MATA31 - Assignment #4

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A

Consider the linear function
$$f(x) = 3x + 1.$$
We know intuitively that
$$\lim_{x \to -1} f(x) = -2.$$

A. How close to -1 does x have to be such that $f(x)$ differs from -2 by less than 0.1?

B. How close to -1 does x have to be such that $f(x)$ differs from -2 by less than 0.01?

C. How close to -1 does x have to be such that $f(x)$ differs from -2 by less than 0.001?

$$\forall \varepsilon > 0 \, \exists \, \delta > 0 \quad \text{s.t} \quad 0 < |x - (-1)| < \delta \implies |3x + 1 - (-2)| < \varepsilon$$

When solving the limit, we get $\delta = \frac{\varepsilon}{3}$.

A. If $\varepsilon = 0.1$, x has to be within $0.1 \div 3 = 0.0\overline{3}$ of -1

B. If $\varepsilon = 0.01$, x has to be within $0.01 \div 3 = 0.00\overline{3}$ of -1

C. If $\varepsilon = 0.001$, x has to be within $0.001 \div 3 = 0.000\overline{3}$ of -1

B

Provide the formal definition of the limit
$$\lim_{x \to a} f(x) = L$$
in two ways: one using intervals and one using absolute value inequalities. Use this definition to prove that
$$\lim_{x \to 3} (2x + 4) = 10$$

Interval Definition

$$\forall \varepsilon > 0 \, \exists \, \delta > 0 \quad \text{s.t} \quad x \in (c - \delta, c) \cup (c, c + \delta) \implies f(x) \in (L - \varepsilon, L + \varepsilon)$$

Absolute value inequalities

$$\forall \varepsilon > 0 \, \exists \, \delta > 0 \quad \text{s.t} \quad 0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$$

We will use the absolute value inequality definition to prove this limit.
Want to show:
$$\forall \varepsilon > 0 \, \exists \, \delta > 0 \quad \text{s.t} \quad 0 < |x - 3| < \delta \implies |(2x + 4) - 10| < \varepsilon$$

Proof.
Let $\varepsilon > 0$ be arbitrary.

Choose $\delta = \frac{\varepsilon}{2}$. Note $\delta > 0$.

Assume $0 < |x-3| < \delta$. Then,

$$\begin{aligned}
|(2x+4)-10| &= |2x-6| && \text{(by algebra)} \\
&= |2(x-3)| && \text{(by algebra)} \\
&= |2|\,|x-3| && \text{(by properties of } |\cdot|\text{)} \\
&= 2\,|x-3| && \text{(since } 2>0\text{)} \\
&< 2\delta && \text{(by assumption)} \\
&= 2\left(\tfrac{\varepsilon}{2}\right) && \text{(by choice of } \delta\text{)} \\
&= \varepsilon && \text{(by algebra).}
\end{aligned}$$

As required to show ■.

C

Provide the formal definition of the limit

$$\lim_{x\to a^+} f(x) = \infty$$

in two ways: one using intervals and one using absolute value inequalities. Use this definition to prove that

$$\lim_{x\to 1^+} \frac{1}{x-1} = \infty$$

Interval Definition

$$\forall M>0\,\exists\,\delta>0 \quad \text{s.t} \quad x\in(a,a+\delta) \Longrightarrow f(x)\in(0,\,M)$$

Absolute value inequalities

$$\forall M>0\,\exists\,\delta>0 \quad \text{s.t} \quad 0<x-a<\delta \Longrightarrow f(x)>M$$

We will use the absolute value inequality definition to prove this limit.
Want to show:

$$\forall M>0\,\exists\,\delta>0 \quad \text{s.t} \quad 0<x-1<\delta \Longrightarrow \frac{1}{x-1}>M$$

Proof.
Let $\varepsilon > 0$ be arbitrary.

Choose $\delta = \frac{1}{M}$

Assume $0 < x-1 < \delta$. Then:

$$\begin{aligned}
& x-1<\delta && \text{(by algebra)} \\
\Longrightarrow\ & x-1 = \frac{1}{M} && \text{(by our choice of } \delta\text{)} \\
\Longrightarrow\ & \frac{1}{x-1} > M && \text{(since } x>0 \text{ and } M>0\text{)}
\end{aligned}$$

As required to show. ■.

