

## 1.2 Subspaces

Some vector spaces might be “too big” and be unwieldy to deal with. Therefore, it is often convenient to look at “smaller vector spaces” inside a given vector space. These are called **subspaces**.

**Definition 1.2.1** (Subspace). Let  $V$  be a vector space over  $F$ . A subset  $W$  of  $V$  is called a **subspace** of  $V$ , if  $W$  is also a vector space over  $F$  with addition and scalar multiplication inherited from  $V$ .

*Remark.* Let  $(V, +, \cdot)$  be a vector space over  $F$ , and  $W \subseteq V$ . If  $(W, +, \cdot)$  is also a vector space over  $F$  then  $W$  is a subspace of  $V$ .

**Example** (Trivial Subspaces). Given a vector space  $V$ , then

- $V$  is a subspace of  $V$  itself;
- $\{\mathbf{0}\}$  is a subspace of  $V$  (also known as the “zero subspace”);
- $\emptyset$  is **not** a subspace of  $V$ .

**Theorem 1.2.1 (Subspace Test).** Let  $V$  be a vector space over  $F$  and  $W \subseteq V$ . Then  $W$  is a subspace of  $V$  if and only if the following are satisfied:

- $W$  is non-empty;
- $W$  is closed under addition:  $\mathbf{w}_1 + \mathbf{w}_2 \in W$  for all  $\mathbf{w}_1, \mathbf{w}_2 \in W$ ;
- $W$  is closed under scalar multiplication:  $\alpha \mathbf{w} \in W$  for all  $\alpha \in F$  and  $\mathbf{w} \in W$ .

*Proof.* ( $\Rightarrow$ ) If  $W$  is a subspace, then  $W$  is a vector space, so it contains the zero vector (hence non-empty) and is closed under addition and scalar multiplication by definition.

( $\Leftarrow$ ) Assume the three conditions hold. Since  $W \subseteq V$ , the operations on  $W$  are inherited from  $V$ , so associativity, commutativity, and distributivity are automatically satisfied. It remains to verify the existence of identity elements and inverses.

- **Zero vector:** Since  $W \neq \emptyset$ , there exists  $\mathbf{w} \in W$ . By closure under scalar multiplication,  $0 \cdot \mathbf{w} = \mathbf{0} \in W$ .
- **Additive inverse:** For any  $\mathbf{w} \in W$ , closure under scalar multiplication gives  $(-1) \cdot \mathbf{w} = -\mathbf{w} \in W$ .

Thus  $W$  is a vector space under the inherited operations, i.e., a subspace of  $V$ . ■

*Remark.* Note that from the proof, the first condition of the Subspace Test can be replaced by the requirement that  $W$  contains the zero vector of  $V$ .

*Remark.* In practice, to test whether a given subset of a vector space is a subspace, we use Theorem 1.2.1 instead of checking all eight vector space axioms from scratch, which is much less tedious. Moreover, some introductory textbooks use Theorem 1.2.1 as the definition of a subspace. While this is logically equivalent, it can be somewhat less intuitive than defining a subspace as a subset that is itself a vector space.

**Example** (Line as Subspace).  $W = \{(x, y) \in \mathbb{R}^2 : x + 2y = 0\}$  is a subspace of  $\mathbb{R}^2$ . Indeed:

- **Zero vector:**  $(0, 0) \in W$  since  $0 + 2(0) = 0$ .
- **Addition:** If  $(x_1, y_1), (x_2, y_2) \in W$ , then  $(x_1 + x_2) + 2(y_1 + y_2) = (x_1 + 2y_1) + (x_2 + 2y_2) = 0 + 0 = 0$ , so their sum is in  $W$ .
- **Scalar multiplication:** If  $(x, y) \in W$  and  $c \in \mathbb{R}$ , then  $(cx) + 2(cy) = c(x + 2y) = c(0) = 0$ , so  $c(x, y) \in W$ .

**Example** (Translated Line is Not a Subspace).  $W = \{(x, y) \in \mathbb{R}^2 : x + 2y = 3\}$  is **not** a subspace of  $\mathbb{R}^2$ . This is because it does not contain the zero vector:  $0 + 2(0) = 0 \neq 3$ , so  $(0, 0) \notin W$ .

**Example** (Polynomials of Bounded Degree). Let  $F[x]$  be the vector space of all polynomials over a field  $F$ . For a fixed  $n \in \mathbb{N}$ , the set  $F[x]_{\leq n}$  of polynomials of degree at most  $n$  is a subspace of  $F[x]$ .

