

# CSCA67 - Assignment #4

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## Simple Induction

1.1

Prove that  $\sum_{i=1}^n F_i^2 = F_n F_{n+1}$  for all  $n > 0$ .

### Base Case

$$\sum_{i=0}^n F_i^2 = F_n F_{n+1}$$

$$\sum_{i=0}^1 F_i^2 = F_1 F_{1+1}$$

$$F_1^2 = F_1 F_2$$

$$0^2 = 0 \cdot 1$$

$$0 = 0$$

$T$

Therefore,  $P(1)$ .

### Induction Step

Let  $k$  be arbitrary.

(1)

Suppose  $k \geq 1$

(2)

Suppose  $P(k)$

Induction Hypothesis (3)

$$\sum_{i=0}^{k+1} F_i^2 = \sum_{i=0}^k F_i^2 + F_{k+1}^2$$

(4)

$$= F_k F_{k+1} + F_{k+1}^2$$

IH

$$= F_{k+1} (F_k + F_{k+1})$$

$$= F_{k+1} F_{k+2}$$

Def of  $F$

$$P(k+1)$$

Definition of  $P$  (5)

$$P(k) \rightarrow P(k+1)$$

3, 5, Implication (6)

$$k \geq 1 \rightarrow (P(k) \rightarrow P(k+1))$$

2, 6, Implication (7)

$$\forall k \geq 1, P(k) \rightarrow P(k+1)$$

1, 7, Implication (8)

As required to show. ■.

1.2

Prove that  $\sum_{i=1}^n F_{2i-1} = F_{2n}$  for all  $n > 0$ .

### Base Case

$$\begin{aligned}
\sum_{i=1}^n F_{2i-1} &= F_{2n} \\
\sum_{i=1}^1 F_{2i-1} &= F_2 \\
F_{2-1} &= F_2 \\
F_1 &= F_2 \\
1 &= 1 \\
T
\end{aligned}$$

Therefore,  $P(1)$ .

### Induction Step

Let  $k$  be arbitrary. (1)

Suppose  $k \geq 1$  (2)

Suppose  $P(k)$

Induction Hypothesis (3)

$$\begin{aligned}
\sum_{i=0}^{k+1} F_{2i+1} &= \sum_{i=0}^k F_{2i+1} + F_{2k+1} \\
&= F_{2k} + F_{2k+1} \\
&= F_{2k+2} \\
&= F_{2(k+1)}
\end{aligned}$$

IH

Def of  $F$

(5)

$P(k+1)$

Definition of  $P$  (6)

$P(k) \rightarrow P(k+1)$

3, 5, Implication (7)

$k \geq 1 \rightarrow (P(k) \rightarrow P(k+1))$

2, 6, Implication (8)

$\forall k \geq 1, P(k) \rightarrow P(k+1)$

1, 7, Implication (9)

## 2. Simple and Strong Induction

,

1

Determine which amounts can be formed using just \$2 bills and \$5 bills. You don't need to provide an explanation, simply state your result.

Any amount starting from 4 dollars can be made. Also, 2 dollars can be made.

