - \lambda_l z^{-1}}$$

$$\rho_l = (1 - \lambda_l z^{-1}) G(z)|_{z=\lambda_l}$$

then using closed form of sequence in time domain

- By using Long division
  - Power series of z-transform

- Rational form by using Partial-fraction expansion
- $Ex. H(z) = \frac{z(z+2.0)}{(z-0.2)(z+0.6)}$, $R.O.C \quad |z| > 0.6$
  - Solution : $H(z) = \frac{1+2.0z^{-1}}{(1-0.2z^{-1})(1+0.6z^{-1})} = \frac{\rho_1}{(1-0.2z^{-1})} + \frac{\rho_2}{(1+0.6z^{-1})}$

$$\rho_1 = (1-0.2z^{-1})H(z)|_{z=0.2} = \left.\frac{1+2.0z^{-1}}{1+0.6z^{-1}}\right|_{z=0.2} = 2.75,$$

$$\rho_2 = (1+0.6z^{-1})H(z)|_{z=0.6} = \left.\frac{1+2.0z^{-1}}{1-0.2z^{-1}}\right|_{z=-0.6} = -1.75.$$

$$H(z) = \frac{2.75}{(1-0.2z^{-1})} + \frac{-1.75}{(1+0.6z^{-1})}.$$

based on $ROC \quad |z| > 0.6, h[n] = 2.75(0.2)^n\mu[n] - 1.75(-0.6)^n\mu[n]$

- Inverse z-transform using long division
- $Ex. H(z) = \frac{z(z+2.0)}{(z-0.2)(z+0.6)}, \quad R.O.C \quad |z| > 0.6$
  - solution:

$$H(z) = \frac{z(z+2.0)}{(z-0.2)(z+0.6)}$$
$$= \frac{z^2 + 2.0z}{z^2 + 0.4z - 0.12} \cdot \frac{z^{-2}}{z^{-2}}$$
$$= \frac{1 + 2.0z^{-1}}{1 + 0.4z^{-1} - 0.12z^{-2}}$$

based on $ROC \quad |z| > 0.6$, then long division can yields

$$1 + 1.6z^{-1} - 0.52z^{-2} + 0.4z^{-3} - 0.222z^{-4} + \cdots$$
$$h[n] = \{1, 1.6, -0.52, 0.4, -0.222, \cdots\}, 0 \le n < \infty$$

5.5 Transfer function for LTI system

- Contents
  - Definition of transfer function
  - Analyze the stability and causality of LTI system
  - Transfer function implementation

Definition of transfer function

$$H(z) = \frac{Y(z)}{X(z)} \text{ or } H(z) = \sum h[n]\, z^{-n}$$

- Geometric interpretation of transfer function

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} \qquad H(z) = H(e^{j\omega})\Big|_{\omega=\frac{1}{j}lnz}$$

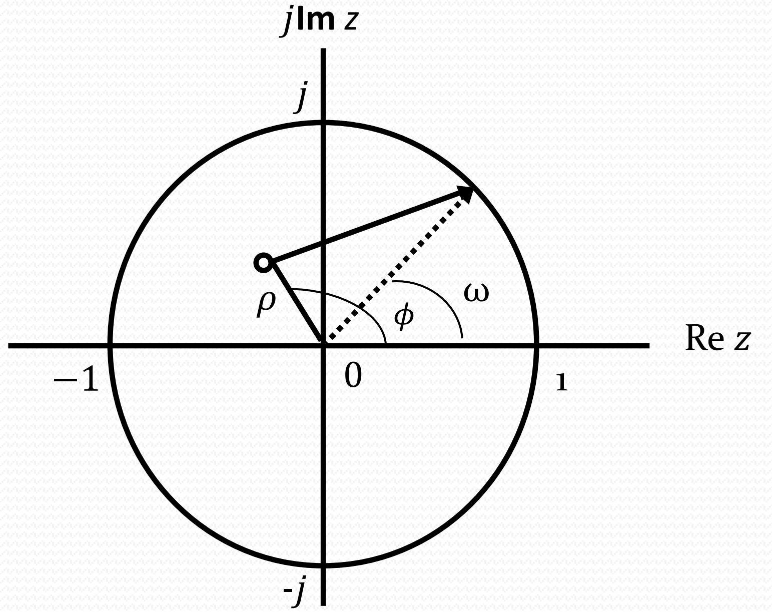
$$H(z) = \frac{\sum_n b[n]z^{-n}}{\sum_m a[m]z^{-m}} = \frac{K\prod_n (z-z_1)(z-z_2)\cdots(z-z_n)}{\prod_m (z-p_1)(z-p_2)\cdots(z-p_m)}$$

$$H(e^{j\omega}) = \frac{\rho_0}{d_0} e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega}-\xi_k)}{\prod_{k=1}^{N}(e^{j\omega}-\lambda_k)} = \frac{\prod_{k=1}^{M}(e^{j\omega}-\rho_k e^{j\phi_k})}{\prod_{k=1}^{N}(e^{j\omega}-d_k e^{j\varphi_k})}$$

$$\left|H(e^{j\omega})\right| = \frac{\prod_{k=1}^{M}(|e^{j\omega}-\rho_k e^{j\phi_k}|)}{\prod_{k=1}^{N}(|e^{j\omega}-d_k e^{j\varphi_k}|)},$$

$$e^{j\omega} = \rho_k e^{j\phi_k}, \text{ zero vector}$$

$$e^{j\omega} = d_k e^{j\varphi_k}, \text{ pole vector}$$



If the digital filter is to be designed to be highly attenuate in a specified range of frequncies, then place zeros of the transfer function very close to or on the unit circle in this range.

