离散数学

Discrete Mathematics

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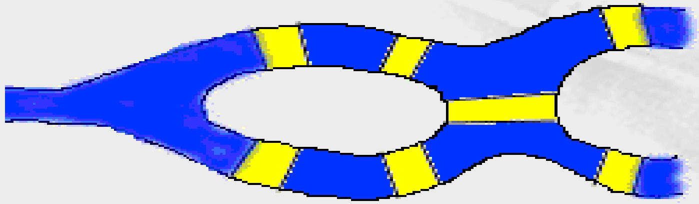
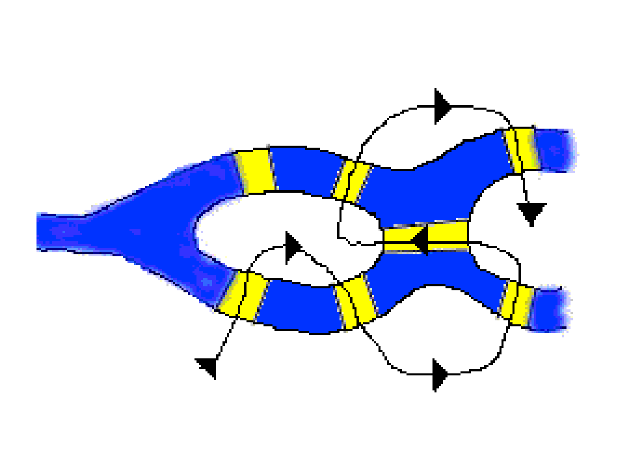
CHAPTER 10 Graphs

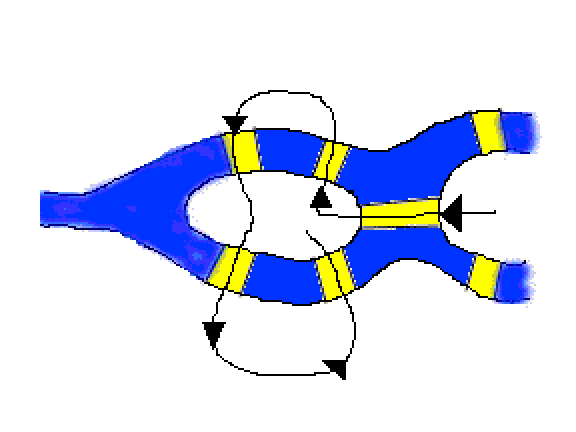
图论起源：哥尼斯堡七桥问题

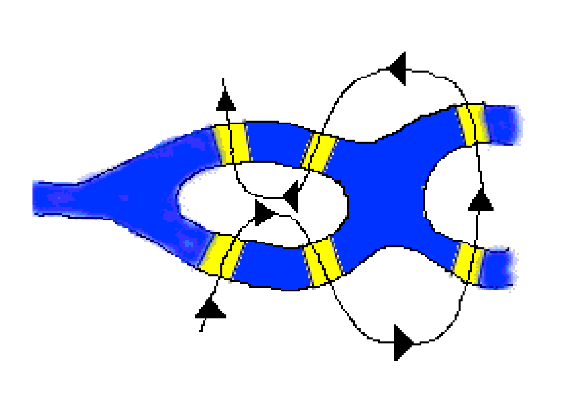
In Konigsberg, (原属普鲁士，现在称加里宁格勒，属俄罗斯), a river ran through the city. After passing an island in its center, the river broke into two branches. Seven bridges were built so that the people of the city could get from one part to another.



People wondered whether or not one could walk around the city in a way that would involve crossing each bridge exactly once.

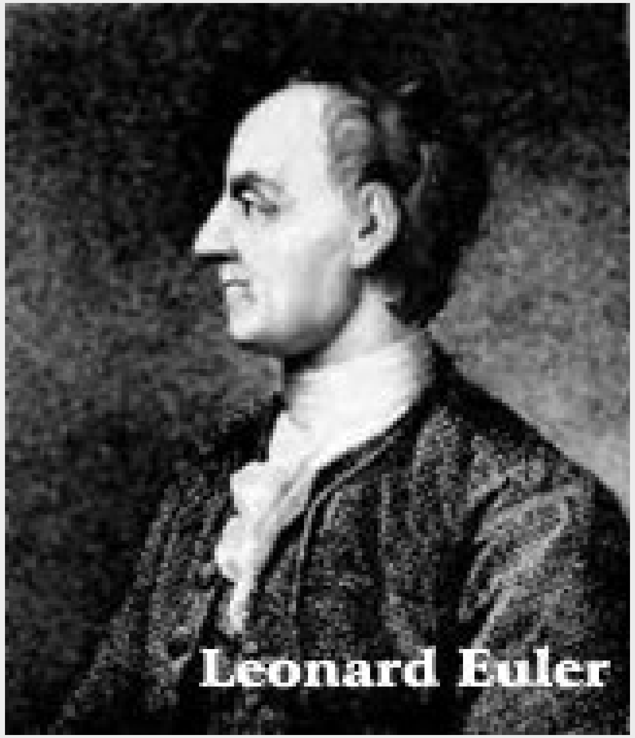




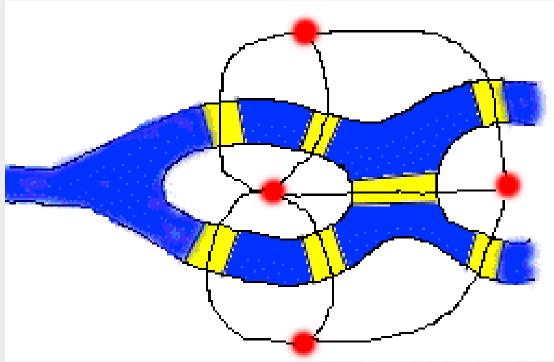


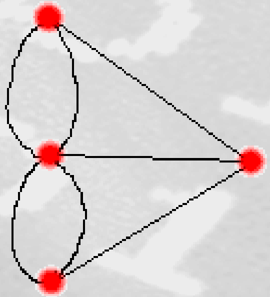
图论起源：哥尼斯堡七桥问题

Leonhard Euler (1707 -1783)