D

Provide the formal definition of the limit

$$\lim_{x\to\infty} f(x) = L$$

in two ways: one using intervals and one using absolute value inequalities. Use this definition to prove that

$$\lim_{x\to\infty} \frac{2}{x+1} = 0$$

Interval Definition

$$\forall\varepsilon>0\,\exists N>0 \quad \text{s.t} \quad x\in(0,N) \implies f(x)\in(L-\varepsilon, L+\varepsilon)$$

Absolute value inequalities

$$\forall\varepsilon>0\,\exists N>0 \quad \text{s.t} \quad x>N \implies |f(x)-L|<\varepsilon$$

We will use the absolute value inequality definition to prove this limit. Want to show:

$$\forall\varepsilon>0\,\exists N>0 \quad \text{s.t} \quad x>N \implies \left|\tfrac{2}{x+1}-0\right|<\varepsilon$$

Proof.

Let $\varepsilon>0$ be arbitrary.

Choose $N=\max\left\{1, \frac{2}{\varepsilon}-1\right\}$. Note that $N>0$.

Assume $x>N$. Then

$$\begin{aligned}
x &> \frac{2}{\varepsilon}-1 && \text{(by our choice of N)}\\
\implies x+1 &= \frac{2}{\varepsilon} && \text{(by algebra)}\\
\implies \frac{1}{x+1} &< \frac{\varepsilon}{2} && \text{(since } x>0 \text{ and } \varepsilon>0\text{)}\\
\implies \frac{2}{x+1} &< \varepsilon && \text{(by algebra)}\\
\implies \frac{2}{x+1}-0 &< \varepsilon && \text{(by algebra)}\\
\implies \left|\frac{2}{x+1}-0\right| &< \varepsilon && \text{(since } x>0\text{)}
\end{aligned}$$

As required to show ■.

E

Provide the formal definition of the limit

$$\lim_{x\to\infty} f(x) = \infty$$

in two ways: one using intervals and one using absolute value inequalities. Use this definition to prove that

$$\lim_{x\to\infty} \left(x^2+1\right) = \infty$$

Interval Definition

$$\forall M > 0 \, \exists N > 0 \quad \text{s.t} \quad x \in (0, N) \implies f(x) \in (0, M)$$

Absolute value inequalities

$$\forall M > 0 \, \exists N > 0 \quad \text{s.t} \quad x > N \implies f(x) > M$$

We will use the absolute value inequality definition to prove this limit.
Want to show:

$$\forall M > 0 \, \exists N > 0 \quad \text{s.t} \quad x > N \implies x^2 + 1 > M$$

Proof.
Let $M > 0$ be arbitrary

Choose $N = \sqrt{M}$. Note that N > 0.

Assume $x > N$. Then

$$\begin{aligned} x^2 + 1 &> N^2 + 1 && \text{(by algebra)} \\ &= M + 1 && \text{(by our choice of } N\text{)} \\ &> M && \text{(by properties of inequalities)} \end{aligned}$$

As required to show ■.

F

Find the equation of a possible function f with $f(0) = 5$, $\lim_{x\to 1^+} f(x) = \infty$ and $\lim_{x\to 1^-} f(x) = \infty$

The equation that we will choose is:

$$f(x) = \frac{1}{(x-1)^2} + 4$$

The reason I chose this function is because I know $\frac{1}{(x-1)^2}$ has an asymptote at $x = 1$, so I squared the denominator to make the function approach $+\infty$ on both sides. The value of this function is 1 at $x = 0$, so I then added 4.

Prove f(0) = 5

$$\begin{aligned} f(x) &= \frac{1}{(x-1)^2} + 4 \\ f(\mathbf{0}) &= \frac{1}{(\mathbf{0}-\mathbf{1})^2} + 4 \\ &= \frac{1}{(-1)^2} + 4 \\ &= \frac{1}{1} + 4 \\ &= 1 + 4 \\ &= 5 \end{aligned}$$

As required to show. ■.

