MATA31 - Assignment #3

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Textbook Questions

Question 26.

Determine whether the statement: There exist real numbers x and y such that $x + y = 4$ is true Justify your answer.

Since this statement uses the phrase "there exists", only one solution is necessary to make this statement true. Let $x = 1, y = 3$, then:

$$\begin{aligned} x + y &= 4 \\ \Rightarrow 1 + 3 &= 4 \\ \Rightarrow 4 &= 4 \end{aligned}$$

$\therefore$ The statement is true. $\blacksquare$

Question 50.

Suppose A and B represent logical statements, write the converse and contrapositive of: $\neg A \Rightarrow \neg B$.

$$\begin{aligned} \neg A &\Rightarrow \neg B \\ \text{Converse: } \neg B &\Rightarrow \neg A \\ \text{Contrapositive: } B &\Rightarrow A \end{aligned}$$

Question 62.

Write the converse and contrapositive of: If $x < -2$, then $|x| = -x$. Provide counterexamples if the original, the converse, and/or the contrapositive statements are false.

Original: If $x < -2$, then $|x| = -x$.

Converse: If $|x| = -x$, then $x < -2$.

Contrapositive: If $|x| \neq -x$, then $x \geq -2$.

The original statement is true, since if $x < -2$, then x is negative, and the absolute value of a negative number is its negation.
The converse is false, since if $x = -1$, then $|x| = -x$ but $x \not< -2$.
The contrapositive is true, since if $|x| \neq -x$, then x is positive or zero, which means $x \geq -2$.

Question 64.

Write the converse and contrapositive of: If x is positive and rational, then $x - 1$ is positive and rational.

Original: If x is positive and rational, then $x - 1$ is positive and rational.

Converse: If $x - 1$ is positive and rational, then x is positive and rational.

Contrapositive: If $x - 1$ is not positive or not rational, then x is not positive or not rational.

The original statement is true, since if x is positive and rational, then $x-1$ is also positive and rational. The converse is false, since if $x-1=0$, then $x=1$, which is positive and rational, but if $x-1=-1$, then $x=0$, which is not positive.
The contrapositive is true, since if $x-1$ is not positive, then $x \leq 1$, and if $x-1$ is not rational, then x is not rational.

Question 86.

Using the definition of absolute value and systems of inequalities, prove that for any real numbers x and c, and for any positive real number δ, the statement: $|x-c|<\delta \iff x \in (c-\delta, c+\delta)$

$$\text{We must prove that } |x-c|<\delta \iff x \in (c-\delta, c+\delta),$$

For all real numbers x, c, and all positive real numbers δ.

Proof.
Let $\delta > 0$ be arbitrary.

($\Rightarrow$) Assume $|x-c|<\delta$.

$$\begin{aligned} |x-c|<\delta &\equiv -\delta < x-c < \delta && \text{(by properties of absolute values)} \\ &\equiv c-\delta < x < c+\delta && \text{(by algebra)} \end{aligned}$$

Thus $x \in (c-\delta, c+\delta)$.

($\Leftarrow$) Assume $x \in (c-\delta, c+\delta)$.

$$\begin{aligned} x \in (c-\delta, c+\delta) &\equiv c-\delta < x < c+\delta && \text{(by definition of open interval)} \\ &\equiv -\delta < x-c < \delta && \text{(by algebra)} \\ &\equiv |x-c|<\delta && \text{(by properties of absolute values)} \end{aligned}$$

Thus $|x-c|<\delta$.

Since both directions hold, we conclude that

$$|x-c|<\delta \iff x \in (c-\delta, c+\delta).$$

As required to show. ■

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2. Non-textbook Questions

1.A

Prove that $\log_k(ab) = \log_k(a) + \log_k(b), k>0, k \neq 1$ and a, b are any two positive numbers.

$$\text{We must prove that } \log_k(ab) = \log_k(a) + \log_k(b)$$

For any real positive numbers a, b, and any k, such that $k>0, k \neq 1$.

