1.2 Linear Maps

$$ % Sets

% Linear Algebra

% Operators $$

1 Linear Maps

1.1 Introduction

Linear maps are functions between vector spaces that preserve the linear structure. They are the natural morphisms in the category of vector spaces.

1.2 Definition

Definition 1.1 (Linear Map)**.** Let V and W be vector spaces over $\mathbb{R}$. A function $T : V \to W$ is called a **linear map** (or linear transformation) if for all $\mathbf{u}, \mathbf{v} \in V$ and $a \in \mathbb{R}$:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (additivity)
2. $T(a\mathbf{v}) = aT(\mathbf{v})$ (homogeneity)

Equivalently, T is linear if and only if $T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$ for all vectors and scalars.

1.3 Fundamental Subspaces

Definition 1.2 (Kernel and Image)**.** Let $T : V \to W$ be a linear map.

- The **kernel** (or null space) of T is $\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}$
- The **image** (or range) of T is $\mathrm{im}(T) = \{T(\mathbf{v}) : \mathbf{v} \in V\}$

Theorem 1.1 (Rank-Nullity Theorem)**.** *Let $T : V \to W$ be a linear map where V is finite-dimensional. Then:*

$$\dim(V) = \dim(\ker(T)) + \dim(im(T))$$

1.4 Exercises

Exercise 1.1 (Composition of Linear Maps)**.** Prove that the composition of two linear maps is linear.