0.3 Polynomials

Definition 0.3.1 (Polynomial)**.** A **polynomial** over a field F in the indeterminate x is an expression of the form:

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = \sum_{k=0}^{n} a_k x^k$$

where $a_0, a_1, \dots, a_n \in F$ are called the **coefficients** of $p(x)$.

Definition 0.3.2 (Degree)**.** Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a nonzero polynomial with $a_n \neq 0$. The **degree** of $p(x)$, denoted $\deg p(x)$, is n. The coefficient a_n is called the **leading coefficient**.

By convention, the zero polynomial $p(x) = 0$ has degree $-\infty$.

Definition 0.3.3 (Polynomial Ring)**.** The set of all polynomials over a field F is denoted $F[x]$:

$$F[x] = \left\{ \sum_{k=0}^{n} a_k x^k : n \in \mathbb{N},\ a_k \in F \right\}$$

This set forms a **ring** under polynomial addition and multiplication. (You can omit what ring means here.)

Example (Elements of the Polynomial Ring). The following are elements of $\mathbb{R}[x]$:

$$3x^2 - 2x + 1, \quad x^5 + \pi x^3 - \sqrt{2}, \quad 7, \quad 0$$

Note that constant polynomials (including 0) are also polynomials.

0.3.1 Polynomial Spaces

In linear algebra, we often work with polynomials of bounded degree.

Definition 0.3.4 (Polynomials of Degree at Most n)**.** Let F be a field and $n \in \mathbb{N}$. The set of all polynomials over F of degree at most n is denoted:

$$F[x]_{\leq n} = \{p(x) \in F[x] : \deg p(x) \leq n\}$$

Equivalently:

$$F[x]_{\leq n} = \{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 : a_k \in F\}$$

Example (Polynomials of Degree at Most 2). The set $\mathbb{R}[x]_{\leq 2}$ consists of all polynomials of degree at most 2:

$$\mathbb{R}[x]_{\leq 2} = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$$

Examples include $3x^2 - 2x + 1$, $5x + 7$, and -4.

Definition 0.3.5 (Polynomials of Degree Exactly n)**.** We use $F[x]_{=n}$ to denote polynomials of degree **exactly** n:

$$F[x]_{=n} = \{p(x) \in F[x] : \deg p(x) = n\}$$

Equivalently:

$$F[x]_{=n} = \{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 : a_n \neq 0,\ a_k \in F\}$$

Remark. Note the distinction:

- $F[x]_{\leq n}$ contains polynomials of degree $\leq n$ (including the zero polynomial)
- $F[x]_{=n}$ contains polynomials of degree **exactly** n (so $a_n \neq 0$)

Thus $F[x]_{\leq n} \supsetneq F[x]_{=n}$, and in fact $F[x]_{\leq n} = F[x]_{=n} \cup F[x]_{\leq n-1}$.

Example (Comparing Notations).

- $\mathbb{R}[x]_{\leq 2} = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$ includes $x^2 + 1$, $3x - 2$, and 5
- $\mathbb{R}[x]_{=2} = \{ax^2 + bx + c : a \neq 0,\ a, b, c \in \mathbb{R}\}$ includes $x^2 + 1$ but **not** $3x - 2$ or 5