Chapter 4 Discrete Fourier Transform

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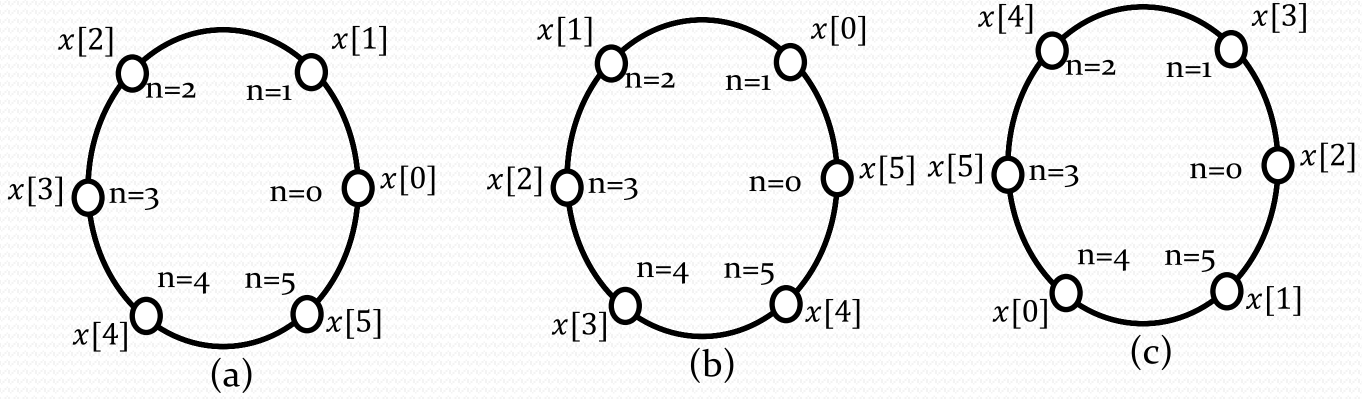
4.1 Basic mathematics in DFT

- The modulo arithmetic
  $$\langle m\rangle_N = m \bmod N = l\cdot N + m' = m'$$
- Ex. $\langle 6\rangle_5 = 1; \langle 7\rangle_3 = ?$
- $x[\langle n\rangle_N]$
  - Ex. $x[\langle 5\rangle_3] = x[?]$
- Circular shift of sequence
  $$x[\langle n-n_0\rangle_N]$$
  $n_0 > 0$ means the shifting direction is forward; otherwise means backward;

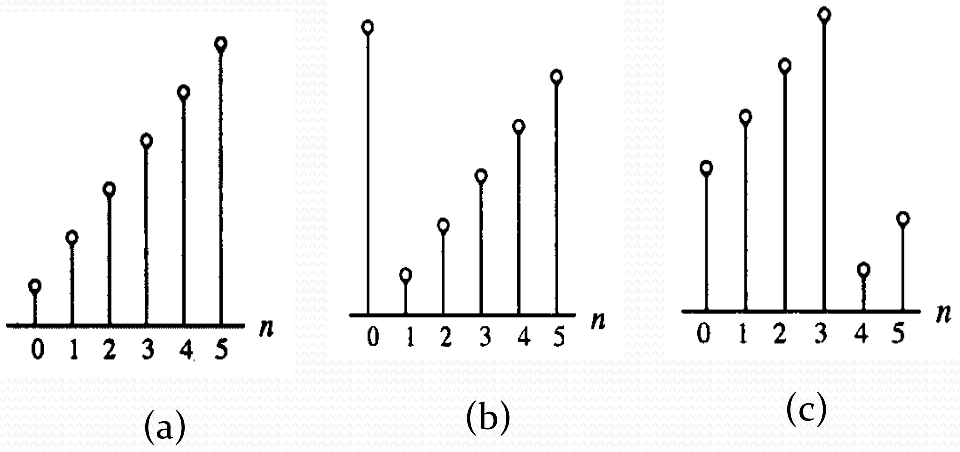
$$x_c[n] = x[\langle n-n_0\rangle_N]\ or\, x[\langle n+n_0\rangle_N]$$
$$= \begin{cases} x[n-n_0], for\ 0 < n_0 \le n \le N-1, \\ x[n-n_0+N], for\ 0 \le n \le n_0. \end{cases}$$

$$\text{or } = \begin{cases} x[n+n_0], for\ 0 < n_0 + n \le N-1, \\ x[n+n_0-N], for\ N \le n+n_0. \end{cases}$$

- Ex. $x[\langle n-1\rangle_6] = x[?]$



Alternate illustration of a circular shift of a finite-length sequence. (a) $x[n]$, (b) $x[\langle n-1\rangle_6] = x[\langle n+5\rangle_6]$, and $(c) x[\langle n-4\rangle_6] = x[\langle n+2\rangle_6]$



Alternate illustration of a circular shift of a finite-length sequence. (a) $x[n]$, (b) $x[\langle n-1\rangle_6] = x[\langle n+5\rangle_6]$, and $(c) x[\langle n-4\rangle_6] = x[\langle n+2\rangle_6]$

4.2 M-points DFT and N-length IDFT

- M-point DFT from *N*-length signal

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{M}} \quad ,$$

$$\text{If } \omega_k = \frac{2\pi k}{M}, \quad \text{then } X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\omega_k n}$$

$$\text{Let } W_M = e^{-j\frac{2\pi}{M}} \;, \quad X[k] = \sum_{n=0}^{N-1} x[n]\, W_M^{nk}$$

- *N*-length IDFT from M-point DFT

$$x[n] = \frac{1}{M}\sum_{k=0}^{M-1} X[k]\, e^{j\frac{2\pi kn}{M}}$$

$$\text{if } \omega_k = \frac{2\pi k}{M}, \quad \text{then } x[n] = \frac{1}{M}\sum_{k=0}^{M-1} X[k]\, e^{j\omega_k n}$$

$$\text{Let } W_M = e^{-j\frac{2\pi}{M}}, \qquad x[n] = \frac{1}{M}\sum_{k=0}^{M-1} X[k]\, W_M^{-nk}$$

- Some special equations about W_M
  - $\sum_{k=0}^{M-1} W_M^{k} = 0;$
  - $\sum_{k=0}^{M-1} W_M^{km} = \begin{cases} M, if\ m\ is\ a\ multiple\ of\ M \\ 0, \quad else \end{cases} \rightarrow \quad \sum_{k=0}^{M-1} W_M^{km} = M\delta[\langle m\rangle_M]$

- The matrix form of DFT and IDFT

$$\mathbf{x} = [x[0] \quad x[1] \quad \cdots \quad x[M-1]]^T \leftrightarrow \mathbf{X} = [X[0] \quad X[1] \quad \cdots \quad X[M-1]]^T$$

$$\mathbf{D}_M = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_M^1 & W_M^2 & \cdots & W_M^{M-1} \\ 1 & W_M^2 & W_M^4 & \cdots & W_M^{2(M-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_M^{M-1} & W_M^{2(M-1)} & \cdots & W_M^{(M-1)(M-1)} \end{bmatrix}$$

DFT

$$\mathbf{X} = \mathbf{D_M x}$$

$$\mathbf{D}_\mathrm{M}^{-1} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_M^{-1} & W_M^{-2} & W_M^{-3} & W_M^{-(M-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_M^{-(M-1)} & W_M^{-2(M-1)} & \cdots & W_M^{-(M-1)(M-1)} \end{bmatrix}$$

IDFT

$$\mathbf{x} = \frac{\mathbf{1}}{\mathbf{M}} \mathbf{D_M^{-1} X}$$

$\mathbf{D}_4$ and $\mathbf{D}_4^{-1}$?

$$\mathbf{D}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\mathbf{D}_4^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

N-point DFT for some basic signals

- $x[n] = \delta[n]\,, X[k] = ?$
  - Solution: $\sum_{n=0}^{N-1} \delta[n] W_N^{nk} = 1$
  - Proof: $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} W_N^{-nk} = \delta[n]_N$ only n=0,$x[n]$ =1.
- $x[n] = a^n(u[n] - u[n - N1]), 0 \le n \le N1 - 1 \le N,, X[k] = ?$
  - Solution: $X[k] = \sum_{n=0}^{N1-1} a^n W_N^{nk} = \sum_{n=0}^{N1-1} \left(aW_N^k\right)^n = \frac{1-\left(aW_N^k\right)^{N1}}{1-aW_N^k}$

The effects of M-point and N-length

- Normaly, M =N,   that is $x[n]$ can be recovered from $X[k]$

$$X[k] = \sum_{n=0}^{M-1} x[n]e^{-j\frac{2\pi kn}{M}} \leftrightarrow x[n] = \frac{1}{M}\sum_{k=0}^{M-1} X[k]\, e^{j\frac{2\pi kn}{M}}$$

- Proof:

Let $W_M = e^{-j\frac{2\pi}{M}}$, then

$$x[n] = \frac{1}{M}\sum_{k=0}^{M-1} X[k]\, W_M^{-nk} = \frac{1}{M}\sum_{k=0}^{M-1}\left(\sum_{m=0}^{M-1} x[m]\, W_M^{mk}\right) W_M^{-nk}$$

$$=\frac{1}{M}\sum_{m=0}^{M-1} x[m]\sum_{k=0}^{M-1} W_M^{(m-n)k} = \begin{cases} \sum_{k=-\infty}^{\infty} x[n+kM], \text{when } m\text{=}n\text{+}kM \\ 0, \quad \text{elese} \end{cases},$$

$$(\text{since } when\ m = n + \text{kM},\quad \sum_{k=0}^{M-1} W_M^{(m-n)k} = M)$$

