CSCA67 - Assignment #3

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1. Proof Strategies

1.1

We write a | b, read "a divides b", to stand for $\exists k, b = ak$. Prove that if 4 divides $a - b$, then it also divides $a^2 - b^2$.

<u>Proof</u>

$$
\begin{aligned}
&\text{Suppose } 4 \mid (a+b) && (1)\\
&\quad \exists k, a+b = 4k && \text{1, Def of } x \mid y \ (2)\\
&\quad a+b = 4k && \text{2, E.I } (3)\\
&\quad a^2 - b^2 = (a+b)(a-b) && (4)\\
&\qquad\qquad = 4k(a-b) && \text{Math}\\
&\quad \exists k, a^2 - b^2 = 4k && \text{4, E.G } (5)\\
&\quad 4 \mid \left(a^2 - b^2\right) && \text{5, Def of } x \mid y \ (6)\\
&4 \mid (a+b) \to 4 \mid \left(a^2 + b^2\right) && \text{1, 6, Implication } (7)
\end{aligned}
$$

As required to prove. ■.

1.2

For all real numbers x and y, $min(x, y) = (x + y - |x - y|)/2$ and $\max(x, y) = (x + y + |x - y|)/2$

1.3

Recall that x is irrational if there are no integers a and b, such that $x = \frac{a}{b}$. Prove that the third power of any real number is irrational only if the number itself is irrational.

WTS: $x^3 \notin \mathbb{Q} \to x \notin \mathbb{Q}$

<u>Proof.</u>

Suppose $x \in \mathbb{Q}$ (1)

$\exists a, b \in \mathbb{Z} \left(x = \frac{a}{b}\right) \wedge (b \neq 0)$ 1, Def^n of $\mathbb{Q}$ (2)

$\left(x = \frac{a}{b}\right) \wedge (b \neq 0)$ 2, E.I (3)

$x = \frac{a}{b}$ 3, simp. (4)

$b \neq 0$ 3, simp. (5)

$x^3 = \frac{a^3}{b^3}$ 4, math (6)

$c = a^3$ 2, 6 (7)

$d = b^3$ 2, 6 (8)

$d \neq 0$ 7, 5 (9)

$x^3 = \frac{c}{d}$ 6, 7, 8 (10)

$\left(x^3 = \frac{c}{d}\right) \wedge (d \neq 0)$ 9, 10, conj. (11)

$\exists a, b \in \mathbb{Z}, \left(x^3 = \frac{a}{b}\right) \wedge (b \neq 0)$ 11, E.G (12)

$x^3 \in \mathbb{Q}$ 12, Def^n of $\mathbb{Q}$ (13)

$x \in \mathbb{Q} \rightarrow x^3 \in \mathbb{Q}$ 1, 13, imp. (14)

$x^3 \notin \mathbb{Q} \rightarrow x \notin \mathbb{Q}$ 14, contr. (15)

As required to show. ■.

1.4

Prove that if x is rational, then $x + \sqrt{2}$ is not. You may use the fact that $\sqrt{2}$ is irrational, as we proved in class.

Suppose $x \in \mathbb{Q}$		(1)
Suppose $x + \sqrt{2} \in \mathbb{Q}$	For contradiction	(2)
$\exists a, b \in \mathbb{Z}, \left(x = \frac{a}{b}\right) \wedge (b \neq 0)$	1, Def^n of $\mathbb{Q}$	(3)
$\exists a, b \in \mathbb{Z}, \left(x + \sqrt{2} = \frac{a}{b}\right) \wedge (b \neq 0)$	2, Def^n of $\mathbb{Q}$	(4)
$\left(x = \frac{a}{b}\right) \wedge (b \neq 0)$	3, E.I	(5)
$\left(x + \sqrt{2} = \frac{c}{d}\right) \wedge (d \neq 0)$	4, E.I	(6)
$x = \frac{a}{b}$	5, simp.	(7)
$x + \sqrt{2} = \frac{c}{d}$	6, simp.	(8)
$b \neq 0$	5, simp	(9)
$d \neq 0$	6, simp	(10)
$x + \sqrt{2} - x = \frac{a}{b} - \frac{c}{d}$	7, 8	(11)
$= \frac{ad - bc}{bd}$		
$\sqrt{2} = \frac{ad - bc}{bd}$	11, math	(12)
$bd \neq 0$	9, 10	(13)
$\left(\sqrt{2} = \frac{ad - bc}{bd}\right) \wedge (bd \neq 0)$	12, 13, conj.	(14)
$\exists f, g \in \mathbb{Z}, \left(\sqrt{x} = \frac{f}{g}\right) \wedge (g \neq 0)$	14, E.G	(15)
$\sqrt{2} \in \mathbb{Q}$	15, Def^n of $\mathbb{Q}$	(16)
A contradiction		(17)
$x + \sqrt{2} \notin \mathbb{Q}$	2, 16, contradiction	(18)
$x \in \mathbb{Q} \rightarrow x + \sqrt{2} \notin \mathbb{Q}$	1, 17, imp.	(19)

As required to show. ■.

