

Transport through chaos

This content has been downloaded from IOPscience. Please scroll down to see the full text.

1991 Nonlinearity 4 583

(<http://iopscience.iop.org/0951-7715/4/2/017>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 128.138.73.68

This content was downloaded on 05/12/2014 at 21:34

Please note that [terms and conditions apply](#).

Transport through chaos

Robert W Easton†

Department of Mathematics, University of Colorado, Boulder, CO 80309–0426, USA

Received 10 October 1989, in final form 3 April 1990

Accepted by R S Mackay

Abstract. Certain orbits of area preserving maps of the plane appear to wander randomly and to densely fill regions of the plane having positive Lebesgue measure. These regions are called ergodic zones. Trellises (or homoclinic tangles) embedded within these zones are shown to guide the transport of ensembles of points. Trellises or tangles can be localized within what are called *resonance zones*. Transport through resonance zones is calculated using the MacKay–Meiss–Percival action principle.

AMS classification scheme numbers: 58F05, 58F13

1. Introduction

Numerical simulations of area preserving maps of the plane reveal the presence of invariant curves and ergodic zones. The presence of invariant curves is predicted by the KAM theorem. Ergodic zones are populated by orbits which appear to wander randomly in some region possibly bounded by invariant curves. One way to generate an ergodic zone is to plot an orbit which starts near an unstable fixed point. Often the orbit will seem to be distributed densely throughout a region having positive area. This region is then called an ergodic zone. Of course an orbit generated by computer simulation is not likely to be a true orbit of the map and further, since only finitely many points are plotted one can only speculate what set the closure of this orbit might be.

Hidden inside ergodic zones are structures called trellises which guide the transport of ensembles of points. According to Poincaré [5] a *trellis* is the figure formed by the stable and unstable manifolds of a saddle point. More generally, a trellis is the figure formed by the stable and unstable manifolds of a collection of hyperbolic periodic points. Trellis structure is not revealed by the common numerical experiment which plots many points on an orbit. Rather, one must find and plot segments of stable and unstable manifolds. A basis for the systematic study of trellises is found in [3].

Poincaré observed that these manifolds weave across one another forming a grid which partitions the plane. The geometry and combinatorics of the trellis then

† e-mail address: easton@boulder.colorado.edu

determine how ensembles of points are transported. The action principle of MacKay, Meiss and Percival [4] can be used to compute areas of pieces of the grid. Thus knowledge of trellis geometry together with area computations will form the basis for an analysis of ergodic zones.

The MacKay–Meiss–Percival action principle is reformulated and developed in section one. In section two resonance zones and their exit time decompositions are defined and investigated. An example is discussed which illustrates how the internal trellis of a resonance zone and its exit time decomposition are related. Section three summarizes the preceding discussion and indicates directions for further research.

2. Flux calculations

MacKay, Meiss and Percival [4] developed a procedure for calculating areas of regions in the plane bounded by pieces of stable and unstable manifolds. Their procedure requires that the dynamical system f have a generating function, and satisfy a twist condition. Here a new ‘flux’ calculation will be derived without these assumptions.

Suppose that $f: R^2 \rightarrow R^2$ is a diffeomorphism with constant Jacobian $1/\alpha$ (the Henon map for example). Let (q, p) be coordinates in R^2 and let $dp \wedge dq$ denote the area 2-form on R^2 . Since f has Jacobian $1/\alpha$,

$$dp \wedge dq - \alpha f^* dp \wedge dq = 0$$

where f^* denotes the pull-back of $dp \wedge dq$. Since exterior differentiation commutes with the pull-back operation $d(p \, dq - \alpha f^* p \, dq) = 0$, and consequently

$$p \, dq - \alpha f^* p \, dq = dF \quad (2.1)$$

for some smooth function $F: R^2 \rightarrow R^1$.

One can now use (2.1) to compute the line integral of the 1-form $p \, dq$ along a curve $\gamma: [0, 1] \rightarrow R^2$:

$$\int_{\gamma} p \, dq - \int_{\gamma} \alpha f^* p \, dq = \int_{\gamma} dF = \Delta F(\gamma)$$

where $\Delta F(\gamma) = F(\gamma(1)) - F(\gamma(0))$. However, from the definition of the pull-back operation,

$$\int_{\gamma} f^* p \, dq = \int_{f \circ \gamma} p \, dq.$$

Thus $\int_{\gamma} p \, dq = \alpha \int_{f \circ \gamma} p \, dq + \Delta F(\gamma)$. Now replace $\int_{f \circ \gamma} p \, dq$ with

$$\alpha \int_{f^2 \circ \gamma} p \, dq + \Delta F(f \circ \gamma).$$

Then $\int_{\gamma} p \, dq = \alpha^2 \int_{f^2 \circ \gamma} p \, dq + \alpha \Delta F(f \circ \gamma) + \Delta F(\gamma)$.

