

# CSCA67 - Exercises #3

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## 1. Conditionals

Analyse the logical forms of the following statements. Construct a converse and a contrapositive for each conditional statement: provide your answers both as logical expressions and English sentences.

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1. If Alice is at the party, then so is Bob.

Converse:  $B \rightarrow A$  (If Bob is at the party, then so is Alice.)

Contrapositive:  $\neg B \rightarrow \neg A$  (If Bob is not at the party, then Alice is not at the party.)

2. Charlie is at the party, only if both Alice and Bob are.

Original:  $C \rightarrow (A \wedge B)$

Converse:  $(A \wedge B) \rightarrow C$  (Alice and Bob are both at the party, only if Charlie is.)

Contrapositive:  $\neg(A \wedge B) \rightarrow \neg C$  (If both Alice and Bob are not at the party, then Charlie is not at the party.)

3. David is not at the party, if Alice is.

Original:  $A \rightarrow \neg D$

Converse:  $\neg D \rightarrow A$  (If David is not at the party, then Alice is.)

Contrapositive:  $D \rightarrow \neg A$  (If David is at the party, then Alice is not.)

4. If Bob is not at the party, then Alice is.

Original:  $\neg B \rightarrow A$

Converse:  $A \rightarrow \neg B$  (If Alice is at the party, then Bob is not.)

Contrapositive:  $\neg A \rightarrow B$  (If Alice is not at the party, then Bob is.)

5. If Bob is not at the party, then neither is Alice.

Original:  $\neg B \rightarrow \neg A$

Converse:  $\neg A \rightarrow \neg B$  (If Alice is not at the party, then neither is Bob.)

Contrapositive:  $A \rightarrow B$  (If Alice is at the party, then so is Bob.)

6. Alice is not at the party, unless Bob is.

Original:  $A \rightarrow B$

Converse:  $B \rightarrow A$  (If Bob is at the party, then Alice is.)

Contrapositive:  $\neg B \rightarrow \neg A$  (If Bob is not at the party, then neither is Alice.)

7. Neither Alice nor Bob being at the party is a sufficient condition for Charlie to be at the party.

Original:  $\neg(A \wedge B) \rightarrow C$

Converse:  $C \rightarrow \neg(A \wedge B)$  (If Charlie is at the party, then neither Alice nor Bob is.)

Contrapositive:  $\neg C \rightarrow (A \wedge B)$  (If Charlie is not at the party, then both Alice and Bob are.)

8. Both Alice and Bob being at the party is a necessary condition for Charlie to be at the party.

Original:  $C \rightarrow (A \wedge B)$

Converse:  $(A \wedge B) \rightarrow C$  (If both Alice and Bob are at the party, then so is Charlie.)

Contrapositive:  $\neg(A \wedge B) \rightarrow \neg C$  (If both Alice and Bob are not at the party, then neither is Charlie.)

## 2. Logical Equivalences

For each pair of expressions, either prove that the two are equivalent or prove that they are not.

1.  $\neg(a \rightarrow b)$  and  $\neg a \wedge b$

When  $a$  is True, and  $b$  is False, then:

$$\begin{array}{ll} \neg(a \rightarrow b) & \neg a \wedge b \\ \Rightarrow \neg(T \rightarrow F) & \Rightarrow \neg T \wedge F \\ \Rightarrow \neg(F) & \Rightarrow F \wedge F \\ \Rightarrow T & \Rightarrow F \end{array}$$

$\therefore$  The statements are not equivalent. ■

2.  $\neg(a \rightarrow b)$  and  $a \wedge \neg b$

$$\begin{array}{ll} \neg(a \rightarrow b) & \boxed{a \wedge \neg b} \\ \Rightarrow \neg(\neg a \wedge b) & \text{(Conditional Law)} \\ \Rightarrow \neg\neg a \vee \neg b & \text{(De Morgan's Law)} \\ \Rightarrow \boxed{a \vee \neg b} & \text{(Double Negation Law)} \end{array}$$

$\therefore$  The statements are equivalent. ■

3.  $a \iff \neg b$  and  $(a \wedge \neg b) \vee (\neg a \wedge b)$

$$\begin{array}{ll} a \iff \neg b & \boxed{(a \wedge \neg b) \vee (\neg a \wedge b)} \\ \Rightarrow (\neg b \rightarrow a) \wedge (a \rightarrow \neg b) & \text{(Biconditional Law)} \\ \Rightarrow (\neg b \vee a) \wedge (\neg a \vee \neg b) & \text{(Conditional Law)} \\ \Rightarrow (b \vee a) \wedge (\neg a \vee \neg b) & \text{(Double Negation Law)} \\ \Rightarrow ((b \vee a) \wedge \neg a) \vee ((b \vee a) \wedge \neg b) & \text{(Distributive Law)} \\ \Rightarrow ((b \wedge \neg a) \vee (a \wedge \neg a)) \vee ((b \wedge \neg b) \vee (a \wedge \neg b)) & \text{(Distributive Law)} \\ \Rightarrow \boxed{(b \wedge \neg a) \vee (a \wedge \neg b)} & \text{(Negation Law)} \end{array}$$

$\therefore$  The statements are equivalent. ■

## 3. Logical Inference

Use the rules of inference from class, to prove validity of the following arguments.

