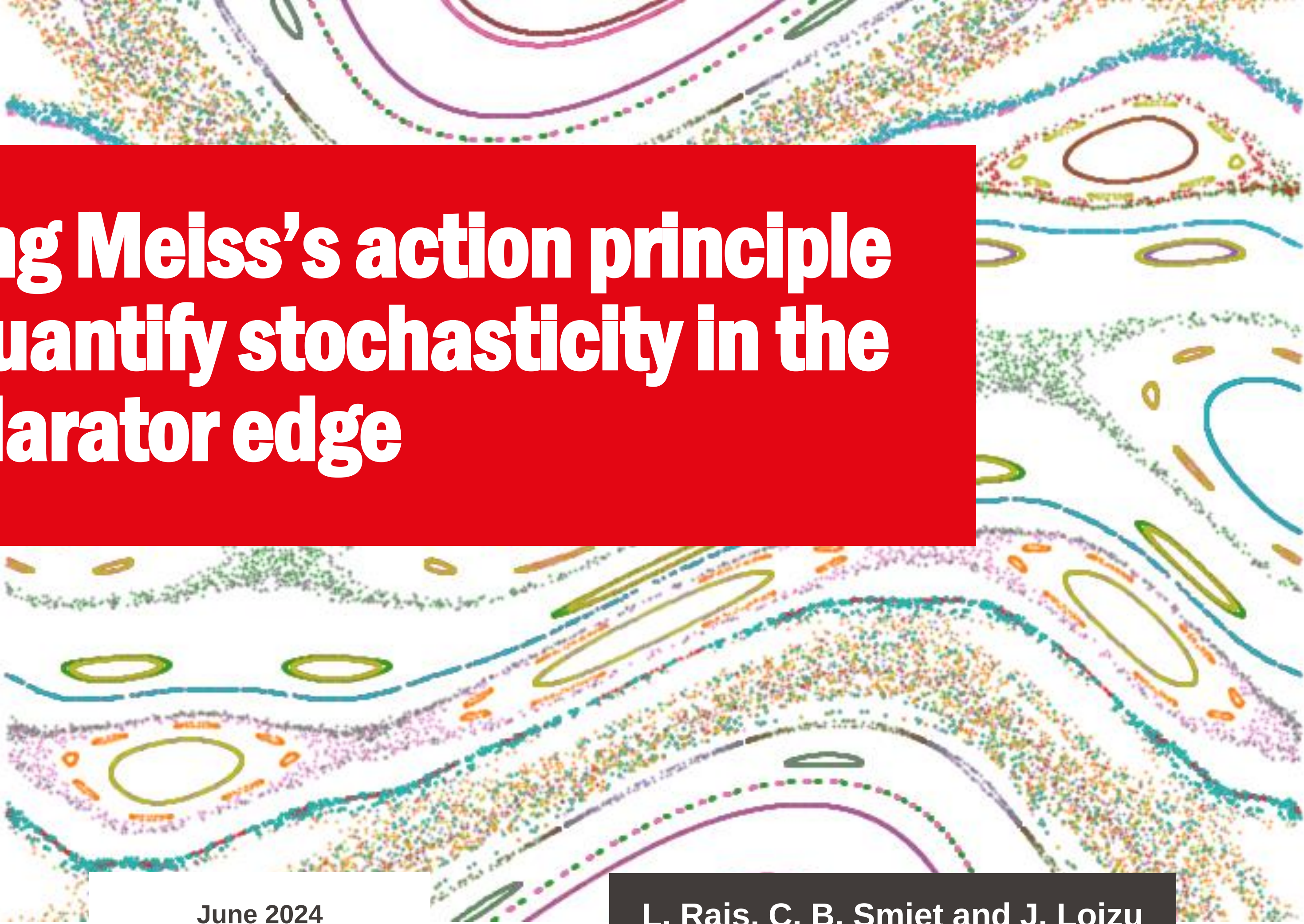


Using Meiss's action principle to quantify stochasticity in the stellarator edge

June 2024

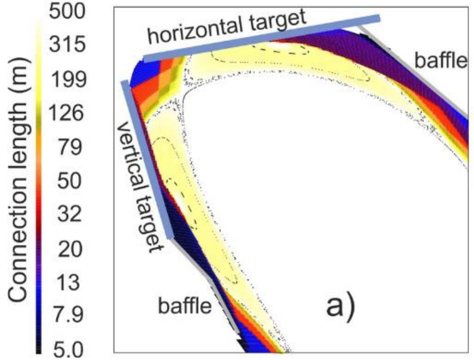
L. Rais, C. B. Smiet and J. Loizu

Ecole Polytechnique Fédérale de Lausanne

There are different divertor concepts with more or less chaos

- The magnetic edge structure relates to the strike points, thus the heat flux, on divertor
- Island/Resonant divertors, use low order islands, sensitive to the iota profile
- Non-resonant divertor ? Trough regions
- Ergodic divertor, creating controled chaos in the edge (TEXTOR, WEST)

Island divertor



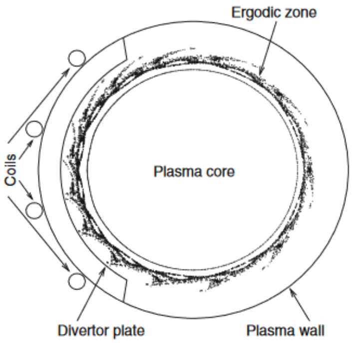
Feng and W7-X-team, 'Review of Magnetic Islands from the Divertor Perspective and a Simplified Heat Transport Model for the Island Divertor'.

NRDs ?

Fig. 9.1 Schematic view of the section of an ergodic divertor tokamak

Abdullaev, Magnetic Stochasticity in Magnetically Confined Fusion Plasmas. Ch.9.1

Ergodic divertor



EPFL

Chaotic configurations in W7X revealed by tweaking the currents

- Default configuration :

5/5 island chain and is representative of a resonant divertor concept

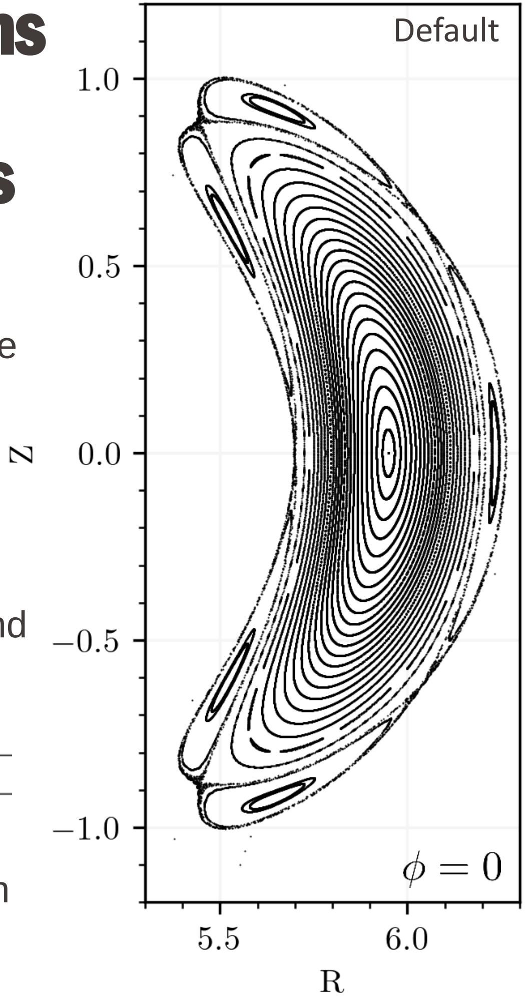
- GYM00-1750 (***Robert Davies***) :

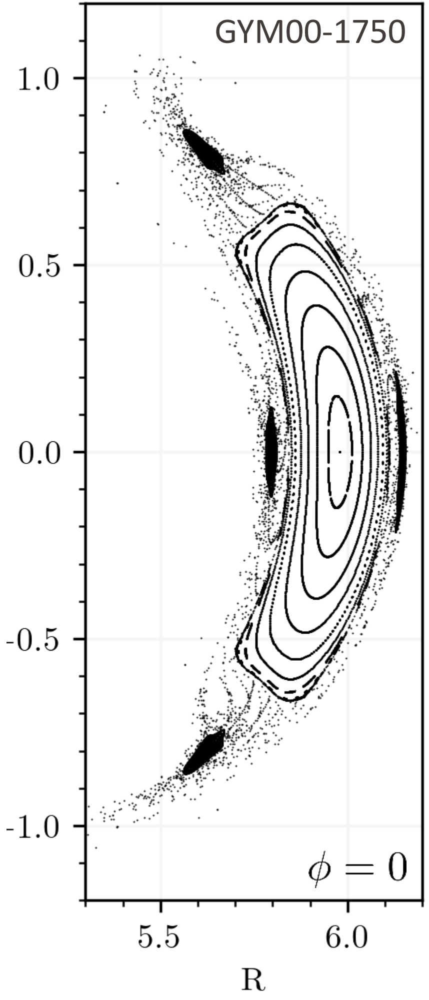
Chaotic edge and a 5/4 island explores the non-resonant divertor concept.

- The current values for modular and planar coils are

Configuration	modular	planar
Default [MA]	1.62	0.
GYM00-1750 [MA]	1.1095	-0.3661

- How to quantify the stochasticity in the edge ?





Ludovic Rais

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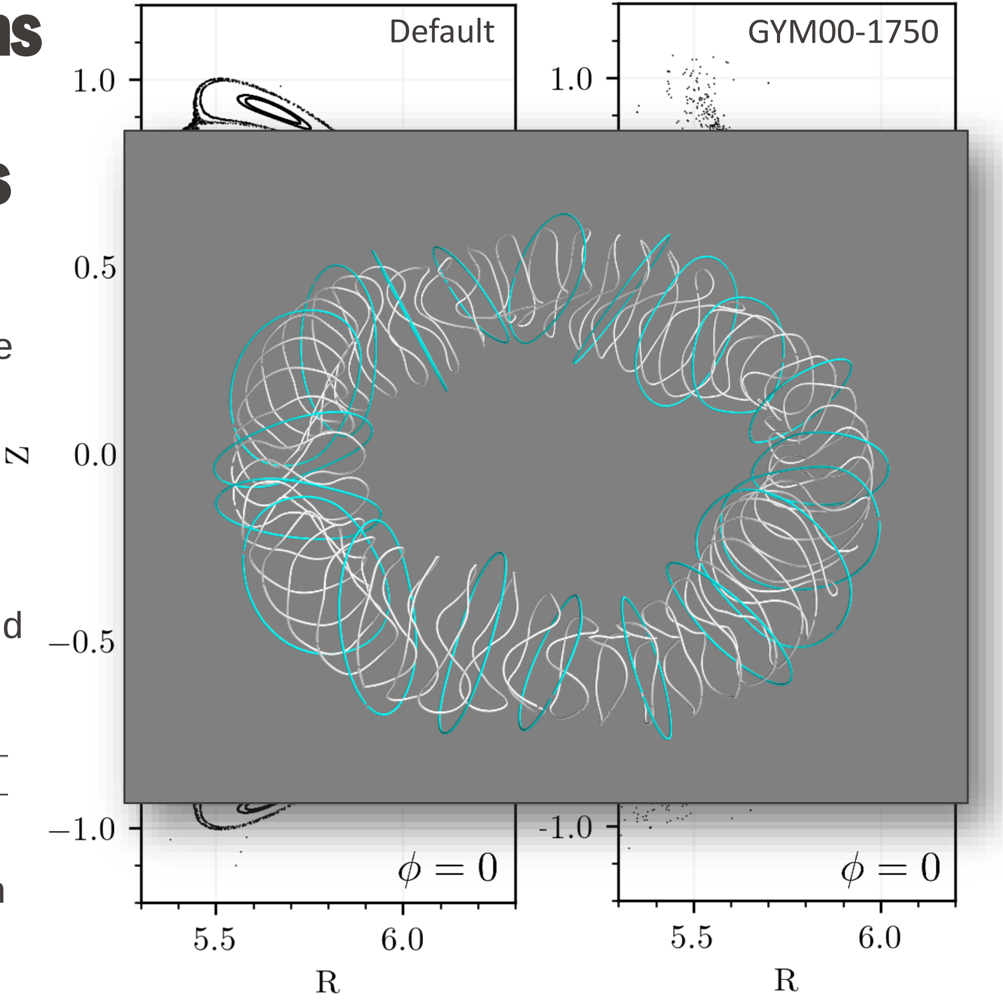

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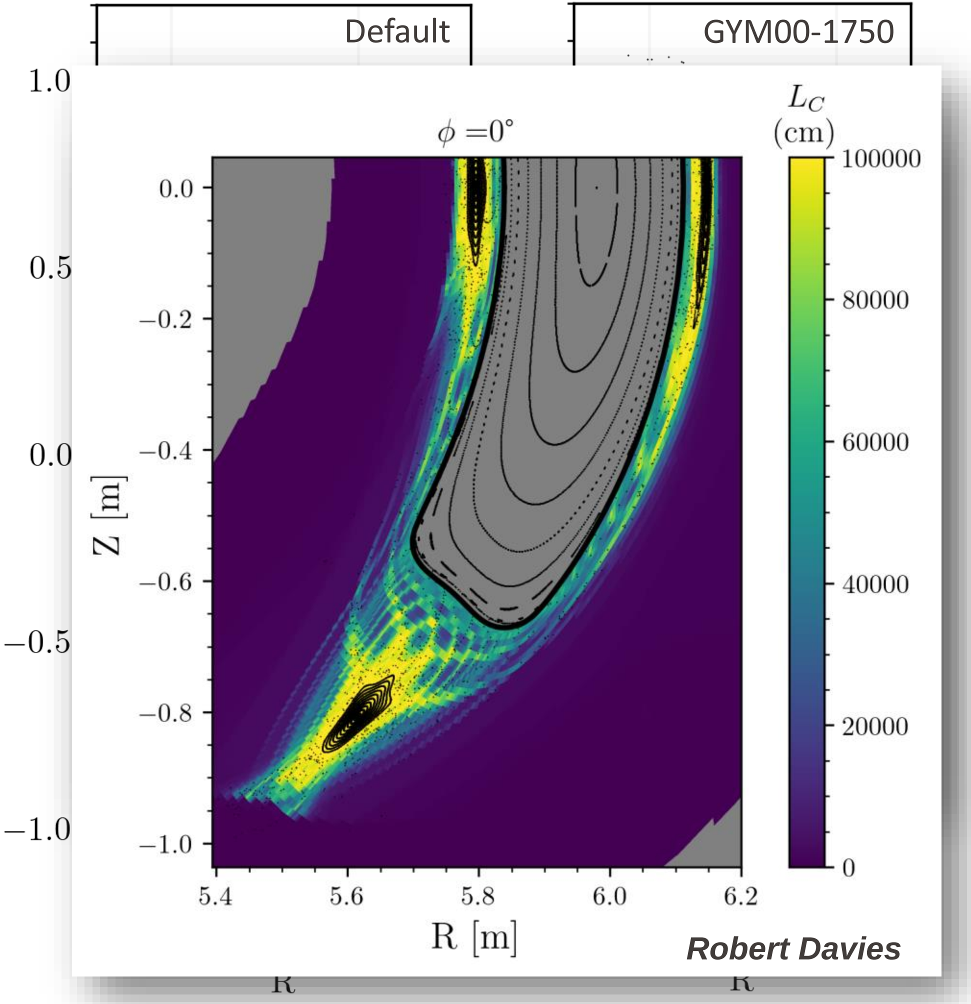


Robert Davies

Poincare map sends points from ϕ_i to $\phi_i + T$ and conserves the flux of B

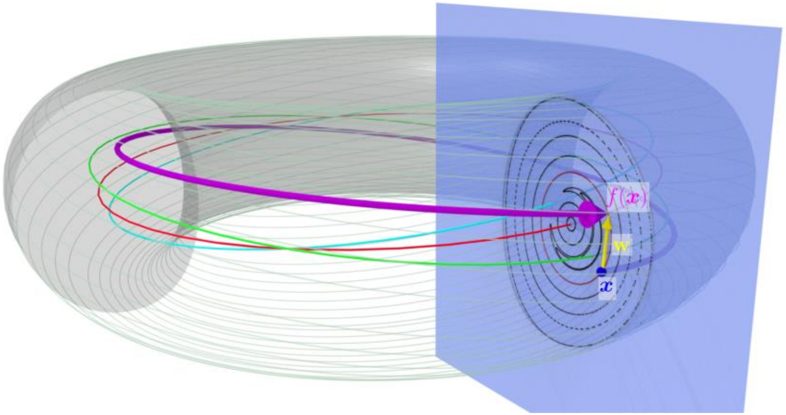
- Following the magnetic field and recording their intersection with constant ϕ planes
- The poincare map sends points from the ϕ_i section to the $\phi_i + 2\pi/n_{fp} = \phi_i + T$ section

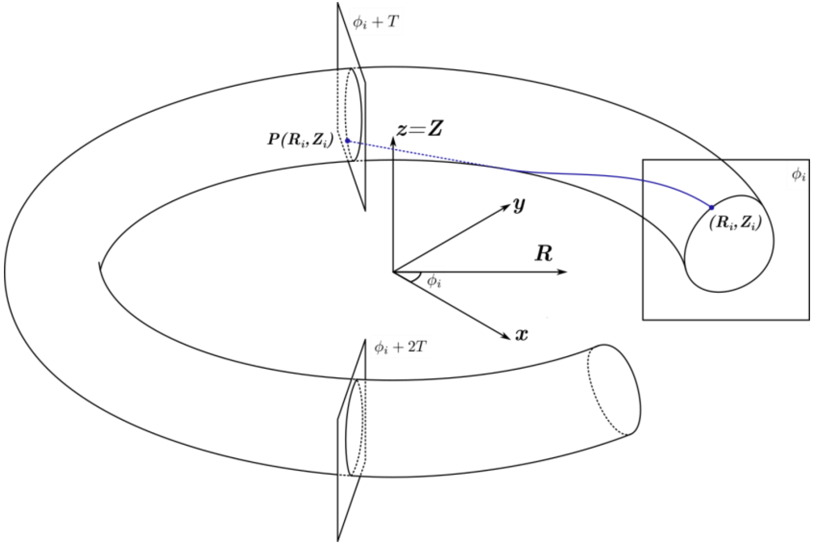
$$\begin{pmatrix} R \\ Z \end{pmatrix} \mapsto \mathcal{P}(R,\, Z) = \int_{\phi_i}^{\phi_i+T} \begin{pmatrix} B^R/B^\phi \\ B^Z/B^\phi \end{pmatrix} ds + \begin{pmatrix} R \\ Z \end{pmatrix}$$

- As the magnetic field is divergence free, the poincare map preserves the flux (the flux form $d\Omega = B^\phi dr \wedge dz$)

$$\nabla \cdot \mathbf{B} = 0 \implies \iint_A \mathbf{B} \cdot \mathbf{dS} = \iint_{\mathcal{P}(A)} \mathbf{B} \cdot \mathbf{dS}$$

Smiet, C. B., G. J. Kramer, and S. R. Hudson. "Mapping the sawtooth."

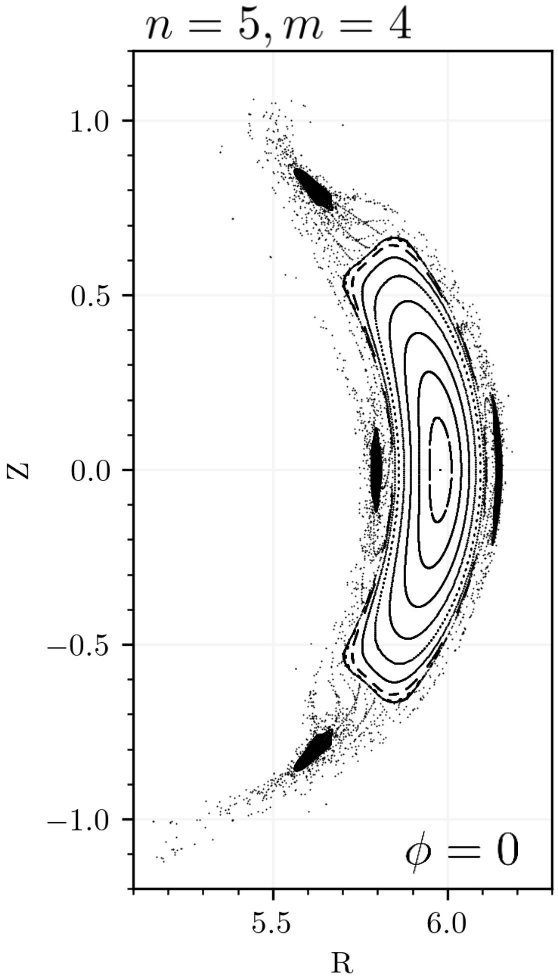




Ludovic Rais

Fixed points of $\mathcal{P}^m$ are O/X points of $\iota = n/m$ magnetic islands

- A fixed point of $\mathcal{P}$ of order k is such that
$$\mathcal{P}^k(x^\star) = x^\star$$
- Helicity $\iota = n/m$ gives the fraction of poloidal turns around the axis parcoured after one toroidal turn
- An X or O-point of an n/m island (chain) comes back to itself after m toroidal turns, thus $k = m\, n_{fp}$
- There are m intersections with a constant ϕ-section and using the field periodicity $k = m$



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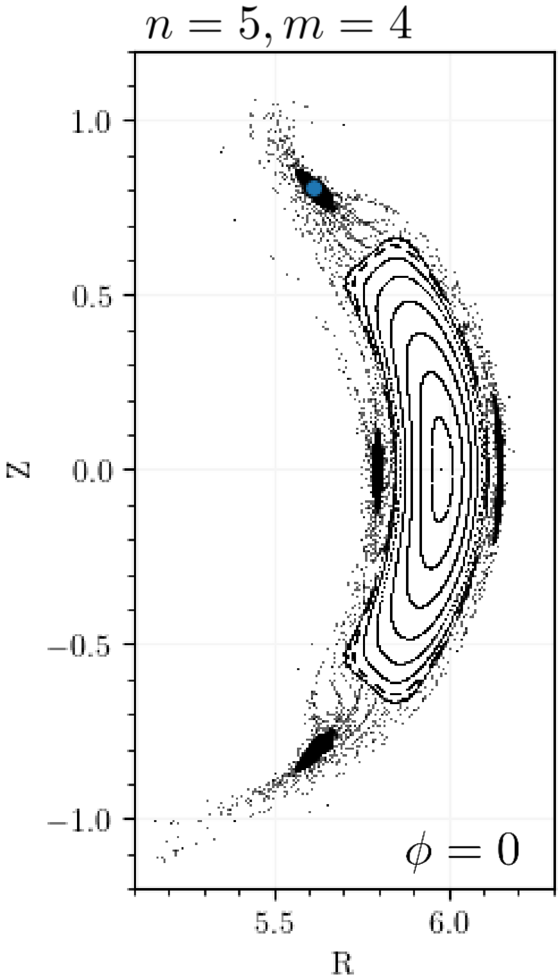

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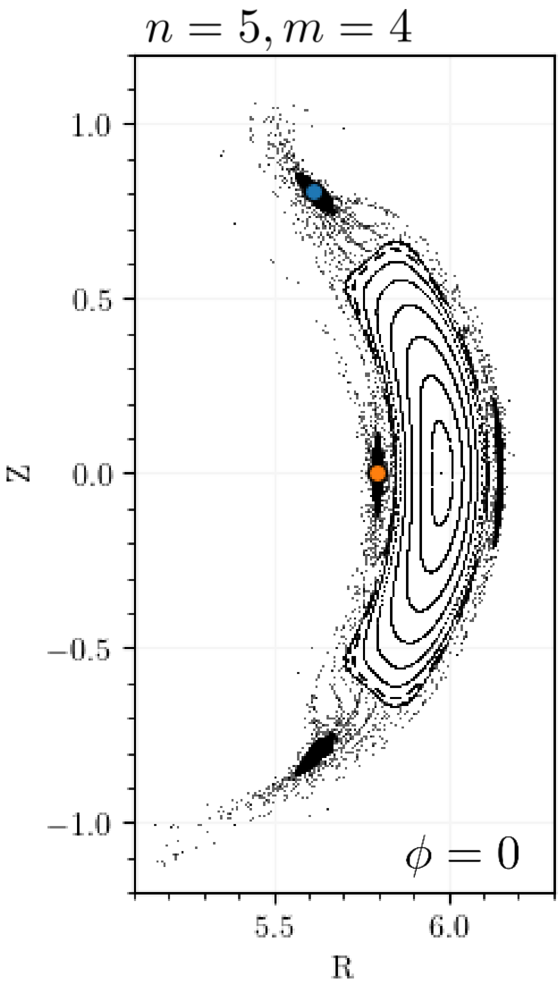

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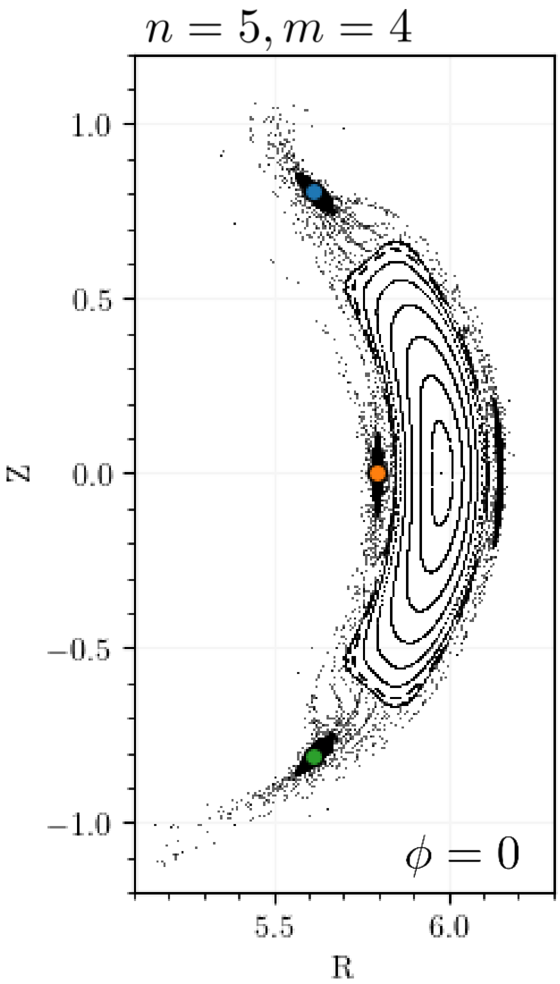

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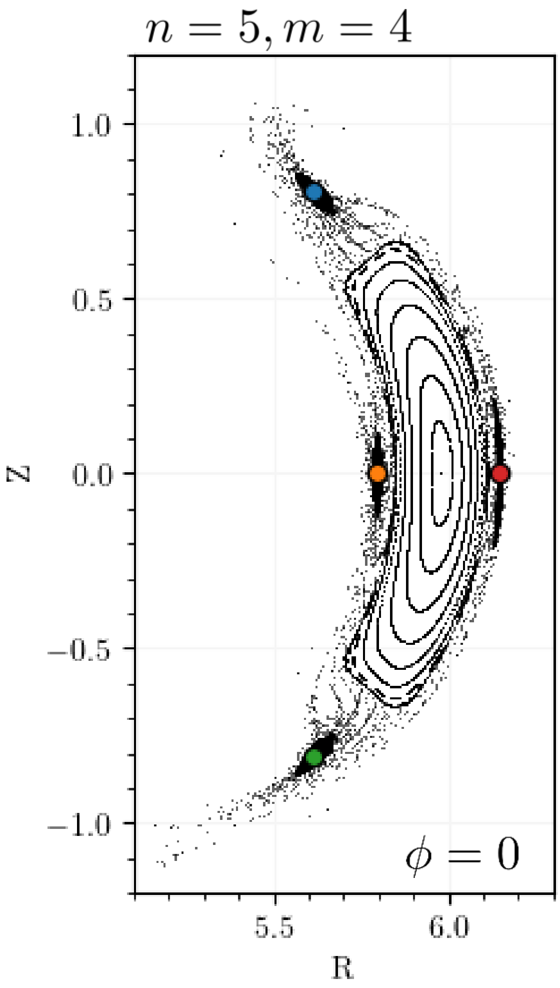

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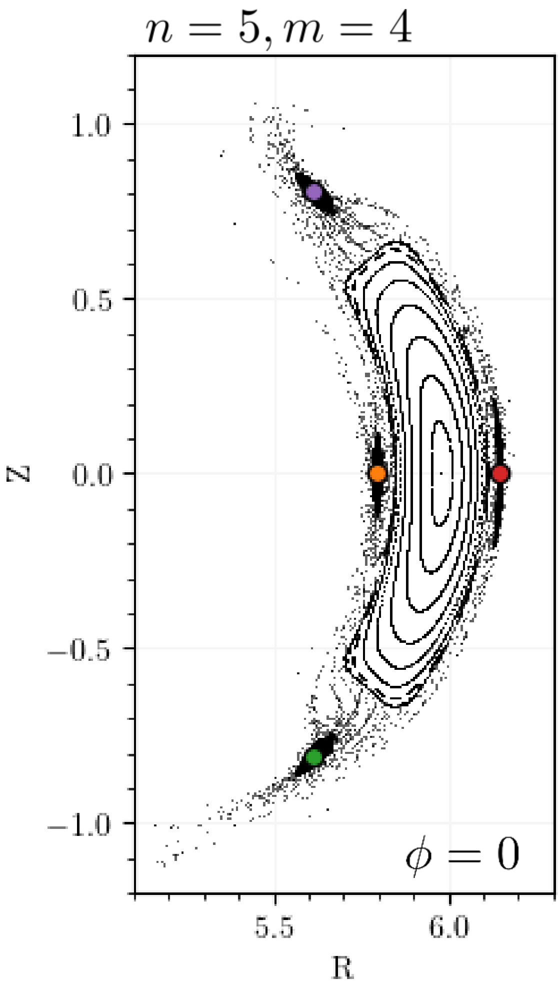

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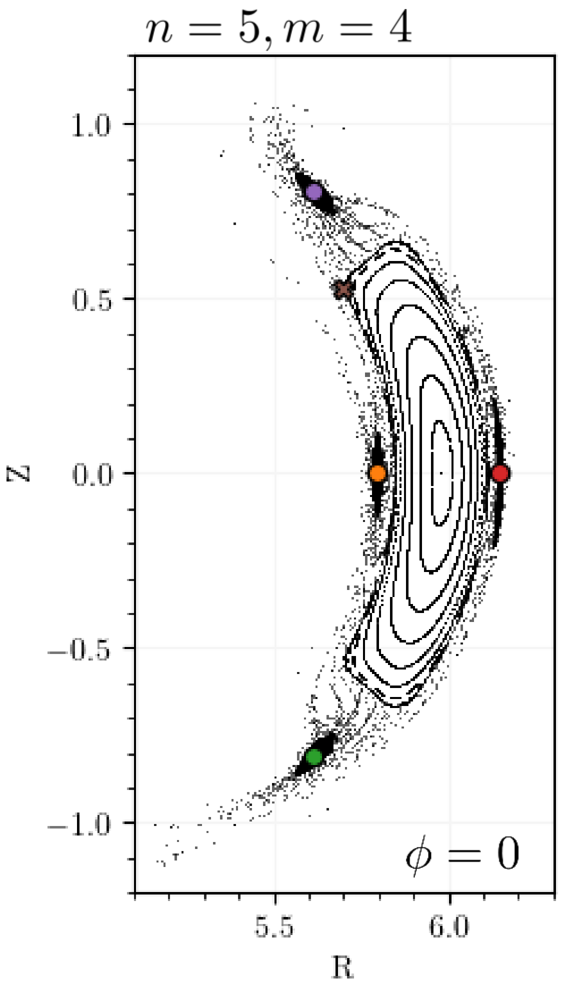

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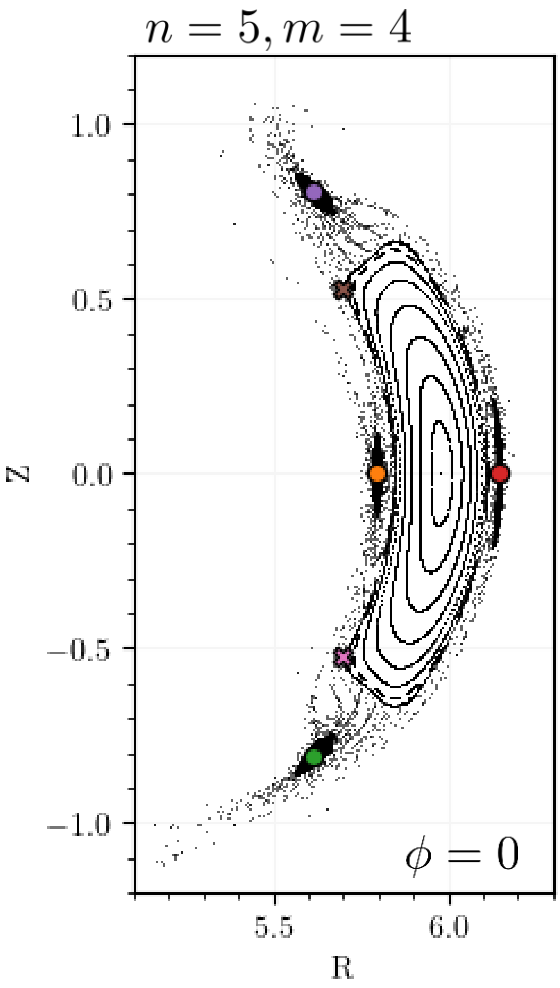

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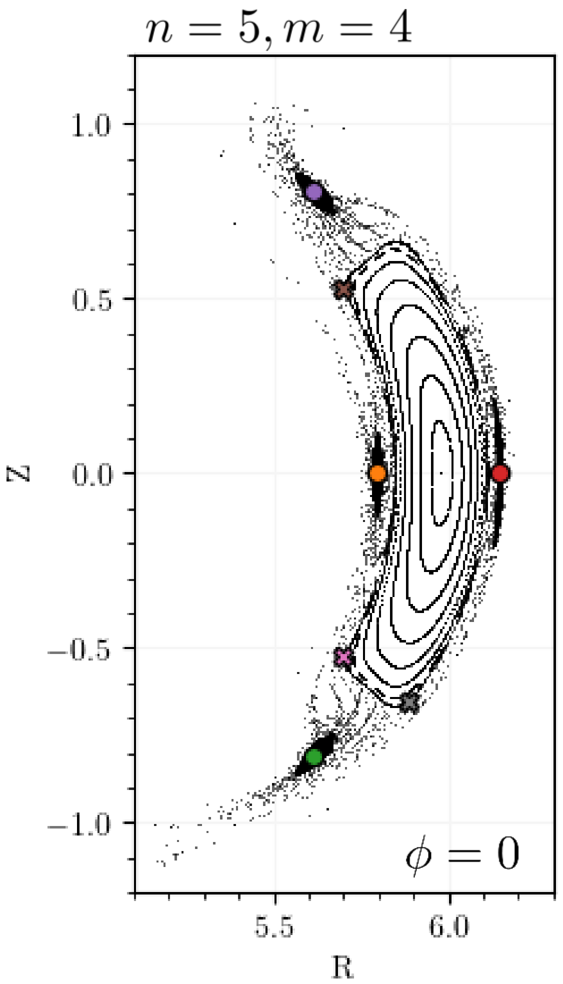

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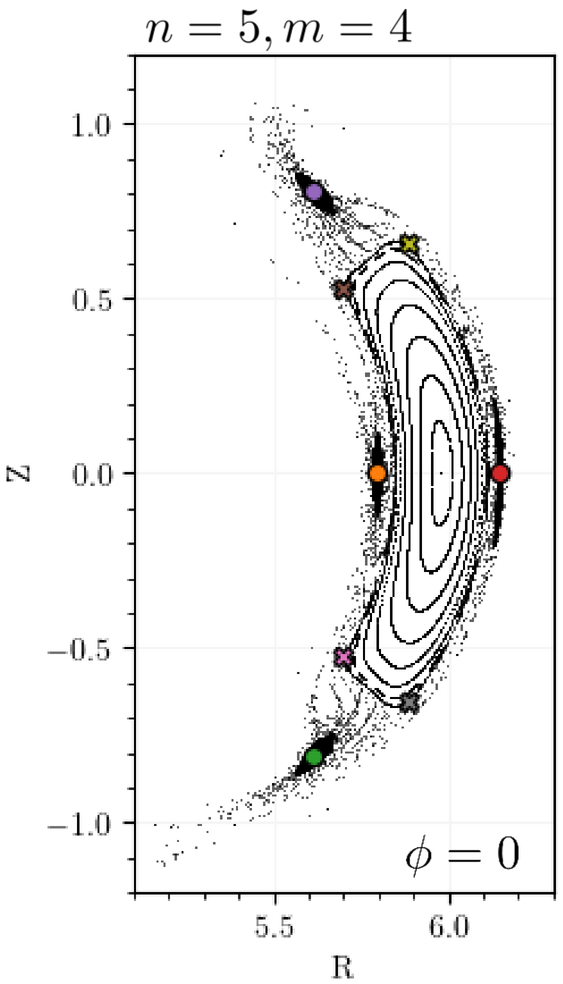

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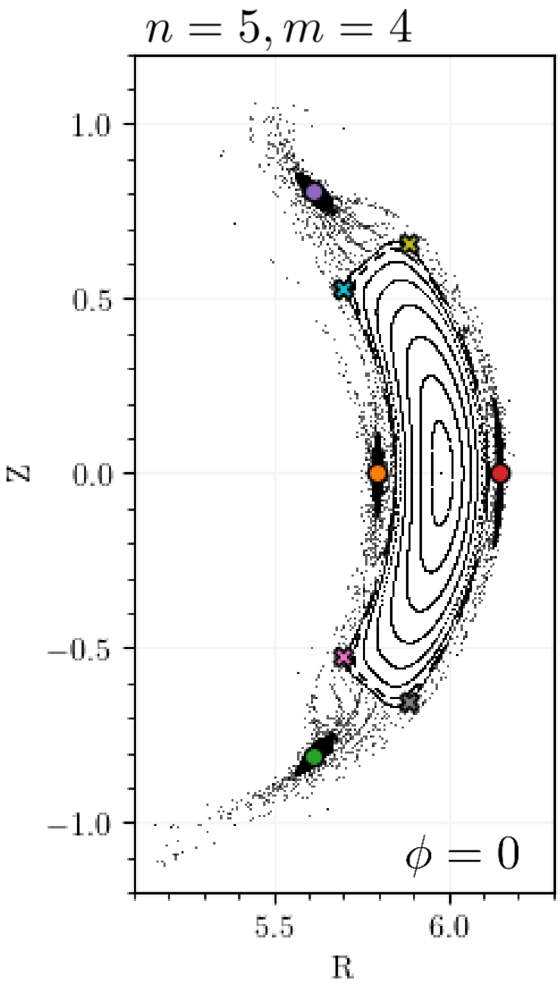


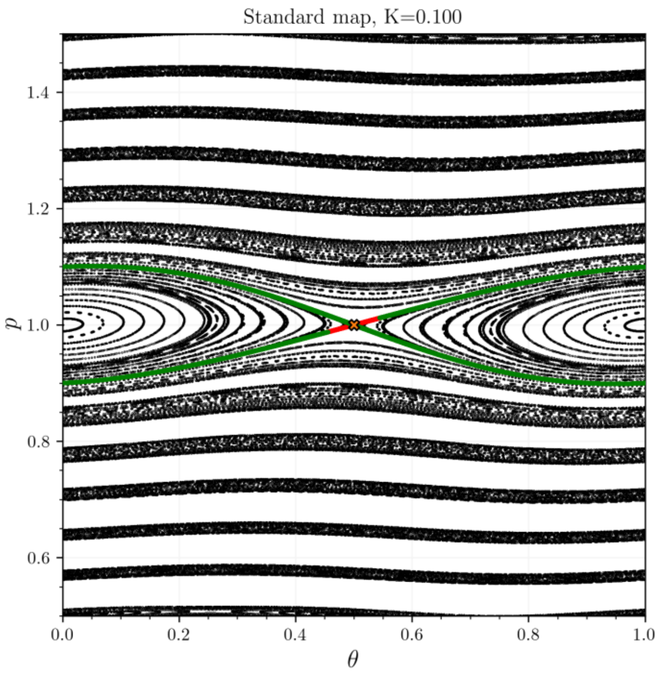
Separatrix of an island can break and indicates Chaos by brokeness

- Poincare map can be difficult to study because it requires field line tracing which is slow
- Well studied Chirikov standard map on the cylinder

$$p_{n+1} = p_n - \frac{K}{2\pi}\sin(2\pi\theta_n)$$
$$\theta_{n+1} \equiv \theta_n + p_{n+1} \quad (\mathrm{mod}\ 1)$$

Meiss, 'Thirty Years of Turnstiles and Transport'.

- It preserves the area directly, similar to the poincare map which preserves the flux.
- As the parameter K increases, the system becomes more chaotic, playing the role of a perturbation.
- The separatrix divides in two



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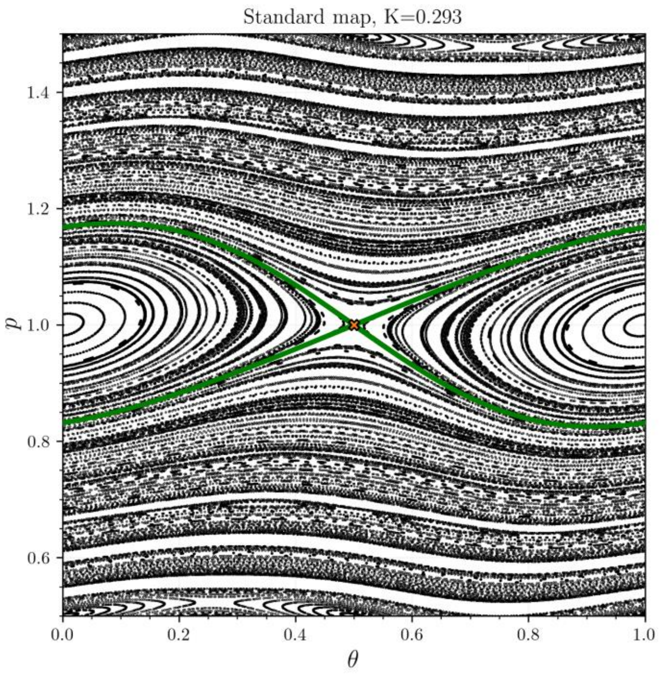

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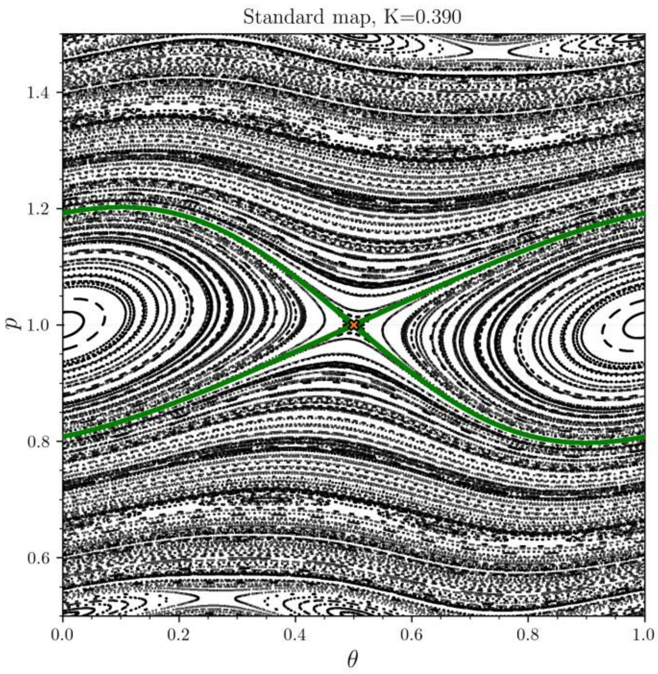

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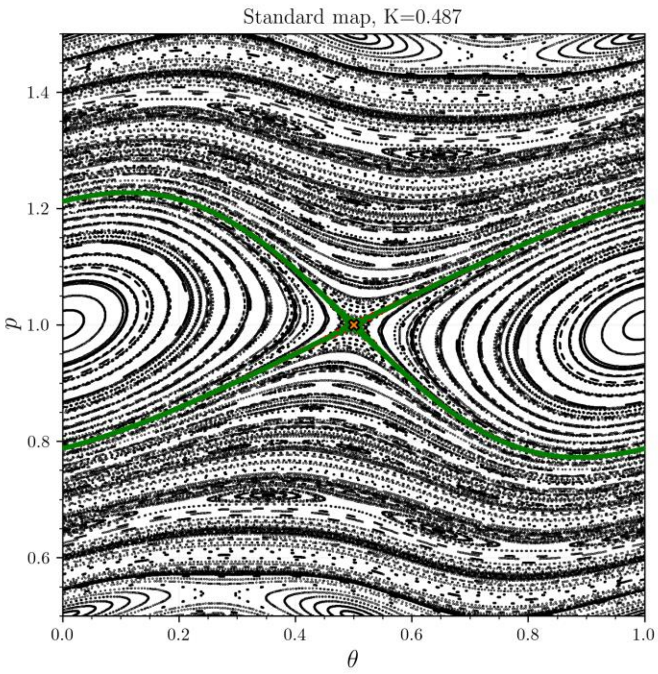

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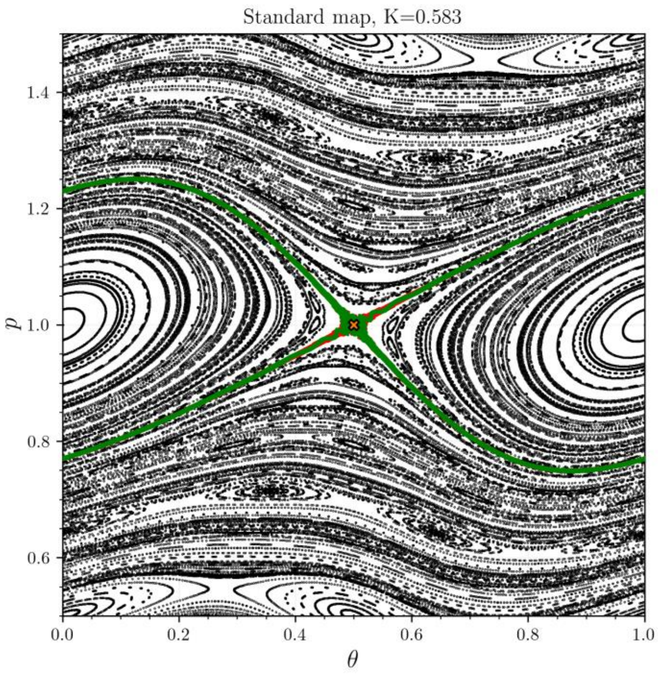

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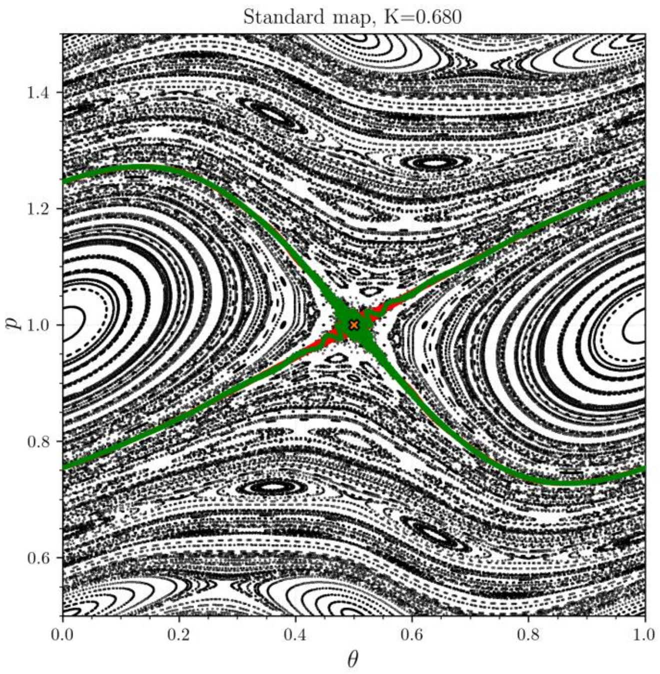

