CSCA67 - Exercises #8

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2.1

For every integer $n \geq 1, n^3 - n$ is divisible by 3.

Let $P(n)$ be $\exists z \in \mathbb{N}, n^3 - n = 3z$.
Base Case

$$\begin{aligned} & n^3 - n \\ \Rightarrow & 1^3 - 1 \\ = & 0 \\ = & 3(0) \end{aligned}$$

Therefore, $P(1)$.

Induction Step

$$\begin{array}{llr}
\text{Let k be arbitrary.} & & (1) \\
\quad \text{Suppose } k \geq 1 & & (2) \\
\quad\quad \text{Suppose } P(k) & \text{Induction Hypothesis} & (3) \\
\quad\quad\quad \exists z \in N, k^3 - k = 3z & \text{Defintion of } P & (4) \\
\quad\quad\quad k^3 - k = 3z & \text{4, E.I} & (5) \\
\quad\quad\quad (k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 & & (6) \\
\quad\quad\quad\quad = \left(k^3 - k\right) + 3k^2 + 3k + 1 - 1 & & \\
\quad\quad\quad\quad = \left(k^3 - k\right) + 3k^2 + 3k & & \\
\quad\quad\quad\quad = 3z + 3k^2 + 3k & \text{IH} & \\
\quad\quad\quad\quad = 3\left(z + k^2 + k\right) & & \\
\quad\quad\quad (k+1)^3 - (k+1) = 3j & \left(z + k^2 + k\right) \in \mathbb{N} & (7) \\
\quad\quad\quad \exists z \in N, (k+1)^3 - (k+1) = 3z & \text{7, E.G} & (8) \\
\quad\quad\quad P(k+1) & \text{Definition of } P & (9) \\
\quad\quad P(k) \to P(k+1) & \text{3, 9, Implication} & (10) \\
\quad k \geq 1 \to (P(k) \to P(k+1)) & \text{2, 10, Implication} & (11) \\
\forall k, k \geq 1 \to (P(k) \to P(k+1)) & \text{1, 12, Implication} & (12)
\end{array}$$

Therefore, $\forall n \geq 1, P(n) \Longrightarrow \forall n \geq 1, \exists z \in \mathbb{N} \quad \text{s.t} \quad n^3 - n = 3z \blacksquare$.

2.2

For every integer $n \geq 4, 2^n < n!$.

Let $P(n)$ be $2^n < n!$.
Base Case

$$\begin{aligned} & 2^n < n! \\ \Rightarrow & 2^4 < 4! \\ & 16 < 24 \\ & T \end{aligned}$$

Therefore, $P(4)$.

Induction Step

$$\begin{array}{llr}
\text{Let k be arbitrary.} & & (1) \\
\quad \text{Suppose } k \geq 4 & & (2) \\
\quad\quad \text{Suppose } P(k) & \text{Induction Hypothesis} & (3) \\
\quad\quad\quad 2^k < k! & \text{Defintion of } P & (4) \\
\quad\quad\quad \left(2^k\right)(k) < (k!)(2) & & (5) \\
\quad\quad\quad 2^{k+1} < (k!)(2) & & (6) \\
\quad\quad\quad\quad\quad < (k!)(k+1) & (k \geq 4) & \\
\quad\quad\quad\quad\quad < (k+1)! & & \\
\quad\quad\quad P(k+1) & \text{Definition of } P & (7) \\
\quad\quad P(k) \to P(k+1) & \text{3, 7, Implication} & (8) \\
\quad k \geq 4 \to (P(k) \to P(k+1)) & \text{2, 8, Implication} & (9) \\
\forall k, k \geq 4 \to (P(k) \to P(k+1)) & \text{1, 9, Implication} & (10)
\end{array}$$

Therefore, $\forall n \geq 4, P(n) \Longrightarrow \forall n \geq 4, 2^n < n!$ ■

2.3

For every integer $n \geq 1$, if $a, r \in \mathbb{R}$ and $r \neq 1$, then $a + ar + ar^2 + \ldots + ar^n = \frac{ar^n + 1 - a}{r - 1}$. Using summation notation, $\sum_{i=0}^{n} ar^n = \dfrac{ar^{n+1} - a}{r - 1}$

Let $P(n)$ be $\sum_{i=0}^{n} ar^n = \dfrac{ar^{n+1} - a}{r - 1}$

Base Case

$$\begin{aligned}
\sum_{i=0}^{n} ar^n &= \frac{ar^n + 1 - a}{r - 1} \\
\sum_{i=0}^{1} ar^n &= \frac{ar^2 - a}{r - 1} \\
a + ar &= \frac{ar^2 - a}{r - 1} \\
a + ar &= \frac{a\left(r^2 - 1\right)}{r - 1} \\
a + ar &= \frac{a(r+1)(r-1)}{r - 1} \\
a + ar &= a(r+1) \\
a + ar &= ar + a \\
& T
\end{aligned}$$

Therefore, $P(1)$.