- 1735年，有几名大学生写信向欧拉请教(当时正在俄罗斯的彼得斯堡科学院任职)
- 1736年，欧拉提交了《哥尼斯堡七桥》论文，开创图论的研究分支





10.1 Introduction to Graphs

Types of Graphs 图的种类

Undirected Graphs 无向图

- Simple graph 简单图
- Multigraph 多重图
- Pseudograph 伪图

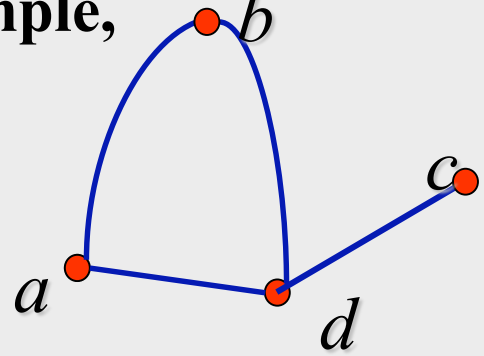
【Definition】 A *simple* graph $G=(V,E)$ consists of vertices, V, and edges, E, connecting distinct elements of V.简单图G=(V,E)是由非空顶点集V和边集E所组成的，V的不同元素的无序对称为边。

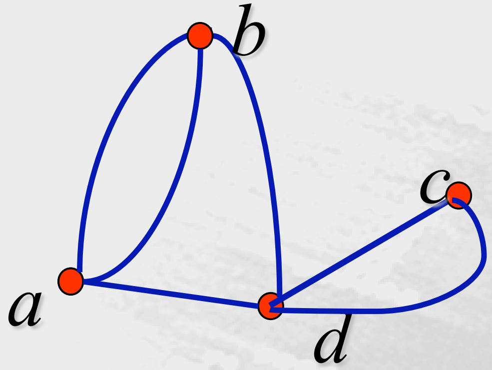
- no loops 没环
- can't have multiple edges joining vertices 两个顶点间最多只有一条边

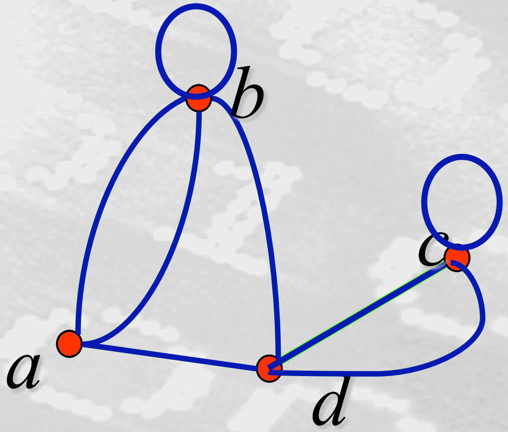
A *multigraph* allows multiple edges for two vertices.多重图允许顶点对之间有多重边