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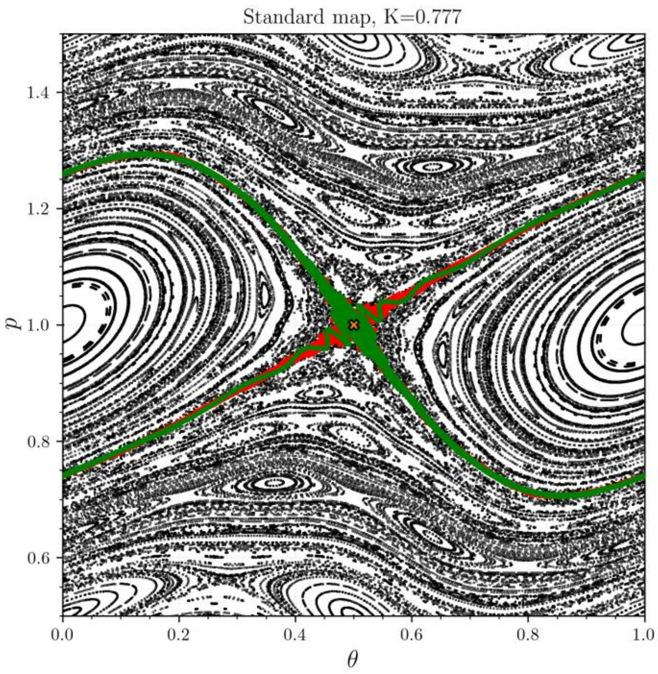

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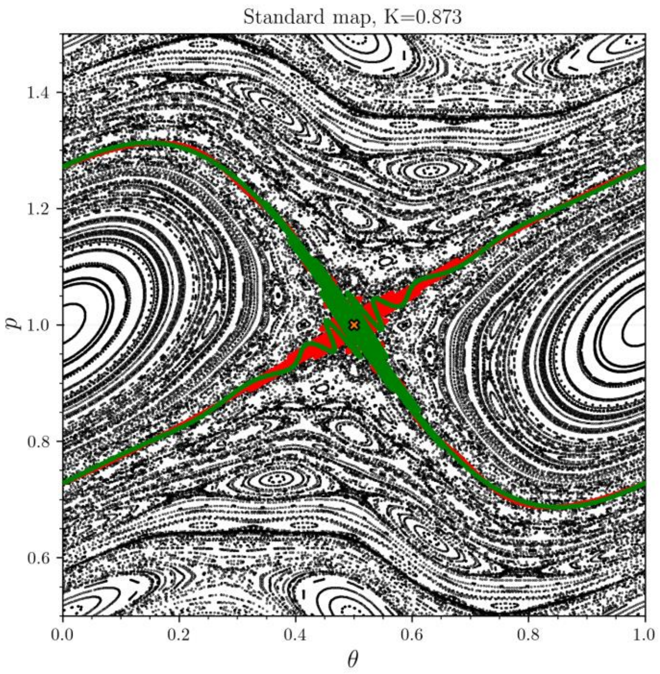

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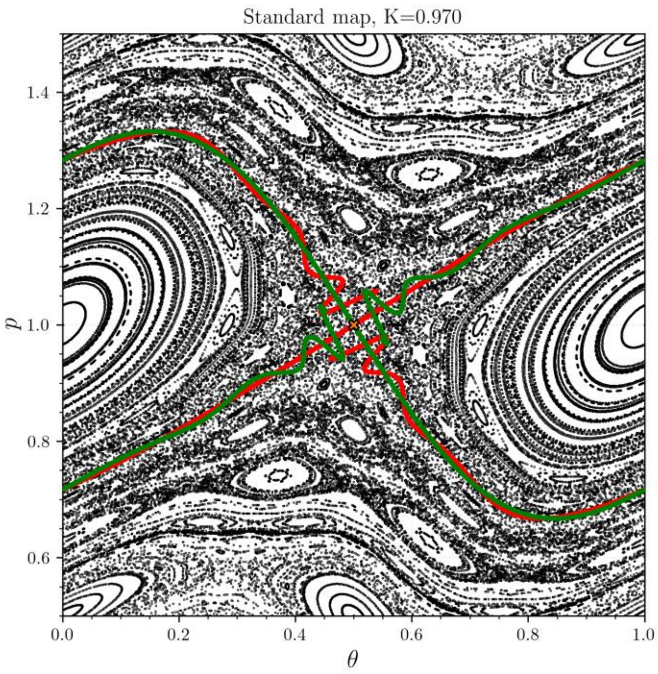
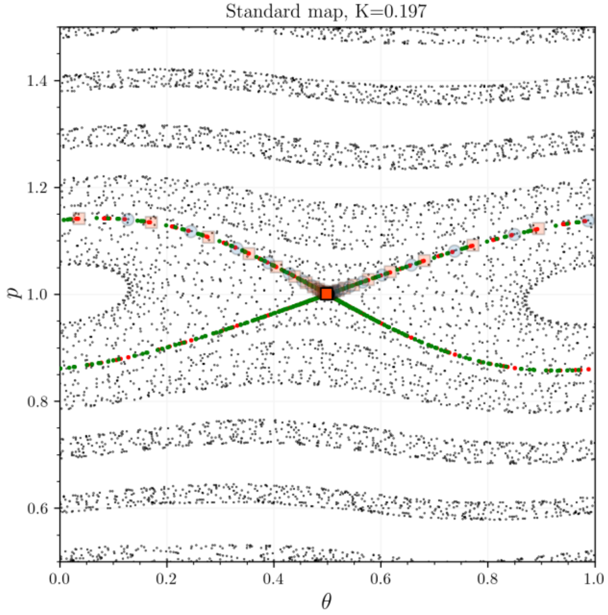


Hyperbolic fixed points have a stable and unstable manifold

- The jacobian indicates the behaviour of nearby lines
- Calling the standard map F, and due to the area conservation, the determinant of its jacobian has to be one

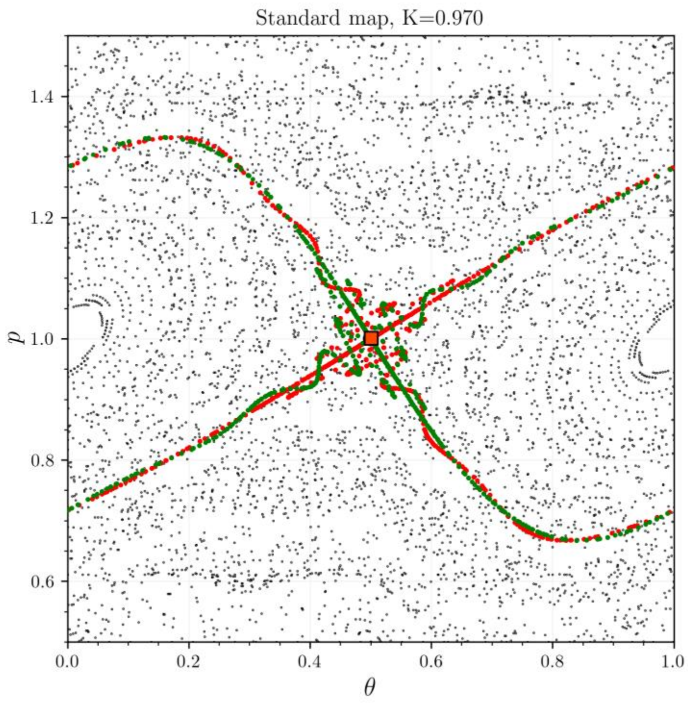
$$J_F(\theta,p) = \begin{pmatrix} \frac{\partial F^\theta}{\partial \theta} & \frac{\partial F^\theta}{\partial p} \\ \frac{\partial F^p}{\partial \theta} & \frac{\partial F^p}{\partial p} \end{pmatrix} \qquad \det(J_F) = \lambda_1 \lambda_2 = 1$$

- For an elliptic point (O-points), λ_1, λ_2 are complex conjugate and indicate rotation
- For a hyperbolic point (X-points), there is a stable ($|\lambda_s| < 1$) and unstable ($|\lambda_u| > 1$) direction
- Initializing a point $x = x^* + \varepsilon_u \boldsymbol{e}_u$ and apply the map
- Initializing a point $x = x^* + \varepsilon_s \boldsymbol{e}_s$ and apply the inverse map



Unstable and stable manifold create a tangle structure when choatic

- Manifolds are
  - Stable W^s : points converging to x^* as $t \rightarrow +\infty$
  - Unstable W^s : points converging to x^* as $t \rightarrow -\infty$
- When the chaos appear, the unstable and stable manifold that were creating the separatrix separates
- As we move on the unstable/stable manifold back towards the X-point they gets strechted out
- The interconnection creates a grid structure with infinite number of intersections, resembling a trellis (*H. Poincaré, Les méthodes nouvelles de la mécanique céleste*. *Ch.31*)



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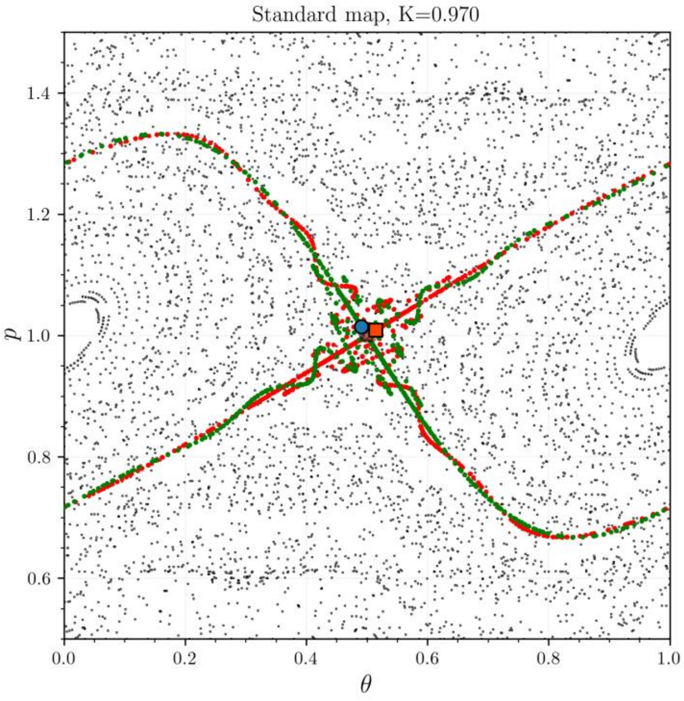

Unstable and stable manifold create a tangle structure when choatic

- Manifolds are
  - Stable W^s : points converging to x^* as $t \rightarrow +\infty$
  - Unstable W^s : points converging to x^* as $t \rightarrow -\infty$
- When the chaos appear, the unstable and stable manifold that were creating the separatrix separates
- As we move on the unstable/stable manifold back towards the X-point they gets strechted out
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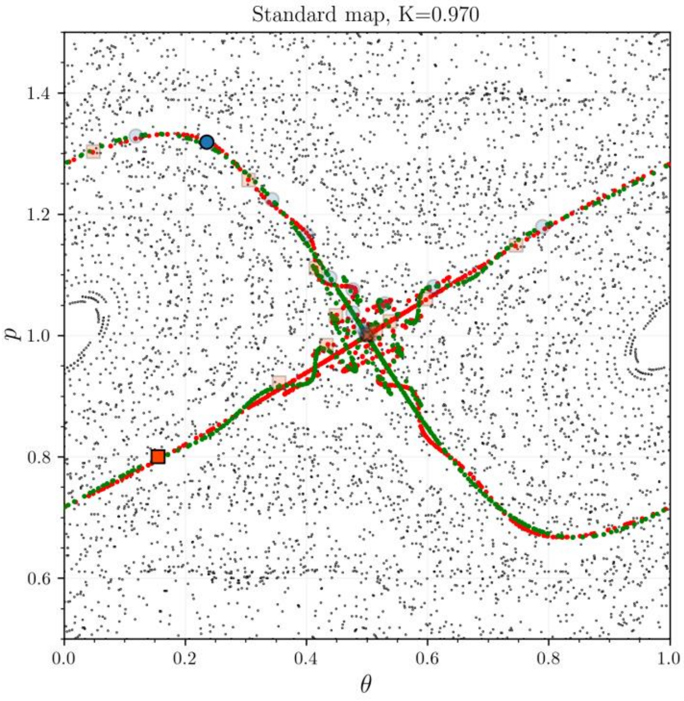


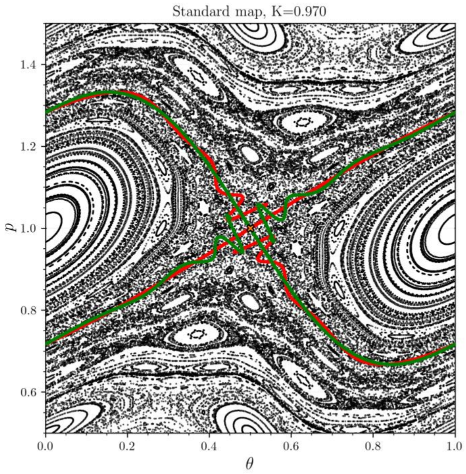
Intersection of W^s and W^u are homo/hetero-clinic points

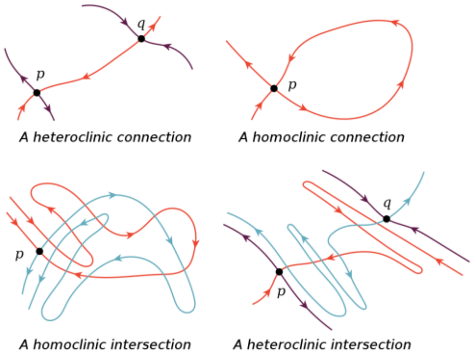
- Heteroclinic orbit : points for which the forward and backward time evolution converge towards fixed points

$$x(t) \to x_0^\star \quad \text{as} \quad t \to -\infty$$
$$x(t) \to x_1^\star \quad \text{as} \quad t \to +\infty$$

- If $x_0^* = x_1^*$ then referred to as a homoclinic orbit
- In fact the space of heteroclinic points can be defined as

$$H := W^s(x_0^\star) \cap W^u(x_1^\star) - \{x_0^\star, x_1^\star\}$$

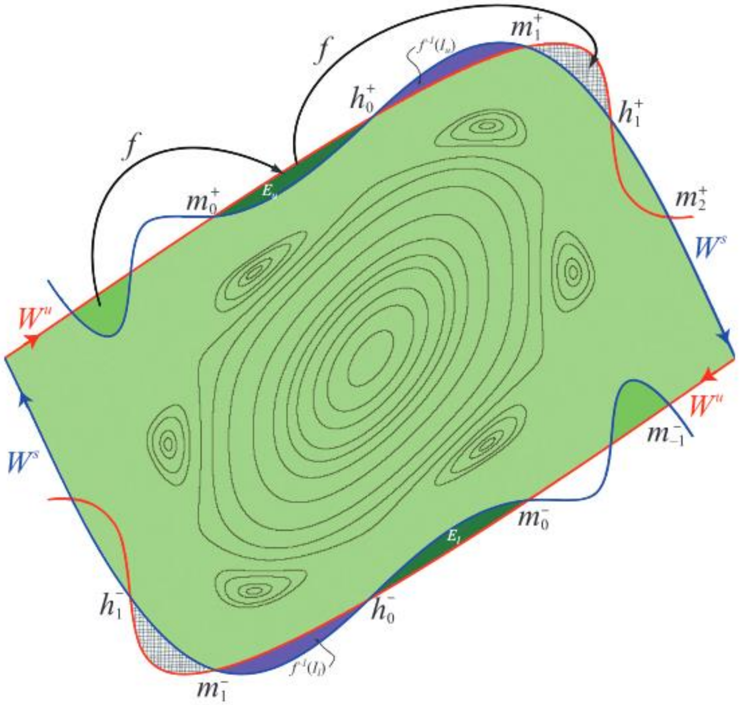




https://en.wikipedia.org/wiki/Homoclinic_connection

Points in the intersection exit or enter the resonance zone

- Points in the inside of the bounded region but in the intersection will end up outside
- On the contrary points catched on the outside but in the intersection will end up inside
- For flux/area preserving map, it implies that the entering flux is equal to the outgoing flux
- Similar to a turnstile



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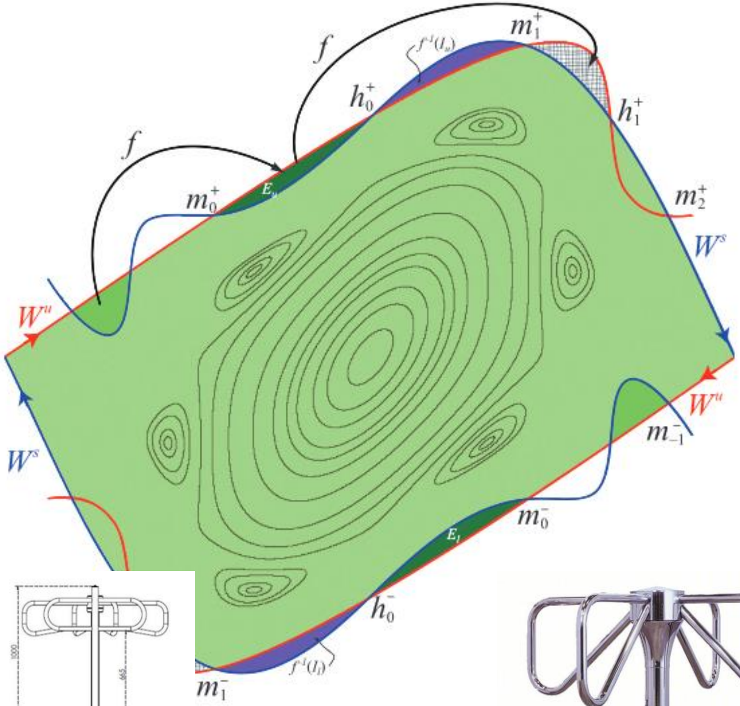
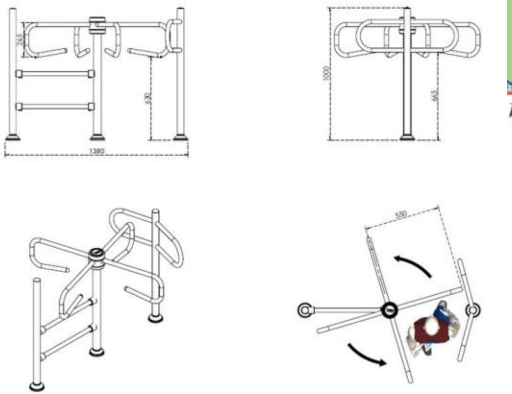
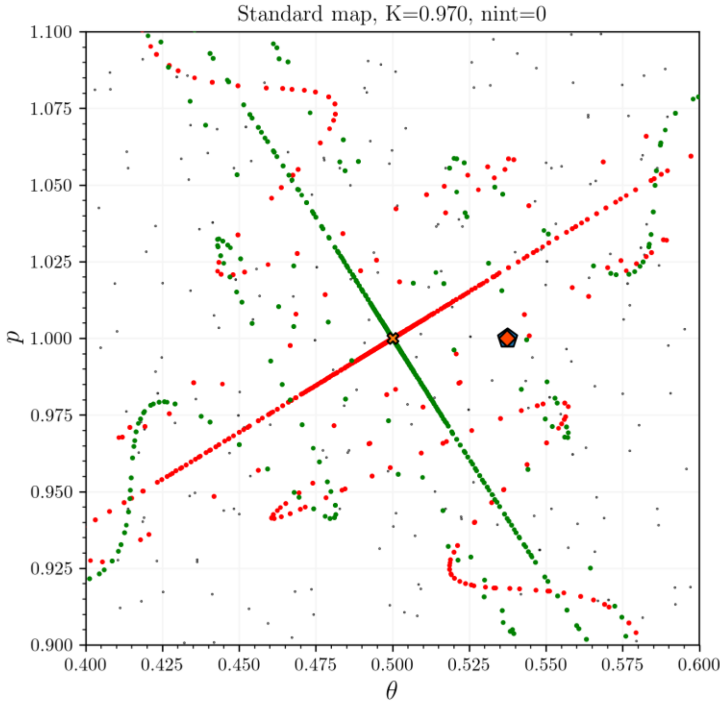




Trellis structure and turnstile behaviour explain why the brokeness is linked to chaos

- Taking two close starting points inside of the resonance zone and manifold intersection
- They start close but are still separated due to the infinitly thin grid structure
- Thus after some amout of iterations they will end up in different region, no matter the initial their initial closeness.
- How to compute this indicative turnstile flux for the poincare map ?



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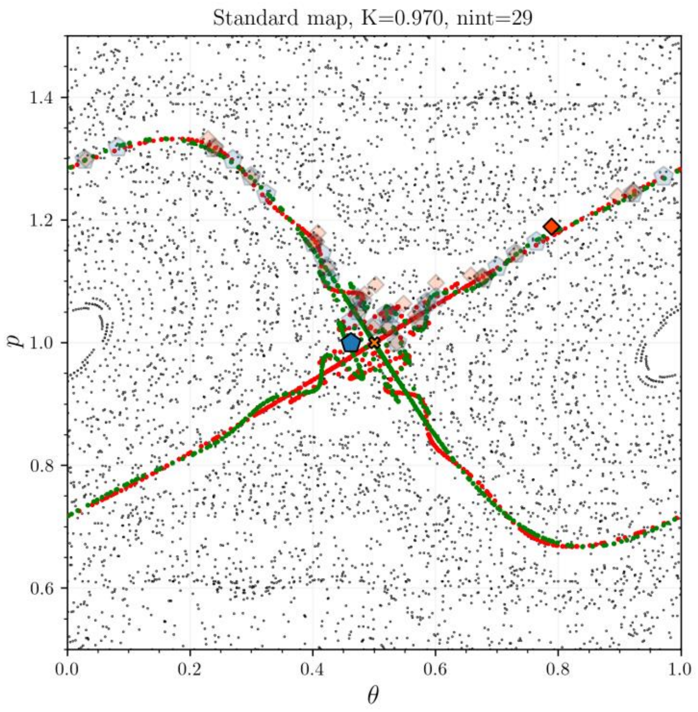


Introducing an analytic high aspect ratio tokamak equilibrium

- Introducing an analytic field in cylindrical geometry to facillitate the field line tracing, verify the latter calculations

$$\mathbf{B} = -\frac{1}{R}\frac{\partial\psi}{\partial Z}\partial_R + \frac{1}{R}\frac{\partial\psi}{\partial R}\partial_Z + \left(\frac{\partial A^R}{\partial Z} - \frac{\partial A^Z}{\partial R}\right)\partial_\phi$$

- Poloidal flux surfaces $\psi \propto \rho^2$ for nested toroids
- In the high aspect ratio $\varepsilon \gg 1$ approximation, the q-profile is quadratic with ρ

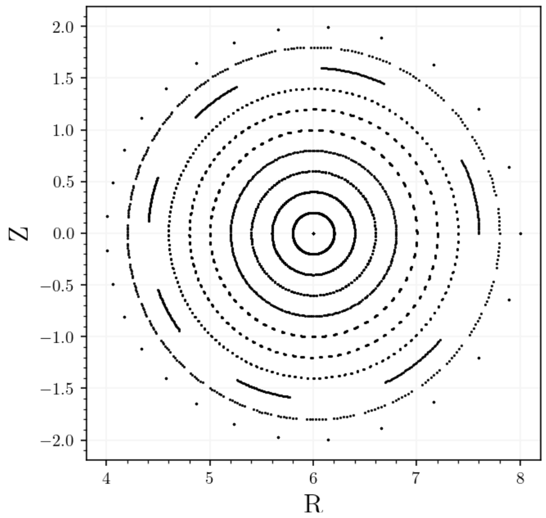
$$q \approx \frac{\Delta\phi}{\Delta\theta} = \frac{\rho\tilde{B}^\phi}{R_0\tilde{B}^\theta} = \frac{2\pi}{\mu_0\tilde{I}^\phi R_0}\rho^2$$

- Adding the A^R related to

$$B^\phi = 2\sqrt{R^2-\rho^2}\,(\mathrm{sf} + \frac{\mathrm{shear}}{2}\rho^2)/R^2$$

- Gives a q-profile

$$q(\rho) = \mathrm{sf} + \frac{\mathrm{shear}}{2}\rho^2$$



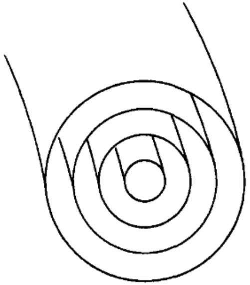
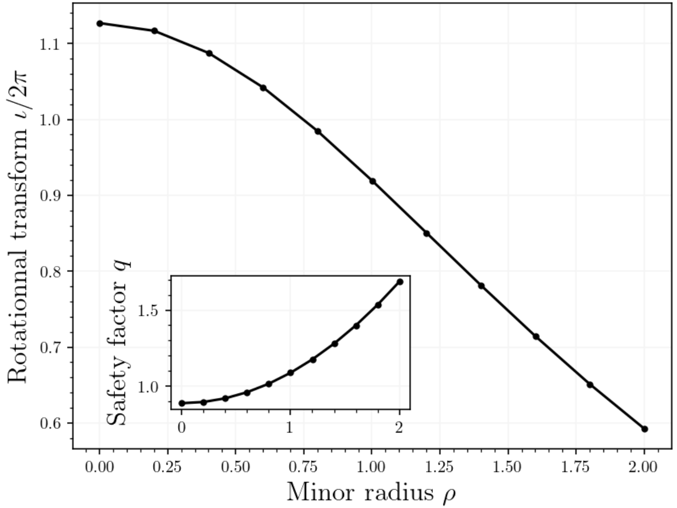


Fig. 3.2.1 Magnetic flux surfaces forming a set of nested toroids

Wesson, Tokamaks.



Adding the field of a circular current loop to create an X-point

- The vector potential of a circular current loop at position $(R_l, 0)$

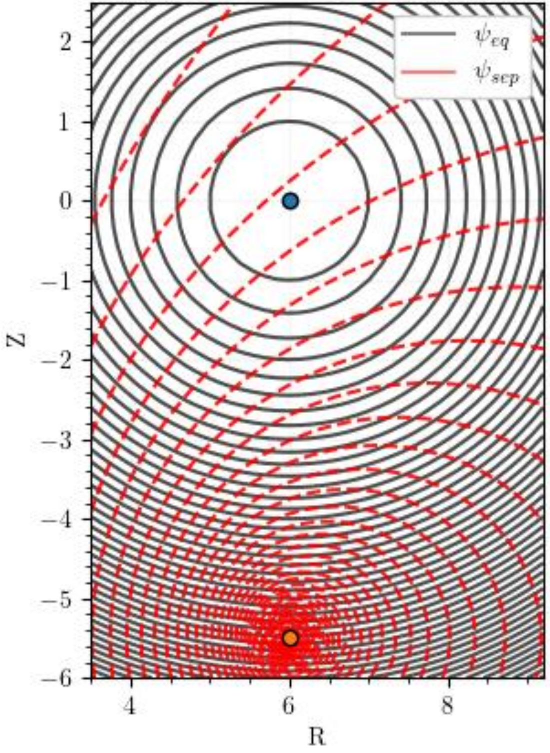
$$A^{\phi} = \frac{\mu_0}{4\pi}\frac{4IR_l}{\beta R}\left(\frac{(2-k^2)K(k^2) - 2E(k^2)}{k^2}\right)$$

Simpson et al., 'Simple Analytic Expressions for the Magnetic Field of a Circular Current Loop'

where E, K are the complete elliptic integral of the first and second kind and

$$\alpha^2 = (R_l - R)^2 + Z^2$$
$$\beta^2 = (R_l + R)^2 + Z^2$$
$$k = 1 - \alpha^2/\beta^2$$

- Substituting Z for $Z - Z_l$ makes it general and then take



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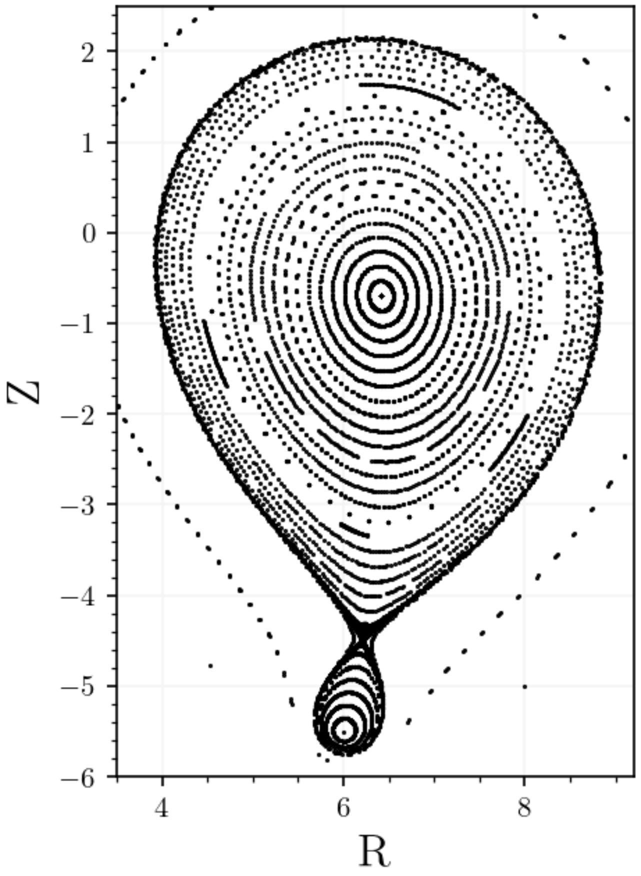

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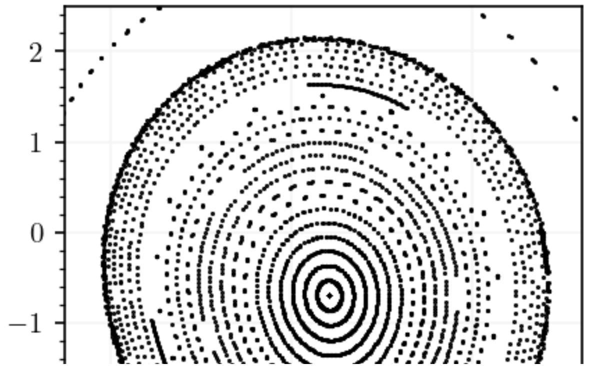

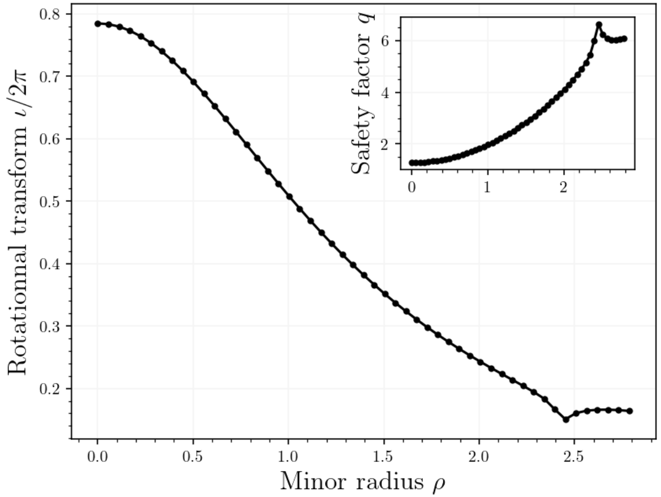

Adding perturbations

- To see Chaos emerge, a perturbation field is required
- Noting the angle periodicity in toroidal coordinates

$$A^i(\rho,\phi,\theta) = \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} A^i(\rho)\cos(n\phi + \varphi_n)\cos(m\theta + \varphi_m)$$

- Maxwell–Boltzmann and Normal distributions for $A^i(\rho)$

1. $A^i(\rho) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(\rho-\mu)^2}{2\sigma^2}\right)$
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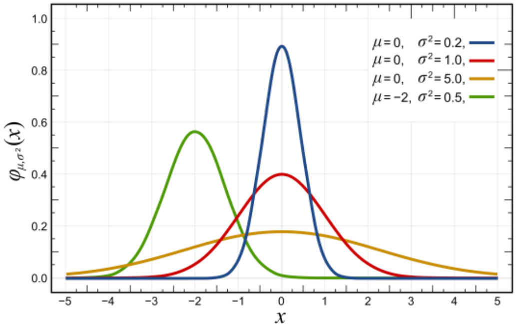
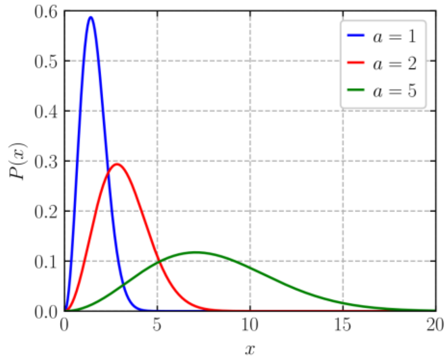
- Adding an $\psi_{pert} = R^2 A^{\phi}_{pert}$ Maxwell-Boltzmann with the $m/n = 6/1$ to target the edge





EPFL

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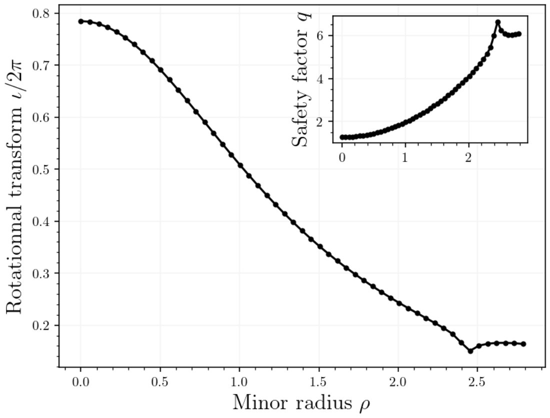

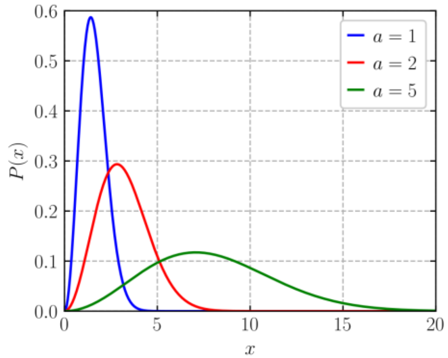

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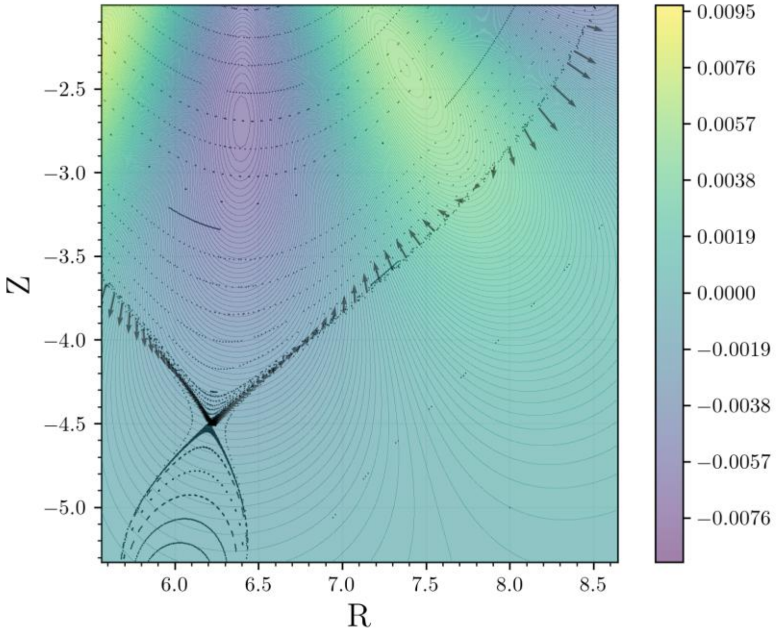
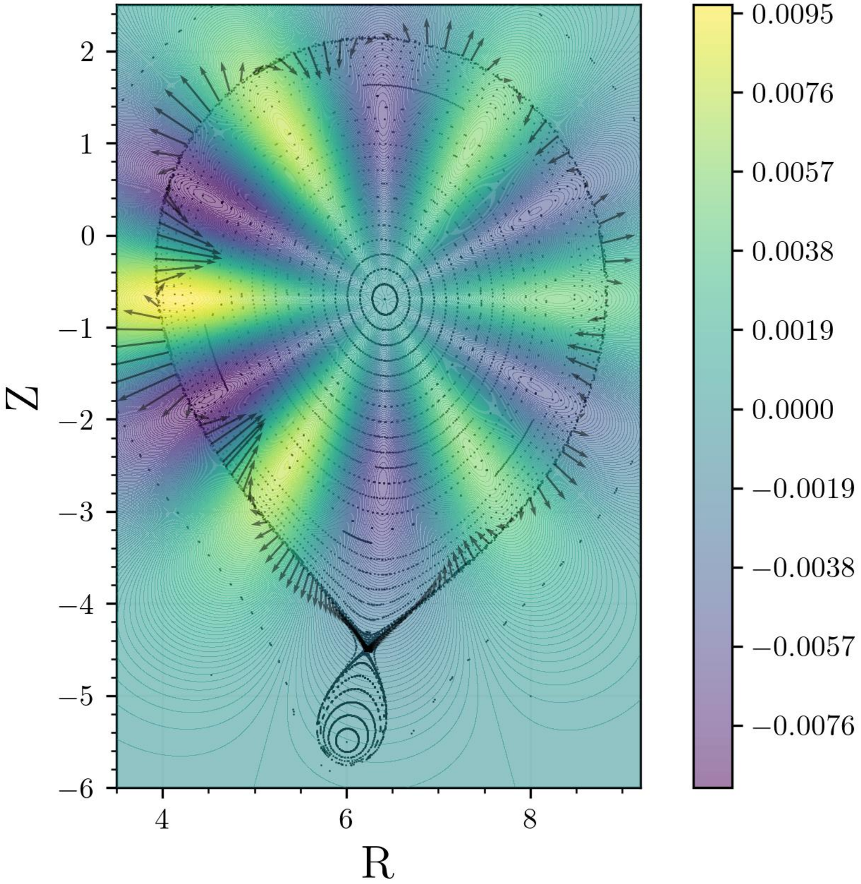
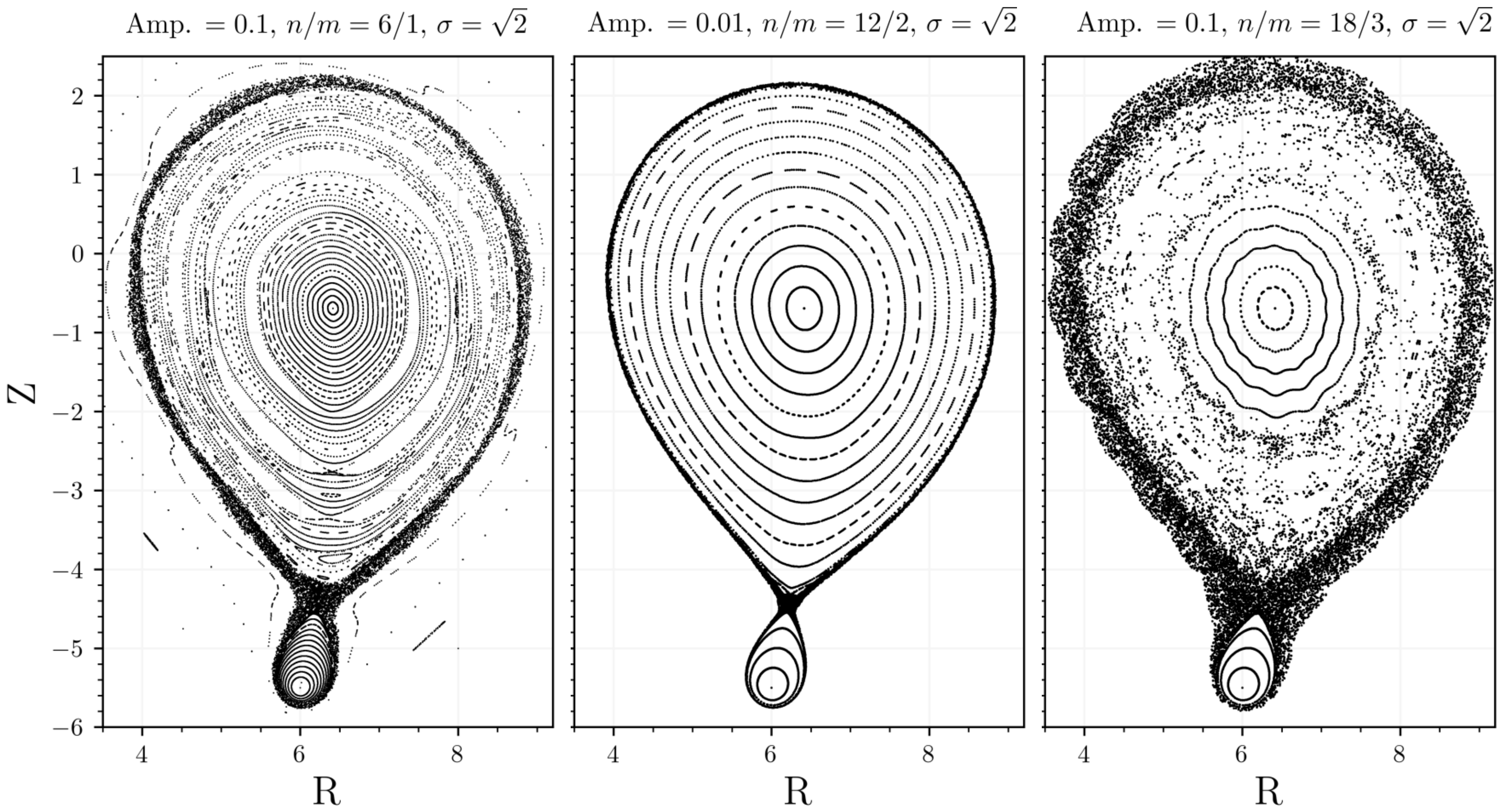




Poincare sections of the ToyTok with perturbations



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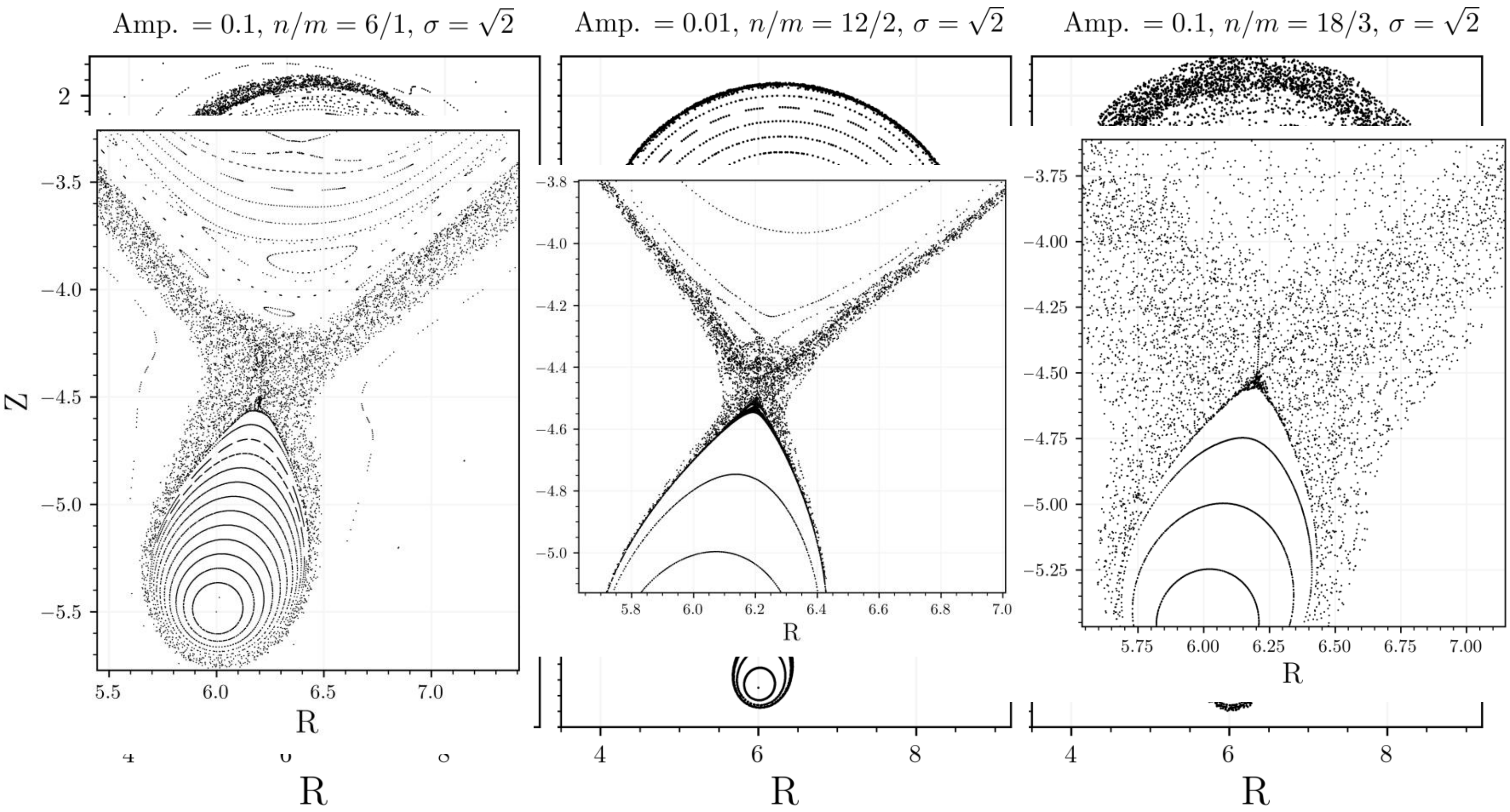
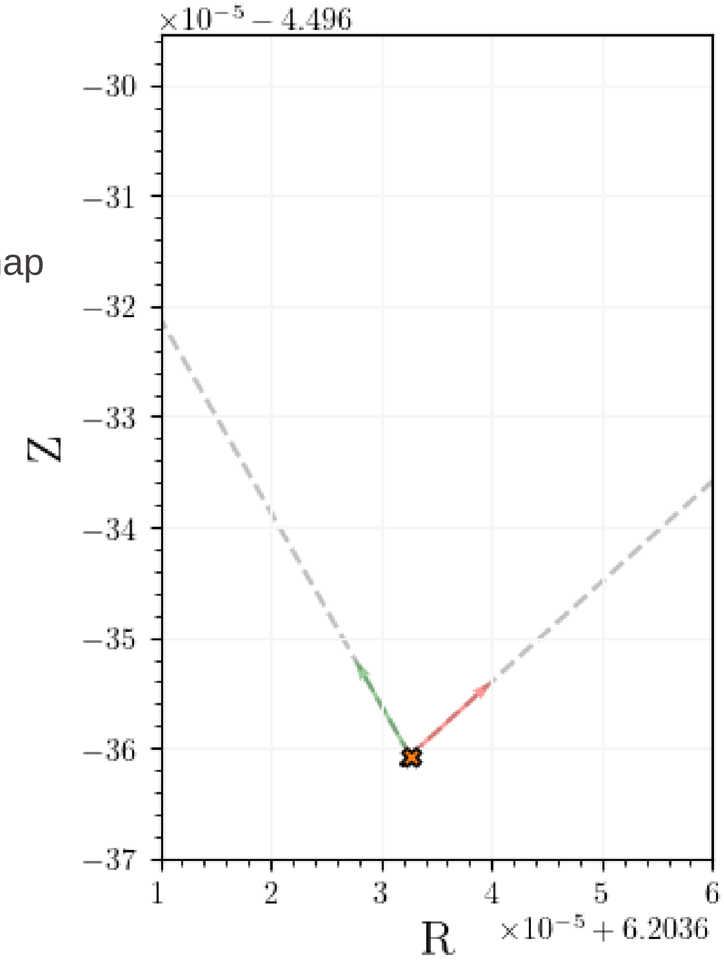


Tracing the manifold for the poincare map

- Adapting the computation for the poincare map
  - $x = x^* + \varepsilon_u \boldsymbol{e}_u$ and forward integration
  - $x = x^* + \varepsilon_s \boldsymbol{e}_s$ and backward integration
- Initializing points between x and $\mathcal{P}(x)$