If highly emphasize signal components in a specified range of frequencies, then place poles very close to the unit circle

- The transform function of FIR system

$$h[n] = \begin{cases} \frac{1}{M}, 0 \leq n \leq M-1 \\ 0, otherwise \end{cases}$$

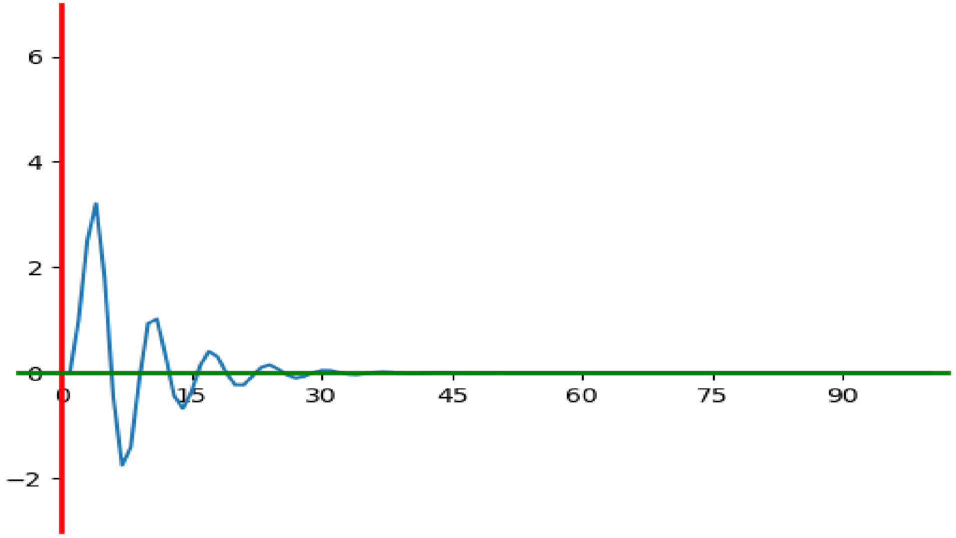
$$H(z) = \frac{1}{M}\sum_{n=0}^{M-1} z^{-n} = \frac{1-z^{-M}}{M(1-z^{-1})} = \frac{z^{M}-1}{M(z^{M-1}(z-1))}$$

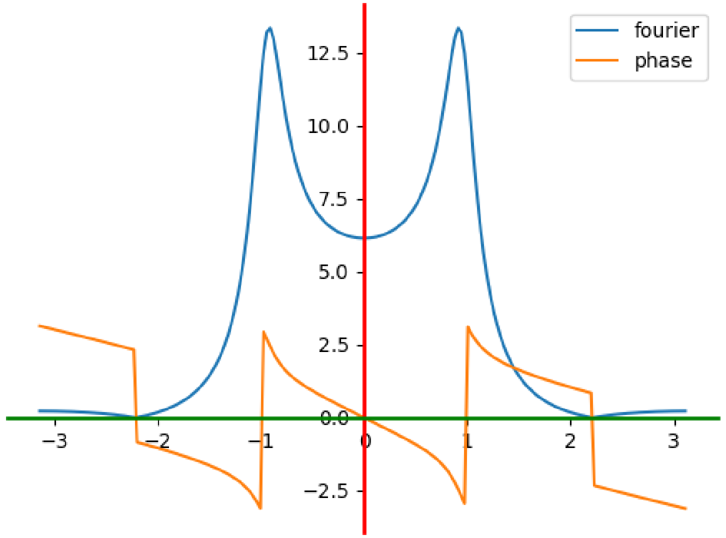
  - M zeros on the unit circle at $z = e^{j2\pi k/M}, k = 0,1,2, \cdots M-1$
  - $(M-1)\mathrm{th} - \mathrm{order\ pole\ at\ the\ z} = 0$
  - Single pole at $\mathrm{z} = 1$ which is cancled by a zero at $\mathrm{z} = 1$, so all poles are at the orignal.
  - It is stable since no pole on unit circle

- The transform function of IIR system

$$y[n] = x[n-1] - 1.2x[n-2] + x[n-3] + 1.3y[n-1] - 1.04y[n-2] + 0.22y[n-3]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} - 1.2z^{-2} + z^{-3}}{1 - 1.3z^{-1} + 1.04z^{-2} - 0.222z^{-3}}$$





Analyze causality of LTI system

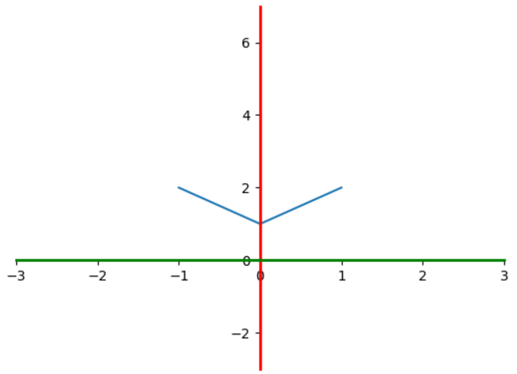
- R.O.C should out of a circle
- A causal LTI FIR digital filter with bounded impluse response coefficients is always stable as all its poles are at the origin in the z-plane
- A causal LTI IIR filter may be unstable .

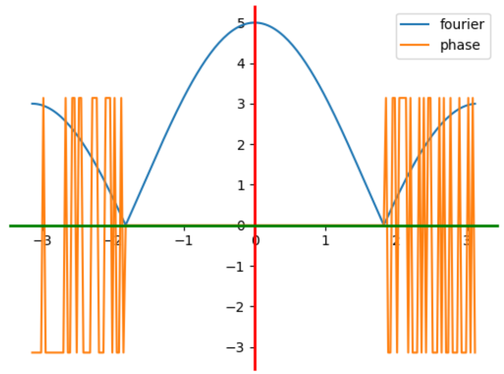
Analyze stability of LTI system

- R.O.C includes unit circle
  - if right-side signal, all the poles should be inside the unit circle

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\sum_r b_r z^{-r}}{\sum_k a_k z^{-k}}=\frac{A\prod_r(1-c_r z^{-1})}{\prod_k(1-d_k z^{-1})}, \text{ poles } |d_k|<1$$

  - if left-side signal, all the poles should be outside the unit circle
- If stable, then z-transform=>system has $H(e^{j\omega})$
  - Ex. $H(z)=2z+1+2z^{-1}\Rightarrow H(e^{j\omega})=2e^{j\omega}+1+2e^{-j\omega}$

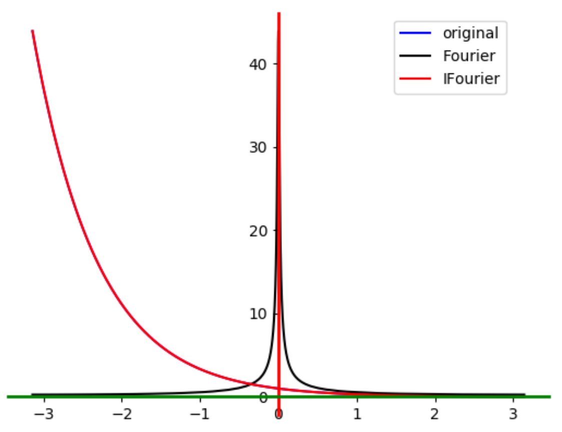




- Ex. Analyze stability of a LTI system
  - $h_1[n] = \alpha^n u[n],$
    - Solution : $H_1(\mathrm{z}) = \frac{1}{1-\alpha z^{-1}}, ROC\ |\mathrm{z}| > \alpha,$

