University of Kelaniya, Sri Lanka

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BBFE 22452 : Differential Equations

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Higher Order Differential Equations
Chapter 3

Higher order linear differential equations

- Definition

A **linear ordinary differential equation of order *n*,** in the dependent variable y and the independent variable x is an equation that is in, or can be expressed in the form,

$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{(n-1)}y}{dx^{(n-1)}} + \cdots + a_{(n-1)}\frac{dy}{dx} + a_n(x)y = b(x)$ where a_0 is not identically zero.

Examples.

- $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$
- $\frac{d^4y}{dx^4} + x^2\frac{d^3y}{dx^3} + x^3\frac{dy}{dx} = xe^x$

Exercises

- Find the order and the degree of the following differential equations.

i. $x\frac{dy}{dx} = a\sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)}$ order= 1 degree =2

ii. $\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}} = a\frac{d^2y}{dx^2}$ order= 2 degree =2

• Explicit Methods of Solving Higher Order Linear Differential Equations

Definition 1

A linear ordinary differential equation of order n in the dependent variable y and the independent variable x is an equation that is in, or can be expressed in the form,

$$a_n(x)\frac{d^n y}{dx^n} + a_{(n-1)}(x)\frac{d^{(n-1)}y}{dx^{(n-1)}} + \cdots + a_1\frac{dy}{dx} + a_0(x)y = g(x) \quad [1]$$

Where a_n is not identically zero. We shall assume that $a_n, a_{n-1}, \dots \dots, a_0$ and g are continuous real functions on a real interval $a \leq x \leq b$ and that $a_n(x) \neq 0$ for any x on $a \leq x \leq b$, if $g(x) \neq 0$,

Then [1] is called linear non- homogeneous ordinary differential equation.

And if $g(x) = 0$,

$$a_n(x)\frac{d^n y}{dx^n} + a_{(n-1)}(x)\frac{d^{(n-1)}y}{dx^{(n-1)}} + \cdots + a_1\frac{dy}{dx} + a_0(x)y = 0$$

Then [1] is called linear homogeneous ordinary differential equation.

Note: $y^{(n)} = \frac{d^n y}{dx^n},\ example : y'' = \frac{d^2 y}{dx^2}$

Examples:

- $y'' + 3xy' + x^3 y = e^x$
- $y''' + xy'' + 3x^2 y' - 5y = \sin x$
- $y''' + 2y'' + 4xy' + x^2 y = 0$

Linear Homogeneous Differential Equations

We now consider the fundamental results concerning the homogeneous equation.

$a_n(x)y^n + a_{(n-1)}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$ (1)

Theorem 1

Any linear combinations of solutions of the homogeneous linear differential equation (1) is also a solution of (1).

Definition 2

The n functions $f_1, f_2, \dots \dots .., f_n$ are called **linearly dependent** on $a \le x \le b$ if there exist constants, $c_1, c_2, \dots \dots .., c_n$ not all zero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

For all x such that $a \le x \le b$

Definition 3

The n functions $f_1, f_2, \dots \dots .., f_n$ are called **linearly independent** on $a \le x \le b$ if

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

if and only if, $c_1 = c_2 =, \dots \dots \dots = c_n = 0$,For all x such that $a \le x \le b$

Theorem 2

The n^{th} order homogeneous linear differential equation(1) always possesses n solutions that are linearly independent.

Further , if $f_1, f_2, \dots \dots .., f_n$ are n linearly independent solutions of(1). Then every solution of (1) can be expressed as a linear combination

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$$

Definition 3

If $f_1, f_2, \dots \dots .., f_n$ are n linearly independent solutions of (1) on $a \le x \le b$ then the set $f_1, f_2, \dots \dots .., f_n$ is called a **fundamental set of solutions** of (1) and the function f defined by

$f(x) = c_1\, f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)$ where $c_1, c_2, \dots \dots .., c_n$ are arbitrary constants is called a **general solution of** (1) $a \le x \le b$.

Definition 4

Let $f_1, f_2, \dots \dots .., f_n$ be n real functions each of which has $(n-1)^{st}$ derivative on a real interval $a \le x \le b$.

The determinant,

$$W(f_1, f_2, \dots \dots .., f_n) = \begin{vmatrix} f_1 & f_2 \dots \dots & f_n \\ f_1' & f_2' \dots \dots & f_n' \\ & \dots \dots \dots \dots \dots .. & \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots \dots f_n^{(n-1)} \end{vmatrix}$$ is called the Wronskian of these n functions .