A *pseudograph* is a multigraph which permits loops.伪图也是多重图，它可以存在环。

For example,

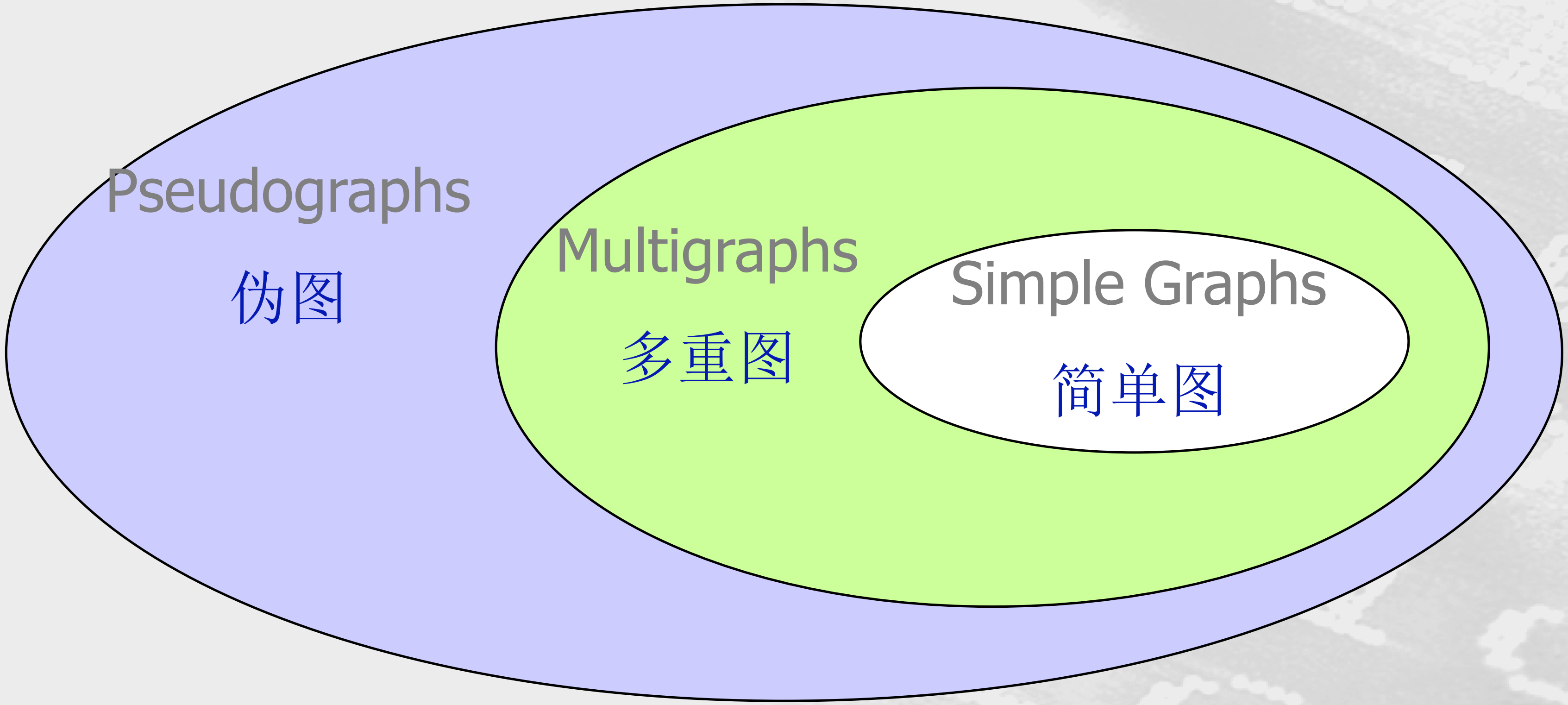






The relations of different undirected graphs

各种无向图之间的关系

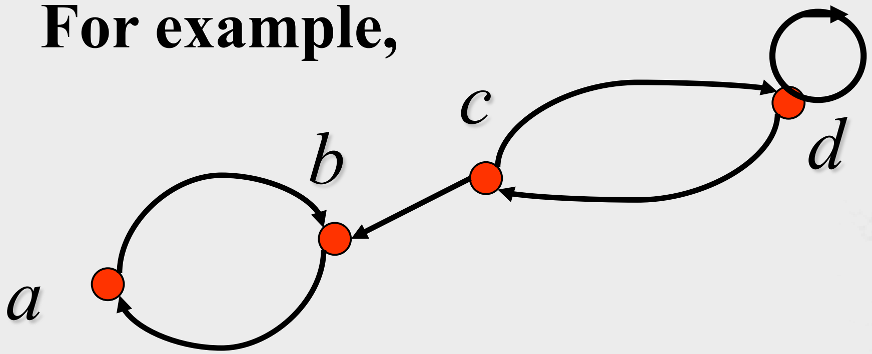


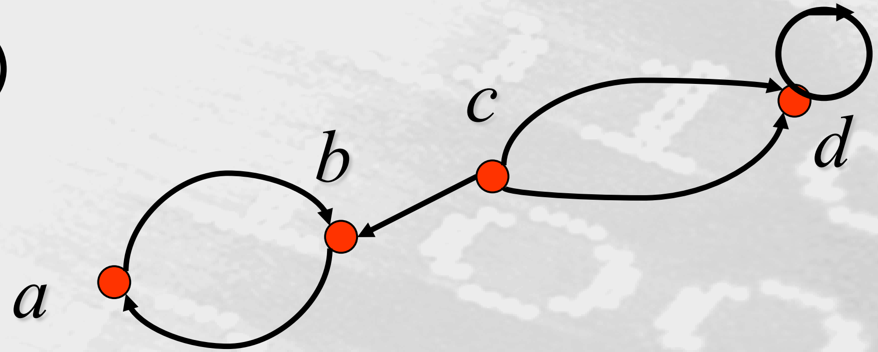
Directed Graph:

In a ***directed graph*有向图** $G = (V, E)$ **the edges are ordered pairs （有序对）of (not necessarily distinct) vertices.**有向图(V,E)是由非空顶点集V、边集E所组成的，边V中元素的有序对。允许有环(即相同元素的有序对)，但不允许在两个顶点之间有同向的多重边。

In a *directed multigraph* 有向多重图 $G = (V, E)$ **the edges are ordered pairs of (not necessarily distinct) vertices, and in addition there may be multiple edges.** 有向多重图G=(V,E)是由非空顶点集V、边集E组成的,其中可以存在多重边。

For example,





Types of Graphs and Their Properties图的类型及其性质

类型 Type	边 Edges	允许多重边 Multiple Edges?	允许环 Loops?
simple graph 简单图	Undirected 无向	No 否	No 否
Multigraph 多重图	Undirected 无向	Yes 是	No 否
Pseudograph 伪图	Undirected 无向	Yes 是	Yes 是
directed graph 有向图	directed 有向	no 否	Yes 是
dir. Multigraph 有向多重图	Directed 有向	Yes 是	Yes 是

Graph Models图模型

〖Example 1〗 How can we represent a network of (bi-directional) railways connecting a set of cities?

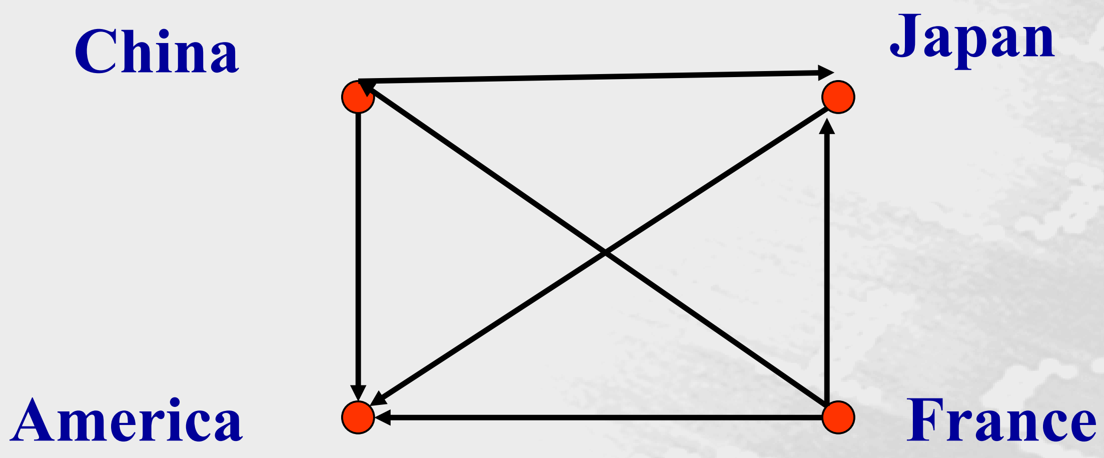
Solution:

We should use a simple graph with an edge $\{a, b\}$ indicating a direct train connection between cities a and b.