By induction one obtains the formula

$$\int_{\gamma} p \, dq = \alpha^n \int_{f^n \circ \gamma} p \, dq + \sum_{j=0}^{n-1} \alpha^j \Delta F(f^j \circ \gamma). \quad (2.2)$$

Lemma 2.1. Suppose that the curve γ is a piece of the stable manifold of some saddle point. Then

$$\lim_{n \rightarrow \infty} \alpha^n \int_{f^n \circ \gamma} p \, dq = 0.$$

Proof. Since the range of γ is a segment of the stable manifold of a saddle point, there is a positive constant K such that the arc length of the curve $f^n \circ \gamma$ is bounded by $K\lambda^n$ where $0 < \lambda < 1 < \mu$ are the eigenvalues of the saddle point. $|\int_{f^n \circ \gamma} p \, dq|$ is bounded by a constant times the arc length of $f^n \circ \gamma$. Thus $|\alpha^n \int_{f^n \circ \gamma} p \, dq| \leq \alpha^n \lambda^n C$ for a positive constant C . Since f has Jacobian $1/\alpha$, $1/\alpha = \lambda\mu$ and hence $\alpha^n \lambda^n C = C/\mu^n$. Since $\mu > 1$ it follows that

$$\alpha^n \int_{f^n \circ \gamma} p \, dq \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Corollary 2.2.

$$\int_{\gamma} p \, dq = \sum_{j=0}^{\infty} \alpha^j \Delta F(f^j \circ \gamma).$$

If one refers to F as the *action function* and $\Delta F(\gamma)$ as an *action difference*, then $\int_{\gamma} p \, dq$ is the sum of the action differences along the orbits of the endpoints of γ .

The curve γ may also be transported by f^{-1} . Let $\delta = f^{-n} \circ \gamma$ and apply formula (2.2) to f . Then

$$\int_{\delta} p \, dq = \alpha^n \int_{\gamma} p \, dq + \sum_{j=0}^{n-1} \alpha^j \Delta F(f^j \circ \delta).$$

Solving for $\int_{\gamma} p \, dq$ one obtains the formula

$$\int_{\gamma} p \, dq = \alpha^{-n} \int_{f^{-n} \circ \gamma} p \, dq - \sum_{j=1}^n \alpha^{-j} \Delta F(f^{-j} \circ \gamma). \quad (2.3)$$

Lemma 2.3. If γ is a piece of the unstable manifold of a saddle point, then

$$\int_{\gamma} p \, dq = - \sum_{j=1}^{\infty} \alpha^{-j} \Delta F(f^{-j} \circ \gamma).$$

Proof. The argument is essentially the same as the ones for lemma 2.1 and corollary 2.2.

These formulae can be used with Stokes' theorem to compute areas of regions bounded by segments of stable and unstable manifolds.

Suppose that D is a disc bounded by segments of stable and unstable manifolds $S[a_0, b_0]$ and $U[a_0, b_0]$. By Stokes' theorem $\int_D dp \wedge dq = \int_{\partial D} p \, dq$. Thus using formulae (2.1) and (2.3)

$$\begin{aligned} \int_D dp \wedge dq &= \int_{S[a_0, b_0]} p \, dq + \int_{U[a_0, b_0]} p \, dq \\ &= \sum_{j=0}^{\infty} \alpha^j [F(b_j) - F(a_j)] - \sum_{j=1}^{\infty} \alpha^{-j} [F(a_{-j}) - F(b_{-j})]. \end{aligned}$$

Rearranging the sum gives

$$\int_D dp \wedge dq = \sum_{j=-\infty}^{\infty} \alpha^j [F(b_j) - F(a_j)].$$

In general suppose that D is a disc bounded by alternating segments of stable and unstable manifold. Suppose that the endpoints of these segments are indexed a^0, a^1, \dots, a^{2m} (with $a^{2m} = a_0$) in a counterclockwise order around the boundary of D . Suppose that the segment joining a^0 and a^1 is contained in a stable manifold. Then by the preceding argument

$$\int_D dp \wedge dq = \sum_{j=-\infty}^{\infty} \alpha^j \sum_{k=0}^{m-1} [F(a_j^{2k+1}) - F(a_j^{2k})]. \quad (2.4)$$

This formula expresses the MacKay–Meiss–Percival action principle.