Theorem 3

The n solutions $f_1, f_2, \dots \dots .., f_n$ of the n^{th} homogeneous linear differential equation (1) are linearly independent on $a \le x \le b$ if and only if the Wronskian of $f_1, f_2, \dots \dots .., f_n$ is different from zero for some x on $a \le x \le b$.

Exercises

1.Show that the solutions $\sin x$ and $\cos x$ of $y'' + y = 0$ are linearly independent.

2.Show that the solutions e^x, e^{-x} and e^{2x} of $y''' - 2y'' - y' + 2y = 0$ are linearly independent.

Solution: 1.

$$w(\sin x\,, cosx) = \begin{vmatrix} \sin x & cosx \\ \cos x & -\sin x \end{vmatrix} = -sin^2x - cos^2x = -1 \neq 0$$

Therefore, solutions $\sin x$ and $\cos x$ of $y'' + y = 0$ are linearly independent.

Theorem 4

Let f be a nontrivial solution of the second order homogeneous linear differential equation.

$a_2(x)y'' + \ a_1(x)y' + a_0(x)y = 0$ (2)

The transformation $y = f(x)v$ reduces equation (2) to the first homogeneous linear differential equation.

$$a_2(x)f(x)\frac{dw}{dx} + [2a_2(x)f'(x) + a_1(x)f(x)]w = 0$$

In the dependent variable w, where $w = v$'.

The function g defined by $g(x) = f(x)v(x)$ is then a solution of the second order equation (2)

Example:

Given that $y = x$ is a solution of

$(x^2 + 1)y'' - 2xy' + 2y = 0, \ x > 0$

Find a linearly independent solution by reducing the order.

Solution: $(x^2 + 1)y'' - 2xy + 2y = 0$ [1]

Let $y = xv$ [2]

Differentiate [2] with respect to x,

$$y' = x\frac{dv}{dx} + v = xv' + v \quad [3] \qquad (\frac{dv}{dx} = v')$$

$y'' = xv'' + v' + v' = 2v' + xv''$ [4]

Substitute[2], [3], [4] in [1]

$(x^2+1)(2v'+xv'')-2x(xv'+v)+2(xv)=0$

$$2x^2v'+2v'+x^3v''+xv''-2x^2v'-2xv+2(xv)=0$$

$$2v'+x^3v''+xv''=0$$

$2v'+(x^3+x)\,v''=0$

Let $w=v', w'=v''$

$$2w+(x^3+x)w'=0$$

Now we have first order differential equation.

$$x(x^2+1)\frac{dw}{dx}=-2w$$

$$\frac{1}{w}dw=\frac{-2}{x(x^2+1)}dx$$

After integration,

$$\ln w=\ln(x^2+1)+\ln\frac{1}{x^2}+\ln k$$

$$w=\frac{(x^2+1)k}{x^2}$$

$$v'=\left(1+\frac{1}{x^2}\right)k$$

$$\frac{dv}{dx}=\left(1+\frac{1}{x^2}\right)k$$

$dv=\left(1+\frac{1}{x^2}\right)kdx$

After integration,

$$v=kx-k\frac{1}{x}+c$$

$$\frac{y}{x} = kx - k\frac{1}{x} + c$$

$$y = \left(kx - k\frac{1}{x} + c\right)x$$

$$g(x) = f(x)v(x)$$

$$g(x) = x\left(kx - k\frac{1}{x} + c\right)$$

$g(x) = k(x^2 - 1) + cx\,,$ where c and k are arbitrary constants.

Now lets check whether $y_1 = (x^2 - 1)$ *and* $y_2 = x$ are linearly independent,

$$w\big((x^2-1), x\big) = \begin{vmatrix} (x^2-1) & x \\ 2x & 1 \end{vmatrix} = (x^2 - 1) - 2x^2 = -x^2 < 0$$

Therefore ,$y_1 = (x^2 - 1)$ *and* $y_2 = x$ are linearly independent solutions of [1]. $g(x)$ is the general solution of [1].

Exercise ;

Find the general solution to ,

$$xy'' - (4x+1)y' + (4x+2)y = 0, \qquad x > 0$$

Given that $y_1 = e^{2x}$ is a solution.

Answer: $g(x) = \mathrm{c}\, e^{2x}\frac{x^2}{2} + c'\, e^{2x}$ c and c’ are arbitrary constants.

Linear Nonhomogeneous Differential Equations

We now return briefly to the nonhomogeneous equation.

$$a_n(x)y^n + a_{(n-1)}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x) \quad (2)$$

Theorem 5

Let v be any solution of the linear differential equation

$$a_n(x)y^n + a_{(n-1)}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

Let u be any solution of the corresponding homogeneous equation

$$a_n(x)y^n + a_{(n-1)}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0 \quad [3]$$

Then $u + v$ is also a solution of the given nonhomogeneous equation.