2

Prove this result using simple induction

$$(2 \geq 0) \wedge (0 \geq 0) \wedge (4 = 2(2) + 5(0)) = P(4)$$

Let $k$ be arbitrary	(1)
Suppose $k \geq 4$	(2)
Suppose $P(k)$	Induction Hypothesis (3)
$\exists x, y, (x \geq 0) \wedge (y \geq 0) \wedge (k = 2x + 5y)$	3, def (4)
$(a \geq 0) \wedge (b \geq 0) \wedge (k = 2a + 5b)$	4, E.I (5)
$a \geq 0$	5, simp. (6)
$b \geq 0$	5, simp. (7)
$k = 2a + 5b$	5, simp. (8)
Suppose $b = 1$	Case 1, no 5 dollar bills. (9)
$2a = k - 5b \geq 4 - 0 = 4$	2, 7 8 (10)
$a \geq 2$	10 (11)
$a - 2 \geq 0$	11 (12)
$b + 1 \geq 0$	7 (13)
$2(a - 2) + 5(b + 1) = 2a + 5b + 1 = k + 1$	8 (14)
$\exists x, y, (x \geq 0) \wedge (y \geq 0) \wedge (k + 1 = 2x + 5y)$	12-14, conj, E.G (15)
$P(k + 1)$	15, def (16)
Suppose $b \geq 1$	Case 2, at least 1 5 dollar bills. (17)
$b - 1 \geq 0$	17 (18)
$a + 2 \geq 0$	6 (19)
$3(a + 2) + 5(b - 1) = k - 1$	8 (20)
$\exists x, y, (x \geq 0) \wedge (y \geq 0) \wedge (k + 1 = 2x + 5y)$	18-20, conj, E.G (21)
$P(k + 1)$	9, 16, 17, 22, cases (22)
$P(k) \rightarrow P(k + 1)$	3, 23, implication (23)
$(k \geq 4) \rightarrow P(k) \rightarrow P(k + 1)$	2, 24, implication (24)
$\forall k \geq 4, P(k) \rightarrow P(k + 1)$	1, 25, U.G (25)
	(26)

3

Prove this result using strong induction

Let $k$ be arbitrary.	(1)
Suppose $k \geq 4$	(2)
Suppose $\forall 4 \leq i < k, P(i)$	Induction Hypothesis (3)
Suppose $k \leq 5$	(4)
$(2 \geq 0) \wedge (0 \geq 0) \wedge (4 = 2 \cdot 2 + 0 \cdot 5)$	(5)
$(0 \geq 0) \wedge (1 \geq 0) \wedge (5 = 0 \cdot 2 + 1 \cdot 5)$	(6)
$\exists x, y, (x \geq 0) \wedge (y \geq 0) \wedge (k = 2x + 5y)$	5, 6, Cases (7)
$P(k)$	7, def (8)
Suppose $k \geq 6$	(9)
$4 \leq k - 2 \leq k$	(10)
$P(k - 3)$	3, 10, UMP, IH (11)
$\exists x, y, (x \geq 0) \wedge (y \geq 0) \wedge (k - 2 = 2x + 5y)$	11, def (12)
$(x \geq 0) \wedge (y \geq 0) \wedge (k - 2 = 2x + 5y)$	12, E.I (13)
$(x + 1 \geq 0) \wedge (y \geq 0) \wedge (k = 2(x + 1) + 5y)$	Add 2 dollar bill (14)
$\exists x, y, (x \geq 0) \wedge (y \geq 0) \wedge (k = 2x + 5y)$	14, E.G (15)
$P(k)$	15, def (16)
$P(k)$	4, 8, 9, 16, Cases (17)
$\forall 4 \leq i < k, P(i) \rightarrow P(k)$	3, 17, implication (18)
$(k \geq 4) \rightarrow \forall 4 \leq i < k, P(i) \rightarrow P(k)$	2, 18, implication (19)
$\forall k \geq 4, P(k)$	1, 19, U.G, Strong Induction (20)

### 3. Strong Induction

#### 3.1

Consider the following sequence definition:

$$a_0 = 1$$

$$a_1 = 2$$

$$a_n = 5a_{n-1} - 6a_{n-2} \text{ for } n \geq 2$$

Prove that  $a_n = 2^n$  for all integers  $n \geq 0$

Let $k$ be arbitrary.	(1)
Suppose $k \geq 0$	(2)
Suppose $\forall 0 \leq i < k, P(i)$	Induction Hypothesis (3)
Assume $k = 0$	(4)
$a_0 = 1$	(5)
$= 2^0$	
$P(k)$	5, definition of P (6)
Assume $k = 0$	(7)
$a_1 = 2$	(8)
$= 2^1$	
$P(k)$	8, definition of P (9)
Assume $k \geq 2$	(10)
$a_k = 5a_{k-1} - 6a_{k-2}$	(11)
$= 5(2^{k-1}) - 6(2^{k-2})$	IH
$= 2^{k-2}(5(2) - 6)$	
$= 2^{k-2}(4)$	
$= 2^{k-2}(2^2)$	
$= 2^k$	
$P(k)$	11, definition of P (12)
$P(k)$	4, 6, 7, 9, 10, 12, Cases (13)
$\forall 0 \leq i < k, P(i) \rightarrow P(k)$	3, 14, implication (14)
$(k \geq 0) \rightarrow \forall 0 \leq i < k, P(i) \rightarrow P(k)$	2, 15, implication (15)
$\forall k \geq 0, \forall 0 \leq i < k, P(i) \rightarrow P(k)$	1, 16, U.G (16)
$\forall k \geq 0, P(k)$	17, Strong Induction (17)
	(18)

As required to show. ■.

