

离散数学

Discrete Mathematics



CHAPTER 10 Graphs

10.1 Introduction to Graphs

10.2 Graph Terminology 图的术语

10.3 Representing Graphs and Graph Isomorphism

10.4 Connectivity

10.5 Euler and Hamilton Paths

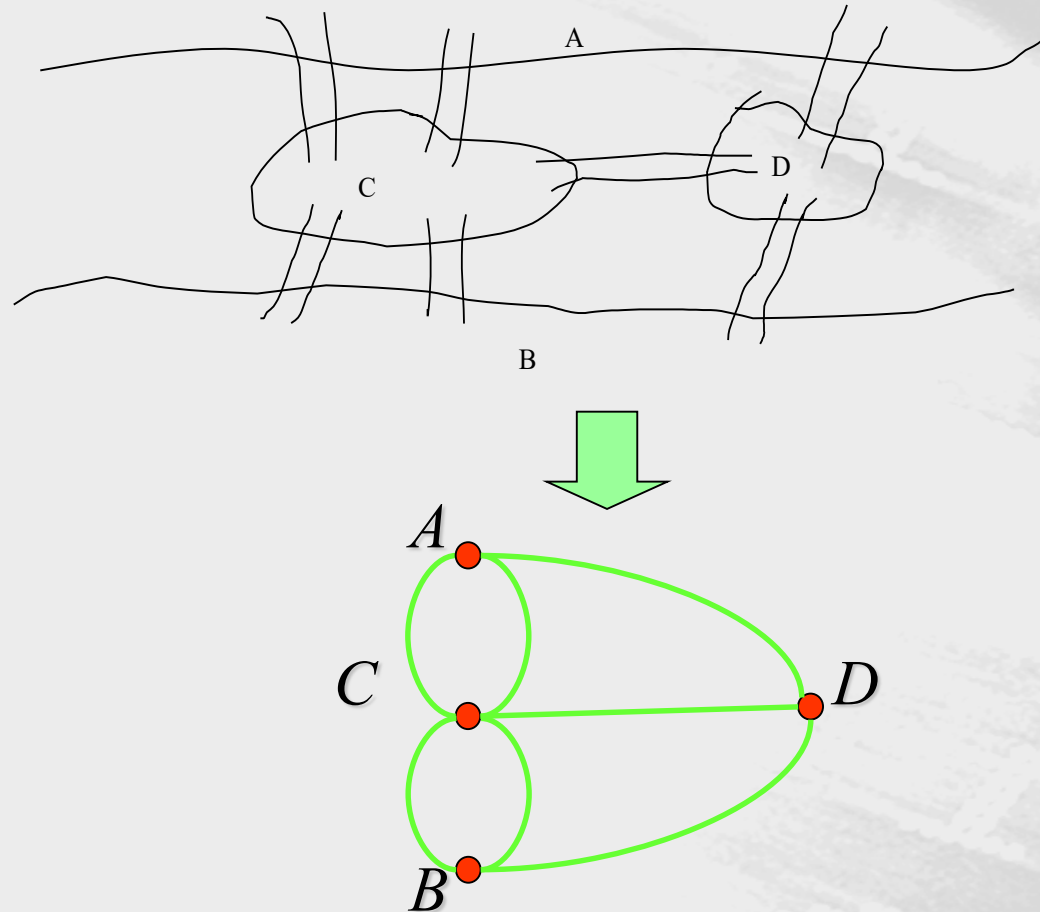
10.6 Shortest Path Problems

10.7 Planar Graphs

10.8 Graph Coloring

9.5.1 Euler Paths

Konigsberg Seven Bridge Problem 哥尼斯堡七桥问题



◆ 定义1 设 G 是一个图， G 中包含所有边的迹(即每条边恰好出现一次的路径)称为Euler迹，闭的Euler迹称为Euler闭迹或Euler回路，具有Euler回路的图称为Euler图，开的Euler迹称为Euler开迹，具有Euler开迹的图称为半Euler图。

Terminologies:

- ***Euler Circuit***

图 G 里的欧拉回路是包含着 G 的每一条边的简单回路。

- ***Euler Path***

图 G 里的欧拉通路是包含着 G 的每一条边的简单通路

- ***Euler Graph***

A graph contains an Euler circuit.