Definition 5

Consider the nonhomogeneous linear differential equation (2) and the corresponding homogeneous equation (3).

1.The general solution of (3) is called the complementary function y_c of the equation (2).

2. Any particular solution of (2) involving no arbitrary constants is called a particular integral y_p of (2)

3.The solution $y_c + y_p$ is called the general solution of the equation (2).

Solving higher order linear differential equations

Type 1: Homogeneous Linear Equations with Constant Coefficients

Consider the equation

$a_n y^{(n)} + a_{(n-1)} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$ (4)

Where $a_n, a_{n-1}, \dots \dots, a_0$ are real constants.

We shall seek a solution of the form $y = e^{mx}$, where it will be chosen such that e^{mx} satisfies the equation. Assuming that $y = e^{mx}$ is a solution of (4).

$$
\begin{aligned}
y' &= me^{mx} \\
y'' &= m^2 e^{mx} \\
&\vdots \\
y^{(n)} &= m^n e^{mx}
\end{aligned}
$$

Substituting in (4) we obtain,

$a_n m^n e^{mx} + a_{(n-1)} m^{(n-1)} e^{mx} + \cdots + a_1 m e^{mx} + a_0 e^{mx} = 0$

Or $\quad e^{mx}(a_n m^n + a_{(n-1)} m^{(n-1)} + \cdots + a_1 m + a_0) = 0$

The polynomial equation

$a_n m^n + a_{(n-1)} m^{(n-1)} + \cdots + a_1 m + a_0 = 0$ (5)

Is called the **auxiliary equation** or the **characteristic equation** of the given differential equation.

Three cases arise according to the roots of (5) are real and distinct, real and repeated or complex.

Case 1:Distinct Real Roots

Theorem 6

Consider the n^th order homogeneous linear differential equation (4). If the auxiliary equation (5) has the n distinct real roots, $m_1, m_2,\ldots\ldots m_n$, then the general solution of (4) is,

$$y = c_1e^{m_1x} + c_2e^{m_2x} + \cdots + c_ne^{m_nx}$$

Where are $c_1, c_2, \ldots \ldots .., c_n$ are arbitrary constants.

Examples

i. $y''' - 4y'' + y' + 6y = 0$

ii. $y'' - 3y' + 2y = 0$

Solution:

$y''' - 4y'' + y' + 6y = 0$ (A)

Let $y = e^{mx}$ [1]

$y' = me^{mx}$ [2]

$y'' = m^2e^{mx}$ [3]

$y''' = m^3e^{mx}$ [4]

Substitute , [1],[2],[3],[4] in (A)

$m^3e^{mx} - 4\,m^2e^{mx} + m\,e^{mx} + 6e^{mx} = 0$

$e^{mx}(m^3 - 4\,m^2 + m + 6)=0$

Auxiliary equation,

$(m^3 - 4\,m^2 + m + 6)=0$

$$(m + 1)(m - 2)(m - 3) = 0$$

$m = -1, m = 2, m = 3$

$e^{-x},\ e^{2x},\ e^{3x}$ are solutions of (A)

General solution $y = c_1\,e^{-x} + c_2e^{2x} + c_3e^{3x}$, c_1, c_2, c_3 are arbitrary constants.

Case 2 :Repeated Real Roots

Examples

i. $y'' - 6y' + 9y = 0$

ii. $y''' - 4y'' - 3y' + 18y = 0$

iii. $y^{iv} - 5y''' + 6y'' + 4y' - 8y = 0$

Solution I. $y'' - 6y' + 9y = 0$ (A)

Let $y = e^{mx}$ [1]
$y' = me^{mx}$ [2]
$y'' = m^2 e^{mx}$ [3]

Substitute , [1],[2],[3] in (A)

$m^2 e^{mx} - 6\, me^{mx} + 9\, e^{mx} = 0$

$e^{mx}(m^2 - 6\, m + 9) = 0$

Auxiliary equation,

$(m^2 - 6\, m + 9) = 0$

$(m-3)^2 = 0$
$m = 3, m = 3$

$y_1 = e^{3x}$ is a solution of (A)

Let $y = e^{3x}v$ [4]

$y' = e^{3x}v' + 3ve^{3x}$ [5]

$y'' = e^{3x}\mathrm{v}'' + 3v'^{e^{3x}} + 9ve^{3x} + 3v'^{e^{3x}}$ = $e^{3x}(\mathrm{v}'' + 6v' + 9v)$[6]

Substitute , [4],[5],[6] in (A)

$$e^{3x}(\mathrm{v}'' + \ 6v' + 9v) - 6(e^{3x}v' + 3ve^{3x}) + 9(e^{3x}v) = 0$$
$$e^{3x}v'' = 0$$
$$v'' = 0$$

Let $w = v', w' = v''$

$$w' = 0$$

$$\frac{dw}{dx} = 0$$

$$dw = 0$$

$$w = c$$

$$\frac{dv}{dx} = \mathrm{c}$$

$$dv = cdx$$

$$\int dv = \int cdx$$

$$v = cx + k$$

$$g(x) = vf(x)$$

$g(x) = e^{3x}(cx + k)$, where c and k are arbitrary constants.