Proof.
Let $a>0, b>0$ and $k>0, k \neq 1$ be arbitrary.

Let $x = \log_k(a)$, and let $y = \log_k(b)$
Then by definitions of logarithms:

$$k^x = a \quad \text{and} \quad k^y = b.$$

Therefore,

$$\begin{aligned}
ab &= k^x \cdot k^y && \text{(Definition of } x = \log_k(a) \text{ and } y = \log_k(b)\text{)} \\
&= k^{x+y} && \text{(By properties of exponents.)} \\
\log_k(ab) &= \log_k\left(k^{x+y}\right) && \text{(By algebra)} \\
&= x + y && \text{(By properties of logarithms)} \\
&= \log_k(a) + \log_k(b) && \text{(Substitute } x = \log_k(a),\ y = \log_k(b)\text{)}
\end{aligned}$$

As required to show. ■.

A.2

Provide a counter-example to show that the relation

$$\log_k(a+b) = \log_k(a) + \log_k(b)$$

is not true.

Proof.
Choose $a = 625, b = 25, k = 5$.
Then,

$$\begin{aligned}
\log_k(a+b) &= \log_5(625+25) && = \log_5(650) \approx 4.02437, \\
\log_k(a) + \log_k(b) &= \log_5(625) + \log_5(25) && = 4 + 2 = 6.
\end{aligned}$$

Therefore, the relation

$$\log_k(a+b) = \log_k(a) + \log_k(b)$$

is not true, as required to show. ■

2

Given that $0 \le a \le b$, show that

$$a \le \sqrt{ab} \le \tfrac{a+b}{2} \le b$$

Proof.
Let $0 \le a \le b$
Then:

$$\begin{aligned}
a \le b \ &\equiv\ a^2 \le ab && \text{(by algebra)} \\
&\equiv\ \sqrt{a^2} \le \sqrt{ab} && \text{(by algebra)} \\
&\equiv\ a \le \sqrt{ab} && \text{(by algebra)}
\end{aligned}$$

Note that since $\sqrt{x}$ is a positive increasing function and $a, b \ge 0$, then $\sqrt{ab} \ge 0$, meaning the inequality is preserved.

Next:

$$\begin{aligned}
a \le b &\equiv b - a \ge 0 && \text{(by algebra)}\\
&\equiv (b-a)^2 \ge 0 && \text{(squaring both sides)}\\
&\equiv b^2 - 2ab + a^2 \ge 0 && \text{(binomial expansion)}\\
&\equiv b^2 + a^2 \ge 2ab && \text{(by algebra)}\\
&\equiv b^2 + 2ab + a^2 \ge 4ab && \text{(adding 2ab to both sides)}\\
&\equiv (a+b)^2 \ge 4ab && \text{(factoring)}\\
&\equiv \frac{(a+b)^2}{4} \ge ab && \text{(by algebra)}\\
&\equiv \sqrt{\frac{(a+b)^2}{4}} \ge \sqrt{ab} && \text{(square root of both sides)}\\
&\equiv \frac{a+b}{2} \ge \sqrt{ab} && \text{(by algebra)}\\
&\equiv \sqrt{ab} \le \frac{(a+b)}{2}
\end{aligned}$$

Note that since $\sqrt{x}$ is a positive increasing function and $a, b \ge 0$, then the square root is nonnegative, meaning the inequality is preserved.

Finally:

$$\begin{aligned}
a \le b &\equiv a + b \le b + b && \text{(by algebra)}\\
&\equiv a + b \le 2b && \text{(by algebra)}\\
&\equiv \frac{a+b}{2} \le b && \text{(by algebra)}
\end{aligned}$$

Given that $a \le \sqrt{ab}$, $\sqrt{ab} \le \frac{(a+b)}{2}$, and $\frac{a+b}{2} \le b$: By properties of inequalities we can state that:

$$a \le \sqrt{ab} \le \tfrac{a+b}{2} \le b$$

As required to prove. ■.