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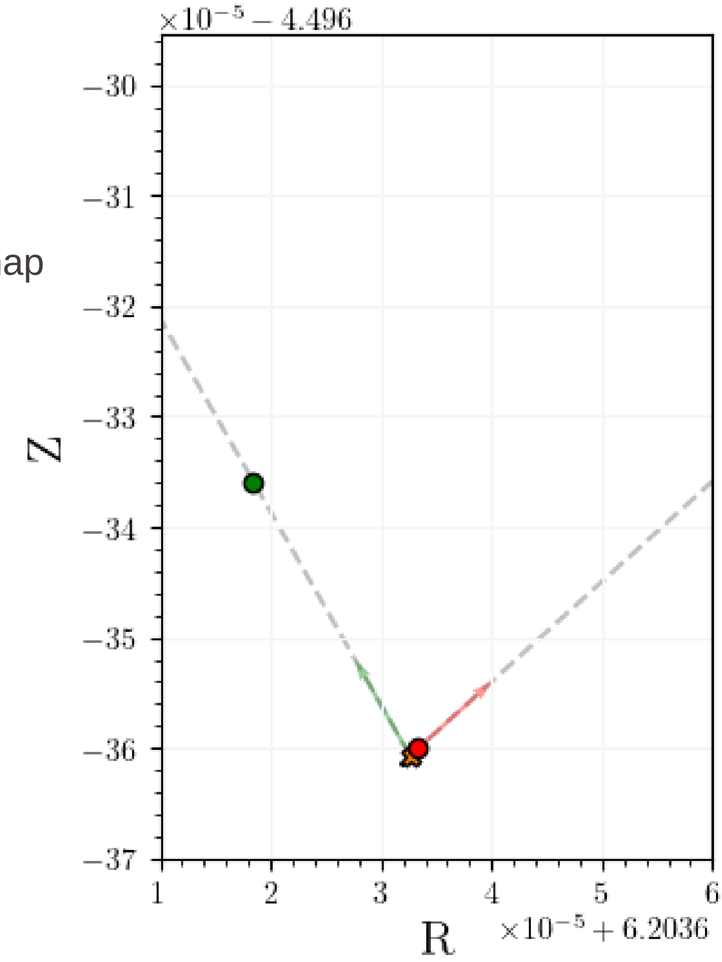

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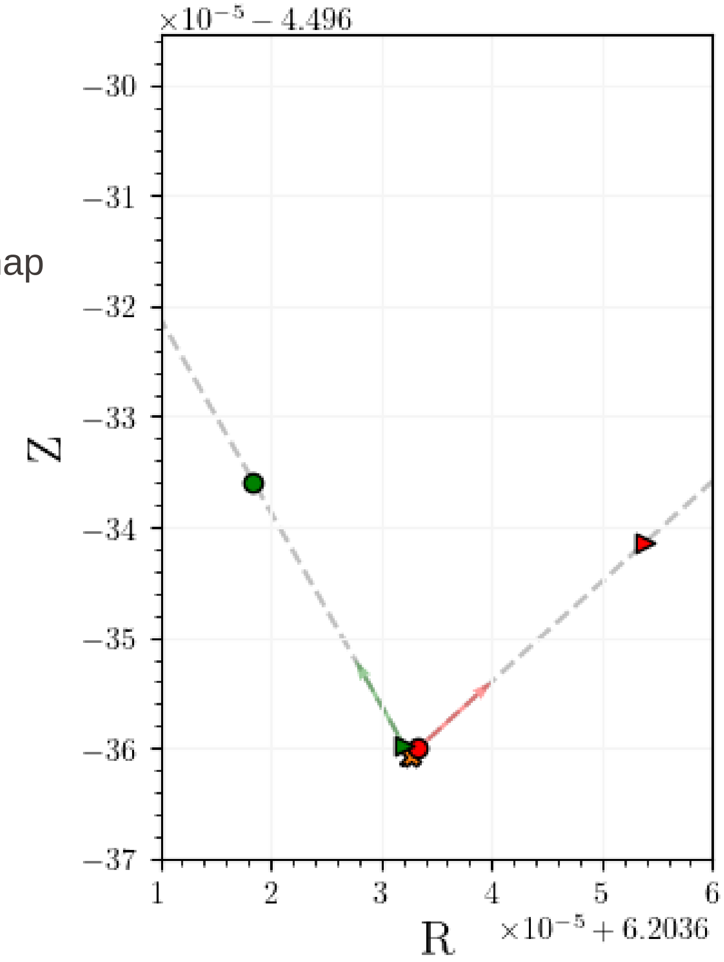

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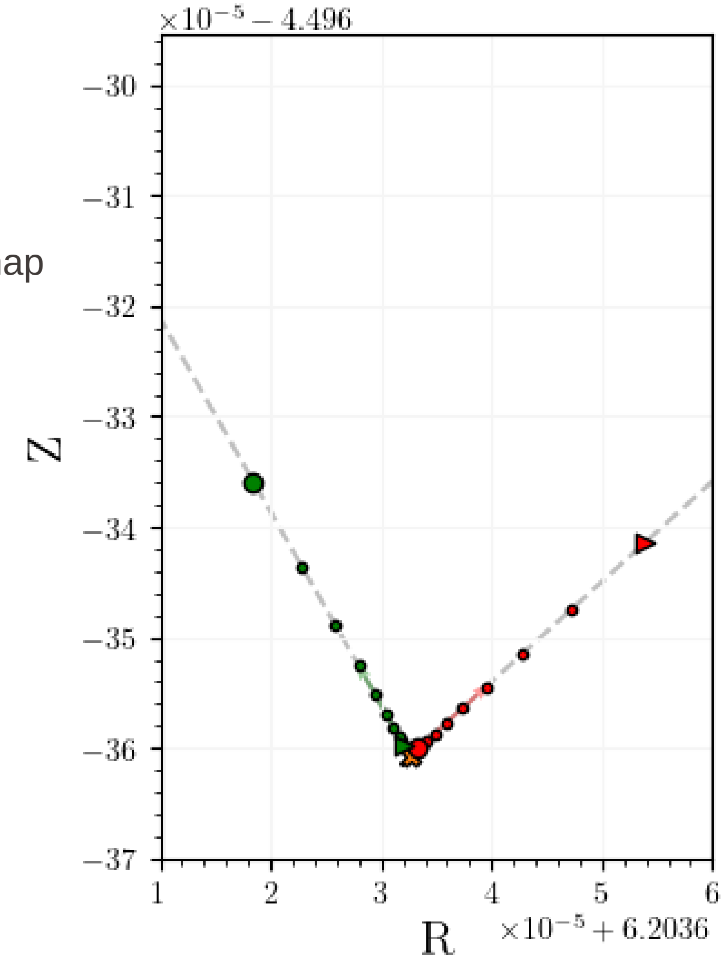

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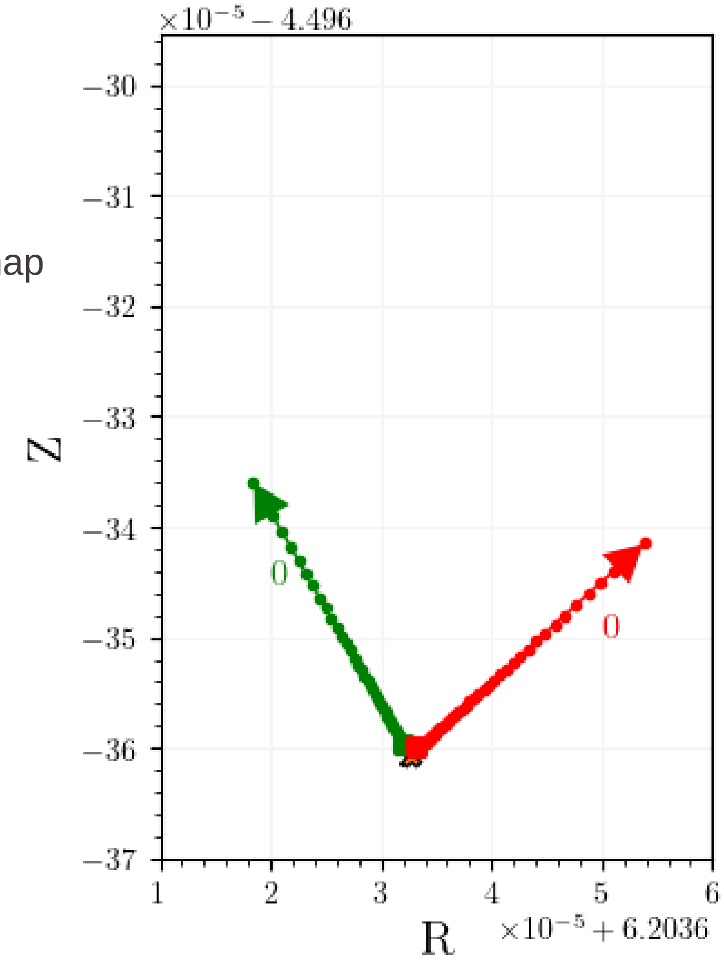

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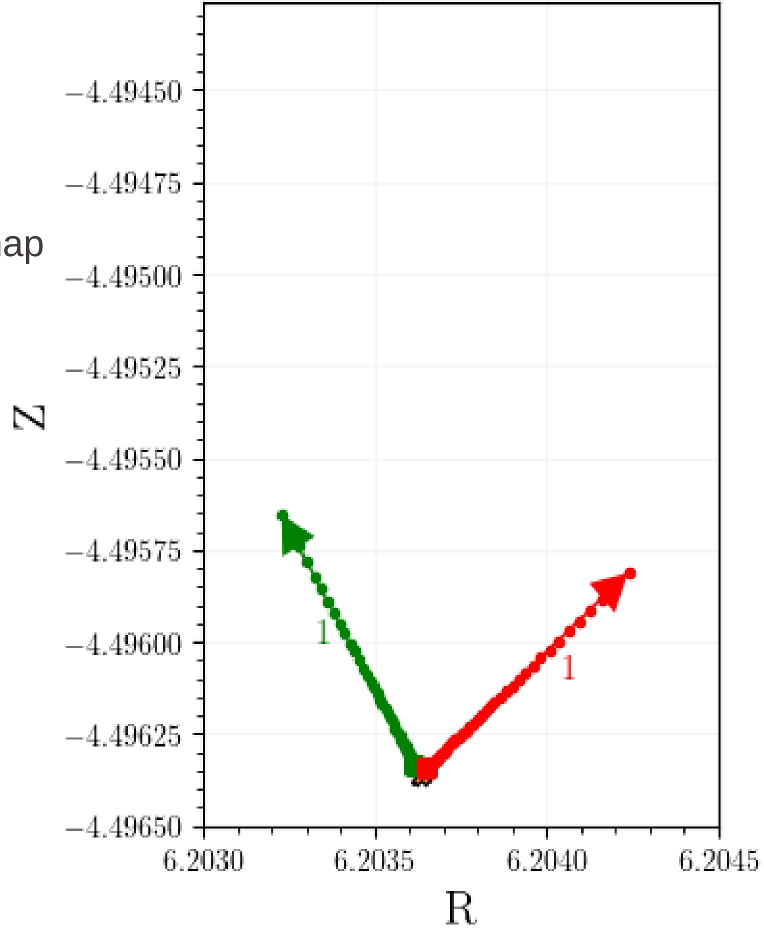

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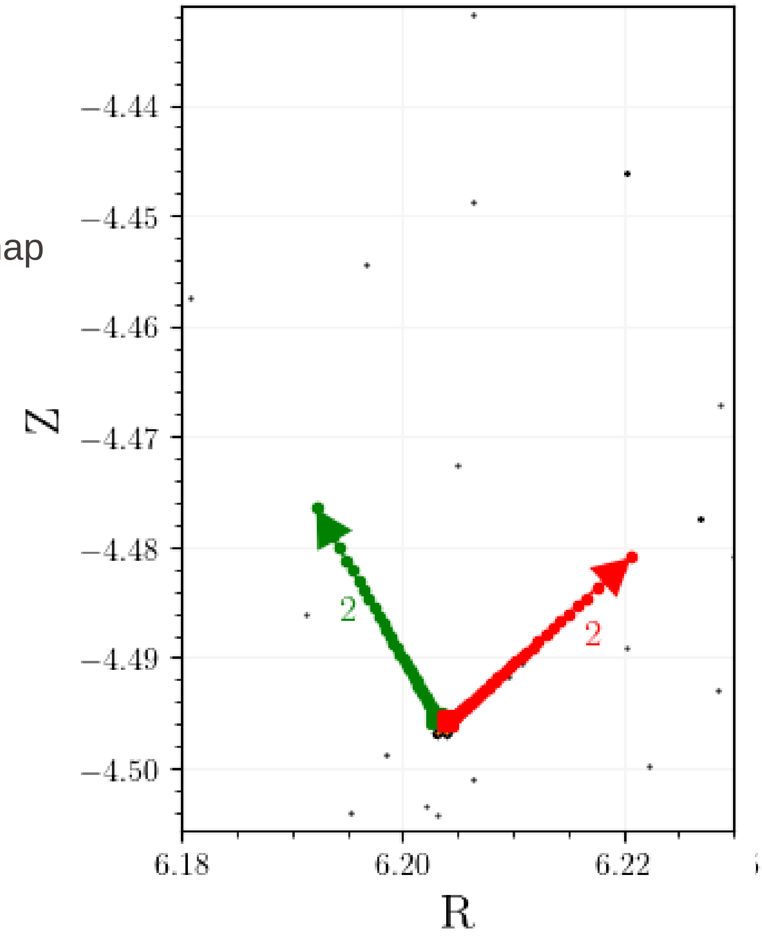

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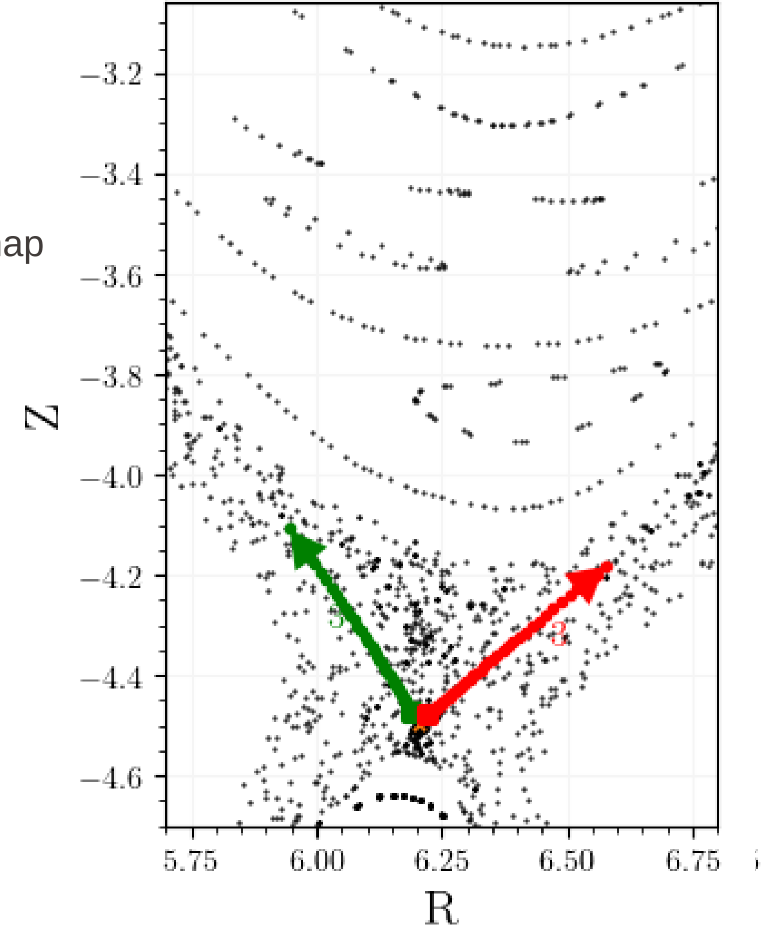

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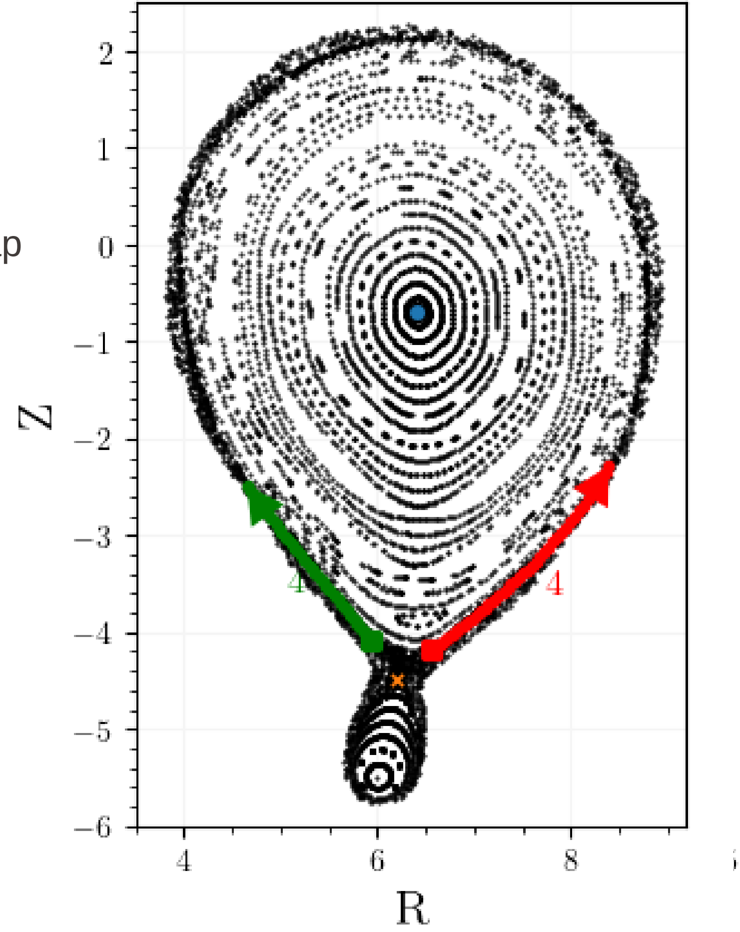

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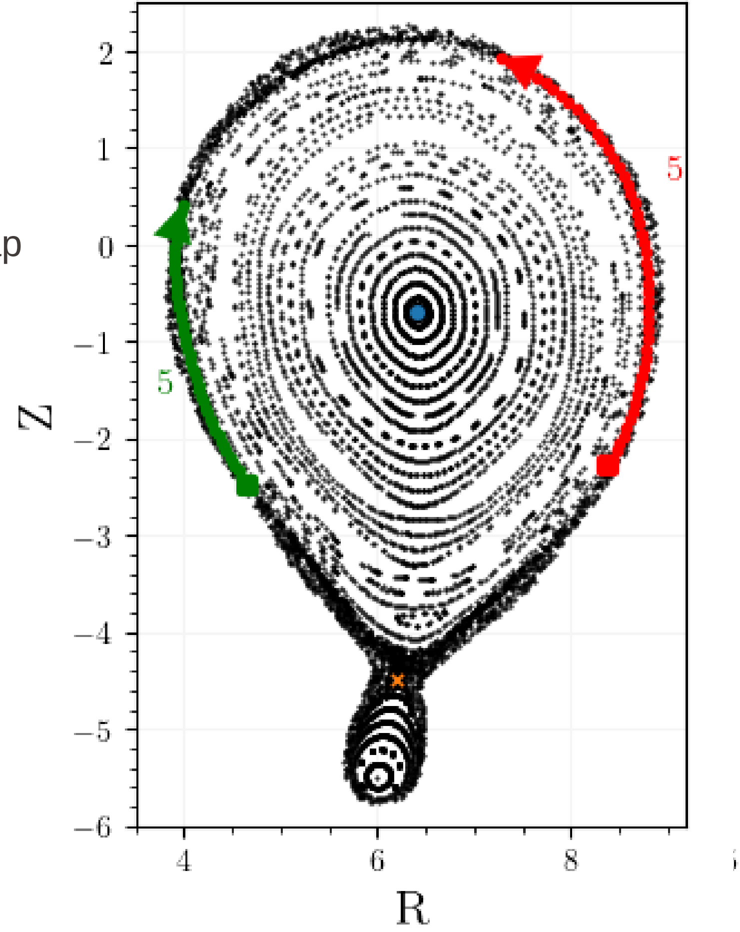

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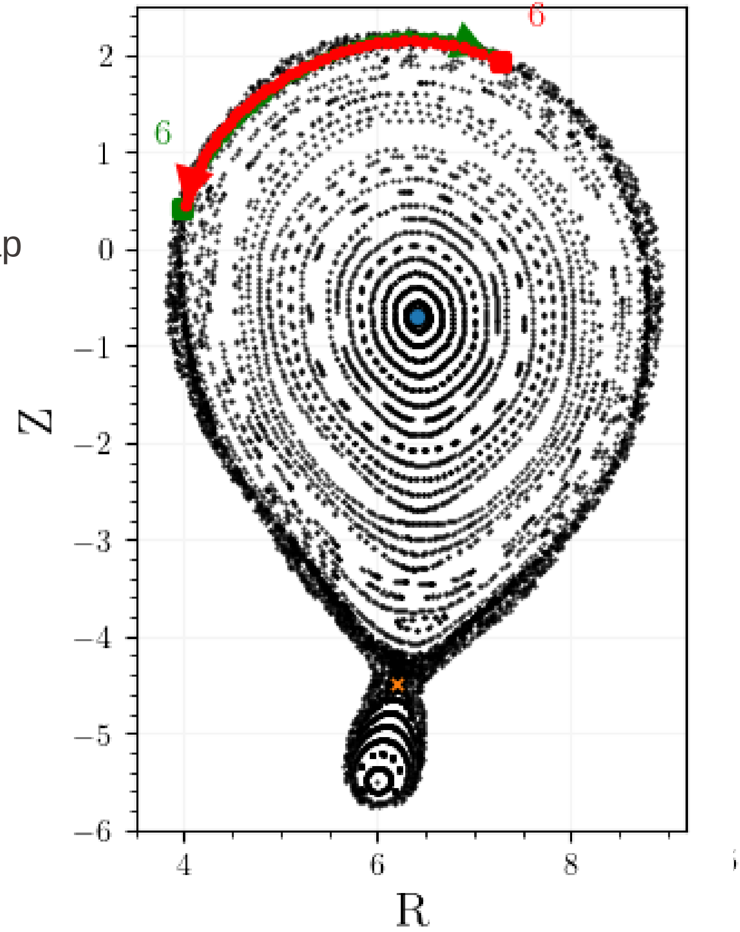

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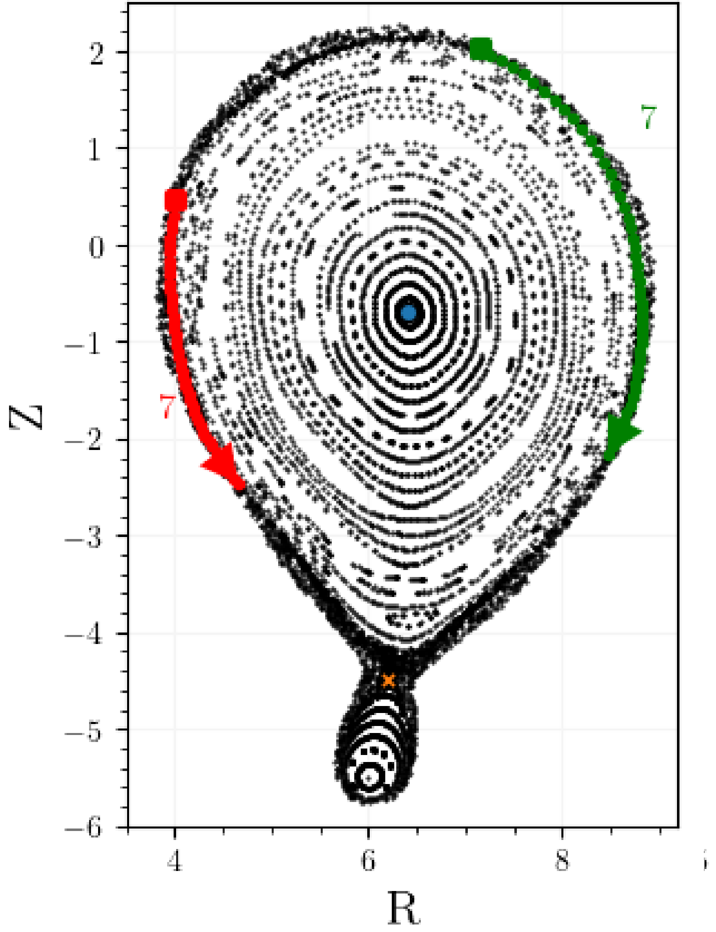

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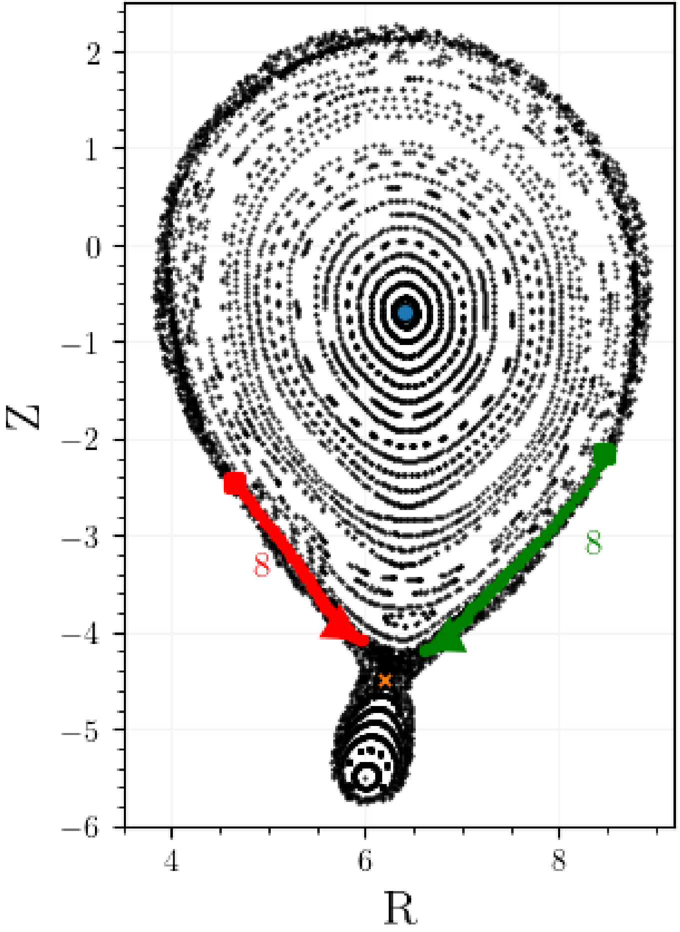

EPFL

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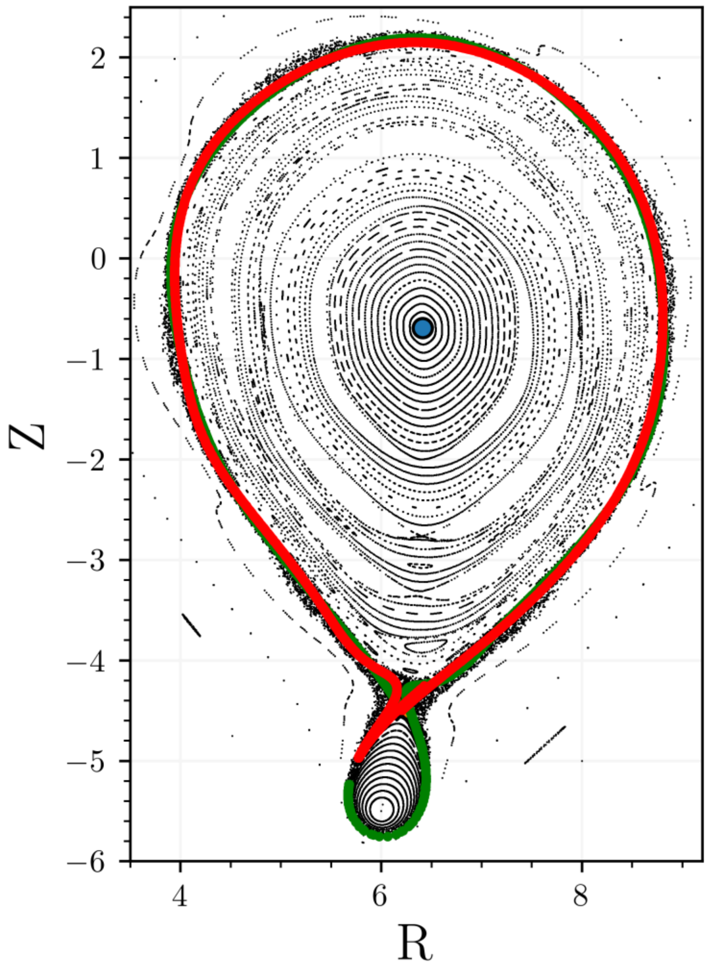


Finding a homoclinic point by searching for a root

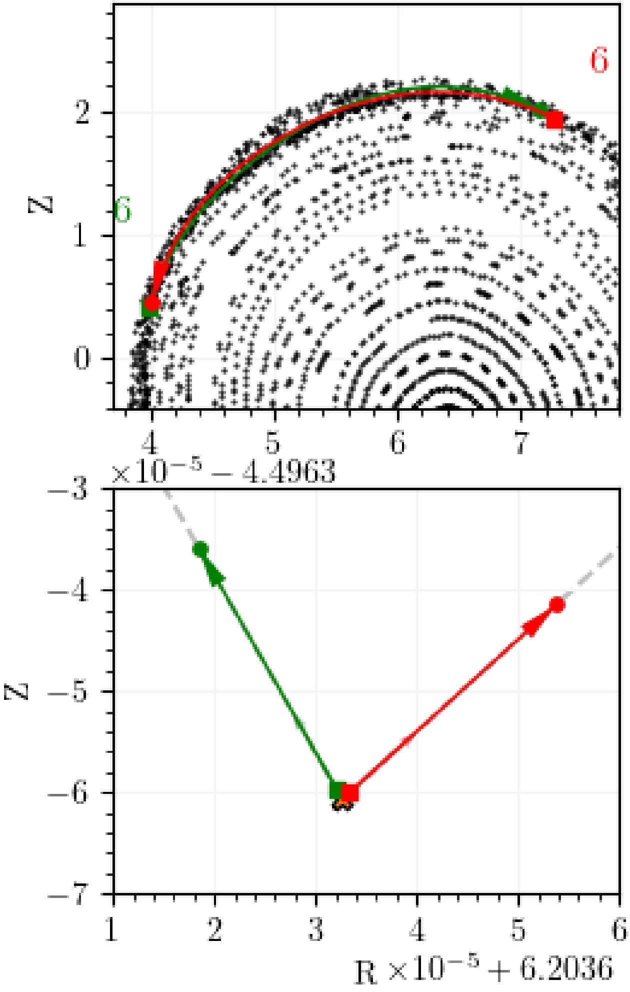
- The intersection of the stable and unstable manifold are homoclinic points, they come back to the X-point
- In this context, we say that homoclinic points differ if they are not in the same orbit. There must be at least two different homoclinic orbit
- Set a starting point on each manifold

$$x^s = x^\star + \varepsilon_s \mathbf{e}_s \quad x^u = x^\star + \varepsilon_u \mathbf{e}_u$$

- Find a tuple of forward n_u and backward iteration n_s such that the point cross each other
- Minimize the resulting difference

$$\mathcal{P}^{-n_s}(x^s) - \mathcal{P}^{n_u}(x^u)$$

Thanks to ***Matt Landreman*** for the general idea



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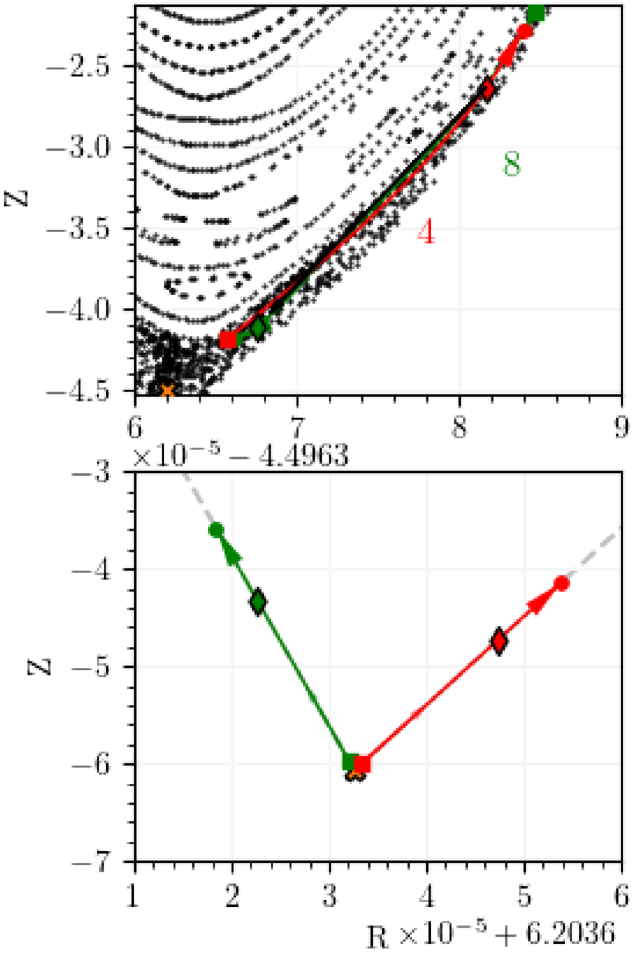

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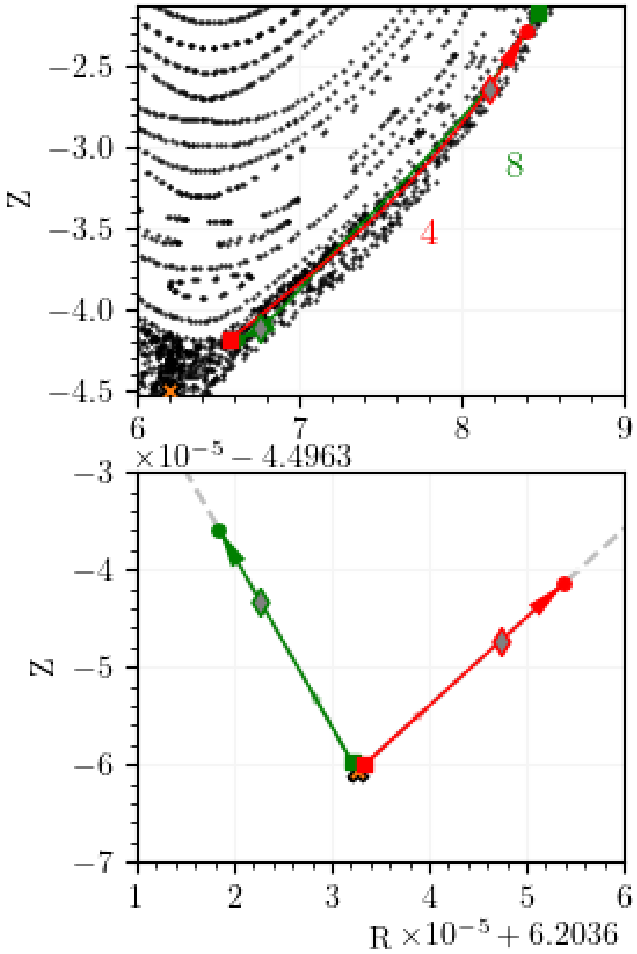

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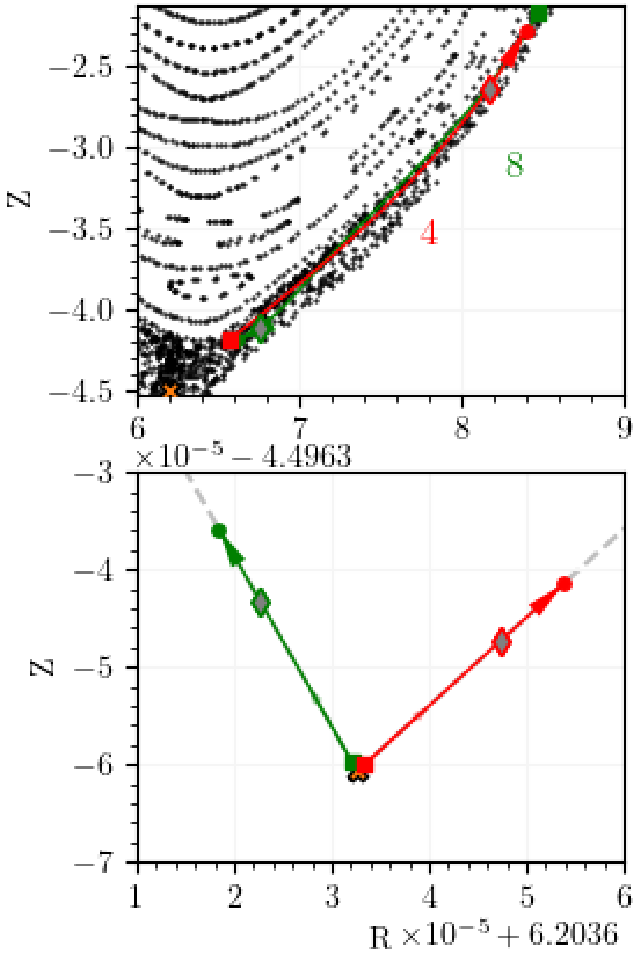

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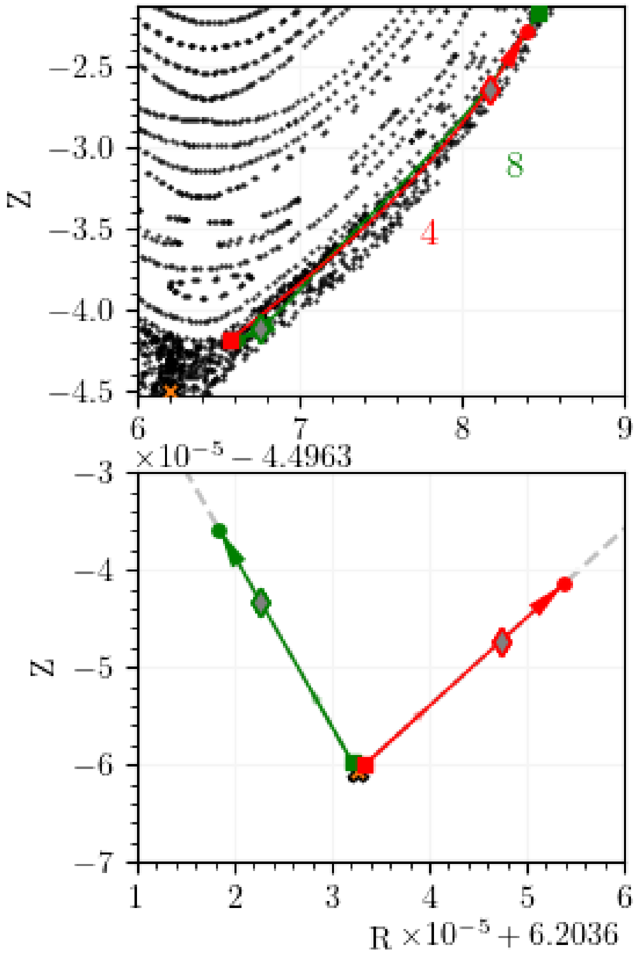

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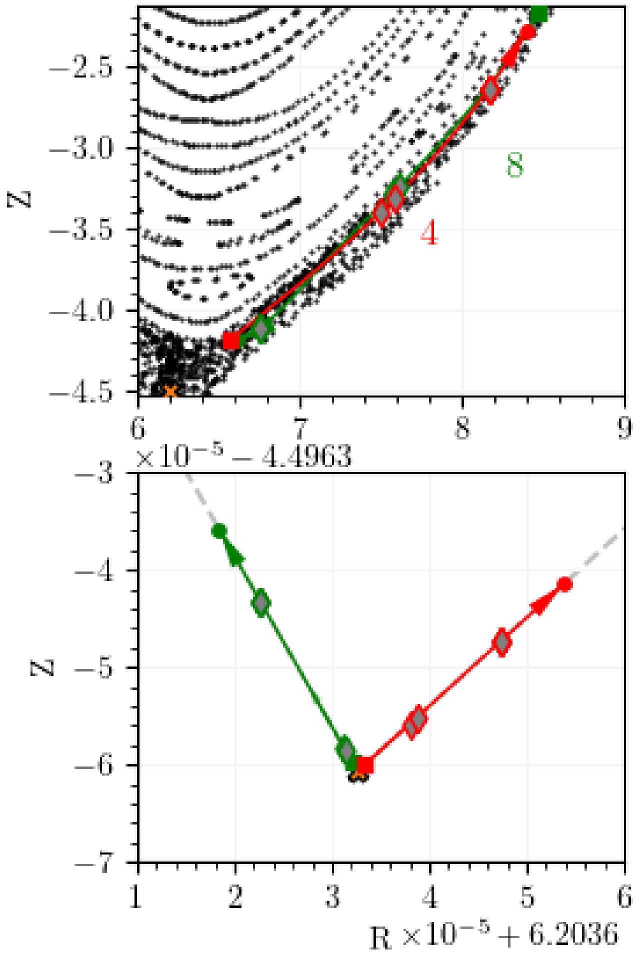

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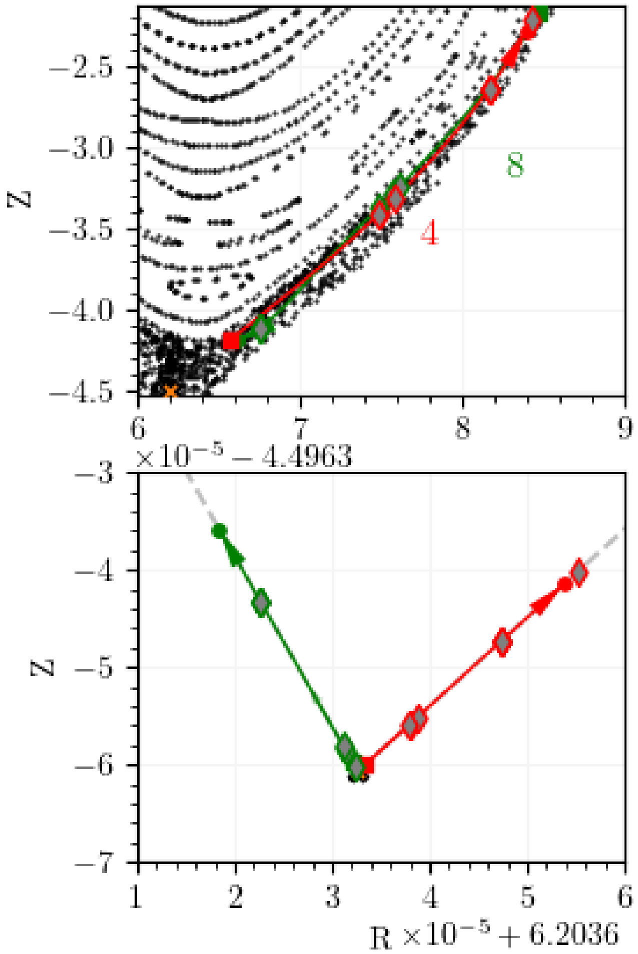

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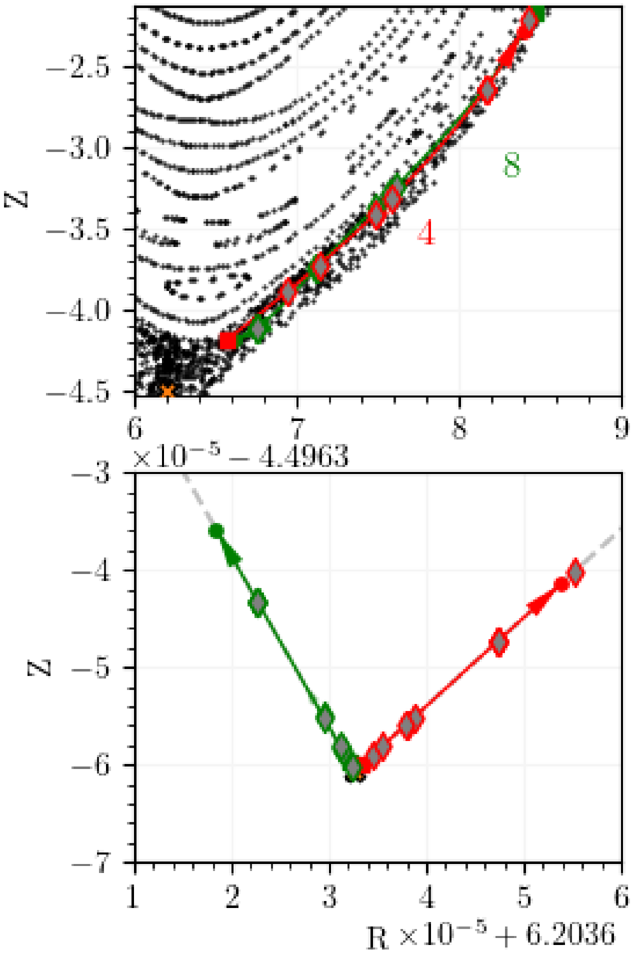

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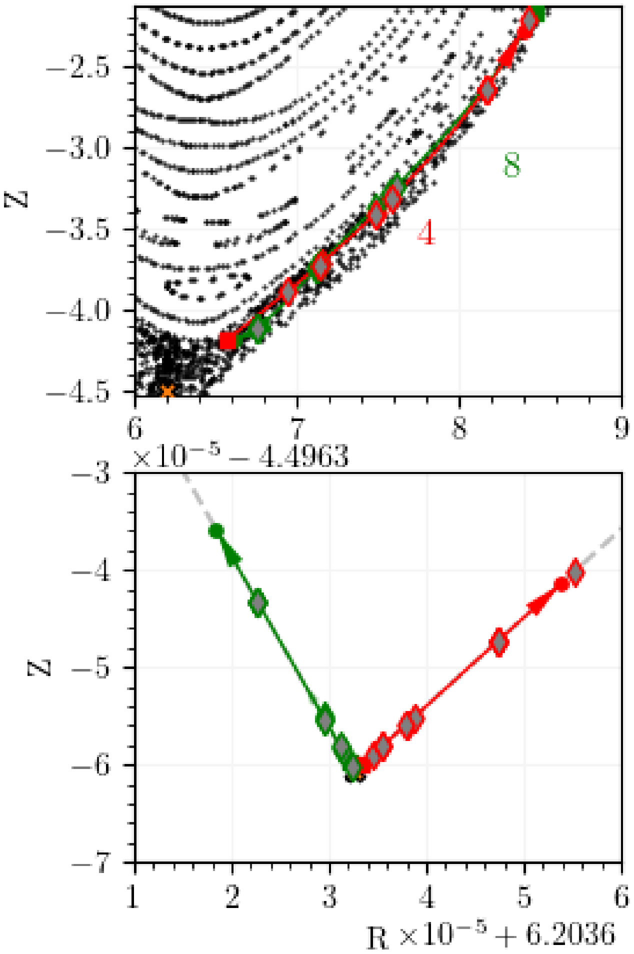

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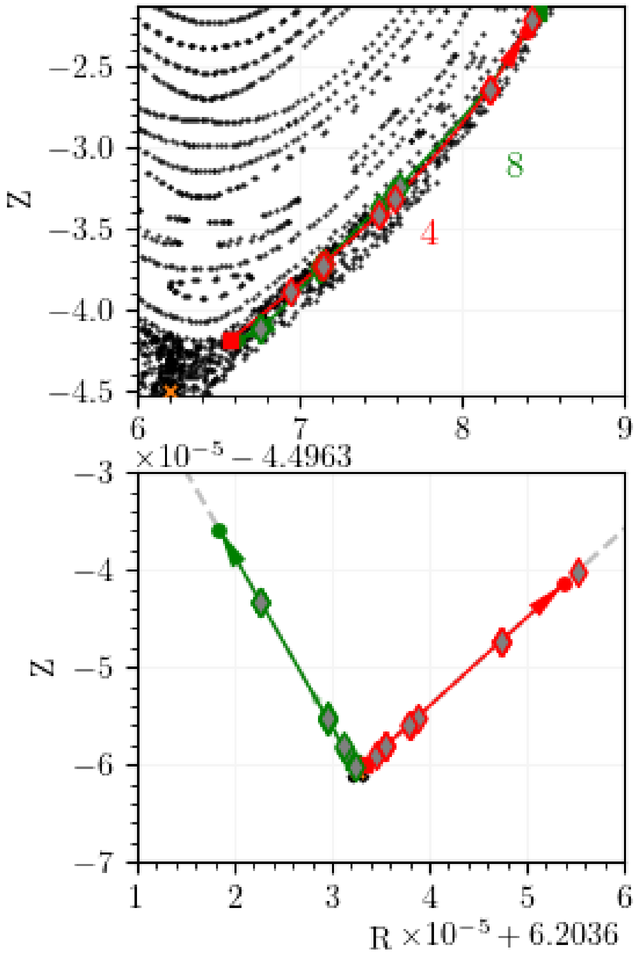