$g(x) = cxe^{3x} + k\ e^{3x}$

Solutions are xe^{3x} and e^{3x}

$g(x)$ is the general solution.

Example 2.

Solve the equation $y''' - 7y'' + 11y' - 5y = 0$.

Solution.

The corresponding characteristic equation is

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0.$$

It is easy to see that one of the roots is the number $\lambda = 1$. Then, factoring the term $(\lambda - 1)$ from the equation, we obtain

$$\lambda^3 - \lambda^2 - 6\lambda^2 + 6\lambda + 5\lambda - 5 = 0, \quad \Rightarrow \lambda^2 (\lambda - 1) - 6\lambda (\lambda - 1) + 5 (\lambda - 1) = 0,$$
$$\Rightarrow (\lambda - 1) \cdot \left(\lambda^2 - 6\lambda + 5\right) = 0, \quad \Rightarrow (\lambda - 1) \cdot (\lambda - 1) \cdot (\lambda - 5) = 0,$$
$$\Rightarrow (\lambda - 1)^2 (\lambda - 5) = 0.$$

Thus, the equation has two roots $\lambda_1 = 1$, $\lambda_2 = 5$, the first of which has multiplicity 2. Then the general solution of differential equations can be written as follows:

$$y(x) = (C_1 + C_2 x) e^x + C_3 e^{5x},$$

where C_1, C_2, C_3 are arbitrary numbers.

Case 3 :Conjugate Complex Roots

Suppose that the auxiliary equation has the complex number $a + bi(a, b \in \mathcal{R}, i^2 = -1, b \neq 0)$ as a non-repeated root. Then $a - bi$ is also a non-repeated root. The corresponding part of the general solution is,

$$k_1 e^{(a+bi)x} + k_2 e^{(a-bi)x}$$

Where k_1 and k_2 are arbitrary constants.

By using Euler's formula,

$$e^{i\theta} = cos\theta + i\ sin\theta$$

$$k_1 e^{(a+bi)x} + k_2 e^{(a-bi)x}$$

$$k_1 e^{ax}(\cos bx + i \sin bx) + k_2 e^{ax}(\cos(-\,bx) + i \sin(-bx)) = e^{ax}[(k_1 + k_2)cosbx + i\ (k_1 - k_2)\ sinbx]$$
$$= e^{ax}\ (c_1\ \ \sin bx + c_2\ \ \cos bx)$$

Where $c_1 = i(k_1 - k_2)$ and $c_2 = (k_1 + k_2)$

Examples

i. $y'' + y = 0$

ii. $y'' - 6y' + 25y = 0$

iii. $y^{iv} - 4y''' + 14y'' - 20y' + 25y = 0$

Solution: $y'' - 6y' + 25y = 0$ (A)

Let $y = e^{mx}$ [1]

$y' = me^{mx}$ [2]

$y'' = m^2 e^{mx}$ [3]

Substitute , [1],[2],[3] in (A)

$m^2 e^{mx} - 6me^{mx} + 25\, e^{mx}$=0

Auxiliary equation, $m^2 - 6m + 25$ =0

$(m-3)^2 = -16$

$(m-3)^2 = 4^2 i^2$

$(m-3) \;= \pm 4i$

$m_1 = 3 + 4i \qquad m_2 = 3 - 4i$

$y = k_1 e^{(3+4i)x} + k_2 e^{(3-4i)x}$

$y = e^{3x}(c_1 \; \sin 4x + c_2 \; \cos 4x)$

Type 2: Non-Homogeneous Linear Equations

Method 1: The method of Undetermined Coefficient

Consider the nonhomogeneous differential equation,

$a_n y^n + a_{(n-1)} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x)$ (6)

Where the coefficients $a_n, a_{n-1}, \ldots \ldots, a_0$ are real constants.

General solution of (6) can be written as,

$$y = y_c + y_p$$

Where y_c is the complementary solution and y_p is a particular integral.

Definition 6

We shall call a function **UC function** if it is either

1.A function defined by one of the following.

