Mathematics misc.

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Question.

Let A, B, C be 3×3 matrices such that A is symmetric and B and C are skew-symmetric. Consider the statements.
Statement 1 : $A^{13}B^{26} - B^{26}A^{13}$ is symmetric.
Statement 2 : $A^{26}C^{13} - C^{13}A^{26}$ is symmetric.

Solution:
Let,

$$P = A^{13}B^{26} - B^{26}A^{13}$$

Then,

$$P^T = (A^{13}B^{26} - B^{26}A^{13})^T$$

$$P^T = (B^{26})^T(A^{13})^T - (A^{13})^T(B^{26})^T$$

$$P^T = B^{26}A^{13} - A^{13}B^{26}$$

Therefore,

$$P^T = -P$$

Hence, P is skew-symmetric and **Statement 1 is false.**
Now,

$$P = A^{26}C^{13} - C^{13}A^{26}$$

$$P^T = (A^{26}C^{13} - C^{13}A^{26})^T$$

$$P^T = (C^{13})^T(A^{26})^T - (A^{26})^T(C^{13})^T$$

$$P^T = -C^{13}A^{26} - A^{26}(-C^{13})$$

$$P^T = A^{26}C^{13} - C^{13}A^{26}$$

Therefore, $P^T = P$ and hence P is symmetric and Statement 2 is true.

Question.

A unit vector is orthogonal to $5\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$ and is coplanar to $2\hat{\imath} + \hat{\jmath} + \hat{k}$ and $\hat{\imath} - \hat{\jmath} + \hat{k}$ then the vector is ?

Let the vector $\vec{a}$ be,

$$\vec{a} = \alpha(2\hat{\imath} + \hat{\jmath} + \hat{k}) + \beta(\hat{\imath} - \hat{\jmath} + \hat{k})$$

Since $\vec{a}$ is a unit vector.

$$|\vec{a}| = (2\alpha + \beta)^2 + (\alpha - \beta)^2 + (\alpha + \beta)^2 = 1$$

$$6\alpha^2 + 4\alpha\beta + 3\beta^2 = 1 \tag{1}$$

Now, since $\vec{a}$ is orthogonal to $5\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$

$$\vec{a}.(5\hat{\imath} + 2\hat{\jmath} + 6\hat{k}) = 0$$

$$5(2\alpha + \beta) + 2(\alpha - \beta) + 6(\alpha + \beta) = 0$$

$$18\alpha + 9\beta = 0$$

$$\beta = -2\alpha$$

Therefore,

$$6\alpha^2 - 8\alpha^2 + 6\alpha^2 = 1$$

$$\alpha = \pm\frac{1}{\sqrt{10}}$$

$$\beta = \mp\frac{2}{\sqrt{10}}$$

Hence, the required vector is,

$$\boxed{\vec{a} = \frac{3\hat{\imath} + \hat{k}}{\sqrt{10}} \text{ or } \frac{3\hat{\jmath} - \hat{k}}{\sqrt{10}}}$$

Question.

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors such that $\vec{a} + 2\vec{b} + 2\vec{c} = 0$, then $|\vec{a} \times \vec{c}|$ is equal to ?

Here,

$$\vec{a} + 2\vec{c} = -2\vec{b}$$

Squaring both sides,

$$|\vec{a}|^2 + 4|\vec{c}|^2 + 4\vec{a}.\vec{c} = 4|\vec{b}|$$

Since, they are all unit vectors,

$$1 + 4 + 4\cos\theta = 4$$

$$\cos\theta = -\frac{1}{4}$$

Therefore,

$$\sin\theta = \sqrt{1-(-\frac{1}{4})^2}$$

$$\sin\theta = \frac{\sqrt{15}}{4}$$

Hence,

$$\boxed{|\vec{a}\times\vec{c}| = \frac{\sqrt{15}}{4}}$$

Question.

If $z = 4 + i\sqrt{7}$, then find the value of $z^3 - 4z^2 - 9z + 91$.