Induction Step

$$
\begin{aligned}
&\text{Let n be arbitrary.} && (1)\\
&\quad \text{Suppose } n \geq 1 && (2)\\
&\quad\quad \text{Suppose } P(n) & \text{Induction Hypothesis} \quad & (3)\\
&\quad\quad \sum_{i=0}^{n} ar^n = \frac{ar^{n+1}-a}{r-1} & \text{Defintion of } P \quad & (4)\\
&\quad\quad \sum_{i=0}^{n} ar^n + ar^{n+1} = \frac{ar^{n+1}-a}{r-1} + ar^{n+1} && (5)\\
&\quad\quad\quad = \frac{ar^{n+1}-a}{r-1} + \frac{\left(ar^{n+1}\right)(r-1)}{r-1}\\
&\quad\quad\quad = \frac{ar^{n+1}-a+ar^{n+1}(r-1)}{r-1}\\
&\quad\quad\quad = \frac{ar^{n+1}-a+ar^{n+2}-ar^{n+1}}{r-1}\\
&\quad\quad\quad = \frac{ar^{n+2}-a}{r-1}\\
&\quad\quad\quad = \frac{ar^{(n+1)+1}-a}{r-1}\\
&\quad\quad \sum_{i=0}^{n+1} ar^n = \frac{ar^{(n+1)+1}-a}{r-1} && (6)\\
&\quad\quad P(n+1) & \text{Definition of } P \quad & (7)\\
&\quad P(n) \to P(n+1) & 3, 7, \text{Implication} \quad & (8)\\
&\quad n \geq 1 \to (P(n) \to P(n+1)) & 2, 8, \text{Implication} \quad & (9)\\
&\forall n, n \geq 1 \to (P(n) \to P(n+1)) & 1, 9, \text{Implication} \quad & (10)
\end{aligned}
$$

Therefore, $\forall n \geq 1, P(n) \implies \forall n \geq 1, \sum_{i=0}^{n} ar^n = \frac{ar^{n+1}-a}{r-1}$ ■.

2.4

For every integer $n \geq 2, \sum_{i=0}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}$

Let $P(n)$ be $\sum_{i=0}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}$.

Base Case

$$
\begin{aligned}
&\sum_{i=0}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}\\
&\sum_{i=0}^{2} \frac{1}{i^2} < 2 - \frac{1}{2}\\
&1 + \frac{1}{2^2} < 2 - \frac{1}{2}\\
&1 + \frac{1}{4} < 2 - \frac{1}{2}\\
&\frac{5}{4} < \frac{3}{2}\\
&T
\end{aligned}
$$

Therefore, $P(2)$.

Induction Step

$$
\begin{aligned}
&\text{Let n be arbitrary.} && && (1)\\
&\quad \text{Suppose } n \geq 2 && && (2)\\
&\quad\quad \text{Suppose } P(n) && && \text{Induction Hypothesis} \quad (3)\\
&\quad\quad\quad \sum_{i=0}^{n} \frac{1}{i^2} < 2 - \frac{1}{n} && && \text{Defintion of } P \quad (4)\\
&\quad\quad\quad \sum_{i=0}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2} < 2 - \frac{1}{n} + \frac{1}{(n+1)^2} && && (5)\\
&\quad\quad\quad\quad = 2 - \frac{(n+1)^2 + n}{n(n+1)^2} && &&\\
&\quad\quad\quad\quad < 2 - \frac{(n+1)}{n(n+1)^2} && && n \geq 2\\
&\quad\quad\quad\quad = 2 - \frac{1}{n(n+1)} && &&\\
&\quad\quad\quad\quad < 2 - \frac{1}{n+1} && && n \geq 2\\
&\quad\quad\quad \sum_{i=0}^{n+1} \frac{1}{i^2} < \frac{1}{n+1} && && (6)\\
&\quad\quad\quad P(n+1) && && \text{Definition of } P \quad (7)\\
&\quad\quad P(n) \rightarrow P(n+1) && && 3, 7, \text{ Implication} \quad (8)\\
&\quad n \geq 2 \rightarrow (P(n) \rightarrow P(n+1)) && && 2, 8, \text{ Implication} \quad (9)\\
&\forall n, n \geq 2 \rightarrow (P(n) \rightarrow P(n+1)) && && 1, 9, \text{ Implication} \quad (10)
\end{aligned}
$$

Therefore, $\forall n \geq 1, P(n) \Longrightarrow \forall n \geq 2, \sum_{i=0}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2}$ ■.