CSCA67 - Exercises #6

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For each of the following arguments, either prove the argument is valid by using Inference Rules or prove the argument is invalid by providing a counter-example world.

1.1

All birds eat at least one species of insect. All species of insects can fly. Therefore, all birds eat at least one flying species.
Let $I(x)$ be "x is a species of insect", $B(x)$ be "x is a bird", $F(x)$ be "x flies", and $E(x,y)$ be "x eats y". Universe of discourse is live beings.

$\forall x, B(x) \to (\exists y, I(y) \land E(x,y))$	(1)
$\forall x, I(x) \to F(x)$	(2)
$\forall x, B(x) \to (\exists y, F(y) \land E(x,y))$	(Conclusion)
Take an arbitrary being c	(3)
$B(c) \to (\exists y, I(y) \land E(c,y))$	(4) (1,3, U.I)
Suppose $B(c)$	(5)
$\exists y, I(y) \land E(c,y)$	(5) (3,4, Implication)
Choose d such that $I(d) \land E(c,d)$	(6)
$I(d) \land E(c,d)$	(7) (5, E.I)
$I(d)$	(8) (7, Simp.)
$F(d)$	(9) (2, 8, U.M.P)
$E(c,d)$	(10) (7, Simp.)
$F(d) \land E(c,d)$	(11) (9, 10, Conj.)
$\exists y, F(y) \land E(c,y)$	(12) (6, 11, E.G)
$B(c) \to (\exists y, F(y) \land E(c,y))$	(13) (5, 12, Implication)
$\forall x, B(x) \to (\exists y, F(y) \land E(x,y))$	(14) (3, 13, U.G)

Therefore the argument is valid. ■

1.2

Some smart people make lots of money. Some people who make lots of money buy very big houses. Therefore, some smart people buy very big houses.
Let $S(x)$ stand for "x is smart", $M(x)$ be "x makes lots of money", and $H(x)$ be "x buys a big houses", universe of discourse is people.

The argument given is:

$\exists x\, S(x) \land M(x)$	(1)
$\exists x, M(x) \land H(x)$	(2)
$\overline{\exists x, S(x) \land H(x)}$	(Conclusion)

Let $W \subseteq U$ s.t. $W = \{a, b, c\}$

$\underline{a}$	$\underline{b}$	$\underline{c}$
$S(a) = \text{True}$	$S(b) = \text{False}$	$S(c) = \text{True}$
$M(a) = \text{True}$	$M(b) = \text{True}$	$M(c) = \text{False}$
$H(a) = \text{False}$	$H(b) = \text{True}$	$H(c) = \text{False}$

Premise (1)

$$\exists x, S(x) \wedge M(x)$$

Let $x = a$

$S(a) \wedge M(a)$

$T \wedge T$

$\mathbf{T}$

Premise (2)

$$\exists x, M(x) \wedge H(x)$$

Let $x = b$:

$M(b) \wedge H(b)$

$T \wedge T$

$\mathbf{T}$

Conclusion

$$\exists x, S(x) \wedge H(x)$$

Let $x = c$:

$S(c) \wedge H(c)$

$T \wedge F$

$\mathbf{F}$

Thus, we have proved that there is a case such that all the premises are true, while the conclusion is false, making this argument invalid. ■

1.3

Everybody knows at least one song. Every song is sung by someone. People only sing songs they like. Therefore, everyone knows at least one song someone likes.
Let $S(x)$ be "x is a song", $P(x)$ be "x is a person", $K(x,y)$ be "x knows y", $L(x,y)$ be "x likes y", $S(x,y)$ be "x sings y". Universe of discourse is people and songs.

$\forall x,\ P(x) \to (\exists y, S(y) \wedge K(x,y))$ (1)

$\forall x,\ S(x) \to (\exists y, P(y) \wedge S(y,x))$ (2)

$\forall x \forall y,\ (P(x) \wedge S(y) \wedge S(x,y)) \to L(x,y)$ (3)

$\forall x, P(x) \to (\exists y, S(y) \wedge K(x,y) \wedge \exists z, (P(z) \wedge L(z,y)))$ (Conclusion)

Let a be arbitrary. (4)

Suppose $P(a)$ (5)

$\exists y, S(y) \wedge K(a,y)$ (6) (1,5, Implication)

Choose b such that $S(b) \wedge K(a,b)$ (7)

$S(b) \wedge K(a,b)$ (8) (7, E.I)

$S(b)$ (9) (8, Simp.)

$K(a,b)$ (10) (8, Simp.)

$\exists y, P(y) \wedge S(y,b)$ (11) (2, 9, U.M.P)

Choose c such that $P(c) \wedge S(c,b)$ (12) (2, 9, U.M.P)

$P(c) \wedge S(c,b)$ (13) (12, E.I)

$S(b) \wedge P(c) \wedge S(c,b)$ (14) (9, 13, Conj.)

$L(c,b)$ (15) (3, 14, U.M.P)

$P(c)$ (16) (13, Simp.)

$P(c) \wedge L(c,b)$ (17) (15, 16, Conj.)

