CSCA67 - Exercises #3

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1. Conditionals

Analyse the logical forms of the following statements. Construct a converse and a contrapositive for each conditional statement: provide your answers both as logical expressions and English sentences.

1. If Alice is at the party, then so is Bob.

   Converse: $B \to A$ (If Bob is at the party, then so is Alice.)
   Contrapositive: $\neg B \to \neg A$ (If Bob is not at the party, then Alice is not at the party.)

2. Charlie is at the party, only if both Alice and Bob are.

   Original: $C \to (A \wedge B)$
   Converse: $(A \wedge B) \to C$ (Alice and Bob are both at the party, only if Charlie is.)
   Contrapositive: $\neg(A \wedge B) \to \neg C$ (If both Alice and Bob are not at the party, then Charlie is not at the party.)

3. David is not at the party, if Alice is.

   Original: $A \to \neg D$
   Converse: $\neg D \to A$ (If David is not at the party, then Alice is.)
   Contrapositive: $D \to \neg A$ (If David is at the party, then Alice is not.)

4. If Bob is not at the party, then Alice is.

   Original: $\neg B \to A$
   Converse: $A \to \neg B$ (If Alice is at the party, the Bob is not.)
   Contrapositive: $\neg A \to B$ (If Alice is not at the party, then Bob is.)

5. If Bob is not at the party, then neither is Alice.

   Original: $\neg B \to \neg A$
   Converse: $\neg A \to \neg B$ (If Alice is not at the party, then neither is Bob.)
   Contrapositive: $A \to B$ (If Alice is at the party, then so is Bob.)

6. Alice is not at the party, unless Bob is.

   Original: $A \to B$
   Converse: $B \to A$ (If Bob is at the party, then Alice is.)
   Contrapositive: $\neg B \to \neg A$ (If Bob is not at the party, then neither is Alice.)

7. Neither Alice nor Bob being at the party is a sufficient condition for Charlie to be at the party.

   Original: $\neg(A \wedge B) \to C$
   Converse: $C \to \neg(A \wedge B)$ (If Charlie is at the party, then neither Alice nor Bob is.)
   Contrapositive: $\neg C \to (A \wedge B)$ (If Charlie is not at the party, then both Alice and Bob are.)

8. Both Alice and Bob being at the party is a necessary condition for Charlie to be at the party.

   Original: $C \to (A \wedge B)$
   Converse: $(A \wedge B) \to C$ (If both Alice and Bob is at the party, then so is Charlie.)
   Contrapositive: $\neg(A \wedge B) \to \neg C$ (If both Alice and Bob are not at the party, then neither is Charlie.)

2. Logical Equivalences

For each pair of expressions, either prove that the two are equivalent or prove that they are not.

1. $\neg(a \to b)$ and $\neg a \wedge b$

When a is True, and b is False, then:

$$
\begin{aligned}
&\neg(a \to b) && \neg a \wedge b \\
&\Rightarrow \neg(T \to F) && \Rightarrow \neg T \wedge F \\
&\Rightarrow \neg(F) && \Rightarrow F \wedge F \\
&\Rightarrow T && \Rightarrow F
\end{aligned}
$$

$\therefore$ The statements are not equivalent. ■

2. $\neg(a \to b)$ and $a \wedge \neg b$

$$
\begin{aligned}
&\neg(a \to b) && && \boxed{a \wedge \neg b} \\
&\Rightarrow \neg(\neg a \wedge b) && \text{(Conditional Law)} \\
&\Rightarrow \neg\neg a \vee \neg b && \text{(De Morgan's Law)} \\
&\Rightarrow \boxed{a \vee \neg b} && \text{(Double Negation Law)}
\end{aligned}
$$

$\therefore$ The statements are equivalent. ■

3. $a \iff \neg b$ and $(a \wedge \neg b) \vee (\neg a \wedge b)$

$$
\begin{aligned}
&a \iff \neg b && && \boxed{(a \wedge \neg b) \vee (\neg a \wedge b)} \\
&\Rightarrow (\neg b \to a) \wedge (a \to \neg b) && \text{(Biconditional Law)} \\
&\Rightarrow (\neg\neg b \vee a) \wedge (\neg a \vee \neg b) && \text{(Conditional Law)} \\
&\Rightarrow (b \vee a) \wedge (\neg a \vee \neg b) && \text{(Double Negation Law)} \\
&\Rightarrow ((b \vee a) \wedge \neg a) \vee ((b \vee a) \wedge \neg b) && \text{(Distributive Law)} \\
&\Rightarrow ((b \wedge \neg a) \vee (a \wedge \neg a)) \vee ((b \wedge \neg b) \vee (a \wedge \neg b)) && \text{(Distributive Law)} \\
&\Rightarrow \boxed{(b \wedge \neg a) \vee (a \wedge \neg b)} && \text{(Negation Law)}
\end{aligned}
$$

$\therefore$ The statements are equivalent. ■

3. Logical Inference

Use the rules of inference from class, to prove validity of the following arguments.