- $M \neq N$, $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{M}}$, $x[n] = \frac{1}{M} \sum_{k=0}^{M-1} X[k]\, e^{j\frac{2\pi kn}{M}}$
  - $M <$ N, then IDFT will have aliasing

Proof: if $M <$ N, $x[n]$ which $n >$ M will be calculated by $e^{j\frac{2\pi kn}{M}}$ will circle into $e^{j\frac{2\pi k(n-\mathrm{M})}{M}}$

  - $M \geq$ N , then IDFT will have no-aliasing since $e^{j\frac{2\pi kn}{M}}$ will not circle

4.3 The relation between X($e^{j\omega}$) and X[k]

- X[k] is the subset of X($e^{j\omega}$)
- Proof:

$$X\left(e^{j\omega}\right) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n} = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}\right]e^{-j\omega n}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X[k]\sum_{n=0}^{N-1} W_N^{-kn}e^{-j\omega n}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X[k]\frac{\sin\left(\dfrac{\omega N-2\pi k}{2}\right)}{N\cdot\sin\left(\dfrac{\omega N-2\pi k}{2N}\right)}\cdot e^{-j[\omega-2\pi k/N]((N-1)/2]}$$

$$\because\ only\ \ if\ \ \omega N-2\pi k=0, etc.\ \omega=2\pi\, k/N$$

$$\frac{\sin\left(\frac{\omega N-2\pi k}{2}\right)}{N\sin\left(\frac{\omega N-2\pi k}{2N}\right)}\cdot e^{-j[\omega-2\pi k/N][(N-1)/2]}=1, else\ 0,$$

$$\therefore\ in\ this\ case, X\left(e^{j\omega}\right)=X[k], with\ \omega=2\pi\, k/N$$

- Determine M ,which can make $\omega_k = \frac{2\pi k}{M}$ approximate to ω, make DFT to approximate to DTFT.
  - Ex. We have a sequence $x[n]$, $0 \leq n \leq N-1$ and we want to calculate $X(e^{j\omega})$ for $\omega = k\Delta\omega$, where $\Delta\omega \ll \pi$.

(a)What value of M is necessary if this is to be done by using DFT and IDFT?

- Solution: $\frac{2\pi}{M} = \Delta\omega$, so $M = \text{round}(\frac{2\pi}{\Delta\omega})$

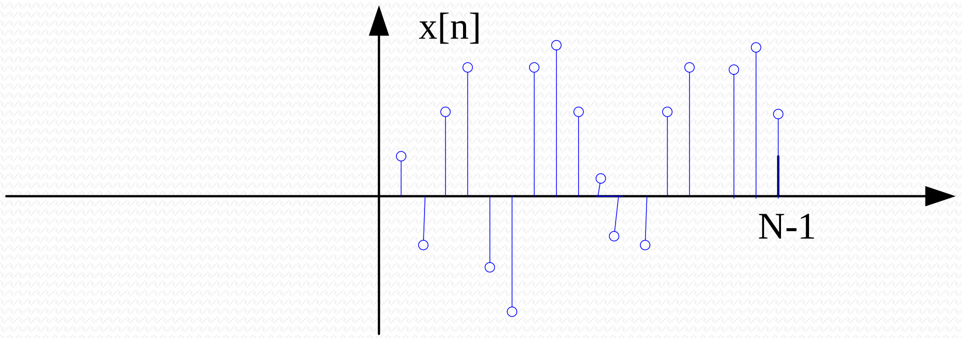
(b)What is the restriction on M ?

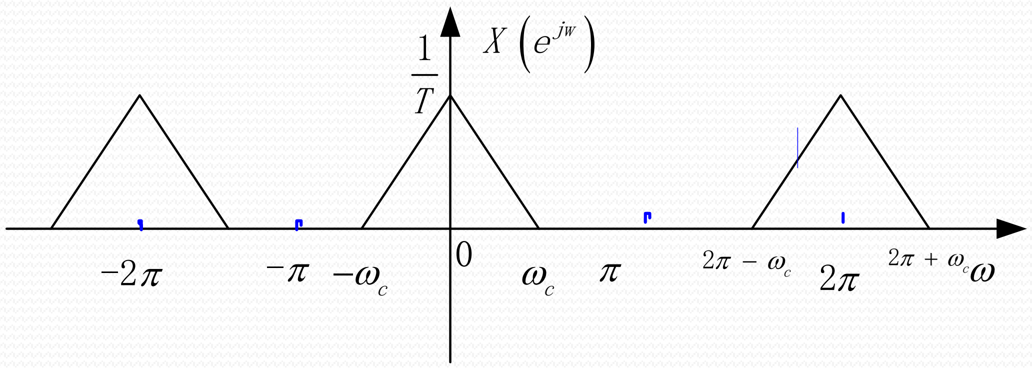
- Solution: normally, $M > N$, but if $M < N$, we can use M'=m M to makeM'$>$ N,and $\Delta\omega = m\Delta\omega'$, so we can get $\omega = km\Delta\omega'$

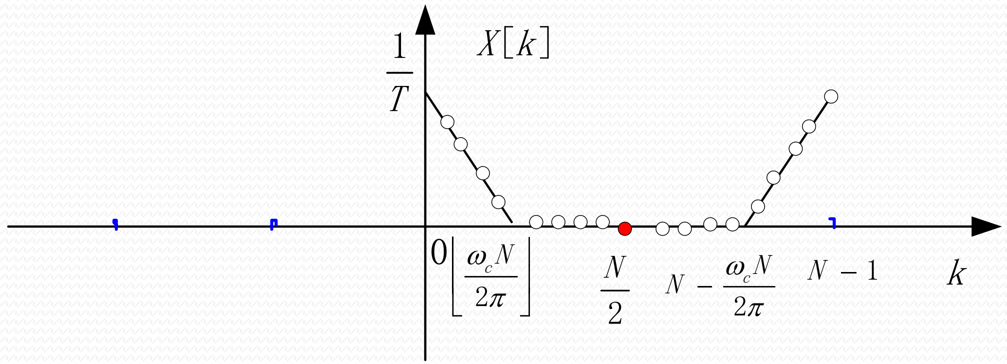
(c)Suppose we want to examine $X(e^{j\omega})$ $\omega_1 \leq \omega \leq \omega_2$, what is the corresponding range of k in $X[k]$.

- Solution: $k_1 = round(\frac{M\omega_1}{2\pi})$,$k_2 = round(\frac{M\omega_2}{2\pi})$

- Some changes from DTFT to DFT
  - Causality and period are different
  - Filter design is different from DTFT
  - Convolution computation is different







4.4 Properties of DFT

- Linearity theorem
- Circular Time-shifting theorem(Delay)
- Circular Frequency-Shifting Theorem
- Circular convolution Theorem
- Modulation Theorem
- Parseval's Relation
- Duality theorem

$$G[n] \overset{DFT}{\leftrightarrow} Ng[\langle k\rangle_N]$$

$$\alpha g[n] + \beta h[n] \overset{DFT}{\leftrightarrow} \alpha G[k] + \beta H[k]$$

$$g[\langle n - n_0\rangle_N] \overset{DFT}{\leftrightarrow} W_N^{kn\cdot} G[k]$$

$$W_N^{-kn_0} g[n] \overset{DFT}{\leftrightarrow} G[\langle k - k_0\rangle_N]$$

$$\sum_{m=0}^{N-1} g[n]h[\langle n - m\rangle_N] \overset{DFT}{\leftrightarrow} G[k]H[k]$$

$$g[n]h[n] \overset{DFT}{\leftrightarrow} \frac{1}{N}\sum_{l=0}^{N-1} G[l]h[\langle k - l\rangle_N]$$

$$\sum_{n=0}^{N-1} |g[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |G[k]|^2$$

$$\sum_{n=0}^{N-1} g[n]h^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} G[k]H^*[k]$$

Using Circular Time-shifting theorem(Delay) and Circular Frequency-Shifting Theorem

- Ex. Let G[k] and H[k], $0 \le k \le 7$, denote the 8-point DFTs of two length-8 sequences, $g[n]$ and $h[n]$, respectively.
  - (a) If $G[k] = \{2.6 + j4.1,\ \ 3 - j2.7,\ \ -4.2 + j1.4,\ \ 3.5 - j2.6,\ \ 0.5,\ \ 1.3 + j4.4,\ \ 2.4 - j1.6\}$ and $h[n] = g[\langle n-5 \rangle_8]$,determine H[k] without forming h[n].
  - (b) If $g[n] = \{-0.1 - j0.7,\ \ 1.3 + j,\ \ 2 + j0.7,\ \ 1.1 + j2.2,\ \ -0.8 + j0.2,\ \ 3.4 - j0.1,\ \ -1.2 + j\}$ and $H[k] = G[(\langle k + 3 \rangle_8]$,determine h[n] without computing the DFT G[k].