- x^n, where n is positive integer or zero.
- e^{ax}, where $a \neq 0$.
- $\sin(bx + c)$ where b and c are constants and $b \neq 0$.
- $\cos(bx + c)$ where b and c are constants and $b \neq 0$.

Or

2. A function defined as a finite product of two or more functions of these four types.

Definition 7

Consider a UC function f. The set of functions consisting of f itself and all linearly independent UC functions of which the successive derivatives of f are either constant multiples or linear combinations will be called UC set of f.

We now outline the method of undetermined coefficients for finding a particular integral y_p of

$$a_n y^n + a_{(n-1)} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x)$$

Where g is a finite linear combination

$$g = A_1 u_1 + A_2 u_2 + \cdots + A_m u_m$$

Of UC functions $u_1, u_2, \ldots \ldots, u_m$, the A_i being known constants.

Assuming the complementary function y_c has already being obtained. We proceed as follows.

1. For each of the UC functions $u_1, u_2, \ldots \ldots, u_m$ form the corresponding UC set $S_1, S_2, \ldots \ldots, S_m$.
2. Suppose that one of the UC set so formed (S_i) is identical with or completely included in another set (S_k) omit the identical or smaller set (S_i).
3. Now consider each of the UC sets (S_i) which includes one or more members which are solutions of the corresponding homogeneous differential equation. If this is the case, we multiply each member of (S_i) by the lowest positive integral power of x so that the resulting revised set will contain no members that are solutions of the corresponding homogeneous differential equation. Replace S_i by this revised set.
4. Now form a linear combination of all the sets with unknown constant coefficients.
5. Determine these unknown coefficients by substituting the linear combination formed in step 4 into the differential equation.

Example : $y'' - 2y' - 3y = x^2 + e^x + xe^x$, Let $g(x) = x^2 + e^x + xe^x$

$u_1 = x^2 \quad u_2 = e^x \quad u_3 = xe^x$

$u_1' = 2x \quad u_2' = e^x \quad u_3' = xe^x + e^x$

$u_1'' = 2 \quad u_2'' = e^x \quad u_3'' = xe^x + 2\,e^x$

$S_1 = \{1, x, x^2\}, S_2 = \{e^x\}, S_3 = \{xe^x, e^x\}$

S_2 is already included in , S_3. So we can use S_1 and S_3

Examples:

i. $y'' - 2y' - 3y = 2e^{4x}$
ii. $y'' - 3y' + 2y = x^2 e^x$
iii. $y'' - 2y' - 3y = 2e^{2x} - 10\sin x$
iv. $y'' - 3y' + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$

Solution: ii. $y'' - 3y' + 2y = x^2 e^x$ (A)

First lets solve the homogeneous equation and obtain the complementary solution

$$y'' - 3y' + 2y = \mathbf{0}$$

Let $y = e^{mx}$ [1]

$y' = me^{mx}$ [2]

$y'' = m^2 e^{mx}$ [3]

Substitute , [1],[2],[3] in (A)

Auxiliary equation $m^2 - 3m + 2 = 0$

$$(m-1)(m-2) = 0$$

$m = 1 \; or \; m = 2,\; y_c = c_1 e^x + c_2 e^{2x}$

$u_1 = x^2\, e^x$

$u_1' = x^2\, e^x + 2\, xe^x$

$u_1'' = x^2\, e^x + 2\, xe^x + 2\, e^x + \ 2\, xe^x = x^2\, e^x + 4\, xe^x + 2\, e^x$

$$S_1 = \{x^2\, e^x, xe^x, e^x\}$$

Multiply S_1 by lowest power of x, that is x. Since member e^x

Of UC set S_1 is a solution of homogeneous equation of (A), multiply S_1 by x.

Then $S' \quad = \{x^3 e^x, x^2 e^x, xe^x\}$

Suppose that $y_p = A_1\, xe^x + A_2\, x^2 e^x + A_3\, x^3 e^x$

$y_p{}' = A_1\, (xe^x + e^x) + A_2\, (x^2 e^x + 2\, xe^x) + A_3\, (x^3 e^x + 3x^2 e^x) = e^x (A_1) + xe^x (A_1 + 2A_2) + x^2 e^x\, (A_2 + 3A_3) + x^3 e^x (A_3)$

$y_p{}'' = A_1\, (xe^x + 2e^x) + A_2\, (x^2 e^x + 4\, xe^x + 2\, e^x) + A_3\, (x^3 e^x + 6x^2 e^x + 6xe^x)$

$y_p{}'' = e^x (2A_1 + 2A_2) + xe^x (A_1 + 4A_2 + 6A_3) + x^2 e^x\, (A_2 + 6A_3) + x^3 e^x (A_3)$

