

0.0 Fields

Definition 0.4.1 (Field). A set F is called a **field** if we can define two binary operations addition $+$, and multiplication \cdot

$$+ : F \times F \rightarrow F, \quad \cdot : F \times F \rightarrow F$$

such that:

$$(F1) \ a + b = b + a$$

$$(F2) \ (a + b) + c = a + (b + c)$$

$$(F3) \ \text{There exists } 0 \in F \text{ such that } a + 0 = a.$$

$$(F4) \ \text{For every } a \in F, \text{ there exists } -a \in F \text{ such that } a + (-a) = 0.$$

$$(F5) \ (a \cdot b) \cdot c = a \cdot (b \cdot c)$$

$$(F6) \ a \cdot b = b \cdot a$$

$$(F7) \ \text{There exists } 1 \in F \text{ such that } 1 \neq 0 \text{ and } a \cdot 1 = a.$$

$$(F8) \ \text{For every } a \in F \setminus \{0\}, \text{ there exists } a^{-1} \in F \text{ such that } a \cdot a^{-1} = 1.$$

$$(F9) \ a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

Remark. We can shorten “ F is a field with operations $+$ and \cdot ” to “ $(F, +, \cdot)$ is a field”. Moreover when the context of $+$ and \cdot is clear, we simply say “ F is a field”, which we will do most of the time in this book.

Example (Examples of Fields).

1. \mathbb{R} is a field.
2. \mathbb{C} is a field.
3. $\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, \gcd(p, q) = 1 \right\}$ is a field.
4. \mathbb{Z} is **not** a field since **(F8)** fails.
5. $M_n(\mathbb{R})$ is **not** a field since **(F6)** fails.

Example (Finite Fields). Let \mathbb{F}_n be the finite field with n elements. Then $(\mathbb{F}_2, +, \cdot)$ is defined to be:

$$F_2 = \{0, 1\}, \quad \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}, \quad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}.$$

Then one can verify that $(\mathbb{F}_2, +, \cdot)$ is a field.