Necessary and sufficient condition for Euler circuit and paths

欧拉回路和欧拉通路的充要条件

【 Theorem 1 】 连通多重图具有欧拉回路当且仅当它的每个顶点都有偶数度

Proof:

(1) Necessary condition 必要条件

G has an Euler circuit \Rightarrow Every vertex in V has even degree

Consider the Euler circuit.

- ◆ the vertex a which the Euler circuit begins with
- ◆ the other vertex

(2) sufficient condition

We will **form a simple circuit** that begins at an arbitrary vertex a of G .

- Build a simple path $x_0=a, x_1, x_2, \dots, x_n=a$.
- An Euler circuit has been constructed if all the edges have been used. otherwise,
- Consider the **subgraph H** obtained from G .

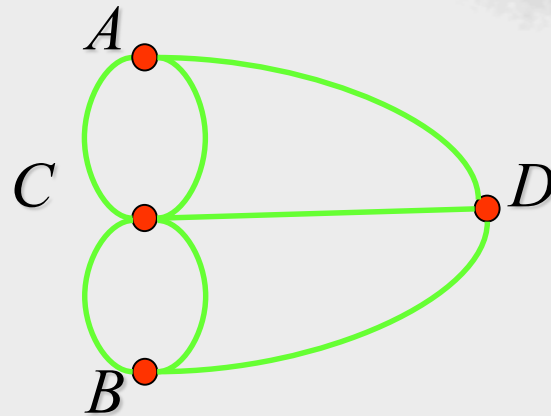
Let w be a vertex which is the common vertex of the circuit and H .

Beginning at w , construct a simple path in H .

【Theorem 2】 连通多重图具有欧拉通路而无欧拉回路，当且仅当它恰有两个奇数度顶点

【Example 1】 Königsberg Seven Bridge Problem

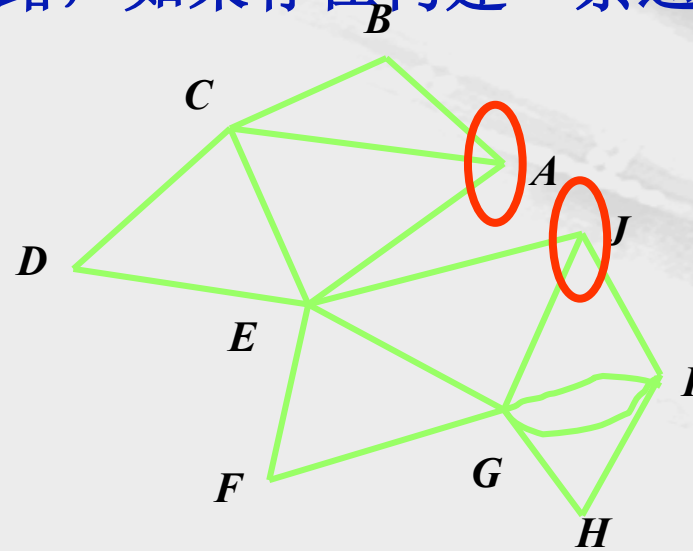
哥尼斯堡七桥问题



Solution:

The graph has four vertices of odd degree. Therefore, it does not have an **Euler circuit**. 欧拉回路

