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Elimination of stochasticity in stellarators

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A method for optimizing stellarator vacuum magnetic fields is introduced. Application of this method shows that the stochasticity of vacuum magnetic fields can be made negligible by proper choice of the coil configuration. This optimization is shown to increase the equilibrium β limit by factors of two or more over that of the simple, "straight" coil winding law. This method is general and ought to be applicable to other systems in which stochasticity (1) presents a problem, yet (2) is affected by the design parameters.

For purposes of plasma confinement, a stellarator should have vacuum magnetic fields which lie on toroidal surfaces and have rotational transform but have no island structure, stochastic regions, or coils inside the outermost flux surface.[1] While these properties are desired, they have not been fully obtained in present-day stellarators because of the lack of a technique for finding the appropriate coil designs. As a result, in typical stellarators, stochasticity causes a significant loss of vacuum flux surfaces in the outer region. In this letter we show that stochasticity can be made negligible in toroidal stellarators, while retaining other desirable features of the vacuum magnetic field, such as substantial rotational transform and inverse aspect ratio.

Magnetic field line flow is a Hamiltonian system with one and one-half degrees of freedom.[2,3] The Hamiltonian is given by the vector potential, which, in turn, is determined by the coils. If the field lines lie on flux surfaces, the equivalent Hamiltonian system is integrable. Whether nonaxisymmetric systems can have dense, toroidal flux surfaces is a longstanding problem. As stated by Grad in 1967 (Ref. 2, p. 137), "... it is extremely difficult to find acceptable toroidal equilibria..., except for certain configurations of great geometrical symmetry... ." In general terms, one is given a class of Hamiltonians (as generated by coils), which have no obvious symmetry, and, therefore, no obvious integral of the motion. The problem is to select the Hamiltonian from this class that is most nearly integrable (i.e., stochastic motion is minimized).

The fact that magnetic field line flow is a Hamiltonian system has many useful consequences. In particular, the destruction of flux surfaces[4,5] fits into the paradigm of resonance overlap.[6] In Refs. 4 and 5, a method for splitting the vacuum field into an integrable part plus a perturbation was developed. This splitting allowed one to identify the stochasticity inducing perturbations and to calculate the magnetic field changes needed to counteract them. However, this calculation led to only modest improvements because it was perturbative, and it could be carried only through first order given available computers. Furthermore, this calculation was limited in that it did not directly yield a coil winding law.

In this letter we describe a method for finding toroidal vacuum magnetic fields without significant islands or stochastic regions within the separatrix. This method is direct; it produces an actual coil winding law. Moreover, it is numerically implemented, and it does not require excessive computer time. For example, the complete optimization of the case shown here required roughly 30 minutes of computing time on a CRAY 1. Applications have yielded nearly integrable vacuum fields with large increases in rotational transform and/or plasma volume. This translates to an increase in the equilibrium β limit by a factor of two or more.

Simply put, the method is to minimize certain measures of nonintegrability with respect to the parameters specifying the Hamiltonian. Of course, minimization is often used in stellarator design. The important point of the present contribution is the definition of nonintegrability measures that make the minimization of nonintegrability practicable.

Describing the nonintegrability measures requires an understanding of the return map. In cylindrical geometry magnetic field lines are curves $[R(\phi), Z(\phi)]$ that satisfy the differential equations

$$\frac{dR}{d\phi} = \frac{B^R}{B^\phi}, \quad \frac{dZ}{d\phi} = \frac{B^Z}{B^\phi}, \tag{1}$$

where (B^R, B^ϕ, B^Z) are the contravariant components of the magnetic field. The return map, $\overline{R}(R,Z)$ and $\overline{Z}(R,Z)$, is obtained by integrating Eqs. (1) from $\phi = 0$ to $\phi = \phi_p$ with initial conditions R,Z, where the field period ϕ_p is defined by $\mathbf{B}(R,Z,\phi) = \mathbf{B}(R,Z,\phi + \phi_p)$. For convenience we refer to (R,Z) collectively as X and the mapping as $\overline{X} = N(X)$.

The motion of neighboring orbits is described by the tangent map. The tangent map T is found by linearization:

$$N(X + \delta) = N(X) + T\delta + O(\delta^2). \tag{2}$$

With this definition, T is a linear operator and can be represented as a 2×2 matrix with components, $T_{ij} = \partial\overline{X}_i/\partial X_j$.