〖Example 2〗 In a round-robin循环赛制 tournament锦标赛, each team plays against each other team exactly once. How can we represent the results of the tournament (which team beats which other team)?

Solution:

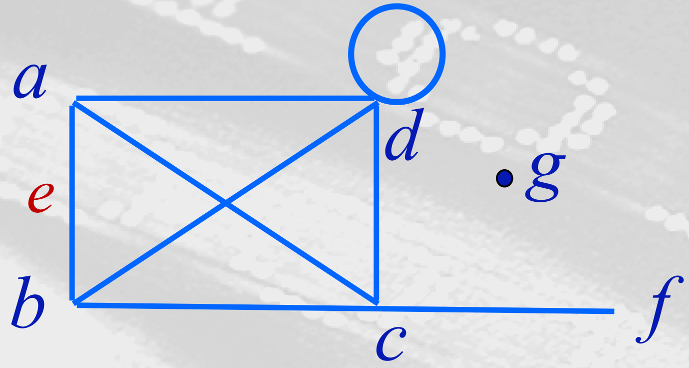
We should use a ***directed graph*** with an edge (a, b) indicating that team a beats team b.



CHAPTER 10 Graphs

Basic Terminology基本术语

Undirected Graphs $G=(V, E)$无向图



- *Vertex, edge*
- Two vertices, *u* and *v* in an undirected graph *G* are called *adjacent* (or *neighbors*) in *G*, if {*u*, *v*} is an edge of *G*. 若{u,v}是无向图G的边，则两个顶点u和v称为在G里邻接(或相邻)。
- An edge e connecting *u* and *v* is called *incident with vertices u and v*, or is said to connect *u* and *v*. 边e称为关联点u和v，也可以说边e连接u和v
- The vertices *u* and *v* are called *endpoints* of edge {*u*, *v*}.顶点u和v称为边{*u*, *v*}的端点

- *loop*
- **The *degree of a vertex* （顶点的度）in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex** 在无向图里顶点的度是与该顶点关联的边的数目，例外的情形是，顶点上的环为顶点的度做出双倍贡献

Notation: deg(v)

- **If deg(v) = 0, v is called *isolated*.孤立的**
- **If deg(v) = 1, v is called *pendant*.悬挂的**

【 Theorem 1】 The Handshaking Theorem握手理论

Let $G = (V, E)$ be an undirected graph G with e edges.

Then设G=(V,E)是e条边的无向图，则

$$\sum_{v\in V}\deg(v) = 2e$$

Proof:

Each edge represents contributes twice to the degree count of all vertices.每条边都为顶点的度之和贡献2

Note:

This applies even if multiple edges and loops are present.

注意即使出现多重边和环，这个式子也仍然成立

Application:

The sum, over the set of people at a party, of the number of people a person has shaken hands with, is even.

Qusetion:

If a graph has 5 vertices, can each vertex have degree 3? 4?

- The sum is $3 \cdot 5 = 15$ which is an odd number.

  Not possible.

- The sum is $20 = 2\,|\,E\,|$ and $20/2 = 10$.

  May be possible.

【 Theorem 2】 An undirected graph has an even number of vertices of odd degree. 无向图有偶数个奇数度顶点

Proof:

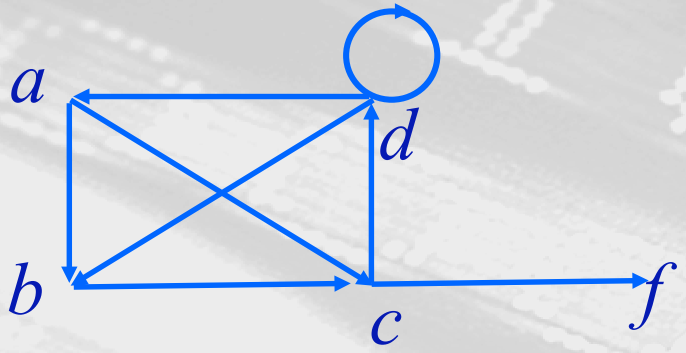
Let V_1, V_2 **be the set of vertices of even degree and the set of vertices of odd degree, respectively.设v1和v2分别是偶数度顶点和奇数度顶点的集合，于是**

$$\sum_{v \in V_1} d(v) + \sum_{v \in V_2} d(v) = 2m$$

Qusetion:

Is it possible to have a graph with 3 vertices each of which has degree 3?

Directed Graphs $G=(V, E)$



Let (*u*, *v*) be an edge in *G*. Then *u* is an *initial vertex起点* and is *adjacent to* *v* and *v* is a *terminal vertex终点* and is *adjacent from* *u*.

The *in degree入度* of a vertex *v*, denoted deg-(*v*) is the number of edges which terminate at *v*. 顶点v的入度是以v作为终点的边数。