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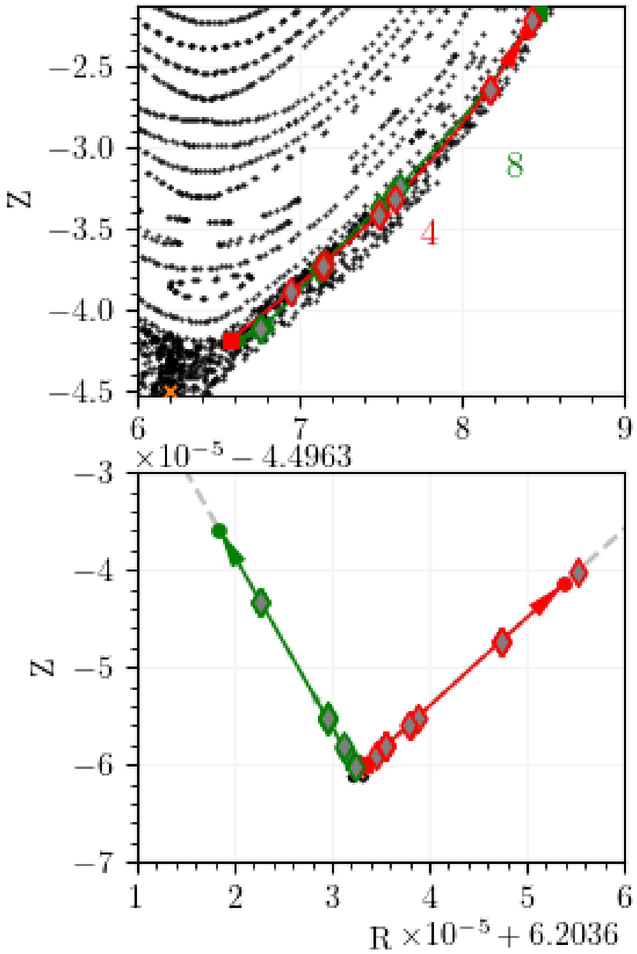

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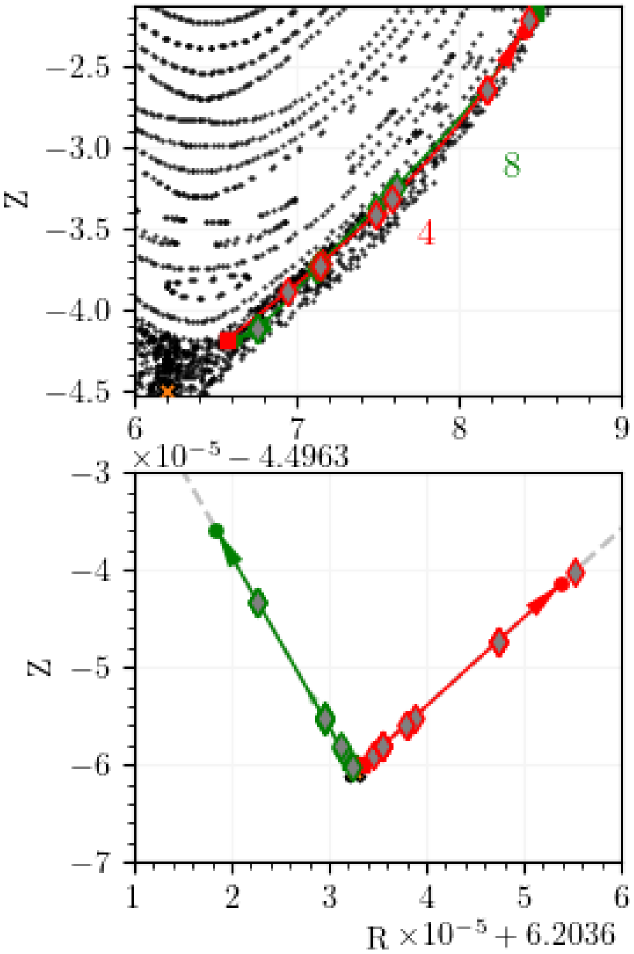

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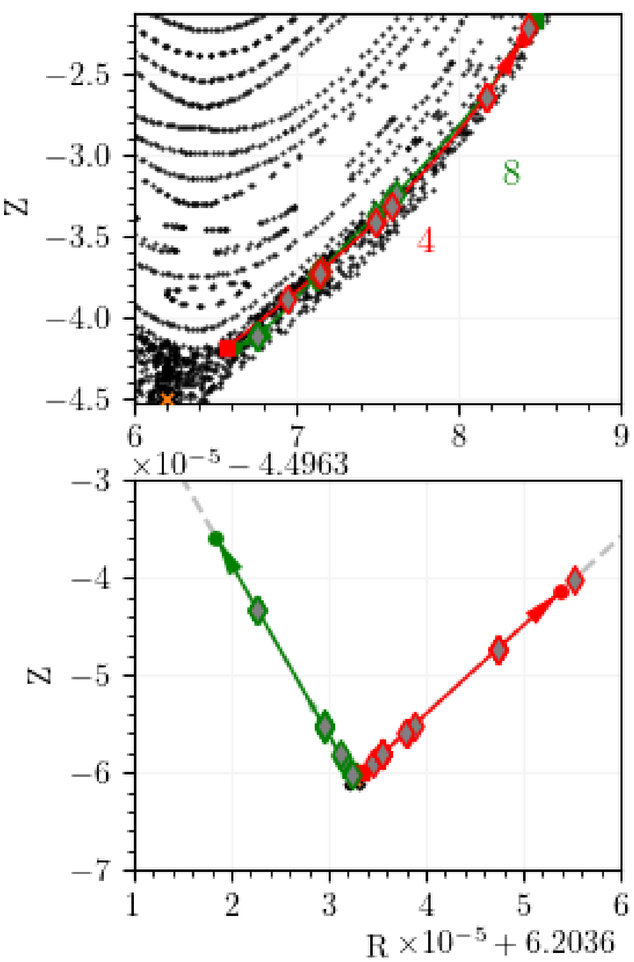

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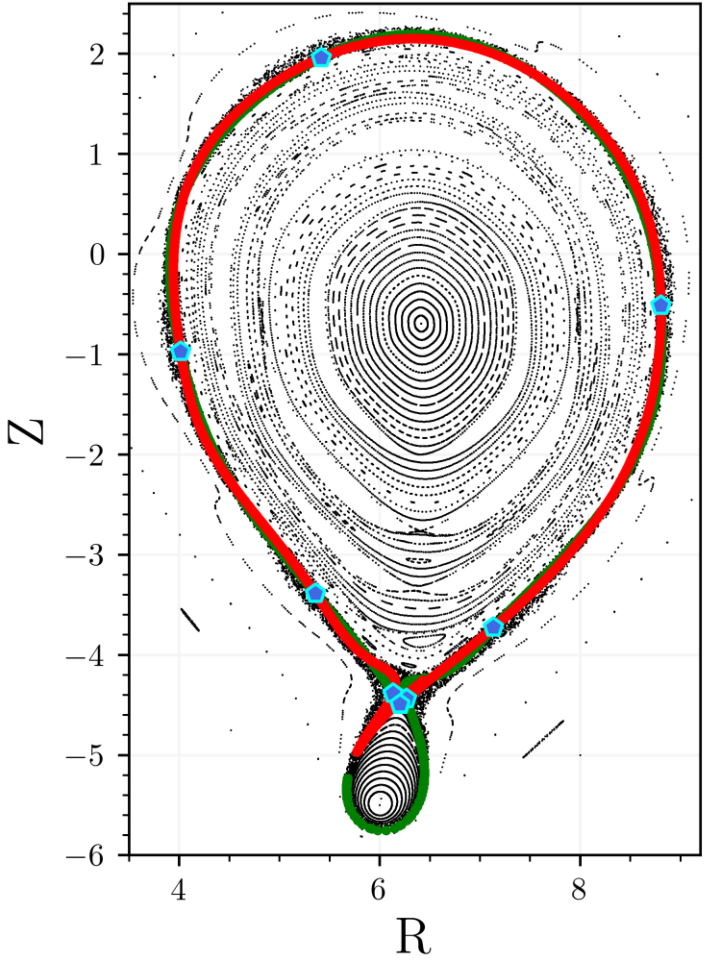
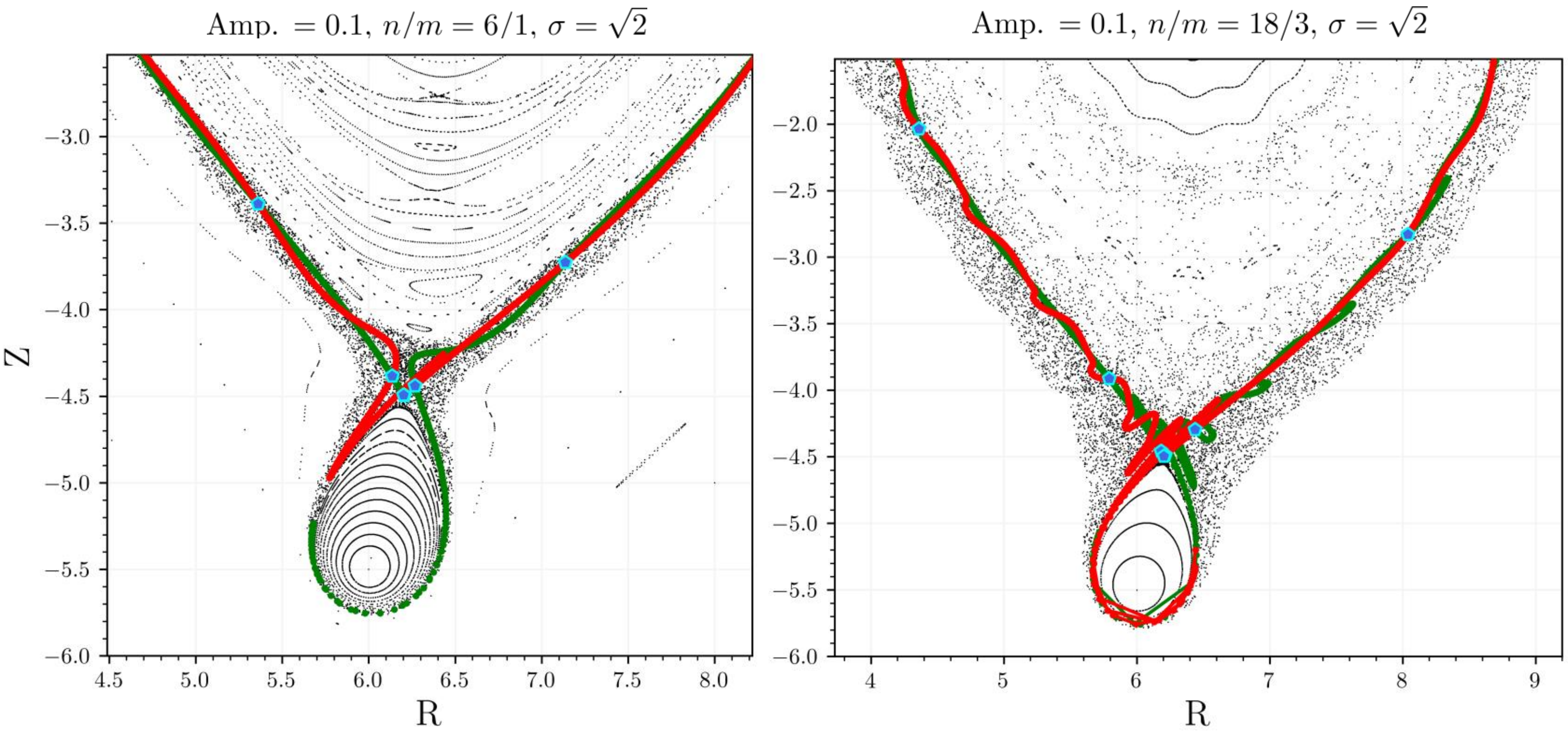


Some ToyTok configuration have more than two homo/hetero-clinic points



Using the same algorithm with different initial guess to find the other homoclincs

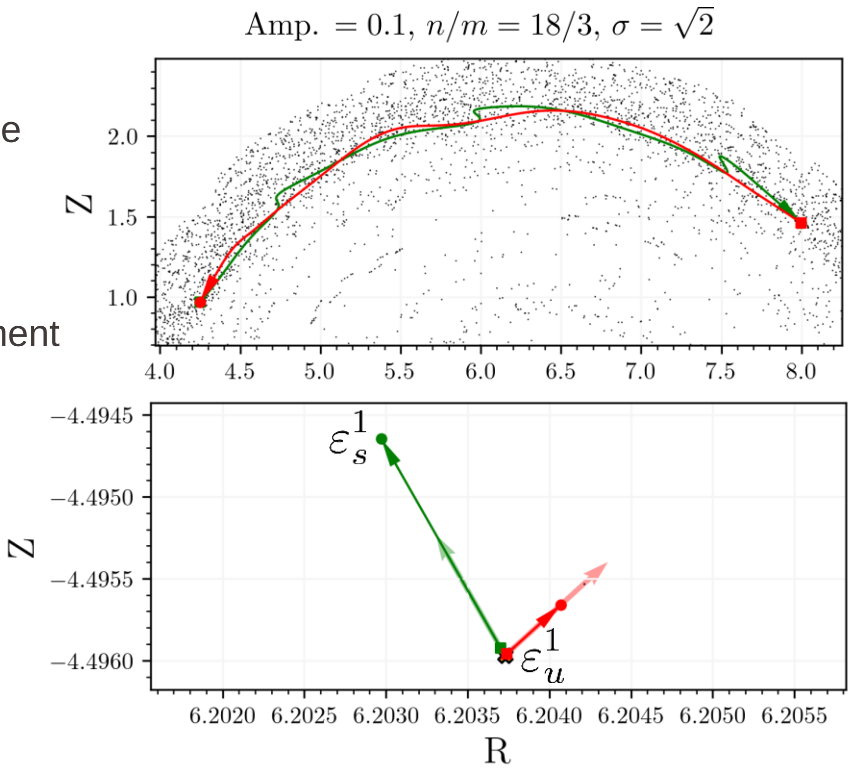
- Once a first homoclinic has been found, all remaining $n_h - 1$ homoclinics have to be in the initial segments
- The intervals are parcourred in opposite directions
- If we write ε_s^1 the end of the stable initial segment and ε_u^1 the beginning of the unstable initial segment, then we choose

$$x^s = x^\star + \lambda_s^{i/n_h} \varepsilon_s^1 \mathbf{e}_s \quad x^u = x^\star + \lambda_u^{i/n_h} \varepsilon_u^1 \mathbf{e}_u$$

$$\text{with} \quad i = 1, \ldots, n_h$$

- And try again to find the root of

$$\mathcal{P}^{-n_s}(x^s) - \mathcal{P}^{n_u}(x^u)$$



Meiss's action principle provides a way to calculate the turnstile area for the standard map

- As mentionned, the standard map preserves the area, now we write the area as

$$\nabla \times (x\,\mathbf{e}_y) = 1 \quad \leftrightarrow \quad \Omega = dx \wedge dy = d(xdy) = d\alpha$$

- Then use Stokes theorem on R, which has as boundary $\partial R = \mathcal{U} - \mathcal{S}$

$$\mu(R) = \int_R \Omega = \int_{\partial R} \alpha = \int_{\mathcal{U}} \alpha - \int_{\mathcal{S}} \alpha$$

- Finally using the *exactness property* we can write those integrals as an infinte some of an associated *lagrangian* function evaluted at the heteroclinic points

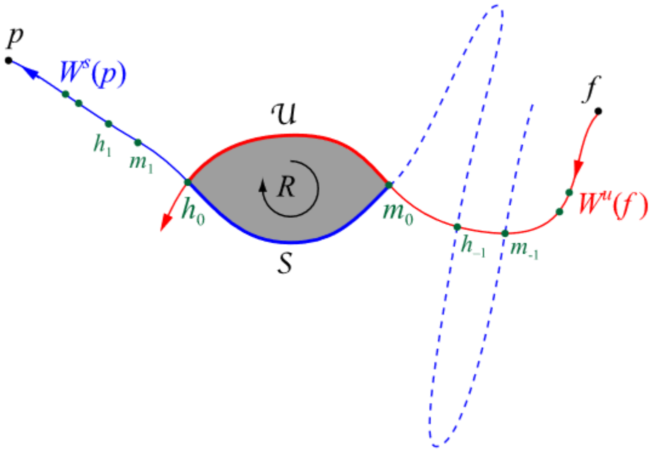


FIG. 8. Lobe formed from segments of stable, $\mathcal{S}$ and unstable, $\mathcal{U}$, manifold of hyperbolic fixed points p and f, respectively, with heteroclinic orbits $\{h_t\}$ and $\{m_t\}$. The lobe area $\mu(R)$ is the difference in between the actions of heteroclinic orbits (21).

Meiss, 'Thirty Years of Turnstiles and Transport'.

$$\mu(R) = \sum_{t=-\infty}^{\infty} (\lambda(m_t) - \lambda(h_t)) = \Delta W[m,h]$$

(heteroclinic lobe area).

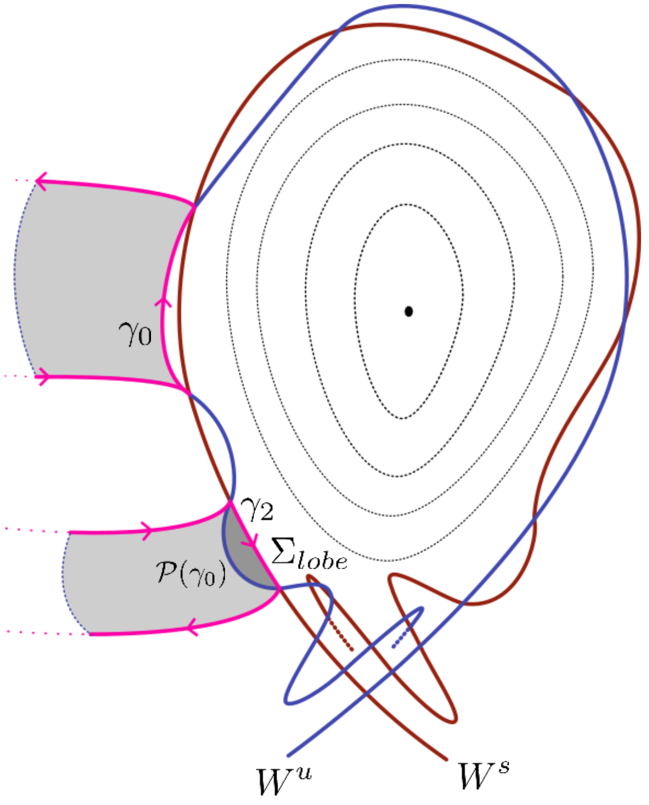
Turnstile flux is computed by integrating A along a specific path

- How to calculate the flux Φ_{lobe} through the surface Σ_{lobe}
- Take the curve γ_0 which is part of the unstable manifold and do one forward iteration. It brings it to $\mathcal{P}(\gamma_0)$
- The boundary of Σ_{lobe} is $\partial\Sigma_{lobe} = \mathcal{P}(\gamma_0) + \gamma_2$
- The flux on the Σ_{ev} surface is null because $\boldsymbol{B}$ lies in it, and therefore

$$\Phi_{lobe} = \int_{\Sigma_{lobe}} \mathbf{B} \cdot \mathbf{dS} = \int_{\mathcal{P}(\gamma_0)+\gamma_2} \mathbf{A} \cdot \mathbf{dl}$$

with $\Gamma = \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3$

- The γ_1 and $-\gamma_3$ describe are the trajectory of homoclinic points



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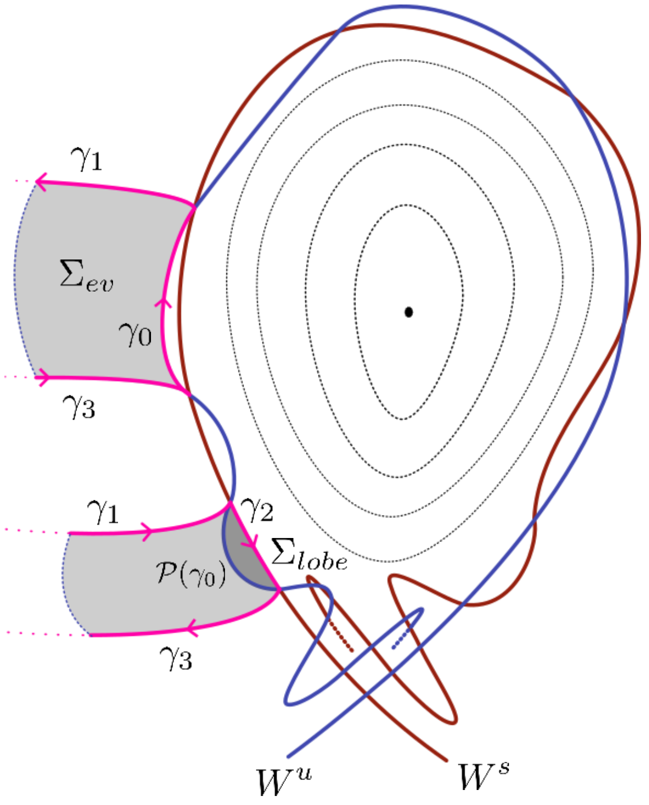
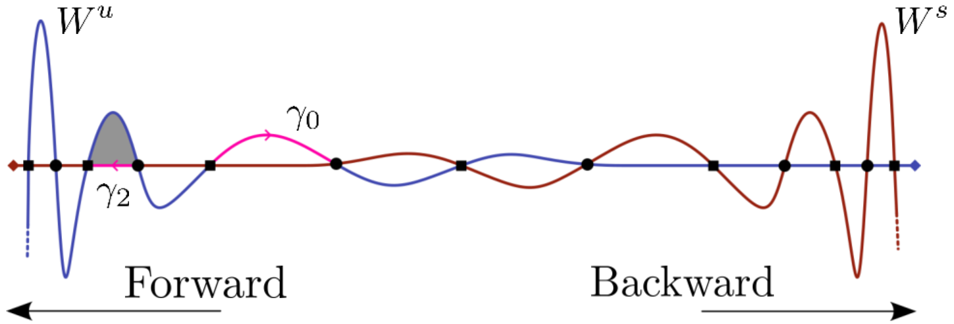
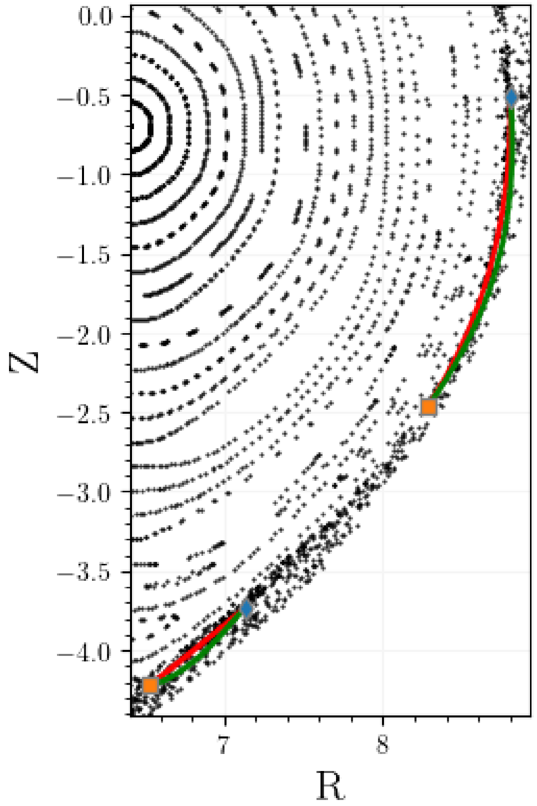


Recursively applying the map reduces the remaining integrals to zero

- The flux is conserved, same flux through Σ_{lobe} as through $\mathcal{P}^k\,(\Sigma_{lobe})$ for any k
- Taking γ_0 mapping it backward will make its length go to zero
- Similarly taking γ_2 mapping it forward will have a same effect
- Therefore, as the integrals are bounded, they converges to zero





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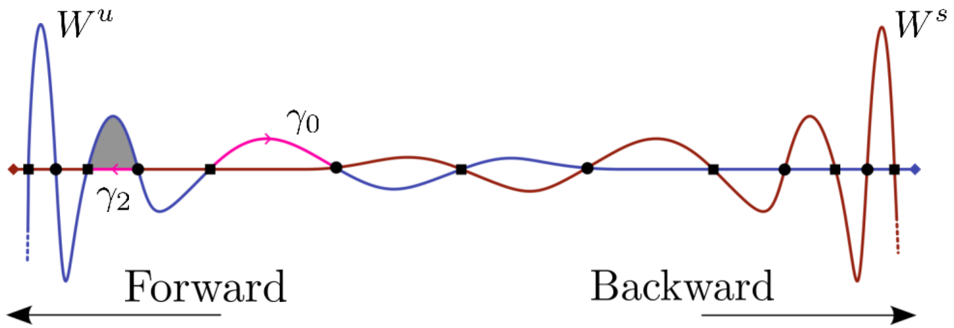

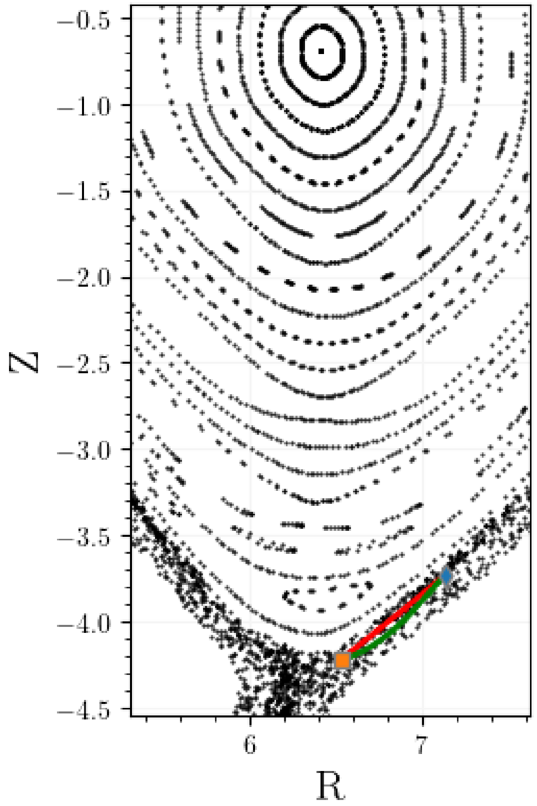

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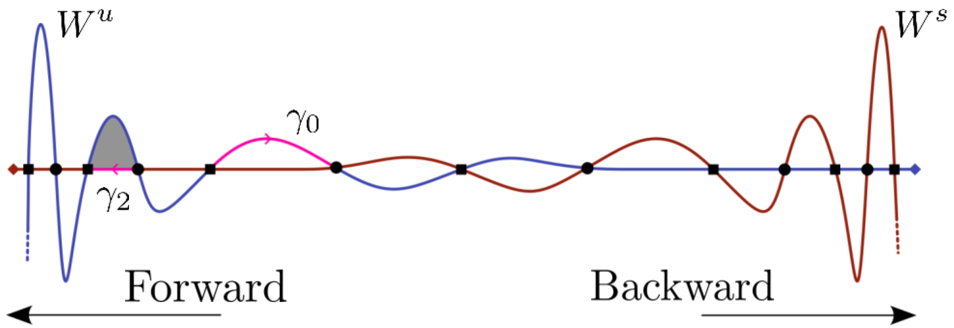

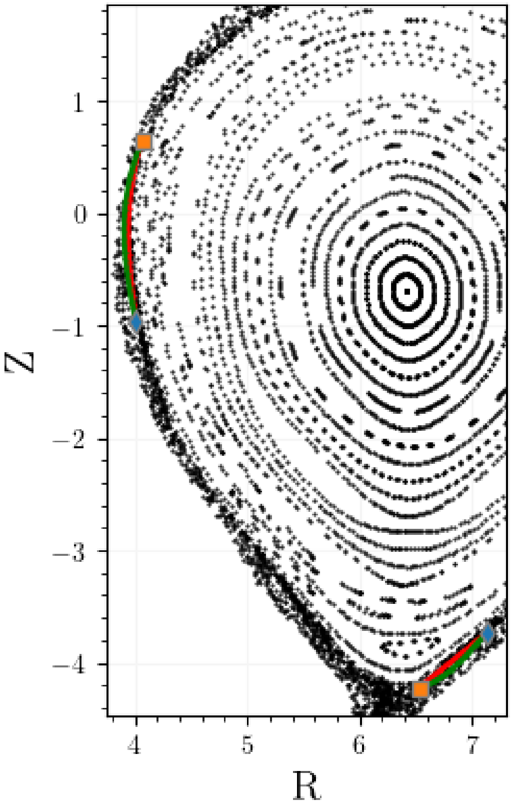

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Ludovic Rais

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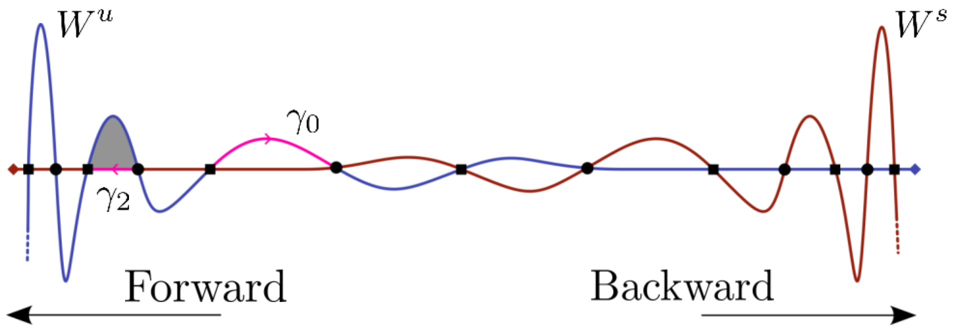

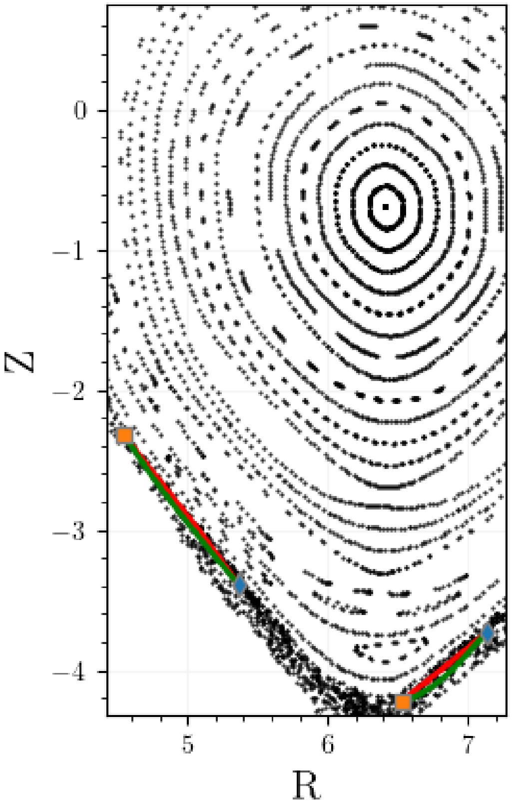

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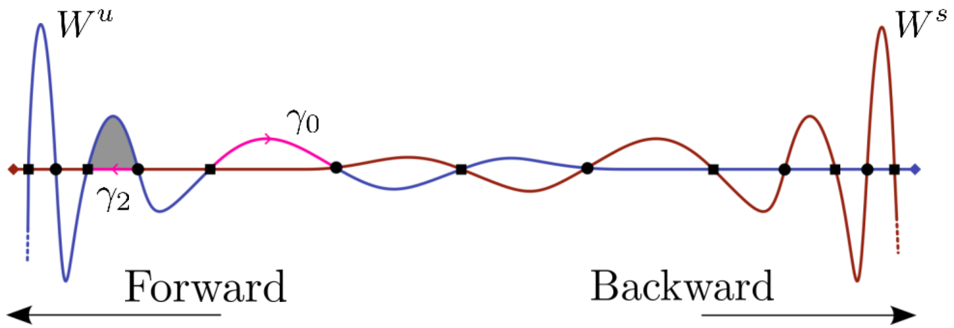

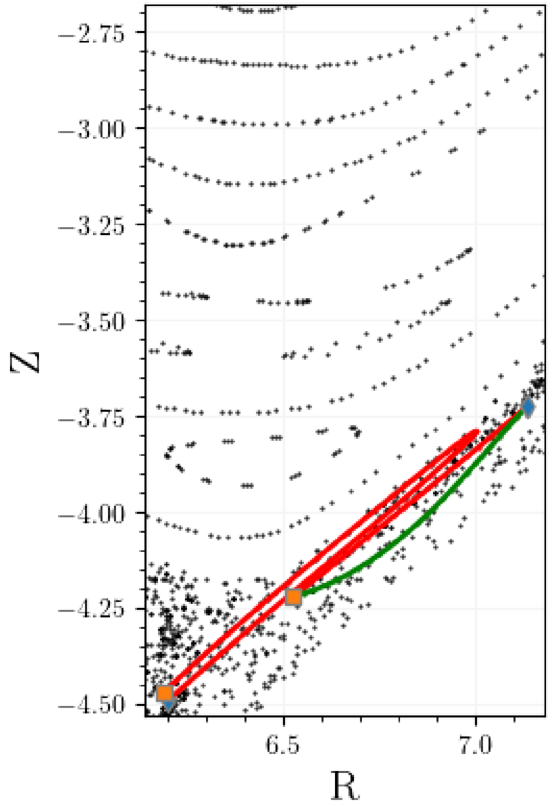

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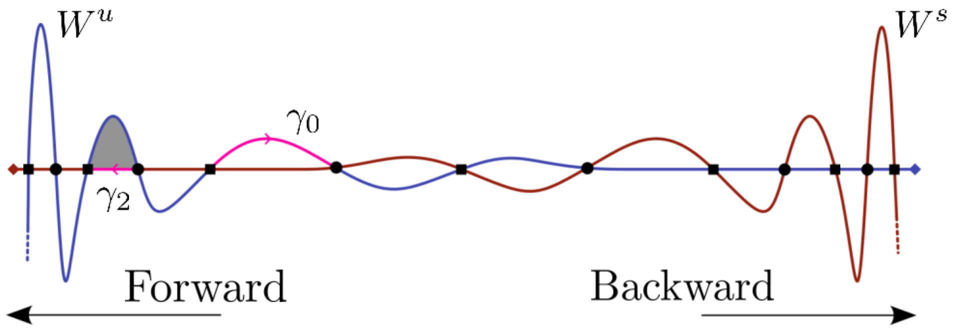

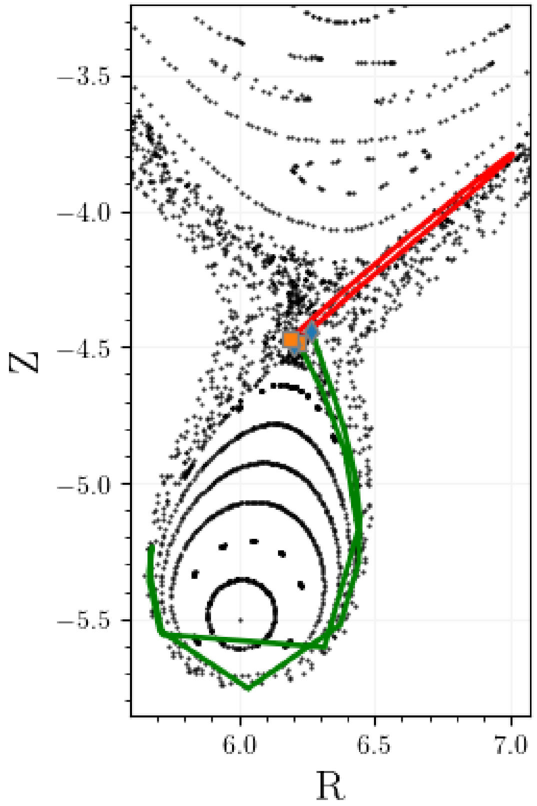

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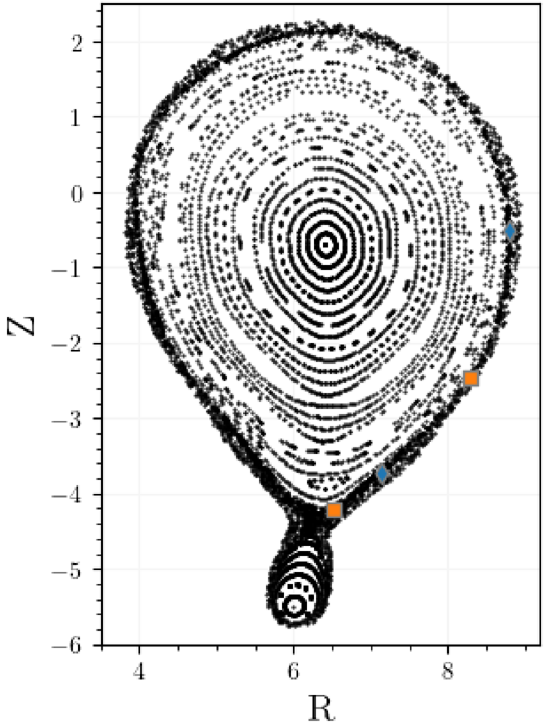

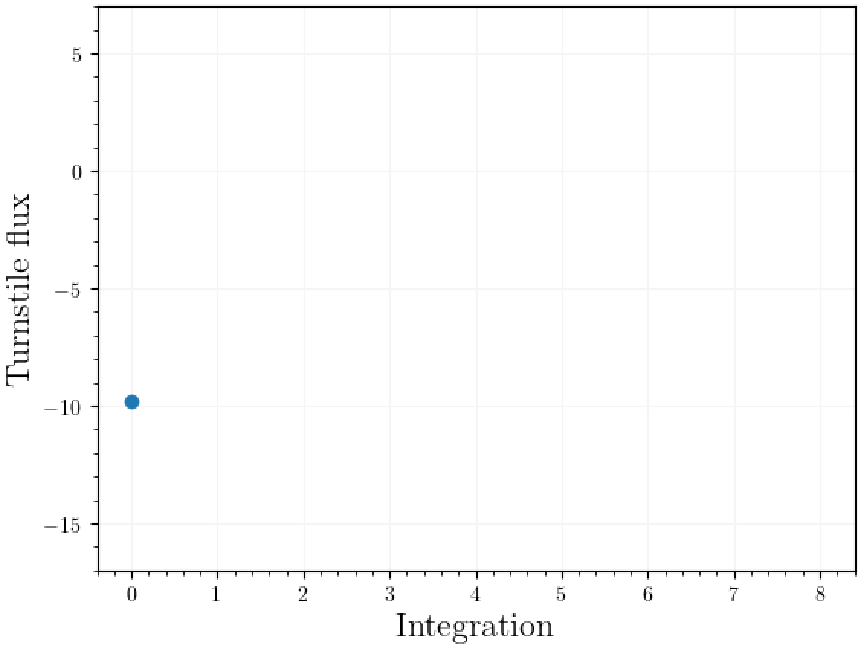

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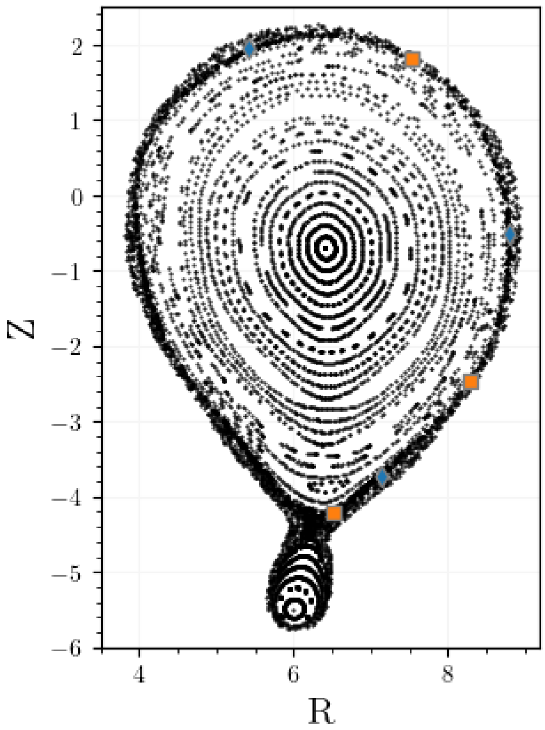

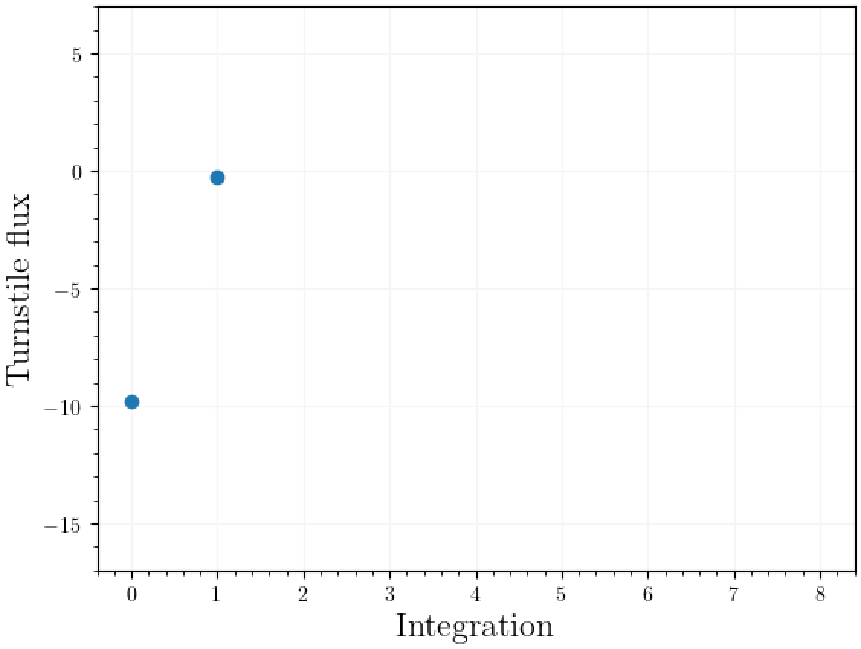

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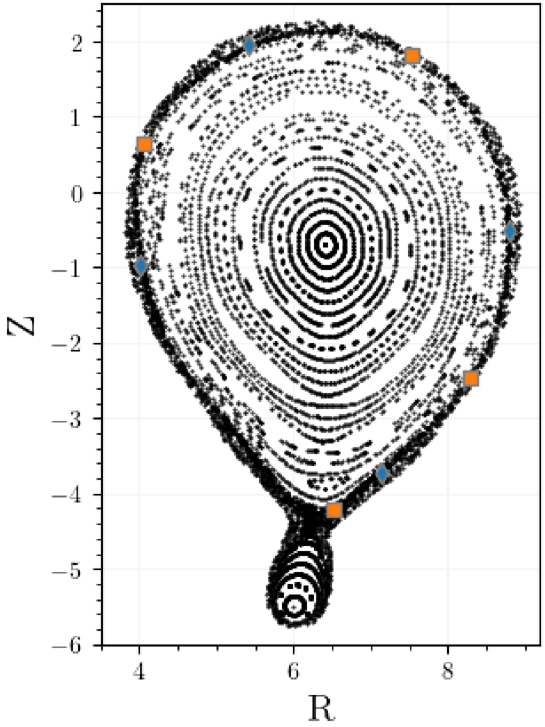

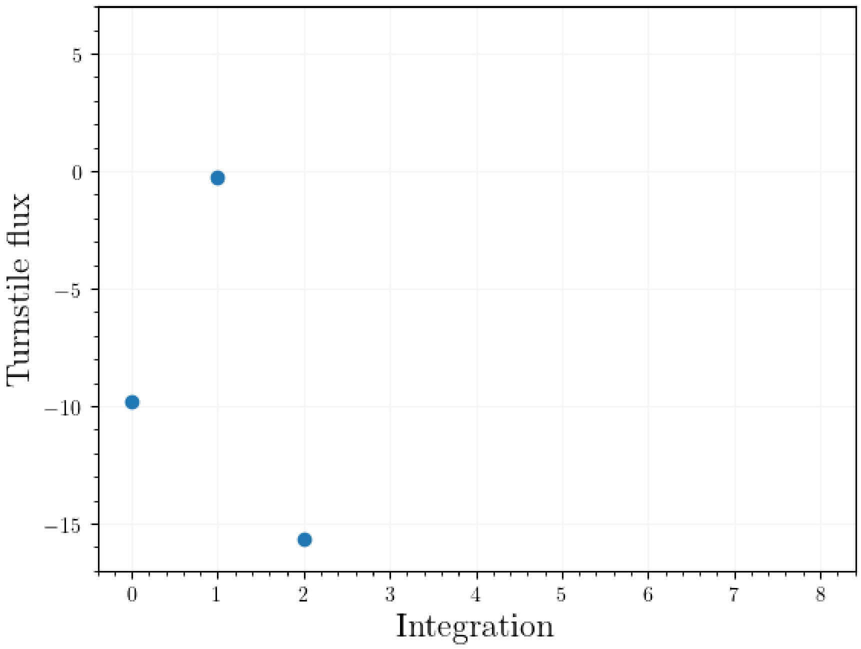

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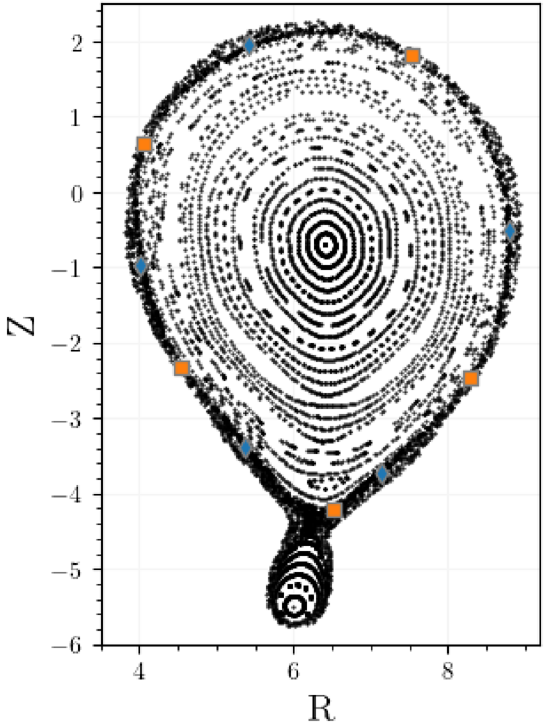

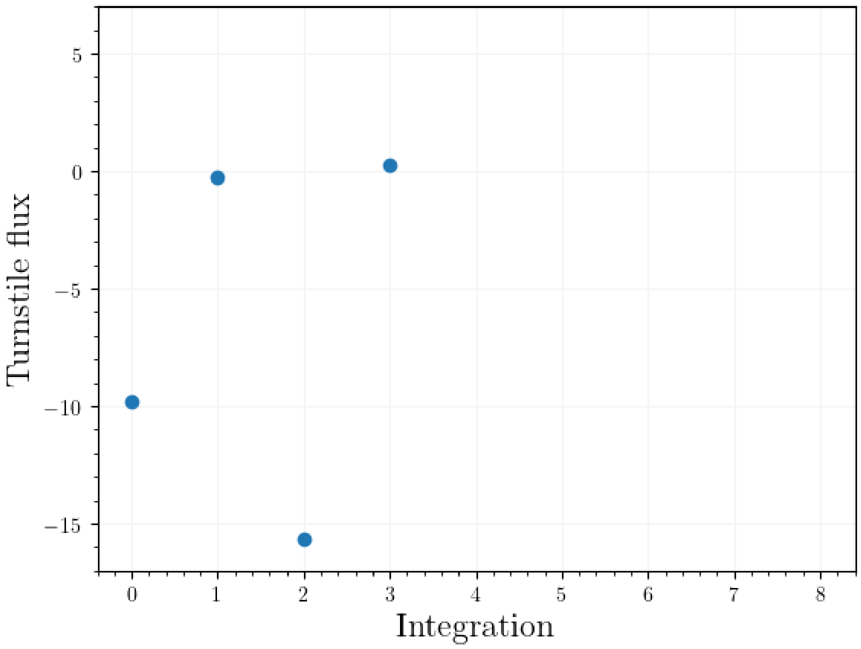

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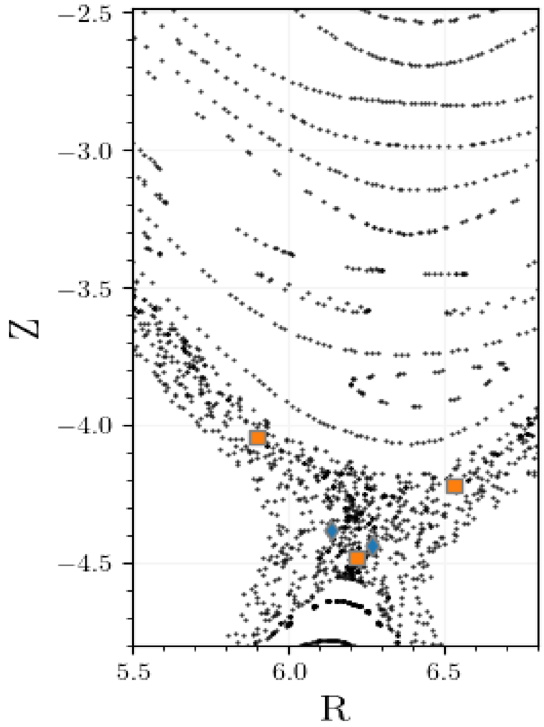