## 4. Counting

a

Exactly 6 characters, each character is a lowercase letter or a digit?

Each character: 26 letters + 10 digits = 36 choices.

Total 6-character strings:  $36^6$

Product Rule

b

At least 5 characters and at most 7 characters, each character is a lowercase letter or a digit?

Each character: 26 letters + 10 digits = 36 choices.

Total 5-character strings:  $36^5$

Product Rule

Total 6-character strings:  $36^6$

Product Rule

Total 7-character strings:  $36^7$

Product Rule

Total strings:  $36^5 + 36^6 + 36^7$

Sum Rule

c

Exactly 6 characters, each character is a lowercase letter or a digit, cannot start with a digit?

Each character: 26 letters + 10 digits = 36 choices.

Total 4-character strings:  $36^4$

Product Rule

Number of letters: 26

Total strings:  $26 \cdot 36^4$

Product Rule

d

Exactly 6 characters, each character is a lowercase letter or a digit, must start with a letter and end with a digit?

Each character: 26 letters + 10 digits = 36 choices.

Total 4-character strings:  $36^4$

Product Rule

Number of letters: 26

Number of numbers: 10

Total strings:  $26 \cdot 36^4 \cdot 10$

Product Rule

Total strings:  $260 \cdot 36^4$

e

Exactly 6 characters, each character is a lowercase letter or a digit, with no repeated characters?

Each character: 26 letters + 10 digits = 36 choices.

r: 6

r-permutations  $P(36, 6)$

r-permutations  $\frac{36!}{(36-6)!}$

r-permutations  $36 \cdot 35 \cdot 34 \cdot 33 \cdot 32 \cdot 31$

Product Rule

f

Exactly 6 characters, each character is a lowercase letter or a digit, palindromes are not allowed?

Each character: 26 letters + 10 digits = 36 choices.

Total 6-character strings:  $36^6$

Product Rule

Palindromes  $36^3$

No palindromes  $36^6 - 36^3$

No palindromes  $36^3 (36^3 - 1)$

g

Exactly 6 characters, each character is a lowercase letter or a digit, starts with "a67"?

Each character: 26 letters + 10 digits = 36 choices.

Total 6-character strings starting with a67:  $36^3$

Product Rule

h

Exactly 6 characters, each character is a lowercase letter or a digit, starts with "a67" and ends with "a67"?

Each character: 26 letters + 10 digits = 36 choices.

Total 6-character strings starting and ending with a67: 1

a67a67

i

Exactly 6 characters, each character is a lowercase letter or a digit, starts with "a67" or ends with "a67"?

Each character: 26 letters + 10 digits = 36 choices.

Total 6-character strings starting with a67:  $36^3$

Product Law

Total 6-character strings ending with a67:  $36^3$

Product Law

Total strings  $2(36^3)$

Sum Law

j

Exactly 6 characters, each character is a lowercase letter or a digit, contains "a67" (any number of times, anywhere)?

Each character: 26 letters + 10 digits = 36 choices.

Positions that a67 could be 4

1, 2, 3, 4

Total strings:  $4(36^3) - 1$

Product Law, Difference Law (position 1 and 4 both contain a67a67)

k

Exactly 6 characters, each character is a lowercase letter or a digit, contains “a67” exactly once?

Each character: 26 letters + 10 digits = 36 choices.

Positions that a67 could be 4

1, 2, 3, 4

Total strings:  $4(36^3) - 2$

Product Law, Difference Law (position 1 and 4 both contain a67a67)

k

Exactly 6 characters, each character is a lowercase letter or a digit, does not contain “a67”?