Associated with the islands of the return map are stable and unstable fixed points. A fixed point X_0 of order q satisfies $N^q(X_0) = X_0$. The tangent map describes the motion in the vicinity of a fixed point. With the definition, $\bar{\delta} \equiv N^q(X_0 + \delta) - N^q(X_0)$, one has

$$\bar{\delta} = T^q\delta + O(\delta^2), \tag{3}$$

where

$$T^q \equiv T(N^{q-1}(X_0)) \cdots T(N(X_0))T(X_0). \tag{4}$$

The mapping (3) can be characterized by the eigenvalues of T^q. Since the system of Eq. (1) is Hamiltonian, the map N is measure preserving. Therefore, the determinant of

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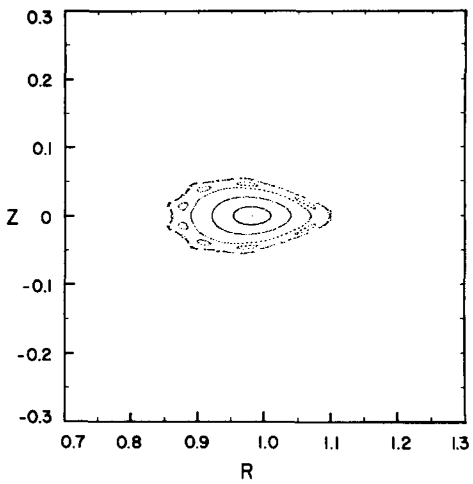

FIG. 1. Surface of section for an unoptimized stellarator with $l_0 = 2$ and $m_0 = 5$.

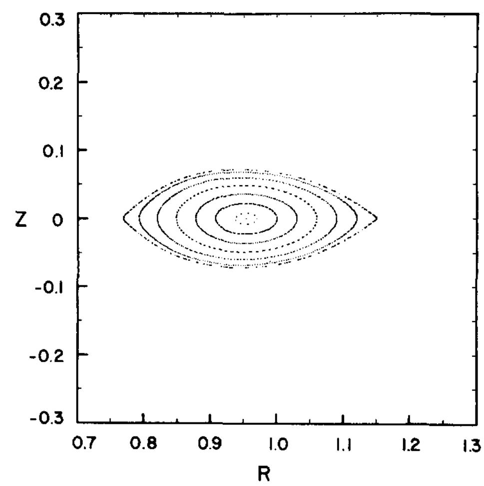

FIG. 2. Surface of section for the optimized stellarator. The amplitudes of the harmonics of the coil winding law are given in Table I.

T^q is unity at a fixed point. Thus the eigenvalues of T^q depend only on the trace of T^q.

Following Greene,[7] we introduce the residue

$$\mathscr{R} \equiv \tfrac{1}{2} - \mathrm{Tr}(T^q)/4. \tag{5}$$

When the eigenvalues are unity, the residue vanishes. In particular, for an integrable system without island structure the fixed points on a rational surface have zero residue. This leads us to an empirical rule: to make a system more integrable, one should minimize the residues of the fixed points. Naturally, this idea is directly applicable to any class of mappings $N(X;p_j)$ which depend continuously on a set of parameters $\{p_j\}$.

In the stellarator case the map is parameterized by the coil currents and the parameters of the coil winding law. For the examples discussed below, the coils are wound on a torus with major radius $R_0 = 1$ and minor radius $r = 0.3$. The coil curve is given by specifying a relation between the toroidal angle ϕ and poloidal angle η, which is defined to measure ordinary angle in the poloidal plane about $(R = 1, Z = 0)$ with $\eta = 0$ on the outside of the torus and (r,η,ϕ) having right-handed orientation. In the present case, the relation between η and ϕ is given by

$$\eta = \frac{m_0\phi}{l_0} + \sum_{k=0}^{\infty}\left[A_k \cos\left(\frac{km_0\phi}{l_0}\right) + B_k \sin\left(\frac{km_0\phi}{l_0}\right)\right]. \tag{6}$$

The quantity m_0 is the number of field periods, and l_0 is the dominant poloidal field harmonic number. The quantities A_k and B_k are the free parameters used in the minimization. The total magnetic field is produced by superimposing a purely toroidal field, $\mathbf{B}_T = B_0\hat{e}_\phi/R$, upon the field produced by the helical coils.

The numerical techniques used to determine the map and the residues were straightforward. The coils were represented by filamentary straight line segments with endpoints lying on the curve specified by Eq. (6). The magnetic field and its derivatives were calculated via the Biot–Savart law. The field line equations were integrated numerically to obtain the return map and the tangent map. Fixed points were found by Newton's method.

Figure 1 shows a surface of section for an unoptimized stellarator with $l_0 = 2$, $m_0 = 5$, $B_0 = 1$, and two helical coils. The first coil has current $I = -0.021$ in unrationalized units with the speed of light equal to unity. For this coil $A_k = B_k = 0$ for all k. The second coil has $I = 0.021$, and $A_k = B_k = 0$ for $k \neq 0$, but $A_0 = \pi/2$. Near the magnetic axis the rotational transform per field period is $\omega = 0.116$. Moving away from the magnetic axis, the rotational transform increases out to the last closed surface where $\omega_{\max} = 0.133$. Just inside this surface there are sizable eighth-order islands.