  $if\ |\alpha| < 1, then\ ROC\ is\ inside\ the\ unit\ circle$, and $h_1[n]\ will\ be\ stable$
  - $h_2[n] = -\beta^n u[-n-1],$
    - Solution :$H_2(\mathrm{z}) = \frac{-\beta^{-1}z}{1-\beta^{-1}z} = \frac{1}{1-\beta z^{-1}}, ROC\ |z| < \beta,$

  $if\ |\beta| > 1, then\ ROC\ is\ inside\ the\ unit\ circle$, and $h_2[n]\ will\ be\ stable$



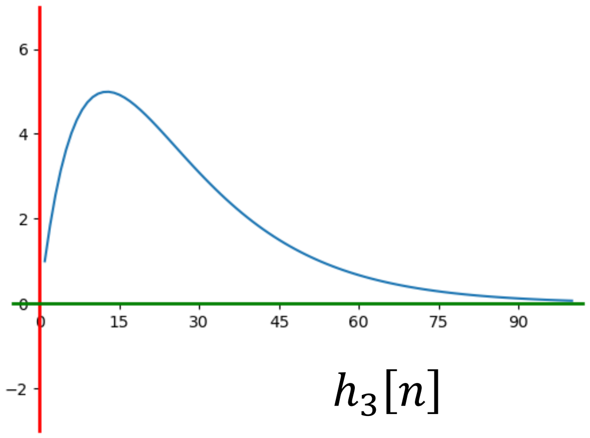
- Quantization of all coefficients cause a stable system to be unstable
  - EX.

$$H_3(z) = \frac{1}{1 - 1.845z^{-1} + 0.850586z^{-2}} = \frac{1}{(1 - 0.902z^{-1})(1 - 0.943z^{-1})}$$

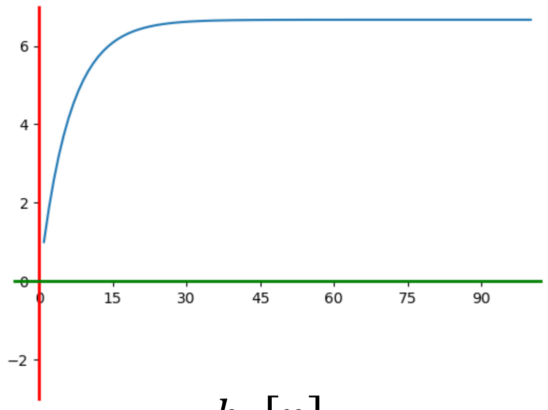
- the poles z=0.902 and z=0.943 are inside the unit circle

$$H_4(z) = \frac{1}{1 - 1.85z^{-1} + 0.85z^{-2}} = \frac{1}{(1 - z^{-1})(1 - 0.85z^{-1})}$$

- the poles z=1 is on the unit circleand z=0.85 is inside the unit circle



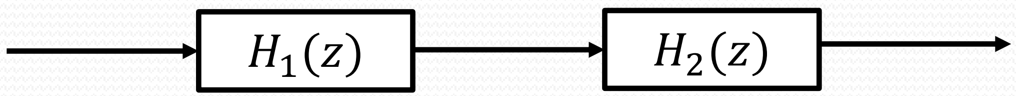
$h_3[n]$

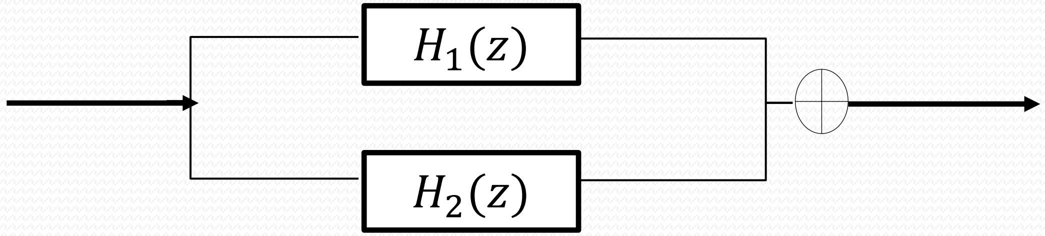


$h_4[n]$

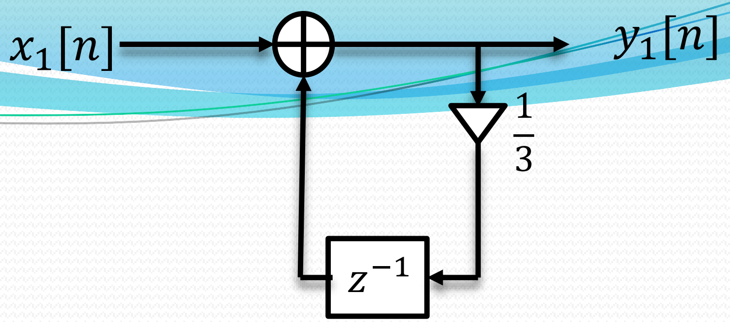
Construct a LTI system

- Write out the transfer function LTI system
  - Cascaded form
  - Parallel form
- Write out the differential equation corresponding to each part of H(z) in three forms

