1.2 Subspaces

Some vector spaces might be "too big" and be unwieldy to deal with. Therefore, it is often convenient to look at "smaller vector spaces" inside a given vector space. These are called **subspaces**.

Definition 1.2.1 (Subspace)**.** Let V be a vector space over F. A subset W of V is called a **subspace** of V, if W is also a vector space over F with addition and scalar multiplication inherited from V.

Remark. Let $(V, +, \cdot)$ be a vector space over F, and $W \subseteq V$. If $(W, +, \cdot)$ is also a vector space over F then W is a subspace of V.

Example (Trivial Subspaces). Given a vector space V, then

- V is a subspace of V itself;
- $\{\mathbf{0}\}$ is a subspace of V (also known as the "zero subspace");
- $\emptyset$ is **not** a subspace of V.

Theorem 1.2.1 (Subspace Test). Let V be a vector space over F and $W \subseteq V$. Then W is a subspace of V if and only if the following are satisfied:

- W is non-empty;
- W is closed under addition: $\mathbf{w}_1 + \mathbf{w}_2 \in W$ for all $\mathbf{w}_1, \mathbf{w}_2 \in W$;
- W is closed under scalar multiplication: $\alpha\mathbf{w} \in W$ for all $\alpha \in F$ and $\mathbf{w} \in W$.

Proof. ($\Rightarrow$) If W is a subspace, then W is a vector space, so it contains the zero vector (hence non-empty) and is closed under addition and scalar multiplication by definition.

($\Leftarrow$) Assume the three conditions hold. Since $W \subseteq V$, the operations on W are inherited from V, so associativity, commutativity, and distributivity are automatically satisfied. It remains to verify the existence of identity elements and inverses.

- **Zero vector:** Since $W \neq \emptyset$, there exists $\mathbf{w} \in W$. By closure under scalar multiplication, $0 \cdot \mathbf{w} = \mathbf{0} \in W$.
- **Additive inverse:** For any $\mathbf{w} \in W$, closure under scalar multiplication gives $(-1) \cdot \mathbf{w} = -\mathbf{w} \in W$.

Thus W is a vector space under the inherited operations, i.e., a subspace of V. ■

Remark. Note that from the proof, the first condition of the Subspace Test can be replaced by the requirement that W contains the zero vector of V.

Remark. In practice, to test whether a given subset of a vector space is a subspace, we use Theorem 1.2.1 instead of checking all eight vector space axioms from scratch, which is much less tedious. Moreover, some introductory textbooks use Theorem 1.2.1 as the definition of a subspace. While this is logically equivalent, it can be somewhat less intuitive than defining a subspace as a subset that is itself a vector space.

Example (Line as Subspace). $W = \{(x, y) \in \mathbb{R}^2 : x + 2y = 0\}$ is a subspace of $\mathbb{R}^2$. Indeed:

- **Zero vector:** $(0, 0) \in W$ since $0 + 2(0) = 0$.
- **Addition:** If $(x_1, y_1), (x_2, y_2) \in W$, then $(x_1 + x_2) + 2(y_1 + y_2) = (x_1 + 2y_1) + (x_2 + 2y_2) = 0 + 0 = 0$, so their sum is in W.
- **Scalar multiplication:** If $(x, y) \in W$ and $c \in \mathbb{R}$, then $(cx) + 2(cy) = c(x + 2y) = c(0) = 0$, so $c(x, y) \in W$.

Example (Translated Line is Not a Subspace). $W = \{(x, y) \in \mathbb{R}^2 : x + 2y = 3\}$ is **not** a subspace of $\mathbb{R}^2$. This is because it does not contain the zero vector: $0 + 2(0) = 0 \neq 3$, so $(0, 0) \notin W$.

Example (Polynomials of Bounded Degree). Let $F[x]$ be the vector space of all polynomials over a field F. For a fixed $n \in \mathbb{N}$, the set $F[x]_{\leq n}$ of polynomials of degree at most n is a subspace of $F[x]$.