- Solution
  - (a) $h[n] = g[\langle n-5\rangle_8]$.

$$Hence, \mathrm{H}[k] = W_8^{-5k} G[k] = e^{j10\pi k/8} G[k] = e^{j5\pi k/4} \mathrm{G}[k]$$

$=\{2.6 + j4.1, e^{\frac{j5\pi}{4}}(3 - j2.7), e^{\frac{j5\pi}{2}}(-4.2 + j1.4), e^{\frac{j15\pi}{4}}(3.5 - j2.6), e^{j5\pi}(0.5), e^{\frac{j25\pi}{4}}(1.3 + j4.4), e^{\frac{j15\pi}{2}}(2.4 - j1.6), e^{\frac{j35\pi}{4}}(-3 + j1.6)\}$.

  - (b)$H[k] = G[\langle k+3\rangle_8]$.

$Hence, h[n] = W_8^3 g[n] = e^{-j6\pi n/8} g[n] = e^{-j3\pi n/4} g[n]$

$= \left\{-0.1 - j0.7, e^{\frac{-j3\pi}{4}}(1.3 + j), e^{\frac{-j3\pi}{2}}(2 + j0.7), e^{\frac{-j9\pi}{4}}(1.1 + j2.2), e^{-j3\pi}(-0.8 + j0.2), e^{\frac{-j15\pi}{4}}(3.4 - j0.1), e^{\frac{-j9\pi}{2}}(-1.2 + j3.1), e^{\frac{-j21\pi}{4}}(j1.5)\right\}$.

- Ex. A length-9 sequence is given by $\{x[n]\} = \{3,5,1,4,-3,5,-2,-2,4\}, 0 \leq \text{n} \leq 8$,with an 9-point DFT given by $X[k], 0 \leq k \leq 8$.Without computing the IDFT, determine the sequence $y[n]$ whose 9-point DFT is given by $Y[k] = W_3^{-2k} X[k]$.
- Solution:

$\text{Y[k]=}W_3^{-2k} X[k] = W_9^{-6k} X[k]. \text{Therefore, y[n]} = \text{x[}\langle n - 6\rangle_9]. Thus, y[0] = x[3] = 4, y[1] = x[4] = -3, y[2] = x[5] = 5, y[3] = x[6] = -2, y[4] = x[7] = -2, y[5] = x[8] = 4, y[6] = x[0] = 3, y[7] = x[1] = 5, y[8] = x[2] = 1.$

- The first 5 samples of the 9-point DFT H[k],0 ≤k ≤ 8,of a length-9 real sequence h[n],0 ≤n≤ 8,given by

H[k] ={156.8414- j6.0572 6.0346- j1.957 j8.6603-6.876-j11.4883}

Determine the 9-point DFT G[k] of the length-9 sequence $e^{j2\pi n/3}h[n]$ without computing h[n],forming the sequence g[n],and then taking its DFT.

solution :

$H[k]=H^*[\langle -k\rangle_9]=H^*[9-k]$. Hence, H[5]= H*[4]= -6.876 -j11.4883,

H[6]=H*[3]=-j8.6603,H [7]=H*[2]=6.0346+ j1.957,

H[8]=H*[1]=6.8414+ j6.0572.

Now g[n]= ej2πn/3h[n]= ej6πn/9h[n]= W9-6nh[n].

Therefore G[k]= H[<k-6>9],0 <k <8.

Symmetry properties of DFT

- for complex sequence

Length-N Sequence	N-point DFT
$x[n = x_{re}[n] + jx_{im}[n]$	$X[n] = X_{re}[n] + jX_{im}[n]$
$x^*[-n]$ $x^*[\langle -n \rangle_N]$ $x_{re}[n]$ jx_{im} $x_{cs}[n]$ $x_{ca}[n]$	$X^*[\langle -k \rangle_N]$ $X^*[k]$ $X_{cs}[k]=\frac{1}{2}\{X[k] + X^*[\langle -k \rangle_N]\}$ $X_{ca}[k] = \frac{1}{2}\{X[k] - X^*[\langle -k \rangle_N]\}$ $X_{re}[k]$ $jX_{im}[k]$

- for real sequence

Length-N Sequence	N-point DFT
$x[n] = x_{ev}[n] + x_{od}[n]$	$X[k] = X_{re}[k] + jX_{im}[k]$
$x_{ev}[n]$ $x_{od}[n]$	$X_{re}[k]$ $jX_{im}[k]$
Symmetry ralations	$X[k] = X^*[\langle -k \rangle_N]$ $X_{re}[k] = X_{re}[\langle -k \rangle_N]$ $X_{im}[k] = -X_{im}[\langle -k \rangle_N]$ $\lvert X[k] \rvert = \lvert X[\langle -k \rangle_N \rvert$ $arg\lvert X[k] \rvert == arg\lvert X[\langle -k \rangle_N \rvert$

Using symmetry properties to compute DFT or IDFT

- Ex. The following 7 samples of a length 12 real sequence $x[n]$ with a real−valued 12−point DFT $X[k]$ are given by :

$x[0] = 3.8, x[2] = 0.7, x[3] = -3.25, x[5] = 4.1, x[6] = 2.87, x[8] = 9.3, and\ x[11] = -2$. $\quad Find\ the\ remaining\ 5\ samples\ of\ x[n]$.

Solution:

Since the DFT $X[k]$ is real-valued, $x[n]$ is a circularly even sequence, i.e.

$x[n] = x[\langle -n\rangle_{12}]$. Therefore,

$$
\begin{gathered}
x[1] = x[\langle -1\rangle_{12}] = x[11] = -2, \\
x[4] = x[\langle -4\rangle_{12}] = x[8] = 9.3, \\
x[7] = x[\langle -7\rangle_{12}] = x[5] = 4.1, \\
x[9] = x[\langle -9\rangle_{12}] = x[3] = -3.25, \\
x[10] = x[\langle -10\rangle_{12}] = x[2] = 0.7.
\end{gathered}
$$

- Using DFT symmetry relations for DFT
- Ex. Without computing the DFT, determine which one of the following length-9 sequences defined for has a real-valued 9-point DFT and which one has an imaginary-valued 9-point DFT.

(a) $\{x_1[n]\}=\{4\ \ 3\ \ -5\ \ 1\ \ -2\ \ -2\ \ 1\ \ -5\ \ 3\}$,

(b) $\{x_2[n]\}=\{0\ 5\ 1\ 4\ -3\ 3\ -4\ -1\ -5\}$,

(c) $\{x_3[n]\}=\{0\ -5\ 2\ 4\ -3\ 3\ -4\ -1\ -5\}$,

(d) $\{x_4[n]\}=\{-5\ 5\ \ -2\ 2\ 4\ 4\ 2\ -2\ 5\}$.

(a) $x1[\langle -n\rangle_9] = x1[n]$.
$Thus, x1[n]\ is\ a\ circular\ even\ sequence\ and\ hence, it\ has\ a\ real - valued\ 9 - point\ DFT$.

(b) $x2[\langle -n\rangle_9] = -x2[n]$.
$Thus, x2[n]\ is\ a\ circular$ odd $sequence\ and\ hence, it\ has$ an imaginar $- valued\ 9 - point\ DFT$.

(c) $x3[\langle -n\rangle_9]\ is\ \ neither\ equal\ to\ x3[n]\ nor\ equal\ to\ - x3[n]. Thus, x3[n]\ has\ a\ complex - valued\ 9 - point\ DFT$.

(d) $x4[\langle -n\rangle_9] = x4[n]$.
$Thus\ x4[n]\ is\ a\ circular\ even\ sequence\ and\ hence, it\ has\ a\ real - valued\ 9 - point\ DFT$.

- Ex. The even samples of the 9-point DFT of a length-9 real sequence are given by $X[0] = -5.7$, $X[2] = 1.2 - j4.1$, $X[4] = -3.5 + j5.3$, $X[6] = 8.6 - j9.6$, and $X[8] = -7.7 - j3.2$. Determine the missing odd samples of the DFT.
- solution:

Since $x[n]$ is a length-9 real sequence, $X[k] = X^*[\langle -k \rangle_9]$.

Therefore, $X[1] = X^*[\langle -1 \rangle_9] = X^*[8] = -7.7 + j3.2$

$$X[3] = X^*[\langle -3 \rangle_9] = X^*[6] = 8.6 + j9.6,$$

$$X[5] = X^*[\langle -5 \rangle_9] = X^*[4] = -3.5 - j5.3,$$

$$X[7] = X^*[\langle -7 \rangle_9] = X^*[2] = 1.2 + j4.1.$$

- Ex. The 8-point DFT of a length-8 complex sequence $\mathrm{v}[n] = \mathrm{x}[n] + j\mathrm{y}[n]$ is given by

$\mathrm{V}[0] = 3 + \mathrm{j}7$, $\mathrm{V}[1] = -2 + \mathrm{j}6$, $\mathrm{V}[2] = 1 - \mathrm{j}5$, $\mathrm{V}[3] = 4 - \mathrm{j}9$,

$\mathrm{V}[4] = 5 + \mathrm{j}2$, $\mathrm{V}[5] = 3 - \mathrm{j}2$, $\mathrm{V}[6] = \mathrm{j}4$, $\mathrm{V}[7] = -3 - \mathrm{j}8$,

Where $\mathrm{x}[n]$ and $\mathrm{y}[n]$ are, respectively, the real and imaginary parts of $\mathrm{v}[n]$. Without computing the IDFT of $\mathrm{V}[k]$, determine the 8-point DFTs $\mathrm{X}[k]$ and $\mathrm{Y}[k]$ of the real sequences $\mathrm{x}[n]$ and $\mathrm{y}[n]$, respectively.