Substituting $y_p,\ y_p',\ y_p{}''$ in (A),

$$e^x (2A_1 + 2A_2) + xe^x (A_1 + 4A_2 + 6A_3) + x^2 e^x\, (A_2 + 6A_3) + x^3 e^x (A_3) - 3[e^x (A_1) + xe^x (A_1 + 2A_2) + x^2 e^x\, (A_2 + 3A_3) + x^3 e^x (A_3)] + 2[A_1\, xe^x + A_2\, x^2 e^x + A_3\, x^3 e^x] = x^2 e^x$$

By comparing coefficients:

e^x: $2A_1 + 2A_2 - 3\, A_1 = 0$

$2A_2 - A_1 = 0 \quad (1)$

xe^x: $A_1 + 4A_2 + 6A_3 - 3(A_1 + 2A_2) + 2A_1 = 0$

$3A_3 = A_2 \qquad (2)$

x^2e^x: $(A_2 + 6A_3\ ) - 3(A_2 + 3A_3\ ) + 2A_2 = 1$

$$-3A_3 = 1$$

$$A_3 = -\frac{1}{3}$$

From (2)

$$A_2 = -1$$

$$A_1 = -2$$

$\therefore y_p = -2xe^x - x^2e^x - \frac{1}{3}x^3e^x$

General solution is,

$$y = y_c + y_p$$

$y = c_1e^x + c_2\, e^{2x} - 2xe^x - x^2e^x - \frac{1}{3}x^3e^x$

Note:

$$a_n y^n + a_{(n-1)} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x)$$

- For those g(x) that are not one of the above, the method of variation of parameters should be used to solve the nonhomogeneous equation. Indeed, the method of variation of parameters is a more general method and works for arbitrary nonhomogeneous term g(x) (including the types that can be solved by the method of undetermined coefficients).

Method 2: Variation of parameters

We thus seek a method of finding a particular integral that applies in all cases in which the complementary function is known. Consider the general second order linear differential equation with variable coefficients.

$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ (7)

Suppose that $y_1$and y_2 are linearly independent solutions of the corresponding homogeneous equation.

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

Then the complementary function of (7) is, $c_1y_1(x) + c_2y_2(x)$ where $c_1$and c_2 are arbitrary constants.

The procedure of the method of variation of parameters is to replace arbitrary constants $c_1$and c_2 in the complementary function by respective functions v_1 and v_2 which will be determined so that the resulting function,

$v_1(x)y_1(x) + v_2(x)y_2(x)$ (8)

will be particular integral of equation (7).

We have at our disposal the two functions v_1 and v_2 with which to satisfy the one condition that (8) be a solution of (7).

We thus assume a solution of the form (8) and write,

$y_p = v_1(x)y_1(x) + v_2(x)y_2(x)$ (9)

Differentiating (9), we have

$$y_p{}'(x) = v_1(x)y_1{}'(x) + v_1{}'(x)y_1(x) + v_2(x)y_2{}'(x) + v_2{}'(x)y_2(x)$$

We simplify y_p' by demanding that $v_1{}'(x)y_1(x) + v_2{}'(x)y_2(x) = 0$

With this condition imposed, $y_p{}'(x) = v_1(x)y_1{}'(x) + v_2(x)y_2{}'(x)$

Now differentiating we obtain,

$$y_p{}''(x) = v_1(x)y_1{}''(x) + v_1{}'(x)y_1{}'(x) + v_2(x)y_2{}''(x) + v_2{}'(x)y_2{}'(x)$$

Thus we substitute y, y' and y'' into the equation (7).

$a_2(x)[\, v_1(x)y_1{}''(x) + v_1{}'(x)y_1{}'(x) + v_2(x)y_2{}''(x) + v_2{}'(x)y_2{}'(x)]$
$+a_1(x)[\, v_1(x)y_1{}'(x) + v_2(x)y_2{}'(x)] + a_0(x)[\, v_1(x)y_1(x) + v_2(x)y_2(x)] = g(x)$

This can be written as,

$$v_1(x)[\, a_2(x)y_1''(x) + a_1(x)y_1'(x) + a_0(x)y_1(x)] + v_2(x)[\, a_2(x)y_2''(x) + a_1(x)y_2'(x) + a_0(x)y_2(x)]$$
$$+ a_2(x)[\, v_1'(x)y_1{}'(x) + v_2{}'(x)y_2{}'(x)] = g(x)$$

Since y_1 and y_2 are the solutions of the corresponding homogeneous differential equation, the expressions of the first two brackets are identically zero. Hence

$$v_1'(x)y_1{}'(x) + v_2{}'(x)y_2{}'(x) = \frac{g(x)}{a_2(x)}$$

Solving this system, we can obtain $v_1'(x)$ and $v_2'(x)$.

