0.0 Matrices

Definition 0.5.1 (Matrix)**.** An $m \times n$ **matrix** over a field F is a rectangular array of elements from F arranged in m rows and n columns:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Remark. $M_{m\times n}(F)$ is the set of all matrices of size $m \times n$ and have entries in F. In the above definition, we can write $\mathbf{A} \in M_{m\times n}(F)$.

Moreover, when $m = n$, we abbreviate $M_{m\times n}(F)$ to $M_n(F)$.

Example (Matrices). Here are some examples of matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \in M_2(\mathbb{R}), \quad \begin{bmatrix} 1 & 0 & -1 \\ 2 & 5 & 3 \end{bmatrix} \in M_{2\times 3}(\mathbb{R}), \quad \begin{bmatrix} i & 1+i \\ 0 & 2-i \end{bmatrix} \in M_2(\mathbb{C}).$$

0.0.1 Special Matrices

Definition 0.5.2 (Zero Matrix)**.** The **zero matrix** $\in M_{m\times n}(F)$ is the matrix with all entries equal to zero:

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Definition 0.5.3 (Identity Matrix)**.** The **identity matrix** $\mathbf{I}_n \in M_n(F)$ is the $n \times n$ square matrix with 1s on the diagonal and 0s elsewhere:

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

When the size is clear from context, we simply write $\mathbf{I}$.

Definition 0.5.4 (Transpose)**.** Let $\mathbf{A} = (a_{ij}) \in M_{m\times n}(F)$. The **transpose** of $\mathbf{A}$, denoted

$\mathbf{A}^\top$, is the $n \times m$ matrix obtained by swapping rows and columns:

$$(\mathbf{A}^\top)_{ij} = a_{ji}.$$

Visually, the rows of $\mathbf{A}$ become the columns of $\mathbf{A}^\top$:

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}^\top = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}.$$

Definition 0.5.5 (Symmetric Matrix)**.** A square matrix $\mathbf{A} \in M_n(F)$ is **symmetric** if $\mathbf{A} = \mathbf{A}^\top$. This means it is symmetric across its main diagonal:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}.$$

Definition 0.5.6 (Diagonal Matrix)**.** A square matrix $\mathbf{A} \in M_n(F)$ is **diagonal** if all entries off the main diagonal are zero:

$$\mathbf{A} = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}.$$

We often write $\mathbf{A} = \mathrm{diag}(d_1, d_2, \dots, d_n)$.

Definition 0.5.7 (Upper Triangular Matrix)**.** A square matrix $\mathbf{A} \in M_n(F)$ is **upper triangular** if all entries below the main diagonal are zero:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}.$$

Example (Zero and Identity Matrices)**.**

$${}_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example (Examples of Special Matrices).

- **Transpose:** $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^\top = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.
- **Symmetric:** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ is symmetric since $a_{ij} = a_{ji}$.
- **Diagonal:** $\operatorname{diag}(3, -1, 0) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
- **Upper Triangular:** $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$.

0.0.2 Matrix Addition

Definition 0.5.8 (Matrix Addition)**.** Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ be matrices in $M_{m\times n}(F)$. Their **sum** $\mathbf{A} + \mathbf{B}$ is the matrix $\mathbb{C} = (c_{ij}) \in M_{m\times n}(F)$ where:

$$c_{ij} = a_{ij} + b_{ij}.$$

Definition 0.5.9 (Additive Inverse)**.** The **additive inverse** (or **negation**) of a matrix $\mathbf{A} = (a_{ij}) \in M_{m\times n}(F)$ is the matrix $-\mathbf{A} = (-a_{ij}) \in M_{m\times n}(F)$.

