1.1 Vector Spaces

Why do we bother with a list of eight axioms? To a student first encountering Linear Algebra, this can feel like a bureaucratic exercise in "checking boxes." However, there is a profound beauty hidden in this formality.

In our preliminary chapter, we saw three very different worlds: the world of geometric arrows ($\mathbb{R}^n$), the world of functions ($F[x]$), and the world of matrices ($M_{m\times n}$). On the surface, an arrow is not a polynomial, and a polynomial is not a matrix. But if you squint, they all behave the same: you can add them, and you can scale them.

By defining a **Vector Space** through these axioms, we are choosing to ignore what the objects *are* and focus entirely on how they *act*. If we prove a theorem using only these eight axioms, that theorem becomes a "universal law" that applies to arrows, functions, and matrices all at once. This is the power of abstraction: solve the problem once, and you solve it for every universe that obeys these rules.

Definition 1.1.1 (Vector Space)**.** A set V over a field F with two operations: addition $+$ and scalar multiplication $\cdot$

$$+ : V \times V \to V, \quad \cdot : F \times V \to V$$

such that satisfy the following axioms for all $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in V$ and $\alpha, \beta \in F$:

(VS1) $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1$

(VS2) $(\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3 = \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3)$

(VS3) There exists $\mathbf{0} \in V$ such that $\mathbf{v}_1 + \mathbf{0} = \mathbf{v}_1$

(VS4) There exists $\mathbf{v}_1' \in V$ such that $\mathbf{v}_1 + \mathbf{v}_1' = \mathbf{0}$

(VS5) $\alpha(\mathbf{v}_1 + \mathbf{v}_2) = \alpha\mathbf{v}_1 + \alpha\mathbf{v}_2$

(VS6) $(\alpha + \beta)\mathbf{v}_1 = \alpha\mathbf{v}_1 + \beta\mathbf{v}_1$

(VS7) $\alpha(\beta\mathbf{v}_1) = (\alpha\beta)\mathbf{v}_1$

(VS8) $1\mathbf{v}_1 = \mathbf{v}_1$

Remark. Instead of writing $(V, +, \cdot)$ is a vector space over F, we usually simplify it to V is a vector space over F, or even just V is a vector space if the context is clear.

Remark. We sometimes will abuse the name and refer to $\mathbf{0} = \mathbf{0}_V \in V$ as the "zero vector" of the vector space V. Readers should not confuse $\mathbf{0}_V$ with the zero scalar $0 = 0_F \in F$ from the field F.

Example (Examples of Vector Spaces).

1. Most defaultly, F^n with the usual operations of $+$ and $\cdot$ is a vector space over F.
2. Working with polynomials is common too: $F[x]$ with the usual operations of $+$ and $\cdot$ is a vector space over F.
3. Let $\mathcal{D}$ be any open interval. Let

$$C(\mathcal{D}) := \{f : \mathcal{D} \to \mathbb{R} : f \text{ is continuous}\}.$$

Then $C(\mathcal{D})$ is a vector space over $\mathbb{R}$. The zero vector of this vector space is given by the zero polynomial $(x \mapsto 0)$.
4. Let F be any field and fix $n \in \mathbb{N}$. Then $M_n(F)$ is a vector space over F.
5. The set $V = \mathbb{R}_{>0}$ of positive real numbers forms a vector space over $F = \mathbb{R}$ under the following operations: for $x, y \in V$ and $\alpha \in F$, define

$$x \oplus y := xy, \quad \alpha \odot x := x^\alpha = e^{\alpha \log x}.$$

Under these operations, the zero vector is $\mathbf{0} = 1$, and the additive inverse of x is x^{-1}.

You can check if addition and scalar multiplication make sense and follow all axioms of vector spaces.

Example ($\mathbb{Q}$ Adjoin $\sqrt{2}$). We verify that $\mathbb{Q}(\sqrt{2}) := \left\{a + b\sqrt{2} : a, b \in \mathbb{Q}\right\}$ is a vector space over $\mathbb{Q}$ by checking all eight axioms. Let $\mathbf{v}_1 = a_1 + b_1\sqrt{2}$, $\mathbf{v}_2 = a_2 + b_2\sqrt{2}$, $\mathbf{v}_3 = a_3 + b_3\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ and $\alpha, \beta \in \mathbb{Q}$.

