FUNGSI GAMMA

Improper integral tertentu

Mempermudah perhitungan improper integral

DEFINISI FUNGSI GAMMA

- Untuk setiap n, didefinisikan

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$$

Dengan uji konvergensi, fungsi gamma terdefinisi untuk setiap $n > 0$.

$$\boxed{\Gamma(n+1)} = \int_0^\infty x^n e^{-x}\,dx$$

$$= -\left[x^n e^{-x} \Big|_0^\infty - n\int_0^\infty x^{n-1} e^{-x}\,dx \right]$$

$$= n\int_0^\infty x^{n-1} e^{-x}\,dx \quad \boxed{= n\,\Gamma(n)}$$

Rumus lain yang melibatkan fungsi gamma

$$\boxed{\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}}, \quad 0 < x < 1 \ldots$$

CONTOH 1

Buktikan $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$

Ambil $x = \frac{1}{2}$ maka dengan rumus sebelumnya

$$\Gamma(1/2)\Gamma(1 - 1/2) = \frac{\pi}{\sin\left(\frac{\pi}{2}\right)}$$

atau $\Gamma(1/2)^2 = \pi$

atau $\Gamma(1/2) = \sqrt{\pi}$

IMPROPER INTEGRAL KHUSUS

$$\boxed{\text{Prove that } \int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}\,/\,2.}$$

Let $I_M = \int_0^M e^{-x^2}\,dx = \int_0^M e^{-y^2}\,dy$ and let $\lim\limits_{M\to\infty} I_M = I$, Then

$$\begin{aligned} I_M^2 &= \left(\int_0^M e^{-x^2}\,dx\right)\left(\int_0^M e^{-x^2}\,dy\right) \\ &= \int_0^M \int_0^M e^{-(x^2+y^2)}\,dx\,dy \\ &= \iint_M e^{-(x^2+y^2)}\,dx\,dy \end{aligned}$$

Since the integrand is positive, we have

$$\iint_M e^{-(x^2+y^2)}\,dx\,dy \leqq I_M^2 \leqq \iint_M e^{-(x^2+y^2)}\,dx\,dy$$

IMPROPER INTEGRAL KHUSUS

Using polar coordinates, we have

$$\int_{\phi=0}^{\pi/2}\int_{p=0}^{M} e^{-\rho^2}\rho\, d\rho\, d\phi \leqq I_M^2 \leqq \int_{\phi=0}^{\pi/2}\int_{\pi=0}^{M\sqrt{2}} e^{-\rho^2}\rho\, d\rho\, d\phi$$

or

$$\frac{\pi}{4}(1-e^{M^2}) \leqq I_M^2 \leqq \frac{\pi}{4}(1-e^{-2M^2})$$

Then, taking the limit as $M \to \infty$, we find

$$\lim_{M\to\infty} I^2{}_M = I^2 = \pi/4 \quad \text{and} \quad \boxed{I = \sqrt{\pi}\,/\,2}$$

CONTOH 1 (Bukti alternatif)

Buktikan $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$

Misalkan $x = u^2$ dan $\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} x^{-1/2} e^{-x}\, dx.$

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^{\infty} e^{-u^2}\, du = 2\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi}$$

CONTOH 2

Hitung

$$\int_0^\infty \sqrt{x}\ e^{-x^3} dx$$

Misalkan $y = x^3$ maka integral di atas menjadi

$$\int_0^\infty y^{1/6}\ e^{-y}\left(\frac{1}{3}y^{-2/3}dy\right) = \frac{1}{3}\int_0^\infty y^{-1/2}e^{-y}dy$$

Atau

$$\frac{1}{3}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{3}$$

CONTOH 3

Hitung

$$\Gamma\left(-\frac{1}{2}\right)$$

Untuk $x < 0$, gunakan rumus

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}$$

Sehingga

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{\left(-\frac{1}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)} = \frac{\pi}{\left(-\frac{1}{2}\right)} = -2\sqrt{\pi}$$

Contoh Kasus

Suatu partikel di posisi awal x = a dan bermassa m.

Waktu yang dibutuhkan partikel berpindah ke suatu posisi tertentu dinyatakan dengan

$$T = \sqrt{m} \int_0^a \frac{1}{\sqrt{\frac{\ln(a)}{x}}} dx$$

Dengan fungsi gamma, tentukan waktu yang dibutuhkan partikel untuk pindah. Nyatakan dalam m dan a.

Solusi :

Misalkan $y = \ln\left(\frac{a}{x}\right)$ maka $x = ae^{-y}$ dan $dx = -\, ae^{-y}dy$.

Batas integralnya $[0, a]$ menjadi $[\infty, 0]$. Sehingga

$$T = \sqrt{m}\int_0^a \frac{1}{\sqrt{\frac{\ln(a)}{x}}}dx = \sqrt{m}\int_0^\infty \left(\frac{1}{y^{1/2}}\right)ae^{-y}dy$$

$$= a\sqrt{m}\int_0^\infty y^{\frac{1}{2}-1}e^{-y}dy = a\sqrt{m}\,\Gamma\left(\frac{1}{2}\right) = a\sqrt{m\pi}$$