MATA31 - Assignment #8

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2.3.64

Find the derivative of

$$f(x) = \frac{(x-2)^2}{\left(x^2+1\right)(x-3)}$$

$$\begin{aligned} f(x) &= \frac{(x-2)^2}{\left(x^2+1\right)(x-3)} \\ &= \frac{x^2-2x+4}{x^3-3x^2+x-3} \end{aligned}$$

$$\begin{aligned} \frac{d}{dx} f(x) &= \frac{x^2-2x+4}{x^3-3x^2+x-3} \\ &= \frac{\left(x^3-3x^2+x-3\right)(2x-2)-\left(x^2-2x+4\right)\left(3x^2-6x+1\right)}{\left(x^3-3x^2+x-3\right)^2} \\ &= \frac{\left(2x^4-8x^3+8x^2-8x+6\right)-\left(3x^4-12x^3+25x^2-26x+4\right)}{\left(x^3-3x^2+x-3\right)^2} \\ &= \frac{-x^4+8x^3-23x^2+18x+8}{\left(x^3-3x^2+x-3\right)^2} \end{aligned}$$

2.3.88

Use the definition of the derivative to prove the following special case of the product rule:

$$\frac{d}{dx}\left(x^2 f(x)\right) = 2xf(x) + x^2 f'(x)$$

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{\left(x+h^2\right) f(x+h) - x^2 f(x)}{h} \\ &= \lim_{h\to 0} \frac{\left(x^2+2xh+h^2\right) f(x+h) - x^2 f(x)}{h} \\ &= \lim_{h\to 0} \frac{x^2 f(x+h) - x^2 f(x)}{h} + \lim_{h\to 0} \frac{2xhf(x+h)}{h} + \lim_{h\to 0} \frac{h^2 f(x+h)}{h} \\ &= \lim_{h\to 0} \frac{x^2 (f(x+h) - f(x))}{h} + \lim_{h\to 0} 2xf(x+h) + \lim_{h\to 0} hf(x+h) \\ &= x^2 \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + 2xf(x) \\ &= x^2 f'(x) + 2xf(x) \end{aligned}$$

As required to show. ■.

2.3.92

Consider the piecewise-defined function

$$f(x) = \begin{cases} g(x) & \text{if } x \leq c \\ h(x) & \text{if } x > c \end{cases}$$

Prove that if $g(x)$ and $h(x)$ are continuous and differentiable at $x = c$, and if $g(c) = h(c)$ and $g(c) = h(c)$, then f is differentiable at $x = c$.

Note that since $f(x) = g(x)$ at $x = c$, and $g(c) = h(c)$, then $f(c) = g(c) = h(c)$ **Lemma 1**
Suppose $g(x)$ is differentiable at c, and $h(x)$ is differentiable at c, and $g'(c) = h'(c) = L$. Then:

$$\begin{aligned} \lim_{x \to c} \frac{g(x) - g(c)}{x - c} &= L \\ \text{and } \lim_{x \to c} \frac{h(x) - h(c)}{x - c} &= L \end{aligned}$$

In turn, by limit definition, this means that:

$$\begin{aligned} \forall \varepsilon > 0, \exists \delta_1 > 0 \quad \text{s.t.} \quad 0 < |x - c| < \delta_1 &\implies \left| \frac{g(x) - g(c)}{x - c} - L \right| < \varepsilon \\ \text{and } \forall \varepsilon > 0, \exists \delta_2 > 0 \quad \text{s.t.} \quad 0 < |x - c| < \delta_2 &\implies \left| \frac{h(x) - h(c)}{x - c} - L \right| < \varepsilon \end{aligned}$$

Let $\delta = \min(\delta_1, \delta_2)$
Suppose $0 < |x - c| < \delta$
Suppose $x \leq c$. Then $f(x) = g(x)$, therefore:

$$\left| \frac{f(x) - f(c)}{x - c} - L \right| = \left| \frac{g(x) - g(c)}{x - c} - L \right|$$
$$< \varepsilon$$

Suppose $x > c$. Then $f(x) = h(x)$, therefore:

$$\left| \frac{f(x) - f(c)}{x - c} - L \right| = \left| \frac{h(x) - h(c)}{x - c} - L \right|$$
$$< \varepsilon$$

Therefore, for all x,

$$\left| \frac{f(x) - f(c)}{x - c} - L \right| < \varepsilon$$

Which means that:

$$\forall \varepsilon > 0, \exists \delta > 0 \quad \text{s.t.} \quad 0 < |x - c| < \delta \implies \left| \frac{f(x) - f(c)}{x - c} - L \right| < \varepsilon$$

Which in turn, means:

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = L$$

Which implies f is differentiable at $x = c$, as required to show. ■.

