CSCA67 - Exercises #9

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2.1

The famous Fibonacci sequence is defined as follows.

$$F_0 = 0$$
$$F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2}$$

Perhaps surprisingly, the n^{th} Fibonacci number can be calculated for any n by using the following formula:

$$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}} \text{ where } \phi = \frac{1+\sqrt{5}}{2} \text{ and } \psi = \frac{1-\sqrt{5}}{2}$$

Prove this formula.

Let $P(n)$ be $F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$

Note the following:
$F_0 = 0$
$F_1 = 1$
$\phi = \frac{1+\sqrt{5}}{2}$
$\psi = \frac{1-\sqrt{5}}{2}$
WTS: $\forall k \geq 0, (\forall 0 \leq i < k, (P(i)) \rightarrow P(k))$
Proof.

Let k be arbitrary. (1)

Suppose $k \geq 0$ (2)

Suppose$\forall 0 \leq i < k, P(i)$ (3)

Suppose $k = 0$ (4)

$$\frac{\phi^0 - \psi^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} \quad \text{Math} \quad (5)$$

$$= \frac{0}{\sqrt{5}} \quad (6)$$

$$= 0 \quad (7)$$

$$\frac{\phi^0 - \psi^0}{\sqrt{5}} = F_0 \quad (8)$$

$P(0)$ (9)

Suppose $k = 1$ (10)

$$\frac{\phi - \psi}{\sqrt{5}} = \frac{\frac{2\sqrt{5}}{2}}{\sqrt{5}} \quad \text{Math} \quad (11)$$

$$= \frac{2\sqrt{5}}{2\sqrt{5}} \quad (12)$$

$$= 1 \quad (13)$$

$$\frac{\phi - \psi}{\sqrt{5}} = F_1 \quad (14)$$

$P(1)$ (15)

Suppose $k \geq 2$ (16)

$$F_k = F_{k-1} + F_{k-2} \quad (17)$$

$$= \frac{\phi^{k-1} - \psi^{k-1} + \phi^{k-2} - \psi^{k-2}}{\sqrt{5}} \quad (18)$$

$$F_n = \frac{\phi^k - \psi^k}{\sqrt{5}} \quad (19)$$

$$F_n = \frac{\phi^k - \psi^k}{\sqrt{5}} \quad (20)$$

(21)