Invers transformasi Laplace

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TUJUAN PEMBELAJARAN

- Mampu membentuk factor linear/kuadratik pada fungsi rasional
- Mampu menggunakan sifat invers transformasi Laplace
- Mampu mensintesis factor lienar/kuadratik untuk menyelesaikan invers transformasi Laplace

FUNGSI RASIONAL

$$\mathrm{F}(\mathrm{x}) = \frac{N(x)}{\boldsymbol{D}(\boldsymbol{x})}$$

FUNGSI RASIONAL : BENTUK UMUM

Bentuk Umum : $\frac{N(x)}{D(x)}$ dengan $N(x)$ dan $D(x)$ dua polinomial serta $\deg(N) < \deg(D)$

Contoh

- $\frac{1}{s-2}$
- $\frac{s(s+1)}{(s-1)(s+2)(s-9)}$

$D(x)$ Hanya mempunyai faktor linear

Bentuk

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s-p_1)\cdot(s-p_2)\cdot(s-p_3)\cdot\ldots\cdot(s-p_n)}$$

Dpt diubah menjadi **jumlahan fungsi rasional**

$$F(s) = \frac{r_1}{(s-p_1)} + \frac{r_2}{(s-p_2)} + \frac{r_3}{(s-p_3)} + \ldots + \frac{r_n}{(s-p_n)}$$

dengan

$$\boxed{r_k = \lim_{s \to p_k} (s-p_k)F(s) = (s-p_k)F(s)\Big|_{s=p_k}}$$

Sederhanakan bentuk berikut :

Contoh 1

$$F_1(s) = \frac{3s+2}{s^2+3s+2}$$

Misalkan

$$F_1(s) = \frac{3s+2}{s^2+3s+2} = \frac{3s+2}{(s+1)(s+2)} = \frac{r_1}{(s+1)} + \frac{r_2}{(s+2)}$$

dengan

$$r_1 = \lim_{s \to -1} (s+1)F(s) = \left.\frac{3s+2}{(s+2)}\right|_{s=-1} = -1$$

dan

$$r_2 = \lim_{s \to -2} (s+2)F(s) = \left.\frac{3s+2}{(s+1)}\right|_{s=-2} = 4$$

sehingga

$$F_1(s) = \frac{3s+2}{s^2+3s+2} = \frac{-1}{(s+1)} + \frac{4}{(s+2)}$$

$D(x)$ mempunyai faktor linear dan tidak linear

Bentuk

$$F(s) = \frac{N(s)}{(s-p_1)^m (s-p_2)\ldots(s-p_{n-1})(s-p_n)}$$

Dapat diubah menjadi **jumlahan fungsi rasional**

$$F(s) = \frac{r_{11}}{(s-p_1)^m} + \frac{r_{12}}{(s-p_1)^{m-1}} + \frac{r_{13}}{(s-p_1)^{m-2}} + \ldots + \frac{r_{1m}}{(s-p_1)}$$
$$+ \frac{r_2}{(s-p_2)} + \frac{r_3}{(s-p_3)} + \ldots + \frac{r_n}{(s-p_n)}$$

$D(x)$ mempunyai faktor linear dan tidak linear

Dengan

$$r_k = \lim_{s \to p_k} (s - p_k)F(s) = (s - p_k)F(s)\Big|_{s = p_k}$$

untuk k = 2, 3, 4, ..., n

dan

$$r_{1k} = \lim_{s \to p_1} \frac{1}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}} [(s - p_1)^m F(s)]$$

untuk k = 1, 2, ..., m

Contoh 2 :

Ubah bentuk berikut ke bentuk jumlahan rasional

$$F_4(s) = \frac{s+3}{(s+2)(s+1)^2}$$

Misalkan

$$F_4(s) = \frac{s+3}{(s+2)(s+1)^2} = \frac{r_1}{(s+2)} + \frac{r_{21}}{(s+1)^2} + \frac{r_{22}}{(s+1)}$$

$$r_{21} = \lim_{s\to-1} \frac{1}{(2-1)!}\frac{d^{2-1}}{ds^{2-1}}\left((s+1)^2\, F_4(s)\right) = \lim_{s\to-1} \frac{-1}{(s+2)^2} = -1$$

Contoh 2 : lanjutan ...