Thus, we obtained the functions v_1 and v_2 be chosen such that the system of equations

$$v_1{}'(x)y_1(x) + v_2{}'(x)y_2(x) = 0$$

$$v_1'(x)y_1{}'(x) + v_2{}'(x)y_2{}'(x) = \frac{g(x)}{a_2(x)}$$

Solving this system, we can obtain $v_1'(x)$ and $v_2{}'(x)$.

Thus we obtained the functions v_1 and v_2 defined by integrating.

- Examples

i. $y'' + y = \tan x$
ii. $y''' - 6y'' + 11y' - 6y = e^x$
iii. $(x^2 + 1)y'' - 2xy' + 2y = 6(x^2 + 1)^2$

Example: $y''+y = \tan x$ (A)

Let $y = e^{mx}$ [1]

$y' = me^{mx}$ [2]

$y'' = m^2e^{mx}$ [3]

Substitute , [1],[2],[3] in (A)

$m^2e^{mx} + e^{mx}$=0

Auxiliary equation, $m^2 + 1$=0

$$m^2 = -1$$

$$m^2 = i^2$$

$$m = \pm i$$

$$m_1 = i \qquad m_2 = -i$$

$$y = k_1e^{ix} + k_2e^{-ix}$$

$$y_c = c_1 \sin x + c_2 \cos x$$

Assume that $y_p = v_1(x)\sin x + v_2(x)\cos x$

$y_p' = v_1(x)\cos x + v_1'(x)\sin x - v_2(x)\sin x + v_2'(x)\cos x$

Since $v_1'(x)\sin x + v_2'(x)\cos x$=0

$y_p' = v_1(x)\cos x - v_2(x)\sin x$

$y_p'' = -v_1(x)\sin x + v_1'(x)\cos x - v_2(x)\cos x - v_2{}'(x)\sin x$

Substituting $y_p,\ y_p',\ y_p''$ in (A),

$$-v_1(x)\sin x + v_1'(x)\cos x - v_2(x)\cos x - v_2'(x)\sin x + v_1(x)\sin x + v_2(x)\cos x = \tan x$$

$v_1'(x)\cos x - v_2'(x)\sin x = \tan x$

System of equations is

$v_1'(x)\sin x + v_2{}'(x)\cos x = 0$

$v_1'(x)\cos x - v_2'(x)\sin x = \tan x$

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$v_1'(x) = \frac{\begin{vmatrix} 0 & \cos x \\ \tan x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & +\cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{0 - \sin x}{-sin^2x - cos^2x} = sinx$$

$$v_2'(x) = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \tan x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{\frac{sin^2x}{\cos x}}{-sin^2x - cos^2x} = -\frac{sin^2x}{\cos x}$$

By integrating $v_1'(x) = \sin x$

$\int dv_1'(x) = \int \sin x \, dx$

$$v_1 \ (x) = -\cos x + c_3$$

By integrating $v_2'(x) = -\frac{sin^2 x}{\cos x}$

$\int dv_2'(x) = \int -\frac{sin^2 x}{\cos x} \, dx$

$v_2 \ (x) = \int -\frac{(1-cos^2 x)}{\cos x} \, dx$=$\int (\cos x - \sec x \ )dx$

$$v_2 \ (x) = sin\, x - ln \,|\, tan\, x + sec\, x \ | + c_4$$

Hence $y_p = (-cos\ x + c_3)\, sin\, x + (sin\ x - ln\,|\, tan\, x + sec\ x\ | + c_4) \cos x$

$= -\sin x. cos\ x + c_3 \sin x + \sin x. cos\ x - (ln\,|\,tan\, x + sec\ x\ |) \cos x + c_4 \cos x$

$= c_3 \sin x + c_4 \cos x - (ln\,|\,tan\, x + sec\ x\ |) \cos x$

Since a particular integral is a solution free of arbitrary constants, we may assign any particular values A and B to $c_3 and \ \ c_4$ respectively.

Then $y_p = A \sin x + B \cos x - (ln\,|\,tan\, x + sec\ x\ |) \cos x$

$$y = y_c + y_p$$

$y = A \sin x + B \cos x - (\ln|tan\, x + sec\ x\,|) \cos x + c_1 \ \sin x + c_2 \ \cos x$

$y = C_1 \sin x + C_2 B \cos x - (\ln|tan\, x + sec\ x\,|) \cos x$, where$C_1 = c_1 \ + A$, $C_2 = c_2 \ + B$

Type 3: Linear differential Equations with Variable Coefficients

The Cauchy- Euler Equation

Any linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{(n-1)} y}{dx^{(n-1)}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

Where the coefficients $a_n, a_{n-1}, \dots \dots, a_0$ are constants, is known as an Euler equation or Cauchy- Euler Equation.

