Chapter 4

Matrix Decompositions

4.1 Introduction

Matrix decompositions factor a matrix into products of simpler matrices. These factorizations reveal structure and enable efficient computation.

4.2 Spectral Decomposition

Theorem 4.1 (Spectral Theorem)**.** *Let* $\mathbf{A} \in \mathbb{R}^{n \times n}$ *be symmetric. Then* $\mathbf{A}$ *can be decomposed as:*

$$\mathbf{A} = \mathbf{Q}\mathbf{Q}^\top$$

where $\mathbf{Q}$ *is orthogonal (its columns are orthonormal eigenvectors of* $\mathbf{A}$*) and* $=$ $diag(\lambda_1, \ldots, \lambda_n)$ *is diagonal (containing the eigenvalues).*

4.3 Singular Value Decomposition

Definition 4.1 (Singular Value Decomposition (SVD))**.** Every matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ can be factored as:

$$\mathbf{A} = \mathbf{U}\mathbf{V}^\top$$

where:

- $\mathbf{U} \in \mathbb{R}^{m \times m}$ is orthogonal (left singular vectors)
- $\in \mathbb{R}^{m \times n}$ is diagonal with non-negative entries (singular values)
- $\mathbf{V} \in \mathbb{R}^{n \times n}$ is orthogonal (right singular vectors)

Theorem 4.2 (SVD and Rank)**.** *The rank of* $\mathbf{A}$ *equals the number of non-zero singular values.*

4.4 Applications

Example 4.1 (Principal Component Analysis)**.** PCA uses the SVD to find the directions of maximum variance in a dataset. If $\mathbf{X}$ is a centered data matrix, the principal components are the right singular vectors of $\mathbf{X}$.

4.5 Exercises

Exercise 4.1 (Moore-Penrose Pseudoinverse)**.** Using the SVD $\mathbf{A} = \mathbf{U}\mathbf{V}^\top$, show that the pseudoinverse is given by $\mathbf{A}^+ = \mathbf{V}^+\mathbf{U}^\top$.