- solution : $\mathrm{v}[n] = \mathrm{x}[n] + j\mathrm{y}[n]$.

Hence, $\mathrm{X}[k] = \frac{1}{2}\{\mathrm{V}[k] + V * [\langle -k \rangle_8]\}$ and $\mathrm{Y}[k] = \frac{1}{2j}\{\mathrm{V}[k] - V * [\langle -k \rangle_8]\}$

$\mathrm{V}[k] = [3 + j7, -2 + j6, 1 - j5, 4 - j9, 5 + j2, 3 - j2, j4, -3 - j8]$.

$V * [\langle -k \rangle_8] = [3 + j7, -3 + j8, -j4, 3 + j2, 5 - j2, 4 + j9, 1 + j5, -2 - j6]$.

Therefore,

$$\mathrm{X}[k] = \left[3 + j7, -\frac{5}{2} + j7, \frac{1}{2} + j\frac{9}{2}, \frac{7}{2} - j\frac{7}{2}, 5, \frac{7}{2} + j\frac{7}{2}, \frac{1}{2} + j\frac{7}{2}, \frac{1}{2} - j\frac{9}{2}, -\frac{5}{2} - j7\right],$$

$$\mathrm{Y}[k] = \left[0, -1 - j\frac{1}{2}, -\frac{1}{2} - j\frac{1}{2}, -\frac{11}{2} - j\frac{1}{2}, 2, -\frac{11}{2} + j\frac{1}{2}, -\frac{1}{2} + j\frac{1}{2}, -1 + j\frac{1}{2}\right].$$

4.5 DFT for infinite length

- Cut the x[n] into several short N-length $x_n[m]$, then solve $X_n[k]$ by recursive method
- Step1:
  - $X_n[k] = \sum_{m=0}^{N-1} x[n-N+1+m]W_N^{mk}$,
- Step2

$$X_{n+1}[k] = \sum_{m=0}^{N-1} x[n-N+2+m]W_N^{mk} = X_n[k]W_N^{-k} + x[n+1]W_N^{-k} - x[n-N]W_N^{-k}$$

$$= [X_n[k] + x[n+1] - x[n-N+1]]W_N^{-k}$$

- compare to $X_{n+1}(e^{j\omega}) = e^{j\omega}X_n(e^{j\omega}) + \mathrm{x}[\mathrm{n}+1] - \mathrm{x}[\mathrm{n}-\mathrm{N}+1]e^{j\omega N}$

4.6 DFT for LTI system

- **M-points Circular Convolution**

$$y_c[n] = DFT^{-1}\{H[k]X[k]\} = \sum_{m=0}^{M-1} x[m]h[\langle n-m\rangle_M]$$

$$\begin{bmatrix} yc[0] \\ yc[1] \\ yc[2] \\ \vdots \\ yc[M-1] \end{bmatrix} = \begin{bmatrix} h[0] & h[M-1] & h[M-2] & \cdots & h[1] \\ h[1] & h[0] & h[M-1] & \cdots & h[2] \\ h[2] & h[1] & h[0] & \cdots & h[3] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h[M-1] & h[M-2] & h[M-3] & \cdots & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[M-1] \end{bmatrix}$$

- Ex. Consider the two finite-length sequence $g[n] = \{2 \ -1 \ 3\}, 0 \leq n \leq 2$ and $h[n] = \{-2 \ 4 \ 2 \ -1\}, 0 \leq n \leq 3$

(a) Determine $y_L[n] = g[n] \circledast h[n]$.

(b) Extend $g[n]$ to a length-4 sequence $g_4[n]$ by zero-padding and compute

$$y_C[n] = g_4[n] ④ h[n].$$

(c)Extend $g[n]$ and $h[n]$ to a length-6 sequences by zero-padding and compute the 6-point circular convolution

(d)Is $y_C[n]$the same as $y_L[n]$ determined in Part(a)?

- Solution:
  - (a) $y_L[0] = g[0]h[0] = -4,\ \ y_L[1] = g[0]h[1] + g[1]h[0] = 10,$
  $y_L[2] = g[0]h[2] + g[1]h[1] + g[2]h[0] = -6, y_L[3] = g[0]h[3] + g[1]h[2] + g[2]h[1] = 8,$
  $y_L[4] = g[1]h[3] + g[2]h[2] = 7, y_L[5] = g[2]h[3] = -3.$
  - (b) $y_C[0] = g_e[0]h[0] + g_e[1]h[3] + g_e[2]h[2] + g_e[3]h[1] = g[0]h[0] + g[1]h[3] + g[2]h[2] = 3,$

$y_C[1] = g_e[0]h[1] + g_e[1]h[0] + g_e[2]h[3] + g_e[3]h[2] = g[0]h[1] + g[1]h[0] + g[2]h[3] = 7,$
$y_C[2] = g_e[0]h[2] + g_e[1]h[1] + g_e[2]h[0] + g_e[3]h[3] = g[0]h[2] + g[1]h[1] + g[2]h[0] = -6,$
$y_C[3] = g_e[0]h[3] + g_e[1]h[2] + g_e[2]h[1] + g_e[3]h[0] = g[0]h[3] + g[1]h[2] + g[2]h[1] = 8.$

  - (c) $g_e[n] = [2, -1, 3, 0, 0, 0], h_e[n] = [-2, 4, 2, -1, 0, 0],$

$y_C[0] = g_e[0]\ h_e[0] + g_e[1]\ h_e[5] + g_e[2]\ h_e[4] + g_e[3]\ h_e[3] + g_e[4]\ h_e[2] + g_e[5]\ h_e[1] =$
$g[0]\ h[0] = -4 = y_L[0],$
$y_C[1] = g_e[0]\ h_e[1] + g_e[1]\ h_e[0] + g_e[2]\ h_e[5] + g_e[3]\ h_e[4] + g_e[4]\ h_e[3] + g_e[5]\ h_e[2] =$
$g[0]\ h[1] + g[1]\ h[0] = 10 = y_L[1],$
$y_C[2] = g_e[0]\ h_e[2] + g_e[1]\ h_e[1] + g_e[2]\ h_e[0] + g_e[3]\ h_e[5] + g_e[4]\ h_e[4] + g_e[5]\ h_e[3] =$
$g[0]\ h[2] + g[1]\ h[1] + g[2]\ h[0] = -6 = y_L[2],$
$y_C[3] = g_e[0]\ h_e[3] + g_e[1]\ h_e[2] + g_e[2]\ h_e[1] + g_e[3]\ h_e[0] + g_e[4]\ h_e[5] + g_e[5]\ h_e[4] =$
$g[0]\ h[3] + g[1]\ h[2] + g[2]\ h[1] = 8 = y_L[3],$
$y_C[4] = g_e[0]\ h_e[4] + g_e[1]\ h_e[3] + g_e[2]\ h_e[2] + g_e[3]\ h_e[1] + g_e[4]\ h_e[0] + g_e[5]\ h_e[5] =$
$g[1]\ h[3] + g[2]\ h[2] = 7 = y_L[4],$
$y_C[5] = g_e[0]\ h_e[5] + g_e[1]\ h_e[4] + g_e[2]\ h_e[3] + g_e[3]\ h_e[2] + g_e[4]\ h_e[1] + g_e[5]\ h_e[0] =$
$g[2]\ h[3] = -3 = y_L[5].$

(d)

The relation between linear convolution circular convolution

- For N-points circular convolution , how much N should be for each n, $y_c[n] == y_L[n]$?
- solution: $Nx + Nh - 1 \leq N$
  - Proof:

  For N-points circular convolution, by using DFT, all signal should be zero-filled to be N length, so

$$x[n] = \begin{cases} x[n], for\ 0 \leq n \leq Nx - 1 \\ 0, for\ Nx \leq n \leq N - 1 \end{cases}$$

and

$$h[n] = \begin{cases} h[n], for\ 0 \leq n \leq Nh - 1 \\ 0, for\ Nh \leq n \leq N - 1 \end{cases}$$

$$y[n] = \begin{cases} y[n], for\ 0 \leq n \leq Ny - 1 \\ 0, for\ Ny \leq n \leq N - 1 \end{cases},$$

so $Ny = Nx + Nh - 1$ should equal to or less then N , that is $Nx + Nh - 1 \leq N$

- Proof 2 :

if x[n] is N_1 − length, h[n] is N_2 − length

$$y_c[n] = \sum_{m=0}^{n} x[m]h[n-m] + \sum_{m=n+1}^{N-1} x[m]h[n-m+N]$$

If circle convolution is same as linear convolution, $h[n-m+N]$ should be 0,

since x[n] is N_x − length,

$$min(n-m+N) = min(n) - max(m) + N = 0 - (N_x - 1) + N$$

$$max(n-m+N) = max(n) - min(m) + N = N - 1 - (N) + N = N - 1.$$

so when $N - N_x + 1 \leq k \leq N - 1$, $h[k]$ should be 0,

since $h[k]$ is N_h − length, $N_h \leq N - N_x + 1 \leq N - 1$, so $N_h + N_x - 1 \leq N$

- If (Nx and Nh) <N<Nx+N_h-1 , within which range of n that $y_L[n]$== $y_C[n]$?