- **Zero vector:** The zero polynomial has degree  $-\infty$ , so  $0 \in F[x]_{\leq n}$ .
- **Addition:** If  $\deg p \leq n$  and  $\deg q \leq n$ , then  $\deg(p + q) \leq \max(\deg p, \deg q) \leq n$ .
- **Scalar multiplication:** If  $\deg p \leq n$  and  $c \in F$ , then  $\deg(cp) \leq \deg p \leq n$ .

**Example** (Polynomials of Exact Degree). The set  $F[x]_{=n}$  of polynomials of degree **exactly**  $n$  is **not** a subspace of  $F[x]$ .

- It does not contain the zero vector (since  $\deg 0 = -\infty \neq n$ ).
- It is not closed under addition. For example, if  $p(x) = x^n + x$  and  $q(x) = -x^n$ , both have degree  $n$ , but  $p(x) + q(x) = x$ , which has degree  $1 \neq n$  (assuming  $n > 1$ ).

**Example** (Differentiable Functions). Let  $C(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$  be the set of all continuous functions on  $\mathbb{R}$ . We consider the set of differentiable functions:

$$D(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ is differentiable}\}$$

To verify that  $D(\mathbb{R})$  is a subspace of  $C(\mathbb{R})$ :

- **Subset:** Since differentiability implies continuity, we have  $D(\mathbb{R}) \subseteq C(\mathbb{R})$ .

- **Zero vector:** The zero function  $f(x) = 0$  is differentiable, so  $\mathbf{0} \in D(\mathbb{R})$ .
- **Closure under addition:** If  $f, g \in D(\mathbb{R})$ , then  $f+g$  is differentiable with  $(f+g)' = f' + g'$ , so  $f+g \in D(\mathbb{R})$ .
- **Closure under scalar multiplication:** If  $f \in D(\mathbb{R})$  and  $c \in \mathbb{R}$ , then  $cf$  is differentiable with  $(cf)' = cf'$ , so  $cf \in D(\mathbb{R})$ .

Thus,  $D(\mathbb{R})$  is a subspace of  $C(\mathbb{R})$ .

**Example** (Symmetric Matrices). The set of all symmetric matrices in  $M_n(F)$ , denoted by  $S_n(F) = \{\mathbf{A} \in M_n(F) : \mathbf{A} = \mathbf{A}^\top\}$ , is a subspace of  $M_n(F)$ .

- **Zero vector:** The zero matrix is symmetric since  $0^\top = 0$ , so  $0 \in S_n(F)$ .
- **Addition:** If  $\mathbf{A}, \mathbf{B} \in S_n(F)$ , then  $(\mathbf{A} + \mathbf{B})^\top = \mathbf{A}^\top + \mathbf{B}^\top = \mathbf{A} + \mathbf{B}$ , so  $\mathbf{A} + \mathbf{B} \in S_n(F)$ .
- **Scalar multiplication:** If  $\mathbf{A} \in S_n(F)$  and  $c \in F$ , then  $(c\mathbf{A})^\top = c\mathbf{A}^\top = c\mathbf{A}$ , so  $c\mathbf{A} \in S_n(F)$ .

**Example** (Trace-free Matrices). The set of all trace-free matrices in  $M_n(F)$ , denoted by  $W = \{\mathbf{A} \in M_n(F) : \text{tr}(\mathbf{A}) = 0\}$ , is a subspace of  $M_n(F)$ .

- **Zero vector:**  $\text{tr}(0) = 0 + 0 + \cdots + 0 = 0$ , so  $0 \in W$ .
- **Addition:** If  $\mathbf{A}, \mathbf{B} \in W$ , then  $\text{tr}(\mathbf{A} + \mathbf{B}) = \text{tr}(\mathbf{A}) + \text{tr}(\mathbf{B}) = 0 + 0 = 0$ , so  $\mathbf{A} + \mathbf{B} \in W$ .
- **Scalar multiplication:** If  $\mathbf{A} \in W$  and  $c \in F$ , then  $\text{tr}(c\mathbf{A}) = c\text{tr}(\mathbf{A}) = c(0) = 0$ , so  $c\mathbf{A} \in W$ .

**Theorem 1.2.2 (Intersection of Subspaces).** Let  $W_1$  and  $W_2$  be subspaces of a vector space  $V$ . Then the intersection  $W_1 \cap W_2$  is also a subspace of  $V$ .

