MATA31 - Assignment #5

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1 Textbook Questions

1.3.62

Prove.

$$\lim_{x \to -2^-} \frac{1}{x+2} = -\infty$$

Want to show:

$$\forall M < 0 \, \exists \delta > 0 \quad \text{s.t} \quad 0 < -2 - x < \delta \implies \frac{1}{x+2} < M$$

Proof.
Let $M < 0$ be arbitrary.

Choose $\delta = -\frac{1}{M}$. Note $\delta > 0$.

Assume $0 < -2 - x < \delta$. Then,

$$\begin{aligned} 0 < -2 - x < \delta &\implies -2 - x < -\frac{1}{M} && \text{(by our choice of } \delta\text{)} \\ &\implies x + 2 > \frac{1}{M} && \text{(because } x < -2\text{)} \\ &\implies \frac{1}{x+2} < M && \text{(by properties of inequalities)} \end{aligned}$$

As required to prove. ■

1.3.64

Prove.

$$\lim_{x \to -\infty} \frac{2x-1}{x} = 2$$

Want to show:

$$\forall \varepsilon > 0 \, \exists N < 0 \quad \text{s.t} \quad x < N \implies \left| \frac{2x-1}{x} - 2 \right| < \varepsilon$$

Proof.
Let $\varepsilon > 0$ be arbitrary.

Choose $N = -\frac{1}{\varepsilon}$. Note $N < 0$.

Assume $x < N$. Then,

$$
\begin{aligned}
\left|\frac{2x-1}{x}-2\right| &= \left|\frac{2x-1-2x}{x}\right| && \text{(by algebra)}\\
&= \left|\frac{-1}{x}\right| && \text{(by algebra)}\\
&= \frac{1}{|x|} && \text{(by properties of } |\cdot|\text{)}\\
&= \frac{1}{-x} && \text{(since } x<0\text{)}\\
\frac{1}{-x} &< \frac{1}{-N} && \text{(since } x<N<0 \implies -x>-N>0\text{)}\\
&= \frac{1}{-\left(\frac{-1}{\varepsilon}\right)} && \text{(by our choice of } N\text{)}\\
&= \frac{1}{\frac{1}{\varepsilon}} && \text{(by algebra)}\\
&= \varepsilon && \text{(by algebra)}
\end{aligned}
$$

As required to prove. ■

1.3.66

Prove.

$$\lim_{x\to-\infty}(3x-5)=-\infty$$

Want to show:

$$\forall M<0\,\exists N<0 \quad \text{s.t} \quad x<N \implies 3x-5<M$$

Proof.

Let $M<0$ be arbitrary.

Choose $N=\frac{M}{3}$. Note $N<0$.

Assume $x<N$. Then,

$$
\begin{aligned}
x<N &\implies x<\frac{M}{3} && \text{(by our choice of } N\text{)}\\
&\implies 3x<M && \text{(by algebra)}\\
&\implies 3x-5<M-5 && \text{(by algebra)}\\
&\implies 3x-5<M-5<M && \text{(by algebra)}\\
&\implies 3x-5<M && \text{(by properties of inequalities)}
\end{aligned}
$$

As required to prove. ■

1.3.70

Prove.

$$\lim_{x\to 1}\left(x^2-6x+7\right)=2$$

Want to show:

$$\forall\varepsilon>0\,\exists\,\delta>0\quad \text{s.t}\quad 0<|x-1|<\delta\Longrightarrow\left|\left(x^2-6x+7\right)-2\right|<\varepsilon$$

Proof.
Let $\varepsilon>0$ be arbitrary

Choose $\delta=\min\left\{5,\frac{\varepsilon}{9}\right\}$. Note that $\delta>0$.

Assume $0<|x-1|<\delta$.