(VS1) $\mathbf{v}_1 + \mathbf{v}_2 = (a_1 + a_2) + (b_1 + b_2)\sqrt{2} = (a_2 + a_1) + (b_2 + b_1)\sqrt{2} = \mathbf{v}_2 + \mathbf{v}_1$.

(VS2) $(\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3 = (a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)\sqrt{2} = \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3)$.

(VS3) The element $\mathbf{0} = 0 + 0\sqrt{2} = 0 \in \mathbb{Q}(\sqrt{2})$ satisfies $\mathbf{v}_1 + \mathbf{0} = (a_1 + 0) + (b_1 + 0)\sqrt{2} = \mathbf{v}_1$.

(VS4) For $\mathbf{v}_1 = a_1 + b_1\sqrt{2}$, define $-\mathbf{v}_1 := (-a_1) + (-b_1)\sqrt{2} \in \mathbb{Q}(\sqrt{2})$. Then $\mathbf{v}_1 + (-\mathbf{v}_1) = (a_1 - a_1) + (b_1 - b_1)\sqrt{2} = \mathbf{0}$.

(VS5) $\alpha(\mathbf{v}_1 + \mathbf{v}_2) = \alpha(a_1 + a_2) + \alpha(b_1 + b_2)\sqrt{2} = (\alpha a_1 + \alpha a_2) + (\alpha b_1 + \alpha b_2)\sqrt{2} = \alpha\mathbf{v}_1 + \alpha\mathbf{v}_2$.

(VS6) $(\alpha + \beta)\mathbf{v}_1 = (\alpha + \beta)a_1 + (\alpha + \beta)b_1\sqrt{2} = (\alpha a_1 + \beta a_1) + (\alpha b_1 + \beta b_1)\sqrt{2} = \alpha\mathbf{v}_1 + \beta\mathbf{v}_1$.

(VS7) $\alpha(\beta\mathbf{v}_1) = \alpha(\beta a_1 + \beta b_1\sqrt{2}) = \alpha\beta a_1 + \alpha\beta b_1\sqrt{2} = (\alpha\beta)\mathbf{v}_1$.

(VS8) $1 \cdot \mathbf{v}_1 = 1 \cdot a_1 + 1 \cdot b_1\sqrt{2} = a_1 + b_1\sqrt{2} = \mathbf{v}_1$.

Thus $\mathbb{Q}(\sqrt{2})$ is a vector space over $\mathbb{Q}$.

Definition 1.1.2 (Vectors and Scalars)**.** Let V be a vector space over F. Then the elements of V are called **vectors**, and the elements of F are called **scalars**.

Additionally, $\mathbf{0} \in V$ is called the **zero vector**, and $(\mathbf{v}_1')$ in **(VS4)** is called the **inverse element** of $\mathbf{v}_1$.

Theorem 1.1.1 (Left Cancellation Law). Let V be a vector space over F, let $\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2 \in V$. If $\mathbf{u} + \mathbf{v}_1 = \mathbf{u} + \mathbf{v}_2$, then $\mathbf{v}_1 = \mathbf{v}_2$.

Proof. Let $\mathbf{u}'$ be an inverse of $\mathbf{u}$.

$$\begin{aligned}
\mathbf{u} + \mathbf{v}_1 &= \mathbf{u} + \mathbf{v}_2 && \text{(given)} \\
\mathbf{u}' + (\mathbf{u} + \mathbf{v}_1) &= \mathbf{u}' + (\mathbf{u} + \mathbf{v}_2) && \text{(add } \mathbf{u}' \text{ to both sides)} \\
(\mathbf{u}' + \mathbf{u}) + \mathbf{v}_1 &= (\mathbf{u}' + \mathbf{u}) + \mathbf{v}_2 && \text{(by VS2)} \\
\mathbf{0} + \mathbf{v}_1 &= \mathbf{0} + \mathbf{v}_2 && \text{(by VS4)} \\
\mathbf{v}_1 &= \mathbf{v}_2 && \text{(by VS3)}
\end{aligned}$$

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Theorem 1.1.2 (Right Cancellation Law). Let V be a vector space over F, let $\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2 \in V$. If $\mathbf{v}_1 + \mathbf{u} = \mathbf{v}_2 + \mathbf{u}$, then $\mathbf{v}_1 = \mathbf{v}_2$.