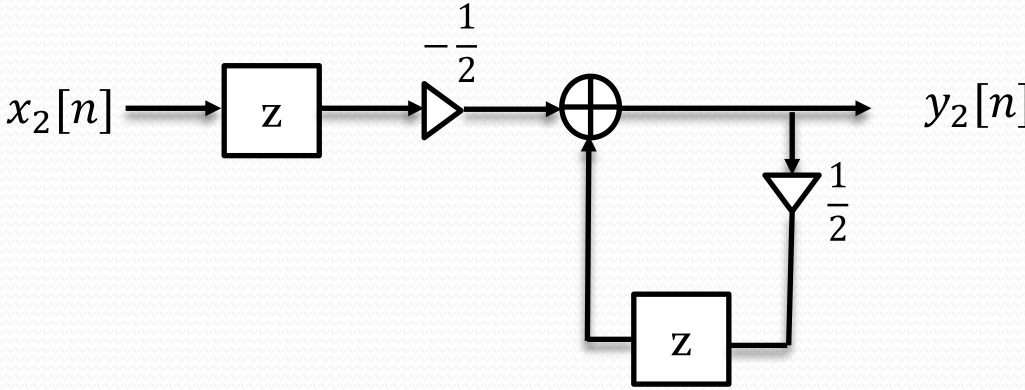




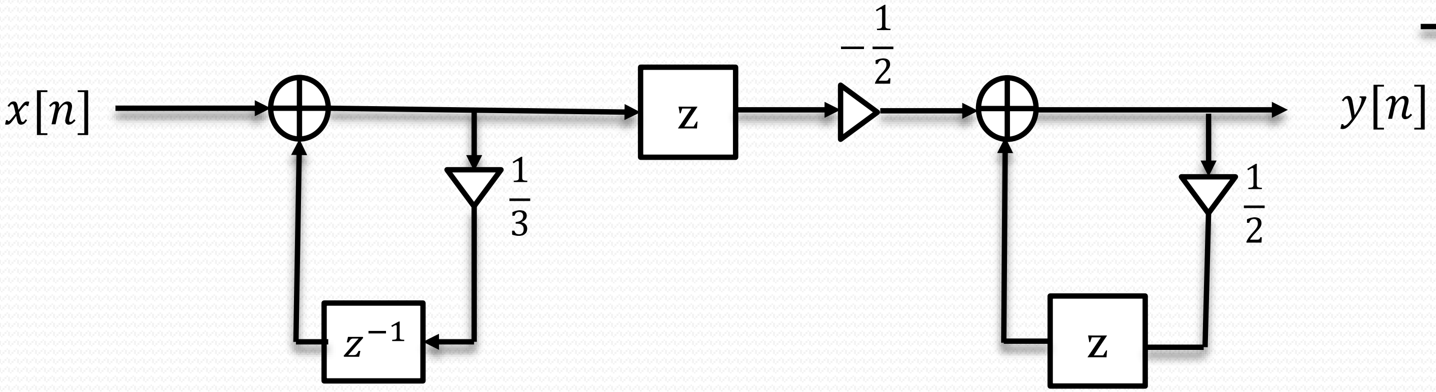
- Ex.
  - $H_1(z) = \frac{Y1(z)}{X1(z)} = \frac{1}{1+\frac{1}{3}z^{-1}}$, $y_1[\text{n}]=\frac{1}{3}\ y_1[\text{n-1}]+ x_1[n]$

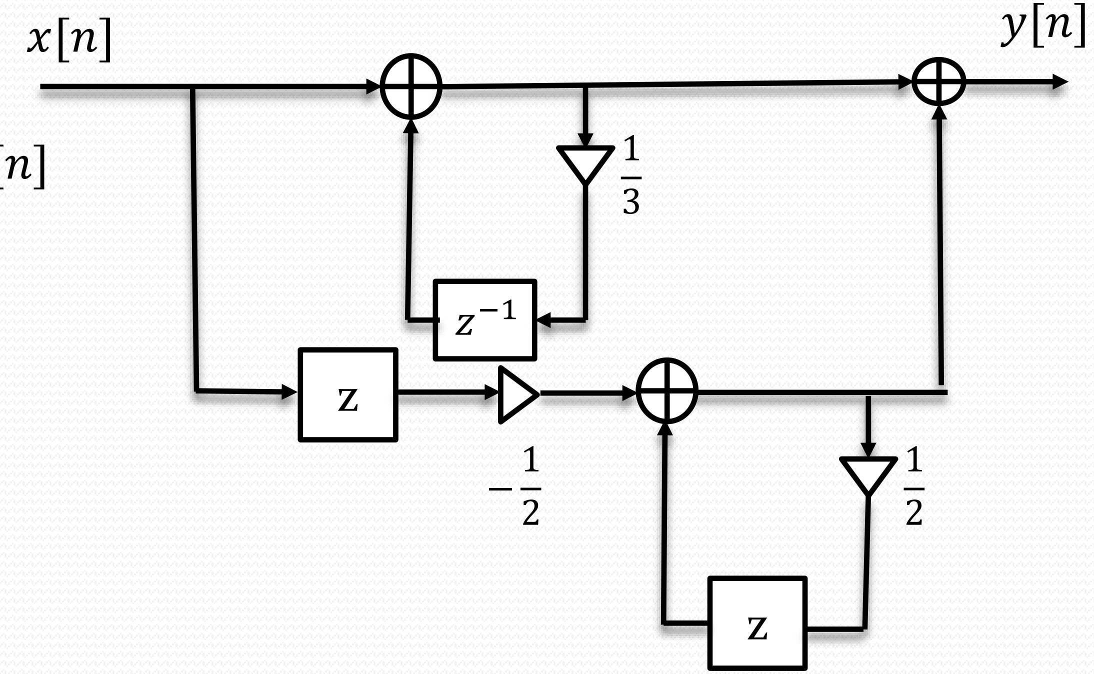


  - $H_2(z)=\frac{Y2(z)}{X2(z)} = -\frac{z}{2}\frac{1}{1-\frac{1}{2}z}$, $y_2[\text{n}]=\frac{1}{2}\ y_2[\text{n+1}]-\frac{1}{2}x_2[n+1]$,



  - $H(z)=H_1(z)H_2(z)$
  - $H(z)=H_1(z)+H_2(z)$





- Ex. $h[n] = 2^n u[n] + \left(\frac{1}{3}\right)^n u[n]$

(a)Find H(z) and R.O.C

(b)Find a stable version of h[n] that has the same H(z), but with a different R.O.C

(c)Find a stable set of differential equations for the system

(d)draw the circuit of the system of (c).

