

2.2 Matrix Decompositions

\$\$ % Sets
% Linear Algebra
% Operators \$\$

1 Matrix Decompositions

1.1 Introduction

Matrix decompositions factor a matrix into products of simpler matrices. These factorizations reveal structure and enable efficient computation.

1.2 Spectral Decomposition

Theorem 1.1 (Spectral Theorem). *Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be symmetric. Then \mathbf{A} can be decomposed as:*

$$\mathbf{A} = \mathbf{Q}\mathbf{Q}^\top$$

where \mathbf{Q} is orthogonal (its columns are orthonormal eigenvectors of \mathbf{A}) and $\mathbf{\Lambda} = \text{diag}(\lambda_1, \dots, \lambda_n)$ is diagonal (containing the eigenvalues).

1.3 Singular Value Decomposition

Definition 1.1 (Singular Value Decomposition (SVD)). Every matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ can be factored as:

$$\mathbf{A} = \mathbf{U}\mathbf{V}^\top$$

where:

- $\mathbf{U} \in \mathbb{R}^{m \times m}$ is orthogonal (left singular vectors)
- $\mathbf{\Lambda} \in \mathbb{R}^{m \times n}$ is diagonal with non-negative entries (singular values)
- $\mathbf{V} \in \mathbb{R}^{n \times n}$ is orthogonal (right singular vectors)

Theorem 1.2 (SVD and Rank). *The rank of \mathbf{A} equals the number of non-zero singular values.*

1.4 Applications

Example 1.1 (Principal Component Analysis). PCA uses the SVD to find the directions of maximum variance in a dataset. If \mathbf{X} is a centered data matrix, the principal components are the right singular vectors of \mathbf{X} .

1.5 Exercises

Exercise 1.1 (Moore-Penrose Pseudoinverse). Using the SVD $\mathbf{A} = \mathbf{U}\mathbf{V}^\top$, show that the pseudoinverse is given by $\mathbf{A}^+ = \mathbf{V}^+\mathbf{U}^\top$.