- **Zero vector:** The zero polynomial has degree $-\infty$, so $0 \in F[x]_{\leq n}$.
- **Addition:** If $\deg p \leq n$ and $\deg q \leq n$, then $\deg(p + q) \leq \max(\deg p, \deg q) \leq n$.
- **Scalar multiplication:** If $\deg p \leq n$ and $c \in F$, then $\deg(cp) \leq \deg p \leq n$.

Example (Polynomials of Exact Degree). The set $F[x]_{=n}$ of polynomials of degree **exactly** n is **not** a subspace of $F[x]$.

- It does not contain the zero vector (since $\deg 0 = -\infty \neq n$).
- It is not closed under addition. For example, if $p(x) = x^n + x$ and $q(x) = -x^n$, both have degree n, but $p(x) + q(x) = x$, which has degree $1 \neq n$ (assuming $n > 1$).

Example (Differentiable Functions). Let $C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$ be the set of all continuous functions on $\mathbb{R}$. We consider the set of differentiable functions:

$$D(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is differentiable}\}$$

To verify that $D(\mathbb{R})$ is a subspace of $C(\mathbb{R})$:

- **Subset:** Since differentiability implies continuity, we have $D(\mathbb{R}) \subseteq C(\mathbb{R})$.
- **Zero vector:** The zero function $f(x) = 0$ is differentiable, so $\mathbf{0} \in D(\mathbb{R})$.
- **Closure under addition:** If $f, g \in D(\mathbb{R})$, then $f + g$ is differentiable with $(f + g)' = f' + g'$, so $f + g \in D(\mathbb{R})$.
- **Closure under scalar multiplication:** If $f \in D(\mathbb{R})$ and $c \in \mathbb{R}$, then cf is differentiable with $(cf)' = cf'$, so $cf \in D(\mathbb{R})$.

Thus, $D(\mathbb{R})$ is a subspace of $C(\mathbb{R})$.

Example (Symmetric Matrices). The set of all symmetric matrices in $M_n(F)$, denoted by $S_n(F) = \{\mathbf{A} \in M_n(F) : \mathbf{A} = \mathbf{A}^\top\}$, is a subspace of $M_n(F)$.

- **Zero vector:** The zero matrix $\mathbf{O}$ is symmetric since $\mathbf{O}^\top = \mathbf{O}$, so $\mathbf{O} \in S_n(F)$.
- **Addition:** If $\mathbf{A}, \mathbf{B} \in S_n(F)$, then $(\mathbf{A}+\mathbf{B})^\top = \mathbf{A}^\top + \mathbf{B}^\top = \mathbf{A}+\mathbf{B}$, so $\mathbf{A}+\mathbf{B} \in S_n(F)$.
- **Scalar multiplication:** If $\mathbf{A} \in S_n(F)$ and $c \in F$, then $(c\mathbf{A})^\top = c\mathbf{A}^\top = c\mathbf{A}$, so $c\mathbf{A} \in S_n(F)$.

Example (Trace-free Matrices). The set of all trace-free matrices in $M_n(F)$, denoted by $W = \{\mathbf{A} \in M_n(F) : \operatorname{tr}(\mathbf{A}) = 0\}$, is a subspace of $M_n(F)$.

- **Zero vector:** $\operatorname{tr}(\mathbf{O}) = 0 + 0 + \cdots + 0 = 0$, so $\mathbf{O} \in W$.
- **Addition:** If $\mathbf{A}, \mathbf{B} \in W$, then $\operatorname{tr}(\mathbf{A}+\mathbf{B}) = \operatorname{tr}(\mathbf{A}) + \operatorname{tr}(\mathbf{B}) = 0+0 = 0$, so $\mathbf{A}+\mathbf{B} \in W$.
- **Scalar multiplication:** If $\mathbf{A} \in W$ and $c \in F$, then $\operatorname{tr}(c\mathbf{A}) = c\operatorname{tr}(\mathbf{A}) = c(0) = 0$, so $c\mathbf{A} \in W$.

Theorem 1.2.2 (Intersection of Subspaces). Let W_1 and W_2 be subspaces of a vector space V. Then the intersection $W_1 \cap W_2$ is also a subspace of V.