Since $x^2-6x+7-2=(x-1)(x-5)$ and $x-1<\delta$, we first need to obtain a bound on $|x-5|$. Then

$$\begin{aligned}
|x-1|<\delta &\Longrightarrow |x-1|<5 && \left(\text{since } \delta=\min\left\{5,\frac{\varepsilon}{9}\right\}\le 5\right)\\
&\Longrightarrow -5<x-1<5 && (\text{by properties of } |\cdot|)\\
&\Longrightarrow -9<x-5<1 && (\text{by algebra})\\
&\Longrightarrow -9<x-5<9 && (\text{by properties of inequalities})\\
&\Longrightarrow |x-5|<9 && (\text{by properties of } |\cdot|)
\end{aligned}$$

Therefore, $|x-5|<9$ $(\star)$.
It now follows that:

$$\begin{aligned}
\left|x^2-6x+5\right| &= |(x-5)(x-1)| && (\text{by algebra})\\
&= |x-5|\,|x-1| && (\text{by properties of } |\cdot|)\\
&< |x-5|\,\delta && (\text{by assumption})\\
&< 9\delta && (\text{by } (\star))\\
&= 9\frac{\varepsilon}{9} && (\text{by our choice of } \delta)\\
&= \varepsilon && (\text{by algebra}).
\end{aligned}$$

As required to prove. ∎

1.4.48

Use graphs to determine if f is continuous at the given point $x=c$.

$$f(x)=\begin{cases} x^2-3, & \text{if } x \text{ rational}\\ 3x+1, & \text{if } x \text{ irrational}\end{cases}$$

$c=4$.

The informal definition of continuity is:

(1) $f(c)$ exists.

(2) $\lim_{x\to c} f(x)$ exists.

(3) $\lim_{x \to c} f(x) = f(c)$

Since $c = 4 \implies c \in \mathbb{Q}$, then $f(4) = 4^2 - 3 = 16 - 3 = 13$. Therefore, $f(c)$ exists, meaning (1) holds.

There are infinite irrational numbers between every rational number, therefore we check if the limit of both pieces of the function are equal to each other.

$$\lim_{x - 4} \left(x^2 - 3\right) = 16 - 3 = 13.$$

$$\lim_{x - 4} (3(4) - 1) = 12 + 1 = 13$$

Note that we can simply substitute c into these equations because they are both polynomials with non-negative exponents.

Therfore, since $\lim_{x \to c} f(x)$ exists. and $\lim_{x \to c} f(x) = f(c)$, the function is continuous at $x = 4$.

2 Assignment Questions

D

Find the supremum and infimum of the following sets, if they exist.

(a) $A = \left\{\frac{1}{n} : n \in \mathbb{Z} \quad \text{and} \quad n \neq 0\right\}$

(b) $B = \left\{x \in \mathbb{Q} : 0 \leq x \leq \sqrt{2}\right\}$

(c) $C = \left\{x \in \mathbb{R} : x^2 + x + 1 \geq 0\right\}$

(a). $A = \left\{\frac{1}{n} : n \in \mathbb{Z} \quad \text{and} \quad n \neq 0\right\}$
As $n \to -\infty, \frac{1}{n} \to 0^-$ As $n \to \infty, \frac{1}{n} \to 0^+$ Since $n \in \mathbb{Z}$ the biggest positive and biggest negative numbers we can obtain are -1 and 1. Therefore, the highest number we can achieve is 1, the lowest is -1.
Therefore, the supremum is 1, and the infimum is -1.

(b). $B = \left\{x \in \mathbb{Q} : 0 \leq x \leq \sqrt{2}\right\}$
The lowest number in this set is 0, making it the infimum. There are infinitely many rational numbers in $[0, \sqrt{2}]$, meaning that there is no biggest rational number in this set. Therefore, the infimum is $\sqrt{2}$.

(c). $C = \left\{x \in \mathbb{R} : x^2 + x + 1 \geq 0\right\}$

$$\Delta = 1^2 - 4(1)(1) = -3$$

Since the discriminant < 0, the function does not touch the x-axis. Also, since $a > 0$, the parabola is entirely above the x-axis, making this set contain all real numbers. Therefore, there is no upper or lower bound, which in turn means there is no supremum or infimum.

E

Let S be a non-empty subset of $\mathbb{R}$, and let $\alpha \in \mathbb{R}$ be an upper bound for S. Prove that α is the supremum of S if and only if for every $\varepsilon > 0$, there exists $x \in S$ such that $x > \alpha - \varepsilon$.
Formulate an analogous characterisation of the infimum.

Want to show:

$$\alpha = \sup(S) \iff \forall \varepsilon > 0, \exists x \in S \quad \text{s.t} \quad x > \alpha - \varepsilon$$

Proof.