Theorem 7

The transformation $x = e^t$ reduces the equation,

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{(n-1)} y}{dx^{(n-1)}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

to a linear differential equation with constant coefficients.

Proof: We shall prove this theorem for the case of second order of Cauchy Euler differential equation,

$$a_2 x^2 \frac{d^2y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y = g(x) \quad [1]$$

Letting $x = e^t$ we have, $t = \ln x$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{1}{x}$$

$$\frac{d^2y}{dx^2} = \frac{d(\frac{dy}{dx})}{dx} = \frac{1}{x}\frac{d(\frac{dy}{dt})}{dx} + \frac{dy}{dt}\frac{d(\frac{1}{x})}{dx} = \frac{1}{x}\frac{d^2y}{dt^2}\frac{dt}{dx} - \frac{1}{x^2}\frac{dy}{dt} = \frac{1}{x}\frac{d^2y}{dt^2}\frac{1}{x} - \frac{1}{x^2}\frac{dy}{dt} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

Thus $x\frac{dy}{dx} = \frac{dy}{dt}$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

Substitute in equation [1]

$$a_2\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + a_1\left(\frac{dy}{dt}\right) + a_0 y = g(e^t)$$

$$a_2\left(\frac{d^2y}{dt^2}\right) + (a_1 - a_2)\left(\frac{dy}{dt}\right) + a_0 y = g(e^t)$$

This is a second order differential equation with constant coefficients.

Theorem 8 (General solution of Euler Equation)

The General solution of Euler Equation,

$ax^2y'' + bxy' + cy = 0$, (a and c real constants) in any interval not containing the origin is determined by the roots m_1 and m_2 of the equation

$$F(m) = am(m-1) + bm + c = 0$$

- If the roots are real and different , then $y = c_1x^{m_1} + c_2x^{m_2}$
- If the roots are real and equal , then $(c_1 + c_2 \ln x)x^{m_1}$
- If the roots are complex , then $y = x^{\alpha}(c_1cos(\beta lnx) + c_2sin(\beta lnx))$; where $m_1, m_2\ = \alpha \mp i\beta$.

Method

1. Assume a solution of the form $y = x^m$ where m is a constant to be determined.
2. Plug the assumed formula for y into the differential equation and simplify.
3. Solve the indicial equation for m.

Examples:

i. $x^2y'' - 2xy' - 4y = 0$
ii. $4x^2y'' + 8xy' + y = 0$
iii. $4x^2y'' + 17y = 0; y(1) = -1, y'(1) = -12$
iv. $x^3y''' + 5x^2y'' + 7xy' + 8y = 0$

i. $x^2y'' - 2xy' - 4y = 0$ [A]

Solution: Letting $x = e^t$ we have, $t = \ln x$

Thus $x\frac{dy}{dx} = \frac{dy}{dt}$

$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$

Substitute in equation [A]

$$\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - 2\left(\frac{dy}{dt}\right) - 4y = 0$$

$\frac{d^2y}{dt^2} - 3\left(\frac{dy}{dt}\right) - 4y = 0$ (B)

Let $y = e^{mt}$ [1]

$$\frac{dy}{dt} = me^{mt} \quad [2]$$

$$\frac{d^2y}{dt^2} = m^2e^{mt} \quad [3]$$

Substitute , [1],[2],[3] in (B)

$m^2e^{mt} - 3me^{mt} - 4\,e^{mt} = 0$

$e^{mt}(m^2 - 3m - 4) = 0$

Auxiliary equation,

$(m^2 - 3m - 4) = 0\ m = 4, m = -1$

$$y_1 = e^{-t} = e^{-lnx} = x^{-1}$$
$$y_2 = e^{4t} = e^{4lnx} = e^{\ln x^4} = x^4$$

$\therefore$ The genral solution is $y = c_1x^{-1} + c_2\,x^4$

Method 2. $x^2y'' - 2xy' - 4y = 0$ [A]

$$ax^2y'' + bxy' + cy = 0,$$

$$F(m) = am(m-1) + bm + c = 0$$

$$a = 1, b = -2, c = -4$$

$$m(m-1) - 2m - 4 = 0$$

$(m^2 - 3m - 4) = 0$ $\quad\quad m = 4, m = -1$

If the roots are real and different , then $y = c_1x^{-1} + c_2x^4$