This configuration was optimized by minimizing the residues of the stable and unstable order 3, 4, and 5 fixed points with rotation numbers 1/3, 1/4, and 1/5. The parameters B_1, B_2, and B_3 of the first coil and A_1, B_2, and A_3 of the second were varied. These parameters were chosen in order to retain an up-down symmetry $(Z \to -Z)$ in the surface of section at $\phi = 0$. This symmetry facilitates the location of fixed points.

Figure 2 shows a surface of section for the optimized stellarator. The coil currents are unchanged. The coil parameters are given in Table I. Near the magnetic axis the rotational transform per field period is $\omega = 0.111$. Moving away from the magnetic axis, the rotational transform increases out to the last closed surface where $\omega_{\max} = 0.3$.

The optimized field is substantially more integrable than the unoptimized field. The primary islands of orders 4, 5, 6, and 7 have been made negligibly small, so that good flux surfaces exist outside of them. The order 8 islands that are sizable in Fig. 1 are not even visible in Fig. 2. The outermost flux surface is much closer to the separatrix as evidenced by the sharp corners at $Z = 0$.

In addition to being more nearly integrable, the optimized field has improved plasma properties. A rough estimate of the equilibrium β limit for a stellarator[8] is

$$\beta_{\mathrm{eq}} \approx \langle \iota^2 \rangle \epsilon, \tag{7}$$

where $\langle \iota^2 \rangle$ is the volume average of the square of the rotational transfrom $\iota = m_0\omega$, ϵ is the inverse aspect ratio, and β_{eq} is the equilibrium β limit. We use $\epsilon \equiv (A/\pi)^{1/2}/R_a$, where

TABLE I. Amplitudes of the harmonics of the coil-winding law for the optimized stellarator having the surface of section shown in Fig. 2.

	Coil 1	Coil 2
A_0	0	$\pi/2$
A_1	0	0.224 859
A_2	0	0
A_3	0	− 0.000 856
B_1	0.243 960	0
B_2	0.026 240	− 0.026 490
B_3	0.000 856	0

A is the cross-sectional area enclosed by the last flux surface, and R_a is the radius of the magnetic axis. For the unoptimized stellarator one finds $\beta_{eq} \lesssim 3\%$; for the optimized stellarator one obtains $\beta_{eq} \approx 5.8\%$. This increase is due to increases of both $\langle \iota \rangle^2$ and ϵ.

The sensitivity of these solutions to coil-winding errors does not appear to be too great. A simple estimate based on the derivatives of the residues with respect to the parameters indicates that an absolute error of approximately 0.002 in the amplitudes of Table I will just destroy the flux surface between the order 4 and order 5 fixed points. However, this effect is not too drastic since these fixed points are near the separatrix. The resulting loss of area would be on the order of 10%, and β_{eq} would be reduced to approximately 5.0%.

This research has shown that one can obtain nearly integrable toroidal stellarators by varying the parameters that define the coil-winding law in order to minimize the residues of fixed points of the return map. Possible future work consists of combining negligible stochasticity with other criteria, such as maximal[9] β_{eq}, minimal transport,[10] retention of flux surfaces in the presence of plasma pressure, and plasma stability. As to stability, we mention that we have been able to obtain optimized configurations with a magnetic well. (Such systems will be magnetohydrodynamically stable for low plasma pressure.)

These methods may help in the design of other systems in which stochasticity presents problems. As possible examples, we mention the radial transport in mirror machines caused by nonaxisymmetric fields[11] and the loss of luminance in colliding beam storage rings due to the beam–beam interaction.[12]

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Coherent nonlinear destabilization of tearing modes

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The linear stability of a helical quasi-equilibrium containing large-amplitude $q = 2$ magnetic islands in a cylinder is investigated. Three-dimensional tearing modes are driven unstable by the strong current gradient associated with this helical equilibrium.

Major disruptions in tokamaks are characterized by a rapid decrease in the toroidal current leading to termination of the discharge. In current theoretical models of this phenomenon, the disruption is initiated by the growth of a large-amplitude $m/n = 2/1$ tearing mode, where m and n are the poloidal and toroidal mode numbers, respectively.[1] When the magnetic island associated with this mode is large enough to overlap the original rational surfaces of the 3/2 and 5/3 modes, these shorter wavelength modes are strongly destabilized.[2] The rapid growth of these secondary modes