Proof. Since we have $\mathbf{u} + \mathbf{v}_1 = \mathbf{u} + \mathbf{v}_2 \implies \mathbf{v}_1 = \mathbf{v}_2$, applying **(VS1)** to it gives $\mathbf{v}_1 + \mathbf{u} = \mathbf{v}_2 + \mathbf{u} \implies \mathbf{v}_1 = \mathbf{v}_2$. ■

Theorem 1.1.3 (Zero Vector is Unique). Let V be a vector space over F. The zero vector $\mathbf{0} \in V$ is unique.

Proof. Suppose that there are two vectors $\mathbf{0}_1, \mathbf{0}_2$.

$$\begin{aligned}
\mathbf{0}_1 + \mathbf{0}_2 &= \mathbf{0}_2 + \mathbf{0}_1 && \text{(by VS1)} \\
\mathbf{0}_1 &= \mathbf{0}_2 && \text{(by VS3)}
\end{aligned}$$

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Theorem 1.1.4 (Additive Inverse is Unique). Let V be a vector space over F. Then for every $\mathbf{v} \in V$, its additive inverse described in **(VS4)**, which is $\mathbf{v}'$, is unique.

Proof. Let $\mathbf{v}_1', \mathbf{v}_2'$ both be inverses of $\mathbf{v}$. Then

$$\begin{aligned}
\mathbf{v}_1' &= \mathbf{v}_1' + \mathbf{0} && \text{(by VS3)} \\
&= \mathbf{v}_1' + (\mathbf{v} + \mathbf{v}_2') && \text{(by VS4)} \\
&= (\mathbf{v}_1' + \mathbf{v}) + \mathbf{v}_2' && \text{(by VS2)} \\
&= \mathbf{0} + \mathbf{v}_2' && \text{(by VS4)} \\
&= \mathbf{v}_2' && \text{(by VS3)}
\end{aligned}$$

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Definition 1.1.3 (Notation of Additive Inverse)**.** The unique inverse of a vector $\mathbf{v} \in V$, for a vector space V over F, will be denoted as $-\mathbf{v}$.

Theorem 1.1.5 (Zero Scalar Annihilates). Let $\mathbf{v} \in V$ for a vector space V over F. Then $0 \cdot \mathbf{v} = \mathbf{0}$.

Proof. Let $\mathbf{w} = 0 \cdot \mathbf{v}$. We show that $\mathbf{w} = \mathbf{0}$.

$$\begin{aligned}
\mathbf{w} + \mathbf{w} &= 0 \cdot \mathbf{v} + 0 \cdot \mathbf{v} \\
&= (0 + 0) \cdot \mathbf{v} && \text{(by VS6)} \\
&= 0 \cdot \mathbf{v} && \text{(arithmetic in } F\text{)} \\
&= \mathbf{w} \\
(\mathbf{w} + \mathbf{w}) + (-\mathbf{w}) &= \mathbf{w} + (-\mathbf{w}) && \text{(add } -\mathbf{w} \text{ to both sides)} \\
\mathbf{w} + (\mathbf{w} + (-\mathbf{w})) &= \mathbf{0} && \text{(by VS2, VS4)} \\
\mathbf{w} + \mathbf{0} &= \mathbf{0} && \text{(by VS4)} \\
\mathbf{w} &= \mathbf{0} && \text{(by VS3)}
\end{aligned}$$

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Theorem 1.1.6 (Negation as Scalar Multiplication). Let V be a vector space over F and $\mathbf{v} \in V$ be any vector. Then $-\mathbf{v} = (-1) \cdot \mathbf{v}$.