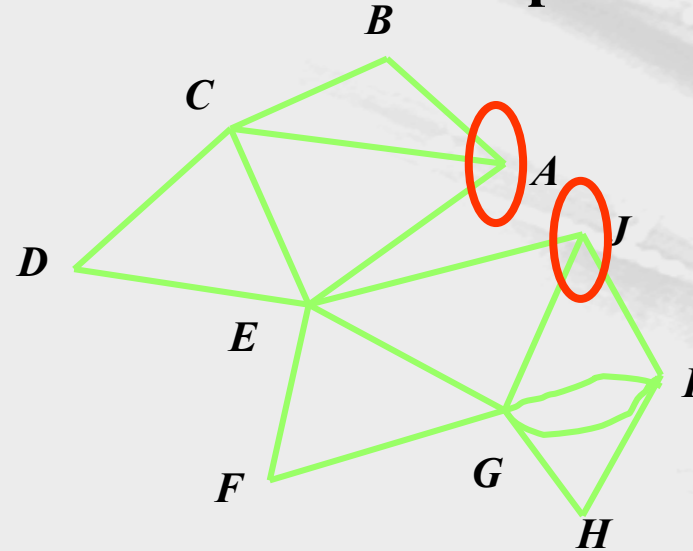
【Example 2】 Determine whether the following graph has an Euler path. Construct such a path if it exists. 判断下图是否具有欧拉通路，如果存在构建一条通路



Solution:

The graph has 2 vertices of odd degree, and all of other vertices have even degree. Therefore, this graph has an Euler path.

[[**Example 2**]] Determine whether the following graph has an Euler path. Construct such a path if it exists.



The Euler path:

*A, C, E, F, G, I, J, E, A, B,
C, D, E, G, H, I, G, J*

Solution:

The graph has 2 vertices of odd degree, and all of other vertices have even degree . Therefore, this graph has an Euler path.

Euler circuit and paths in directed graphs

有向图中的欧拉回路与欧拉通路

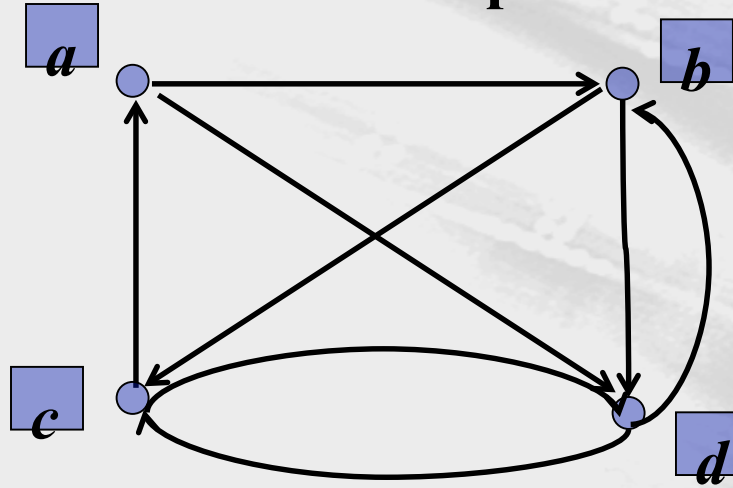
A directed multigraph having no isolated vertices has an Euler circuit if and only if 一个没有孤立顶点的有向多重图含有欧拉回路的充要条件是:

- the graph is weakly connected 弱连通的
- the in-degree and out-degree of each vertex are equal
每个顶点的出度和入度相等

A directed multigraph having no isolated vertices has an Euler path but not an Euler circuit if and only if 一个没有孤立顶点的有向多重图含有欧拉通路但不含欧拉回路的充要条件是:

- the graph is weakly connected 弱连通的
- the in-degree and out-degree of each vertex are equal for all but two vertices, one that has in-degree 1 larger than its out-degree and the other that has out-degree 1 larger than its in-degree. 除去两个顶点外每个顶点的出度和入度相等，其中一个顶点的出度比入度大1，另一个顶点的入度比出度大1.

[[Example 3]] Determine whether the directed graph has an Euler path. Construct an Euler path if it exists.



Solution:

	$\deg^-(v)$	$\deg^+(v)$
<i>a</i>	1	2
<i>b</i>	2	2
<i>c</i>	2	2
<i>d</i>	3	2

Hence, the directed graph has an Euler path.