- Solution：

$$x_2[n] = -\beta^n \mu[-n-1]$$

$$X_2(z) = \frac{-\beta^{-1}z}{1-\beta^{-1}z}, for\ |z| < |\beta|.$$

(a)$\frac{1}{1-2z^{-1}}+\frac{1}{1-\frac{1}{3}z^{-1}}$, R.O.C $|z| > 2$

(b)For first term,H1(z)=$\frac{1}{1-2z^{-1}} \cdot \frac{-\frac{1}{2}z}{-\frac{1}{2}z} = -\frac{z}{2} \cdot \frac{1}{1-\frac{1}{2}z}$

$\Rightarrow$ h1[n] $= -2^n u[-n-1]$, R.O.C $|z| < 2$

For second term, h2[n] $= \left(\frac{1}{3}\right)^n u[n]$, R.O.C $|z| > \frac{1}{3}$

So h[n]=h1[n]+h2[n]= $-2^n u[-n-1] + \left(\frac{1}{3}\right)^n u[n]$, R.O.C $\frac{1}{3}<|z|<2$

(c)From $H_1(z) = \frac{Y_1(z)}{X(z)} = -\frac{z}{2}\frac{1}{1-\frac{1}{2}z}$

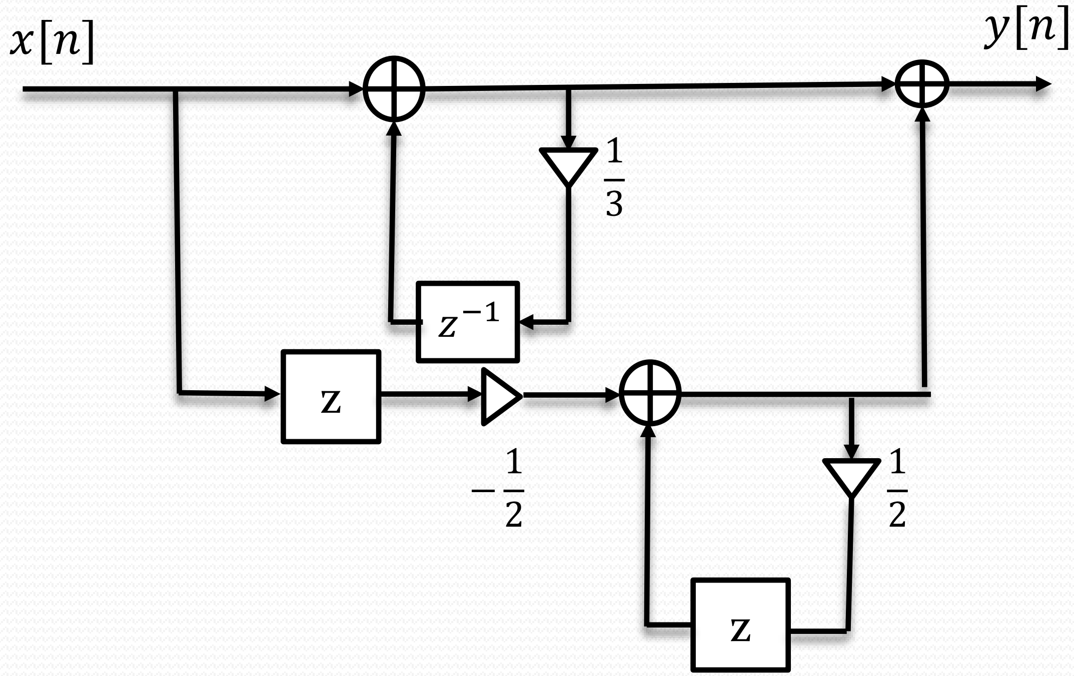
$$\rightarrow 2Y_1(z) - zY_1(z) = -zX(z)$$

$$\rightarrow Y_1(z) = \frac{1}{2}zY_1(z) - \frac{1}{2}zX(z)$$

$$\rightarrow y_1[\mathrm{n}] = \frac{1}{2}y_1[\mathrm{n}+1] - \frac{1}{2}x[n+1]$$

Form $H_2(z) = \frac{Y_2(z)}{X(z)} = \frac{1}{1-\frac{1}{3}z^{-1}} \rightarrow y_2[\mathrm{n}] = \frac{1}{3}y_2[\mathrm{n}-1] + x[n]$

(d)draw as example



5.5 Simple FIR and IIR Filter design by z-transform

- Lowpass FIR digital filter design
  - For moving-average filter with M=2,
    - EX. $H_0(z) = \frac{1}{2}(1 + z^{-1}) = \frac{z+1}{2z} \rightarrow \; H_0(e^{j\omega}) = e^{-j\omega/2} cos\left(\frac{\omega}{2}\right)$
  - Zero is z=-1, pole is z=0, so pole vector has magnitude of unity, for all values of ω, so it is stable;
  - $H_0(e^{j\omega})$ has a magnitude decrease from a value of 1 to 0, as ω increase from 0 to π
- Highpass FIR filter designa
  - Replace z with (–z)
    - *Proof:* $e^{j(\omega-\pi)} \leftrightarrow -z$
  - $H_1(z) = \frac{1}{2}(1 - z^{-1}) = \frac{z-1}{2z} \rightarrow \; H_0(e^{j\omega}) = je^{-j\omega/2} sin\left(\frac{\omega}{2}\right)$
  - Zero is z=1, pole is z=0, so pole vector has magnitude of unity, for all values of ω;
  - $H_0(e^{j\omega})$ has a magnitude increase from a value of 0 to 1, as ω increase from 0 to π

- Low pass IIR filter

$$H_0(z) = \frac{1-\alpha}{2} \cdot \frac{(1+z^{-1})}{1-\alpha z^{-1}}$$

- Proof:

let $H_0(z) = \frac{K}{1-\alpha z^{-1}}, 0 < |\alpha| < 1 \rightarrow \left|H_0\left(e^{j\omega}\right)\right|^2 = H_0(z)H_0(z^{-1})|_{z=e^{j\omega}} = \frac{K^2}{(1+\alpha^2)-2\alpha cos\omega}$

For α>0, , $as\ \omega$ from o to π , $\left|H_0\left(e^{j\omega}\right)\right|^2$ decrese from $\frac{K^2}{(1+\alpha^2)-2\alpha}$to $\frac{K^2}{(1+\alpha^2)+2\alpha}$