The action function F can be easily calculated for a specific map f . For example suppose that f is the standard map given by $f(q, p) = (q + p + a \sin(2\pi q), p + a \sin(2\pi q))$. Then $dF = p \, dq - f^* p \, dq$ can be calculated:

$$dF = [2\pi ap \cos(2\pi q) + a \sin(2\pi q) + \pi a^2 \sin(2\pi q) \cos(2\pi q)] \, dq + [p + a \sin(2\pi q)] \, dp.$$

Furthermore $dF = F_q \, dq + F_p \, dp$ where F_q denotes the partial derivative $\partial/\partial q F$. Since formulae for the partial derivatives of F are known, F is obtained by integration. The end result is that

$$F(q, p) = \frac{1}{2}p^2 + ap \sin(2\pi q) - \cos(2\pi q)/2\pi a - \frac{1}{2}a^2 \cos(4\pi q).$$

It is worth mentioning that the preceding derivation of the action principle can be generalized to higher dimensions. If f is a diffeomorphism of a smooth oriented n -manifold, then one may start with a k -form λ and assume that $\lambda - \alpha f^* \lambda = d\rho$ for some $(k-1)$ -form ρ . Thus $\int_c d\lambda = \int_{\partial c} \lambda$ which can be computed by formulae similar to (2.2) and (2.3) assuming that ∂c is the sum of two pieces (or chains) such that $\lim_{n \rightarrow \infty} \alpha^n \int_{f^n \circ c_1} \lambda$ and $\lim_{n \rightarrow \infty} \alpha^{-n} \int_{f^{-n} \circ c_2} \lambda$ exists.

3. Resonance zones

A general transport problem may be described as follows. Given a probability measure on a set M and a measure preserving transformation f , and disjoint measurable sets E and F , compute the probability that the orbit of a random initial point chosen from E will first reach F at time n . This computation may be difficult in general. However when E is a resonance zone (defined below) and F is its complement, then the transition probabilities can be conveniently calculated. Thus resonance zones provide the means to localize the study of transport and in addition they provide the means to localize the study of trellis geometry.

In this section f is an area and orientation preserving diffeomorphism of the plane. $W^s(p)$ and $W^u(p)$ will denote the stable and unstable manifolds of a hyperbolic fixed point p of f . For a, b in $W^s(p)$ let $S_p[a, b]$ denote the segment of stable manifold between a and b . An *initial segment* of $W^s(p)$ is any segment of the form $S_p[p, b]$. Similarly for c, d in $W^u(p)$ let $U_p[c, d]$ denote the segment of unstable manifold between c and d . An *initial segment* of $W^u(p)$ is any segment of the form $U_p[p, d]$.

A *resonance zone* is a closed subset of the plane which is bounded by alternating initial segments of stable and unstable manifolds of hyperbolic periodic points. These segments are required to intersect only at their endpoints.

The area preserving Henon map $H(x, y) = (y, 1.8 - x - y^2)$ was used to generate the resonance zone pictured in figure 1. H has a saddle point p whose stable and unstable manifolds intersect transversally at the homoclinic point b . The heart-shaped region bounded by the segments $U_p[p, b]$ and $S_p[p, b]$ is a resonance zone. The trellis of p appears to be a trellis of type two as defined in [3].

Let R be a resonance zone. Define *exit time functions* $t^+ : R \rightarrow [0, \infty]$ and $t^- : R \rightarrow [0, \infty]$ by

$$t^+(x) = \begin{cases} \infty & \text{if } f^j(x) \in R \text{ for each } j \geq 0 \\ \text{the least } j & \text{with } f^j(x) \notin R \text{ otherwise} \end{cases}$$

$$t^-(x) = \begin{cases} \infty & \text{if } f^{-j}(x) \in R \text{ for each } j \geq 0 \\ \text{the least } j & \text{with } f^{-j}(x) \notin R \text{ otherwise} \end{cases}.$$

Thus the exit times measure the times to reach the complement of R . Let $R(i, j) = \{x \in R : t^+(x) = i, t^-(x) = j\}$. The collection of sets $R(i, j)$ partitions the resonance zone and forms what is called its *exit time decomposition*. Entry and exit sets are defined as follows: $R_{\text{in}} = \{x \in R : t^-(x) = 1\}$, $R_{\text{out}} = \{x \in R : t^+(x) = 1\}$. Exit times partition these sets into *entry and exit rainbows*. Let $R_{\text{in}}(j) = \{t^-(x) = j\} \cap R_{\text{in}}$ and define P_j to be the measure of $R_{\text{in}}(j)$. The sequence $\{P_j\}$ governs transport through the resonance zone. If P is the measure of the entry set then P_j/P is the probability that a randomly chosen point in the entry set will exit at time n .

