3.1 Convex Optimization

$$ % Sets

% Linear Algebra

% Operators $$

1 Convex Optimization

1.1 Introduction

Convex optimization is the study of minimizing convex functions over convex sets. The theory is elegant and the algorithms are efficient, making convex optimization a cornerstone of modern applied mathematics.

1.2 Convex Sets and Functions

Definition 1.1 (Convex Set)**.** A set $C \subseteq \mathbb{R}^n$ is **convex** if for all $\mathbf{x}, \mathbf{y} \in C$ and $\theta \in [0, 1]$:

$$\theta \mathbf{x} + (1 - \theta)\mathbf{y} \in C$$

Definition 1.2 (Convex Function)**.** A function $f : \mathbb{R}^n \to \mathbb{R}$ is **convex** if for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\theta \in [0, 1]$:

$$f(\theta \mathbf{x} + (1 - \theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1 - \theta) f(\mathbf{y})$$

1.3 Optimality Conditions

Theorem 1.1 (First-Order Optimality)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and differentiable. Then $\mathbf{x}^*$ is a global minimizer if and only if:*

$$\nabla f(\mathbf{x}^*) = \mathbf{0}$$

Theorem 1.2 (Second-Order Condition)**.** *A twice-differentiable function f is convex if and only if its Hessian is positive semidefinite everywhere:*

$$\nabla^2 f(\mathbf{x}) \succeq 0 \quad \textit{for all } \mathbf{x}$$

1.4 Quadratic Programming

Definition 1.3 (Quadratic Program)**.** A **quadratic program** (QP) is an optimization problem of the form:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \frac{1}{2}\mathbf{x}^\top \mathbf{P}\mathbf{x} + \mathbf{q}^\top \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{G}\mathbf{x} \leq \mathbf{h} \end{aligned}$$

The problem is convex if $\mathbf{P} \succeq 0$.

1.5 Exercises

Exercise 1.1 (Least Squares as QP)**.** Show that the least squares problem $\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2$ is a convex QP and derive the normal equations.