

## 1.3 Linear Combinations and Span

In the previous section, we studied subspaces. A natural question arises: given a set of vectors, how can we describe the “smallest” subspace that contains them? This leads us to the concept of linear combinations and the span.

**Definition 1.3.1** (Linear Combination). Let  $V$  be a vector space over  $F$ ,  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$  be a subset of  $V$ . Then the expression

$$a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \dots + a_m \mathbf{v}_m$$

is called a **linear combination** of  $S$ . The scalars  $a_i$  are called the **coefficients** of the linear combination.

**Example** (Examples of Linear Combinations).

1. The vector  $(2, 1, 0)$  in  $\mathbb{R}^3$  is a linear combination of  $S = \{(1, 2, 3), (4, 5, 6)\}$  because

$$(2, 1, 0) = -2 \cdot (1, 2, 3) + 1 \cdot (4, 5, 6).$$

2. The polynomial  $f(x) = x^3 + 2x + 1$  in  $\mathbb{R}[x]$  is a linear combination of  $S = \{1, x, x^2, x^3\}$  because

$$f(x) = 1 \cdot 1 + 2 \cdot x + 0 \cdot x^2 + 1 \cdot x^3.$$

**Definition 1.3.2** (Span). Let  $S$  be a subset of a vector space  $V$  over  $F$ . The **span** of  $S$ , denoted by  $\text{span}_F(S)$ , is the set of all linear combinations of finite subsets of elements of  $S$ .

If  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ , then  $\text{span}_F(S) = \{a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \dots + a_n \mathbf{v}_n : a_i \in F\}$ . By convention,  $\text{span}_F(\emptyset) = \{\mathbf{0}\}$ .

*Remark.* When the underlying field is clear, we abuse the notation of  $\text{span}_F$  and write it as  $\text{span}$ .

**Theorem 1.3.1 (Spanning Set as Subspace).** The span of any subset  $S \subseteq V$  is a subspace of  $V$ . Moreover, it is the smallest subspace of  $V$  containing  $S$ .

*Proof.* Let  $W = \text{span}(S)$ .

- **Zero vector:** Since  $\text{span}(\emptyset) = \{\mathbf{0}\}$ , and for non-empty  $S$ , taking all coefficients as zero gives  $\mathbf{0}$ , we have  $\mathbf{0} \in W$ .
- **Addition:** Let  $\mathbf{x}, \mathbf{y} \in W$ . Then  $\mathbf{x} = \sum a_i \mathbf{v}_i$  and  $\mathbf{y} = \sum b_i \mathbf{v}_i$ . Their sum  $\mathbf{x} + \mathbf{y} = \sum (a_i + b_i) \mathbf{v}_i$  is also a linear combination of elements in  $S$ , so  $\mathbf{x} + \mathbf{y} \in W$ .
- **Scalar multiplication:** For any  $c \in F$ ,  $c\mathbf{x} = \sum (ca_i) \mathbf{v}_i \in W$ .

To see it is the smallest subspace, note that any subspace containing  $S$  must be closed

under addition and scalar multiplication, and thus must contain all linear combinations of elements in  $S$ . ■

**Definition 1.3.3** (Spanning Set). If  $\text{span}(S) = V$ , we say that  $S$  **spans**  $V$ , or that  $S$  is a **spanning set** for  $V$ .

**Example** (Spanning Set for Polynomials). The set  $\{1, x, x^2\}$  spans  $\mathbb{R}[x]_{\leq 2}$ , the space of polynomials of degree at most 2.