1.1 Vector Spaces

$$ % Sets

% Linear Algebra

% Operators $$

1 Vector Spaces

1.1 Introduction

A vector space is one of the most fundamental structures in mathematics. It provides an abstract framework for studying objects that can be added together and scaled.

1.2 Definition

Definition 1.1 (Vector Space)**.** A **vector space** over a field $\mathbb{R}$ (or $\mathbb{C}$) is a set V together with two operations:

1. **Vector addition**: $+ : V \times V \to V$
2. **Scalar multiplication**: $\cdot : \mathbb{R} \times V \to V$

satisfying the following axioms for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $a, b \in \mathbb{R}$:

- Commutativity: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- Associativity: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- Additive identity: There exists $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$
- Additive inverse: For each $\mathbf{v}$, there exists $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
- Scalar associativity: $a(b\mathbf{v}) = (ab)\mathbf{v}$
- Distributivity: $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ and $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$
- Scalar identity: $1\mathbf{v} = \mathbf{v}$

1.3 Examples

Example 1.1 (Euclidean Space)**.** The set $\mathbb{R}^n$ with standard addition and scalar multiplication is a vector space.

1.4 Exercises

Exercise 1.1 (Subspace Criterion)**.** Prove that a non-empty subset $W \subseteq V$ is a subspace if and only if $a\mathbf{u} + b\mathbf{v} \in W$ for all $\mathbf{u}, \mathbf{v} \in W$ and $a, b \in \mathbb{R}$.