0.2 Functions

Definition 0.2.1 (Function)**.** A **function** f from a set A to a set B, denoted $f : A \to B$, is a rule that assigns to each element $a \in A$ exactly one element $f(a) \in B$. The set A is called the **domain** and B is called the **codomain**.

Definition 0.2.2 (Injective, Surjective, Bijective)**.** Let $f : A \to B$ be a function.

- f is **injective** (one-to-one) if $f(a_1) = f(a_2)$ implies $a_1 = a_2$.
- f is **surjective** (onto) if for every $b \in B$, there exists $a \in A$ such that $f(a) = b$.
- f is **bijective** if it is both injective and surjective.

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

Example (Injective, Surjective, and Bijective Functions). Consider the following functions:

1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 1$ is **bijective**.
   - Injective: If $2x_1 + 1 = 2x_2 + 1$, then $x_1 = x_2$.
   - Surjective: For any $y \in \mathbb{R}$, we have $f\left(\frac{y-1}{2}\right) = y$.
2. $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$ is **neither injective nor surjective**.
   - Not injective: $g(1) = g(-1) = 1$, but $1 \neq -1$.
   - Not surjective: There is no $x \in \mathbb{R}$ such that $g(x) = -1$.
3. $h : \mathbb{R} \to [0, \infty)$, $h(x) = x^2$ is **surjective but not injective**.
   - Not injective: $h(1) = h(-1) = 1$.
   - Surjective: For any $y \geq 0$, we have $h(\sqrt{y}) = y$.
4. $k : [0, \infty) \to \mathbb{R}$, $k(x) = x^2$ is **injective but not surjective**.
   - Injective: If $x_1^2 = x_2^2$ with $x_1, x_2 \geq 0$, then $x_1 = x_2$.
   - Not surjective: There is no $x \geq 0$ such that $k(x) = -1$.