- solution : $Nx-1+N_h-N \le n \le N-1$时， $y_L[n]=y_C[n]$
- Proof
  - zero-padding x[n] and h[n]to be N-length, where N>Nx and N> Nh

$y_C[n]=\sum_{m=0}^{N-1}x[m]\,h[\langle n-m\rangle_N]=\sum_{m=0}^{n}x[m]h[n-m]$，此部分不会发生求模运算

$+\sum_{m=n+1}^{N-1}x[m]h[n-m+N]$，此部分当$h[n-m+N]$=0时，$y_C[n]$将和 $y_L[n]$ 相等

由于只有当$n-m+\mathrm{N}\ge N_h$，即$m-N+N_h \le n$时，才会为$h[n-m+N]=0$。

m最大是$Nx-1$，所以$m-N+N_h$ 的最大值为$Nx-1+N_h-N \le n$，且n不会超过N-1，

即，当$Nx-1+N_h-M \le n \le N-1$时，$y_L[n]=y_C[n]$。

Convolution of causal filter with an infinite signal

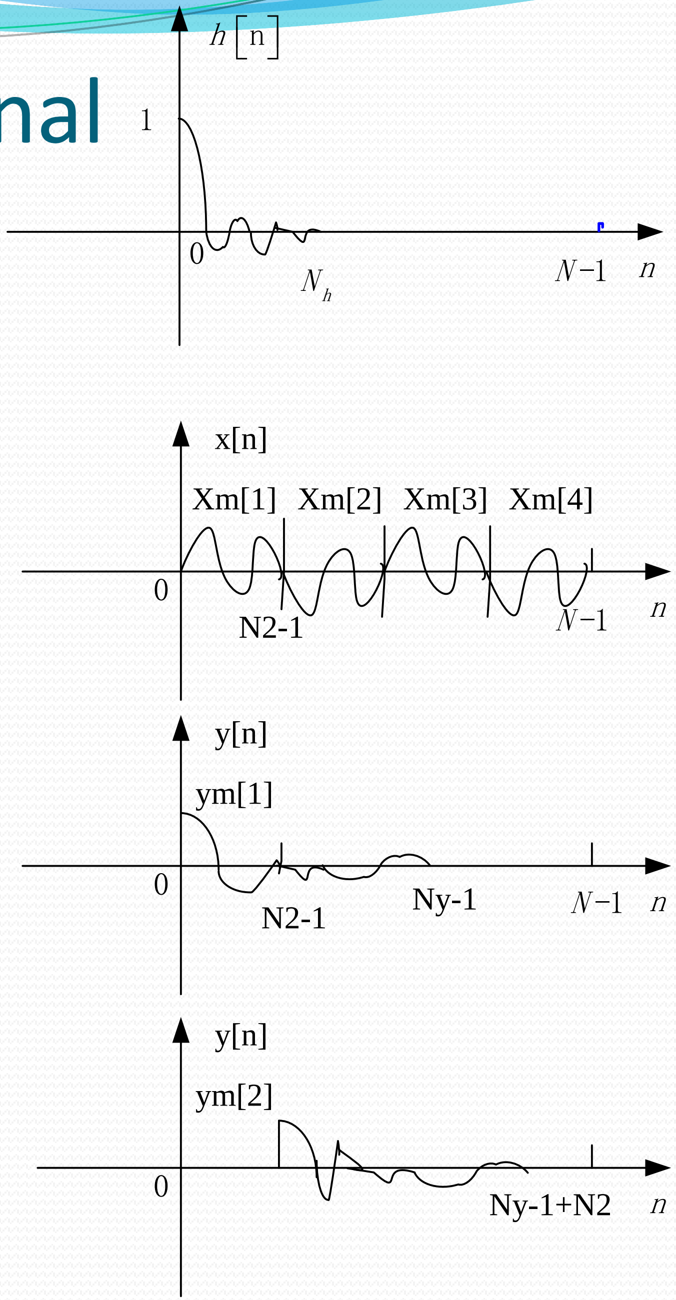
- Overlap-add method using N -point DFT and IDFT ($Nm + Nh - 1 \leq N$，固定y段)
  - Step1:Cut the original sequence x[n] into Nm -length , and zero-fill to N -length $x_m[n]$,

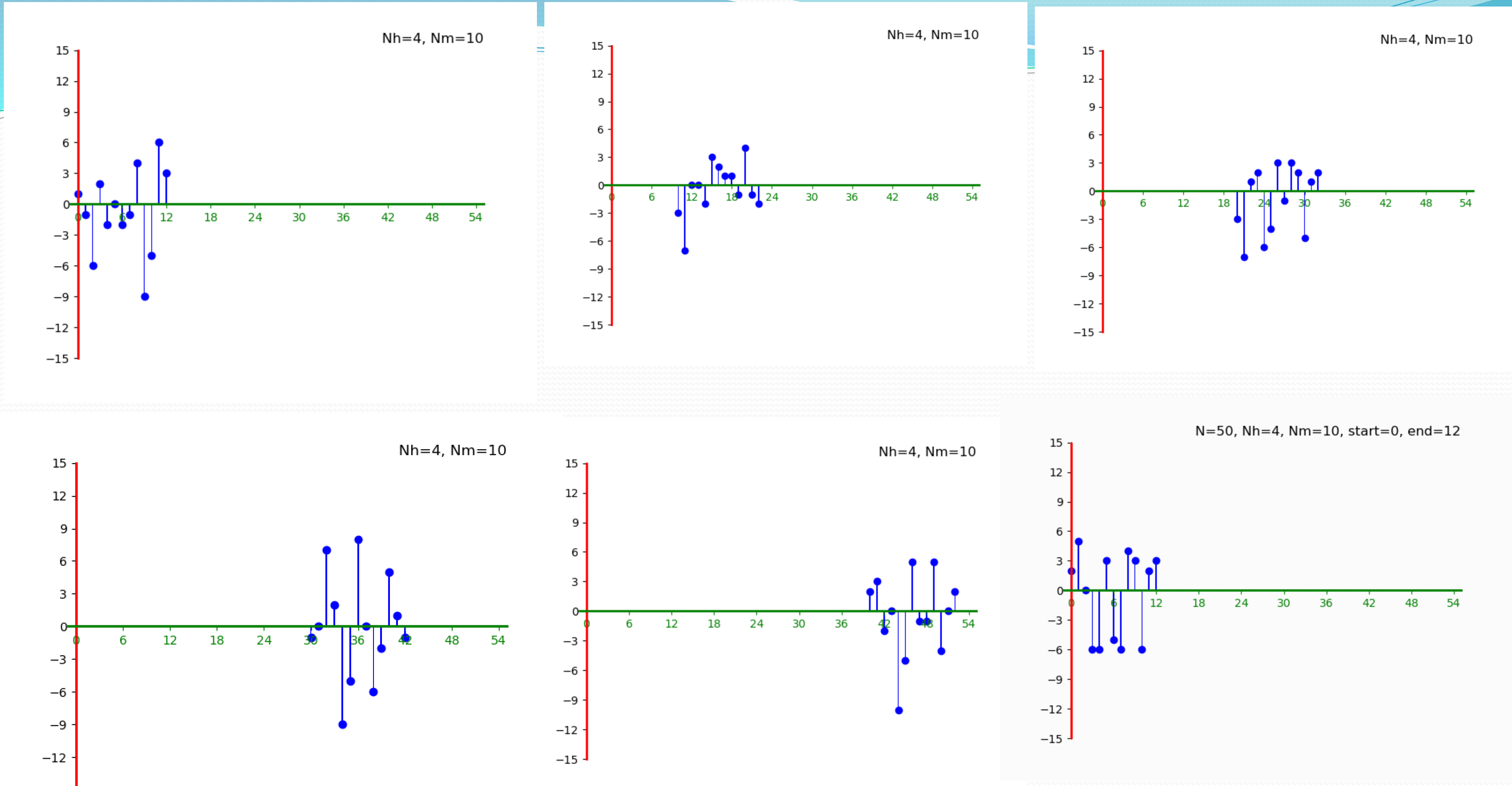
$$x_m[n] = \begin{cases} x[n + (m-1)Nm], for\ 0 \leq n \leq Nm - 1 \\ 0, for\ Nm \leq n \leq N - 1 \end{cases}$$

  - Step2:zero-fill h[n] to N-length

$$h[n] = \begin{cases} h[n], for\ 0 \leq n \leq Nh - 1 \\ 0, for\ Nh \leq n \leq N - 1 \end{cases}$$