We have,

$$z = 4 + i\sqrt{7}$$

$$z - 4 = i\sqrt{7}$$

Squaring,

$$z^2 + 16 - 8z = -7$$

$$z^2 - 8z + 23 = 0 \tag{1}$$

Now, to find the value of $z^3 - 4z^2 - 9z + 91$, we can divide it by (1) and then the remainder will be the answer.

$$z^3 - 4z^2 - 9z + 91 = (z^2 - 8z + 23)(z + 4) - 1$$

Now, since $z^2 - 8z + 23 = 0$,

$$\boxed{z^3 - 4z^2 - 9z + 91 = -1}$$

Question.

Find the value of $(1+i)^6 + (1-i)^6$.

$$z = (1+i)^6 + (1-i)^6 = [(1+i)^2]^3 + [(1-i)^2]^3$$

$$z = (2i)^3 + (-2i)^3$$

Therefore,

$$\boxed{z = (8-8)i^3 = 0}$$

Question.

Consider two A.P.'s:

$$S_1 : 2, 7, 12, 17, \ldots 500 \text{ terms}$$

$$S_2 : 1, 8, 15, 22, \ldots 300 \text{ terms}$$

Find the no.of common terms. Also find the last common term.

Last terms of the respective A.P.'s are,

$$l_1 = 2 + 5(499) = 2497$$

$$l_2 = 1 + 7(299) = 2094$$

And we have $d_1 = 5$ and $d_2 = 7$. Therefore, the common difference of the series of common terms will be, $d = d_1 d_2 = 35$.

By observation, we can notice that the first common term is 22.

$$S_c = 22, 57, 92, \ldots$$

The maximum term can be 2094.

$$t_n = 22 + 35(n-1) = 2094$$

$$n - 1 = \frac{2072}{35}$$

$$n = 60$$

So, there are 60 common terms between the two series . And the last term is $22 + 35(59) = \boxed{2087}$.

Question.

Solve?

$$\int \sqrt{a^2 - x^2}\, dx$$

Let $x = a\sin\theta$, then

$$I = \int a\cos\theta . a\cos\theta \; d\theta$$

$$I = a^2 \int \cos^2\theta \; d\theta$$

$$I = a^2 \int (\frac{1+\cos 2\theta}{2}) \; d\theta$$

$$I = \frac{a^2}{2}\int(1+\cos 2\theta)\, d\theta$$

$$I = \frac{a^2\theta}{2} + \frac{a^2}{4}\sin(2\theta) + c$$

$$I = \frac{a^2}{2}(\sin\theta)(\cos\theta) + \frac{a^2}{2}\sin^{-1}(\frac{x}{a}) + c$$

$$\boxed{I = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}(\frac{x}{a}) + c}$$

Question.

If $3\sin\theta + 5\cos\theta = 5$, then find the value of $5\sin\theta - 3\cos\theta$

Given,

$$3\sin\theta + 5\cos\theta = 5$$

Let, $5\sin\theta - 3\cos\theta = x$

Squaring and adding, you wil get some shit.

Question.

Find the value of a for which $a^2 - 6\sin x - 5a \le 0, \forall\, x \in \mathbb{R}$

$$\frac{a^2-5a}{6} \le \sin x\ \forall\ x \in \mathbb{R}$$

So,

$$\frac{a^2-5a}{6} \le -1$$

$$a^2 - 5a + 6 \le 0$$

$$(a-2)(a-3) \le 0$$

Therefore,

$$a \in [2,3]$$

Question.

Find the range of

$$f(x) = \sin^2 x - 3\sin x + 2$$

$$f(x) = (\sin x - \frac{3}{2})^2 - \frac{1}{4}$$

Now,

$$\sin x \in [-1,1]$$

$$\sin x - \frac{3}{2} \in [-\frac{5}{2}, -\frac{1}{2}]$$

$$(\sin x - \frac{3}{2})^2 \in [\frac{1}{4}, \frac{25}{4}]$$

$$(\sin x - \frac{3}{2})^2 - \frac{1}{4} \in [0,6]$$

$$\boxed{f(x) \in [0,6]}$$

Question.

If $(x+iy)^5 = p+iq$, then prove that $(y+ix)^5 = q+ip$

We have,

$$(x = iy)^5 = p + iq$$

Taking conjugate on both sides,

$$\overline{(x+iy)^5} = \overline{p+iq}$$

$$(x-iy)^5 = p - iq$$

$$i^5(x-iy)^5 = i^5 p - i^6 q$$

$$\boxed{(y+ix)^5 = q+ip}$$

Question.

