1.3 Linear Combinations and Span

In the previous section, we studied subspaces. A natural question arises: given a set of vectors, how can we describe the "smallest" subspace that contains them? This leads us to the concept of linear combinations and the span.

Definition 1.3.1 (Linear Combination)**.** Let V be a vector space over F, $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ be a subset of V. Then the expression

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_m\mathbf{v}_m$$

is called a **linear combination** of S. The scalars a_i are called the **coefficients** of the linear combination.

Example (Examples of Linear Combinations)**.**

1. The vector $(2, 1, 0)$ in $\mathbb{R}^3$ is a linear combination of $S = \{(1, 2, 3), (4, 5, 6)\}$ because
$$(2, 1, 0) = -2 \cdot (1, 2, 3) + 1 \cdot (4, 5, 6).$$

2. The polynomial $f(x) = x^3 + 2x + 1$ in $\mathbb{R}[x]$ is a linear combination of $S = \{1, x, x^2, x^3\}$ because
$$f(x) = 1 \cdot 1 + 2 \cdot x + 0 \cdot x^2 + 1 \cdot x^3.$$

Definition 1.3.2 (Span)**.** Let S be a subset of a vector space V over F. The **span** of S, denoted by $\mathrm{span}_F(S)$, is the set of all linear combinations of finite subsets of elements of S.

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, then $\mathrm{span}_F(S) = \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n : a_i \in F\}$. By convention, $\mathrm{span}_F(\emptyset) = \{\mathbf{0}\}$.

Remark. When the underlying field is clear, we abuse the notation of span_F and write it as span.

Theorem 1.3.1 (Spanning Set as Subspace). The span of any subset $S \subseteq V$ is a subspace of V. Moreover, it is the smallest subspace of V containing S.

Proof. Let $W = \mathrm{span}(S)$.

- **Zero vector:** Since $\mathrm{span}(\emptyset) = \{\mathbf{0}\}$, and for non-empty S, taking all coefficients as zero gives $\mathbf{0}$, we have $\mathbf{0} \in W$.
- **Addition:** Let $\mathbf{x}, \mathbf{y} \in W$. Then $\mathbf{x} = \sum a_i\mathbf{v}_i$ and $\mathbf{y} = \sum b_i\mathbf{v}_i$. Their sum $\mathbf{x} + \mathbf{y} = \sum(a_i + b_i)\mathbf{v}_i$ is also a linear combination of elements in S, so $\mathbf{x} + \mathbf{y} \in W$.

- **Scalar multiplication:** For any $c \in F$, $c\mathbf{x} = \sum (ca_i)\mathbf{v}_i \in W$.

To see it is the smallest subspace, note that any subspace containing S must be closed under addition and scalar multiplication, and thus must contain all linear combinations of elements in S. ∎

Definition 1.3.3 (Spanning Set). If $\text{span}(S) = V$, we say that S **spans** V, or that S is a **spanning set** for V.

Example (Spanning Set for Polynomials). The set $\{1, x, x^2\}$ spans $\mathbb{R}[x]_{\leq 2}$, the space of polynomials of degree at most 2.