CSCA67 - Assignment #1

Satyajit Datta

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1. Logical Equivalence

For each of the following pairs of expressions, either prove that the two expressions are equivalent or prove that they are not. (Clearly state what you are proving!) Do not use truth tables.

Question.

$(a \to b) \wedge (b \to c)$ and $a \to c$

We are proving that the two expressions are not equivalent.
Consider the case where a = True, b = False, c = True.

$$\begin{aligned} & (a \to b) \wedge (b \to c) \\ \Rightarrow\ & (T \to F) \wedge (F \to T) \\ \Rightarrow\ & F \wedge T \\ \Rightarrow\ & \mathbf{F} \end{aligned}$$

$$\begin{aligned} & a \to c \\ \Rightarrow\ & T \to T \\ \Rightarrow\ & \mathbf{T} \end{aligned}$$

$\therefore$ The two expressions are not equivalent. ■

Question.

$a \wedge (a \to b)$ and $a \to b$

We are proving that the two expressions are not equivalent.
Consider the case where a = False, b = True

$$\begin{aligned} & a \wedge (a \to b) \\ \Rightarrow\ & F \wedge (F \to T) \\ \Rightarrow\ & F \wedge T \\ \Rightarrow\ & \mathbf{F} \end{aligned}$$

$$\begin{aligned} & a \to b \\ \Rightarrow\ & F \to T \\ \Rightarrow\ & \mathbf{T} \end{aligned}$$

$\therefore$ The two expressions are not equivalent. ■

Question.

$(a \to b) \wedge (a \to c)$ and $a \to (b \wedge c)$

We are proving that the two expressions **are** equivalent.

$$\begin{aligned} & (a \to b) \wedge (a \to c) \\ \Rightarrow\ & (\neg a \vee b) \wedge (\neg a \vee c) \quad \text{(Conditional Law)} \\ \Rightarrow\ & \neg \mathbf{a} \vee (\mathbf{b} \wedge \mathbf{c}) \quad \text{(Distributive Law)} \end{aligned}$$

$$\begin{aligned} & a \to (b \wedge c) \\ \Rightarrow\ & \neg \mathbf{a} \vee (\mathbf{b} \wedge \mathbf{c}) \quad \text{(Conditional Law)} \end{aligned}$$

$\therefore$ The two expressions are equivalent. ■

Question.

$(a \to c) \lor (b \to c)$ and $(a \lor b) \to c$

We are proving that the two expressions **are** equivalent.

$$\begin{aligned}
&(a \to c) \lor (b \to c) \\
&\Rightarrow (\neg a \lor c) \land (\neg b \lor c) \quad \text{(Conditional Law)} \\
&\Rightarrow (\neg \mathbf{a} \land \neg \mathbf{b}) \lor \mathbf{c} \quad \text{(Distributive Law)}
\end{aligned}$$

$$\begin{aligned}
&(a \lor b) \to c \\
&\Rightarrow \neg(a \lor b) \lor c \quad \text{(Conditional Law)} \\
&\Rightarrow (\neg \mathbf{a} \land \neg \mathbf{b}) \lor \mathbf{c} \quad \text{(De Morgan's Theorem)}
\end{aligned}$$

$\therefore$ The two expressions are equivalent. ■

Question.

$a \iff b$ and $(a \land b) \lor (\neg a \land \neg b)$

$$\begin{aligned}
&a \iff b \\
&\Rightarrow (a \to b) \land (b \to a) \quad \text{(Biconditional Law)} \\
&\Rightarrow (\neg a \lor b) \land (\neg b \lor a) \quad \text{(Conditional Law)} \\
&\Rightarrow (\neg a \land \neg b) \lor (\neg a \land a) \lor (b \land \neg b) \lor (b \land a) \quad \text{(Distributive Law)} \\
&\Rightarrow (\neg \mathbf{a} \land \neg \mathbf{b}) \lor (\mathbf{a} \land \mathbf{b}) \quad \text{(Negation Law)}
\end{aligned}$$

$$(\mathbf{a} \land \mathbf{b}) \lor (\neg \mathbf{a} \land \neg \mathbf{b})$$

$\therefore$ The two expressions are equivalent. ■

Question.

$a \to (b \to (c \to d))$ and $(a \land b \land c) \to d$

$$\begin{aligned}
&a \to (b \to (c \to d)) \\
&\Rightarrow \neg a \lor (\neg b \lor (\neg c \lor d)) \quad \text{(Conditional Law)} \\
&\Rightarrow (\neg a \lor \neg b \lor \neg c) \lor d \\
&\Rightarrow \neg(a \land b \land c) \lor d \quad \text{(De Morgan's)} \\
&\Rightarrow (\mathbf{a} \land \mathbf{b} \land \mathbf{c}) \to \mathbf{d} \quad \text{(Conditional Law)}
\end{aligned}$$

$$(\mathbf{a} \land \mathbf{b}) \lor (\neg \mathbf{a} \land \neg \mathbf{b})$$

$\therefore$ The two expressions are equivalent. ■

Question.

$(a \to b) \lor (b \to a)$ and $a \iff b$

We are proving that the two expressions are not equivalent.
Consider the case where a = True, b = False.

$$\begin{aligned}
&(a \to b) \land (b \to a) \\
&\Rightarrow (T \to F) \lor (F \to T) \\
&\Rightarrow F \lor T \\
&\Rightarrow \mathbf{T}
\end{aligned}$$

$$\begin{aligned}
&a \iff b \\
&\Rightarrow (a \to b) \land (b \to a) \quad \text{(Biconditional Law)} \\
&\Rightarrow (T \to F) \land (F \to T) \\
&\Rightarrow F \land T \\
&\Rightarrow \mathbf{F}
\end{aligned}$$

$\therefore$ The two expressions are not equivalent. ■

Question.

$a \iff b$ and $\neg a \iff \neg b$

$$\begin{aligned} & a \iff b \\ \Rightarrow\ & (a \to b) \land (b \to a) \quad \text{(Biconditional Law)} \\ \Rightarrow\ & (\neg a \lor b) \land (\neg b \lor a) \quad \text{(Conditional Law)} \end{aligned}$$

$$\begin{aligned} & \neg a \iff \neg b \\ \Rightarrow\ & (\neg a \to \neg b) \land (\neg b \to \neg a) \quad \text{(Biconditional Law)} \\ \Rightarrow\ & (\neg\neg a \lor \neg b) \land (\neg\neg b \lor \neg a) \quad \text{(Conditional Law)} \\ \Rightarrow\ & (a \lor \neg b) \land (b \lor \neg a) \quad \text{(Double Negation law)} \end{aligned}$$

$\therefore$ The two expressions are equivalent. ■