The study of a trellis may be localized within a resonance zone in the following way. Let p be a saddle point on the boundary of a resonance zone R . Define the *R-unstable manifold* $W^u(p, R)$ of p to consist of all points in the unstable manifold of p whose backward orbits are contained in R . Define the *R-stable manifold* $W^s(p, R)$ on p to consist of all points in the stable manifold of p whose forward orbits are contained in R . Define $T^u(R)$ and $T^s(R)$ to be the unions of the R -unstable and R -stable manifolds of the saddle points on the boundary of R . Define the *internal trellis* of R to be the union of these two sets.

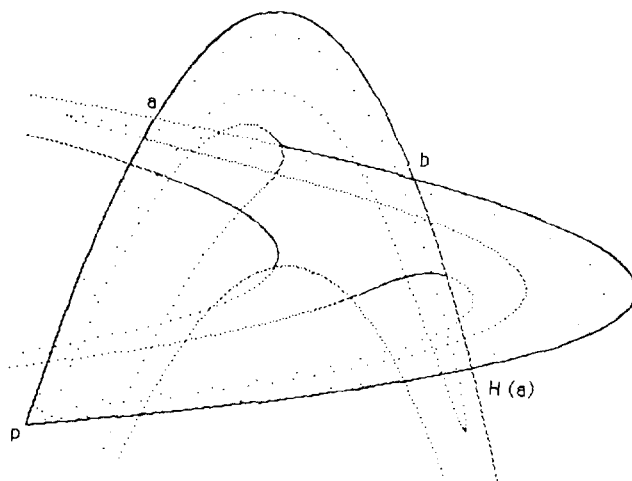


Figure 1. Resonance zone for the Henon map

Proposition. Discontinuity points of t^+ occur on R -stable manifolds. Similarly, discontinuity points of t^- occur on R -unstable manifolds. Hence the internal trellis of the resonance zone partitions the zone into its exit time decomposition.

Proof. For simplicity the proof will be given for the resonance zone pictured in figure 1, but it can be generalized to any resonance zone. Let $x \in R$. Since iterates of H are continuous functions one can choose a neighbourhood of x such that $t^+(y) \leq t^+(x)$ for each y in this neighbourhood. Thus t^+ is discontinuous at x if and only if there exists a sequence of points $\{x_j\}$ converging to x such that $t^+(x_j) = m$ and m is less than $t^+(x)$. Since $H^{m-1}(x_j)$ is in the exit set, $H^{m-1}(x)$ is on the boundary of the exit set. Hence $H^{m-1}(x) \in S_p[a, b]$ and it follows that $x \in T^s(R)$.

Each component of an R -unstable manifold is called a *string*. The endpoints of each string are homoclinic points on the boundary of the resonance zone. Each string produces a shower of fragments as it is clipped into pieces by deleting its intersection with the exit set, and stretched by applying the transformation f . The R -unstable manifold of a saddle point p on the boundary of the zone is constructed by successively clipping and stretching the string containing p . This process is illustrated using the Henon map and the resonance zone pictured in figure 1.

The R -unstable manifold of the saddle point p may be constructed by choosing the string $K = U_p[p, H(a)]$, applying H to K , and deleting the part of $H(K)$ crossing the exit set of R to form the set $K[1]$. Continue this process by defining $K[n+1] = H(K[n] - R_{\text{out}})$. Then the R -unstable manifold of p is the union of the initial segment K , and the sets $K[n]$ for $n \geq 1$.

The endpoints of each string are homoclinic points and they must belong to one of the three segments of stable manifold $S_1 = S_p[H(a), H(b)]$, $S_2 = H(S_1)$ and $S_3 = S_p[p, H^3(a)]$. Strings are said to be of type alpha, beta and gamma respectively if they connect the segments S_3 with S_1 , S_1 with S_2 and S_2 with S_3 . Other types of strings are ignored.

A new symbolic dynamics of strings will replace the usual symbolic dynamics. since $H(S_1) \subset S_2$, $H(S_2) \subset S_3$ and $H(S_3) \subset S_1$, the forward orbits of string endpoints are determined. The fragmentation of strings occurs in the following way. An alpha-string which is clipped and stretched breaks into one alpha and one beta-string. A beta-string is transformed to a gamma-string. A gamma-string is stretched and clipped into two alpha-strings.