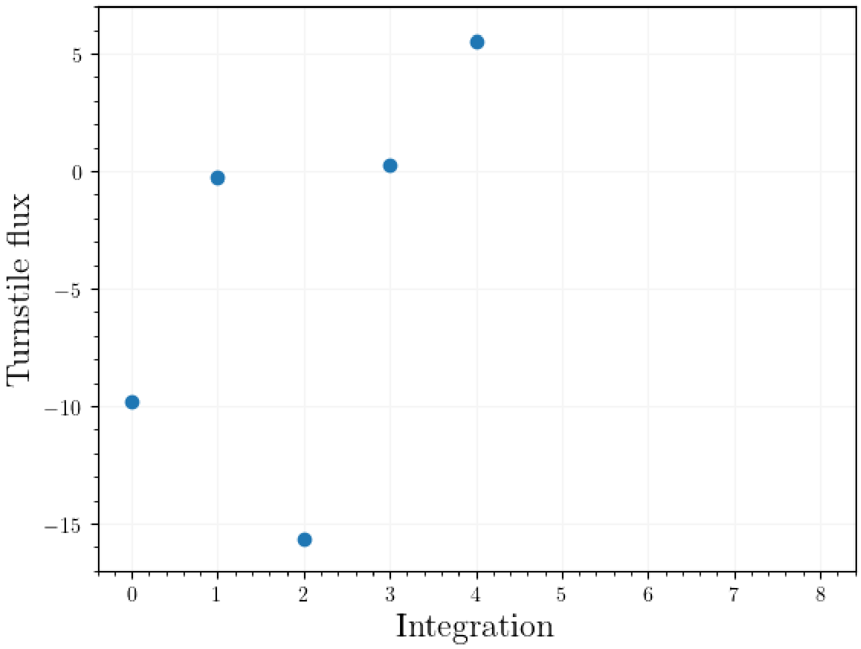

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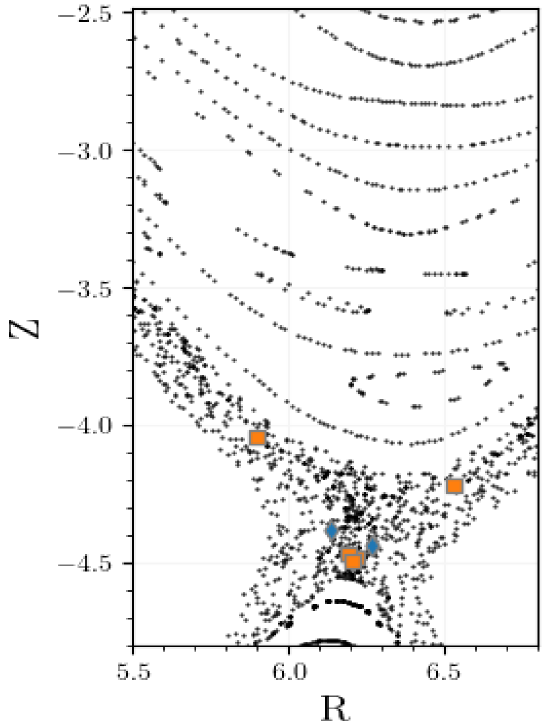

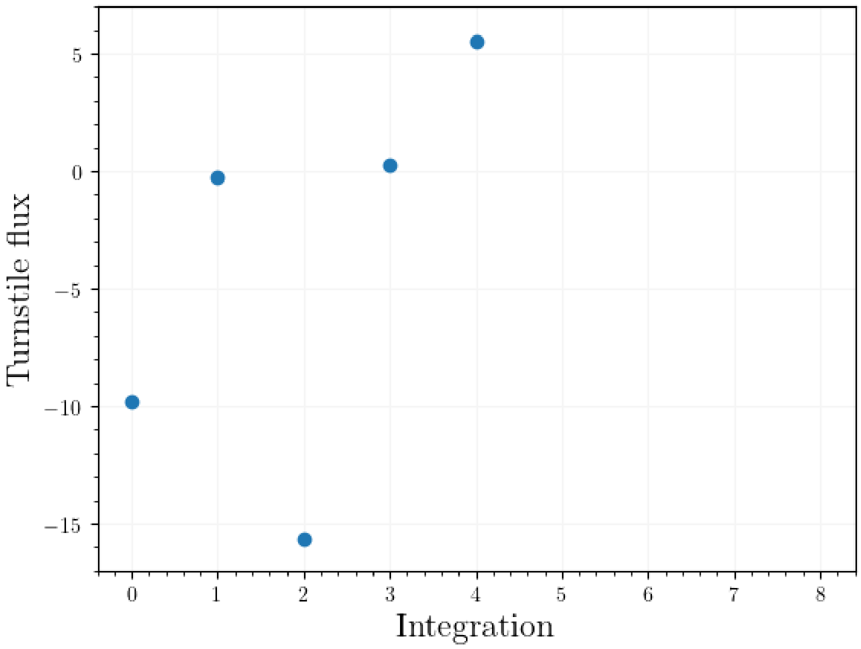

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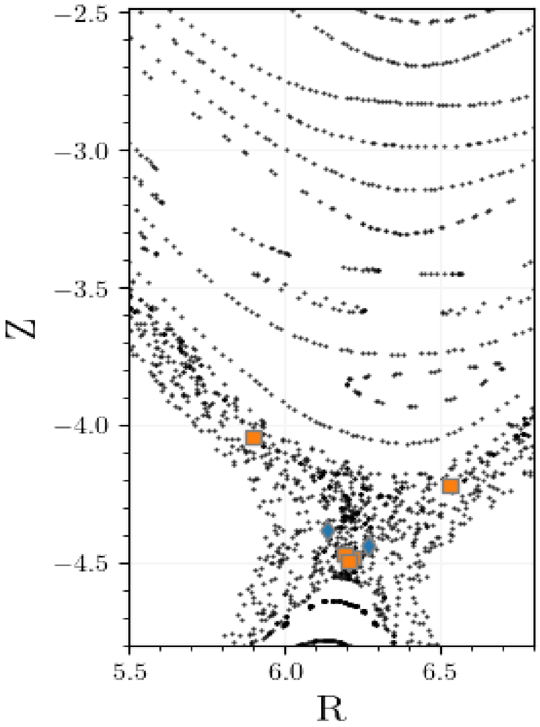

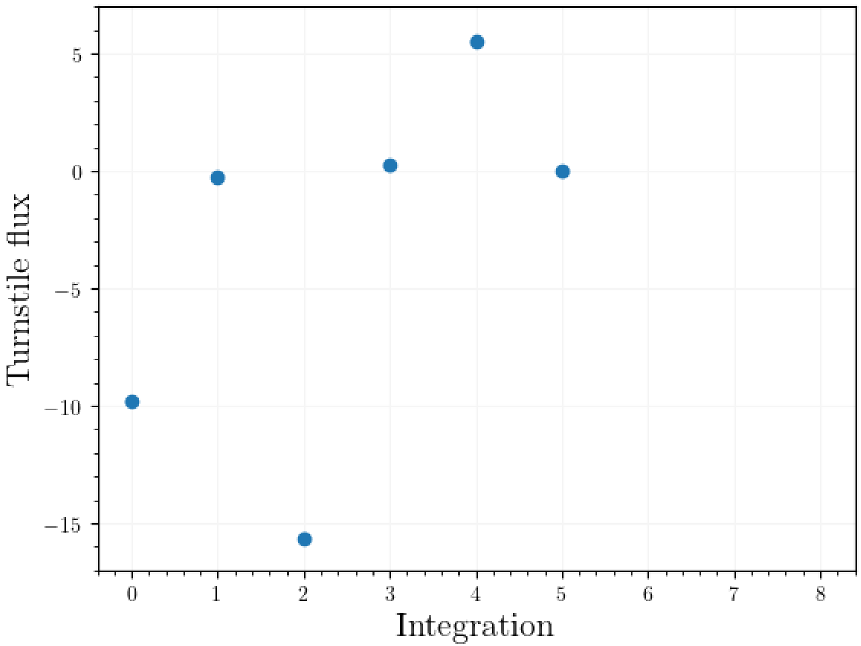

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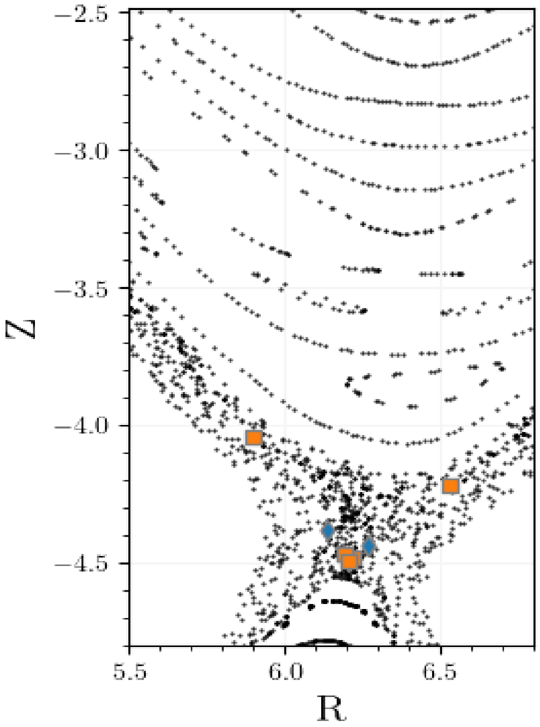

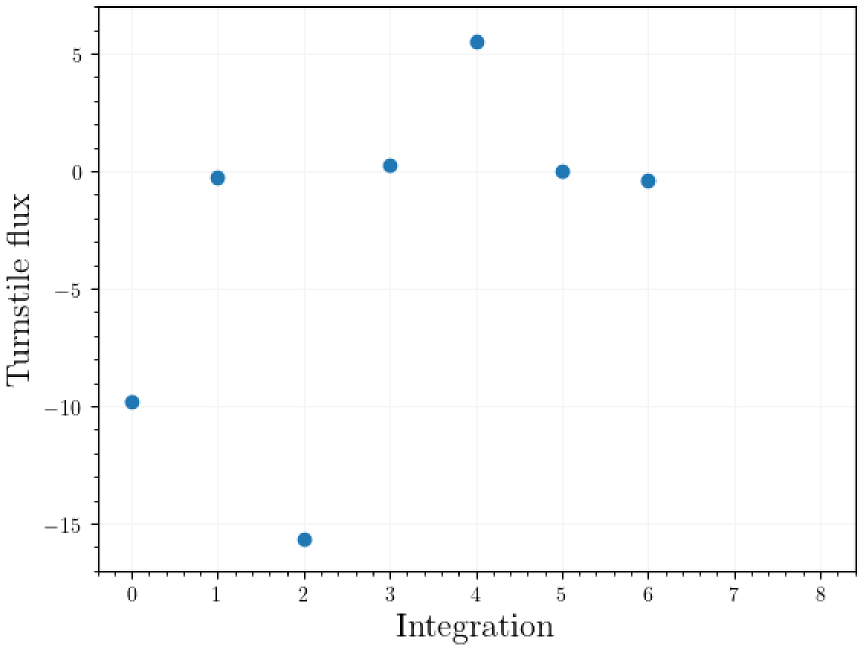

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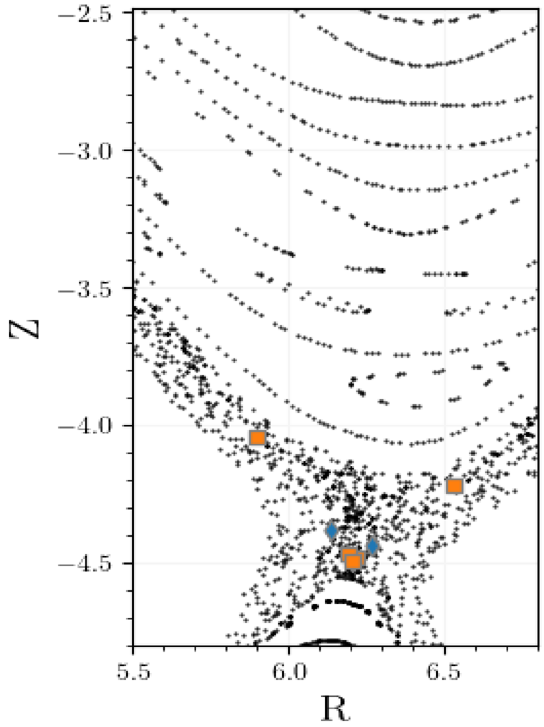

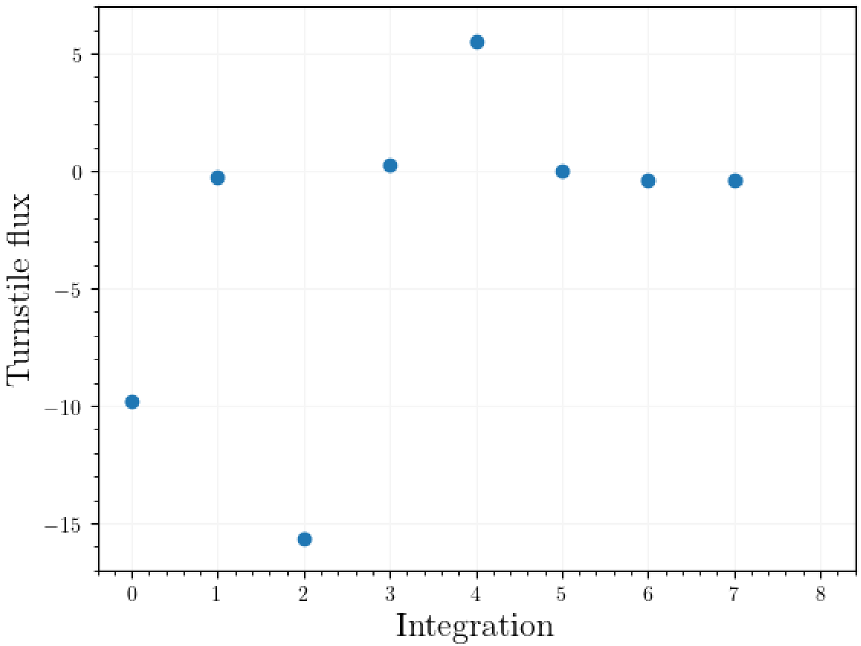

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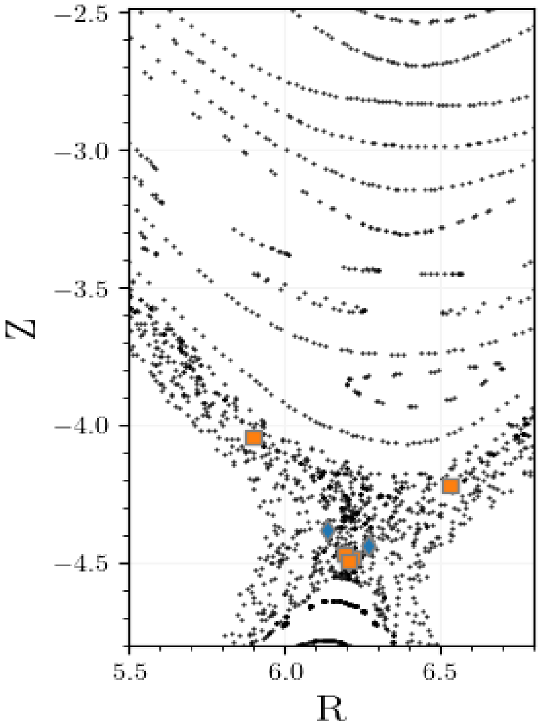

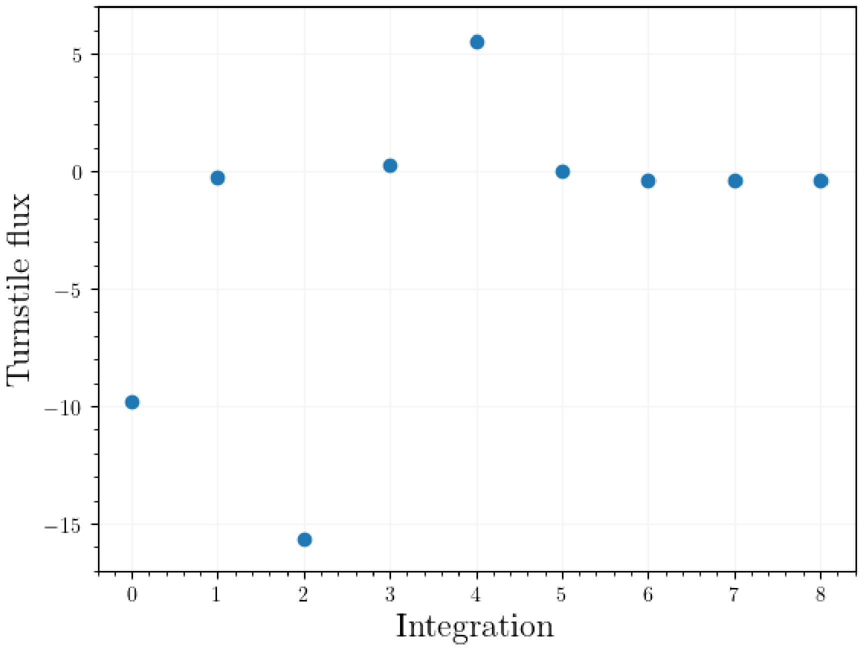

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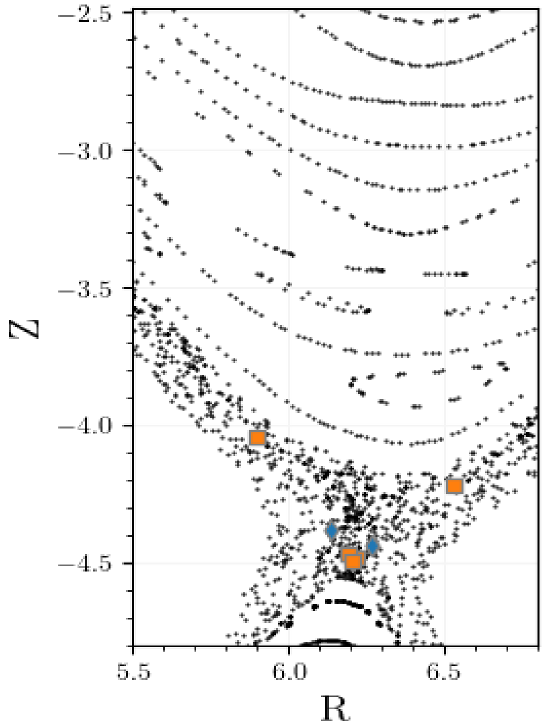
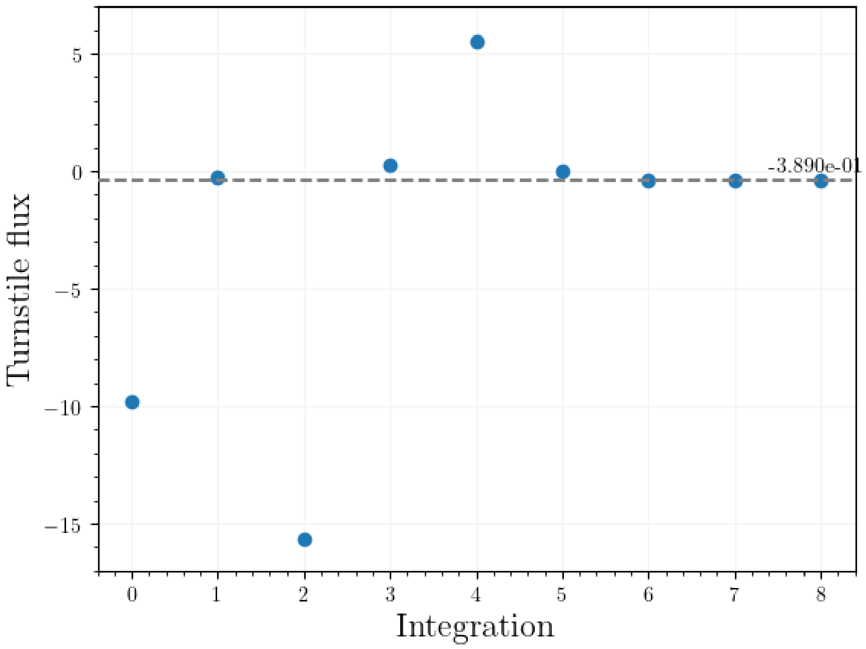




Turnstile flux value is verified by simple approximations

- The calculated flux is $\Phi_{lobe} = -3.8899 \cdot 10^{-1}$
- Taking an estimation of the error due to the non consideration of the joining integrals
  - From one iteration to the next : $5.7 \cdot 10^{-4}$
  - By a evaluating the $A \cdot \Delta l$ at the mid point : $4.16 \cdot 10^{-5}$
- Approximating the flux by

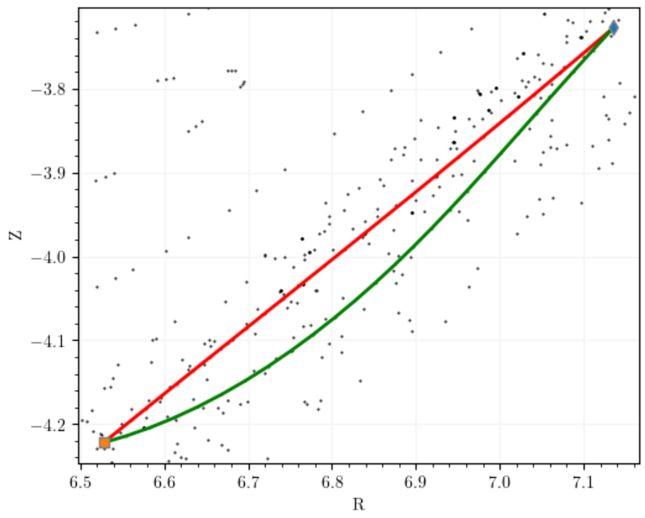
$$\Phi_{lobe} = \int_{\Sigma} \mathbf{B} \cdot \mathbf{dS} \approx |\Sigma| \tilde{B}^{\phi} \quad \tilde{B}^{\phi} \simeq -14.262$$

- Estimating the area by a simple triangle

$$|\Sigma| \simeq 2.9439 \cdot 10^{-2} \implies \Phi_{lobe} \simeq -4.198 \cdot 10^{-1}$$

- Estimating the area by the shoelace formula (area of a polygon by triangulariztion)

$$|\Sigma| \simeq 2.728 \cdot 10^{-2} \implies \Phi_{lobe} \simeq -3.890 \cdot 10^{-1}$$



EPFL

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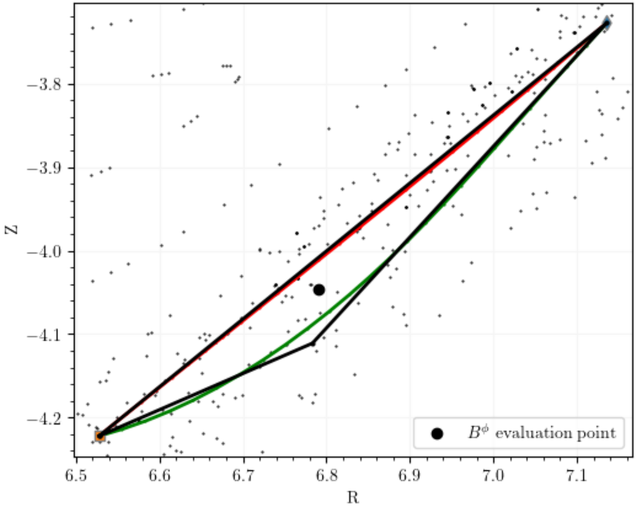
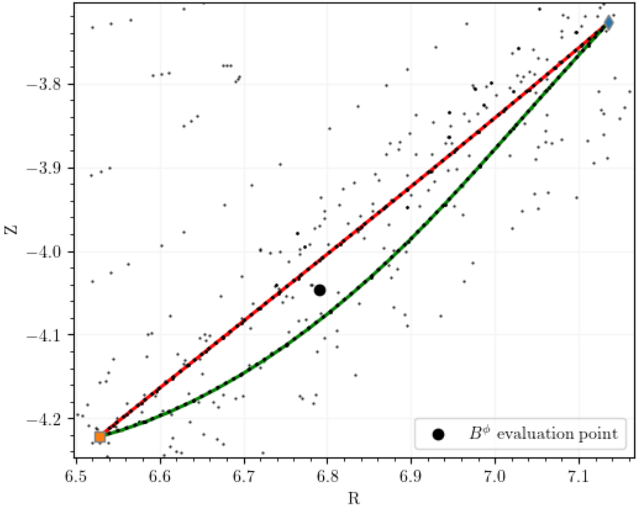
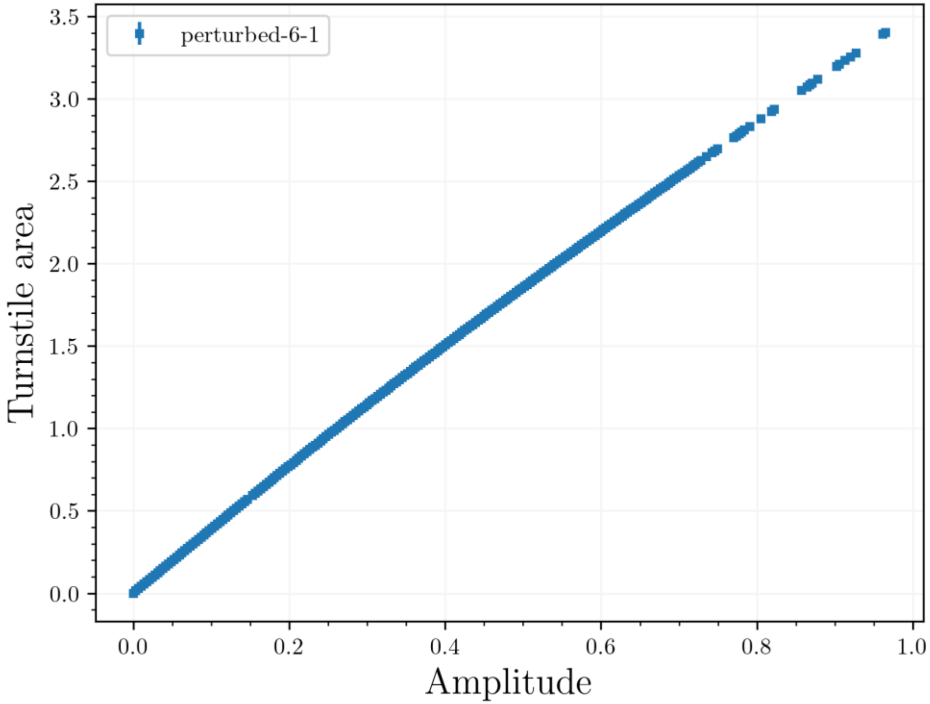
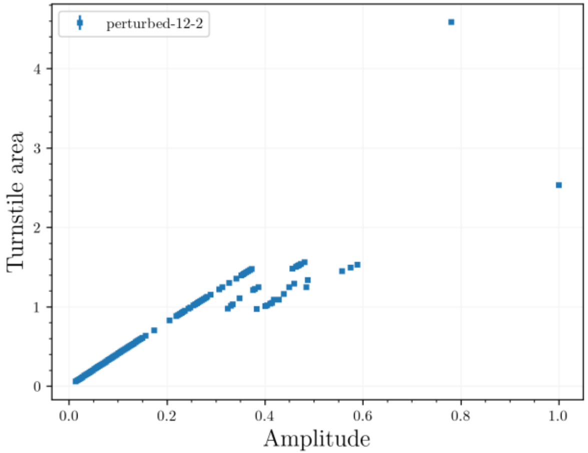
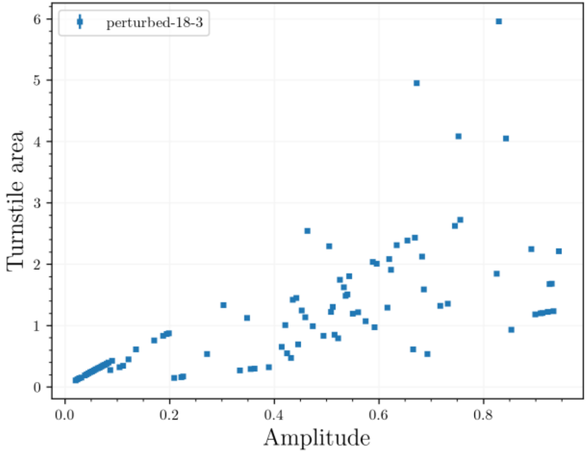




Amplitude scan







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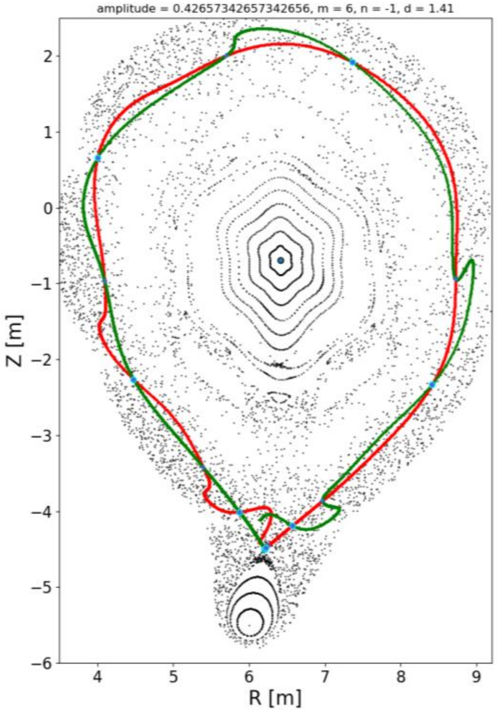

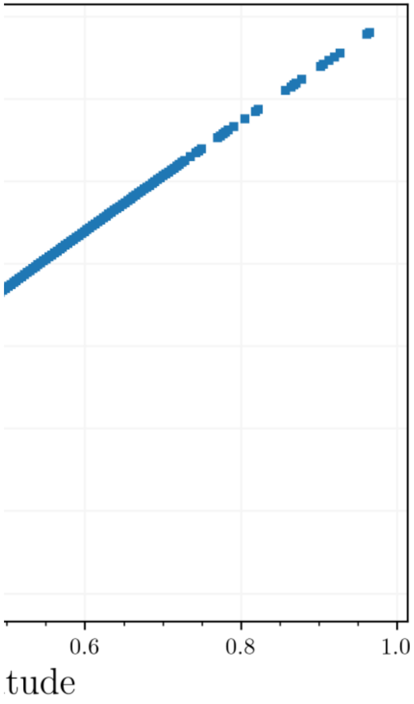

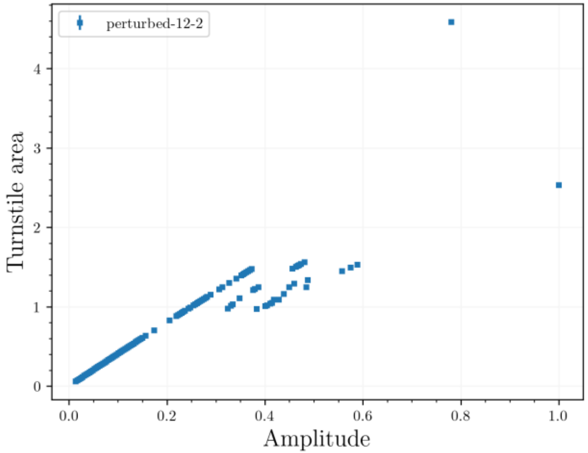

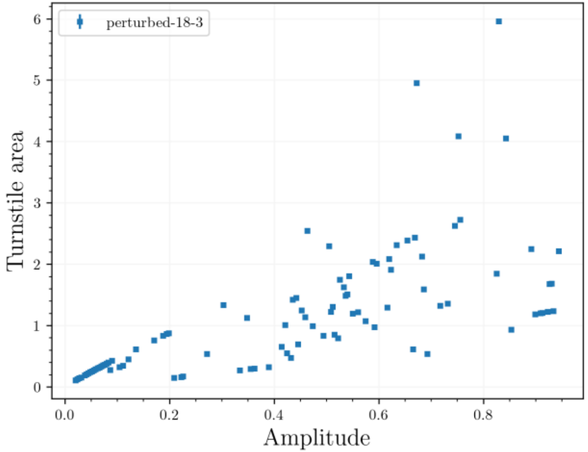

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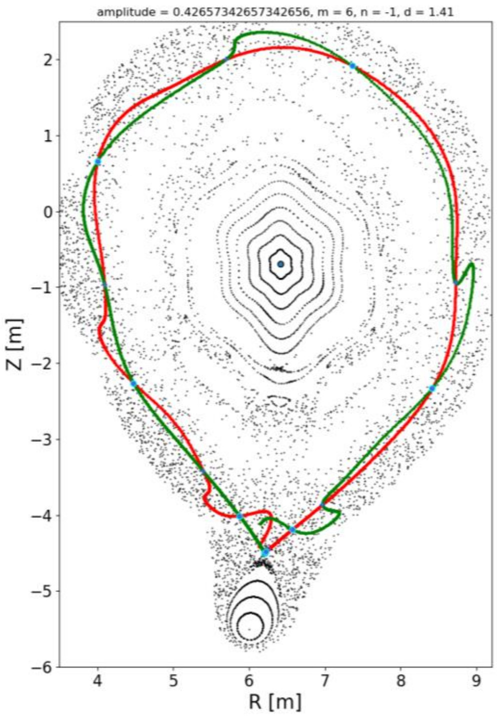

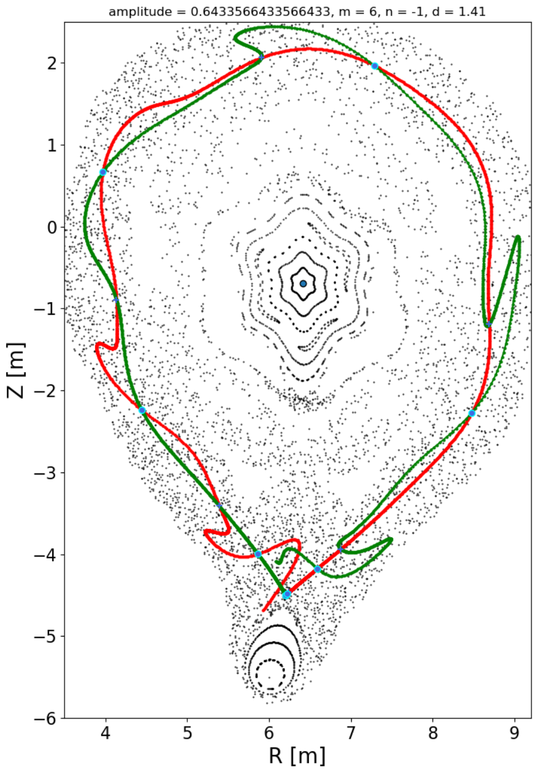

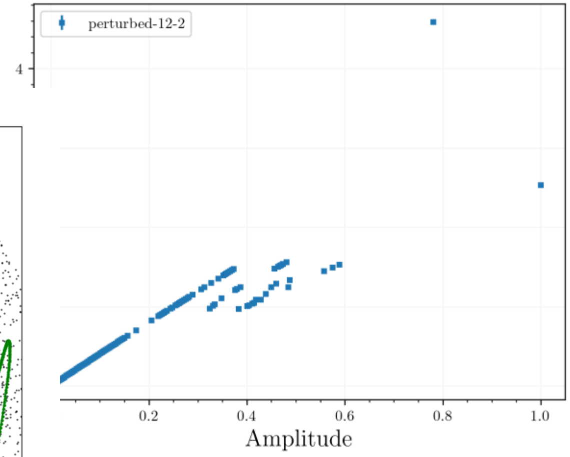

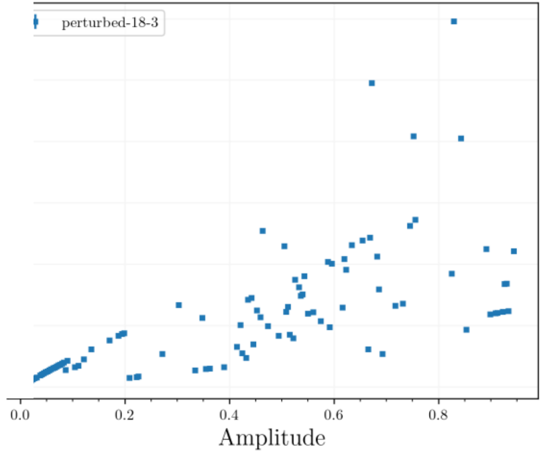

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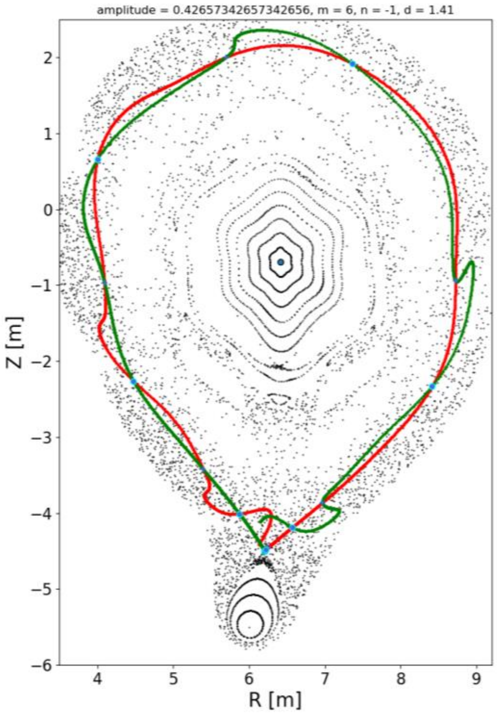

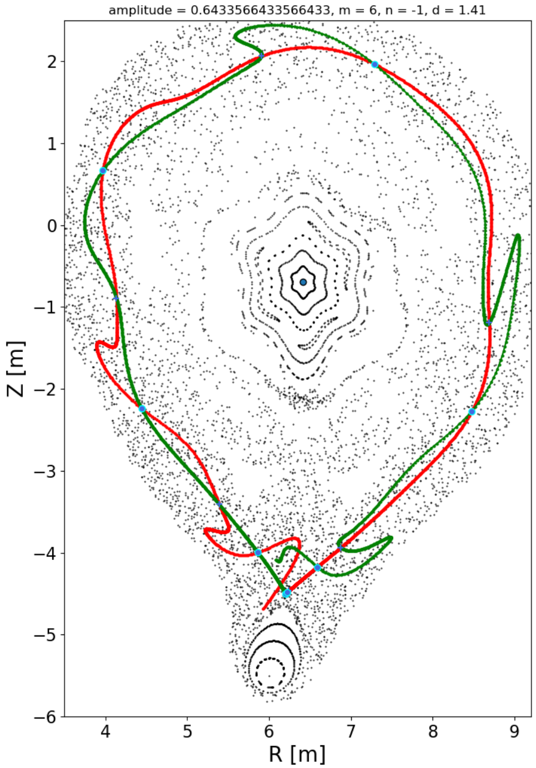

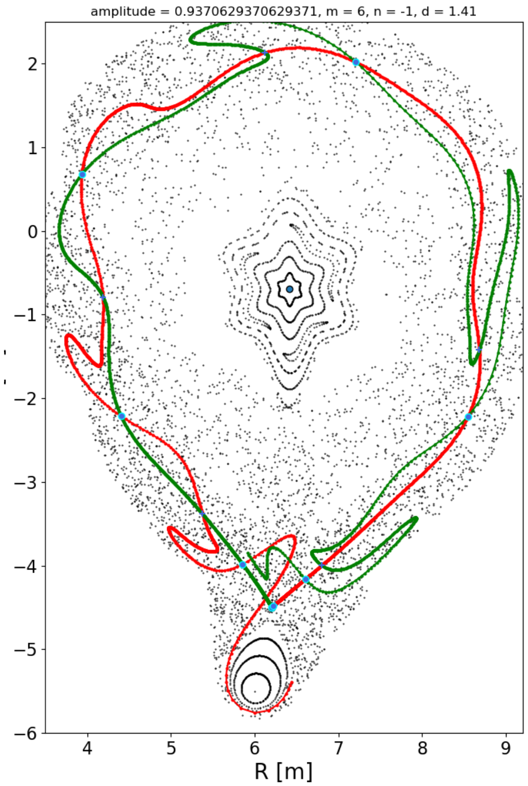

Amplitude

For islands the manifolds link different fixed points and have inner and outer turnstiles

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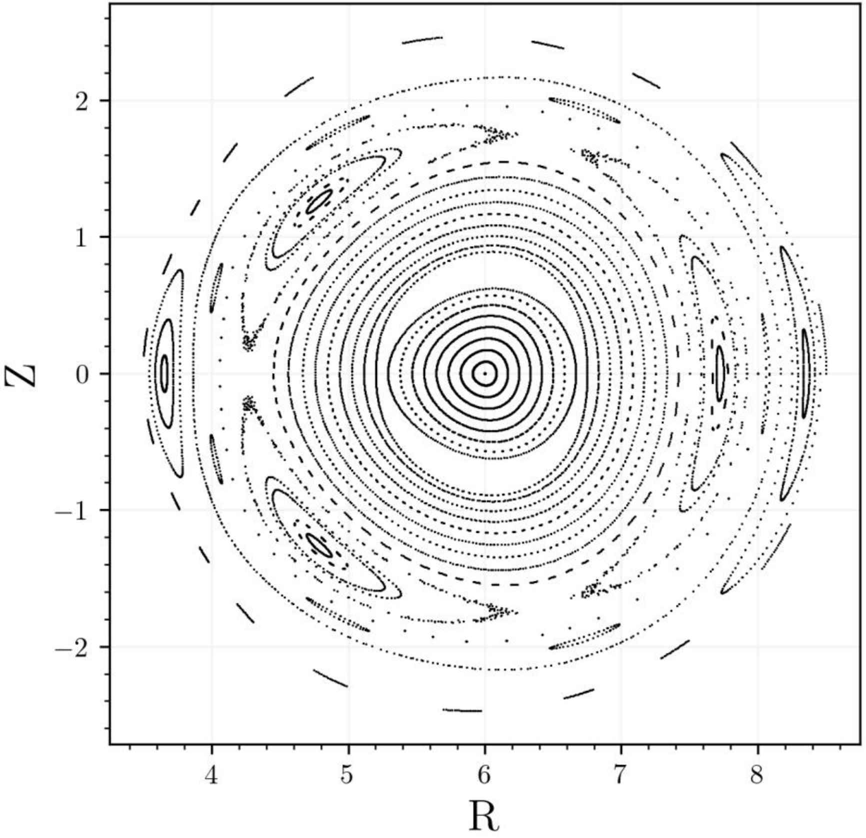
- Toybox quadratic q-profile with a $m/n = 3/2$ perturbation
- There is a tangle structure on the inside and one on the outside
- Four heteroclinic points
- Turnstile calculation gives

	inner	outer
Turnstile flux $[10^{-4}]$	4.17	29.47
Relative Err_{estim}	0.106	0.004
Sum of turnstiles $[10^{-5}]$	0.0	-2.3



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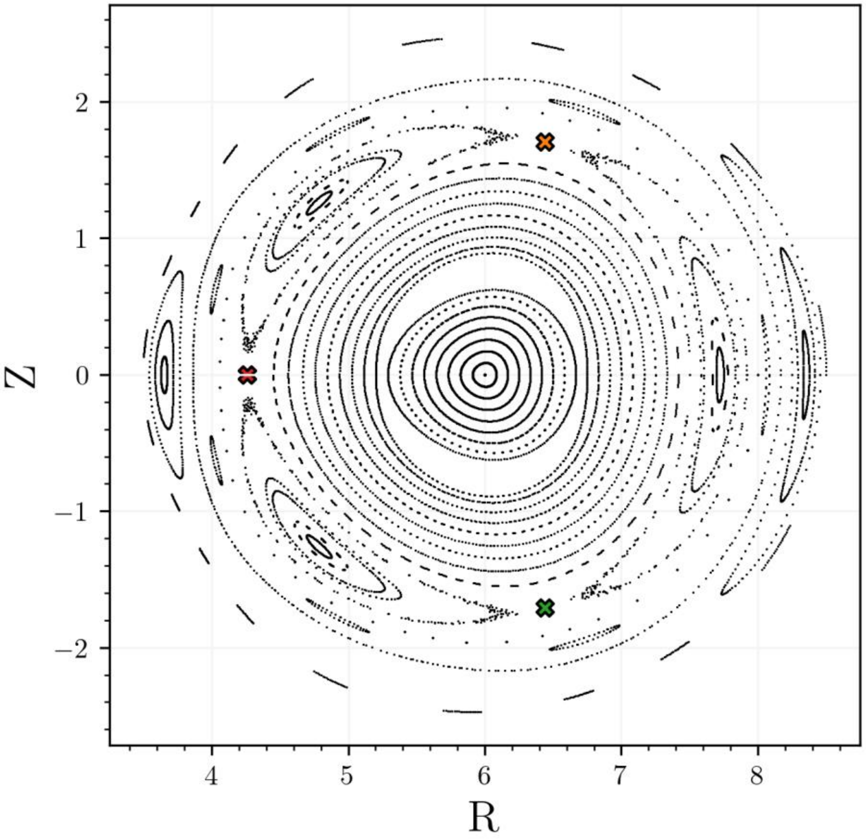

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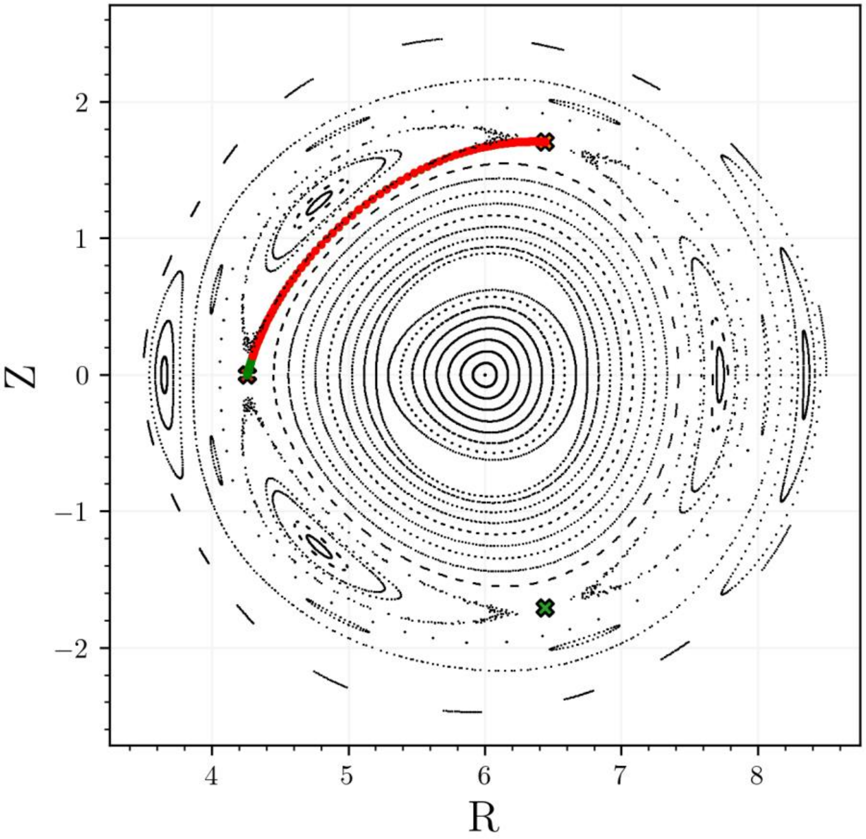