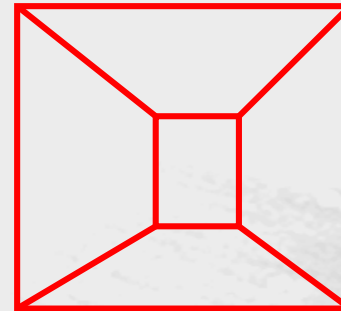
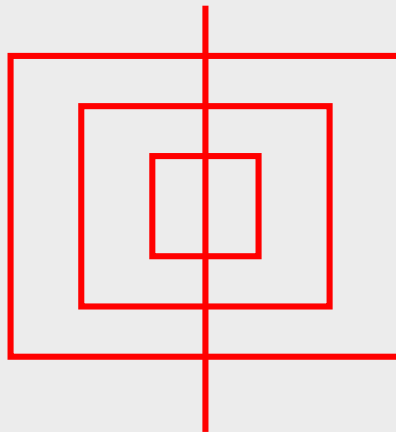
Application

A type of puzzle

Draw a picture in a continuous motion without lifting a pencil so that no part of the picture is retraced.

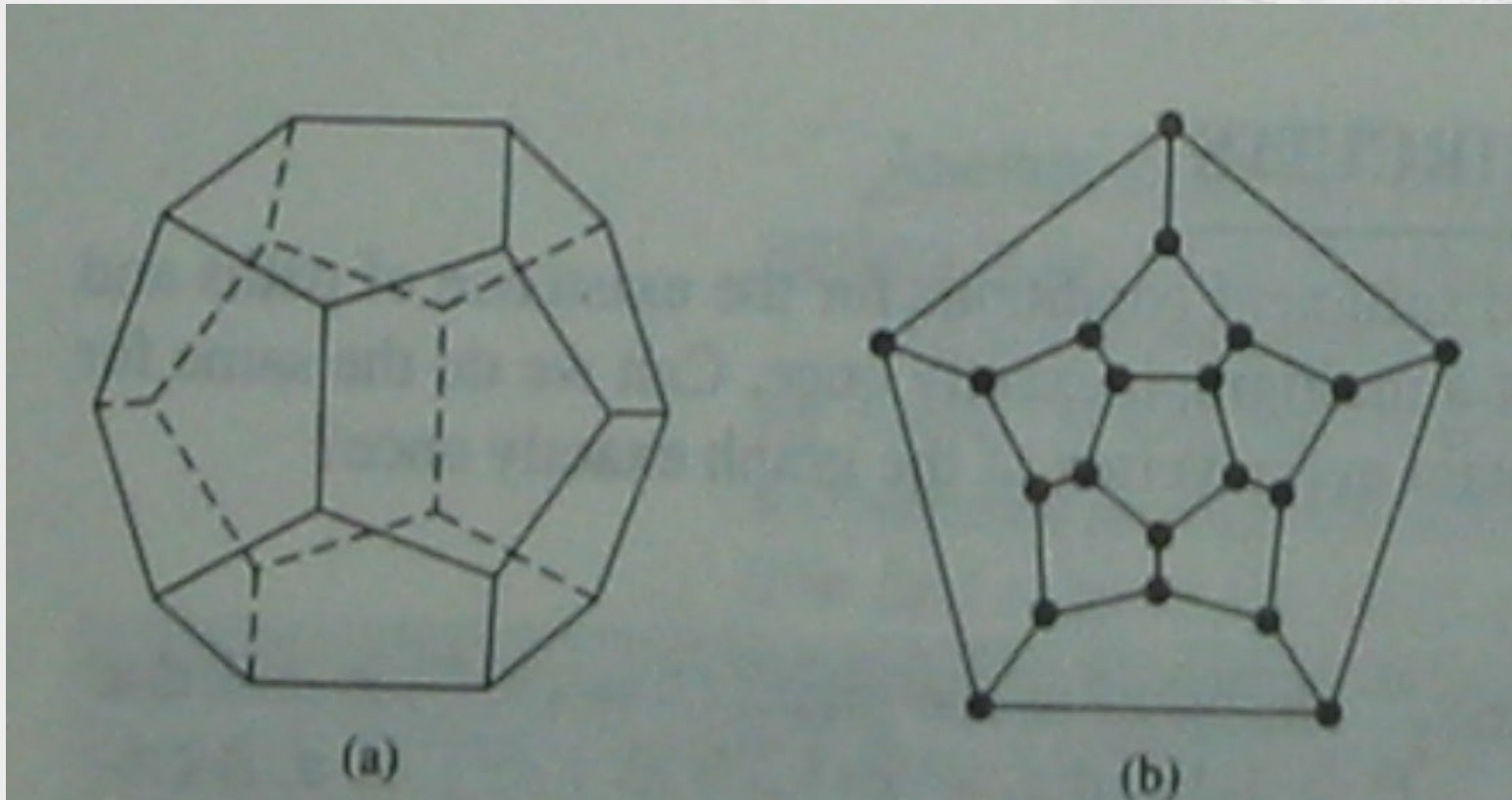
The equivalent problem: Whether the graph exist an Euler path or circuit.