3.A

Prove or disprove (by counterexample):

Let $x, y \in \mathbb{R}$. If $x, y \in \mathbb{Q}$, then $x + y \in \mathbb{Q}$

We will prove this statement.

Proof.
Let $x, y \in \mathbb{Q}$.
Then, by definition of rational numbers, there exist integers $a, b, c, d \in \mathbb{Z}$ with $b \neq 0$ and $d \neq 0$ such that

$$x = \frac{a}{b} \quad \text{and} \quad y = \frac{c}{d}.$$

Consider the sum:

$$
\begin{aligned}
x+y &\equiv \frac{a}{b}+\frac{c}{d} && \text{(substitute x and y)}\\
&\equiv \frac{ad}{bd}+\frac{bc}{bd} && \text{(algebra)}\\
&\equiv \frac{ad+bc}{bd} && \text{(algebra)}
\end{aligned}
$$

Since $a,b,c,d \in \mathbb{Z}$, both the numerator $ad+bc$ and the denominator bd are integers, and $bd \neq 0$.
Thus, $x+y$ is a ratio of two integers, which means $x+y \in \mathbb{Q}$. ■

3.B

Prove or disprove (by counterexample):

Let $x,y \in \mathbb{R}$. If $x \notin \mathbb{Q}$ and $y \notin \mathbb{Q}$, then $x+y \notin \mathbb{Q}$

Proof (by counterexample).
We will disprove the statement by providing a counterexample.
Choose

$$x=\sqrt{2} \quad \text{and} \quad y=-\sqrt{2}.$$

Then,

$$x+y=\sqrt{2}+\left(-\sqrt{2}\right)=0.$$

Since $0 \in \mathbb{Q}$, the sum $x+y$ is rational.
Therefore, the statement "If $x \notin \mathbb{Q}$ and $y \notin \mathbb{Q}$, then $x+y \notin \mathbb{Q}$" is not true. ■

3.C

Prove or disprove (by counterexample):

Let $x,y \in \mathbb{R}$. If $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$, then $xy \notin \mathbb{Q}$

Proof (by counterexample).
We will disprove the statement by providing a counterexample.
Choose

$$x=0 \quad \text{and} \quad y=\sqrt{2}.$$

Then,

$$xy=(0)\left(\sqrt{2}\right)=0.$$

Since $0 \in \mathbb{Q}$, the product xy is rational.
Therefore, the statement "If $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$, then $xy \notin \mathbb{Q}$" is not true. ■

3.D

Prove or disprove (by counterexample):

Let $x,y \in \mathbb{R}$. If $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$, then $x+y \notin \mathbb{Q}$

Proof (by contradiction).
Let $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$ be arbitrary.
For the sake of contradiction, assume

$$x+y \in \mathbb{Q}.$$

By definition of rational numbers, there exist integers $a, b, c, d \in \mathbb{Z}$ with $b \neq 0$ and $d \neq 0$ such that

$$x = \frac{a}{b} \quad \text{and} \quad x + y = \frac{c}{d}.$$

Then, solving for y, we have:

$$\begin{aligned} y &= (x+y) - x && \text{(algebra)} \\ &= \frac{c}{d} - \frac{a}{b} && \text{(substitute } x \text{ and } x+y\text{)} \\ &= \frac{bc - ad}{bd} && \text{(common denominator)} \end{aligned}$$

Since $a, b, c, d \in \mathbb{Z}$, both the numerator $bc - ad$ and the denominator bd are integers, and $bd \neq 0$.
Hence, $y = \frac{bc-ad}{bd} \in \mathbb{Q}$, which contradicts our assumption that $y \notin \mathbb{Q}$.
Therefore, our assumption is false, and the statement

$$\text{If } x \in \mathbb{Q} \text{ and } y \notin \mathbb{Q}, \text{ then } x + y \notin \mathbb{Q}$$

is true. ■