To force $\left|H_0\left(e^{j\omega}\right)\right|^2$ to zero at $\omega = \pi$ and $\left|H_0\left(e^{j\omega}\right)\right|^2$still show low- pass ,even α<0,modify

$$H_0(z) = \frac{K(1+z^{-1})}{1-\alpha z^{-1}}$$

To make DC gain of 0-db, that is 1, K= $\frac{1-\alpha}{2}$,Finally

$$H_0(z) = \frac{1-\alpha}{2} \cdot \frac{(1+z^{-1})}{1-\alpha z^{-1}}$$

- High pass IIR filter
  - Similarly with low pass FIR filter , replace z with –z

$$H_1(z) = \frac{1+\alpha}{2} \cdot \frac{(1-z^{-1})}{1-\alpha z^{-1}}$$

5.6 Filter design without distortion

- Zero phase
  - $H(e^{j\omega})$ is real and nonegative
    - Ex. $H(e^{j\omega}) = \begin{cases} 1, 0 \leq |\omega| \leq \omega_c \\ 0 \end{cases} \rightarrow h[n] = \frac{sin(\pi n)}{\pi n}, -\infty < n < \infty$
    - Ideal LPF, HPF ,BPF and APF
    - It is impossible to make a causal digital filter with zero phase
- Linear phase
  - Linear phase allows the output to be a delayed version of the input
    - $y[n] = x[n - D] \rightarrow Y(e^{j\omega}) = e^{-j\omega D}X(e^{j\omega}) \rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = e^{-j\omega D}$
  - FIR filter can has an exact linear-phase response, whereas it is not possible to design an IIR linear-phase response

Change filter to be zero-phase

- To make a zero-phase filter
  - If a $H_1(z)$ **has no pole on the unit circle**(都是多项式) , then its corresponding zero phase filter is
    $$H(z) = H_1(z)H_1(z^{-1})$$
  - Proof:
    $$H\left(e^{j\omega}\right) = H(z)\big|_{z=e^{j\omega}} = \left|H_1\left(e^{j\omega}\right)\right|^2$$, it is real-value ,so zero phase
  - If $H\left(e^{j\omega}\right)$ is real value, then h[n]=h[-n], for FIR cant be ideal
  - If $\left|H_1\left(e^{j\omega}\right)\right|$=1, then $H(z) =$ is an all-pass IIR filter
- How to achive a zero phase filter in time domain
  - if $x[n]$ is input signal,
    $v[n] = x[n] * h_1[n] \rightarrow \ u[n] = v[-n] \rightarrow \ w[n] = u[n] * h_1[n] \rightarrow y[n] = w[-n]$
  - Proof
    $$Y\left(e^{j\omega}\right) = W\left(e^{-j\omega}\right) = U\left(e^{-j\omega}\right)H_1\left(e^{-j\omega}\right) = V\left(e^{j\omega}\right)H_1\left(e^{-j\omega}\right) = X\left(e^{j\omega}\right)H_1\left(e^{j\omega}\right)H_1\left(e^{-j\omega}\right) = \left|H_1\left(e^{j\omega}\right)\right|^2 X\left(e^{j\omega}\right)$$

- How to determine $H_1(z)$ from zero-phase filter $\left|H_1(e^{j\omega})\right|^2$
  - Given $\left|H_1(e^{j\omega})\right|^2$ in $cos\omega$ with specifications
  - Replace $cos\omega$ with $\frac{(z+z^{-1})}{2}$
  - Assign half of the zeros and poles for $H_1(z)$ and the other half of the zeros and poles at the mirro-image locations for $H_1(z^{-1})$

- Ex. Determine $H_1(z)$ for zero-phase filter by given $\left|H_1(e^{j\omega})\right|^2$

$$\left|H_1(e^{j\omega})\right|^2 = \frac{4(1.09 + 0.6\cos\omega)(1.16 - 0.8\cos\omega)}{(1.04 - 0.2\cos\omega)(1.25 + \cos\omega)}$$

$solution:\ repalce \cos\omega\ with\ \frac{(z+z^{-1})}{2},$

we get $H_1(z)H_1(z^{-1})$

$$= \frac{4(1.09 + 0.3(z+z^{-1}))(1.16 - 0.4(z+z^{-1}))}{(1.04 - 0.1(z+z^{-1}))(1.25 + 0.5(z+z^{-1}))}$$

$$= \frac{4(1 + 0.3z^{-1})(1 + 0.3z)(1 - 0.4z^{-1})(1 - 0.4z)}{(1 - 0.2z^{-1})(1 - 0.2z)(1 + 0.5z^{-1})(1 + 0.5z)}$$

$$H_1(z) = \frac{2(1+0.3z^{-1})(1-0.4z^{-1})}{(1-0.2z^{-1})(1+0.5z^{-1})}$$

$$or\ H_1(z) = \frac{2(1 + 0.3z^{-1})(1 - 0.4z)}{(1 - 0.2z^{-1})(1 + 0.5z^{-1})}$$

$$or\ H_1(z) = \frac{2(0.3 + z^{-1})(1 - 0.4z^{-1})}{(1 - 0.2z^{-1})(1 + 0.5z^{-1})}$$

$$or\ H_1(z) = \frac{2(1 + 0.3z)(0.4 - z^{-1})}{(1 - 0.2z^{-1})(1 + 0.5z^{-1})}$$

Design Linear phase FIR filter

- If FIR H(z) is required have a linear-phase, its frequency response must be of the form

$$H(e^{j\omega}) = e^{j(c\omega+\beta)}\breve{H}(\omega),$$

  - where $\breve{H}(\omega)$ is called *amplitude response or zero phase response*, real function of ω.
  - c is the linear coefficent

- If $h[n]$ is real value, then $|H(e^{j\omega})|$is even function ,

$$H(e^{j\omega}) = H^*(e^{-j\omega}),$$

$$e^{j(c\omega+\beta)}\breve{H}(\omega) = [e^{j(-c\omega+\beta)}\ \breve{H}(-\omega)]^* = e^{j(c\omega-\beta)}\ \breve{H}(-\omega),$$

- Let $\beta = 0\ or\ \ \pi$, then $\breve{H}(\omega) = \breve{H}(-\omega)$ and when c=−N/2, $h[n] = h[N-n], 0 \leq n \leq N$
- Let $\beta = \frac{\pi}{2}\ or\ \ -\frac{\pi}{2}$, then $\breve{H}(\omega) = -\breve{H}(-\omega)$and when c=−N/2 , $h[n] = -h[N-n], 0 \leq n \leq N$

Four types of Linear phase FIR filter

- Since real-value FIR has odd length and even length , so there are four types of linear-phase FIR filter
  - Type1: Odd length with symmetric impulse response
  - Type2: Even length with symmetric impulse response
  - Type3:Odd length with antisymmetrci impulse response
  - Type4: even length with antisymmetric impulse response