Similarly, the *out degree 出度*of *v*, denoted deg+(*v*), is the number of edges which initiate at *v*.顶点v的出度是以v作为起点的边数

underlying undirected graph

【 Theorem 3】 Let $G = (V, E)$ be a graph with direct edges. Then

$$\sum_{v\in V} d^{+}(v) = \sum_{v\in V} d^{-}(v) \quad =|E|$$

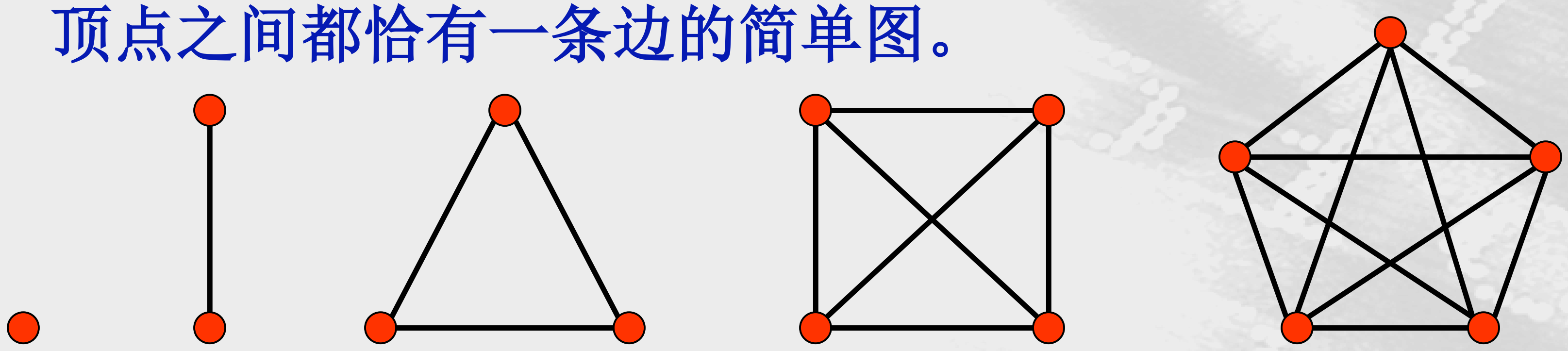
在带有向边的图里，所有顶点的入度之和等于出度之和。这两个和都等于图的边数。

Some Special Simple Graphs 一些特殊的简单图

(1) Complete Graphs 完全图- K_n: the simple graph with

- n vertices
- exactly one edge between every pair of distinct vertices.

The graphs K_n for n=1,2,3,4,5.n个顶点的完全图是在每对不同顶点之间都恰有一条边的简单图。

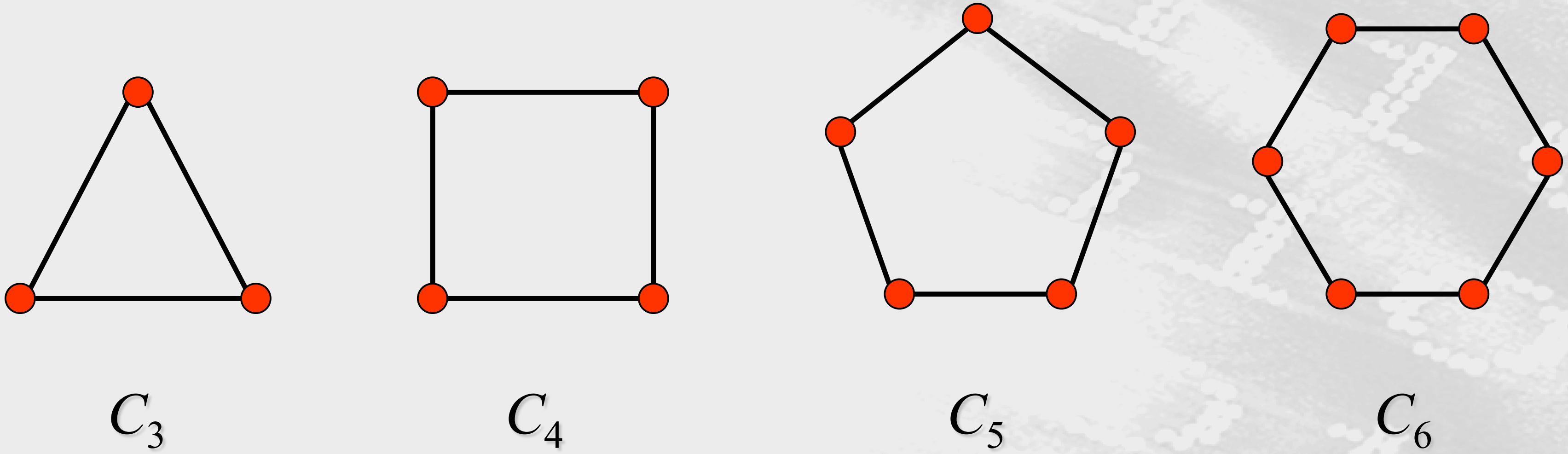


K_1 K_2 K_3 K_4 K_5

Qusetion: The number of edges in K_n ? C(n, 2)

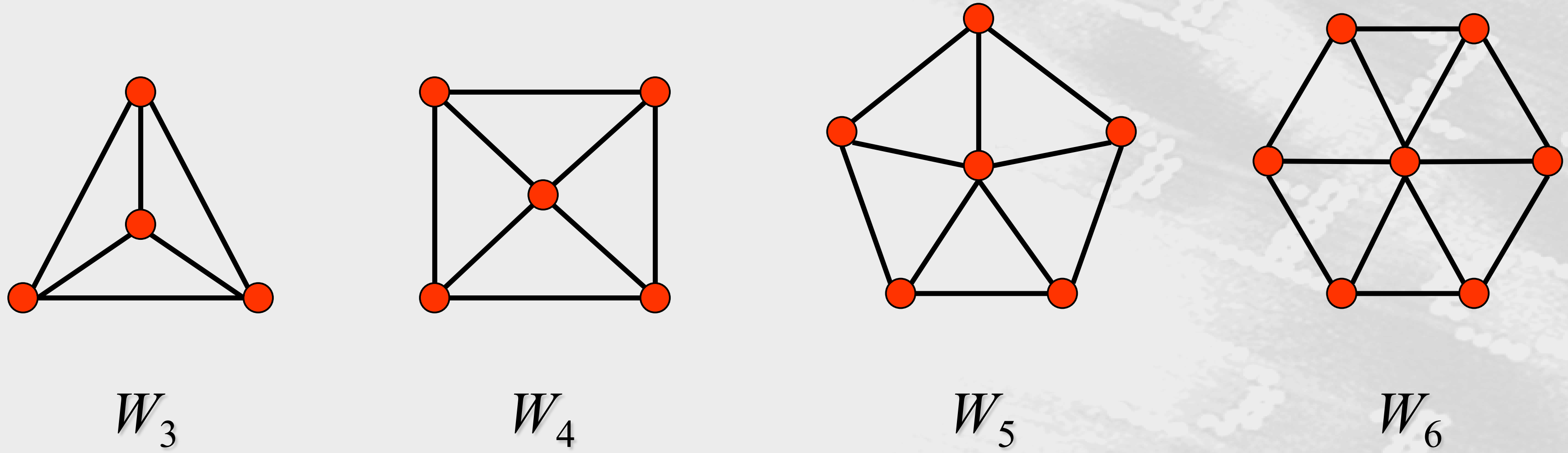
(2) Cycles 圈图C_n ($n>2$)