$$r_1 = \lim_{s \to -2} (s+2) F_4(s) = \lim_{s \to -2} \frac{s+3}{(s+1)^2} = 1$$

dan

$$r_{22} = \lim_{s \to -1} (s+1)^2 \, F_4(s) = \lim_{s \to -1} \frac{s+3}{s+2} = 2$$

Sehingga

$$F_4(s) = \frac{1}{s+2} + \frac{-1}{(s+1)^2} + \frac{2}{s+1}$$

INVERS TRANSFORMASI LAPLACE

$$F(t) = \mathcal{L}^{-1}\big(f(s)\big)$$

INVERS TRANSFORMASI LAPLACE : DEFINISI

Misalkan

$$\mathcal{L}\big(F(t)\big) = f(s)$$

Maka invers transformasinya

$$F(t) = \mathcal{L}^{-1}\big(f(s)\big)$$

TABEL INVERS TRANSFORMASI LAPLACE

No	f(s)	$L^{-1}\{f(s)\} = F(t)$	Syarat
1.	$\frac{1}{s}$	1	s > 0
2.	$\frac{1}{s^{n+1}}$	$\frac{t^n}{\Gamma(n+1)}$	s >0 n > -1
3.	$\frac{1}{s-a}$	e^{at}	s > a
4	$\frac{1}{s^2+a^2}$	$\frac{1}{a}$ sin t	s > 0

5	$\dfrac{s}{s^2+a^2}$	cos at	$s > 0$
6	$\dfrac{1}{s^2-a^2}$	$\dfrac{1}{a}$ sinh at	$s > \lvert a \rvert$
7	$\dfrac{s}{s^2-a^2}$	cosh at	$s > \lvert a \rvert$

SIFAT-SIFAT ITL

1. Sifat Kelinieran

Jika $L^{-1} = \{f_1(s)\} = F_1(t)$ dan $L^{-1} = \{f_2(s)\} = F_2(t)$, maka :

$$L^{-1}\{\, c_1 f_1(s) \pm c_2 f_2(s) \,\} = c_1 F_1(t) \pm c_2 F_2(t)$$

<u>Contoh 1.</u>

Hitunglah $L^{-1}\left\{\frac{5s+4}{s^3} - \frac{2s-18}{s^2+9}\right\} = \ldots\ldots$

Penyelesaian.

$$L^{-1}\left\{\frac{5s+4}{s^3} - \frac{2s-18}{s^2+9}\right\}$$

$$= L^{-1}\left\{\frac{5}{s^2}\right\} + L^{-1}\left\{\frac{4}{s^3}\right\} - 2\,L^{-1}\left\{\frac{s}{s^2+9}\right\} + 18\,L^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= 5\,\frac{t}{\Gamma(2)} + 4\,\frac{t^2}{\Gamma(3)} - 2\cos 3t + \frac{18}{3}\sin 3t$$

$$= 5\,t + 2\,t^2 - 2\cos 3t + 6\sin 3t$$

2. Sifat Translasi

Jika $L^{-1}\{ f(s) \} = F(t)$, maka
$$\begin{cases} a)\ L^{-1}\{f(s-a)\} = e^{at}F(t) \\ b)\ L^{-1}\{e^{-as} f(s)\} = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases} \\ \qquad = F(t-a)\,U(t-a) \end{cases}$$

<u>Contoh 2.</u>

Berapakah $L^{-1}\left\{\dfrac{1}{s^4+4s^3+6s^2+4s+1}\right\} = \ldots\ldots\ldots?$

Penyelesaian.

$$L^{-1}\left\{\frac{1}{s^4+4s^3+6s^2+4s+1}\right\} = L^{-1}\left\{\frac{1}{(s+1)^4}\right\} = e^{-t}\,\frac{t^3}{\Gamma(4)} = \frac{1}{6}\,t^3 e^{-t}$$

3. Sifat Pergantian Skala

Jika $L^{-1}\{f(s)\} = F(t)$, maka $L^{-1}\{f(as)\} = \frac{1}{a}\,F\left(\frac{t}{a}\right)$

<u>Contoh 3.</u>

Berapakah $L^{-1}\left\{\frac{1}{2s^2}\right\}$ = ...?