Proof. We use the Subspace Test on $W = W_1 \cap W_2$:

- **Zero vector:** Since W_1 and W_2 are subspaces, $\mathbf{0} \in W_1$ and $\mathbf{0} \in W_2$. Thus $\mathbf{0} \in W_1 \cap W_2$.
- **Closure under addition:** Let $\mathbf{w}, \mathbf{z} \in W_1 \cap W_2$. Then $\mathbf{w}, \mathbf{z} \in W_1$ and $\mathbf{w}, \mathbf{z} \in W_2$. Since W_1 and W_2 are subspaces, they are closed under addition, so $\mathbf{w} + \mathbf{z} \in W_1$ and $\mathbf{w} + \mathbf{z} \in W_2$. Hence $\mathbf{w} + \mathbf{z} \in W_1 \cap W_2$.
- **Closure under scalar multiplication:** Let $\mathbf{w} \in W_1 \cap W_2$ and $c \in F$. Then $\mathbf{w} \in W_1$ and $\mathbf{w} \in W_2$. Since W_1 and W_2 are closed under scalar multiplication, $c\mathbf{w} \in W_1$ and $c\mathbf{w} \in W_2$. Hence $c\mathbf{w} \in W_1 \cap W_2$.

Since all three conditions of the Subspace Test are satisfied, $W_1 \cap W_2$ is a subspace of V. ■

Remark. Unlike the intersection, the **union** of two subspaces is not necessarily a subspace. For $W_1 \cup W_2$ to be a subspace, one subspace must be contained within the other (i.e., $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$).

Example (Union of Axes). Consider the vector space $V = \mathbb{R}^2$. Let W_1 be the x-axis and W_2 be the y-axis:

$$W_1 = \{(x, 0) : x \in \mathbb{R}\}, \quad W_2 = \{(0, y) : y \in \mathbb{R}\}.$$

Both W_1 and W_2 are subspaces of $\mathbb{R}^2$. However, their union $W_1 \cup W_2$ is **not** a subspace because it is not closed under addition.

Indeed, $(1,0) \in W_1 \subseteq W_1 \cup W_2$ and $(0,1) \in W_2 \subseteq W_1 \cup W_2$, but their sum:

$$(1,0) + (0,1) = (1,1)$$

is not in $W_1 \cup W_2$ since $(1,1)$ is neither on the x-axis nor the y-axis.

Proposition 1.2.1 (Union of Subspaces). Let W_1, W_2 be subspaces of V. Then $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Proof. ($\Leftarrow$) If $W_1 \subseteq W_2$, then $W_1 \cup W_2 = W_2$, which is a subspace. Similarly, if $W_2 \subseteq W_1$, the union is W_1, which is also a subspace.

($\Rightarrow$) We prove the contrapositive: if neither is contained in the other, then the union is not a subspace. Assume $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$.

- Since $W_1 \not\subseteq W_2$, there exists a vector $\mathbf{u} \in W_1$ such that $\mathbf{u} \notin W_2$.
- Since $W_2 \not\subseteq W_1$, there exists a vector $\mathbf{v} \in W_2$ such that $\mathbf{v} \notin W_1$.

Consider the sum $\mathbf{w} = \mathbf{u} + \mathbf{v}$. We show that $\mathbf{w} \notin W_1 \cup W_2$:

- If $\mathbf{w} \in W_1$, then $\mathbf{v} = \mathbf{w} - \mathbf{u}$. Since $\mathbf{w} \in W_1$ and $\mathbf{u} \in W_1$, their difference $\mathbf{v}$ must be in W_1 (by closure). But we chose $\mathbf{v} \notin W_1$, which is a contradiction.
- If $\mathbf{w} \in W_2$, then $\mathbf{u} = \mathbf{w} - \mathbf{v}$. Since $\mathbf{w} \in W_2$ and $\mathbf{v} \in W_2$, their difference $\mathbf{u}$ must be in W_2. But we chose $\mathbf{u} \notin W_2$, which is a contradiction.

Therefore, $\mathbf{u} + \mathbf{v}$ is in neither W_1 nor W_2, so it is not in $W_1 \cup W_2$. Thus, the union is not closed under addition and is therefore not a subspace. ■