Example (Matrix Addition). Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. Then:

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}.$$

Theorem 0.5.1 (Properties of Matrix Addition). Let $\mathbf{A}, \mathbf{B}, \mathbb{C} \in M_{m\times n}(F)$. Then:

1. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ (Commutativity)
2. $(\mathbf{A} + \mathbf{B}) + \mathbb{C} = \mathbf{A} + (\mathbf{B} + \mathbb{C})$ (Associativity)
3. $\mathbf{A} + \quad = \mathbf{A}$ (Additive identity)
4. $\mathbf{A} + (-\mathbf{A}) = \quad$ (Additive inverse)

Proof. We prove commutativity. Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$. Then:

$$(\mathbf{A} + \mathbf{B})_{ij} = a_{ij} + b_{ij} = b_{ij} + a_{ij} = (\mathbf{B} + \mathbf{A})_{ij}$$

where the second equality uses commutativity of addition in the underlying field. Since this holds for all i, j, we have $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

The remaining properties follow similarly. ■

0.0.3 Matrix Multiplication

Definition 0.5.10 (Matrix Multiplication)**.** Let $\mathbf{A} = (a_{ij}) \in M_{m\times n}(F)$ and $\mathbf{B} = (b_{jk}) \in M_{n\times p}(F)$. Their **product AB** is the matrix $\mathbb{C} = (c_{ik}) \in M_{m\times p}(F)$ where:

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

In other words, the (i, k)-entry of **AB** is the dot product of the i-th row of **A** with the k-th column of **B**.

Example (Matrix Multiplication). Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \in M_2(\mathbb{R})$ and $\mathbf{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \in M_2(\mathbb{R})$. We compute **AB** entry by entry:

$$\begin{aligned}
(\mathbf{AB})_{11} &= (1)(5) + (2)(7) = 5 + 14 = 19 \\
(\mathbf{AB})_{12} &= (1)(6) + (2)(8) = 6 + 16 = 22 \\
(\mathbf{AB})_{21} &= (3)(5) + (4)(7) = 15 + 28 = 43 \\
(\mathbf{AB})_{22} &= (3)(6) + (4)(8) = 18 + 32 = 50
\end{aligned}$$

Therefore:

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Example (Matrix Multiplication with Different Dimensions). Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in M_{2\times 3}(\mathbb{R})$ and $\mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \in M_{3\times 1}(\mathbb{R})$. The product $\mathbf{AB} \in M_{2\times 1}(\mathbb{R})$:

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} (1)(1) + (2)(0) + (3)(-1) \\ (4)(1) + (5)(0) + (6)(-1) \end{bmatrix} = \begin{bmatrix} 1 + 0 - 3 \\ 4 + 0 - 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

Note that **BA** is **not defined** since the number of columns of **B** (which is 1) does not equal the number of rows of **A** (which is 2).

Theorem 0.5.1 (Properties of Matrix Multiplication). Let $\mathbf{A}, \mathbf{B}, \mathbb{C}$ be matrices of compatible dimensions. Then:

1. $(\mathbf{AB})\mathbb{C} = \mathbf{A}(\mathbf{B}\mathbb{C})$ (Associativity)
2. $\mathbf{A}(\mathbf{B} + \mathbb{C}) = \mathbf{AB} + \mathbf{A}\mathbb{C}$ (Left distributivity)
3. $(\mathbf{A} + \mathbf{B})\mathbb{C} = \mathbf{A}\mathbb{C} + \mathbf{B}\mathbb{C}$ (Right distributivity)
4. $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ (Multiplicative identity)

Proof. We prove associativity. Let $\mathbf{A}$ be $m \times n$, $\mathbf{B}$ be $n \times p$, and $\mathbb{C}$ be $p \times q$. For any entry (i, ℓ):

$$\begin{aligned}
((\mathbf{AB})\mathbb{C})_{i\ell} &= \sum_{k=1}^{p} (\mathbf{AB})_{ik} c_{k\ell} = \sum_{k=1}^{p} \left(\sum_{j=1}^{n} a_{ij} b_{jk} \right) c_{k\ell} \\
&= \sum_{k=1}^{p} \sum_{j=1}^{n} a_{ij} b_{jk} c_{k\ell} = \sum_{j=1}^{n} \sum_{k=1}^{p} a_{ij} b_{jk} c_{k\ell} \\
&= \sum_{j=1}^{n} a_{ij} \left(\sum_{k=1}^{p} b_{jk} c_{k\ell} \right) = \sum_{j=1}^{n} a_{ij} (\mathbf{B}\mathbb{C})_{j\ell} \\
&= (\mathbf{A}(\mathbf{B}\mathbb{C}))_{i\ell}
\end{aligned}$$

■

Remark. Matrix multiplication is **not** commutative in general. That is, $\mathbf{AB} \neq \mathbf{BA}$ even when both products are defined.