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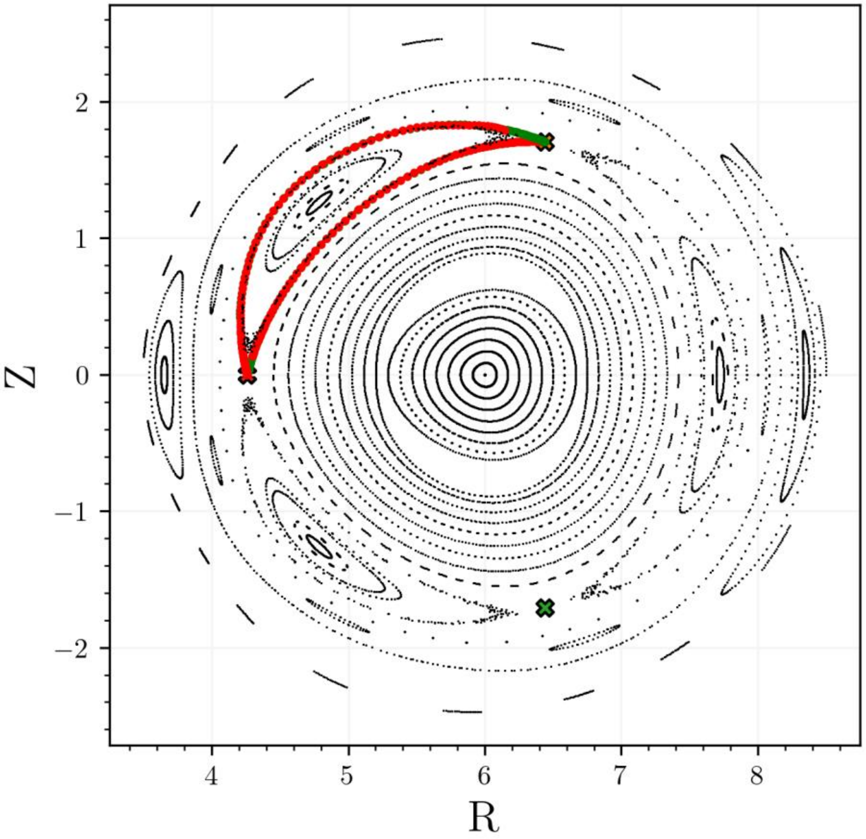

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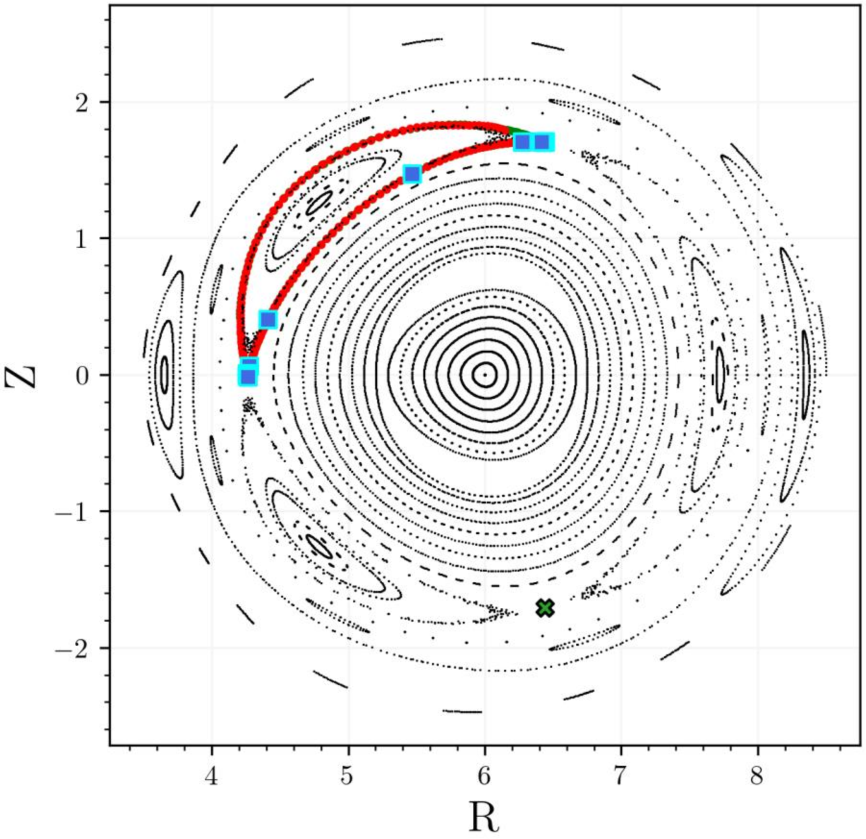

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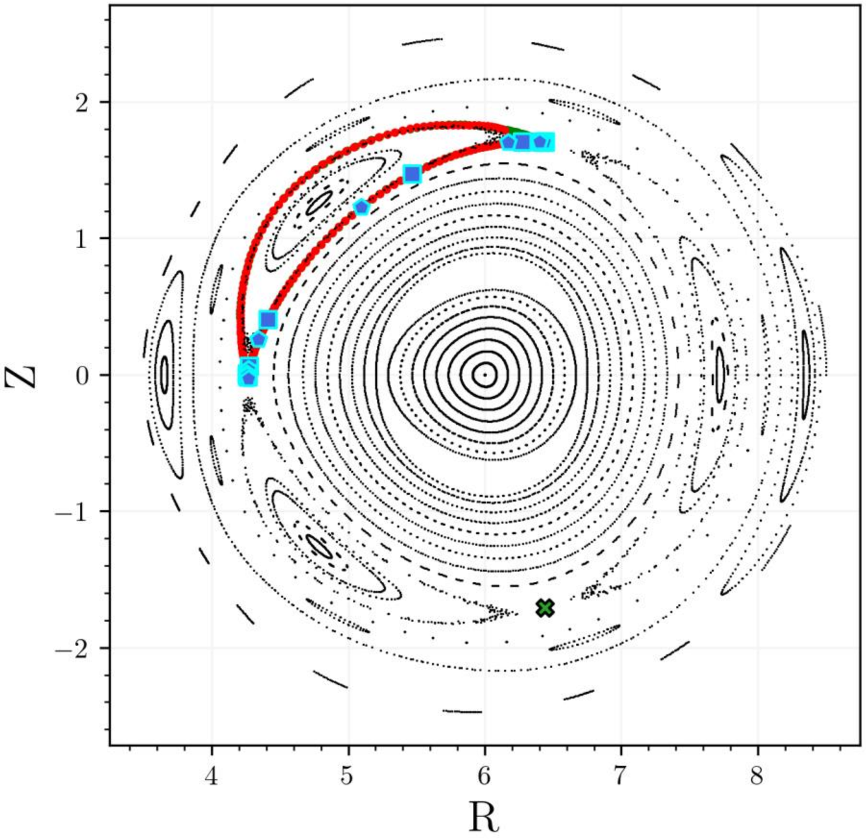

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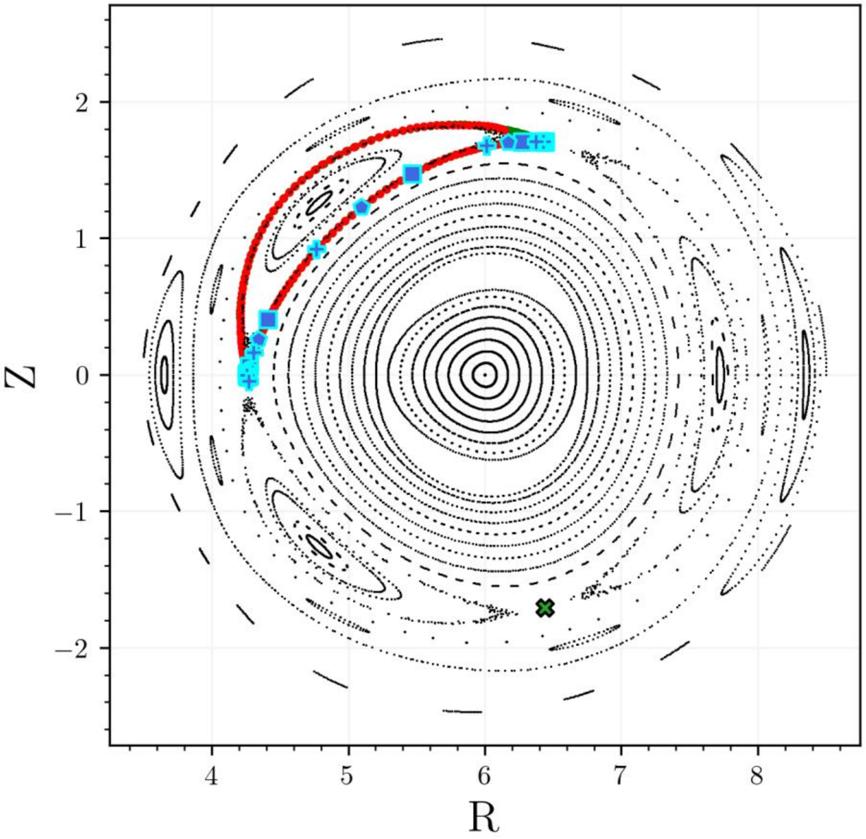

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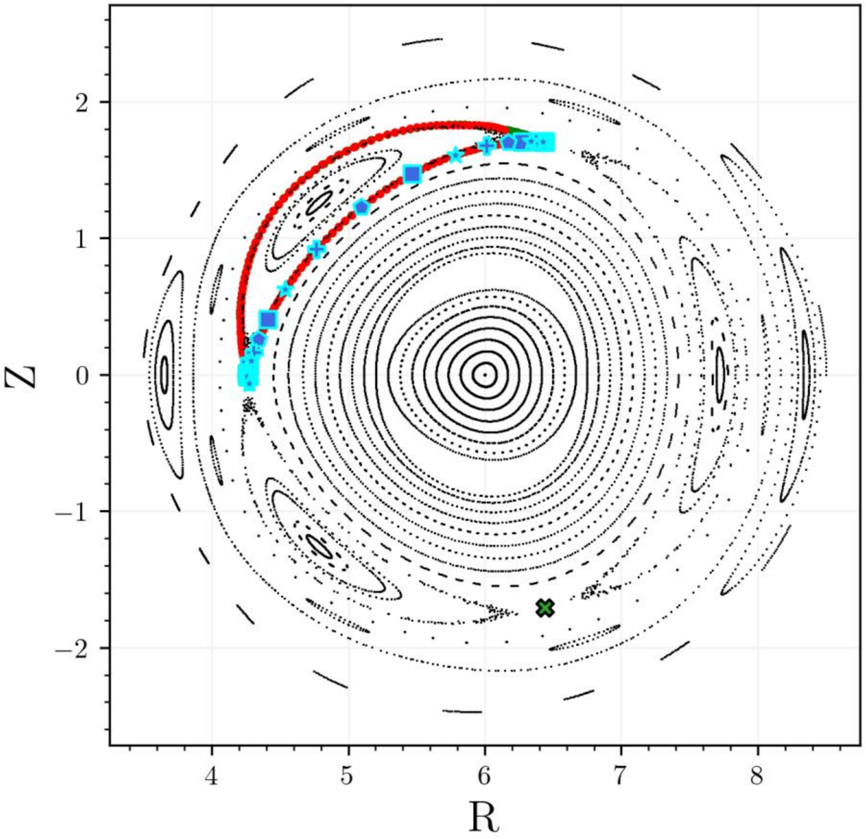

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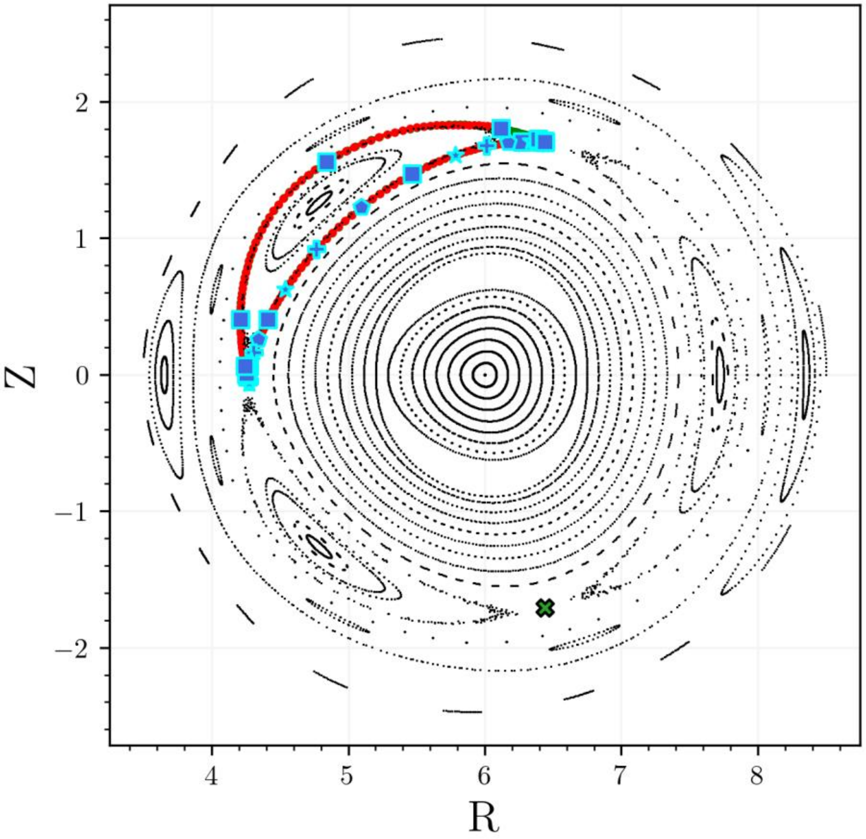

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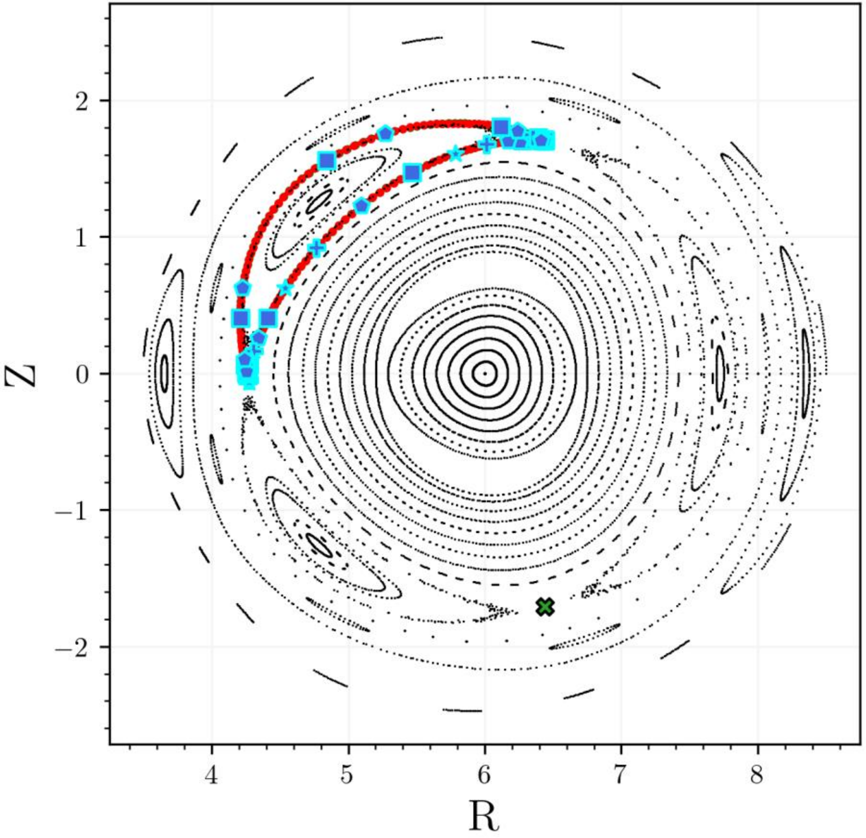

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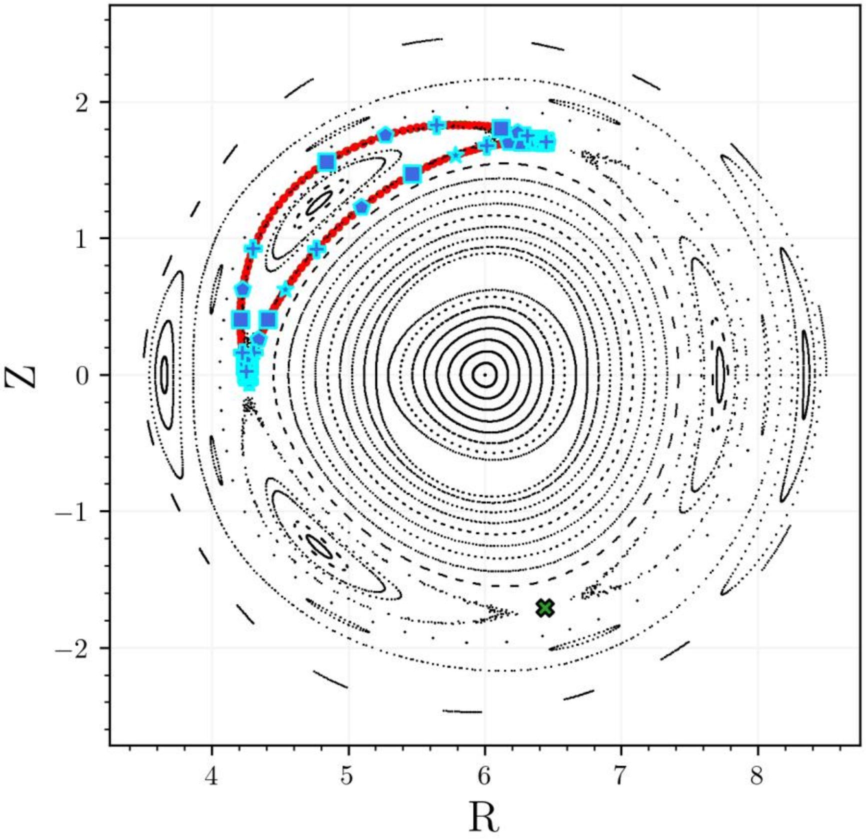

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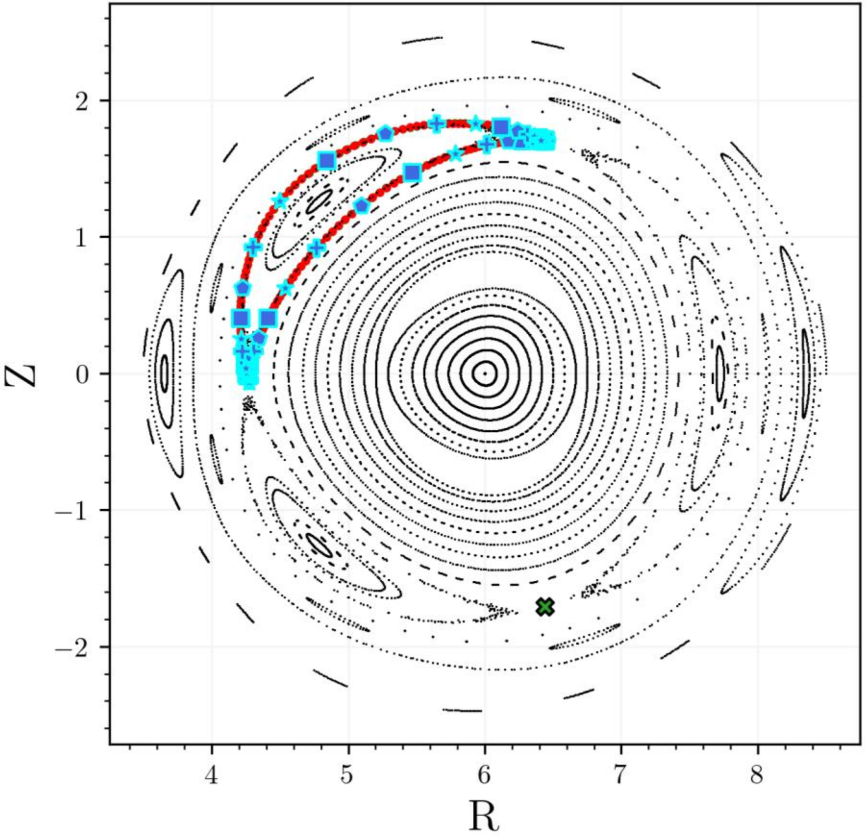

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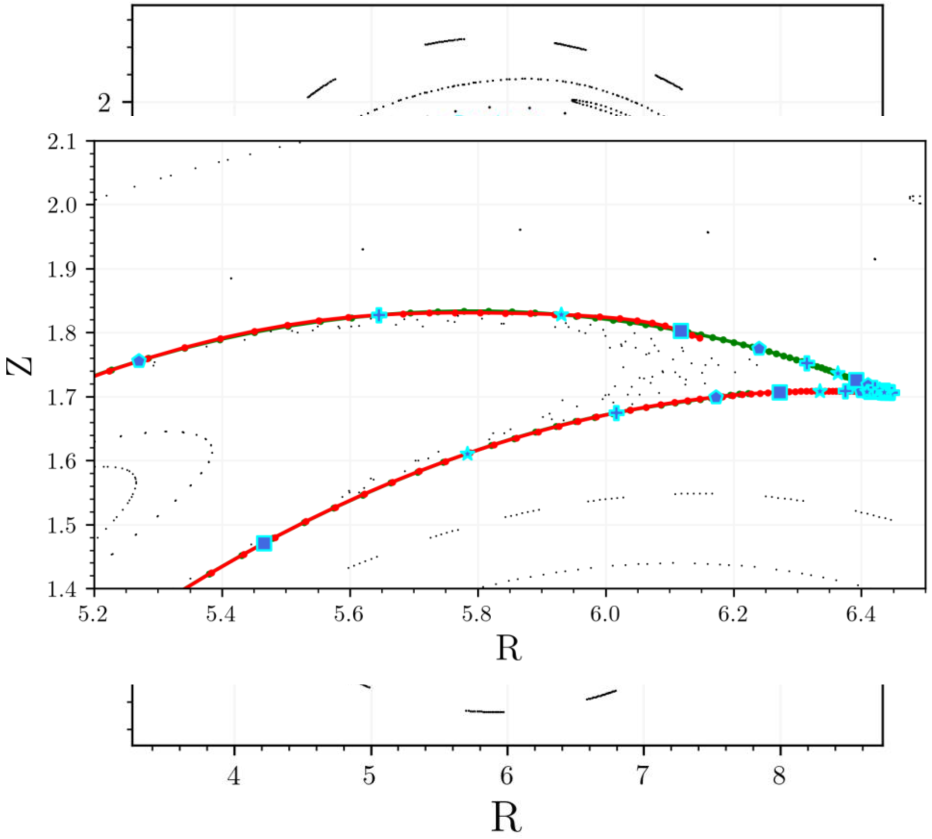
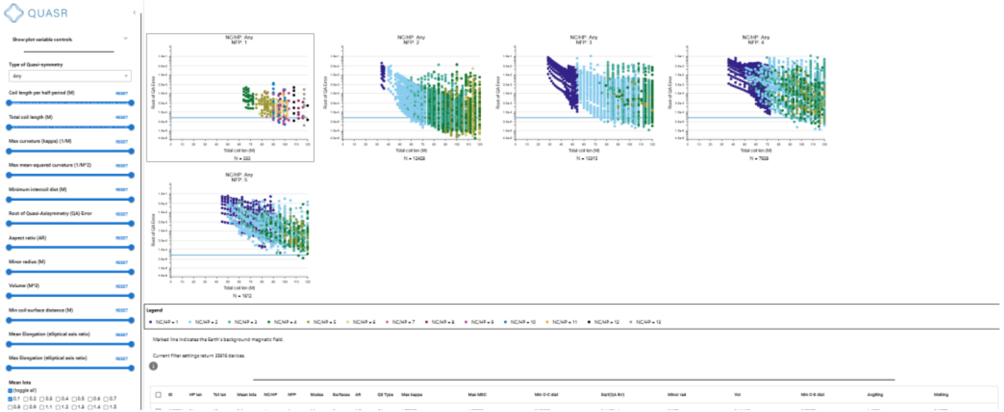


QUASR, an open source database of stellarartors

- ***Andrew Giuliani***, Flatiron Institute (NYU)
- Optimized for volume quasi-symmetry : QA and QH
- Quasr database contains over 320’000 coil sets
- Ability to fine tune the search and select disered configurations



	ID	HP l...	Tot len	Mean Iota	NC/HP	NFP	Modes	Surfaces	AR	QS Type	Max kappa	Max MSC	Min C-C di
☑	71451	7	56	0.6	1	4	16	3	6.7	QA	4.82728	5.00002	0.09999
☐	941453	7	56	0.3	1	4	16	3	6.7	QA	4.40566	5.00003	0.09999
☐	943581	7	56	0.5	1	4	16	3	6.7	QA	4.94169	5.00003	0.10000

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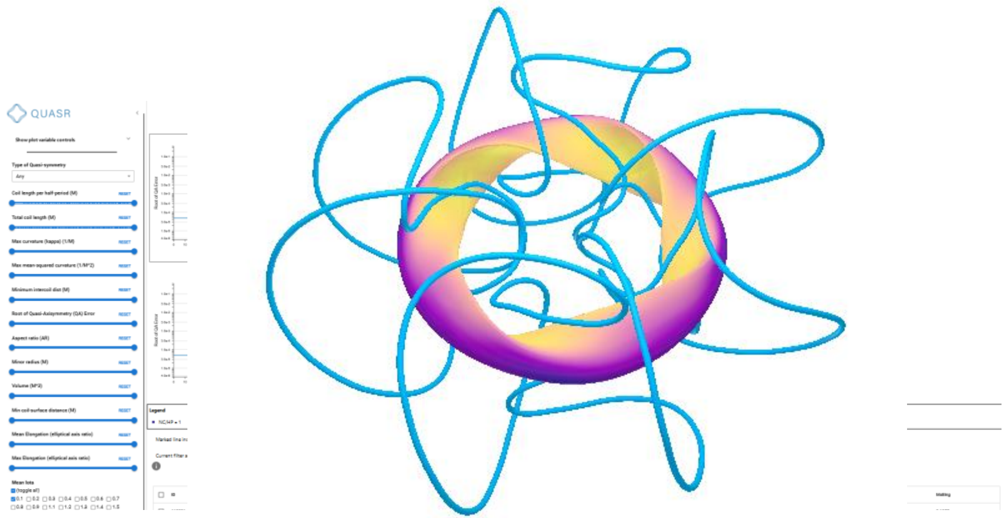

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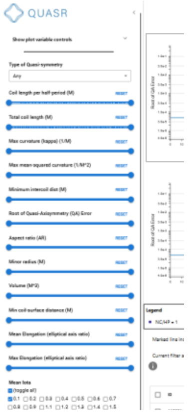

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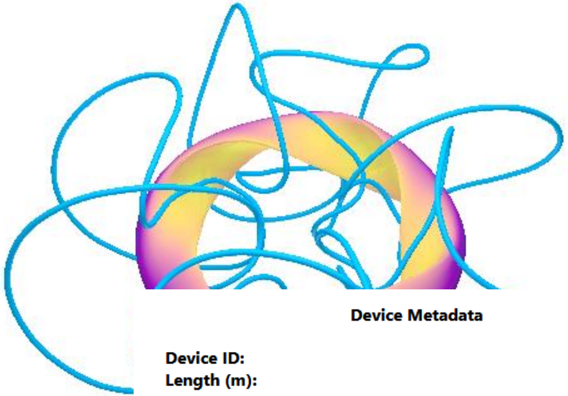


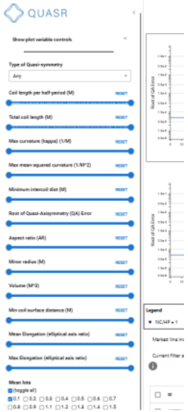
Device Metadata

Device ID:	71451
Length (m):	56.000192167464334 (7.000024020933042 per half-period)
Root of QA Error:	1.1831950762e-2
Mean Iota:	0.6
Coils per HP:	1
Number of Field Periods (NFP):	4
Volume (M^3):	0.444132198
Max curvature (kappa) (1/M):	4.8272845368
Max mean-sq curve (1/M^2):	5.0000180835
Minor radius (M):	0.149999334
Minimum intercoil dist (M):	0.0999870181
Min coil-surface distance (M):	0.1986511655
Mean Elongation (elliptical axis ratio):	3.0583866359
Max Elongation (elliptical axis ratio):	4.5015394681
Aspect ratio (AR):	6.6667554715
Type of Quasi-symmetry:	QA

EPFL

QUASR, an open source database of stellarartors

- ***Andrew Giuliani***, Flatiron Institute (NYU)
- Optimized for volume quasi-symmetry : QA and QH
- Quasr database contains over 320’000 coil sets
- Ability to fine tune the search and select disered configurations



	ID	HP l... ↓	Tot len	
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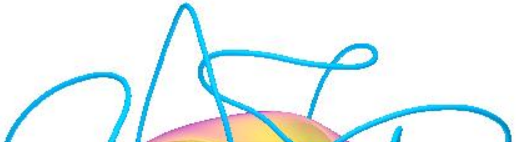
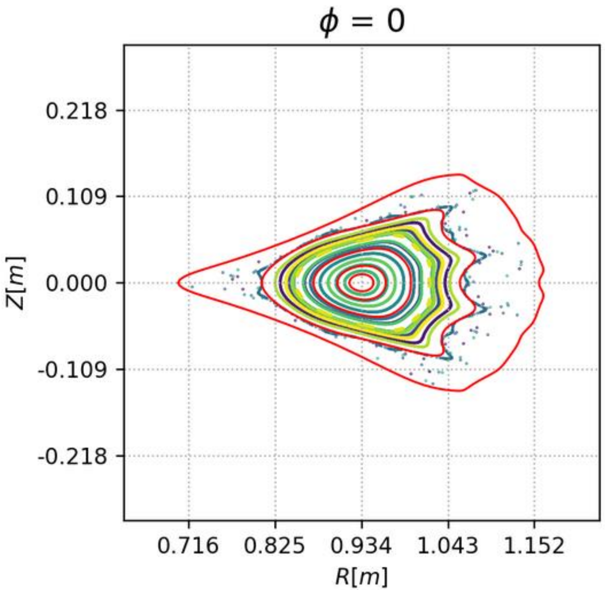
☐	941453	7	56	0.
☐	943581	7	56	0.





71451
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831950762e-2
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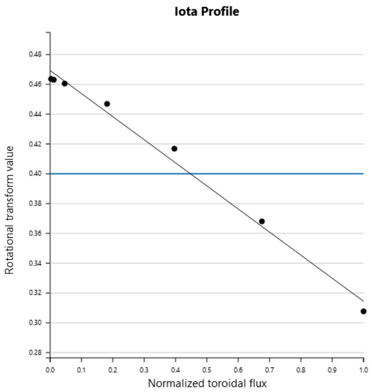
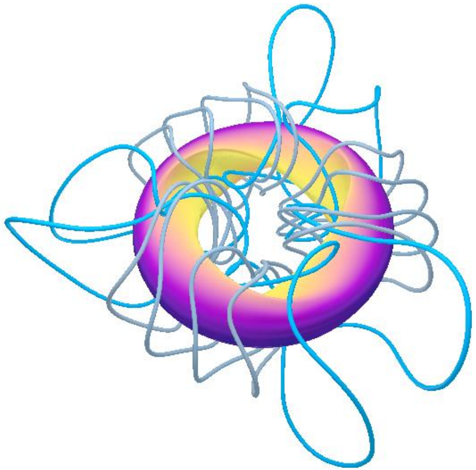
0.1986511655
3.0583866359
4.5015394681
6.6667554715
QA

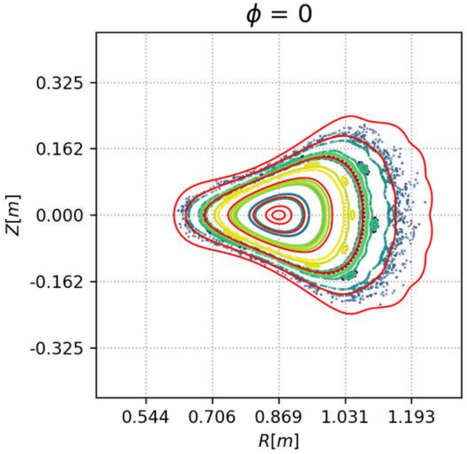
QUASR #0229079

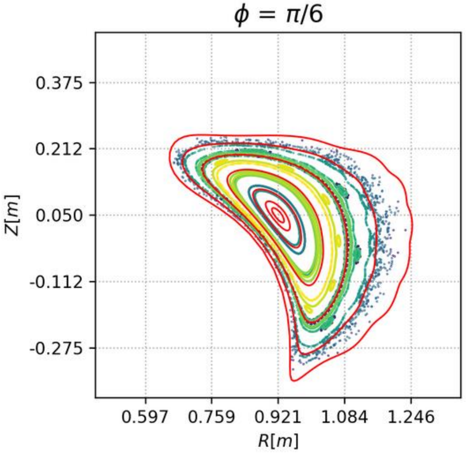
Device Metadata

Device ID:	229079
Length (m): 99.0000108105973 (16.50001801766217 per half-period)	
Root of QA Error:	3.9733904059e-3
Mean Iota:	0.4
Coils per HP:	3
Number of Field Periods (NFP):	3
Volume (M^3):	1.2337005501
Max curvature (kappa) (1/M):	4.9999496827
Max mean-sq curve (1/M^2):	5.0000055232
Minor radius (M):	0.2499997859
Minimum intercoil dist (M):	0.0999892129
Min coil-surface distance (M):	0.1718310121
Mean Elongation (elliptical axis ratio):	2.9119590714
Max Elongation (elliptical axis ratio):	4.4885757826
Aspect ratio (AR):	4.0000102777
Type of Quasi-symmetry:	QA









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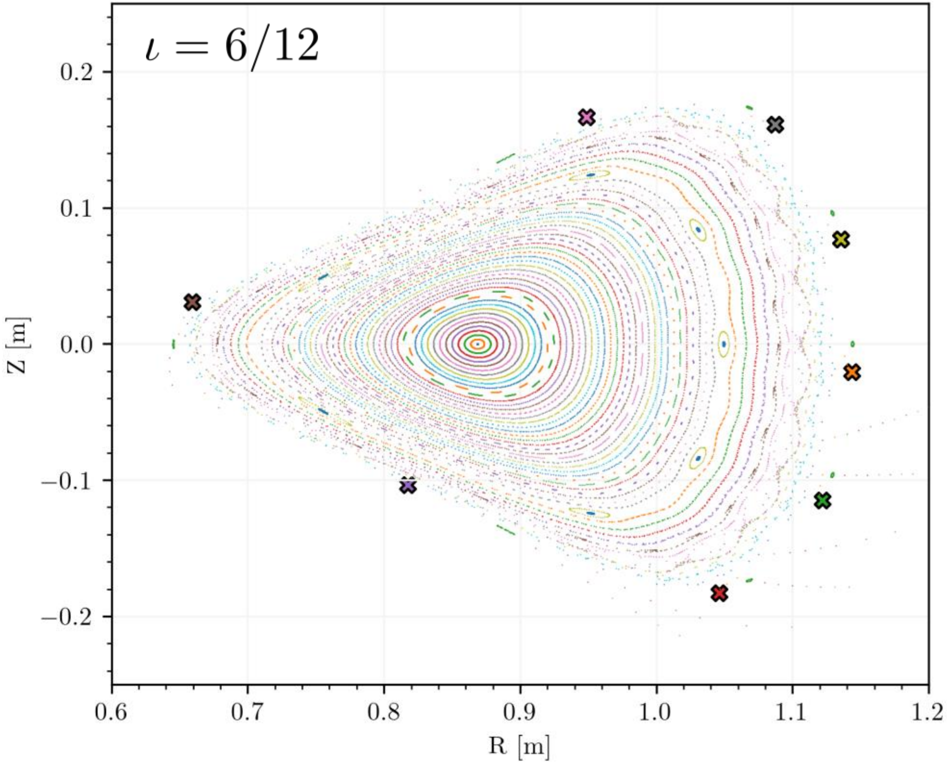

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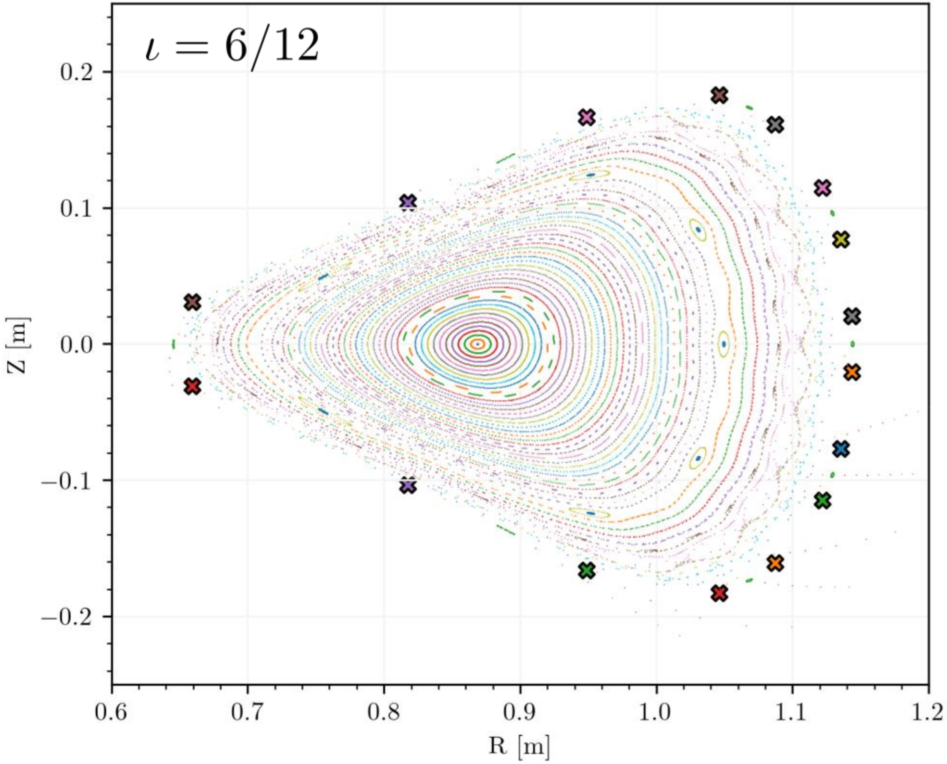

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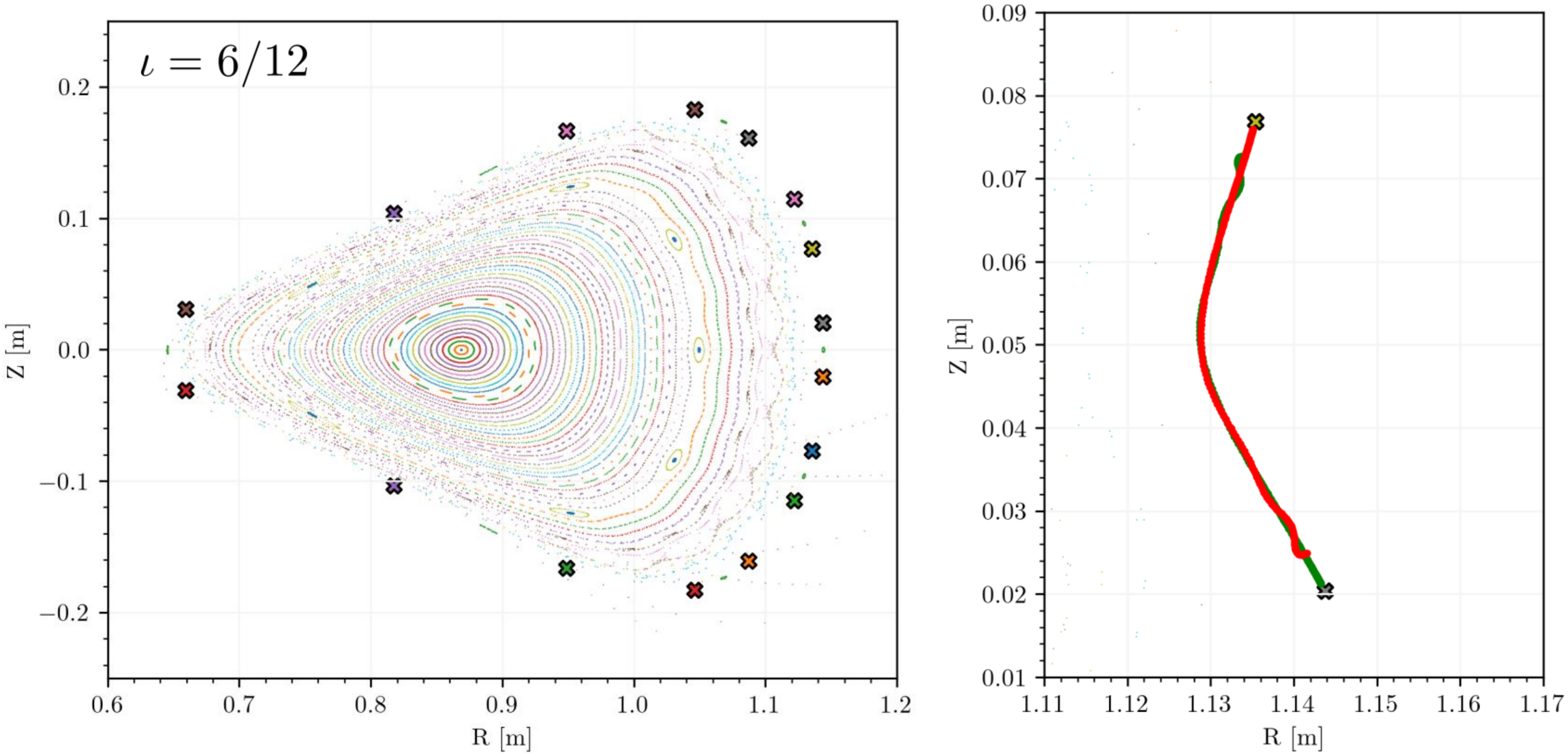

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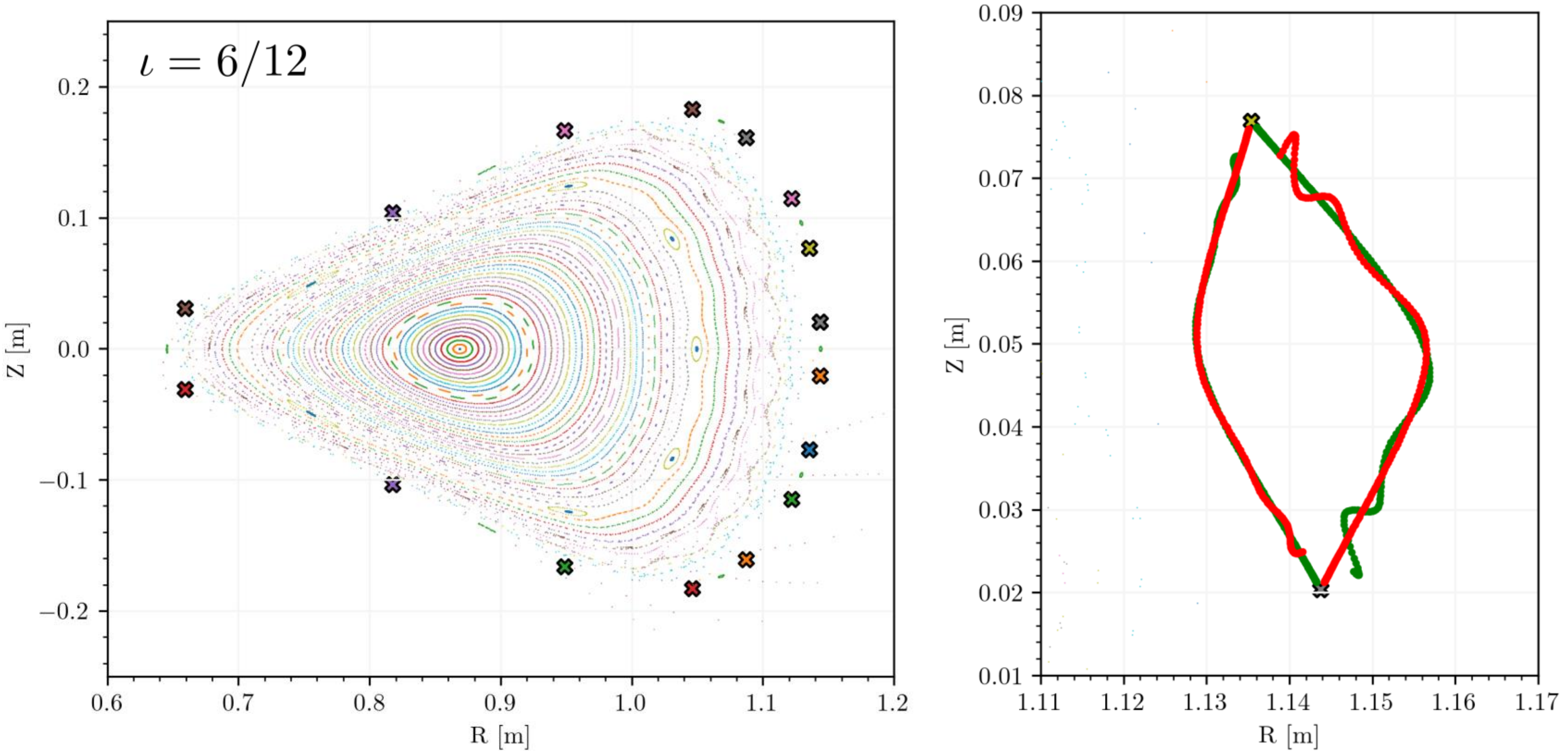

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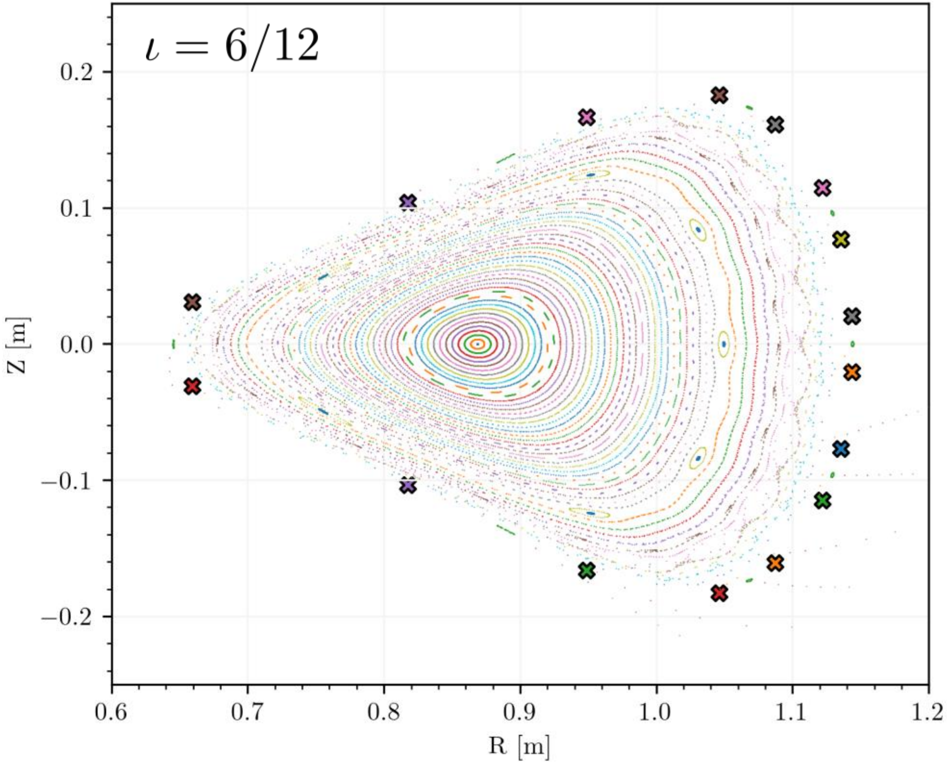