Type 1 Odd length , order N is even with symmetric

$$h[n] = h[N-n], 0 \le n \le N, with\ c = -\frac{N}{2}, \text{N is even}$$

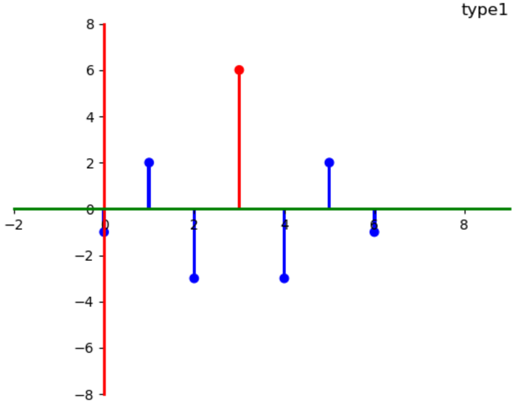
$$\breve{H}(e^{j\omega}) = \mathrm{h}\left[\frac{N}{2}\right] + 2\sum_{n=1}^{N/2} h\left[\frac{N}{2} - n\right] cos(\omega n),$$

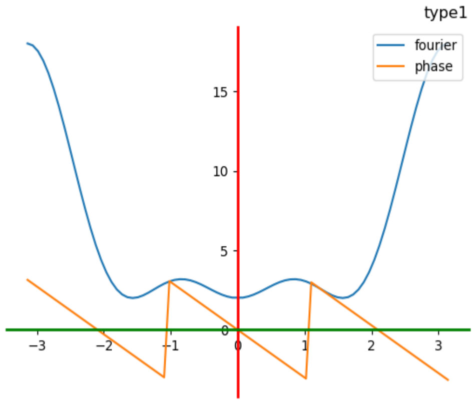
$$\theta(\omega) = -\frac{N}{2}\ \omega, or -\frac{N}{2}\ \omega + \pi$$

- Ex. $\breve{H}(e^{j\omega}) = 6 - 6cos(\omega) + 4cos(2\omega) - 2cos(3\omega)$, N=6
  - $for\ \ \theta(\omega) = -3\ \omega,$

    $$H(z) = -1 + 2z^{-1} - 3z^{-2} + 6z^{-3} - 3z^{-4} + 2z^{-5} - z^{-6},$$
  - $for\ \theta(\omega) = -3\ \omega + \pi,$

    $$H(z) = 1 - 2z^{-1} + 3z^{-2} - 6z^{-3} + 3z^{-4} - 2z^{-5} + z^{-6}$$
- Properties
  - Don't have obvious zeros for $\omega = 0$ and π
  - This a delayed zero-phase filter, which is good for construct type 1 of LP, HP. BP and BR



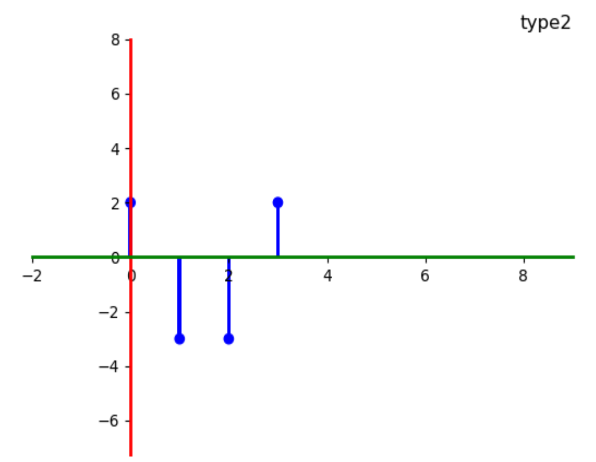


Type 2 even length , N is odd with symmetric

$$h[n] = h[N-n], 0 \le n \le N, with\ c = -N/2, \text{N is odd}$$

$$\breve{H}(e^{j\omega}) = 2\sum_{n=1}^{(N+1)/2} h\left[\frac{N+1}{2} - n\right] cos\left(\omega\left(n - \frac{1}{2}\right)\right),$$

$$\theta(\omega) = -\frac{N}{2}\ \omega, or -\frac{N}{2}\ \omega + \pi$$

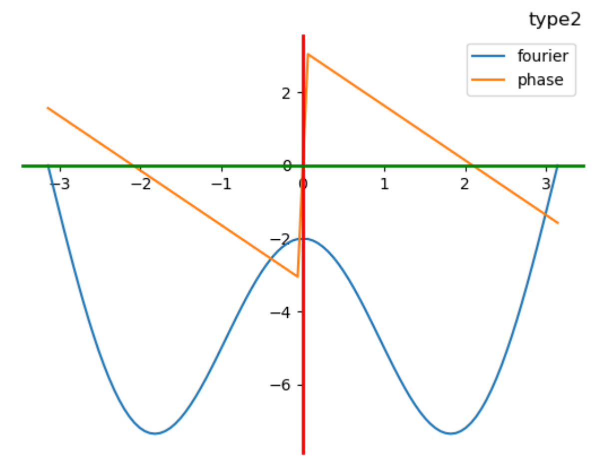


- Ex. $\breve{H}(e^{j\omega}) = 2 * (-3)cos\left(\frac{\omega}{2}\right) + 2 * 2cos\left(\frac{3\omega}{2}\right)$, N=3
  - for $\theta(\omega) = -\frac{3}{2}\omega$,

$$H(z) = 2 - 3z^{-1} - 3z^{-2} + 2z^{-3}$$

  - for $\theta(\omega) = -\frac{3}{2}\omega + \pi$

$$H(z) = -2 + 3z^{-1} + 3z^{-2} - 2z^{-3}$$



- Propeties
  - $\omega = \pi: H(e^{j\omega}) = 0$

    Bad for HP

Type 3 odd length, order N is even, antisymmetric

$$h[n] = -h[N-n], 0 \le n \le N, with\ c = -\frac{N}{2}, \text{N is even}$$

$$\breve{H}\left(e^{j\omega}\right) = 2\sum_{n=1}^{N/2} h\left[\frac{N}{2} - n\right] sin(\omega n),$$