For example,



9.5.2 Hamilton paths and circuit 哈密顿通路和回路

Hamilton's puzzle



A ***Hamilton path*** in a graph G is a path which visits every vertex in G exactly once.

哈密顿通路是一个访问图 G 中每个顶点次数有且仅有一次的通路

A ***Hamilton circuit*** (or ***Hamilton cycle***) is a cycle which visits every vertex exactly once, *except for the first vertex*, which is also visited at the end of the cycle.

哈密顿回路，仅访问每个顶点一次，但除去始点，这个始点同样也是终点。

If a connected graph G has a Hamilton circuit, then G is called a ***Hamilton graph***.

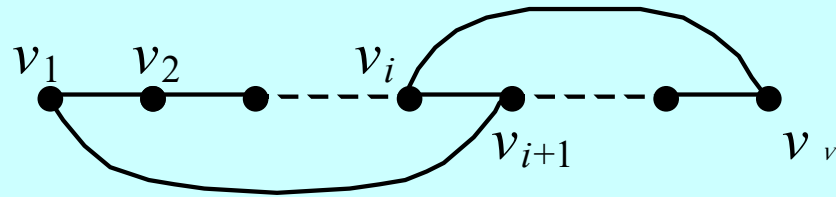
如果一个连通图 G 含有哈密顿回路，那么 G 是哈密顿图

Note: 定义适用与所有类型的有向图和无向图.

The sufficient condition for the existence of Hamilton path and Hamilton circuit 哈密顿通路和哈密顿回路存在的充分条件

【Theorem 3】 DIRAC'THEOREM 狄拉克定理
 如果 G 是带 n 个顶点的连通简单图，其中 $n \geq 3$ ，则 G 有哈密顿回路的充分条件是每个顶点的度都至少为 $n/2$

证明 假设 G 不是Hamilton图，设 G 为极大非Hamilton图；
 $G + uv$ 是Hamilton图；
 $d(u) + d(v) = |S| + |T| = |S \cup T| < n$ 。



引理 设 G 是简单图，顶点 u 和 v 一对不相邻的顶点，且满足 $\deg(u) + \deg(v) \geq n$ 则 G 是哈密顿图当且仅当 $G + uv$ 是哈密顿图

The sufficient condition for the existence of Hamilton path and Hamilton circuit 哈密顿通路和哈密顿回路存在的充分条件

引理 设 G 是简单图，顶点 u 和 v 一对不相邻的顶点，且满足 $\deg(u)+\deg(v) \geq n$ 则 G 是哈密顿图当且仅当 $G+uv$ 是哈密顿图

【 Theorem 4】 ORE'THEOREM 奥尔定理

If G is a simple graph with n vertices with $n \geq 3$ such that $\deg(u)+\deg(v) \geq n$ for every pair of nonadjacent vertices

u and v in G , then G has a Hamilton circuit.