C_n is an n vertex graph which is a cycle.



(3) Wheels 轮图 W_n (n>2)

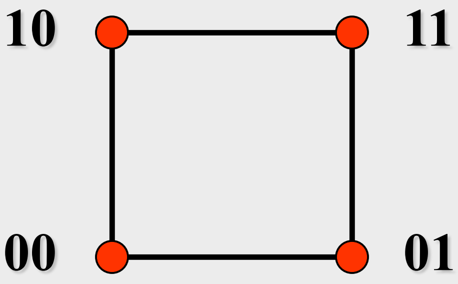
Add one additional vertex to the cycle C_n and add an edge from each vertex to the new vertex to produce W_n.当给圈图添加另一个顶点，而且把这个顶点与圈图里n个顶点逐个连接时，就得出轮图。



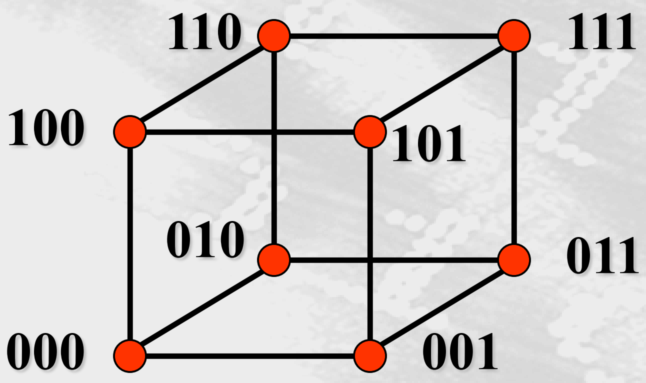
(4) *n-Cubes* Q_n (n>0) n立方体

Q_n is the graph with 2^n vertices representing bit strings of length n. An edge exists between two vertices that differ by one bit position.n立方体图是用顶点表示 2^n 个长度为n的位串的图。两个顶点相邻，当且仅当他们所表示的位串恰恰相差一位。

Q_1



Q_2



Q_3

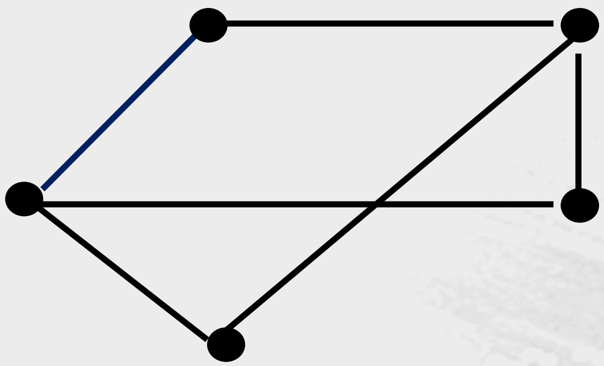
(5) Bipartite Graphs 偶图（二分图）

A simple graph *G* is *bipartite* if *V* can be partitioned into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2. 若把简单图G的顶点集分成两个不相交的非空集合V_1和V_2，使得图里的每一条边都连接着V_1里的一个顶点与V_2里的一个顶点，则G称为偶图

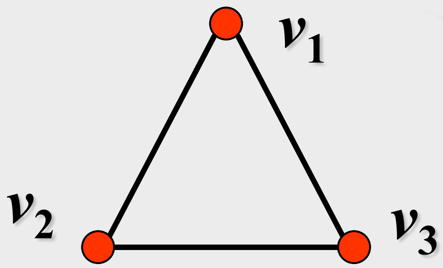
Note:

There are no edges which connect vertices in V_1 or in V_2.

For example,

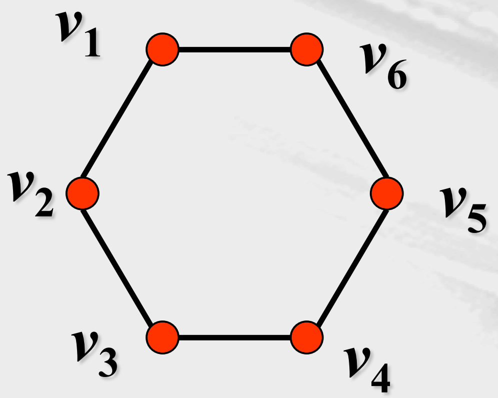


〖Example 1〗 Is C_3 bipartite?

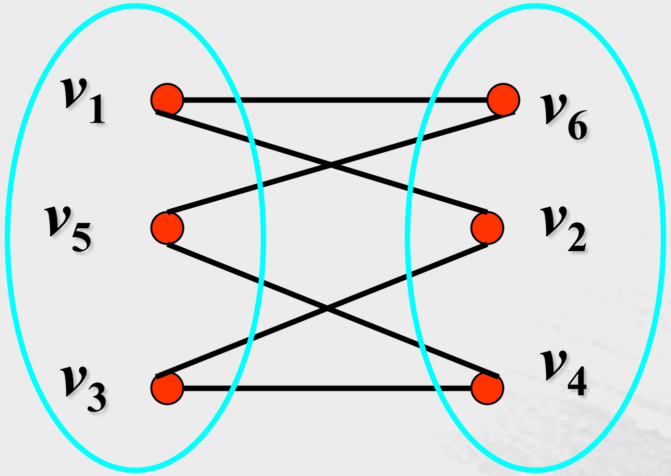


No.

