

## 0.2 Functions

**Definition 0.2.1** (Function). A **function**  $f$  from a set  $A$  to a set  $B$ , denoted  $f : A \rightarrow B$ , is a rule that assigns to each element  $a \in A$  exactly one element  $f(a) \in B$ . The set  $A$  is called the **domain** and  $B$  is called the **codomain**.

**Definition 0.2.2** (Injective, Surjective, Bijective). Let  $f : A \rightarrow B$  be a function.

- $f$  is **injective** (one-to-one) if  $f(a_1) = f(a_2)$  implies  $a_1 = a_2$ .
- $f$  is **surjective** (onto) if for every  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$ .
- $f$  is **bijective** if it is both injective and surjective.

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

**Example** (Injective, Surjective, and Bijective Functions). Consider the following functions:

1.  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = 2x + 1$  is **bijective**.
  - Injective: If  $2x_1 + 1 = 2x_2 + 1$ , then  $x_1 = x_2$ .
  - Surjective: For any  $y \in \mathbb{R}$ , we have  $f\left(\frac{y-1}{2}\right) = y$ .
2.  $g : \mathbb{R} \rightarrow \mathbb{R}$ ,  $g(x) = x^2$  is **neither injective nor surjective**.
  - Not injective:  $g(1) = g(-1) = 1$ , but  $1 \neq -1$ .
  - Not surjective: There is no  $x \in \mathbb{R}$  such that  $g(x) = -1$ .
3.  $h : \mathbb{R} \rightarrow [0, \infty)$ ,  $h(x) = x^2$  is **surjective but not injective**.
  - Not injective:  $h(1) = h(-1) = 1$ .
  - Surjective: For any  $y \geq 0$ , we have  $h(\sqrt{y}) = y$ .
4.  $k : [0, \infty) \rightarrow \mathbb{R}$ ,  $k(x) = x^2$  is **injective but not surjective**.
  - Injective: If  $x_1^2 = x_2^2$  with  $x_1, x_2 \geq 0$ , then  $x_1 = x_2$ .
  - Not surjective: There is no  $x \geq 0$  such that  $k(x) = -1$ .