Two vertices of a triangle are (3,-2) and (-2, 3) and its orthocentre is (-6, 1). The coordinates of its third vertex are-

Let, $B \equiv (3,-2)$ and $C \equiv (-2,3)$ and the orthocentre be H and the third vertex be, $A \equiv (\alpha,\beta)$. Since, $AH \perp BC$,

$$\frac{\beta-1}{\alpha=6} = 1$$

$$\beta - \alpha = 7 \qquad \text{(i)}$$

Now, since orthocentre is the concurrency of all the altitudes of the triangle, then $CH \perp AB$.

$$\frac{\beta+2}{\alpha-3} = -(\frac{-4}{-2})$$

$$\beta + 2 = -2\alpha - 6$$

$$2\alpha + \beta = 4 \qquad \text{(ii)}$$

Now, solving (i) and (ii),

$$\alpha = -1\,;\beta = 6$$

Therefore the third vertex is, $\boxed{A \equiv (-1, 6)}$

Question.

Possible value of $\tan(\frac{1}{4}\sin^{-1}(\frac{\sqrt{63}}{8}))$.

Let

$$T = \tan(\frac{1}{4}\sin^{-1}(\frac{\sqrt{63}}{8}))$$

$$T = \tan(\frac{1}{4}\cos^{-1}(\frac{1}{8}))$$

Let, $\cos^{-1}(\frac{1}{8}) = \theta$

$$T = \tan(\frac{\theta}{4})$$

Now,

$$\cos\theta = \frac{1}{8}$$

$$2\cos^2\frac{\theta}{2} - 1 = \frac{1}{8} \implies \cos(\frac{\theta}{2}) = \frac{9}{16}$$

$$\cos(\frac{\theta}{2}) = \frac{1 - \tan^2\frac{\theta}{4}}{1 + \tan^2\frac{\theta}{4}} = \frac{9}{16}$$

Therefore,

$$\boxed{T = \tan(\frac{\theta}{4}) = \frac{1}{\sqrt{7}}}$$

Question.

Let $\vec{a} = 4\hat{\imath} + 3\hat{\jmath} + 5\hat{k}$ and $\vec{b} = \hat{\imath} + 2\hat{\jmath} - 4\hat{k}$. Let $\vec{b}_1$ be parallel to $\vec{a}$ and $\vec{b}_2$ be perpendicular to $\vec{a}$. If $\vec{b} = \vec{b}_1 + \vec{b}_2$, then the value of $5\vec{b}_2.(\hat{\imath} + \hat{\jmath} + \hat{k})$ is-

Since, $\vec{b}_1$ is parallel to $\vec{a}$, we can write

$$\vec{b}_1 = \lambda\vec{a}$$

Now,

$$\vec{b}.\vec{a} = (\vec{b}_1 + \vec{b}_2).\vec{a} \implies \vec{b}_1.\vec{a} + \vec{b}_2.\vec{a}$$

$$4 + 6 - 20 = \lambda|\vec{a}|^2 + 0$$

$$\lambda = -\frac{10}{50} = -\frac{1}{5}$$

Now,

$$\vec{b} = -\frac{1}{5}\vec{a} + \vec{b}_2$$

$$5\vec{b}_2 = 5\vec{b} + \vec{a} = 9\hat{\imath} + 13\hat{\jmath} - 15\hat{k}$$

Therefore,

$$\boxed{5\vec{b}.(\hat{\imath} + \hat{\jmath} + \hat{k}) = 7}$$

Question.

Let f, g and h be real valued functions defined on $\mathbb{R}$ as, $f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0; \\ 1, & x = 0;. \end{cases}$,

$$g(x) = \begin{cases} \frac{\sin(x+1)}{x+1}, & \text{if } x \neq -1; \\ 1, & \text{if } x = -1; \end{cases}$$

and $h(x) = 2[x] - f(x)$ where $[x]$ if G.I.F. Then the value of $\lim_{x\to 1} g(h(x-1))$ is-

We have,

$$f(x) = \text{sgn}(x)$$

Therefore,

$$h(x) = 2[x] - \text{sgn}(x)$$

And we have to find the value of $h(x) \to 1$.

$$\lim_{x\to 1^-}(h(x-1)) = 2[-1] - (-1) = -1$$

$$\lim_{x\to 1^+}(h(x-1)) = 2[0] - 1 = -1$$

$$\lim_{x\to 1}(h(x-1)) = 2[0] - 1 = -1$$

Since, LHL = RHL = L, limit exists at $x \to 1$

$$\lim_{x\to 1}(g(-1)) = \lim_{x\to 1}\frac{\sin(x+1)}{x+1} = 1$$

$$(\because x + 1 \to 0 \text{ when } x \to -1)$$

Hence,

$$\boxed{\lim_{x\to 1} g(h(x-1)) = 1}$$

Question.