  - Step3:N-point DFT to Xm[k] and Hm[k]
  - Step4:$Y_m[k]= X_m[k]\ {}^*H_m[k]$
  - Step5:Inverse DFT $y_m[n]$
  - Step6:$y[n] = \sum_{m=1}^{\infty} y_m[n + (m-1)Nm]$





overlap-add method

- Overlap-save method using N_m -point circular convolution($N_m > N_h$,固定x段为 N_m length)
  - Step1:Cut the original sequence x[n] into N_m -length
  - Step2: using circular convolution directly.
  - Step3: Keep the last M- N_h +1 samples , but reject previous N_h -1

$$let\ x_m[k] = x[k + m(N_m - N_h + 1)], 0 \le k \le N_m - 1$$

$$Then\ x[n] = \sum_{m=0}^{\infty} x_m[n - m(N_m - N_h + 1)]$$

$$w_m[n] = x_m[n] circular\ convolution\ h[n]$$

$$y_m[n] = \begin{cases} 0, 0 \le n \le N_h - 2, & reject; \\ w_m[n], N_h - 1 \le n \le N_m - 1, accept \end{cases}$$

$$y_L[n + m(M - N_h + 1)] = y_m[n]$$

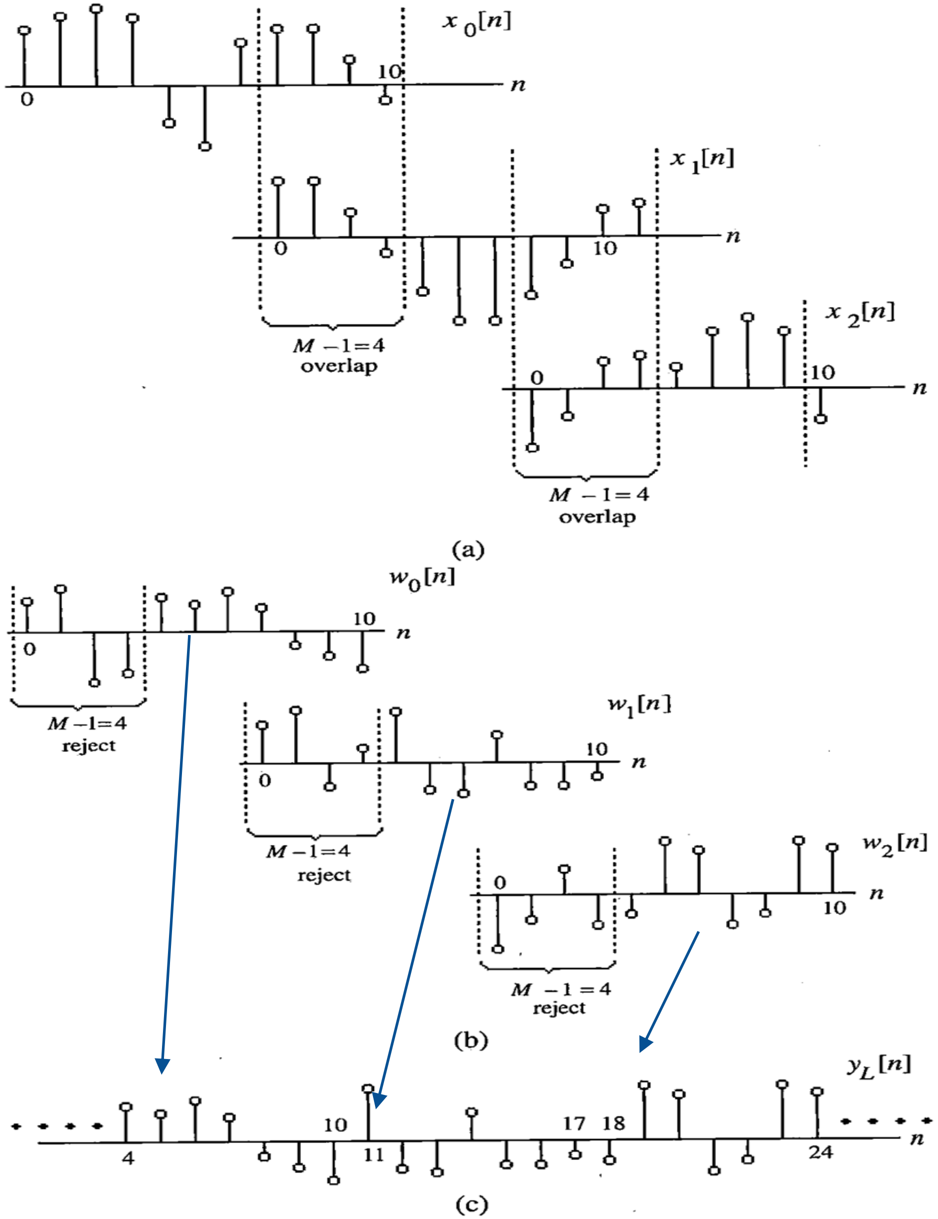
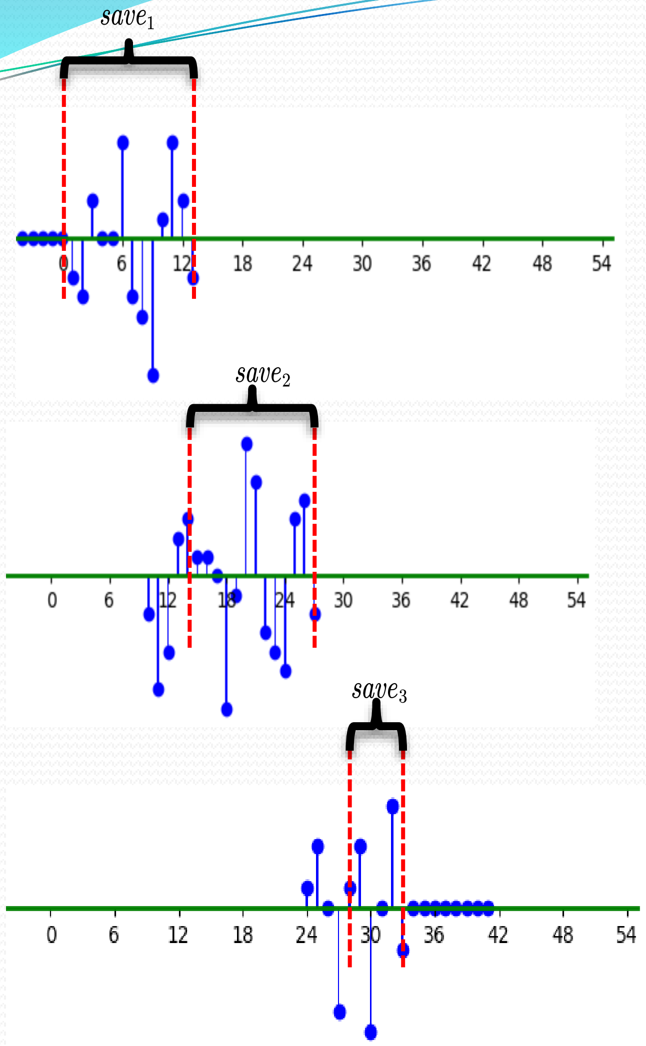
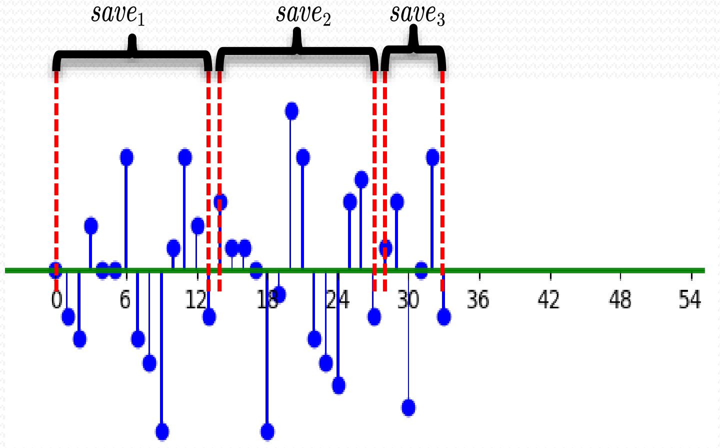
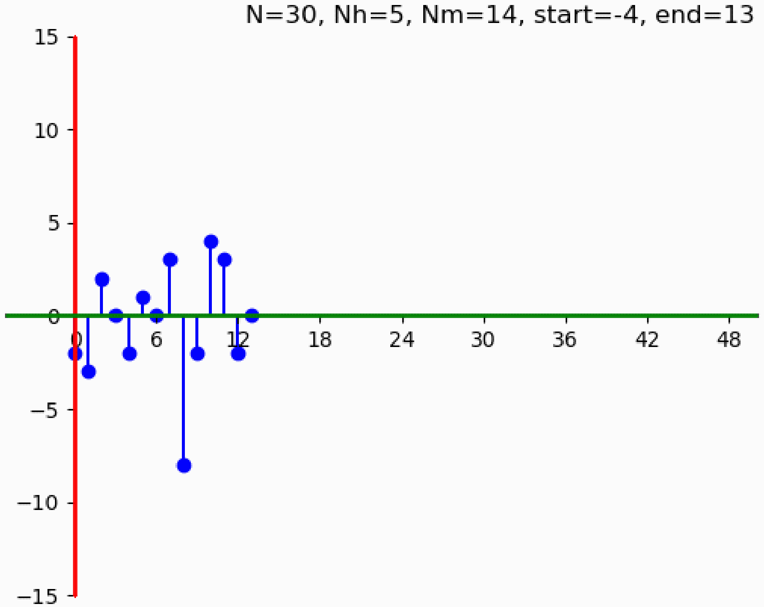


Illustration of the overlap-save method.(a)Overlapped segments of the sequence x[n] (b)Sequences generated by an 11-point circular convolution ,and (c) sequence obtained

N=30, Nh=5, Nm=14







overlap-save

Ex. The linear convolution of a length-110 sequence with a length-1300 sequence is to be computed using 128-point DFT and IDFTs

(a) Determine the smallest number of DFTs and IDFTs needed to compute the above linear convolution using the Overlap-add approach.