〖Example 2〗 Is C_6 bipartite?



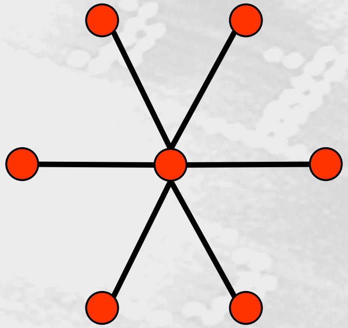
Yes. Because we can display C_6 like this:



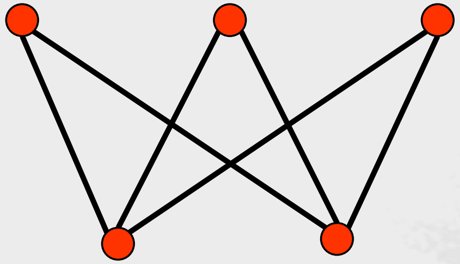
The *complete bipartite graph* is the simple graph that has its vertex set partitioned into two subsets V_1 and V_2 with m and n vertices, respectively, and *every vertex* in V_1 is connected to *every vertex* in V_2, denoted by $K_{m,n}$, where $m = |V_1|$ and $n = |V_2|$.完全偶图 $K_{m,n}$ 是顶点集分成分别含有m和n个顶点的两个子集的图。两个顶点之间有边当且仅当一个顶点属于第一个子集而另一个顶点属于第二个子集。

For example,

(1) A Star network is a $K_{1,n}$ bipartite graph.



(2) $K_{3,2}$



(6) Regular graph (正则图)

A simply graph is called *regular* if every vertex of this graph has the same degree.

A *regular graph* is called n-regular if every vertex in this graph has degree n.

For example,

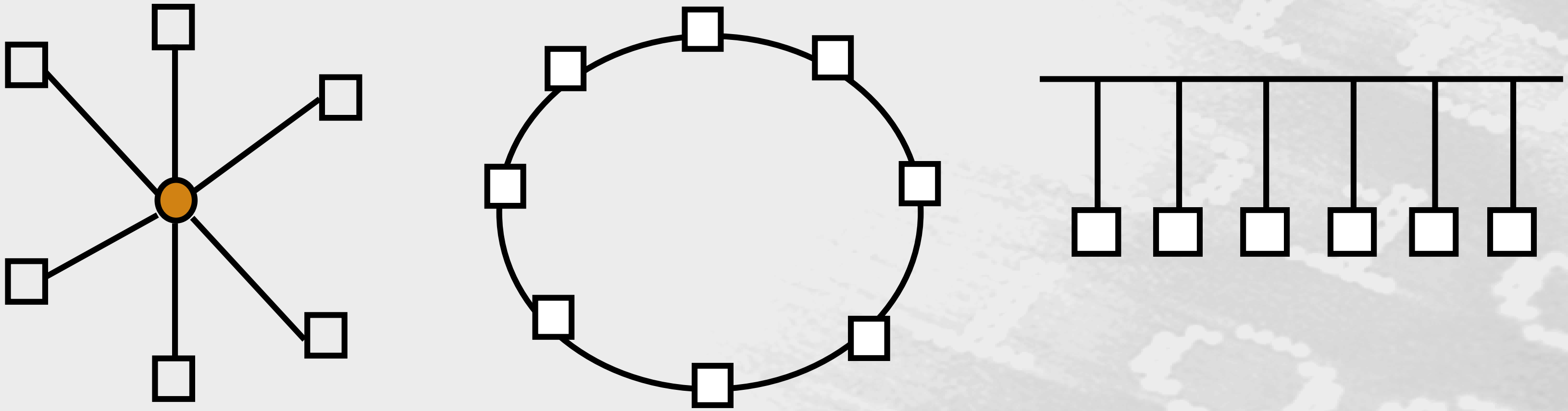
(1) k_n is a (n-1)-regular.

(2) For which values of m and n is $K_{m,n}$ regular?

Some applications of special types of graphs

〖Example 3〗 Local Area Networks.

1. Star topology 星形技术
2. Ring topology 环形技术
3. Bus topology 总线型技术



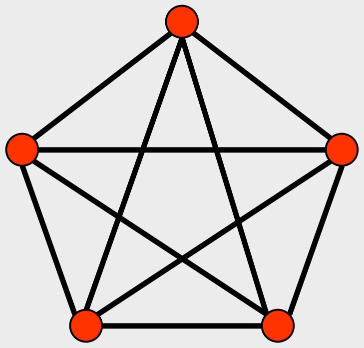
Some New Graphs From Old

【Definition】 $G = (V, E)$, $H = (W, F)$

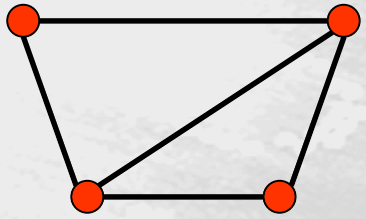
- H is a *subgraph 子图* of G if $W \subseteq V, F \subseteq E$.
- H is a *spanning subgraph 生成子图* of G

if $W = V, F \subseteq E$.