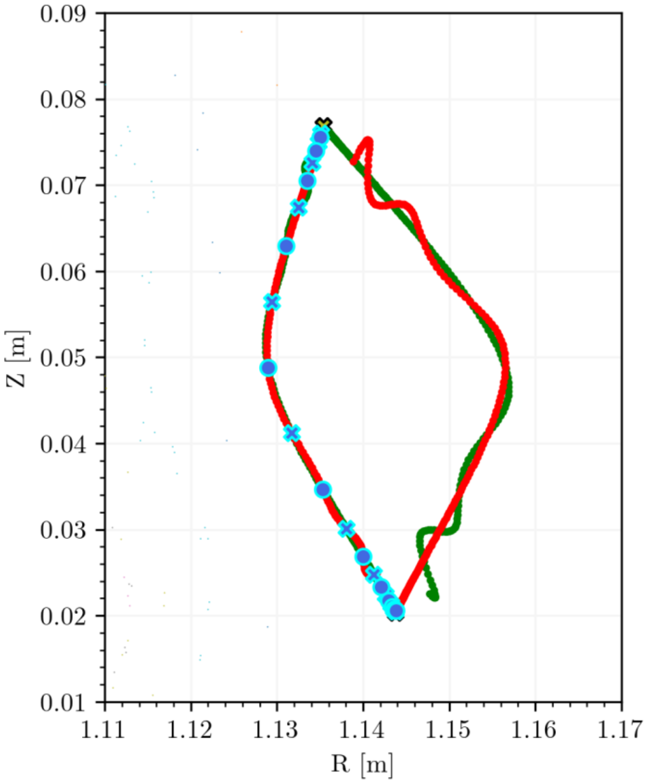

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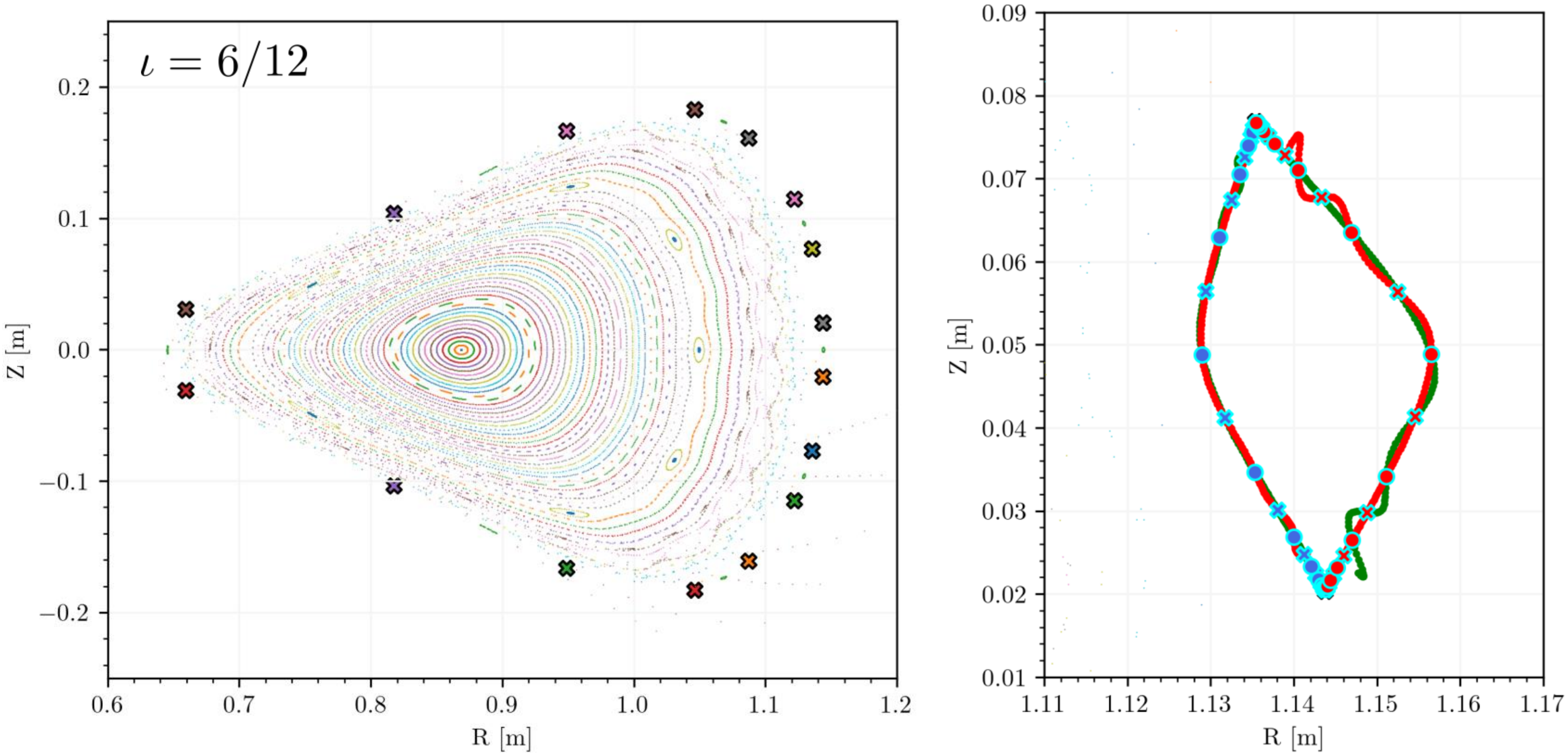

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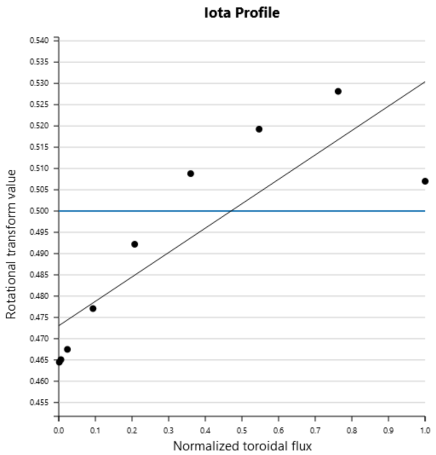
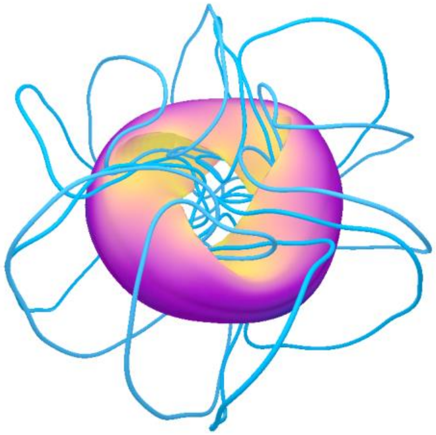
QUASR #0928241

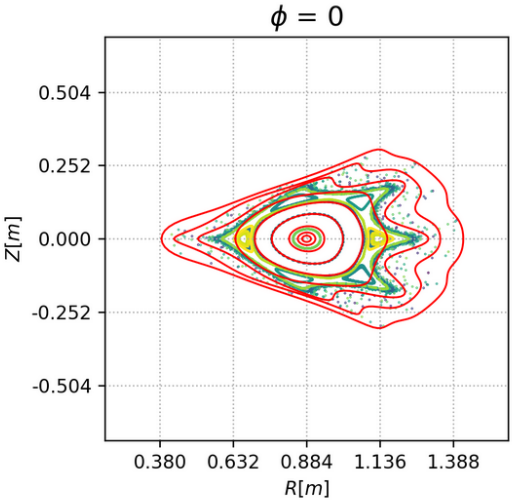
Ludovic Rais

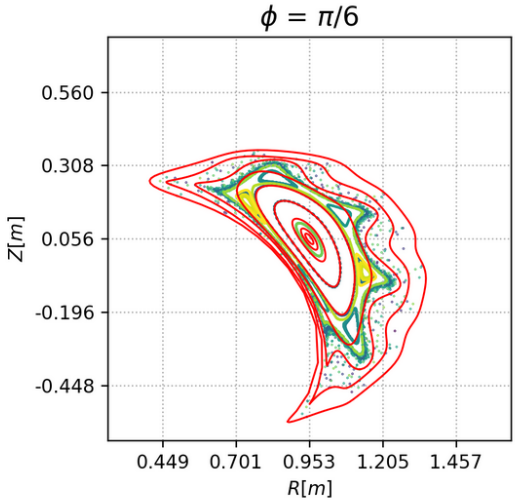
Device Metadata

Device ID:	928241
Length (m):	99.00018437090642 (16.500030728484404 per half-period)
Root of QA Error:	2.6808663110e-2
Mean Iota:	0.5
Coils per HP:	2
Number of Field Periods (NFP):	3
Volume (M^3):	2.4180530783
Max curvature (kappa) (1/M):	4.9985290624
Max mean-sq curve (1/M^2):	5.0001159247
Minor radius (M):	0.3499820441
Minimum intercoil dist (M):	0.1000258344
Min coil-surface distance (M):	0.1707523513
Mean Elongation (elliptical axis ratio):	3.0626593608
Max Elongation (elliptical axis ratio):	4.8744432163
Aspect ratio (AR):	2.8575826391
Type of Quasi-symmetry:	QA









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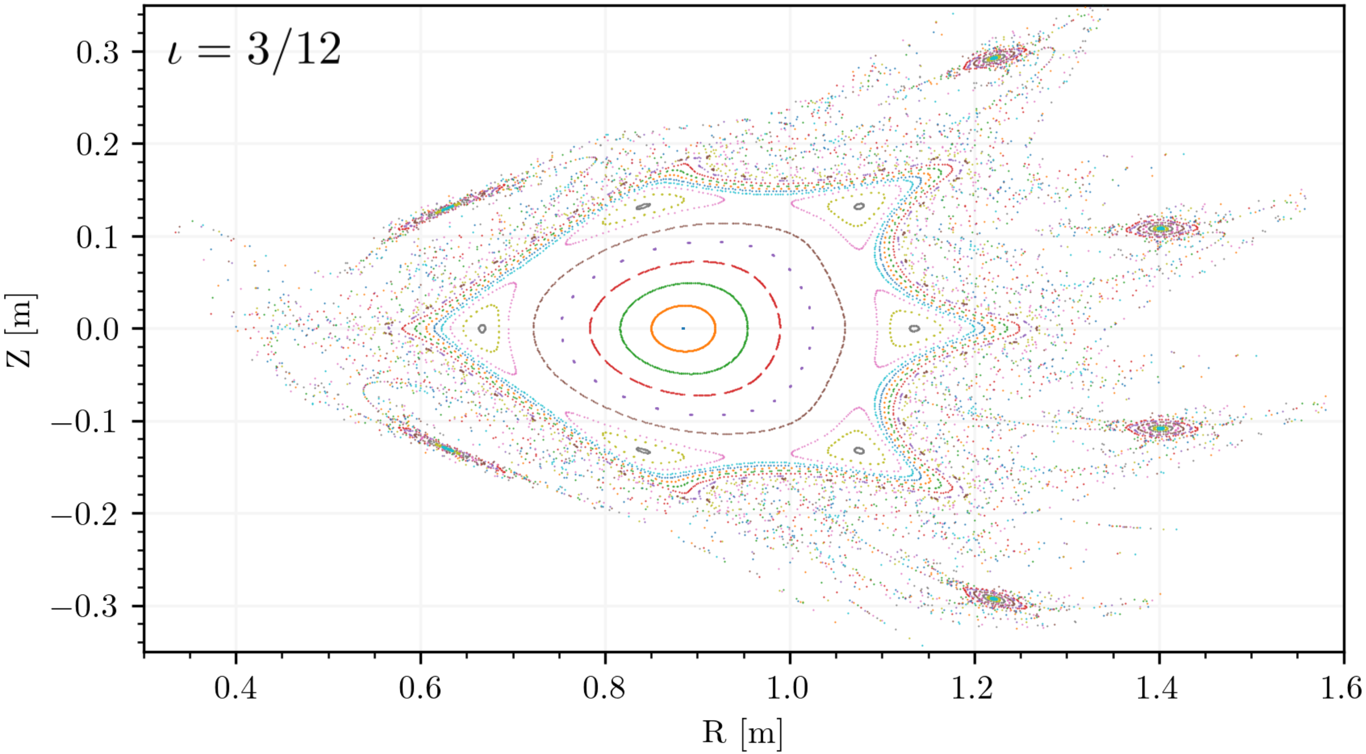

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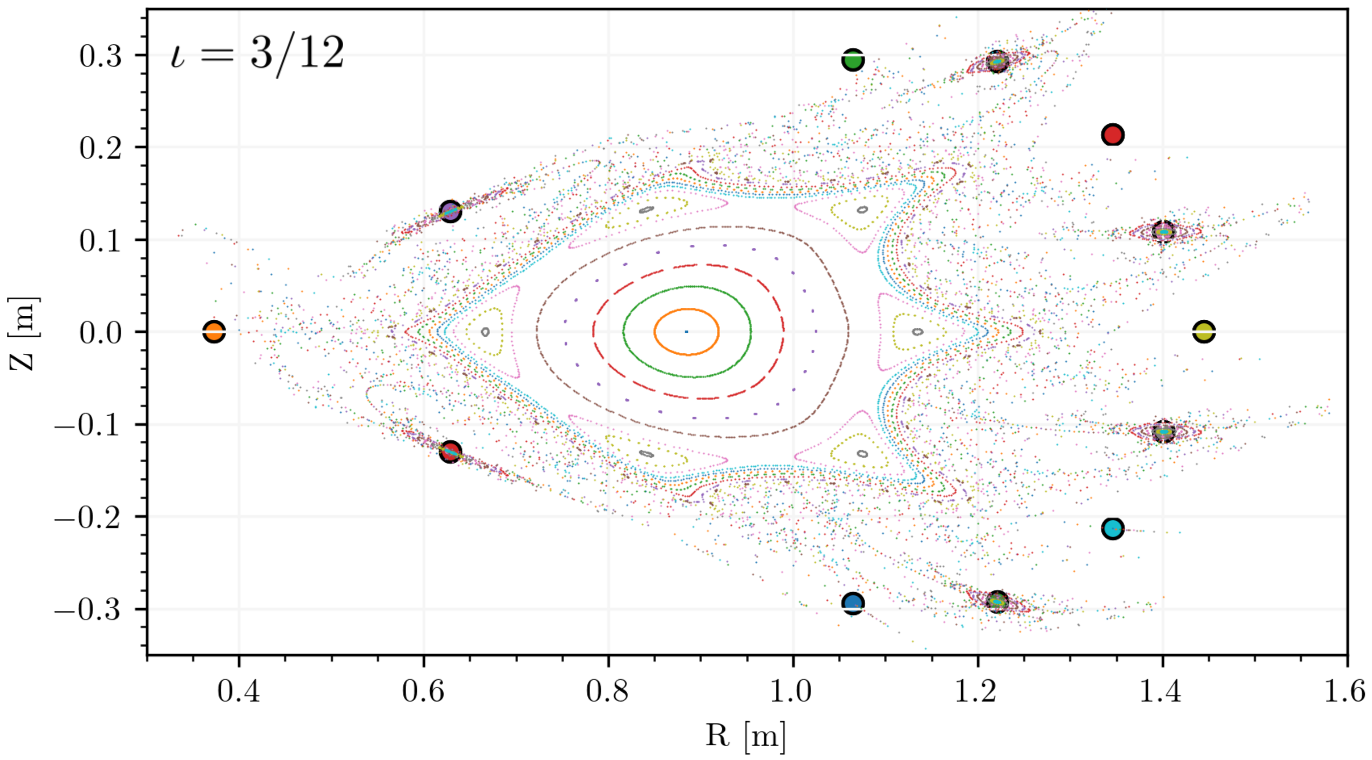

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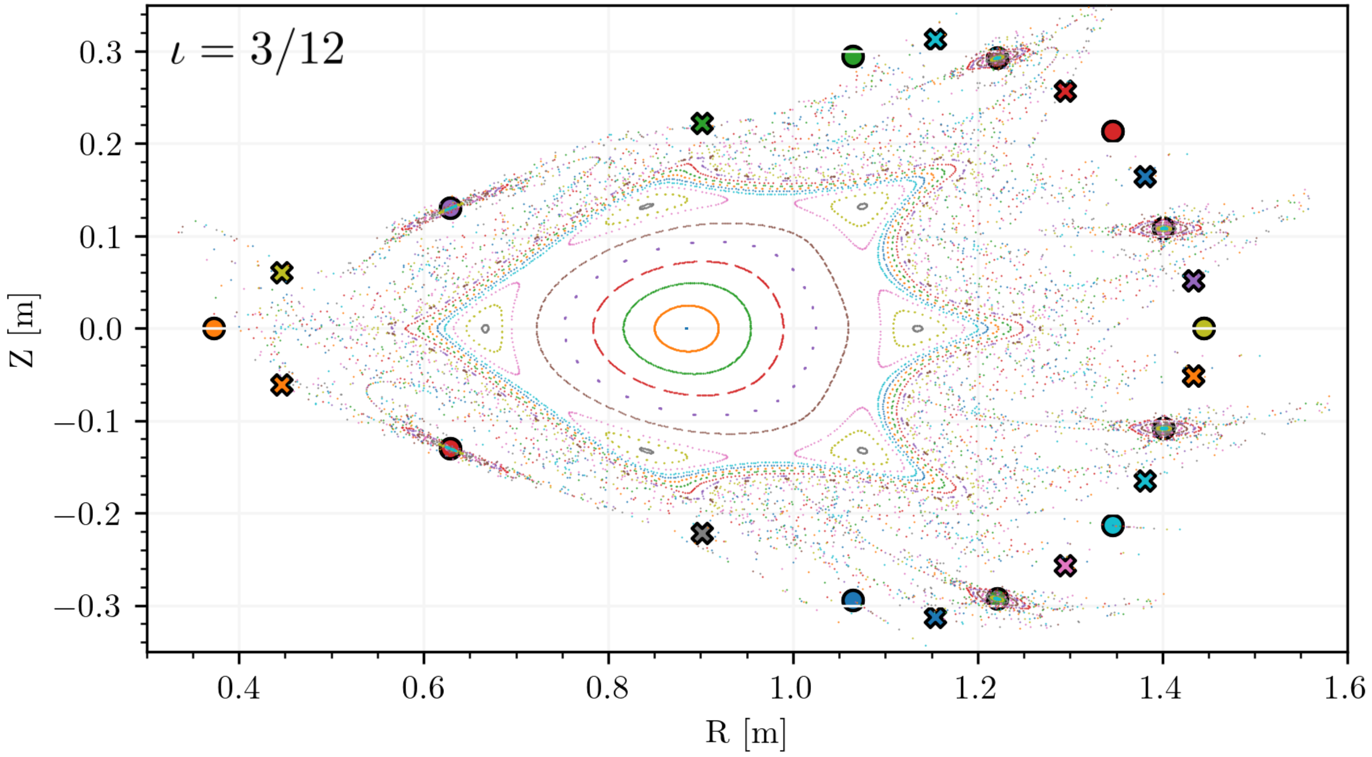

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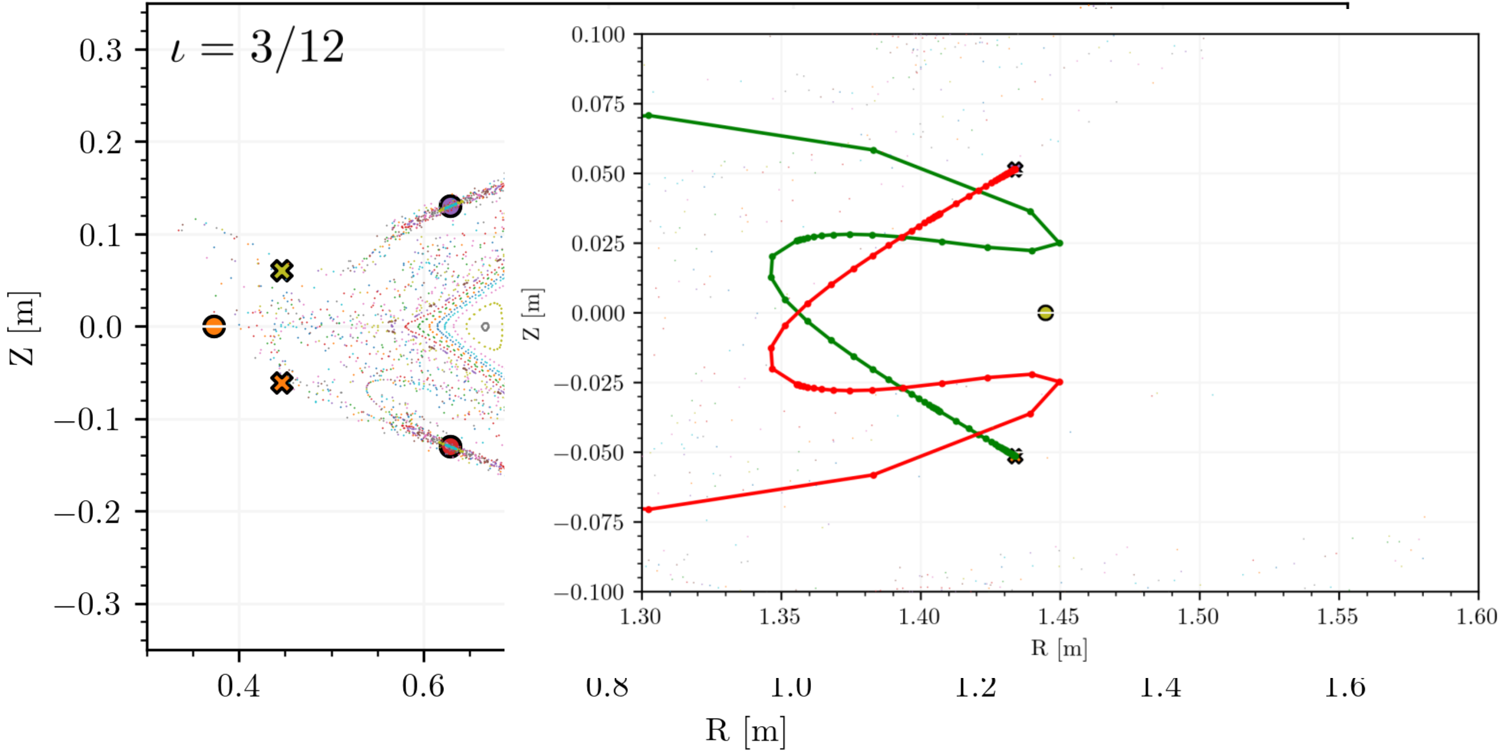

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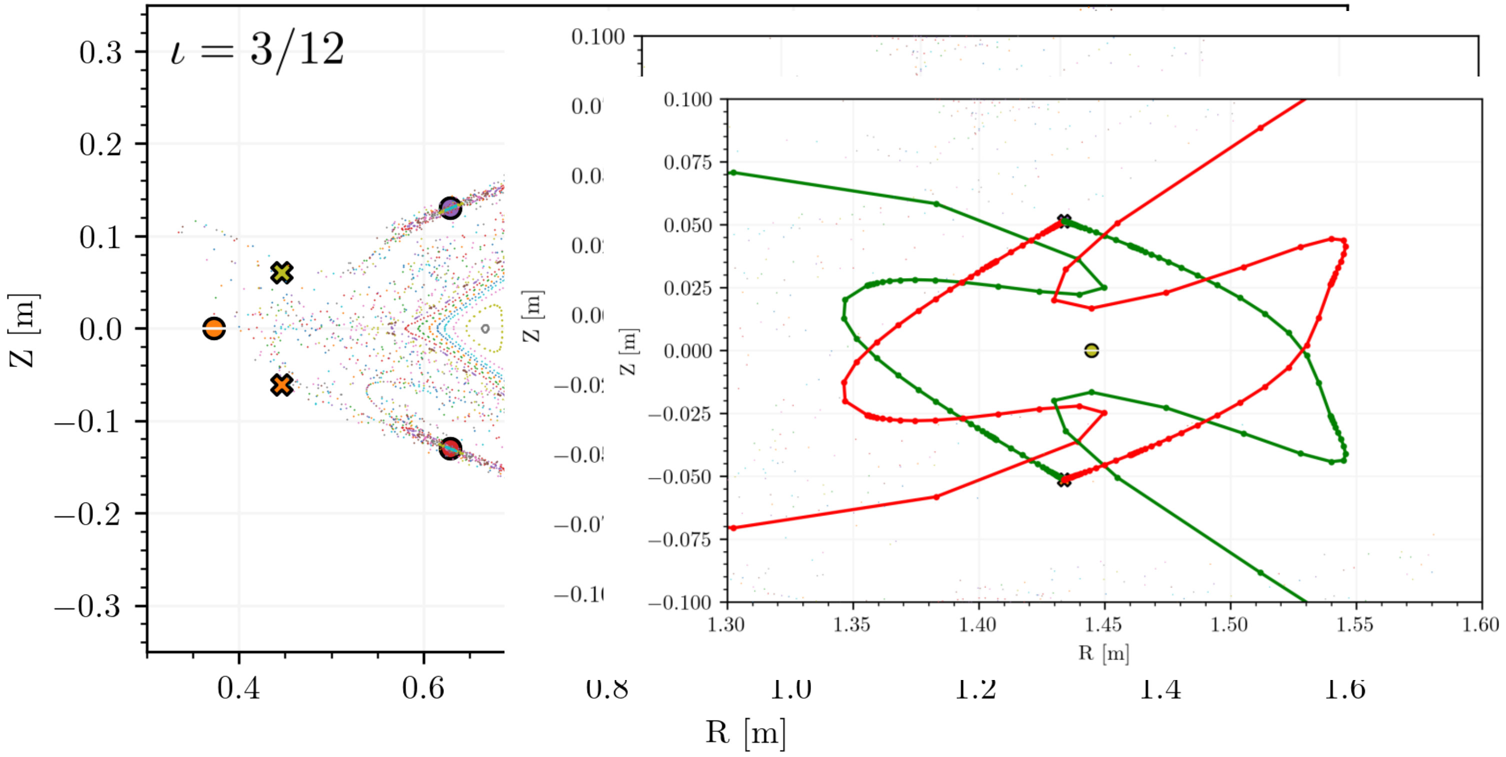

EPFL

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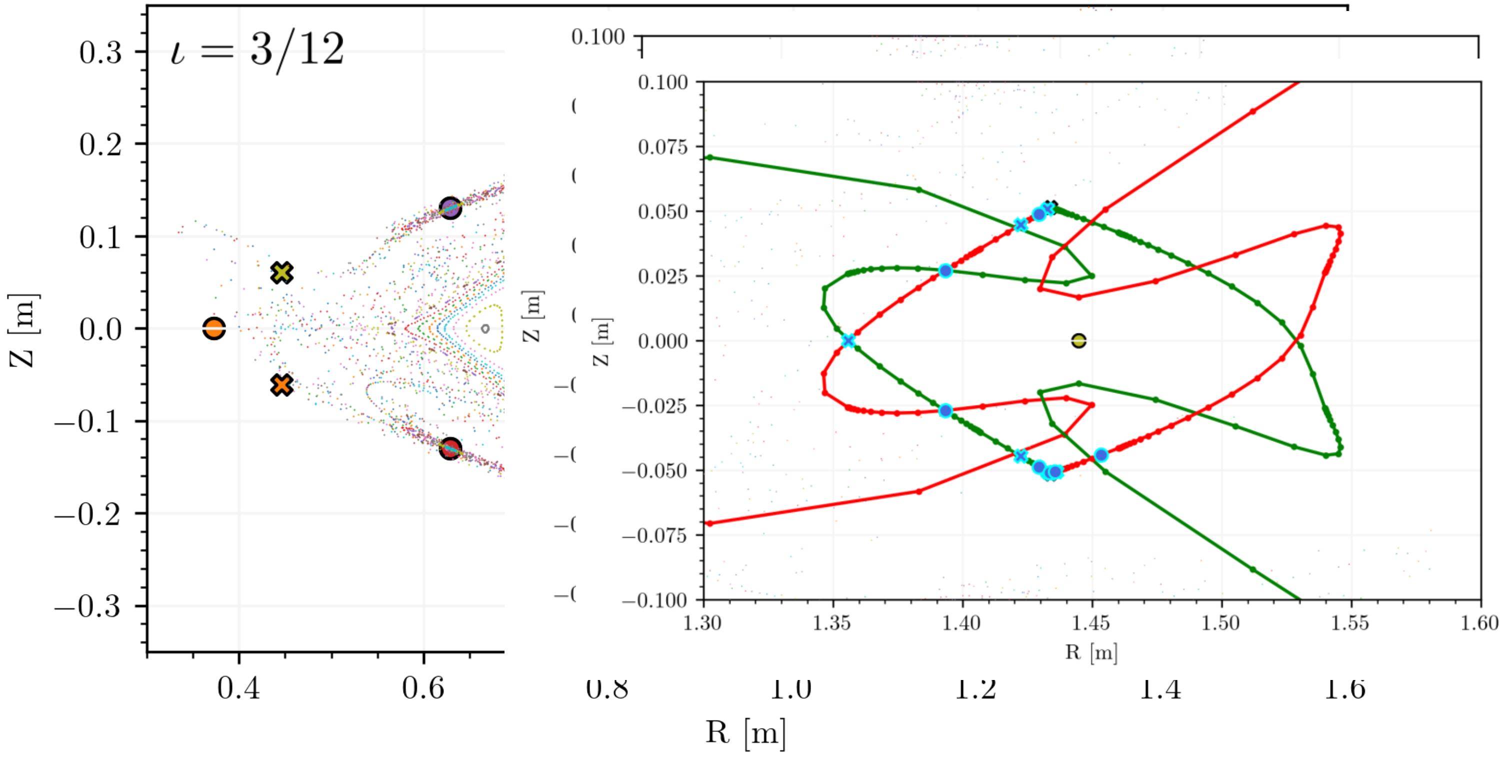

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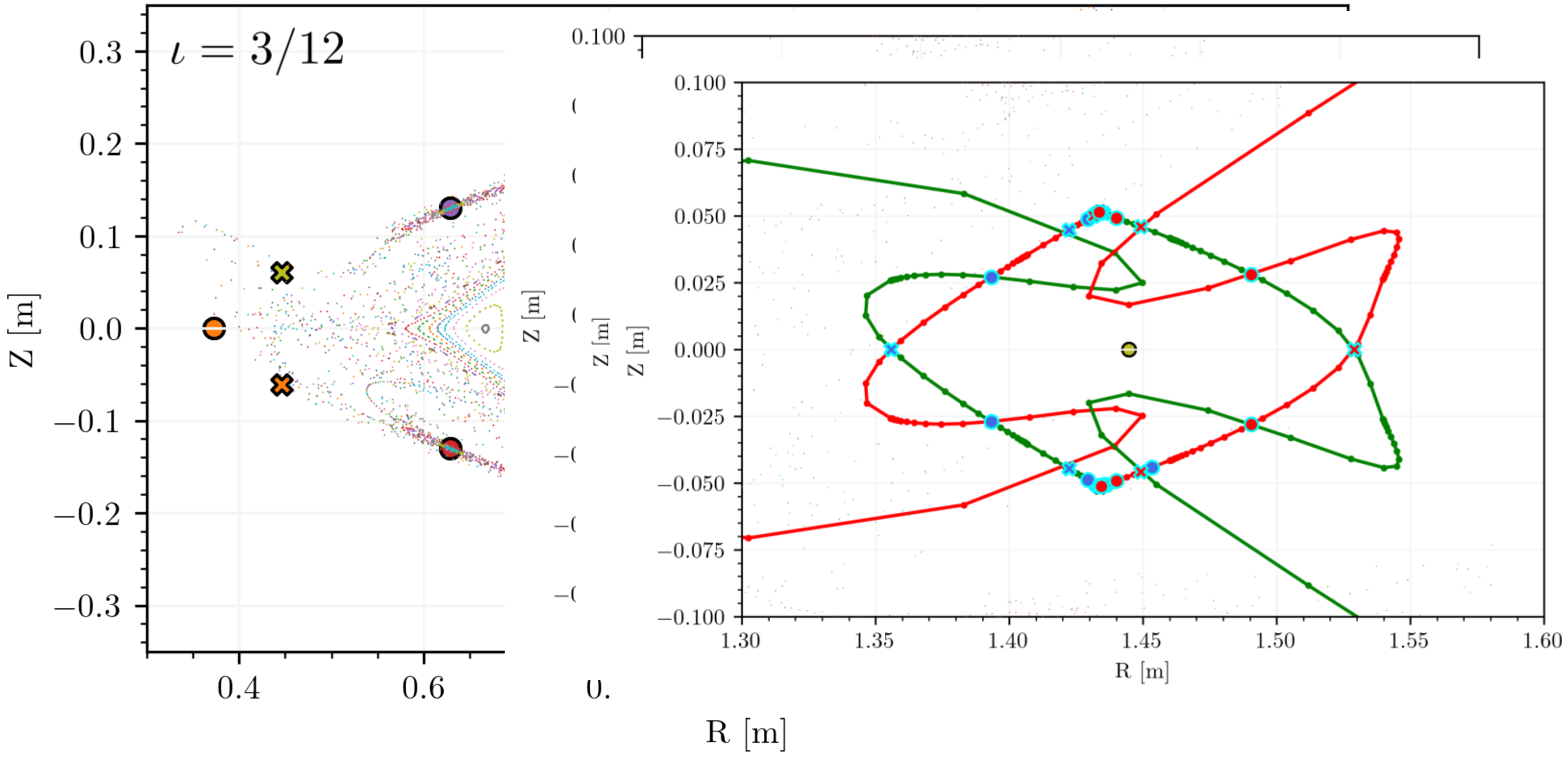
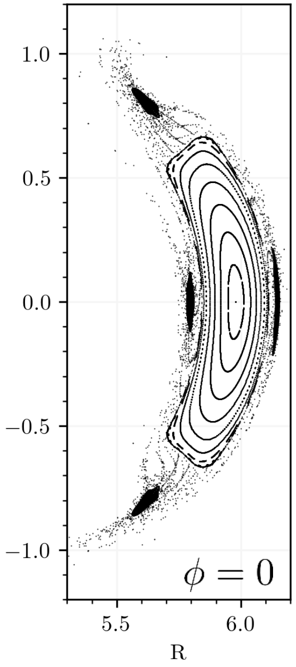
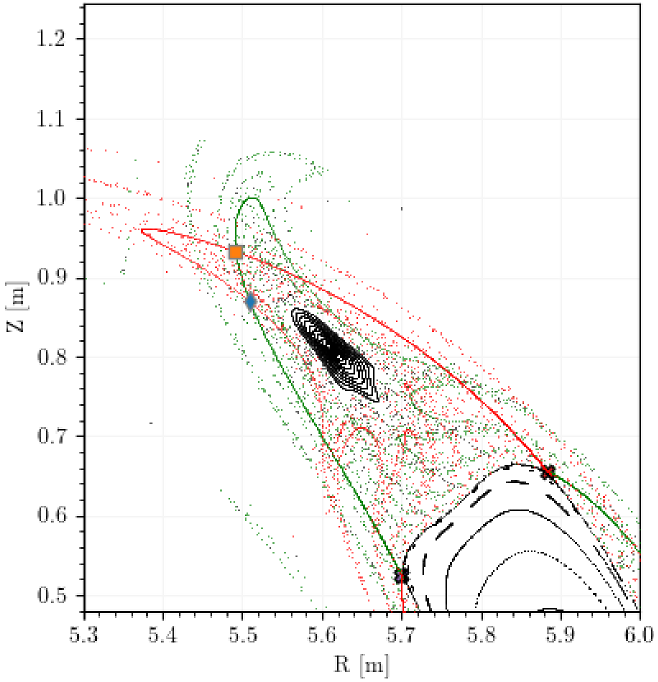
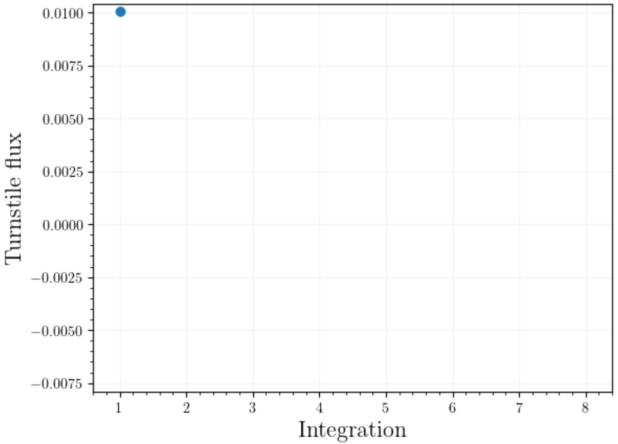


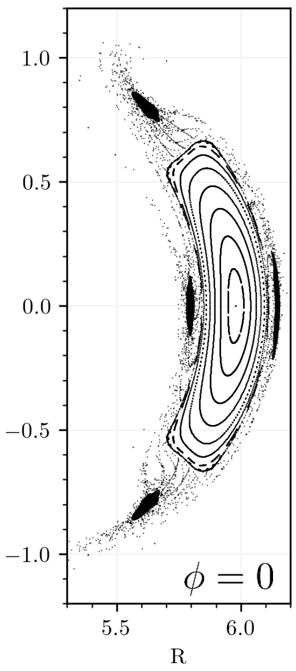
Back on W7X-GYM00-1750 to caculate the turnstile flux

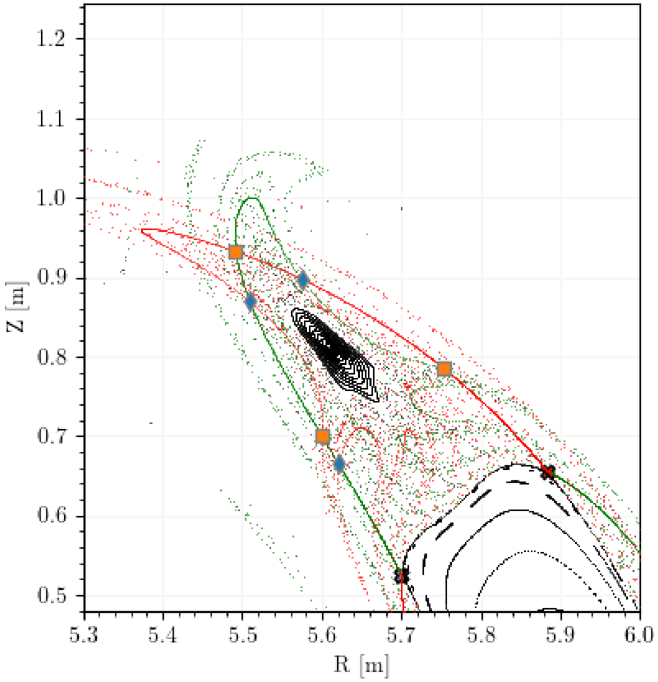


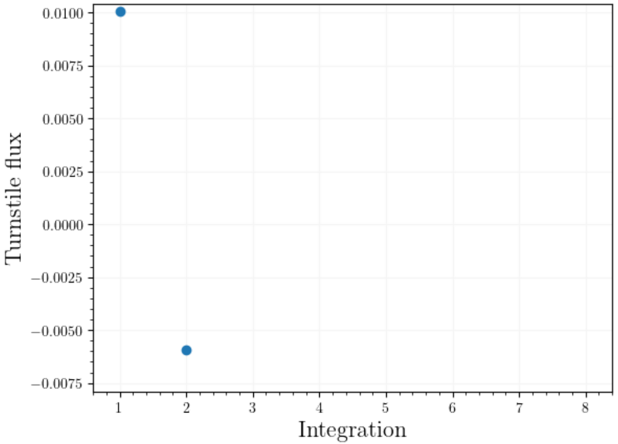




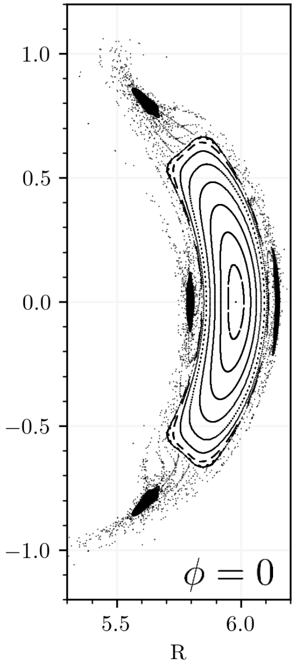
Back on W7X-GYM00-1750 to caculate the turnstile flux

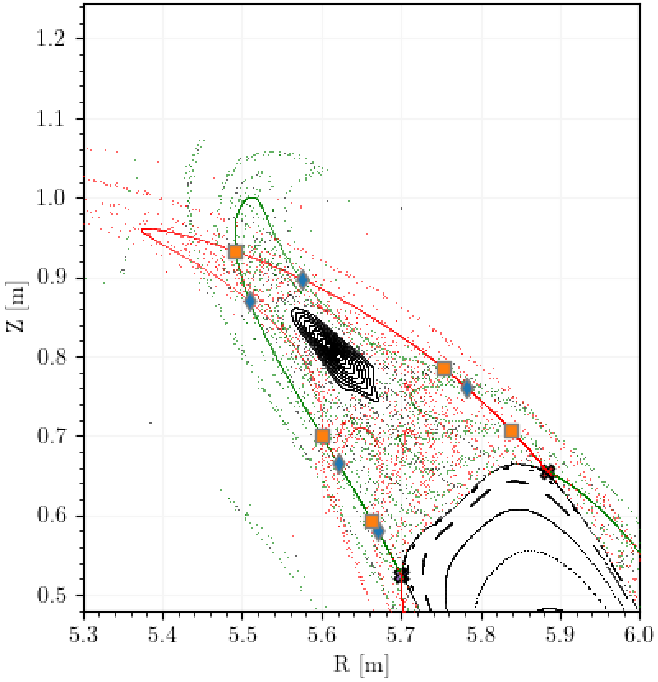


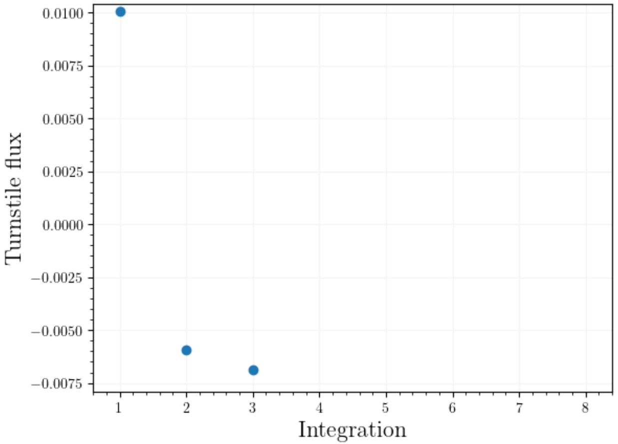




Back on W7X-GYM00-1750 to caculate the turnstile flux







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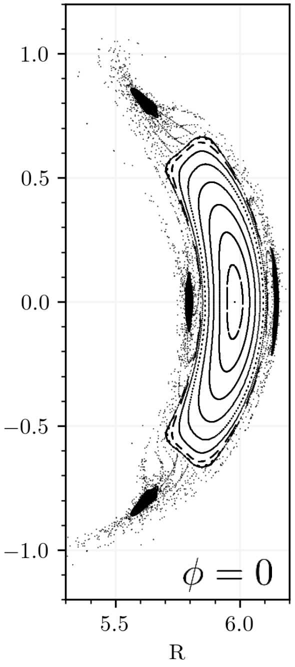
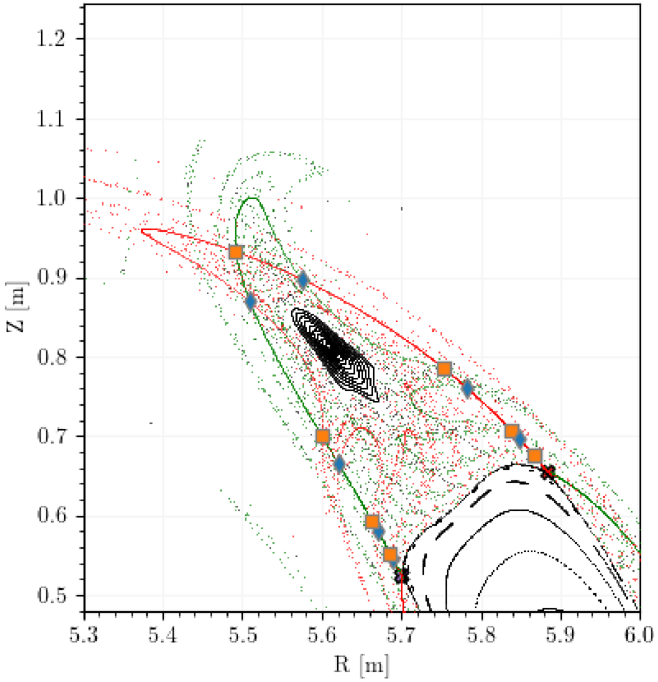
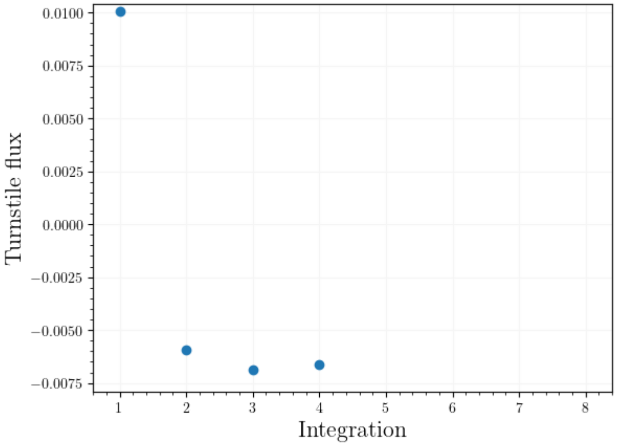
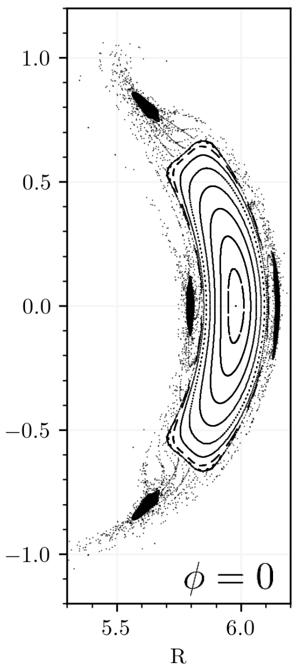
Back on W7X-GYM00-1750 to caculate the turnstile flux

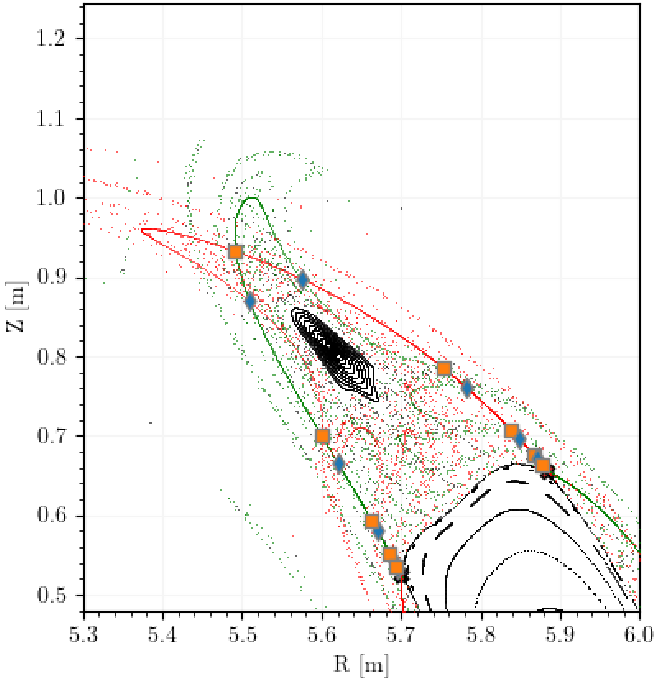


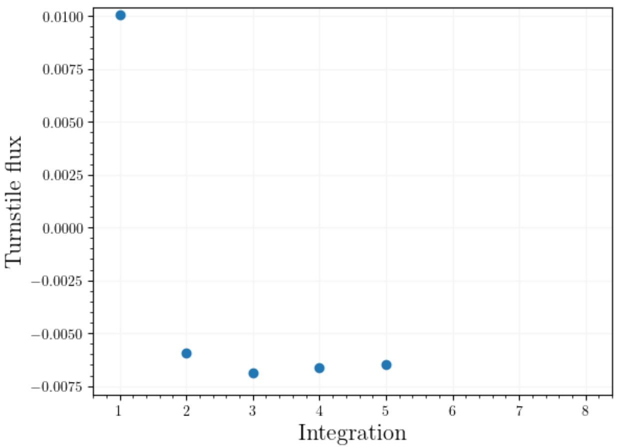




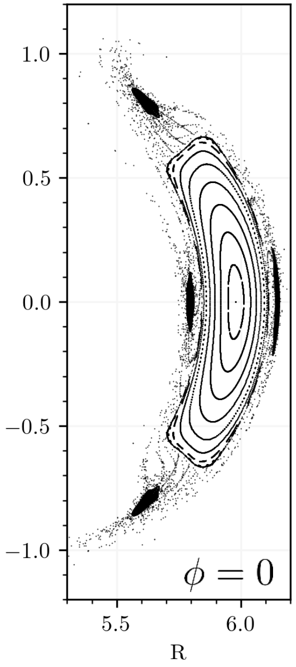
Back on W7X-GYM00-1750 to caculate the turnstile flux