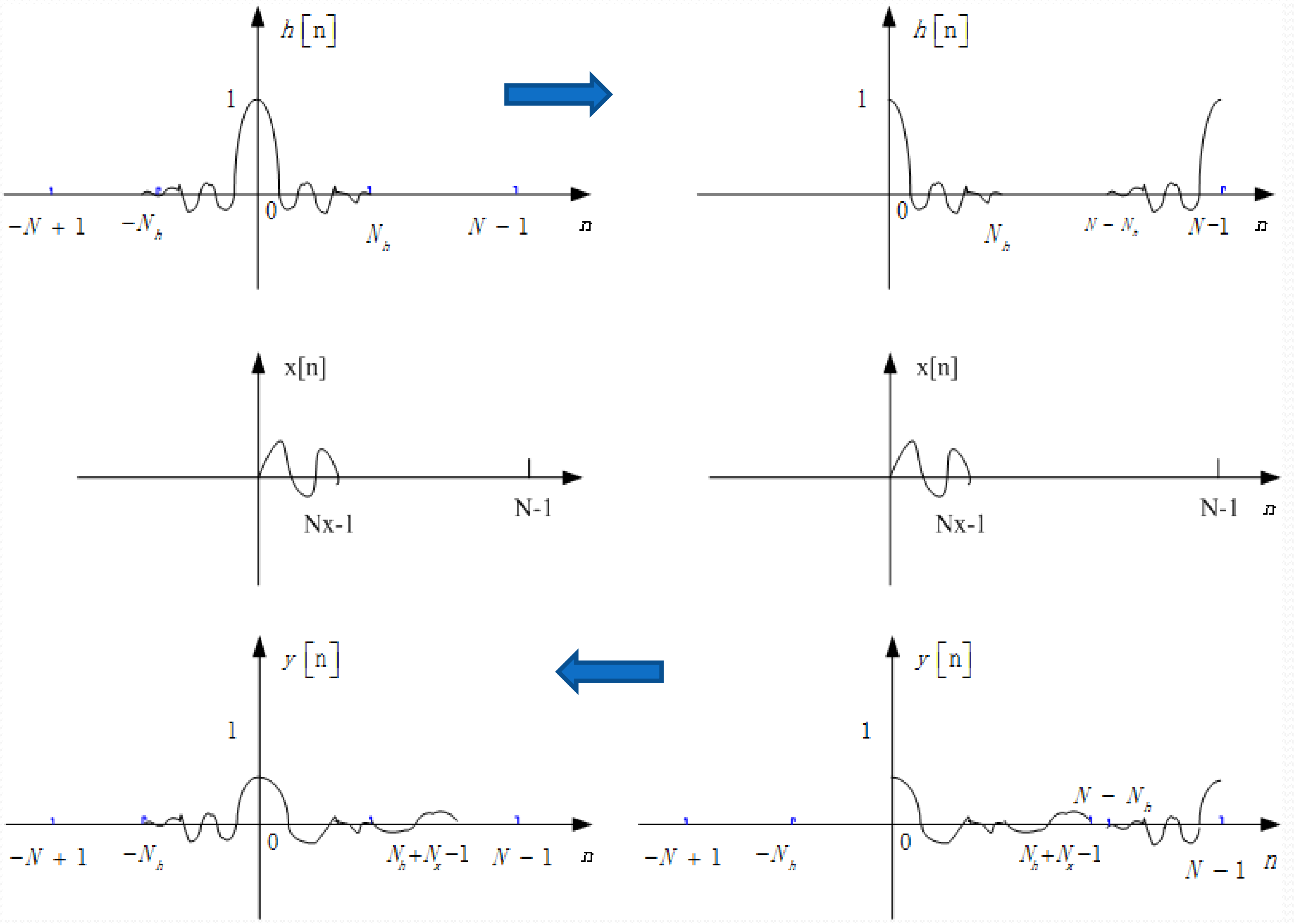
(b) Determine the smallest number of DFTs and IDFTs needed to compute the above linear convolution using the Overlap-save approach.

(a) Overlap-add method : Since the impulse response is of length and the DFT size to be used is 128,hence, the number of data samples required for each convolution will be 128-109=19.Also,the DFTs required for the length-1300 data sequence is $\left\lceil \frac{1300}{13} \right\rceil = 69$.Also , the DFT of the impulse response needs to be computed once. Hence , the total number of DFTs used are=69+1=70.The total number of IDFTs used are =69.

(b) Overlap-save method : In this case ,since the first 110-1=109 points are lost , we need to pad the data sequence with 109 zeros for a total length of 1409.Again,each convolution will result in 128-109=19 correct values .Thus the total number of DFTs required for the data are $\left\lceil \frac{1409}{19} \right\rceil = 75$.Again,1DFT is required for the impulse response .The total number of DFTs used are 75+1=76.The total number of IDFTs used are =75.

Convolution of non-causal filter with causal input

- First, since DFT has no negative part, shift negative part to get Non-negative signal.
- Second, convolution in DFT to get Y[k]，then IDFT of Y[k] need shift back
- After get y[n], then shift back N-Nh-1 <=n<=N-1 part to Nh-1 <=n<0 to get correct y[n]



4.7 DFT filter design

- Goals:
  - Given $\mathrm{H}\left(e^{j\omega}\right)$, design a zero-phase FIR LPF h[n] with cut-off ω_c using DFT and inverse DFT;

$$H_{LP}\left(e^{j\omega}\right) = \begin{cases} 1, 0 \le |\omega| \le \omega_c, \\ 0, \omega_c < |\omega| \le \pi \end{cases} \rightarrow H[k] = \begin{cases} 1, 0 \le k \le k_c, \\ 0, k_c < k \le N - k_c, \\ 1, N - k_c < k \le N - 1 \end{cases}$$

- Method1:
  - pick N, define $\frac{2\pi}{N} = \Delta\omega$, then $\omega_c \rightarrow \frac{2\pi k}{N} = k_c\Delta\omega$.
  - for $0 < \omega < \omega_c, \mathit{that\ is}\ 1 \le k \le k_c, \mathrm{H}\left(e^{j\omega}\right) \rightarrow H[k]$
  - for $-\omega_c < \omega < 0, \quad \mathrm{H}^*\left(e^{j\omega}\right) \rightarrow H[N-k]$.
  - use N-point IDFT， to get h[n] and h[N-k]
  - cascaded to get a completely h[n]

Proof

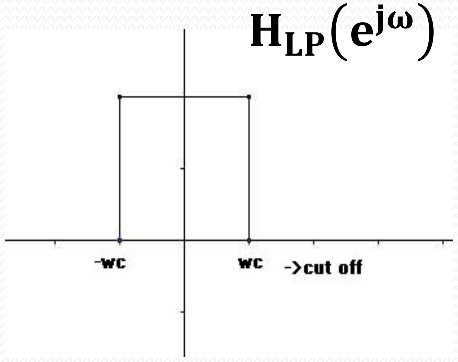
$$h[n] = DFT^{-1}(H[k])$$

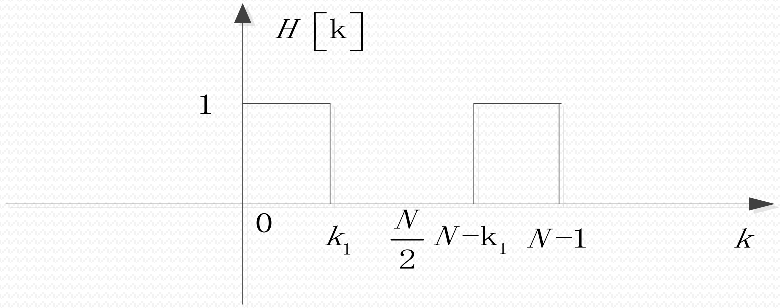
$$= \frac{1}{N}\sum_{k=0}^{k_c}(W_N^{-n})^k + \frac{1}{N}\sum_{k=N-k1}^{k_c}(W_N^{-n})^k$$

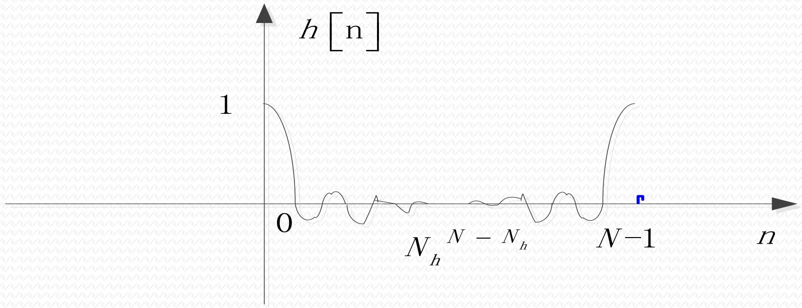
$$= \frac{1}{N}\frac{1-W_N^{-n(k_c+1)}}{W_N^{-n}} + \frac{1}{N}\frac{W_N^{-nk_c}-1}{W_N^{-n}} = \frac{1}{N}\frac{W_N^{-nk_c}-W_N^{-n(k_c+1)}}{W_N^{-n}}\frac{W_N^{-n/2}}{W_N^{-n/2}}$$