如果 G 是带 n 个顶点的连通简单图，其中 $n \geq 3$ ，并且对于 G 中每一对不相邻的顶点 u 和 v 来说，都有 $\deg(u)+\deg(v) \geq n$ ，则 G 有哈密顿回路。

- ◆ 设 G 是一个图，反复连接满足 $d(u)+d(v)\geq n$ 的不相邻顶点 u, v ，直到没有这样的顶点对为止，这样得到的图称作图 G 的闭包，记为 $C(G)$ 。
- ◆ 定理 3 简单图 G 是Hamilton图当且仅当 $C(G)$ 是Hamilton图。

◆ 推论1 若 $C(G)$ 是完全图，则 G 是Hamilton图。

◆ 推论2 若 G 中任意不相邻顶点 u, v 均满足 $d(u) + d(v) \geq n$ ，则 G 是Hamilton图。

The necessary condition for Hamilton path and Hamilton circuit

For undirected graph:

The necessary condition for the existence of Hamilton path:

- G is connected;
- There are at most two vertices whose degree is less than 2.

The necessary condition for the existence of Hamilton circuit:

- The degree of each vertex is larger than 1.

Remark Certain properties can be used to show that a graph has no Hamilton circuit.

- A graph with a vertex of degree 1 cannot have a Hamilton circuit.
- If a vertex in the graph has degree 2, then both edges that are incident with this vertex must be part of any Hamilton circuit.
- A Hamilton circuit cannot contain a smaller circuit within it.

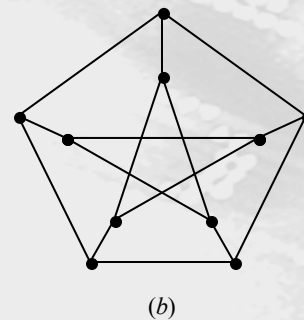
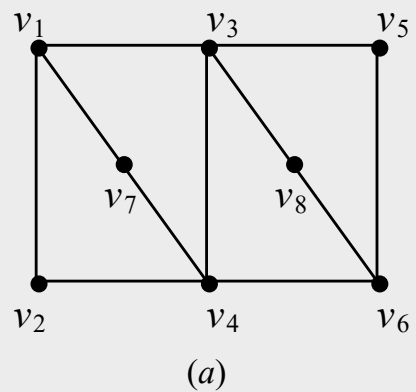
◇ 哈密顿图的必要条件

◇ 定理1 设 G 是Hamilton图，则对于顶点集 V 的任一非空真子集 S ，均有

◇ $\omega(G-S) \leq |S|$ 。

◇ 这里 $G-S$ 表示从图 G 中删去 S 中的所有顶点以及所关联的边

◇ 例1 在下图(a)中, 取 $S = \{v_1, v_4\}$, 则 $G - S$ 有3个连通分支, 故该图不是Hamilton图。



◇ 但图b满足Hamilton图必要条件, 但它不是Hamilton图

【Example 5】 There are seven people denoted by A, B, C, D, E, F, G. Suppose that the following facts are known.

A--English (A can speak English.)