A string population model can be formed by defining a population vector $v(n)$ with three components specifying the number of alpha-, beta- and gamma-strings present at the n th generation. Then $v(n+1) = Av(n)$, where A is the matrix

$$\begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The population of fragments of the alpha-string K which form the components of the set $K[n]$ is given by $A^n v(0)$ where $v(0)$ is the column vector with entries 1, 0, 0. The asymptotic growth rate of the string population is determined by the largest positive eigenvalue λ of the matrix A .

A rough estimate of the asymptotic escape rate of entry points from the resonance zone is $v_1 w_1 + v_2 w_2 + v_3 w_3$ where $v = (v_1, v_2, v_3)$ satisfies the conditions $Av = \lambda v$, and $v_1 + v_2 + v_3 = 1$. w_1, w_2, w_3 are the average fractions (in terms of arc length) of strings of type alpha, beta and gamma respectively that are clipped by the exit set.

Exact calculation of the element P_n of the transport sequence $\{P_n\}$ may be done as follows. Let $K_0 = U_p[b, H(a)]$. A homoclinic point $q \in K_0$ is defined to be of *type* n if it is an endpoint of an open sub-interval of K_0 on which $t^+ = n + 1$. Define $\varphi(q) = q'$ provided that q and q' are the endpoints of a sub-interval of K_0 on which $t^+ = n + 1$. Let X_n denote the set of type n homoclinic points. The function φ produces a *u-pairing* of the points of X_n . X_n is mapped by H^n into $J_0 = S_p[a, b]$. Define the *s-pairing* of points in X_n by $\psi(q) = q'$ provided $H^n(q)$ and $H^n(q')$ are the endpoints of a sub-interval of J_0 on which $t^- = n + 1$. Choose $a_0 \in X_n$ and form the sequence $a_0, a_1 = \varphi(a_0), a_2 = \psi(a_1)$ and so on until $a_{2n} = a_0$. The alternating sequence of stable and unstable manifold segments joining these homoclinic points bounds a set of entry points which exit at time n . The action difference formula (2.4) applied to the sequence a_0, a_1, \dots, a_{2n} gives the measure of this set. The sum of the measures of all such sets is P_n . Thus the transport sequence can be calculated by knowing the type numbers of homoclinic points, and their pairings as subsets of the stable and unstable manifolds.

4. Discussion

A theoretical foundation for the study of transport through resonance zones has been developed. Resonance zones are carefully defined. The new concepts of the internal trellis of a resonance zone, exit times, and entry and exit rainbows were introduced. These concepts were illustrated for an area preserving Henon map. The MacKay–Meiss–Percival action principle was derived without using a twist condition. It was shown that the action principle can be applied to compute areas of pieces of entry and exit rainbows and hence to compute transport through a resonance zone.

Numerical methods for computing transport which use resonance zones and the action principle can be found in [1] and [2]. These studies are limited to systems with two-dimensional phase spaces. There is considerable motivation to develop methods which will work both theoretically and computationally in higher-dimensional phase spaces. Wiggins [6] has made a start in this direction. Based on the observation that the maximal invariant subset of a resonance zone is isolated, it appears that isolating blocks as defined in [8] may be useful generalizations of resonance zones in higher dimensions.

The work of Rom-Kedar [7] extends the study of trellises begun in [3] and relates trellis geometry to topological entropy.

Acknowledgments

I want to thank Rex Skodje and Jim Meiss for showing me the transport problem and for many stimulating discussions while this paper was being written.

References

- [1] Chen Q, Dana I, Meiss J, Murray N and Percival I 1991 Resonances and transport in the sawtooth map *Physica D* in press
- [2] Davis M and Skodje R 1990 Statistical rate theory for transient species: exact decay rates from periodic orbits *J. Chem. Phys. Lett.* **175** 92–100

- [3] Easton R 1986 Trellises formed by stable and unstable manifolds in the plane *Trans. Am. Math. Soc.* **294** 719–31
- [4] MacKay R, Meiss J and Percival I 1984 Transport in Hamiltonian systems *Physica* **13D** 55–81
- [5] Poincaré H 1892 *Methodes Nouvelles de la Mechanique Celeste* (Paris: Gauthier-Villars) ch 33
- [6] Wiggins S 1990 On the geometry of transport in phase space, 1. Transport in k -degree-of-freedom Hamiltonian systems *Physica* **44D** 471–501
- [7] Rom-Kedar V 1990 Transport rates of a class of two-dimensional maps and flows *Physica* **43D** 229–68
- [8] Easton R 1989 Isolating blocks and epsilon chains for maps *Physica* **39D** 95–110