Let $f(x)$ be a differentiable function on $[0,2]$ such that $f'(x) = f'(2-x)$ for all $x \in (0,2)$, $f(0) = 1$ and $f(2) = e^2$. Then the value of $\int_0^2 f(x)\,dx$ is-

We have,

$$f'(x) = f'(2-x)$$

Integrating,

$$f(x) = -f(2-x) + c$$

at $x = 0$

$$f(0) = -f(2) + c$$

$$c = 1 + e^2$$

Now,

$$f(x) + f(2-x) = 1 + e^2$$

$$\int_0^2 f(x)dx = \int_0^1 \{f(x) + f(2-x)\}dx$$

Therefore,

$$\boxed{\int_0^2 f(x)dx = 1 + e^2}$$

Question.

Find the locus of midpoint of family of chords $\lambda x + y = 5$ (λ is a parameter) of the parabola $x^2 = 20y$.

Let the equation of family of chords be rewritten as,

$$(y-5) + \lambda(x-0) = 0$$

Which indicate that this family of lines, intersect at $(0,5)$. Which happens to be the focus of the parabola.

Now, let the midpoint be $M \equiv (h,k)$. Then the equation of the chord will be,

$$ky - 10(h+x) = h^2 - 20k$$

Now, since the chord passes through $(0,5)$

$$5k - 10h = h^2 - 20k$$

Therefore the locus will be,

$$\boxed{x^2 + 10x - 25y = 0}$$

Question.

Equation $y^2 + 2y - x + 5 = 0$ represents a parabola. Find its vertex, equation of axis, equation of latus rectum, coordinates of the focus equation of the directrix, extremeties of latus rectum and the length of the latus rectum.

We have,

$$y^2 + 2y + 5 = x$$

$$(y+1)^2 = (x-4)$$

$$(y+1)^2 = 4(\frac{1}{4})(x-4)$$

Therefore, the vertex is $\boxed{(4,-1)}$.

We know that if, the vertex is (h,k), the focus will be $(h+a,k)$ and the equation of directrix will be, $x = -(h+a),k$. Therefore,

$$\text{Focus} = \boxed{(\frac{17}{4},-1)}$$

$$\text{Directrix}: \boxed{x = -\frac{17}{4}}$$

The equation of its axis is the line passing through focus and the vertex.

$$\text{Vertex}: \boxed{y = -1}$$

Now, the latus rectum is the chord perpendicular to the axis and passing through the focus. So, the equation of L.R can be given by,

$$\text{L.R}\ : \boxed{x = \frac{17}{4}}$$

The length of the latus rectum is $4a$ units. Therefore,

$$\boxed{\text{Length of L.R} = 1}$$

The extremeties of the L.R lies $2a$ units above and below the axis of the parabola. Therefore,

$$\text{Extremeties of L.R} = \boxed{\{(\frac{17}{4},-\frac{1}{2}),(\frac{17}{4},-\frac{3}{2})\}}$$

Question.

The parametric equation of a parabola is $x = t^2+1, y = 2t+1$. Find the equation of directrix.

We have,

$$t = \frac{y-1}{2}$$

Substituting it in the equation for x,

$$x = (\frac{y-1}{2})^2 + 1$$

The equation of parabola is,

$$(y-1)^2 = 4(1)(x-1)$$

The directrix is a line perpendicular to the axis of the parabola and is a units away from the vertex of the parabola. Therefore, the equation of directrix will be,

$$\boxed{\text{Directrix}: x = 1-1 \implies x = 0}$$

Question.

Find the equation of parabola having focus at $(1,1)$ and vertex at $(-3,-3)$

From the information we can see that the axis of the parabola is $x - y = 0$. And since the vertex is the mid-point of focus and directrix, the directrix point of intersection of directrix with the axis is $(-7,-7)$.