Each character: 26 letters + 10 digits = 36 choices.

Positions that a67 could be 4

1, 2, 3, 4

Total strings that contain a67:  $4(36^3) - 1$

Product Law, Difference Law (position 1 and 4 both contain a67a67)

Total possible  $36^6$

Product Law

Not containing a67,  $36^6 - 4(36^3) + 1$

Complement of containing a67

## 5. Counting

A Department has 12 faculty members, 6 staff, 200 undergraduate students, and 50 graduate students. It is forming a committee to decide on a new program in Machine Learning. In how many ways can we form this committee, if we make the following decisions?

1

It contains 2 faculty members, 1 staff member, and nobody else?

Ways to pick 2 faculty:  $C(12, 2)$

Ways to pick 2 faculty:  $\frac{12!}{2!10!}$

Ways to pick 2 faculty: 66

Staff members 6

Combinations  $66 \cdot 6 = 396$

Product Law

2

It contains 2 faculty members, 1 staff member, 1 undergraduate student, and 1 graduate student?

Ways to pick 2 faculty: 66  
 Staff members 6  
 graduate students 200  
 undergraduate students 50  
 Combinations  $66 \cdot 6 \cdot 200 \cdot 50 = 4950000$

Product Law

3

It contains at least 2 faculty members, at least 1 student, and no staff; and it contains 5 members in total?

Suppose the possible faculty members are  $k$ , then the students in that scenario is  $5-k$ , then the possible scenarios for faculty vs student is

2 : 3  
 3 : 2  
 4 : 1

For each scenario, choosing faculty is  $C(12, k)$ , students is  $C(250, 5-k)$ . Then, you add up the scenarios for every  $K$  value. Therefore, the possible scenarios

$$\begin{aligned} & C(12, 2) \cdot C(250, 3) + C(12, 3) \cdot C(250, 2) + C(12, 4) \cdot C(250, 1) \\ &= \frac{12!}{2!10!} \cdot \frac{250!}{3!247!} + \frac{12!}{3!9!} \cdot \frac{250!}{2!248!} + \frac{12!}{4!8!} \cdot \frac{250!}{249!} \\ &= \frac{12 \cdot 11}{2} \cdot \frac{250 \cdot 249 \cdot 248}{6} + \frac{12 \cdot 11 \cdot 10}{6} + \frac{250 \cdot 249}{2} + \frac{12 \cdot 11 \cdot 10 \cdot 9}{24} \cdot 250 \\ &= 6 \cdot 125 \cdot 83 \cdot 248 + 220 \cdot 125 \cdot 249 + 30 \cdot 99 \cdot 250 \end{aligned}$$

4

It contains at least 1 faculty member and at least 1 staff member, and has at least as many students as non-students; and it contains 6 members in total?

Either there are 2 non-student members (1 faculty, 1 staff), or 3 non-student members (2 faculty, 1 staff) or (1 faculty, 2 staff), The non-students is given as  $C(250, 6-k)$ , where  $k$  is the non-student members.

If there are 2 non student members:  $12 \cdot 6 \cdot \frac{250!}{4!246!} = 12 \cdot 6 \cdot 250 \cdot 83 \cdot 31 \cdot 247 = S_1$  If there are (2 faculty, 1 staff):  $C(12, 2) \cdot 6 \cdot \frac{250!}{4!247!} = 66 \cdot 6 \cdot 125 \cdot 83 \cdot 248 = S_2$  If there are (1 faculty, 2 staff):  $12 \cdot C(6, 2) \cdot \frac{250!}{4!247!} = 12 \cdot 15 \cdot 125 \cdot 83 \cdot 248 = S_3$  All of the above are derived from product rule, then from sum rule, the total possibilities are  $S_1 + S_2 + S_3$

5

It contains at least 1 faculty or staff member; and it contains 6 members in total?