2.4.78

Find the derivative of:

$$3y = 5x^2 + \sqrt[3]{y-2}$$

Taking the derivative of both sides, we get:

$$\begin{aligned}
3\frac{dy}{dx} &= 10x + \frac{1}{3}(y-2)^{\frac{2}{3}}\frac{dy}{dx} \\
3\frac{dy}{dx} - \frac{1}{3}(y-2)^{-\frac{2}{3}}\frac{dy}{dx} &= 10x \\
\frac{dy}{dx}\left(3 - \frac{1}{3}(y-2)^{-\frac{2}{3}}\right) &= 10x \\
\frac{dy}{dx} &= \frac{10x}{\left(3 - \frac{1}{3}(y-2)^{-\frac{2}{3}}\right)}
\end{aligned}$$

2.4.84

$$y^3 - 3y - x = 1$$

,

(a):

$$\begin{aligned}
y^3 - 3y - x &= 1 \\
y^3 - 3y &= 0 \\
y\left(y^2 - 3\right) &= 0 \\
y &= 0, \pm\sqrt{3}
\end{aligned}$$

$$\begin{aligned}
3y^2\frac{dy}{dx} - 3\frac{dy}{dx} - 1 &= 0 \\
\frac{dy}{dx}\left(3y^2 - 3\right) &= 1 \\
\frac{dy}{dx} &= \frac{1}{3\left(y^2 - 1\right)} \\
\left.\frac{dy}{dx}\right|_{y=0} &= -\frac{1}{3} \\
\left.\frac{dy}{dx}\right|_{y=\sqrt{3}} &= \frac{1}{6} \\
\left.\frac{dy}{dx}\right|_{y=-\sqrt{3}} &= \frac{1}{6}
\end{aligned}$$

(b):

$$\begin{aligned} y^3 - 3y - x &= 1 \\ 8 - 6 - x &= 1 \\ x &= 1 \end{aligned}$$

$$\begin{aligned} 3y^2 \frac{dy}{dx} - 3\frac{dy}{dx} - 1 &= 0 \\ \frac{dy}{dx}\left(3y^2 - 3\right) &= 1 \\ \frac{dy}{dx} &= \frac{1}{3\left(y^2 - 1\right)} \\ \left.\frac{dy}{dx}\right|_{y=2} &= \frac{1}{9} \end{aligned}$$

(c):

$$\begin{aligned} 3y^2 \frac{dy}{dx} - 3\frac{dy}{dx} - 1 &= 0 \\ \frac{dy}{dx}\left(3y^2 - 3\right) &= 1 \\ \frac{dy}{dx} &= \frac{1}{3\left(y^2 - 1\right)} \\ 0 &= \frac{1}{3\left(y^2 - 1\right)} \\ 0 &= 1 \\ &DNE \end{aligned}$$

(d):

$$\begin{aligned} 3y^2 \frac{dy}{dx} - 3\frac{dy}{dx} - 1 &= 0 \\ \frac{dy}{dx}\left(3y^2 - 3\right) &= 1 \\ \frac{dy}{dx} &= \frac{1}{3\left(y^2 - 1\right)} \\ 0 &= \frac{1}{3\left(y^2 - 1\right)} \\ 3\left(y^2 - 1\right) = 0 y^2 - 1 = 0 y &= \pm 1 \end{aligned}$$

$$\begin{aligned} y^3 - 3y - x &= 1 \\ 1 - 3 - x &= 1 \\ x &= -3 \\ &(-3, 1) \end{aligned}$$

$$\begin{aligned} y^3 - 3y - x &= 1 \\ -1 + 3 - x &= 1 \\ x &= 1 \\ (1, -1) \end{aligned}$$

2.5.26

Find the derivative of

$$f(x) = 3^x + \log_3 x$$

$$f'(x) = \ln 3 \left(3^x\right) + \frac{1}{x \ln 3}$$

2.5.36

Find the derivative of

$$f(x) = \sqrt{\ln\left(x^2+1\right)}$$

$$f(x) = \left(\ln\left(x^2+1\right)\right)^{\frac{1}{2}}$$

$$\begin{aligned} f'(x) &= \frac{1}{2}\left(\ln\left(x^2+1\right)\right)^{-\frac{1}{2}} \cdot \frac{1}{x^2+1} \cdot 2x \quad \text{Chain Rule} \\ &= \frac{1}{2\sqrt{\ln\left(x^2+1\right)}} \cdot \frac{2x}{x^2+1} \\ &= \frac{2x}{2\left(x^2+1\right)\sqrt{\ln\left(x^2+1\right)}} \\ &= \frac{x}{\left(x^2+1\right)\sqrt{\ln\left(x^2+1\right)}} \end{aligned}$$