1. If I play hockey, then I am sore the next day. I use the whirlpool if I am sore. I did not use the whirlpool. Therefore, I did not play hockey.

$$
\text{Statements} = \begin{cases} H : \text{I played hockey.} \\ S : \text{I am sore the next day.} \\ W : \text{I used the whirlpool.} \end{cases}
$$

The argument is:

$$
\begin{array}{ll}
H \to S & (1) \\
S \to W & (2) \\
\neg W & (3) \\
\hline
\therefore \neg H & \text{Conclusion}
\end{array}
$$

New statements that we can make are:

$$\begin{aligned} &\neg S \quad (3), (2), \text{Modus Tollens} && (4) \\ &\neg H \quad (4), (1), \text{Modus Tollens} && (5) \end{aligned}$$

(5) ≡ Conclusion, therefore the argument is true. ■

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2. I am either dreaming or hallucinating. I am not dreaming. If I am hallucinating, I see elephants running down the road. Therefore, I see elephants running down the road.

$$\text{Statements} = \begin{cases} D : \text{I am dreaming.} \\ H : \text{I am hallucinating} \\ E : \text{I see elephants running down the road.} \end{cases}$$

The argument is:

$$\begin{array}{ll} D \vee H & (1) \\ \neg D & (2) \\ H \to E & (3) \\ \hline \therefore E & \text{Conclusion} \end{array}$$

New statements that we can make are:

$$\begin{aligned} &H \quad (2), (1), \text{Disjunctive Syllogism} && (4) \\ &E \quad (4), (3), \text{Modus Ponens} && (5) \end{aligned}$$

(5) ≡ Conclusion, therefore the argument is true. ■

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3. If I go running, I stay in the sun for too long. If I go swimming, I stay in the sun for too long. If I stay in the sun for too long, I get sunburn. I did not get a sunburn. Therefore, I neither went running nor swimming.

$$\text{Statements} = \begin{cases} R : \text{I went running.} \\ S : \text{I stay in the sun for too long.} \\ W : \text{I went swmming.} \\ B : \text{I get sunburn.} \end{cases}$$

The argument is:

$$\begin{array}{ll} R \to S & (1) \\ W \to S & (2) \\ S \to B & (3) \\ \neg B & (4) \\ \hline \therefore (\neg R \wedge \neg W) & \text{Conclusion} \end{array}$$

New statements that we can make are:

$$\begin{aligned} &\neg S \quad (3), (4), \text{Modus Tollens} && (5) \\ &\neg W \quad (5), (2), \text{Modus Tollens} && (6) \\ &\neg R \quad (5), (1), \text{Modus Tollens} && (7) \\ &\neg R \wedge \neg W \quad (6), (7), \text{Conjunction} && (8) \end{aligned}$$

(8) ≡ Conclusion, therefore the argument is true. ■

4. Variables and Sets

If U is the universe of discourse, then the complement of the set A, which we will denote as A, is the set

$$U \setminus A = \{x \in U \mid x \notin A\}$$

Use Venn diagrams to illustrate the following identities.

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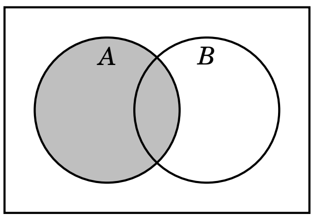
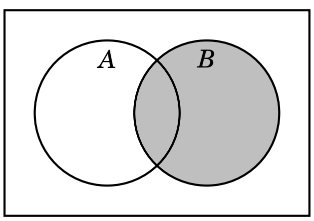
1. $\overline{A \cup B} = \overline{A} \cap \overline{B}$