Take scenarios where there are no faculty or staff (only students)

$$C(250, 6)$$

$$C(268, 6)$$

$$C(268, 6) - C(250, 6)$$

$$\begin{aligned} &= \frac{268!}{6!262!} - \frac{250!}{6!244!} \\ &= \frac{268 \cdot 267 \cdot 266 \cdot 265 \cdot 264 \cdot 263 \cdot 250 \cdot 249 \dots \cdot 245}{6!} \end{aligned}$$

Possible combinations of any 6 people

Complement of only students

6

Two graduate students have a conflict of interest and may not serve on a committee together. The committee contains 1 faculty member, 1 staff member, and 3 students (either graduate or undergraduate).

7

The committee needs a Chair, which must be a faculty member or a staff member. It contains 5 members in total.

18 possibilities for a chair, 5 seats left with 267 possible candidates: Combinations:  $18 \cdot C(267, 5) = 18 \cdot \frac{267!}{5!262!}$

## 6. Counting

1

How many permutations of the letter ABCDEFG contain the string "BCD"?

Treat "BCD" as 1 letter. Then with the 4 other letters, there are  $5!$  ways to arrange the sequence.

2

How many permutations of the letter ABCDEFG contain the string "CFG"?

Treat "CFG" as 1 letter. Then with the 3 other letters, there are  $4!$  ways to arrange the sequence.

3

How many permutations of the letter ABCDEFG contain the strings "ABC" and "GFE"?

Treat "ABC" as 1 letter. Treat "GFE" as another letter. The only remaining letter is D. Therefore there are  $3!$  ways to arrange the sequence.

4

How many permutations of the letter ABCDEFG contain the strings "BC", "AF", and "DE"?

Treat "BC", "AF", "DE" as one letter each. Only remaining letter is G. Therefore there are  $4!$  ways to arrange this sequence

5

How many ways are there for four men and five women to stand in a line, so that all men stand together?

Treat the 4 men as one person. Then, with the 5 other women, there are  $6!$  ways to arrange the people such that then men are together. However, the 4 men can be arranged in a group together, so there are  $4!$  ways to arrange that. In total, by product law, there are  $6!4!$  ways to arrange all the people.

6

How many ways are there for four men and five women to stand in a line, so that all women stand together?

Treat the 5 women as one person. Then, with the 4 other men, there are  $5!$  ways to arrange the people such that then men are together. However, the 5 women can be arranged in a group together, so there are  $5!$  ways to arrange that. In total, by product law, there are  $(5!)^2$  ways to arrange all the people.

7

How many ways are there for four men and five women to stand in a line, so that no two men stand next to each other?

After every man, there is a woman, therefore after using 4 women, there is 1 woman that can be placed anywhere in the sequence. There are 8 people, therefore there are 9 places to put the remaining woman, resulting in 9 possible ways.

8

How many bit strings contain exactly eight 0s and ten 1s, if every 0 must be immediately followed by a 1?

Similarly to the previous question, treating every "01" as one character, we have 8 characters, and 2 extra ones, resulting in 11! ways to arrange this sequence.

9

How many licence plates consisting of 4 uppercase letters followed by 3 numbers, contain no repeated letters and no repeated digits?

There are 26 uppercase letters, 10 numbers.

4 digit sequence of uppercase letters:  $P(26, 4)$  3 digit sequence of numbers:  $P(10, 3)$  Product law:  $P(26, 4) \cdot P(10, 3) = \frac{26!10!}{3!4!7!22!}$

10

How many licence plates consisting of 4 uppercase letters followed by 3 numbers, contain no repeated letters or no repeated digits?

There are 26 uppercase letters, 10 numbers.

4 digit sequence of uppercase letters:  $P(26, 4)$  3 digit sequence of numbers:  $P(10, 3)$

4 uppercase and repeating numbers:  $P(26, 4) \cdot 1000$  ( $10^3$ ) product law repeating letters and 3 numbers:  $26^4 \cdot P(10, 3)$  product law Product law:  $P(26, 4) \cdot P(10, 3) = \frac{26!10!}{3!4!7!22!}$  (Sequences with both not repeating) Difference

Rule:  $P(26, 4) \cdot 1000 + 26^4 \cdot P(10, 3) - P(26, 4) \cdot P(10, 3) = \frac{1000(26!)}{4!22!} + \frac{(26^4)10!}{3!10!} - \frac{26!10!}{3!4!7!22!}$