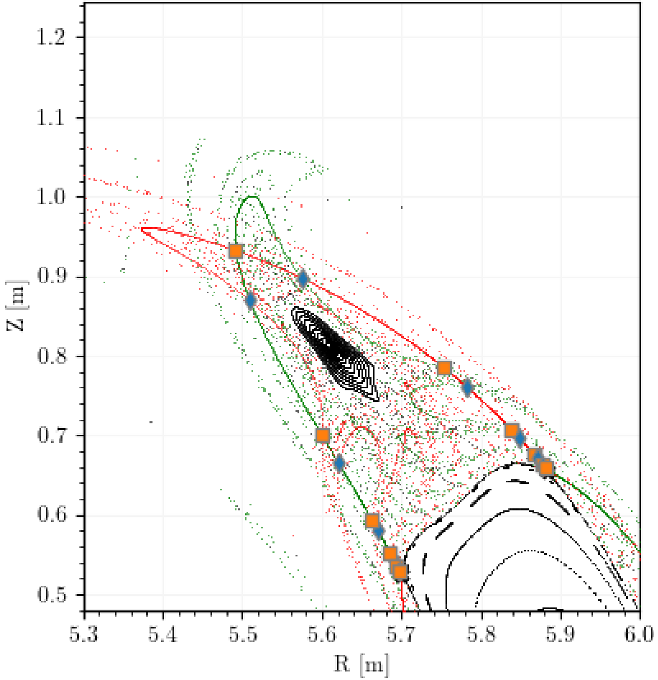


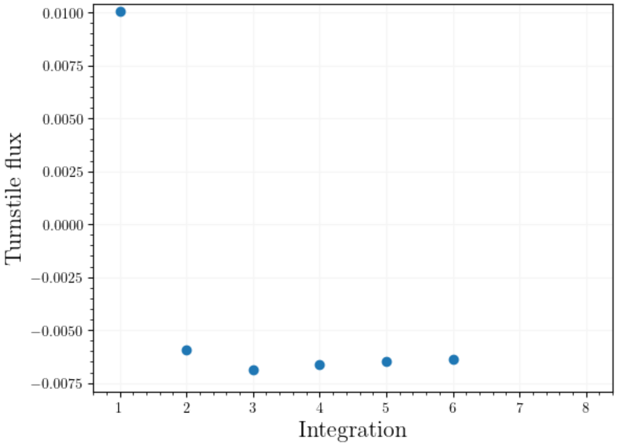




Back on W7X-GYM00-1750 to caculate the turnstile flux

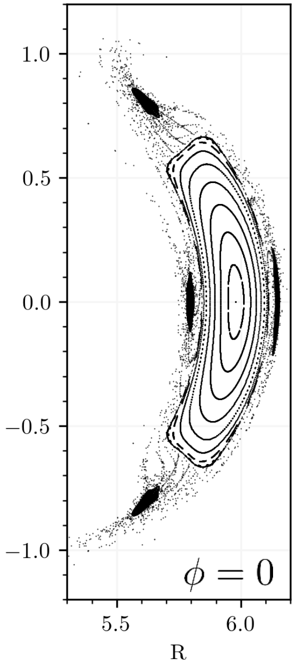


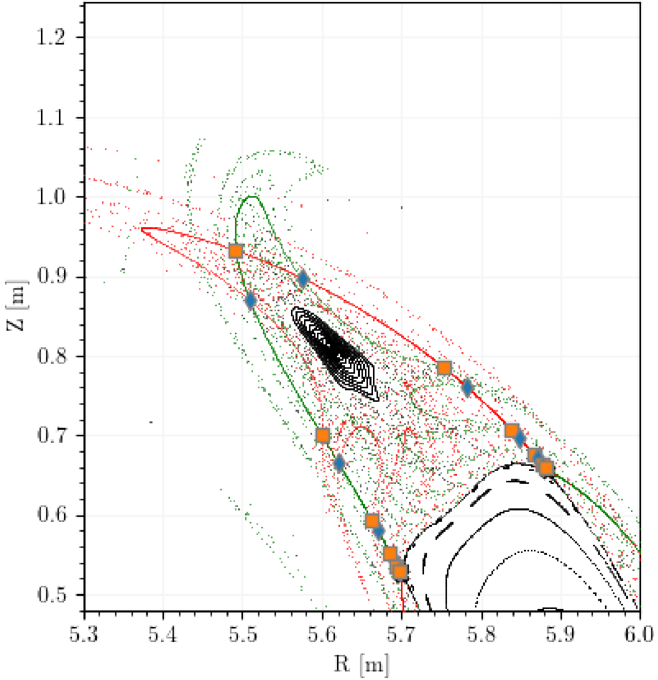


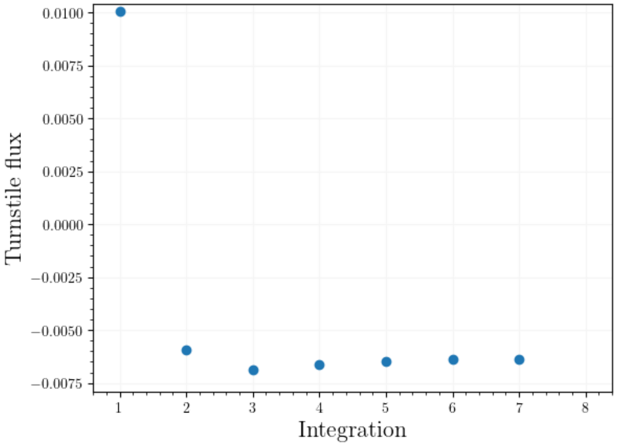


EPFL

Back on W7X-GYM00-1750 to caculate the turnstile flux







EPFL

Back on W7X-GYM00-1750 to caculate the turnstile flux

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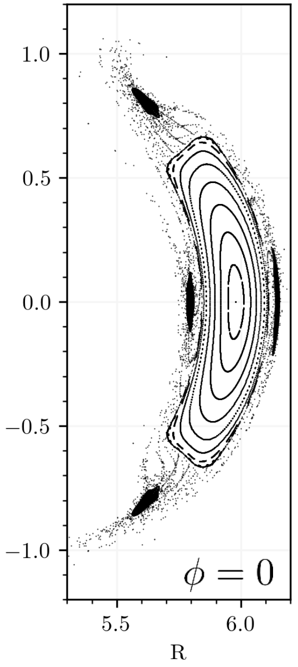
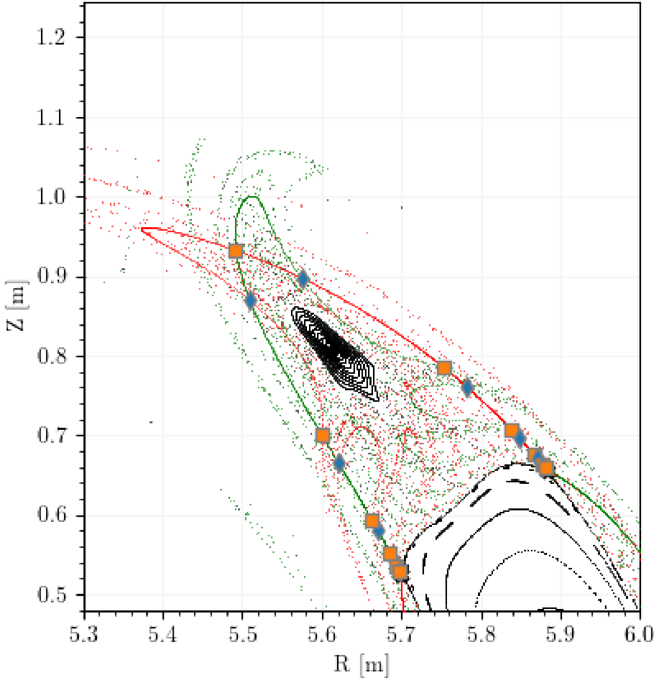
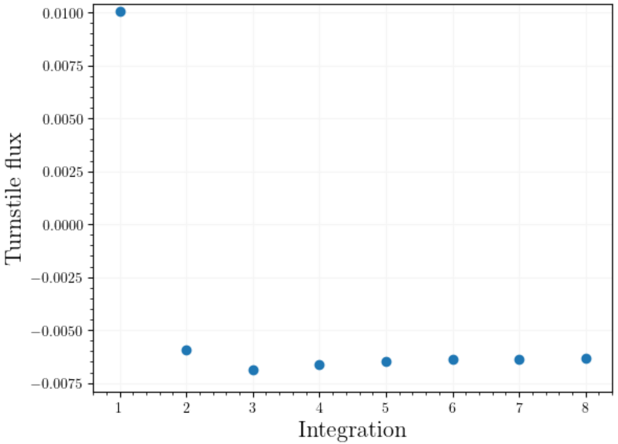
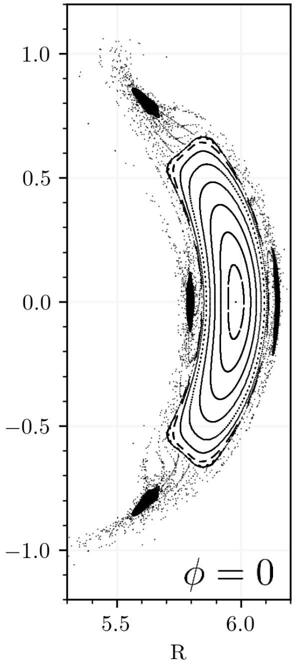
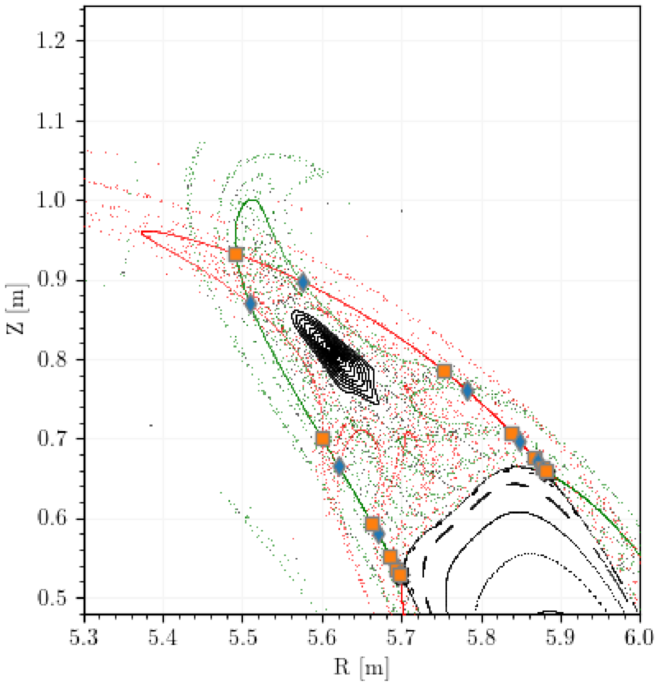


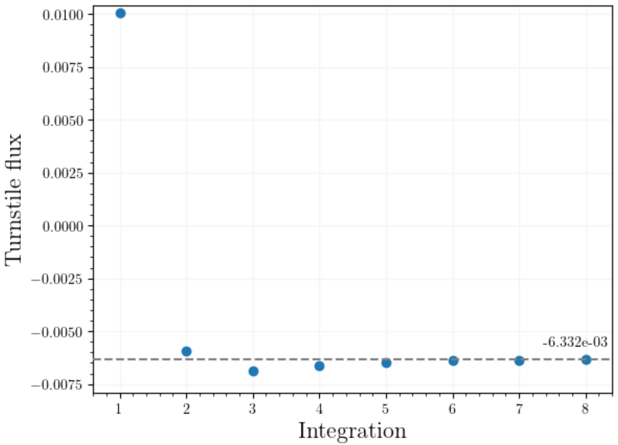




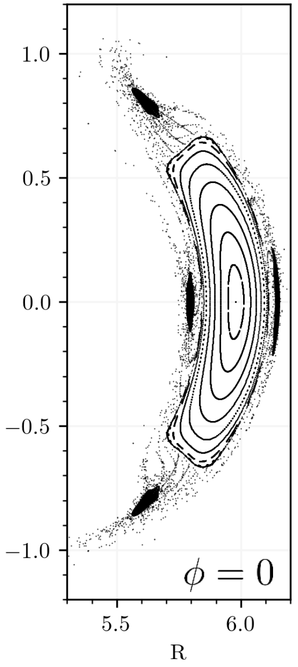
Back on W7X-GYM00-1750 to caculate the turnstile flux

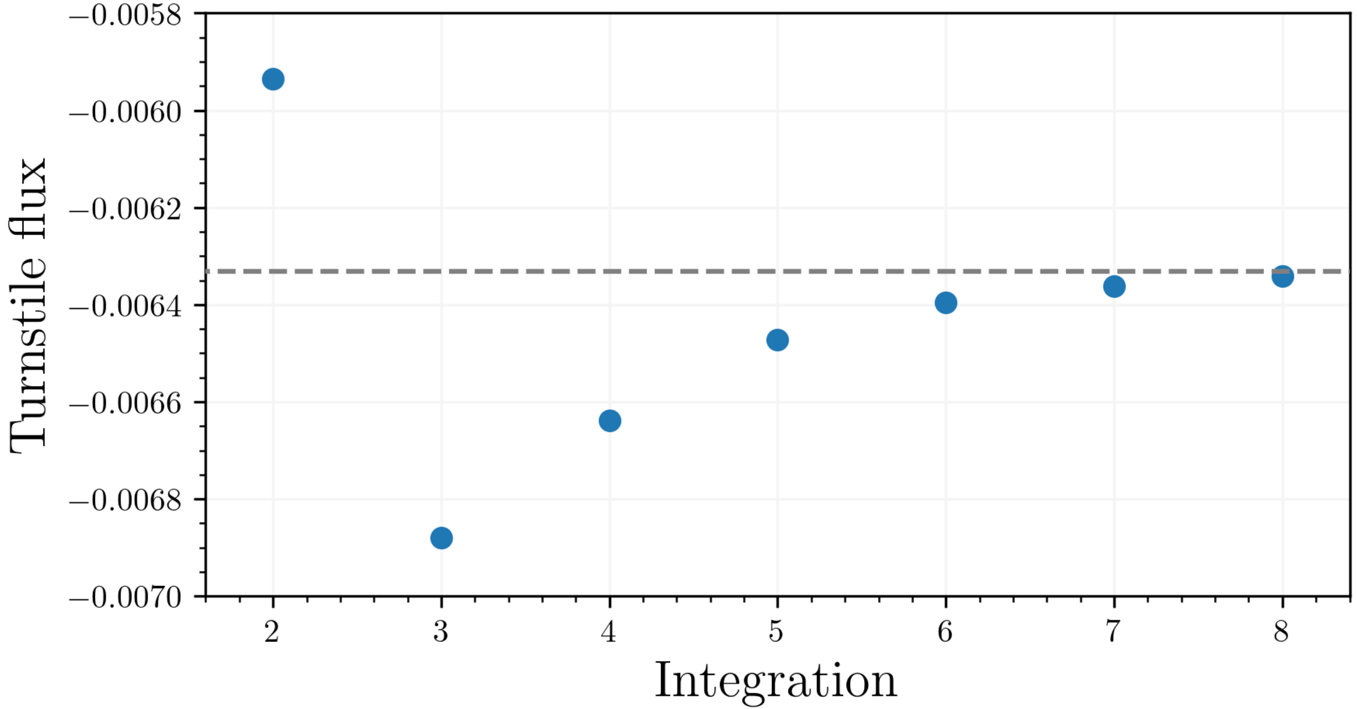




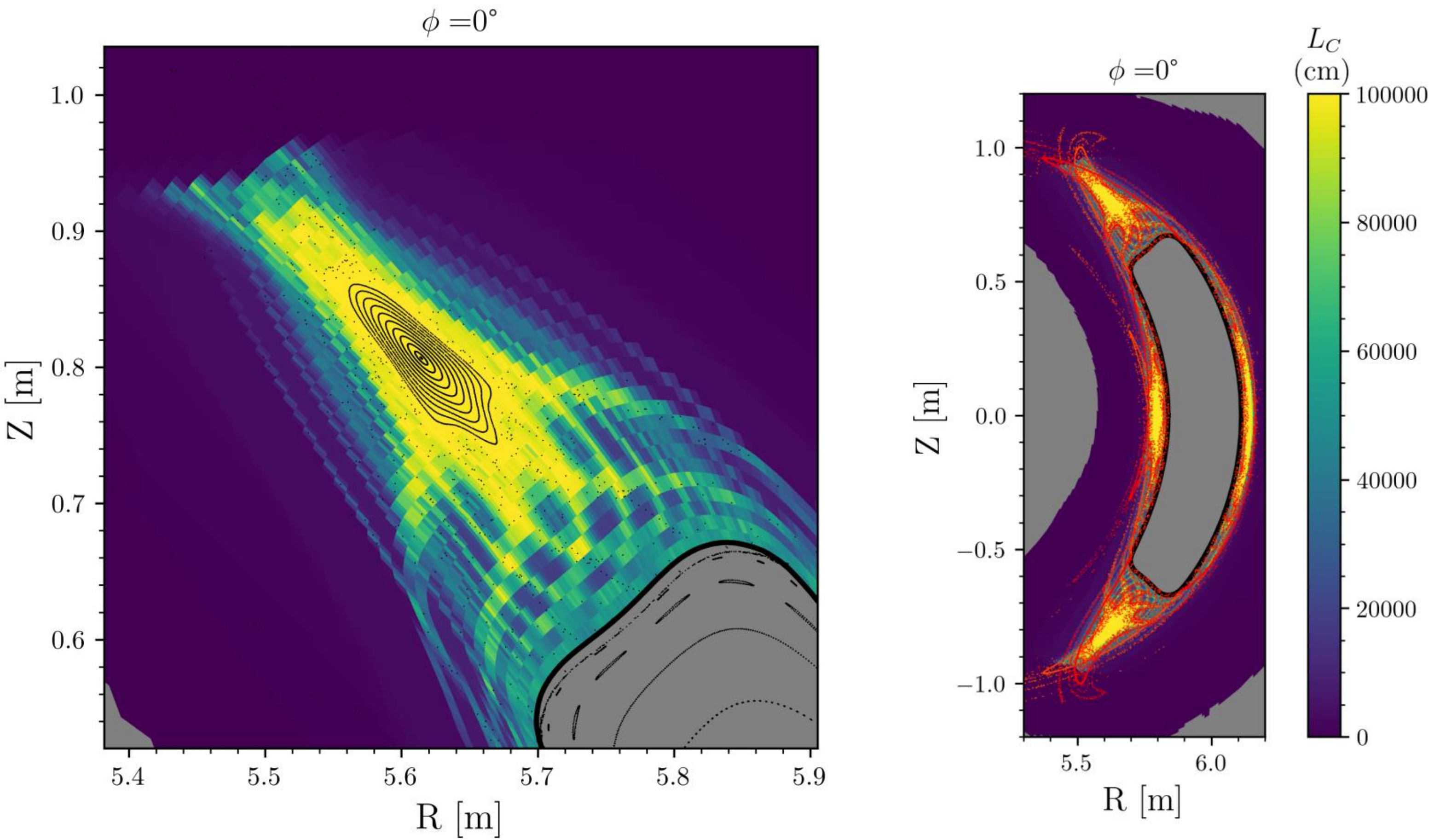


Back on W7X-GYM00-1750 to caculate the turnstile flux

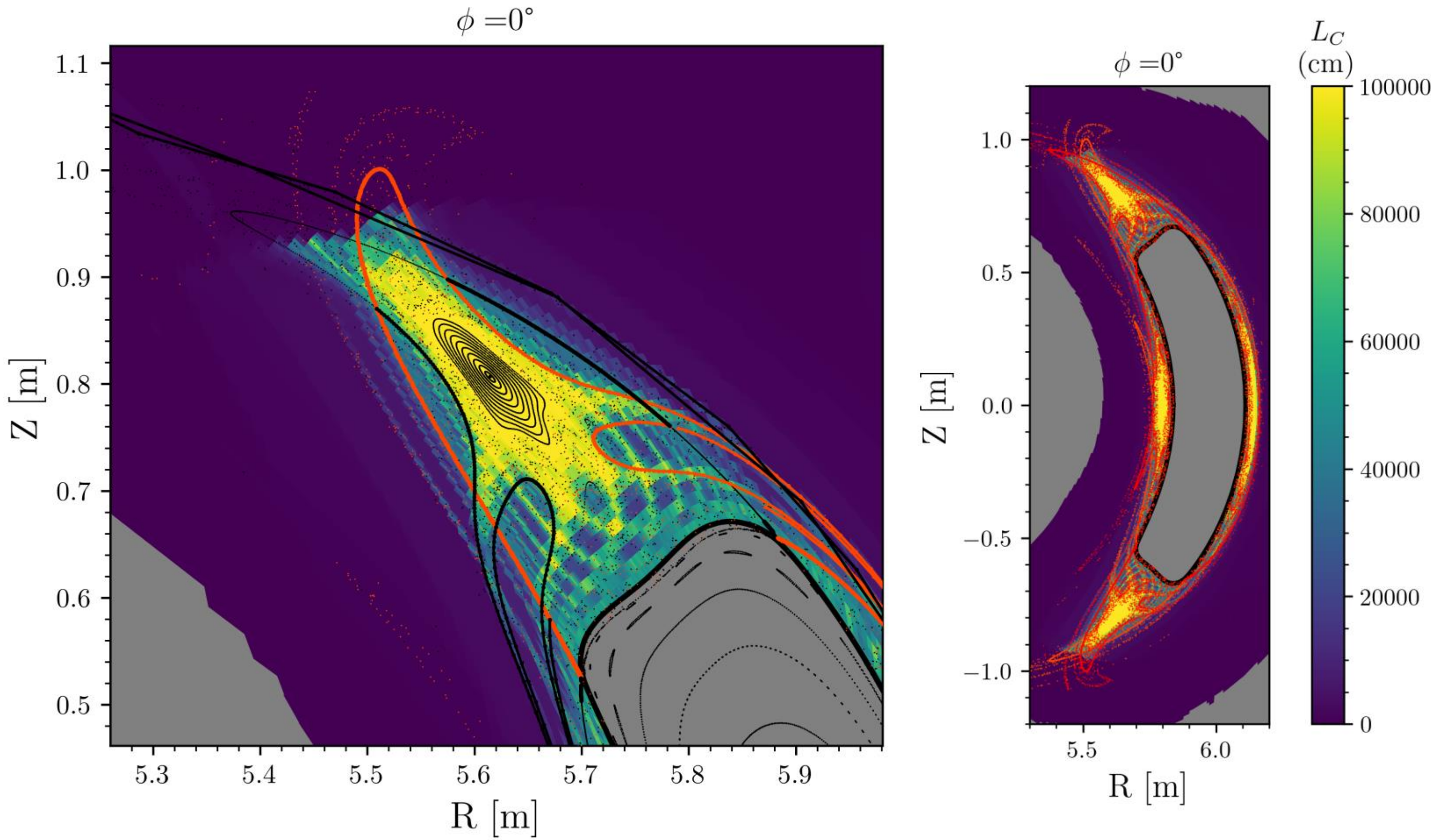




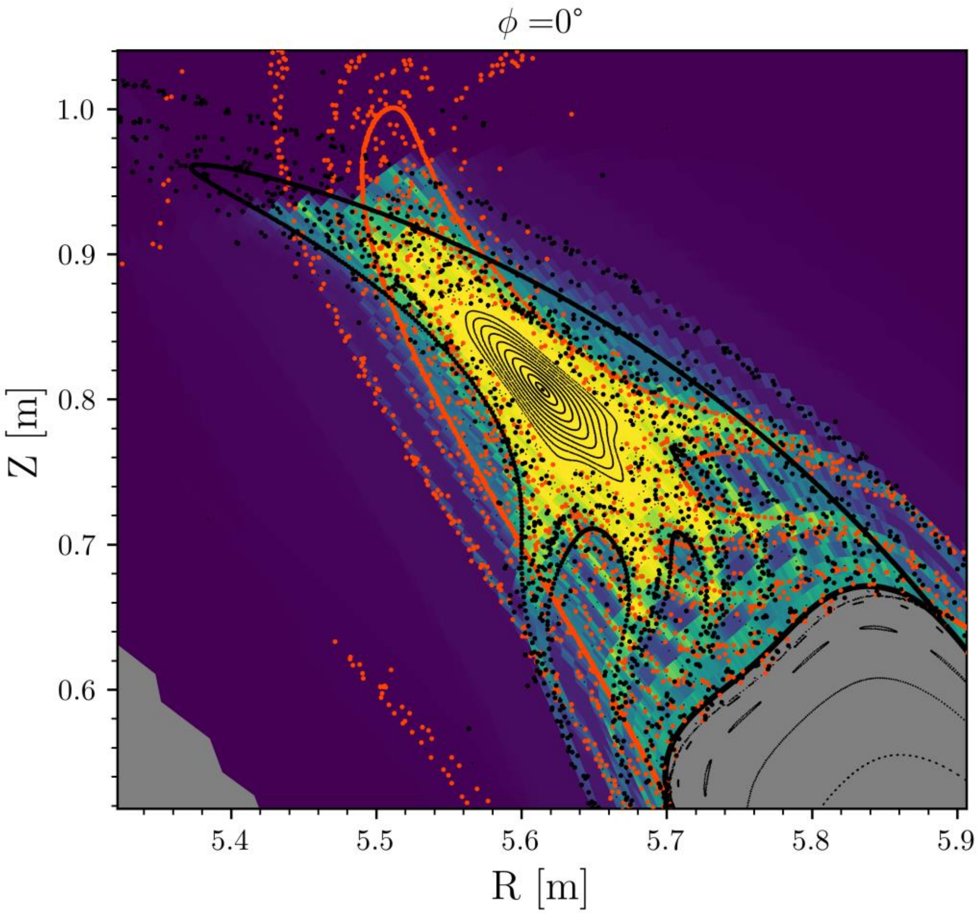
Connection lenght is explained by the turnstile

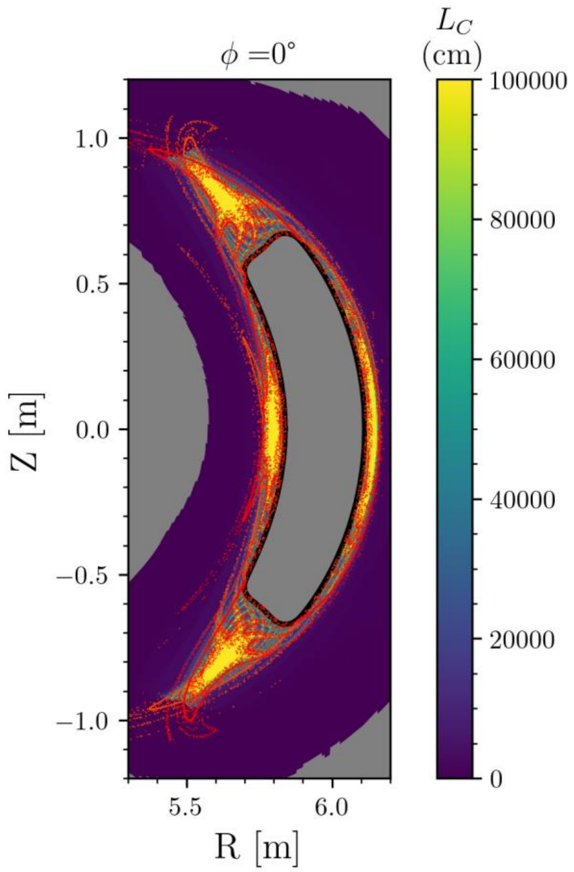


Connection lenght is explained by the turnstile



Connection lenght is explained by the turnstile



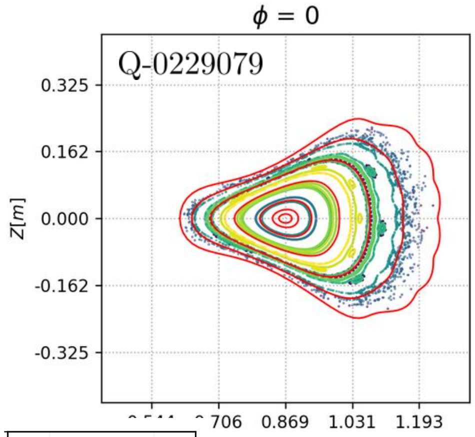


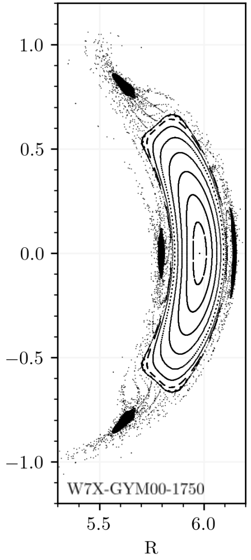
Stochastic transport due to difference between inner and outer turnstiles

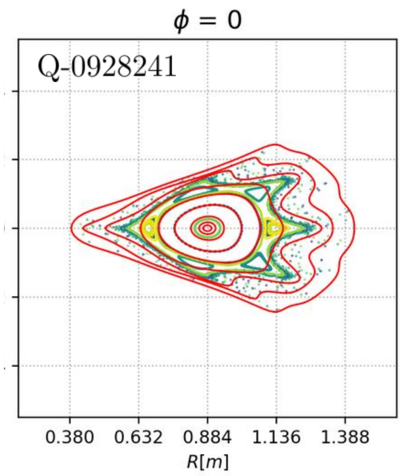
- Inner and outer turnstile fluxes have different values, the inner one is lower

configuration	$\Phi_{inner}/\tilde{B}_0^{\phi}$	$\Phi_{outer}/\tilde{B}_0^{\phi}$
Q-0229079 $[10^{-5}]$	1.66	2.29
Q-0928241 $[10^{-4}]$	6.43	4.43
W7X-GYM00-1750 $[10^{-6}]$	8.15	629.97

- The difference in turnstile flux induces transport
  - From the inside, longer to go inside the island due to the low inner turnstile flux
  - Once inside however, not long until the field line leave due to the higher outer turnstile flux

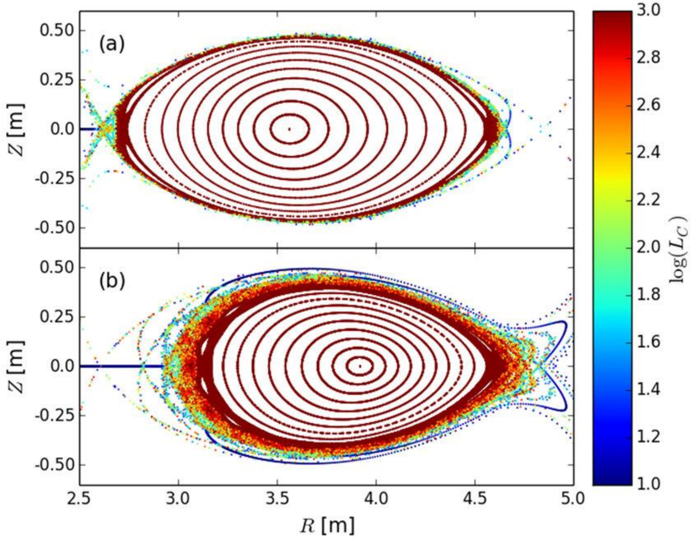






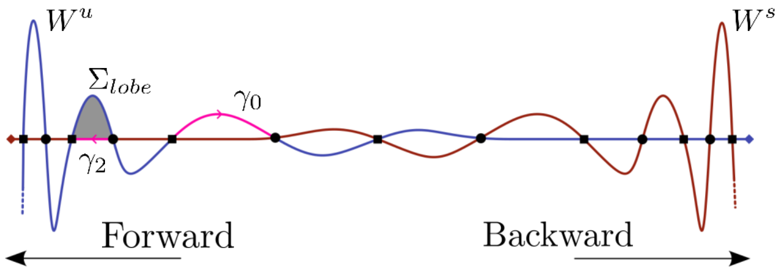
Next steps

- Apply the method in stellarator optimization
  - Reducing the chaos in the LHD edge, coordination with **Todd Elder**
  - Find different chaotic configurations by changing the currents in existing stellarators
- Relate the turnstile flux to relevant transport metrics
- Find a robust method to identify all hetero/homo-clinic points
- If you want us to look at the turnstile flux in your configuration, we're happy to help!



Suzuki et al., 'Impact of Magnetic Topology on Radial Electric Field Profile in the Scrape-off Layer of the Large Helical Device'.

Wrapping things up



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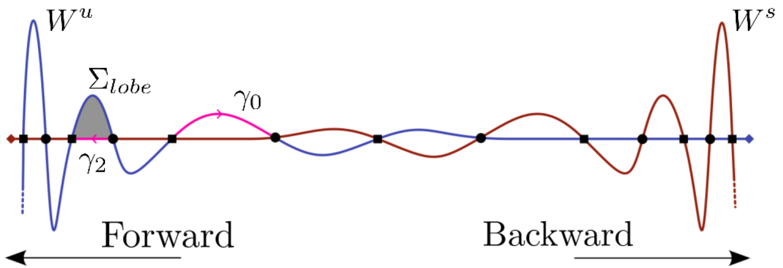

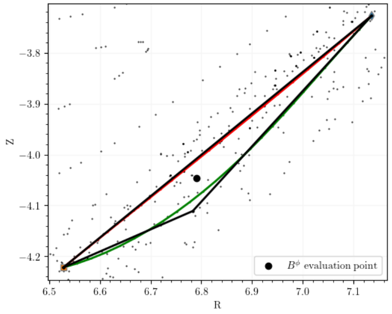

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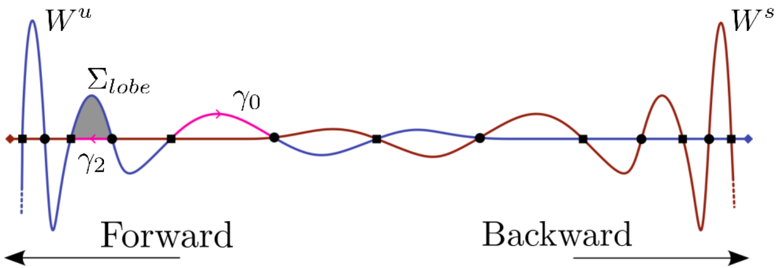

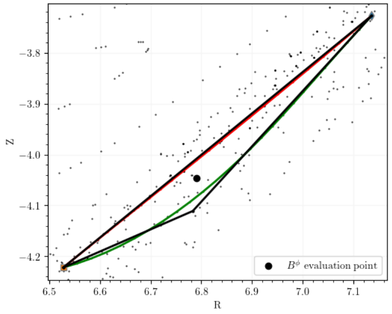

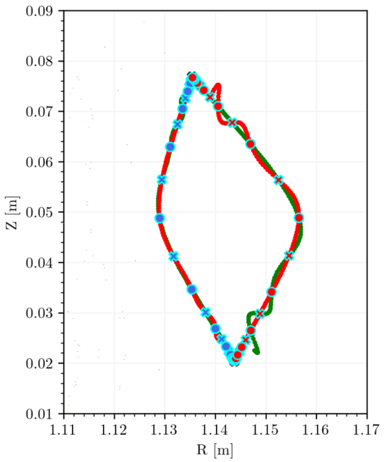

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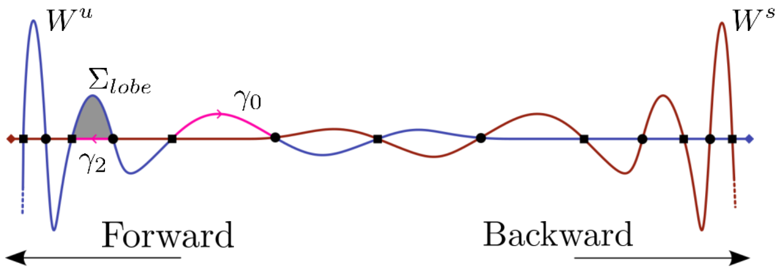
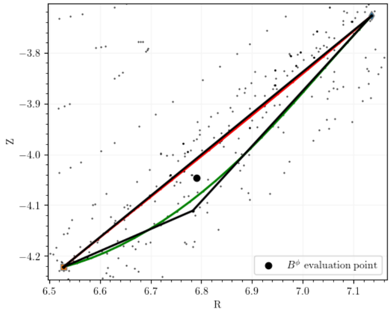
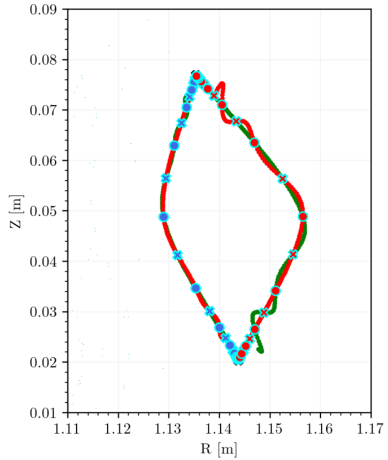
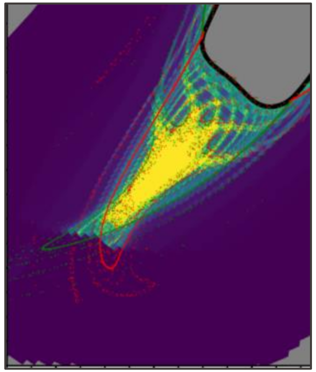


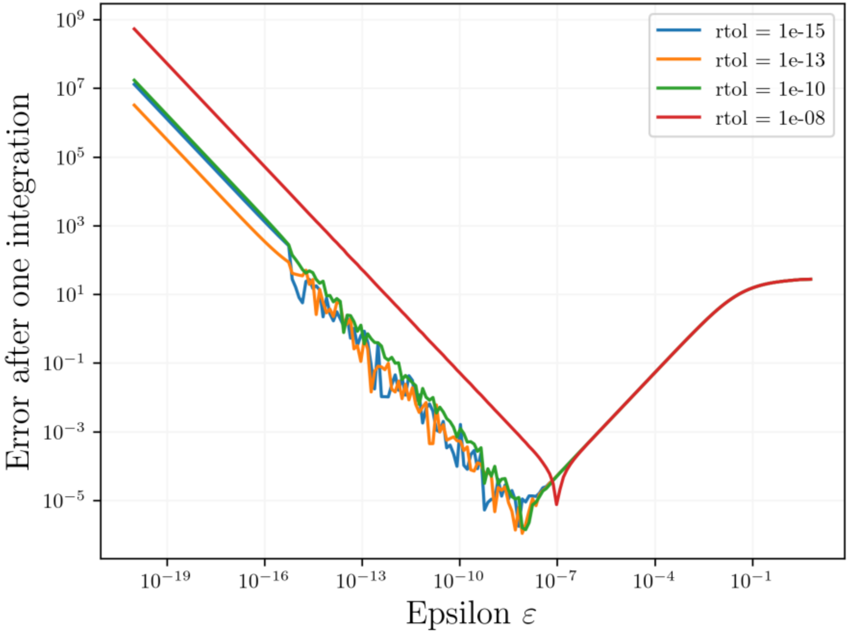






Linear regime lower bound

- Due to integration errors the linear regime is restricted by a lower bound $\varepsilon_{lb} < \varepsilon \ll 1$



Finding many clinic points is a hard problem in general

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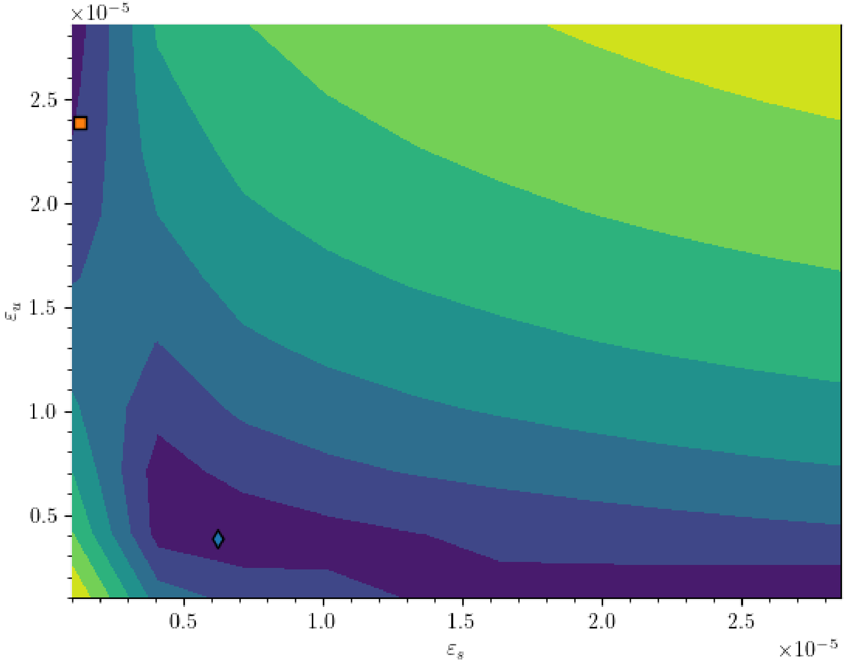

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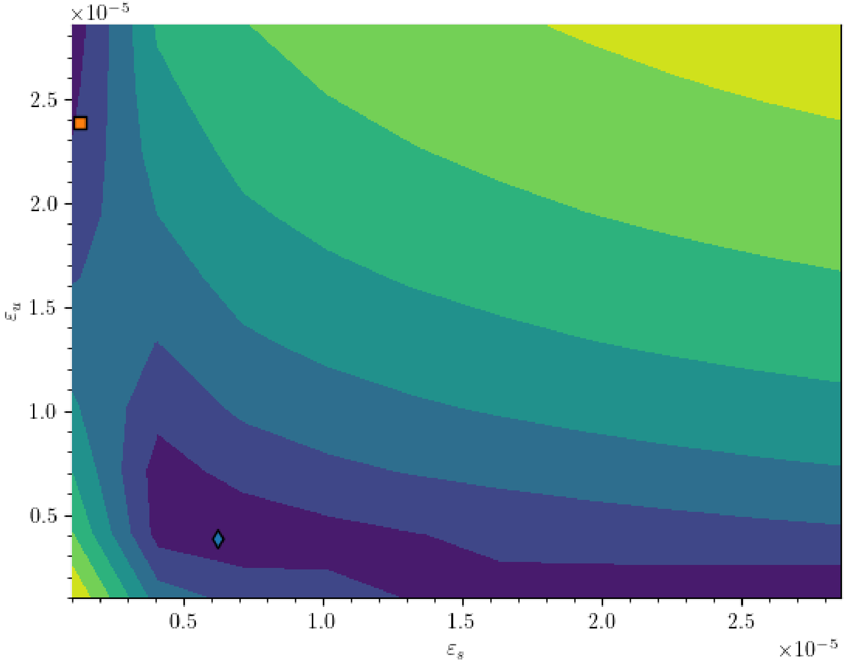

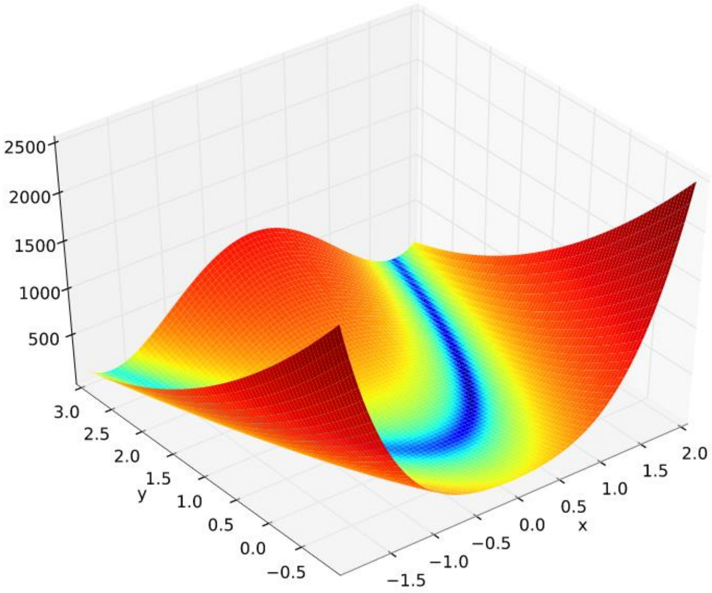

EPFL

Link between the flow map determinant and the divergence free field

$$du \wedge dv = \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) \wedge \left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)$$
$$= \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) dx \wedge dy. \quad (6)$$

As $\boldsymbol{X}_{\mathrm{pol}}(\phi_s, \phi_e)$ is a typical 2D map from the section $\Sigma(\phi_s)$ to $\Sigma(\phi_e)$, the determinant of $\mathcal{D}\boldsymbol{X}_{\mathrm{pol}}(\phi_s, \phi_e)$, denoted by $|\mathcal{D}\boldsymbol{X}_{\mathrm{pol}}(\phi_s, \phi_e)|$, is indeed the same thing as $\partial_x u\, \partial_y v - \partial_y u\, \partial_x v$. One could be curious about the geometric meaning of $|\mathcal{D}\boldsymbol{X}_{\mathrm{pol}}(\phi_s, \phi_e)|$ and conjecture that it must be related to the divergence of the field, since it has been well-known [33] (p. 408) that for $|\mathcal{D}\boldsymbol{X}(\boldsymbol{x}_0, t)|$

$$|\mathcal{D}\boldsymbol{X}(\boldsymbol{x}_0, t)| = \exp\left(\int_0^t \mathrm{tr}\nabla \boldsymbol{B}(\boldsymbol{X}(\boldsymbol{x}_0, \tau))d\tau\right)$$
$$= \exp\left(\int_0^t \nabla \cdot \boldsymbol{B}(\boldsymbol{X}(\boldsymbol{x}_0, \tau))d\tau\right), \quad (7)$$

which indicates that, for a divergence-free field, $|\mathcal{D}\boldsymbol{X}(\boldsymbol{x}_0, t)|$ is always zero. A similar formula for $|\mathcal{D}\boldsymbol{X}_{\mathrm{pol}}(\phi_s, \phi_e)|$ is deduced (proof in appendix A.1) to reveal the relationship between $|\mathcal{D}\boldsymbol{X}_{\mathrm{pol}}(\phi_s, \phi_e)|$ and the divergence along the corresponding trajectory $\boldsymbol{X}_{\mathrm{pol}}(\phi_s, \phi)$, $\phi_s \leqslant \phi \leqslant \phi_e$, as shown below

$$|\mathcal{D}\boldsymbol{X}_{\mathrm{pol}}(\phi_s, \phi_e)| = \exp\left(\int_{\phi_s}^{\phi_e} \frac{R(\nabla \cdot \boldsymbol{B})}{B_\phi} d\phi\right) \frac{B_\phi|_{\phi_s}}{B_\phi|_{\phi_e}}, \quad (8)$$

When the map is a diffeomorphism, volume can be computed using a differential form

$$\Omega = \rho(z)dz^1 \wedge dz^2 \wedge \ldots \wedge dz^n, \quad (2)$$

with density ρ, i.e., $\mu(A) = \int_A \Omega$. Then, f preserves volume when

$$f^*\Omega = \Omega. \quad (3)$$

Here, the *pullback*, f^*, is the local action of the map f on differentials. An easy mnemonic for how this works is to denote the image as a function $z'(z) = f(z)$, and then $f^*dz = dz'$ is the differential of this function, i.e., $f^*dz^i = \sum_j Df_{i,j}(z)dz^j$. Here, $Df_{i,j} = \partial f^i/\partial z^j$ is the Jacobian matrix of f. The implication is that (3) for the form (2) implies that

$$\det(Df(z))\rho(f(z)) = \rho(z).$$

For the common case that $\rho(z) = 1$, this implies that the Jacobian of f has determinant one.

Floer Homology

- Nice papers !

Action principle in detail for exact maps

- Symp
- What is an exact map :

Flux form and the flux conservation

Symplectic maps

- Symp
- What is an exact map :