CSCA67 - Exercises #7

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2.1

Prove that $\sqrt{2}$ is irrational.

Suppose $\sqrt{2}$ is rational	for contradiction	(1)
$\exists a \exists b, \left(\sqrt{2} = \frac{a}{b}\right) \wedge (a, b$ have no common factors $\neq 1.)$	(1, Definition of $\mathbb{Q}$)	(2)
$\left(\sqrt{2} = \frac{x}{y}\right) \wedge (x, y$ have no common factors $\neq 1.)$	(2, E.I)	(3)
$x^2 = 2y^2$	(3, simp. algebra)	(4)
y^2 is an integer		(5)
$\exists k, x^2 = 2k$	(4, 5, E.G)	(6)
x^2 is even	(6, Definition of even)	(7)
x is even	(7, def., U.M.P)	(8)
$x = 2i$	(8, def. E.I)	(9)
$y^2 = \frac{x^2}{2} = \frac{(2i)^2}{2} = 2i^2$	(4, 9)	(10)
y^2 is even	(10, def.)	(11)
y is even	(11, def., U.M.P)	(12)
$y = 2j$	12, def. E.I	(13)
2 is a common factor ofx, y	(9, 13)	(14)
$(a, b$ have no common factors $\neq 1)$	(3, simp.)	(15)
a contradiction	(14, 15)	(16)
$\sqrt{2}$ is irrational.	(1, 16, contradiction.)	(17)

As required to prove. ■

2.2

Prove that the sum of a rational number and an irrational number is irrational.

$$WTS : \forall x \forall y, (x \in \mathbb{Q} \wedge y \notin \mathbb{Q}) \rightarrow ((x + y) \notin \mathbb{Q})$$

$$
\begin{array}{lll}
\text{Let } x, y \text{ be arbitrary} & & (1) \\
\quad \text{Suppose } (x \in \mathbb{Q}) \wedge (y \notin \mathbb{Q}) & & (2) \\
\qquad \text{Suppose } (x+y) \in \mathbb{Q} & \text{(for contradiction)} & (3) \\
\qquad\quad x \in \boldsymbol{Q} & \text{(2, simp.)} & (4) \\
\qquad\quad y \notin \boldsymbol{Q} & \text{(2, simp.)} & (5) \\
\qquad\quad \exists a, b \in \mathbb{Z}, \left((b \neq 0) \wedge \left(x = \frac{a}{b}\right)\right) & \text{(4, def}^n\text{ of } \mathbb{Q}\text{)} & (6) \\
\qquad\quad \left((b \neq 0) \wedge \left(x = \frac{a}{b}\right)\right) & \text{(6, E.I.)} & (7) \\
\qquad\quad x = \frac{a}{b} & \text{(7, simp.)} & (8) \\
\qquad\quad \exists u, v \in \mathbb{Z}, \left((v \neq 0) \wedge \left(x+y = \frac{u}{v}\right)\right) & \text{(3, def}^n\text{ of } \mathbb{Q}\text{)} & (9) \\
\qquad\quad \left((v \neq 0) \wedge \left(x+y = \frac{u}{v}\right)\right) & \text{(9, E.I.)} & (10) \\
\qquad\quad x+y = \frac{u}{v} & \text{(10, simp.)} & (11) \\
\qquad\quad y = (x+y) - x & \text{(math)} & (12) \\
\qquad\quad y = \frac{bu - au}{vb} & \text{(8, 11, math)} & (13) \\
\qquad\quad v \neq 0 & \text{(7, simp.)} & (14) \\
\qquad\quad b \neq 0 & \text{(10, simp.)} & (15) \\
\qquad\quad vb \neq 0 & \text{(14, 15, math)} & (16) \\
\qquad\quad (vb \neq 0) \wedge \left(y = \frac{bu - au}{vb}\right) & \text{(16, 13, conj.)} & (17) \\
\qquad\quad \exists c, d \in \mathbb{Z}, (d \neq 0) \wedge \left(y = \frac{c}{d}\right) & \text{(17, E.G)} & (18) \\
\qquad\quad y \in \mathbb{Q} & \text{(18, def}^n\text{ of } \mathbb{Q}\text{)} & (19) \\
\qquad\quad \text{a contradiction} & \text{(5, 19)} & (20) \\
\qquad x + y \notin \mathbb{Q} & \text{(3, 20, contradiction)} & (21) \\
\quad (x \in \mathbb{Q} \wedge y \notin \mathbb{Q}) \rightarrow (x + y \notin \mathbb{Q}) & \text{(2, 21, imp.)} & (22) \\
\forall x \forall y, (x \in \mathbb{Q} \wedge y \notin \mathbb{Q}) \rightarrow (x + y \notin \mathbb{Q}) & \text{(1, 22, U.G)} & (23)
\end{array}
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As required to prove. ■