2.5.42

Find the derivative of

$$f(x) = \ln\left(x^x\right)$$

$$f(x) = x \ln x$$

$$\begin{aligned} f'(x) &= x\frac{1}{x} + \ln x \quad \text{Product Law} \\ &= 1 + \ln x \end{aligned}$$

2.5.52

Find the derivative of

$$f(x) = \frac{e^{2x}\left(x^3-2\right)^4}{x\left(3e^{5x}+1\right)}$$

$$\ln(y) = \ln\left(\frac{e^{2x}\left(x^3-2\right)^4}{x\left(3e^{5x}+1\right)}\right)$$

$$\begin{aligned}
\ln(y) &= \ln\left(e^{2x}\right) + 4\ln\left(x^3-2\right) - \ln x - \ln\left(3e^{5x}+1\right) \\
\frac{1}{y}y' &= 2 + \frac{12x^2}{x^3-2} - \frac{1}{x} - \frac{15e^{5x}}{3e^{5x}+1} \\
f'(x) &= (y)\left(2 + \frac{12x^2}{x^3-2} - \frac{1}{x} - \frac{15e^{5x}}{3e^{5x}+1}\right) \\
f'(x) &= \left(\frac{e^{2x}\left(x^3-2\right)^4}{x\left(3e^{5x}+1\right)}\right)\left(2 + \frac{12x^2}{x^3-2} - \frac{1}{x} - \frac{15e^{5x}}{3e^{5x}+1}\right)
\end{aligned}$$

2.6.30

Find the derivative of

$$f(x) = \frac{\log_3(3^x)}{\sin^2 x + \cos^2 x}$$

$$\begin{aligned}
f(x) &= \frac{\log_3(3^x)}{\sin^2 x + \cos^2 x} \\
&= \frac{\log_3(3^x)}{1} \\
&= \log_3\left(3^x\right) \\
&= x\log_3(3) \\
&= x(1) \\
&= x
\end{aligned}$$

$$f'(x) = 1$$

2.6.36

Find the derivative of

$$f(x) = \ln(x\sin x)$$

$$
\begin{aligned}
f'(x) &= \frac{1}{x\sin x}\cdot(x\cos x+\sin x)\\
&= \frac{x\cos x+\sin x}{x\sin x}\\
&= \frac{x\cos x}{x\sin x}+\frac{\sin x}{x\sin x}\\
&= \frac{\cos x}{\sin x}+\frac{1}{x}\\
&= \cot x+1
\end{aligned}
$$

2.6.64

Find the derivative of

$$y=(\sec x)^x$$

$$
\begin{aligned}
\ln y &= \ln\left((\sec x)^x\right)\\
&= x\ln(\sec x)
\end{aligned}
$$

$$
\begin{aligned}
\frac{y'}{y} &= \ln(\sec x)+x\cdot\frac{1}{\sec x}\cdot\sec(x)\tan(x)\\
&= \ln(\sec x)+x\tan x\\
y' &= (y)(\ln(\sec x)+x\tan x)\\
y' &= \left((\sec x)^x\right)(\ln(\sec x)+x\tan x)
\end{aligned}
$$

2.6.74

Find a function that has the derivative:

$$f'(x)=\frac{1}{1-9x^2}$$

We notice how this function looks similar to the derivative of the inverse hyperbolic tangent function:

$$\frac{dy}{dx}\tanh^{-1}x=\frac{1}{1-x^2}$$

Let's sub in $3x$ in the argument of $\tanh^{-1}$ to get $1-9x^2$ in the denominator:

$$
\begin{aligned}
\frac{dy}{dx}\tanh^{-1}3x &= \frac{1}{1-(3x)^2}(3)\\
&= \frac{3}{1-9x^2}
\end{aligned}
$$

Finally, lets multiply the function by $\frac{1}{3}$ to get rid of the numerator.

$$\frac{dy}{dx}\left(\frac{1}{3}\tanh^{-1}3x\right) = \frac{1}{3}\frac{1}{1-(3x)^2}(3)$$
$$= \frac{1}{3}\frac{3}{1-9x^2}$$
$$= \frac{1}{1-9x^2}$$

Therefore, a function that has the derivative $\frac{1}{1-9x^2}$ is $\frac{1}{3}\tanh^{-1}3x$, as required to show. ■.

Note that since $\tanh^{-1}x$ is defined strictly on $(-1,1)$, $f(x)$ is defined on solely $\left(-\frac{1}{3},\frac{1}{3}\right)$. The derivative is also only defined on the same interval.