Now, since the directrix is perpendicular to the axis, the equation of the directrix will be,

$$x + y + \lambda = 0$$

Since, directrix passes through, $(-7,-7)$. The equation of directrix will be,

$$x + y + 14 = 0$$

Now the equation of parabola will be the locus of point which is equidistant to the focus as well as directrix.

$$\sqrt{(x-1)^2 + (y-1)^2} = \frac{|x+y+14|}{\sqrt{2}}$$

We get,

$$\boxed{x^2 + y^2 - 2xy - 32x - 32y - 192 = 0}$$

Which is the equation of the required parabola.

Question.

Find the equation of ellipse whose focus is $S(-1,1)$, the corresponding directrix is $x - y + 3 = 0$ and the eccentricity is 1/2. Also, find its centre, the second focus, the equation of the second directrix, and the length of the latus rectum.

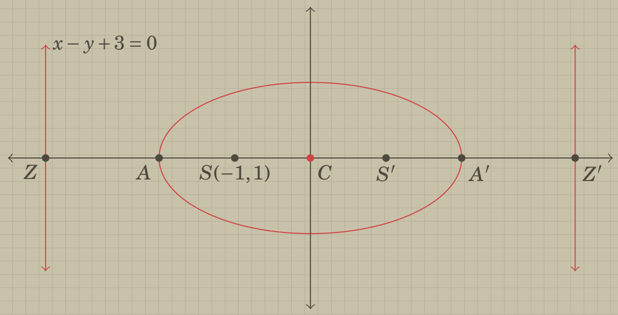
Since the base equation of an ellipse can be defined as,

$$\sqrt{(x-a)^2 + (y-b)^2} = e\frac{|lx+my+n|}{\sqrt{l^2+m^2}}$$

$$(x+1)^2 + (y-1)^2 = \frac{1}{4}\frac{(x-y+3)^2}{2}$$

Therefore, we have,

$$7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0$$



The major axis is perpendicular to the directrix and has the equation $x + y + \lambda = 0$. Since, major axis passes through $(-1,1)$, the equation of major axis will be,

$$x + y = 0 \implies x = -y$$

Therefore, $Z \equiv (-\frac{3}{2}, \frac{3}{2})$

Now, A divides ZS internally in the ratio 2:1,

$$A \equiv (-\frac{7}{6}, \frac{7}{6})$$

And, A' divides ZS externally in the ratio 2:1 i.e $ZA : SA = 2:1$. Therefore,

$$A' \equiv (-\frac{1}{2}, \frac{1}{2})$$

C is the mid-point of AA'.

$$C \equiv (-\frac{5}{6}, \frac{5}{6})$$

C is also the mid-point of ZZ',

$$Z' \equiv (-\frac{1}{6}, \frac{1}{6})$$

Since, the second directrix is parallel to the first directrix and passes through (-1/6, 1/6), its equation will be

$$x - y + \frac{1}{3} = 0$$

And, the length of the Latus rectum is

$$\text{L.R} = 2e \times ZS$$

$$\text{L.R} = 2(\frac{1}{2})\frac{|-1-1+3|}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Question.

Variable complex number z satsifies the equation $|z-1+2i|+|z+3-i| = 10$. Prove that locus of complex number z is an ellipse. Also, find centre, foci and eccentricity of the ellipse.

The expression $|z-(1-2i)|$ represents the distance of z from $1-2i$ in the argand plane. Similarly, $|z-(-3+i)|$ represents the distance of z from $i-3$.

The expression, $|z-1+2i|+|z+3-i| = 10$ conveys that the sum of the distances is constant and is equal to 10, which can only be the case, if z is an ellipse.

Here, the foci are, $S_1(1,-2)$ and $S_2(-3,1)$. And the centre is, $(-1,-\frac{1}{2})$

Distance between foci is $2c = 2ae$. So,

$$2ae = (10)e = 5$$

$$e = \frac{1}{2}$$

Question.