B--English, Chinese

C--English, Italian, Russian

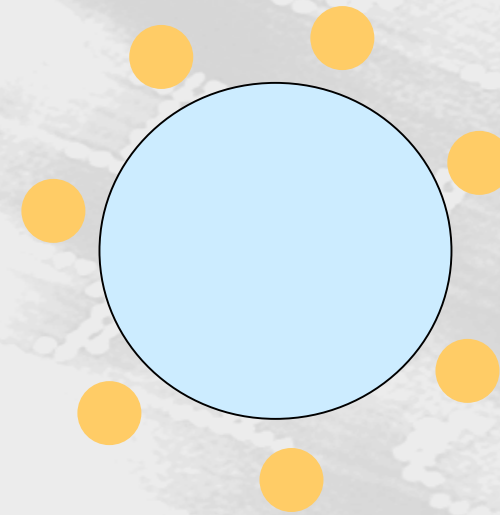
D--Japanese, Chinese

E--German, Italia

F--French, Japanese, Russian

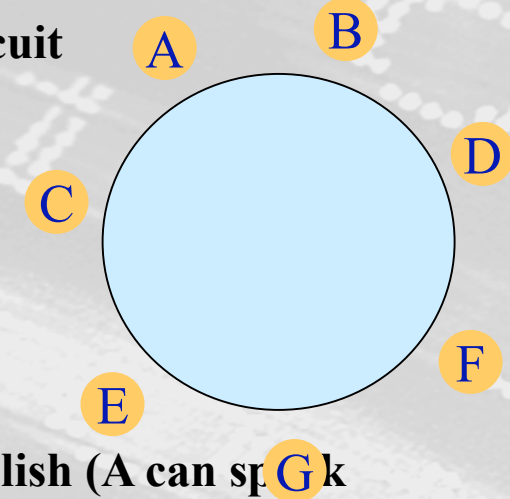
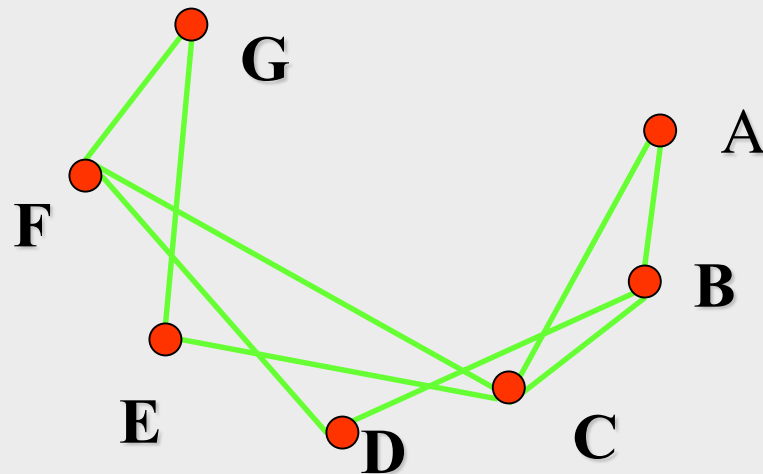
G--French, German

How to arrange seat for the round desk such that the seven people can talk each other?



Solution:**(1) Construct graph**

$V = \{A, B, C, D, E, F, G\}$, $E = \{(u, v) | u, v \text{ can speak at least one common language.}\}$



A--English (A can speak English.)

B--English, Chinese

C--English, Italian, Russian

D--Japanese, Chinese

E--German, Italian

F--French, Japanese, Russian

G--French, German

(2) If there is a H circuit, then we can arrange seat for the round desk such that the seven people can talk each other.

H circuit: A,B,D,F,G,E,C,A

- ◆ 定义 2 设 D 是有向图， D 中包含所有顶点的有向圈称为Hamilton有向圈，含有Hamilton有向圈的有向图称为Hamilton有向图， D 中包含所有顶点的有向路，称为Hamilton有向路，含有Hamilton有向路的有向图称为半Hamilton有向图。
- ◆ Hamilton有向图必定是强连通的。

◆ 若有向图 D 中每两个顶点之间恰有一条弧，则称 D 为竞赛图

◆ D 是竞赛图 $\Leftrightarrow D$ 是完全图的定向图

◆ 下面的两个定理，指出了竞赛图与Hamilton图的关系。

◆ 定理 竞赛图必是半Hamilton有向图。

◆ 定理 强连通的竞赛图必是Hamilton有向图。

Homework

P703 Exercises: 26, 48