1.5

Using the same definition of rational as above, prove or disprove: If x is irrational, then so is $x + \sqrt{2}$

We will disprove this statement.
Let $x = -\sqrt{2}$
Therefore, x is irrational.
$x + \sqrt{2} = -\sqrt{2} + \sqrt{2} = 0$
0 is rational.

Therefore, we have shown a counterexample where the hypothesis is true and the conclusion is false, therefore this statement is not valid.

2. Induction

2.1

Prove that:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

for all positive integers n.

Base Case.

$$\sum_{i=1}^{n} i^2 = 1$$
$$\frac{(1)(1+1)(2+1)}{6} = 1$$

Therefore, P(1)
Let $P(n)$ be $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.
WTS: $\forall k \geq 0, P(k) \to P(k+1)$

Let k be arbitrary	(1)
Suppose $k \geq 1$	(2)
Suppose P(k)	Induction Hypothesis (3)
$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$	3, Def^n of P (4)
$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$	math (5)
$= \frac{k(k+1)(2k+1)}{6} + (n+1)^2$	IH
$= \frac{k(k+1)(2k+1)}{6} +$	
$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$	
$= \frac{(k+1)(k(2k+1)+6(k+1))}{6}$	
$= \frac{(k+1)\left(2k^2+k+6k+6\right)}{6}$	
$= \frac{(k+1)\left(2k^2+7k+6\right)}{6}$	
$= \frac{(k+1)(k+3)(2k+3)}{6}$	
$= \frac{(k+1)(k+2)(2k+3)}{6}$	
$= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$	
$P(k+1)$	5, Def^n of P (6)
$P(k) \to P(k+1)$	3, 6, implication (7)
$k \geq 1 \to (P(k) \to P(k+1))$	2, 7, implication (8)
$\forall k \geq 1, P(k) \to P(k+1)$	1, 8, U.G (9)

Therefore, $\forall n \geq 1, P(n)$, as required to show. ■.

2.2

Prove that

$$n! > 2^n$$

for all $n \geq 4$

Base Case.

$$4! = 24$$
$$2^4 = 16$$

Therefore, P(4)
Let $P(n)$ be $n! > 2^n$.
WTS: $\forall k \geq 4, P(k) \to P(k+1)$

Let k be arbitrary		(1)
Suppose $k \geq 4$		(2)
Suppose P(k)	Induction Hypothesis	(3)
$(k+1)! = k! \cdot (k+1)$	Math	(4)
$< 2^n \cdot (k+1)$	IH	
$< 2^n \cdot 2$	$2 \leq 4 \leq k$	
$= 2^{n+1}$		
$P(k+1)$	4, Def^n of P	(5)
$P(k) \to P(k+1)$	3, 5, implication	(6)
$k \geq 4 \to (P(k) \to P(k+1))$	2, 6, implication	(7)
$\forall k \geq 4, P(k) \to P(k+1)$	1, 7, U.G	(8)

Therefore, $\forall n \geq 4, P(n)$, as required to show. ■.

2.3

Prove that $3^{2n} - 1$ is divisible by 8, for all positive integers n.

Base Case.

$$3^2 - 1 = 8$$
$$8 \mid 8$$

Therefore, P(1)
Let $P(n)$ be $\exists k, 3^{2n} - 1 = 8k$.
WTS: $\forall k \geq 0, P(k) \to P(k+1)$

Let k be arbitrary		(1)
Suppose $k \geq 1$		(2)
Suppose P(k)	Induction Hypothesis	(3)
$\exists m, 3^{2k} - 1 = 8m$	3, Def^n of P	(4)
$3^{2k} - 1 = 8m$	4, E.I	(5)
$3^{2(k+1)} - 1 = 3^{2k+2} - 1$	math	(6)
$= (9)3^{2k} - 1$		
$= (8+1)3^{2k} - 1$		
$= 8\left(3^{2k}\right) + 3^{2k} - 1$		
$= 8\left(3^{2k}\right) + 3^{2k} - 1$		
$= 8\left(3^{2k}\right) + 8m$	5, IH	
$= 8\left(3^{2k} + m\right)$		
$\exists m, 3^{2(k+1)} - 1 = 8m$	6, E.G	(7)
$P(k+1)$	7, Def^n of P	(8)
$P(k) \to P(k+1)$	3, 8, implication	(9)
$k \geq 1 \to (P(k) \to P(k+1))$	2, 9, implication	(10)
$\forall k \geq 1, P(k) \to P(k+1)$	1, 10, U.G	(11)

Therefore, $\forall n \geq 1, P(n)$, as required to show. ■.