Integrate

$$\int \frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}}$$

$$I = \int \frac{dx}{(x-1)^{3/4}(x+2)^{5/4}}$$

Multiplying and dividng by, $(x-1)^{\frac{5}{4}}$

$$I = \int \frac{dx}{(\frac{x+2}{x-1})^{5/4}(x-1)^2}$$

Now, let $t = \dfrac{x+2}{x-1}$

$$I = -\frac{1}{3}\int \frac{dt}{t^{5/4}}$$

$$\boxed{I = \frac{4}{3}(\frac{x-1}{x+2})^{1/4} + c}$$

Question.

The value of $\sum_{r=0}^{22} {}^{22}C_r\, {}^{23}C_r$

Somehow,

$$\sum_{r=0}^{22} {}^{22}C_r\, {}^{23}C_r = \sum_{r=0}^{22} {}^{22}C_r\, {}^{23}C_{23-r}$$

Therefore,

$$\sum_{r=0}^{22} {}^{22}C_r\, {}^{23}C_{23-r} = \boxed{{}^{45}C_{23}}$$

This result is due to an identity called the Van Der Mond Identity. Which is as follows,

$${}^{m+n}C_k = \sum_{r=0}^{k} {}^{m}C_r\, {}^{n}C_{k-r}$$

Question.

Let A, B, C be three angles such that $A+B+C = \pi$. If $\tan A \tan B = 2$, then find the value of $\dfrac{\cos A \cos B}{\cos C}$.

$$\frac{\cos A \cos B}{\cos C} = -\frac{\cos A \cos B}{\cos(A+B)}$$

$$T = -\frac{\cos A \cos B}{\cos A \cos B - \sin A \sin B} = \frac{1}{\tan A \tan B - 1}$$

$$T = \frac{1}{2-1} = 1$$

Question.

If α, β, γ are three consecutive terms of a non-constant G.P such that the equations $\alpha x^2 + 2\beta x + \gamma = 0$ and $x^2 + x - 1 = 0$ have a common root, then $\alpha(\beta+\gamma)$ is equal to ?

Given that,

$$\alpha, \beta, \gamma = a, ar, ar^2$$

$$ax^2 + 2\beta x + \gamma = ax^2 + 2arx + ar^2 = 0$$

$$\implies x^2 + 2rx + r^2 = 0 \qquad \text{(i)}$$

Which has a common root with,

$$x^2 + x - 1 = 0 \qquad \text{(ii)}$$

Subtracting, (ii) from (i)

$$(2r-1)x + r^2 - 1 = 0$$

$$x = \frac{1-r^2}{2r-1}$$

Now, substituting in (ii)

$$(\frac{1-r^2}{2r-1})^2 + \frac{1-r^2}{2r-1} - 1 = 0$$

$$r^4 - 2r^3 - r^2 + 2r + 1 = 0$$

Dividing, by x^2.

$$r^2 + \frac{1}{r^2} - 2(r - \frac{1}{r}) - 1 = 0$$

$$(r - \frac{1}{r})^2 - 2(r - \frac{1}{r}) + 1 = 0$$

And then u do shit.

Question.

If, $y = \tan^{-1}(\frac{1}{1+x+x^2}) + \tan^{-1}(\frac{1}{x^2+3x+3}) + \tan^{-1}(\frac{1}{x^2+5x+7}) + \cdots$ n terms then find the value of $y'(0)$.

Here,

$$y = \tan^{-1}\frac{(x+1)-1}{1+(x+1)(x)} + \tan^{-1}\frac{(x+2)-(x+1)}{1+(x+2)(x+1)} + \cdots$$

$$y = \tan^{-1}(x+1) - \tan^{-1}x + \tan^{-1}(x+2) - \tan^{-1}(x+1) + \cdots + \tan^{-1}(x+n)$$

$$y = \tan^{-1}(x+n) - \tan^{-1}x$$

$$y'(x) = \frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$$

Therefore,

$$\boxed{y'(0) = \frac{-n^2}{1+n^2+1}}$$

Derivate of inverse of a function.

Let $f(x): \mathbb{R} \to \mathbb{R}$ be one-one and differentiable. Then, let

$$f^{-1}(x) = g(x)$$

$$x = f(g(x))$$

Differentiating,

$$f'(g(x)).g'(x) = 1$$

Then,

$$g'(x) = \frac{1}{f'(g(x))}$$

$$\boxed{(\frac{d}{dx}f^{-1}(x))_{x=a} = \frac{1}{f'(g(a))}}$$

Plane