

1.1 Linear Combinations and Span

In the previous section, we studied subspaces. A natural question arises: given a set of vectors, how can we describe the “smallest” subspace that contains them? This leads us to the concept of linear combinations and the span.

Definition 1.3.1 (Linear Combination). Let V be a vector space over F , $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ be a subset of V . Then the expression

$$a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \cdots + a_m \mathbf{v}_m$$

is called a **linear combination** of S . The scalars a_i are called the **coefficients** of the linear combination.

Example (Examples of Linear Combinations).

1. The vector $(2, 1, 0)$ in \mathbb{R}^3 is a linear combination of $S = \{(1, 2, 3), (4, 5, 6)\}$ because

$$(2, 1, 0) = -2 \cdot (1, 2, 3) + 1 \cdot (4, 5, 6).$$

2. The polynomial $f(x) = x^3 + 2x + 1$ in $\mathbb{R}[x]$ is a linear combination of $S = \{1, x, x^2, x^3\}$ because

$$f(x) = 1 \cdot 1 + 2 \cdot x + 0 \cdot x^2 + 1 \cdot x^3.$$

Definition 1.3.2 (Span). Let S be a subset of a vector space V over F . The **span** of S , denoted by $\text{span}_F(S)$, is the set of all linear combinations of finite subsets of elements of S .

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, then $\text{span}_F(S) = \{a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \cdots + a_n \mathbf{v}_n : a_i \in F\}$. By convention, $\text{span}_F(\emptyset) = \{\mathbf{0}\}$.

Remark. When the underlying field is clear, we abuse the notation of span_F and write it as span .

Theorem 1.3.1 (Spanning Set as Subspace). The span of any subset $S \subseteq V$ is a subspace of V . Moreover, it is the smallest subspace of V containing S .

Proof. Let $W = \text{span}(S)$.

- **Zero vector:** Since $\text{span}(\emptyset) = \{\mathbf{0}\}$, and for non-empty S , taking all coefficients as zero gives $\mathbf{0}$, we have $\mathbf{0} \in W$.
- **Addition:** Let $\mathbf{x}, \mathbf{y} \in W$. Then $\mathbf{x} = \sum a_i \mathbf{v}_i$ and $\mathbf{y} = \sum b_i \mathbf{v}_i$. Their sum $\mathbf{x} + \mathbf{y} = \sum (a_i + b_i) \mathbf{v}_i$ is also a linear combination of elements in S , so $\mathbf{x} + \mathbf{y} \in W$.

- **Scalar multiplication:** For any $c \in F$, $c\mathbf{x} = \sum (ca_i)\mathbf{v}_i \in W$.

To see it is the smallest subspace, note that any subspace containing S must be closed under addition and scalar multiplication, and thus must contain all linear combinations of elements in S . ■

Definition 1.3.3 (Spanning Set). If $\text{span}(S) = V$, we say that S **spans** V , or that S is a **spanning set** for V .

Example (Spanning Set for Polynomials). The set $\{1, x, x^2\}$ spans $\mathbb{R}[x]_{\leq 2}$, the space of polynomials of degree at most 2.