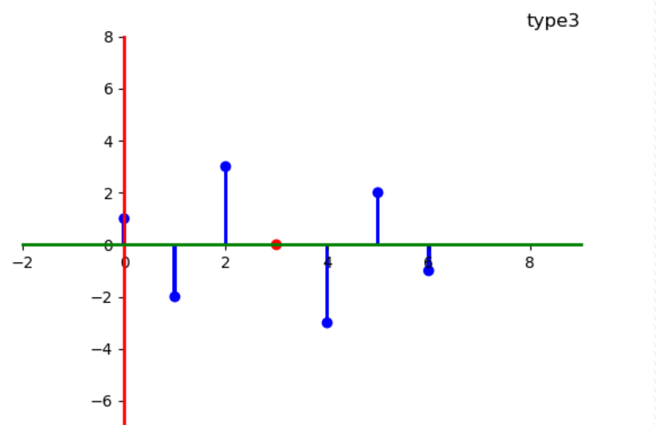
$$\theta(\omega) = -\frac{N}{2}\omega \pm \frac{\pi}{2}$$

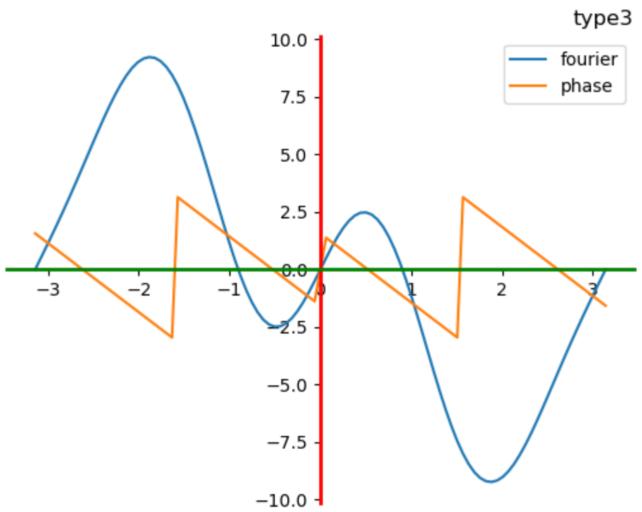
- Ex. $\breve{H}\left(e^{j\omega}\right) = -6sin(\omega) + 4sin(2\omega) + 2sin(3\omega)$
  - for $\theta(\omega) = -3\,\omega + \frac{\pi}{2}$

    $H(z) = 1 - 2z^{-1} + 3z^{-2} + 0 - 3z^{-4} + 2z^{-5} - z^{-6}$
  - for $\theta(\omega) = -3\,\omega - \frac{\pi}{2}$

    $H(z) = -1 + 2z^{-1} - 3z^{-2} + 0 + 3z^{-4} - 2z^{-5} + z^{-6}$

- Propeties
  - bad for LP and HP filter





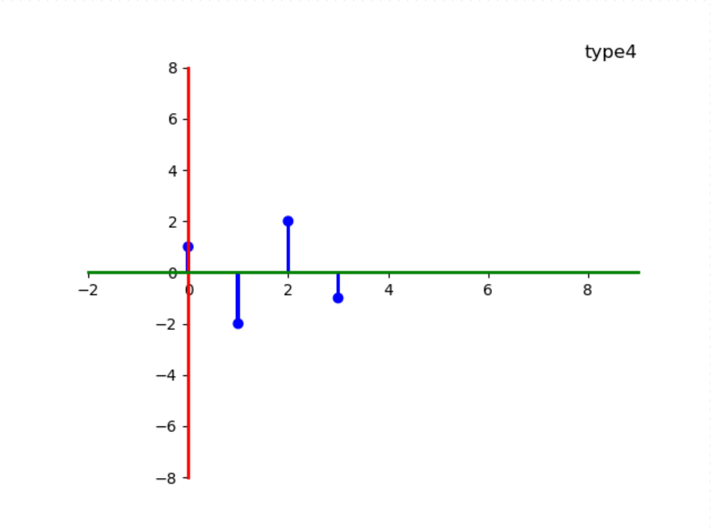
Type 4 even length, order N is odd, antisymmetric

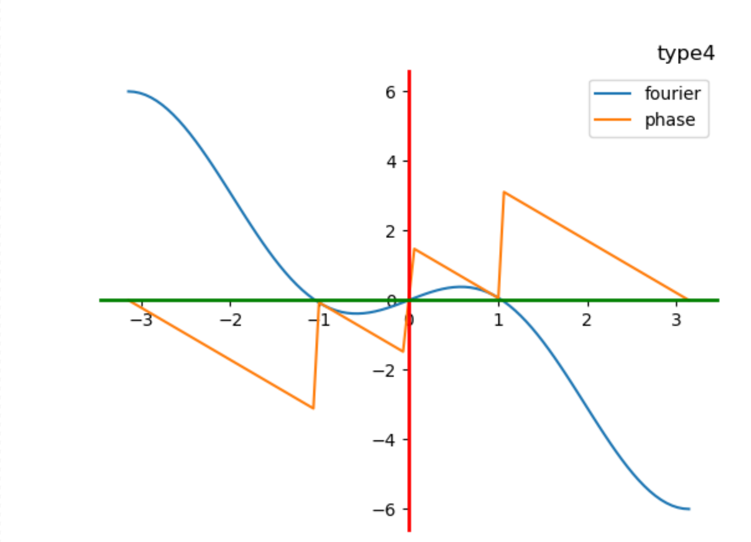
$$h[n] = -h[N-n], 0 \le n \le N, with\ c = -\frac{N}{2}, \text{ N is odd}$$

$$\breve{H}(e^{j\omega}) = 2\sum_{n=1}^{(N+1)/2} h\left[\frac{N+1}{2} - n\right] sin\left(\omega\left(n - \frac{1}{2}\right)\right),$$

$$\theta(\omega) = -\frac{N}{2}\ \omega \pm \frac{\pi}{2}$$

- Ex. $\breve{H}(e^{j\omega}) = 2 * (-2)sin\left(\frac{\omega}{2}\right) + 2 * 1sin\left(\frac{3\omega}{2}\right)$,N=3
  - for $\theta(\omega) = -\frac{3}{2}\omega + \frac{\pi}{2}$
    $$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$
  - for $\theta(\omega) = -\frac{3}{2}\omega - \frac{\pi}{2}$
    $$H(z) = -1 + 2z^{-1} - 2z^{-2} + z^{-3}$$
- Propeties
  - bad for LP filter





Procedure of digital filter design from transfer function

- Step1: Transfer function design based on stability， causality and phase requirements of LTI system
- Step 2: Frequency response design to get H$\left(e^{j\omega}\right)$
- Step 3: DFT frequency response to get H[k]
- Step 4: IDFT to get h[n]