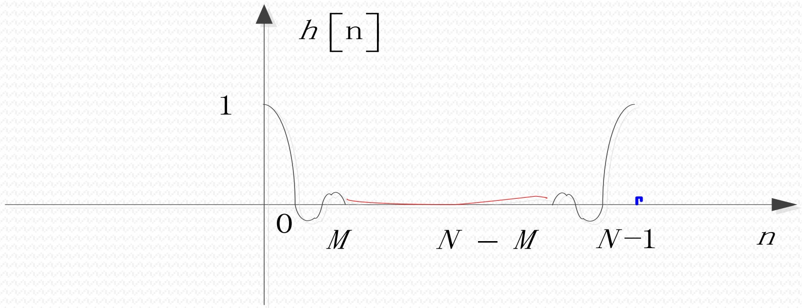
$$= \frac{\sin((^{2\pi}/_N)n(k_c+{}^1/_2))}{\sin((^{\pi}/_N)n)} = \frac{\sin(^{2\pi nk_c}/_N+(^{\pi n}/_{2N}))}{^{\pi}/_N}$$

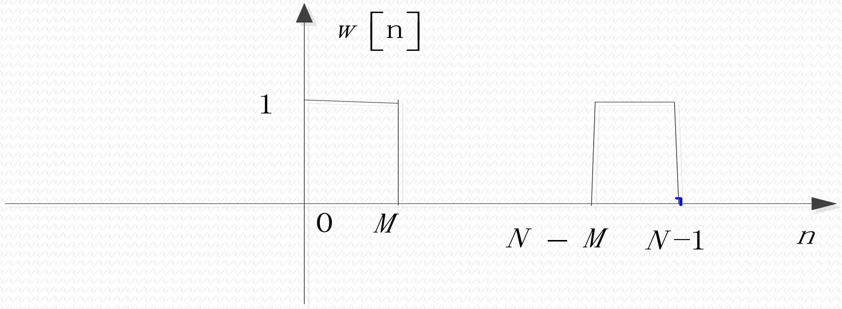
When $N \gg$ n, $^{\pi n}/_{2N} \approx 0$, $\frac{\sin(^{2\pi nk_c}/_N+(^{\pi n}/_{2N}))}{^{\pi}/_N} \approx \frac{\sin(\omega_c n)}{^{\pi}/_N} \Rightarrow DTFT$

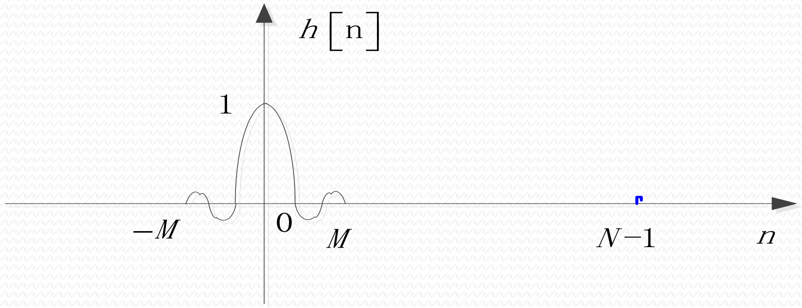












Note: the phase of $\boldsymbol{H[k]}$ at $\omega = 2\pi$ may differ from $\omega = 0$

Procedure of getting h[n] from H[k]

- Step1: Determine the function of H[k], the size of DFT M and cut-off number, k for H[k]
- Step2: IDFT to get h[n]
- Step3: Truncate the h[n] to length of $1+2N_h$ by window function
- Step4: determine the h[n] based on causal or non-causal
  - if h[n] is non-causal, then flip the h[n] for $M-N_h < n<M$ to the $n<0$, so h[n] will be $-N_h <= n <= N_h$

pseudocode

Given N, Based on the above pseudocode for a LPF H[k] with Nh=100 and AM=10, by DFT and IDFT , Plot h[n] and H[k].

$$H\left(e^{j\omega(k)}\right) = H[k]$$

$$\text{when } k= \omega(k)\frac{N}{2\pi}$$

Pseudocode:

```
For 0<=k<=N-1
H[k]=H（e^{jω(k)}）
End
h[n]=IDFT{ H[k]}
for 1<=n<=N/2
h[-n]=h[N-n],
end
stop
```

Program 3

1.write a function signal:

S（w， n）=n.exp(-n/6)·cos（ω， n）

Generate an input signal x(n)

X(n)=s(2,n)+s(1.3,n)+s(2.5,n) ， $N_x = 100$

2. write a function called Amp. With parameter Am,ω， N_m,X and N_x ,which calculate

（1） The amplitude response of x as

$$\omega(k) = \frac{\pi k}{N_m}$$

$A_m = \left| \sum_{n=0}^{N_x} x(n) \cdot z^n \right|$ $z = e^{-j\omega(k)}$,For k=0 to N_m

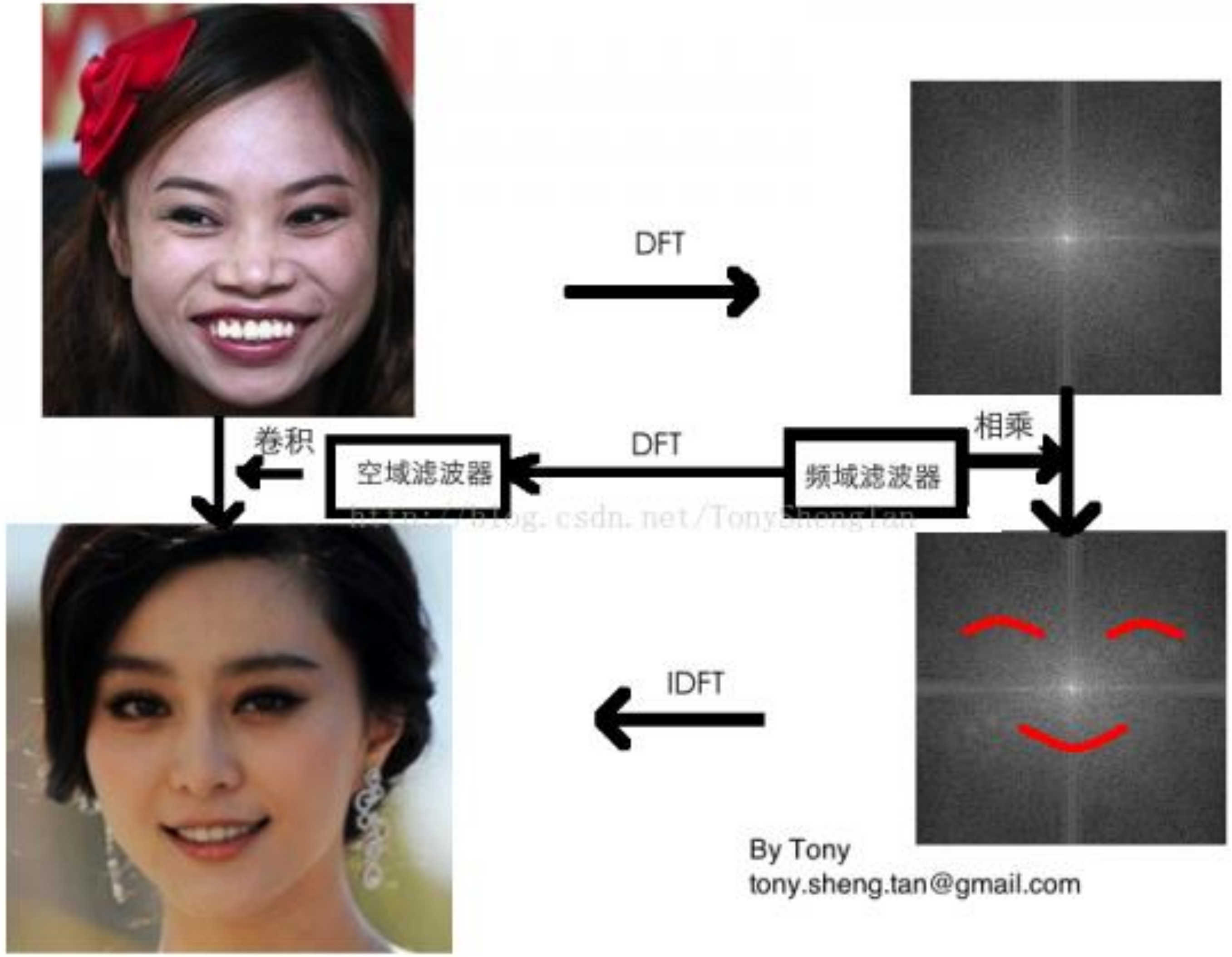
3.write a function called DSINE with parameter h， N_h

ω_{c1} and ω_{c2}， which

designs a band pass FIR digital filter， N_h is its numbers of h， ω_{c1} and ω_{c2} are lower and upper cut-off frequency.

4. use DSINE to pass S（1.3， n） but reject other two component of x(n). Plot the amplitude

$y(n) = h[n] \otimes x(n)$ with CONV of output.


DFT
相乘
卷积
空域滤波器
DFT
频域滤波器
IDFT
By Tony
tony.sheng.tan@gmail.com

Summary for convolution using DFT and IDFT

- circular convolution for causal filter with a causal finite signal
  - Step1:Padding zeroes of signal and filter to M-1 length
  - Step2: do M-1 point circular convolution by straight method or DFT and IDFT method
  - When the result is same as that of linear convolution
    - $N_x - 1 + N_h - M \le n \le M-1$
- Convolution for causal filter with a causal infinite signal
  - Overlap-add method
  - Overlap-save method
- How to do circular convolution for a non-causal filter with a causal signal?
  - Padding zeros for the filter to M-length
  - shift the non-causal part to the end of the length
  - Do the same circular convolution as causal filter
  - Shift back the result