The limit as $x \to 1$ is ∞, making each one-sided limit also ∞.

G

Does

$$\lim_{x\to 2} \frac{|x-2|}{x-2}$$

exist? Explain why or why not.

The numerator of this function is always positive, so we can disregard it, as the sign doesn't change whether $x \to 2^-$ or $x \to 2^+$. There is a vertical asymptote at $x = 2$. If $x < 2 \Rightarrow f(x) < 0$, and if $x > 2 \Rightarrow f(x) > 0$, Therefore the limit does not exist $\blacksquare$.

H

Find $\lim\limits_{x\to 3} f(x)$ if it exists. Otherwise, explain by one-sided limits

$$f(x) = \begin{cases} x^2, & \text{if } x \le 3, \\ 3x+2, & \text{if } x > 3. \end{cases}$$

Lets look at each piece individually.

$$\lim_{x\to 3} x^2 = 9$$

Let us prove this.
Want to show:

$$\forall \varepsilon > 0 \, \exists \, \delta > 0 \quad \text{s.t} \quad 0 < |x-3| < \delta \Longrightarrow \left|x^2 - 9\right| < \varepsilon$$

<u>Proof.</u>
<u>Let</u> $\varepsilon > 0$ be arbitrary

<u>Choose</u> $\delta = min\left\{1, \frac{\varepsilon}{7}\right\}$. Note that N > 0.

<u>Assume</u> $0 < |x-3| < \delta$.

Since $x^2 - 9 = (x+3)(x-3)$ and $x - 3 < \delta$, we first need to obtain a bound on $|x+3|$. Then

$$\begin{aligned}
|x-3| < \delta &\Longrightarrow |x-3| < 1 && \left(\text{since } \delta = \min\left\{1, \frac{\varepsilon}{7}\right\} \le 1\right) \\
&\Longrightarrow -1 < x-3 < 1 && (\text{by properties of } |\cdot|) \\
&\Longrightarrow 5 < x+3 < 7 && (\text{by algebra}) \\
&\Longrightarrow -7 < x+3 < 7 && (\text{by properties of inequalities}) \\
&\Longrightarrow |x+3| < 7 && (\text{by properties of } |\cdot|)
\end{aligned}$$

Therefore, $|x+3| < 7$ $(\star)$.

It now follows that:

$$\begin{aligned}
\left|x^2-9\right| &= |(x+3)(x-3)| && \text{(by algebra)} \\
&= |x+3|\,|x-3| && \text{(by properties of } |\cdot|) \\
&< |x+3|\,\delta && \text{(by assumption)} \\
&< 7\delta && \text{(by } (\star)) \\
&= 7\frac{\epsilon}{7} && \text{(by our choice of } \delta) \\
&= \varepsilon && \text{(by algebra).}
\end{aligned}$$

As required to show. ■.

Now, lets look at

$$\lim_{x\to 3} 3x+2 = 11$$

Let us prove this.
Want to show:

$$\forall \varepsilon > 0\, \exists\, \delta > 0 \quad \text{s.t} \quad 0 < |x-3| < \delta \Longrightarrow |(3x+2)-11| < \varepsilon$$

Proof.
Let $\varepsilon > 0$ be arbitrary

Choose $\delta = \frac{\varepsilon}{3}$. Note that N > 0.

Assume $0 < |x-3| < \delta$.

$$\begin{aligned}
|(3x+2)-11| &= |3x-9| && \text{(by algebra)} \\
&= |3(x-3)| && \text{(by algebra)} \\
&= |3|\,|x-3| && \text{(by properties of } |\cdot|) \\
&= 3\,|x-3| && \text{(since } 2>0) \\
&< 3\delta && \text{(by assumption)} \\
&= 3\frac{\varepsilon}{3} && \text{(by choice of } \delta) \\
&= \varepsilon && \text{(by algebra).}
\end{aligned}$$

As required to show. ■.

Since the left sided limit ≠ the right sided limit, the limit does not exist. ■.