2.4

Prove that

$$\sum_{j=1}^{n} \frac{j}{(j+1)!} \leq 1 - \frac{1}{(n+1)!}$$

for all $n \geq 1$.

Base Case.

$$\sum_{j=1}^{1} \frac{j}{(j+1)!} = \frac{1}{2}$$

$$1 - \frac{1}{(1+1)!} = \frac{1}{2}$$

Therefore, P(1)
Let $P(n)$ be $\sum_{j=1}^{n} \frac{j}{(j+1)!} \leq 1 - \frac{1}{(n+1)!}$.
WTS: $\forall k \geq 0, P(k) \to P(k+1)$

Let k be arbitrary (1)

Suppose $k \geq 1$ (2)

Suppose P(k) Induction Hypothesis (3)

$$\begin{aligned}
\sum_{j=1}^{k+1} \frac{j}{(j+1)!} &= \sum_{j=1}^{k} \frac{j}{(j+1)!} + \frac{k+1}{(k+2)!} && (4)\\
&\leq 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} && \text{IH}\\
&= 1 - \left(\frac{1}{(k+1)!} - \frac{k+1}{(k+2)!}\right)\\
&= 1 - \left(\frac{k+2}{(k+1)!(k+2)} - \frac{k+1}{(k+2)!}\right)\\
&= 1 - \left(\frac{k+2}{(k+2)!} - \frac{k+1}{(k+2)!}\right)\\
&= 1 - \frac{k+2-(k+1)}{(k+2)!}\\
&= 1 - \frac{1}{(k+2)!}\\
&= 1 - \frac{1}{((k+1)+1)!}
\end{aligned}$$

$P(k+1)$ 4, Def^n of P (5)

$P(k) \to P(k+1)$ 3, 5, implication (6)

$k \geq 1 \to (P(k) \to P(k+1))$ 2, 6, implication (7)

$\forall k \geq 1, P(k) \to P(k+1)$ 1, 7, U.G (8)

Therefore, $\forall n \geq 1, P(n)$, as required to show. ■.

3.5

Prove that if $A_1, A_2, \ldots A_n$ and B are sets, then

$$(A_1\ B) \cup (A_2\ B) \cup \ldots \cup (A_n\ B) = (A_1 \cup A_2 \cup \ldots \cup A_n)\ B$$

for all $n \geq 2$

<u>Base Case.</u>

$$\begin{aligned}
(A_1\ B) \cup (A_2\ B) &= x : (x \in A_1 \wedge x \notin B) \vee (x \in A_2 \wedge x \notin B)\\
&= x : (x \notin B) \wedge (x \in A_1 \vee x \in A_2)\\
&= (A_1 \cup A_2)\ B
\end{aligned}$$

Therefore, P(2)
Let $P(n)$ be $(A_1\ B) \cup (A_2\ B) \cup \ldots \cup (A_n\ B) = (A_1 \cup A_2 \cup \ldots \cup A_n)\ B$.
WTS: $\forall k \geq 2, P(k) \to P(k+1)$

Let k be arbitrary	(1)
Suppose $k \geq 2$	(2)
Suppose P(k)	Induction Hypothesis (3)
$(A_1\ B) \cup (A_2\ B) \cup \ldots \cup (A_{k+1}\ B) = (A_1\ B) \cup (A_2\ B) \cup \ldots \cup (A_k\ B) \cup (A_{k+1}\ B)$	(4)
$= (A_1 \cup A_2 \cup \ldots \cup A_k) \cup (A_{k+1}\ B)$	IH
$= x : (x \notin B \wedge (x \in A_1 \vee x \in A_2 \vee \ldots \vee A_k)) \vee (x \notin B \wedge x \in A_{k+1})$	
$= x : (x \notin B) \wedge (x \in A_1 \vee x \in A_2 \vee \ldots \vee x \in A_{k+1})$	
$= (A_1 \cup A_2 \cup \ldots \cup A_{k+1})\ B$	
$P(k+1)$	4, Def^n of P (5)
$P(k) \rightarrow P(k+1)$	3, 5, implication (6)
$k \geq 2 \rightarrow (P(k) \rightarrow P(k+1))$	2, 6, implication (7)
$\forall k \geq 2, P(k) \rightarrow P(k+1)$	1, 7, U.G (8)