$\exists z, P(z) \wedge L(z,b)$ (18) (17, 12, E.G)

$S(b) \wedge K(a,b) \wedge (\exists z, P(z) \wedge L(z,b))$ (19) (18, 8, conj.)

$\exists y, S(y) \wedge K(a,y) \wedge (\exists z, P(z) \wedge L(z,y))$ (20) (19, 7, E.G)

$P(a) \to (\exists y, S(y) \wedge K(a,y) \wedge (\exists z, P(z) \wedge L(z,y)))$ (21) (19, 7, Implication)

$\forall x, P(x) \to (\exists y, S(y) \wedge K(x,y) \wedge \exists z, (P(z) \wedge L(z,y)))$ (22) (4, 21, U.G)

Therefore the argument is valid. ■

2.1

Prove: If x, y, and z are integers and x + y + z is odd, then at least one of x, y, z is odd

Universe of discourse is integers.

$$WTS : O(x+y+z) \to (O(x) \vee O(y) \vee O(z))$$

Lemmas.

$$(**) : E(x) \leftrightarrow \exists k, x = 2k$$

$$(***) : \neg E(x) = O(x)$$

Suppose $E(x) \wedge E(y) \wedge E(z)$	(1)
$(\exists k, x = 2k) \wedge (\exists j, y = 2j) \wedge (\exists i, z = 2i)$	(2) (1, (**))
$x = 2k$	(3) (2, E.I)
$y = 2j$	(4) (2, E.I)
$z = 2i$	(5) (2, E.I)
$x + y + z = 2(k + j + i)$	(6) 3, 4, 5, algebra
$E(x + y + z)$	(7) (6, (**))
$(E(x) \wedge E(y) \wedge E(z)) \to E(x + y + z)$	(8) (1, 7, Implication)
$\neg E(x + y + z) \to \neg(E(x) \wedge E(y) \wedge E(z))$	(9) (8, contra.)
$\neg E(x + y + z) \to (\neg E(x) \vee \neg E(y) \vee \neg E(z))$	(10) (9, DeM.)
$O(x + y + z) \to (O(x) \vee O(y) \vee O(z))$	(11) (10, (***))

2.2

Prove: If n is a positive integer, then n is even if and only if $7n + 4$ is even.

Universe of discourse is $\mathbb{Z}^+$.

$$WTS : \forall n, E(n) \leftrightarrow E(7n + 4)$$

Lemmas.

$$(*) : O(x) \leftrightarrow \exists k, x = 2k + 1$$

$$(**) : E(x) \leftrightarrow \exists k, x = 2k$$

$$(***) : \neg O(x) = E(x)$$

Take an arbitrary positive integer n	(1)
Suppose $E(n)$	(2)
$\exists k, n = 2k$	(3)
$n = 2k$	(4) 3, E.I
$7n+4 = 7(2k)+4$	(5) 4, algebra
$7n+4 = 2(7k+2)$	(6) 5, algebra
$E(7n+4)$	(7) 6, (**)
$E(n) \to E(7n+4)$	(8) (2, 7, Implication)
Assume $O(n)$	(9)
$\exists k, n = 2k+1$	(10) (9, (*))
$n = 2k+1$	(11) (10, E.I)
$7n+4 = 7(2k+1)+4$	(12) (11, algebra)
$7n+4 = 14k+7+4$	(13) (12, algebra)
$7n+4 = 14k+11$	(14) (13, algebra)
$7n+4 = 2(7k+5)+1$	(15) (14, algebra)
$O(7n+4)$	(16) (15, (*))
$O(n) \to O(7n+4)$	(17) (9, 16, Implication)
$\neg O(n) \to \neg O(7n+4)$	(18) (18, Contr.)
$E(n) \to E(7n+4)$	(19) (18, (***))
$(E(n) \to E(7n+4)) \wedge (E(n) \to E(7n+4))$	(20) (8, 20, Conj.)
$(E(n) \leftrightarrow E(7n+4))$	(21) (20, bicond.)
$\forall n, E(n) \leftrightarrow E(7n+4)$	(22) (1, 20, U.G)

2.3

Prove: $(A \cap B \neq \emptyset \ \wedge \ A \subseteq C) \to (B \cap C \neq \emptyset)$

Suppose $A \cap B \neq \emptyset \wedge A \subseteq C$	(1)
$\exists x (x \in A \wedge x \in B)$	(2) (1, Definition of $\cap$)
Choose x such that $x \in A \wedge x \in B$	(3)
$x \in A$	(4) (3, Simp.)
$x \in B$	(5) (3, Simp.)
$\forall y (y \in A \to y \in C)$	(6) (definition of $\subseteq$)
$x \in C$	(7) (4,6, U.M.P)
$x \in B \wedge x \in C$	(8) (5,7, Conj.)
$\exists x (x \in B \wedge x \in C)$	(9) (3,8, E.G)
$B \cap C \neq \emptyset$	(10)(9, definition of $\cap$)
$(A \cap B \neq \emptyset \ \wedge \ A \subseteq C) \to (B \cap C \neq \emptyset)$	(11)(1, 10, Implication)

$\therefore$ As required to show. ■