($\Rightarrow$) Assume $\alpha = \sup(S)$.

Then, by definition, α is the smallest possible upper bound of S.
Assume $\varepsilon > 0$. Then $\alpha - \varepsilon < \alpha$.
However, since α is the smallest possible upper bound, then $\alpha - \varepsilon$ is not an upper bound. In turn, this means that $\exists x \in S \quad \text{s.t} \quad x > \alpha - \varepsilon$

($\Leftarrow$) Solve by contradiction.
Suppose that $\forall \varepsilon > 0, \exists x \in S \quad \text{s.t} \quad x > \alpha - \varepsilon$,
For sake of contradiction, assume that $\alpha \neq \sup(S)$
Then, there exists a b such that b is also an upper bound of S, and $b \leq \alpha$.
Choose $\varepsilon = \alpha - b$
Then,

$$
\begin{aligned}
&\exists x \in S \quad \text{s.t} \quad x > \alpha - \varepsilon \\
\Longrightarrow &\exists x \in S \quad \text{s.t} \quad x > \alpha - (\alpha - b) \\
\Longrightarrow &\exists x \in S \quad \text{s.t} \quad x > b
\end{aligned}
$$

However, we stated that b is an upper bound of S, meaning that there cannot be an element in S that is greater than b. Therefore, our assumption is wrong, and $x = \sup(S)$.

Therefore, $\alpha = \sup(S) \iff \forall \varepsilon > 0, \exists x \in S : x > \alpha - \varepsilon$. ■.

Analogous characterisation of the infimum.

$$\alpha = \sup(S) \iff (\forall \varepsilon > 0,\ \exists x \in S \quad \text{s.t} \quad x < \alpha + \varepsilon)$$

F

Let f be a function defined on an open interval containing a, and suppose that f is continuous at a with $f(a) > 0$. Using the precise definition of the limit, show that there exists an open interval centred at a such that $f(x) > 0$ for all x in that interval.

If $f(x)$ is continuous at a, then:

$$\lim_{x \to a} f(x) = f(a)$$

Therefore, with the definition of a limit, we get:

$$\forall \varepsilon > 0, \exists \delta > 0 \quad \text{s.t} \quad 0 < |x - a| < \delta \Longrightarrow |f(x) - f(a)| < \varepsilon$$

Choose $\varepsilon = f(a)$. (Note $\varepsilon > 0$) Then:

$$
\begin{aligned}
&\exists \delta > 0 \quad \text{s.t} \quad 0 < |x - a| < \delta \Longrightarrow |f(x) - f(a)| < f(a) \\
\Longrightarrow &\exists \delta > 0 \quad \text{s.t} \quad 0 < |x - a| < \delta \Longrightarrow -f(a) < f(x) - f(a) < f(a) \\
\Longrightarrow &\exists \delta > 0 \quad \text{s.t} \quad 0 < |x - a| < \delta \Longrightarrow 0 < f(x) < 2f(a) \\
\Longrightarrow &\exists \delta > 0 \quad \text{s.t} \quad 0 < |x - a| < \delta \Rightarrow f(x) \in (0, 2f(a))
\end{aligned}
$$

Since $f(x)$ is continuous at a, then x can equal a. Therefore:

$$
\begin{aligned}
&\exists \delta > 0 \quad \text{s.t} \quad |x - a| < \delta \Longrightarrow f(x) \in (0, 2fa) \\
\Longrightarrow &\exists \delta > 0 \quad \text{s.t} \quad -\delta < x - a < \delta \Longrightarrow f(x) \in (0, 2f(a)) \\
\Longrightarrow &\exists \delta > 0 \quad \text{s.t} \quad a - \delta < x < a + \delta \Longrightarrow f(x) \in (0, 2f(a)) \\
\Longrightarrow &\exists \delta > 0 \quad \text{s.t} \quad x \in (a - \delta, a + \delta) \Longrightarrow f(x) \in (0, 2f(a))
\end{aligned}
$$

Since $f(a) > 0$, then $(\forall y \in (0, 2f(a)), y > 0)$. Therefore, there exists an open interval centered around a, such that $f(x) > 0$ for all x in that interval.