Penyelesaian.

$$L^{-1}\left\{\frac{1}{2s^2}\right\} = L^{-1}\left\{\frac{2}{(2s)^2}\right\} = 2\,L^{-1}\left\{\frac{1}{(2s)^2}\right\}$$

$$= \frac{2}{2}\cdot\frac{\left(t/2\right)}{\Gamma(2)} = \frac{1}{2}\,t$$

4. Invers Transformasi Laplace dari Derivative

Jika $L^{-1}\{ f(s) \} = F(t)$, maka $\boxed{L^{-1}\{ f^{(n)}(s) \} = (-1)^n t^n}$

Untuk n = 1, maka :

$L^{-1}\{ f'(s) \} = -t\, L^{-1}\{f(s)\}$, sehingga

$$\boxed{L^{-1}\{f(s)\} = -\frac{1}{t}\, L^{-1}\{f'(s)\}}$$

<u>Contoh 4.</u>

Berapakah $L^{-1}\left\{\ln \frac{s+1}{s-1}\right\} = \ldots\ldots\ldots\ldots?$

Pakai Rumus $L^{-1}\{f(s)\} = -\frac{1}{t}\, L^{-1}\{f'(s)\}$, sehingga

$$L^{-1}\left\{\ln \frac{s+1}{s-1}\right\} = -\frac{1}{t}\, L^{-1}\left\{\frac{d}{ds} \ln \frac{s+1}{s-1}\right\} = -\frac{1}{t}\, L^{-1}\left\{\frac{s-\cancel{1}}{s+1} \cdot \frac{s\cancel{-}1-(s\cancel{+}1)}{\cancel{(s-1)^2}}\right\}$$

$$= -\frac{1}{t}\, L^{-1}\left\{\frac{-2}{(s+1)(s-1)}\right\} = \frac{2}{t}\, L^{-1}\left\{\frac{1}{s^2-1}\right\} = \frac{2}{t}\ \sinh t$$

6. ITL UNTUK INTEGRAL

$$\text{If} \quad \mathcal{L}^{-1}\{f(s)\} = F(t), \quad \text{then}$$

$$\mathcal{L}^{-1}\left\{\int_s^\infty f(u)\, du\right\} = \frac{F(t)}{t}$$

CONTOH :

$$\mathcal{L}^{-1}\left\{\ln\left(1+\frac{1}{s}\right)\right\}$$

SOLUSI :

KARENA

$$\mathcal{L}^{-1}\left\{\int_s^\infty \left(\frac{1}{u} - \frac{1}{u+1}\right) du\right\} = \mathcal{L}^{-1}\left\{\ln\left(1+\frac{1}{s}\right)\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} = 1 - e^{-t}$$

MAKA

$$\mathcal{L}^{-1}\left\{\ln\left(1+\frac{1}{s}\right)\right\} = \frac{1-e^{-t}}{t}$$

7. PERKALIAN DENGAN s

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $F(0) = 0$, then

$$\mathcal{L}^{-1}\{s\,f(s)\} = F'(t)$$

CONTOH : DENGAN FAKTA DARI TRANSFORMASI LAPLACAE UNTUK SIN(t) , BUKTIKAN

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t$$

KARENA

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t \quad \text{and} \quad \sin 0 = 0.$$

MAKA DENGAN MENGGUNAKAN SIFAT DI ATAS, DIPEROLEH

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \frac{d}{dt}(\sin t) = \cos t$$

8. PEMBAGIAN OLEH s

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\,du$$

CONTOH : BUKTIKAN

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \frac{1}{4}(1-\cos 2t)$$

KARENA $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t$, **MAKA**

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \int_0^t \frac{1}{2}\sin 2u\,du = \frac{1}{4}(1-\cos 2t)$$

9. SIFAT KONVOLUSI :

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $\mathcal{L}^{-1}\{g(s)\} = G(t)$, then

$$\mathcal{L}^{-1}\{f(s)\,g(s)\} = \int_0^t F(u)\,G(t-u)\,du$$

CONTOH : HITUNG $\mathcal{L}^{-1}\left\{\dfrac{1}{(s-1)(s-2)}\right\}$

KARENA $\mathcal{L}^{-1}\left\{\dfrac{1}{s-1}\right\} = e^t$ and $\mathcal{L}^{-1}\left\{\dfrac{1}{s-2}\right\} = e^{2t}$ **MAKA**

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s-2)}\right\} = \int_0^t e^u\,e^{2(t-u)}\,du = e^{2t} - e^t$$