1. If I play hockey, then I am sore the next day. I use the whirlpool if I am sore. I did not use the whirlpool. Therefore, I did not play hockey.

$$\text{Statements} = \begin{cases} H : \text{I played hockey.} \\ S : \text{I am sore the next day.} \\ W : \text{I used the whirlpool.} \end{cases}$$

The argument is:

$$\begin{array}{ll} H \rightarrow S & (1) \\ S \rightarrow W & (2) \\ \neg W & (3) \\ \hline \therefore \neg H & \text{Conclusion} \end{array}$$

New statements that we can make are:

$$\neg S \quad (3), (2), \text{Modus Tollens} \quad (4)$$

$$\neg H \quad (4), (1), \text{Modus Tollens} \quad (5)$$

(5)  $\equiv$  Conclusion, therefore the argument is true. ■

2. I am either dreaming or hallucinating. I am not dreaming. If I am hallucinating, I see elephants running down the road. Therefore, I see elephants running down the road.

$$\text{Statements} = \begin{cases} D : \text{I am dreaming.} \\ H : \text{I am hallucinating} \\ E : \text{I see elephants running down the road.} \end{cases}$$

The argument is:

$$D \vee H \quad (1)$$

$$\neg D \quad (2)$$

$$H \rightarrow E \quad (3)$$

$$\frac{}{\therefore E} \quad \text{Conclusion}$$

New statements that we can make are:

$$H \quad (2), (1), \text{Disjunctive Syllogism} \quad (4)$$

$$E \quad (4), (3), \text{Modus Ponens} \quad (5)$$

(5)  $\equiv$  Conclusion, therefore the argument is true. ■

3. If I go running, I stay in the sun for too long. If I go swimming, I stay in the sun for too long. If I stay in the sun for too long, I get sunburn. I did not get a sunburn. Therefore, I neither went running nor swimming.

$$\text{Statements} = \begin{cases} R : \text{I went running.} \\ S : \text{I stay in the sun for too long.} \\ W : \text{I went swimming.} \\ B : \text{I get sunburn.} \end{cases}$$

The argument is:

$$R \rightarrow S \quad (1)$$

$$W \rightarrow S \quad (2)$$

$$S \rightarrow B \quad (3)$$

$$\neg B \quad (4)$$

$$\frac{}{\therefore (\neg R \wedge \neg W)} \quad \text{Conclusion}$$

New statements that we can make are:

$$\neg S \quad (3), (4), \text{Modus Tollens} \quad (5)$$

$$\neg W \quad (5), (2), \text{Modus Tollens} \quad (6)$$

$$\neg R \quad (5), (1), \text{Modus Tollens} \quad (7)$$

$$\neg R \wedge \neg W \quad (6), (7), \text{Conjunction} \quad (8)$$

(8)  $\equiv$  Conclusion, therefore the argument is true. ■

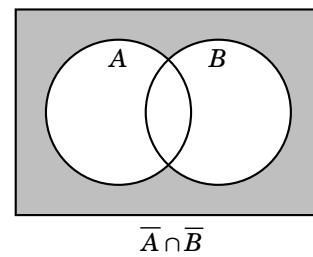
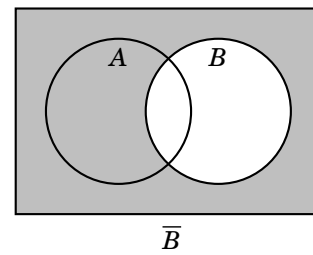
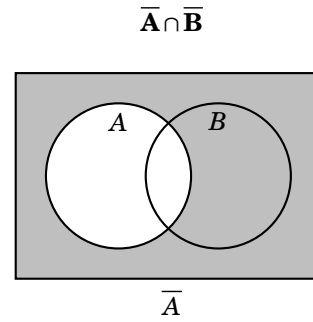
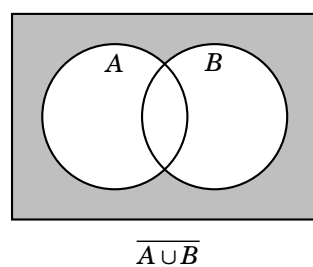
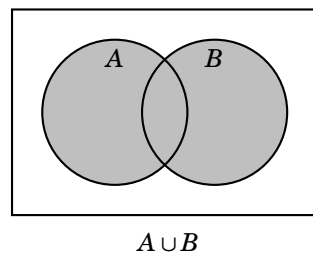
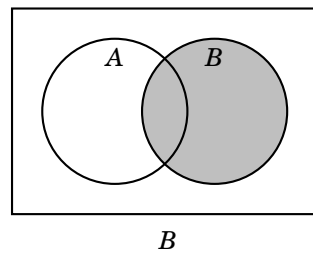
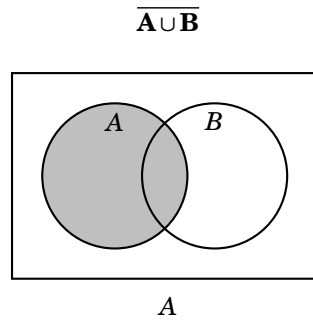
## 4. Variables and Sets

If  $U$  is the universe of discourse, then the complement of the set  $A$ , which we will denote as  $\bar{A}$ , is the set

$$U \setminus A = \{x \in U \mid x \notin A\}$$

Use Venn diagrams to illustrate the following identities.

1.  $\overline{A \cup B} = \bar{A} \cap \bar{B}$



$\therefore \overline{A \cup B} = \bar{A} \cap \bar{B}$