Chapter 3

Eigenvalues and Eigenvectors

3.1 Introduction

Eigenvalues and eigenvectors reveal the intrinsic properties of linear transformations. They tell us about the directions in which a transformation acts by simple scaling.

3.2 Definitions

Definition 3.1 (Eigenvalue and Eigenvector)**.** Let $\mathbf{A} \in \mathbb{R}^{n \times n}$. A scalar $\lambda \in \mathbb{C}$ is called an **eigenvalue** of $\mathbf{A}$ if there exists a non-zero vector $\mathbf{v} \in \mathbb{C}^n$ such that:

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$$

The vector $\mathbf{v}$ is called an **eigenvector** corresponding to λ.

Definition 3.2 (Characteristic Polynomial)**.** The **characteristic polynomial** of $\mathbf{A}$ is:

$$p(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I})$$

The eigenvalues of $\mathbf{A}$ are the roots of $p(\lambda)$.

3.3 Properties

Theorem 3.1 (Real Eigenvalues of Symmetric Matrices)**.** *If $\mathbf{A} \in \mathbb{R}^{n \times n}$ is symmetric, then all eigenvalues of $\mathbf{A}$ are real.*

Proof. Let λ be an eigenvalue with eigenvector $\mathbf{v}$. Taking the conjugate transpose of $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$:

$$\bar{\mathbf{v}}^\top \mathbf{A} = \bar{\lambda}\bar{\mathbf{v}}^\top$$

Multiplying both sides on the right by $\mathbf{v}$ and using symmetry of $\mathbf{A}$:

$$\bar{\mathbf{v}}^\top \mathbf{A}\mathbf{v} = \bar{\lambda}\bar{\mathbf{v}}^\top \mathbf{v} = \bar{\lambda}\left\|\mathbf{v}\right\|^2$$

But also $\bar{\mathbf{v}}^\top \mathbf{A}\mathbf{v} = \bar{\mathbf{v}}^\top(\lambda\mathbf{v}) = \lambda\left\|\mathbf{v}\right\|^2$. Thus $\lambda = \bar{\lambda}$, so $\lambda \in \mathbb{R}$. ■

3.4 Exercises

Exercise 3.1 (Trace and Determinant)**.** Prove that for any $\mathbf{A} \in \mathbb{R}^{n \times n}$:

1. $\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$
2. $\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\mathbf{A}$ (counting multiplicity).