For example,



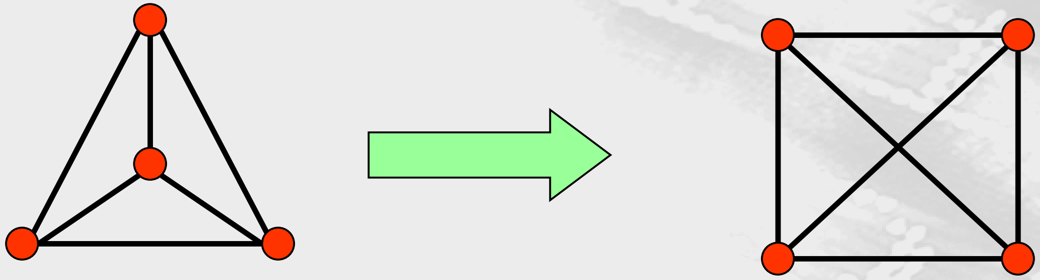
K_5



subgraph of K_5 ?

〖Example 4〗 How many subgraphs with at least one vertex does W_3 have?

Solution:

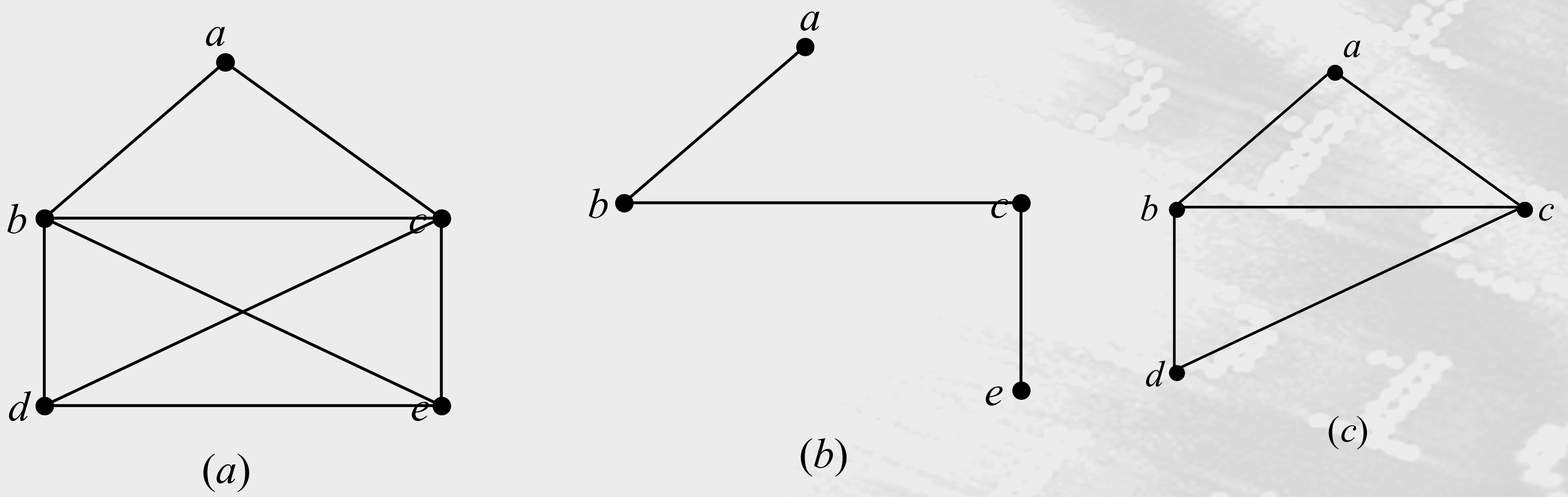


$$C(4,1)+C(4,2)\times 2+C(4,3)\times 2^3+C(4,4)\times 2^6$$

◈ 定义12　设G是一个图，$E_1 \subseteq E(G)$，以E_1为边集，E_1中边的端点全体为顶点集构成的子图，称为由E_1导出的G的子图(边导出子图)，记为$G(E1)$。

又设$V_1 \subseteq V(G)$，以V_1为顶点集，端点均在V_1中的边的全体为边集，构成的子图，称为由V_1导出的G的子图(点导出子图)，记为$G(V_1)$。

- 顶点导出子图不一定是一个边导出子图
  - 如果图中存在孤立点，那么包含这个孤立点的顶点导出子图，不能由任何边的子集导出
- 边导出子图不一定是顶点导出子图
  - (b)是边导出子图，非顶点导出子图(缺少边ac，be)



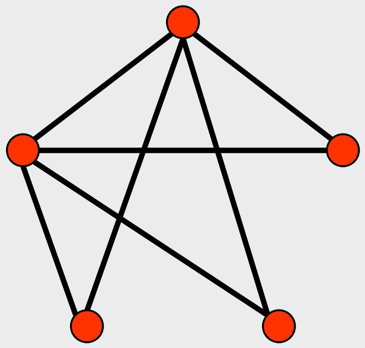
- 定义13　设G是具有n个顶点的简单图，从这n个顶点构成的完全图K_n中删去G的所有边，但保留顶点集$V(G)$所得到的图称为G的补图，简称G的补，记为～G。

The union of $G_1$and G_2 图的并

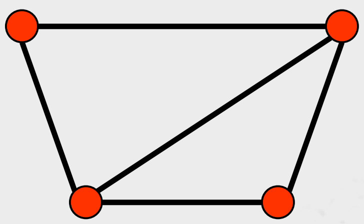
The ***union*** of two simple graphs $G_1 = (V_1 , E_1)$ and $G_2 = (V_2 , E_2)$ is the simple graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$.

Notation: $G_1 \cup G_2$

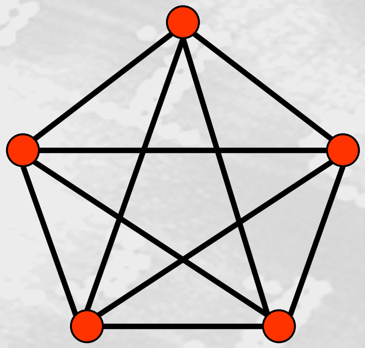
For example,



G_1



G_2



$G_1 \cup G_2 = K_5$

Homework

P650 Exercises: 11

P665 Exercises: 5,26，34