0.0.4 Scalar Multiplication

Definition 0.5.11 (Scalar Multiplication)**.** Let $\mathbf{A} = (a_{ij}) \in M_{m \times n}(F)$ and $c \in F$ be a scalar. The **scalar multiple** $c\mathbf{A}$ is the matrix in $M_{m \times n}(F)$ with entries (ca_{ij}).

0.0.5 Trace of a Matrix

Definition 0.5.12 (Trace)**.** Let $\mathbf{A} = (a_{ij}) \in M_n(F)$ be a square matrix. The **trace** of $\mathbf{A}$, denoted $\mathrm{tr}(\mathbf{A})$, is the sum of the entries on its main diagonal:

$$\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}.$$

Example (Trace). Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Then $\mathrm{tr}(\mathbf{A}) = 1 + 5 + 9 = 15$.

0.0.6 Properties of Transpose and Trace

Theorem 0.5.1 (Properties of Transpose). Let $\mathbf{A}, \mathbf{B}$ be matrices of compatible sizes and $c \in F$. Then:

1. $(\mathbf{A}^\top)^\top = \mathbf{A}$
2. $(\mathbf{A}+\mathbf{B})^\top = \mathbf{A}^\top + \mathbf{B}^\top$
3. $(c\mathbf{A})^\top = c\mathbf{A}^\top$
4. $(\mathbf{AB})^\top = \mathbf{B}^\top\mathbf{A}^\top$ (Reversal rule)

Proof. We prove the reversal rule $(\mathbf{AB})^\top = \mathbf{B}^\top\mathbf{A}^\top$. Let $\mathbf{A} \in M_{m\times n}(F)$ and $\mathbf{B} \in M_{n\times p}(F)$.

$$((\mathbf{AB})^\top)_{ij} = (\mathbf{AB})_{ji} = \sum_{k=1}^{n} a_{jk}b_{ki}$$

Now consider $\mathbf{B}^\top\mathbf{A}^\top$. Its (i,j)-entry is:

$$(\mathbf{B}^\top\mathbf{A}^\top)_{ij} = \sum_{k=1}^{n}(\mathbf{B}^\top)_{ik}(\mathbf{A}^\top)_{kj} = \sum_{k=1}^{n} b_{ki}a_{jk} = \sum_{k=1}^{n} a_{jk}b_{ki}$$

Since the entries are equal for all i, j, we have $(\mathbf{AB})^\top = \mathbf{B}^\top\mathbf{A}^\top$. ∎

Theorem 0.5.2 (Properties of Trace). Let $\mathbf{A}, \mathbf{B} \in M_n(F)$ and $c \in F$. Then:

1. $\mathrm{tr}(\mathbf{A}+\mathbf{B}) = \mathrm{tr}(\mathbf{A}) + \mathrm{tr}(\mathbf{B})$
2. $\mathrm{tr}(c\mathbf{A}) = c\,\mathrm{tr}(\mathbf{A})$
3. $\mathrm{tr}(\mathbf{A}^\top) = \mathrm{tr}(\mathbf{A})$
4. $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$ (Cyclic property)

Proof. We prove $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$.

$$\mathrm{tr}(\mathbf{AB}) = \sum_{i=1}^{n}(\mathbf{AB})_{ii} = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}b_{ji}\right)$$

Rearranging the summation:

$$\mathrm{tr}(\mathbf{AB}) = \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ji}a_{ij} = \sum_{j=1}^{n}(\mathbf{BA})_{jj} = \mathrm{tr}(\mathbf{BA}).$$

∎