*Proof.* We use the Subspace Test on  $W = W_1 \cap W_2$ :

- **Zero vector:** Since  $W_1$  and  $W_2$  are subspaces,  $\mathbf{0} \in W_1$  and  $\mathbf{0} \in W_2$ . Thus  $\mathbf{0} \in W_1 \cap W_2$ .
- **Closure under addition:** Let  $\mathbf{w}, \mathbf{z} \in W_1 \cap W_2$ . Then  $\mathbf{w}, \mathbf{z} \in W_1$  and  $\mathbf{w}, \mathbf{z} \in W_2$ . Since  $W_1$  and  $W_2$  are subspaces, they are closed under addition, so  $\mathbf{w} + \mathbf{z} \in W_1$  and  $\mathbf{w} + \mathbf{z} \in W_2$ . Hence  $\mathbf{w} + \mathbf{z} \in W_1 \cap W_2$ .
- **Closure under scalar multiplication:** Let  $\mathbf{w} \in W_1 \cap W_2$  and  $c \in F$ . Then  $\mathbf{w} \in W_1$  and  $\mathbf{w} \in W_2$ . Since  $W_1$  and  $W_2$  are closed under scalar multiplication,  $c\mathbf{w} \in W_1$  and  $c\mathbf{w} \in W_2$ . Hence  $c\mathbf{w} \in W_1 \cap W_2$ .

Since all three conditions of the Subspace Test are satisfied,  $W_1 \cap W_2$  is a subspace of  $V$ . ■

*Remark.* Unlike the intersection, the **union** of two subspaces is not necessarily a subspace.

For  $W_1 \cup W_2$  to be a subspace, one subspace must be contained within the other (i.e.,  $W_1 \subseteq W_2$  or  $W_2 \subseteq W_1$ ).

**Example** (Union of Axes). Consider the vector space  $V = \mathbb{R}^2$ . Let  $W_1$  be the  $x$ -axis and  $W_2$  be the  $y$ -axis:

$$W_1 = \{(x, 0) : x \in \mathbb{R}\}, \quad W_2 = \{(0, y) : y \in \mathbb{R}\}.$$

Both  $W_1$  and  $W_2$  are subspaces of  $\mathbb{R}^2$ . However, their union  $W_1 \cup W_2$  is **not** a subspace because it is not closed under addition.

Indeed,  $(1, 0) \in W_1 \subseteq W_1 \cup W_2$  and  $(0, 1) \in W_2 \subseteq W_1 \cup W_2$ , but their sum:

$$(1, 0) + (0, 1) = (1, 1)$$

is not in  $W_1 \cup W_2$  since  $(1, 1)$  is neither on the  $x$ -axis nor the  $y$ -axis.

**Proposition 1.2.1 (Union of Subspaces).** Let  $W_1, W_2$  be subspaces of  $V$ . Then  $W_1 \cup W_2$  is a subspace of  $V$  if and only if  $W_1 \subseteq W_2$  or  $W_2 \subseteq W_1$ .

*Proof.* ( $\Leftarrow$ ) If  $W_1 \subseteq W_2$ , then  $W_1 \cup W_2 = W_2$ , which is a subspace. Similarly, if  $W_2 \subseteq W_1$ , the union is  $W_1$ , which is also a subspace.

( $\Rightarrow$ ) We prove the contrapositive: if neither is contained in the other, then the union is not a subspace. Assume  $W_1 \not\subseteq W_2$  and  $W_2 \not\subseteq W_1$ .

- Since  $W_1 \not\subseteq W_2$ , there exists a vector  $\mathbf{u} \in W_1$  such that  $\mathbf{u} \notin W_2$ .
- Since  $W_2 \not\subseteq W_1$ , there exists a vector  $\mathbf{v} \in W_2$  such that  $\mathbf{v} \notin W_1$ .

Consider the sum  $\mathbf{w} = \mathbf{u} + \mathbf{v}$ . We show that  $\mathbf{w} \notin W_1 \cup W_2$ :

- If  $\mathbf{w} \in W_1$ , then  $\mathbf{v} = \mathbf{w} - \mathbf{u}$ . Since  $\mathbf{w} \in W_1$  and  $\mathbf{u} \in W_1$ , their difference  $\mathbf{v}$  must be in  $W_1$  (by closure). But we chose  $\mathbf{v} \notin W_1$ , which is a contradiction.
- If  $\mathbf{w} \in W_2$ , then  $\mathbf{u} = \mathbf{w} - \mathbf{v}$ . Since  $\mathbf{w} \in W_2$  and  $\mathbf{v} \in W_2$ , their difference  $\mathbf{u}$  must be in  $W_2$ . But we chose  $\mathbf{u} \notin W_2$ , which is a contradiction.

Therefore,  $\mathbf{u} + \mathbf{v}$  is in neither  $W_1$  nor  $W_2$ , so it is not in  $W_1 \cup W_2$ . Thus, the union is not closed under addition and is therefore not a subspace. ■