# Chapter 1

# Vector Spaces

## 1.1 Introduction

A vector space is one of the most fundamental structures in mathematics. It provides an abstract framework for studying objects that can be added together and scaled.

## 1.2 Definition

**Definition 1.1** (Vector Space)**.** A **vector space** over a field $\mathbb{R}$ (or $\mathbb{C}$) is a set $V$ together with two operations:

1. **Vector addition**: $+ : V \times V \to V$
2. **Scalar multiplication**: $\cdot : \mathbb{R} \times V \to V$

satisfying the following axioms for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $a, b \in \mathbb{R}$:

- Commutativity: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- Associativity: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- Additive identity: There exists $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$
- Additive inverse: For each $\mathbf{v}$, there exists $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
- Scalar associativity: $a(b\mathbf{v}) = (ab)\mathbf{v}$
- Distributivity: $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ and $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$
- Scalar identity: $1\mathbf{v} = \mathbf{v}$

## 1.3 Examples

**Example 1.1** (Euclidean Space)**.** The set $\mathbb{R}^n$ with standard addition and scalar multiplication is a vector space.

## 1.4 Exercises

**Exercise 1.1** (Subspace Criterion)**.** Prove that a non-empty subset $W \subseteq V$ is a subspace if and only if $a\mathbf{u} + b\mathbf{v} \in W$ for all $\mathbf{u}, \mathbf{v} \in W$ and $a, b \in \mathbb{R}$.