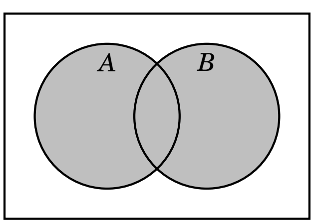
$\overline{\mathbf{A} \cup \mathbf{B}}$



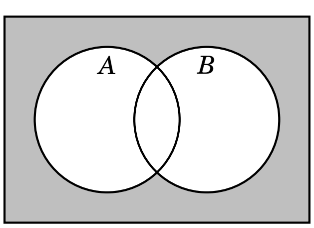
A



B

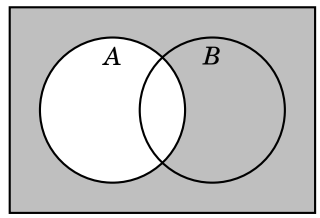


$A \cup B$

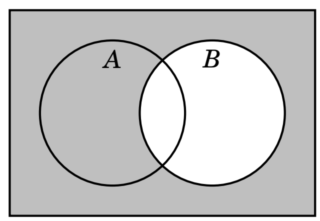


$\overline{A \cup B}$

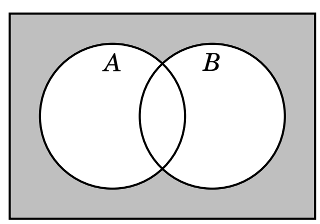
$\overline{\mathbf{A}} \cap \overline{\mathbf{B}}$



$\overline{A}$



$\overline{B}$

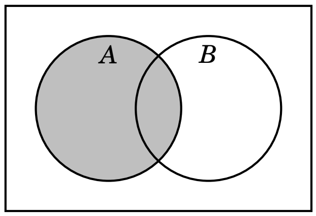


$\overline{A} \cap \overline{B}$

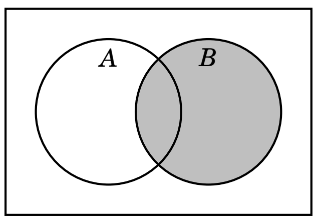
$\boxed{\therefore \overline{A \cup B} = \overline{A} \cap \overline{B}}$

2. $\overline{A \cap B} = \overline{A} \cup \overline{B}$

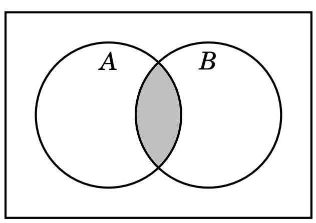
$\overline{\mathbf{A} \cap \mathbf{B}}$



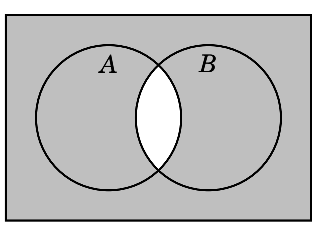
A



B

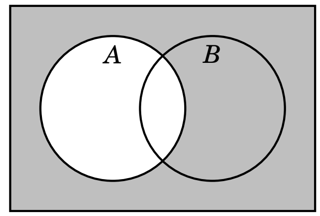


$A \cap B$

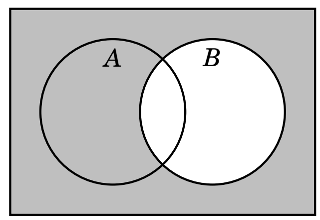


$\overline{A \cap B}$

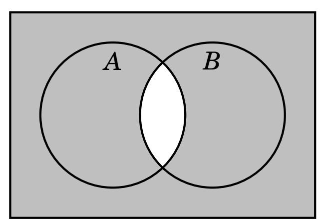
$\overline{\mathbf{A}} \cup \overline{\mathbf{B}}$



$\overline{A}$



$\overline{B}$



$\overline{A} \cup \overline{B}$

$\boxed{\therefore \overline{A \cap B} = \overline{A} \cup \overline{B}}$

Prove the identities in part (a), by writing out (using logical symbols) what it means for anobject x to be an element of each set and then using logical equivalences.

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1. $\overline{A \cup B} = \overline{A} \cap \overline{B}$

$\overline{\mathbf{A} \cup \mathbf{B}}$	$\overline{\mathbf{A}} \cap \overline{\mathbf{B}}$
$\left\{x \mid x \in \left(\overline{A \cup B}\right)\right\}$	$x \in \overline{A} \cap \overline{B}$
$\left\{x \mid x \in \overline{A \cup B}\right\}$	$\Rightarrow x \in \overline{A} \wedge x \in \overline{B}$
$\Rightarrow x \notin A \wedge x \notin B$	$\Rightarrow x \notin A \wedge x \notin B$
$\Rightarrow x \in \overline{A} \wedge x \in \overline{B}$	$\Rightarrow x \in \overline{A} \cap \overline{B}$
$x \in \overline{A} \cap \overline{B}$	$x \in \overline{A} \cap \overline{B}$
	$\therefore \overline{A \cup B} = \overline{A} \cap \overline{B}$