1.5 Bases

We have reached what many consider the "Goldilocks Zone" of Linear Algebra.

If a set is too small, it won't span the whole space—you'll be left with points you can't reach. If a set is too large, it will be linearly dependent—you'll have redundant vectors cluttering your workspace. A **basis** is a set that is "just right." It is large enough to build everything, but small enough that every piece is essential.

Because a basis has no redundancy, it gives us something incredible: a **unique address system**. In an abstract vector space, it's hard to tell someone where a vector is. But once we pick a basis, every vector can be described by a unique list of numbers—its coordinates. A basis is the bridge that allows us to turn abstract geometry into concrete arithmetic.

Definition 1.5.1 (Basis)**.** A subset B of a vector space V is called a **basis** for V if:

1. B is linearly independent;
2. B spans V.

Example (Standard Bases).

- The **standard basis** for F^n is $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$, where $\mathbf{e}_i$ has a 1 in the i-th position and 0 elsewhere.
- The standard basis for $F[x]_{\leq n}$ is $\{1, x, x^2, \dots, x^n\}$.
- The standard basis for $M_{m\times n}(F)$ is the set of matrices $\mathbf{E}_{ij}$ having a 1 at entry (i, j) and 0 elsewhere.

1.5.1 Unique Representation and Coordinates

The most important property of a basis is that every vector in the space has a unique "address" relative to it.

Theorem 1.5.1 (Unique Representation). Let $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for V. Then every vector $\mathbf{v} \in V$ can be written as a linear combination of elements in B in **exactly one way**.

Proof. Since B spans V, there exist scalars a_i such that $\mathbf{v} = \sum a_i\mathbf{v}_i$. Suppose there is another representation $\mathbf{v} = \sum b_i\mathbf{v}_i$. Then:

$$\mathbf{0} = \mathbf{v} - \mathbf{v} = \sum a_i\mathbf{v}_i - \sum b_i\mathbf{v}_i = \sum (a_i - b_i)\mathbf{v}_i.$$

Since B is linearly independent, we must have $a_i - b_i = 0$ for all i, meaning $a_i = b_i$. ■

Definition 1.5.2 (Coordinates)**.** Let $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an **ordered** basis for V. For any $\mathbf{v} \in V$, the unique scalars $c_1, \dots, c_n$ such that $\mathbf{v} = \sum c_i\mathbf{v}_i$ are called the **coordinates**

of $\mathbf{v}$ with respect to B. We write this as a column vector:

$$[\mathbf{v}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Remark. The map $\mathbf{v} \mapsto [\mathbf{v}]_B$ provides a way to treat any abstract n-dimensional vector space as if it were simply F^n. This is the power of a basis.