Proof. We show that $(-1) \cdot \mathbf{v}$ is an additive inverse of $\mathbf{v}$:

$$\begin{aligned}
\mathbf{v} + (-1) \cdot \mathbf{v} &= 1 \cdot \mathbf{v} + (-1) \cdot \mathbf{v} && \text{(by VS8)} \\
&= (1 + (-1)) \cdot \mathbf{v} && \text{(by VS6)} \\
&= 0 \cdot \mathbf{v} && \text{(arithmetic in } F\text{)} \\
&= \mathbf{0} && \text{(by @thm-zero-scalar-mult)}
\end{aligned}$$

Since $(-1) \cdot \mathbf{v}$ is an additive inverse of $\mathbf{v}$, by uniqueness we have $-\mathbf{v} = (-1) \cdot \mathbf{v}$. ■

Theorem 1.1.7 (Negative Scalar Distribution). Let V be a vector space over F. For any $\alpha \in F$ and $\mathbf{v} \in V$, we have $(-\alpha)\mathbf{v} = -(\alpha\mathbf{v})$.

Proof. We show that $(-\alpha)\mathbf{v}$ is an additive inverse of $\alpha\mathbf{v}$:

$$\begin{aligned} \alpha\mathbf{v} + (-\alpha)\mathbf{v} &= (\alpha + (-\alpha))\mathbf{v} && \text{(by VS6)} \\ &= 0 \cdot \mathbf{v} && \text{(arithmetic in } F\text{)} \\ &= \mathbf{0} && \text{(by @thm-zero-scalar-mult)} \end{aligned}$$

Since $(-\alpha)\mathbf{v}$ is an additive inverse of $\alpha\mathbf{v}$, by uniqueness we have $(-\alpha)\mathbf{v} = -(\alpha\mathbf{v})$. ∎

Theorem 1.1.8 (Scalar Multiplication by Zero Vector). Let V be a vector space over F. For any scalar $\alpha \in F$, we have $\alpha\mathbf{0} = \mathbf{0}$.

Proof. Let $\mathbf{w} = \alpha\mathbf{0}$. We show that $\mathbf{w} = \mathbf{0}$.

$$\begin{aligned} \mathbf{w} &= \alpha\mathbf{0} \\ &= \alpha(\mathbf{0} + \mathbf{0}) && \text{(by VS3)} \\ &= \alpha\mathbf{0} + \alpha\mathbf{0} && \text{(by VS5)} \\ &= \mathbf{w} + \mathbf{w} \\ \mathbf{w} + (-\mathbf{w}) &= (\mathbf{w} + \mathbf{w}) + (-\mathbf{w}) && \text{(add } -\mathbf{w} \text{ to both sides)} \\ \mathbf{0} &= \mathbf{w} + (\mathbf{w} + (-\mathbf{w})) && \text{(by VS4, VS2)} \\ \mathbf{0} &= \mathbf{w} + \mathbf{0} && \text{(by VS4)} \\ \mathbf{0} &= \mathbf{w} && \text{(by VS3)} \end{aligned}$$

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Theorem 1.1.9 (Zero Product Law). Let V be a vector space over F, $\mathbf{v} \in V$, and $\alpha \in F$. If $\alpha\mathbf{v} = \mathbf{0}$, then either $\alpha = 0$ or $\mathbf{v} = \mathbf{0}$.

Proof. Suppose $\alpha\mathbf{v} = \mathbf{0}$. We consider two cases:

- **Case** $\alpha = 0$**:** Then the statement holds.
- **Case** $\alpha \neq 0$**:** Since F is a field, α has a multiplicative inverse α^{-1}.

$$\begin{aligned} \alpha\mathbf{v} &= \mathbf{0} && \text{(given)} \\ \alpha^{-1}(\alpha\mathbf{v}) &= \alpha^{-1}\mathbf{0} && \text{(multiply by } \alpha^{-1}\text{)} \\ (\alpha^{-1}\alpha)\mathbf{v} &= \mathbf{0} && \text{(by VS7 and previous Theorem)} \\ 1 \cdot \mathbf{v} &= \mathbf{0} && \text{(field inverse property)} \\ \mathbf{v} &= \mathbf{0} && \text{(by VS8)} \end{aligned}$$

Thus, if $\alpha \neq 0$, we must have $\mathbf{v} = \mathbf{0}$.

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