

## 2.1 Eigenvalues and Eigenvectors

\$\$ % Sets  
% Linear Algebra  
% Operators      \$\$

### 1 Eigenvalues and Eigenvectors

#### 1.1 Introduction

Eigenvalues and eigenvectors reveal the intrinsic properties of linear transformations. They tell us about the directions in which a transformation acts by simple scaling.

#### 1.2 Definitions

**Definition 1.1** (Eigenvalue and Eigenvector). Let  $\mathbf{A} \in \mathbb{R}^{n \times n}$ . A scalar  $\lambda \in \mathbb{C}$  is called an **eigenvalue** of  $\mathbf{A}$  if there exists a non-zero vector  $\mathbf{v} \in \mathbb{C}^n$  such that:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

The vector  $\mathbf{v}$  is called an **eigenvector** corresponding to  $\lambda$ .

**Definition 1.2** (Characteristic Polynomial). The **characteristic polynomial** of  $\mathbf{A}$  is:

$$p(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$$

The eigenvalues of  $\mathbf{A}$  are the roots of  $p(\lambda)$ .

#### 1.3 Properties

**Theorem 1.1** (Real Eigenvalues of Symmetric Matrices). *If  $\mathbf{A} \in \mathbb{R}^{n \times n}$  is symmetric, then all eigenvalues of  $\mathbf{A}$  are real.*

*Proof.* Let  $\lambda$  be an eigenvalue with eigenvector  $\mathbf{v}$ . Taking the conjugate transpose of  $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ :

$$\bar{\mathbf{v}}^\top \mathbf{A} = \bar{\lambda} \bar{\mathbf{v}}^\top$$

Multiplying both sides on the right by  $\mathbf{v}$  and using symmetry of  $\mathbf{A}$ :

$$\bar{\mathbf{v}}^\top \mathbf{A} \mathbf{v} = \bar{\lambda} \bar{\mathbf{v}}^\top \mathbf{v} = \bar{\lambda} \|\mathbf{v}\|^2$$

But also  $\bar{\mathbf{v}}^\top \mathbf{A} \mathbf{v} = \bar{\mathbf{v}}^\top (\lambda \mathbf{v}) = \lambda \|\mathbf{v}\|^2$ . Thus  $\lambda = \bar{\lambda}$ , so  $\lambda \in \mathbb{R}$ . □

## 1.4 Exercises

**Exercise 1.1** (Trace and Determinant). Prove that for any  $\mathbf{A} \in \mathbb{R}^{n \times n}$ :

1.  $\text{tr}(\mathbf{A}) = \sum_{i=1}^n \lambda_i$
2.  $\det(\mathbf{A}) = \prod_{i=1}^n \lambda_i$

where  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $\mathbf{A}$  (counting multiplicity).