

2.1 Eigenvalues and Eigenvectors

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## % Sets  
% Linear Algebra  
% Operators    ##
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1 Eigenvalues and Eigenvectors

1.1 Introduction

Eigenvalues and eigenvectors reveal the intrinsic properties of linear transformations. They tell us about the directions in which a transformation acts by simple scaling.

1.2 Definitions

Definition 1.1 (Eigenvalue and Eigenvector). Let $\mathbf{A} \in \mathbb{R}^{n \times n}$. A scalar $\lambda \in \mathbb{C}$ is called an **eigenvalue** of \mathbf{A} if there exists a non-zero vector $\mathbf{v} \in \mathbb{C}^n$ such that:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

The vector \mathbf{v} is called an **eigenvector** corresponding to λ .

Definition 1.2 (Characteristic Polynomial). The **characteristic polynomial** of \mathbf{A} is:

$$p(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$$

The eigenvalues of \mathbf{A} are the roots of $p(\lambda)$.

1.3 Properties

Theorem 1.1 (Real Eigenvalues of Symmetric Matrices). *If $\mathbf{A} \in \mathbb{R}^{n \times n}$ is symmetric, then all eigenvalues of \mathbf{A} are real.*

Proof. Let λ be an eigenvalue with eigenvector \mathbf{v} . Taking the conjugate transpose of $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$:

$$\bar{\mathbf{v}}^\top \mathbf{A} = \bar{\lambda} \bar{\mathbf{v}}^\top$$

Multiplying both sides on the right by \mathbf{v} and using symmetry of \mathbf{A} :

$$\bar{\mathbf{v}}^\top \mathbf{A}\mathbf{v} = \bar{\lambda} \bar{\mathbf{v}}^\top \mathbf{v} = \bar{\lambda} \|\mathbf{v}\|^2$$

But also $\bar{\mathbf{v}}^\top \mathbf{A}\mathbf{v} = \bar{\mathbf{v}}^\top (\lambda\mathbf{v}) = \lambda \|\mathbf{v}\|^2$. Thus $\lambda = \bar{\lambda}$, so $\lambda \in \mathbb{R}$. \square

1.4 Exercises

Exercise 1.1 (Trace and Determinant). Prove that for any $\mathbf{A} \in \mathbb{R}^{n \times n}$:

1. $\text{tr}(\mathbf{A}) = \sum_{i=1}^n \lambda_i$
2. $\det(\mathbf{A}) = \prod_{i=1}^n \lambda_i$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of \mathbf{A} (counting multiplicity).