IMPROPER INTEGRAL

Motivasi

- Akumulasi Pada Ruang Tak Terbatas
- Pemodelan Radar

TIPE 1 : $\int_a^b f(x)\,dx$

Diberikan f: [a, b] $\to R$ yang memenuhi minimal satu dari tiga keadaan berikut

- $\lim\limits_{x\to a+} f(x) = \infty$
- $\lim\limits_{x\to b-} f(x) = \infty$
- Ada $a < c < b$ sehingga $\lim\limits_{x\to c} f(x) = \infty$

TIPE 2 : DOMAIN TAK-TERBATAS

- $\int_{a}^{\infty} f(x)\,dx = \lim\limits_{b\to\infty} \int_{a}^{b} f(x)\,dx$
- $\int_{-\infty}^{a} f(x)\,dx = \lim\limits_{b\to-\infty} \int_{b}^{a} f(x)\,dx$
- $\int_{-\infty}^{\infty} f(x)\,dx = \lim\limits_{b\to\infty} \int_{-b}^{b} f(x)\,dx$

DEFINISI KONVERGEN

- Improper integral dikatakan **konvergen** jika
  1. Nilai integral / limitnya ada untuk tipe 1, atau
  2. Nilai integral / limitnya ada untuk tipe 2
- Jika tidak demikian, improper integral dikatakan **divergen**.

Selidiki kekonvergenan dari integral tak wajar $\int_1^4 \frac{3}{\sqrt{x-1}}dx$!

Penyelesaian :

Fungsi $f(x) = \frac{3}{\sqrt{x-1}}$ kontinu pada selang (1, 4], integral tak wajar dari fungsi f(x) pada selang [1, 4] adalah

$$\int_1^4 \frac{3}{\sqrt{x-1}}dx = \lim_{a\to 1^+}\int_a^4 \frac{3}{\sqrt{x-1}}dx$$

$$= 3\lim_{a\to 1^+}\left(2\sqrt{x-1}\right)_a^4$$

$$= 3\lim_{a\to 1^+}\left(2\sqrt{3} - 2\sqrt{a-1}\right) = 6\sqrt{3}.$$

Activate Wind

Selidiki kekonvergenan integral tak wajar $\int_{-\infty}^{\infty} \sinh x \, dx$.

Penyelesaian

$$\int_{-\infty}^{\infty} \sinh x \, dx = \int_{-\infty}^{0} \sinh x \, dx + \int_{0}^{\infty} \sinh x \, dx = \lim_{a \to -\infty} \int_{a}^{0} \sinh x \, dx + \lim_{b \to \infty} \int_{0}^{b} \sinh x \, dx$$

$$= \lim_{a \to -\infty} (\cosh x)_a^0 + \lim_{b \to \infty} (\cosh x)_0^b$$

$$= \lim_{a \to -\infty} (1 - \cosh a) + \lim_{b \to \infty} (\cosh b - 1).$$

Karena $\lim_{a \to -\infty} (1 - \cosh a) = -\infty$ dan $\lim_{b \to \infty} (\cosh b - 1) = \infty$, maka kedua limit tersebut tidak ada. Sehingga integral tak wajar ini divergen.

UJI KONV IMPROPER INTEGRAL TIPE 1

TEOREMA 1

1. $\displaystyle\int_a^b \frac{dx}{(x-a)^p}$ converges if $p < 1$ and diverges if $p \geqq 1$.

2. $\displaystyle\int_a^b \frac{dx}{(b-x)^p}$ converes if $p < 1$ and diverges if $p \geqq 1$.

UJI KONV. IMPROPER INT. TIPE 1

TEOREMA 2

(a) *Convergence* Let $g(x) \geq 0$ for $a < x \leq b$, and suppose that $\int_a^b g(x)\,dx$ converges. Then if $0 \leq f(x) \leq g(x)$ for $a < x \leq b$, $\int_a^b f(x)\,dx$ also converges.

(b) *Divergence.* Let $g(x) \geq 0$ for $a < x \leq b$, and suppose that $\int_a^b g(x)\,dx$ diverges. Then if $f(x) \geq g(x)$ for $a < x \geq b$, $\int_a^b f(x)\,dx$ also diverges.

UJI KONV. IMPROPER INT. TIPE 1

TEOREMA 3

Let $\lim\limits_{x \to a+} (x-a)^p f(x) = A$. Then

(i) $\int_a^b f(x)\,dx$ converges if $\boxed{p < 1}$ and A is finite.

(ii) $\int_a^b f(x)\,dx$ diverges if $\boxed{p \geq 1}$ and $A \neq 0$

UJI KONV. IMPROPER INT. TIPE I

TEOREMA4

Let $\lim_{x \to b-} (b-x)^p f(x) = B$. Then

(i) $\int_a^b f(x)\,dx$ converges of $\boxed{p < 1}$ and B is finite.

(ii) $\int_a^b f(x)\,dx$ diverges if $\boxed{p \geq 1}$ and $B \neq 0$

CONTOH 1

Improper integral ini $\int_3^b \frac{\ln x}{(x-3)^4}\,dx$ divergen

karena $\frac{\ln x}{(x-3)^4} > \frac{1}{(x-3)^4}$ dan $\int_3^b \frac{dx}{(x-3)^4}$ divergen

CONTOH 2

$$\int_1^5 \frac{dx}{\sqrt{x^4-1}} \text{ converges, since}$$

$$\lim_{x\to 1+} (x-1)^{1/2} \cdot \frac{1}{(x^4-1)^{1/2}}$$

$$= \lim_{x\to 1+} \sqrt{\frac{x-1}{x^4-1}} = \frac{1}{2}.$$

CONTOH 3

$$\int_0^3 \frac{dx}{(3-x)\sqrt{x^2+1}} \text{ diverges, since}$$

$$\lim_{x\to 3-}(3-x)\cdot\frac{1}{(3-x)\sqrt{x^2+1}} = \frac{1}{\sqrt{10}}.$$

UJI KONV. IMPROPER INT. TIPE II

TEOREMA 1

Misalkan

$$\int_a^{\infty} \frac{dx}{x^p}$$

- Konvergen jika p > 1
- Divergen jika $p \leq 1$

UJI KONV. IMPROPER INT. TIPE II

TEOREMA 2

Let $g(x) \geqq 0$ for all $x \geqq a$, and suppose that

$$\int_a^\infty g(x)\,dx \text{ converges.}$$

if $0 \leqq f(x) \leqq g(x)$ for all $x \geqq a$.

$$\Downarrow$$

$$\boxed{\int_a^\infty f(x)\,dx \text{ also converges.}}$$

UJI KONV. IMPROPER INT. TIPE II

TEOREMA 3

Jika $a \leq g(x) \leq f(x)$ untuk setiap $x \geq a$, dan

$$\int_a^\infty g(x)\, dx$$

Divergen, maka

$$\int_a^\infty f(x)\, dx$$

Divergen juga

UJI KONV. IMPROPER INT. TIPE II

TEOREMA 4

Misalkan

$$\lim_{x\to\infty} x^p f(x) = a$$

- Konvergen, jika p > 1 dan a hingga
- Divergen, jika $p \leq 1$ dan $a \neq 0$

CONTOH 1

Selidiki konv.

$$\int_2^{\infty} \frac{dx}{\ln(x)}$$

Karena $\frac{1}{\ln(x)} \geq \frac{1}{x}$ untuk $\int_2^{\infty} \frac{dx}{x}$ divergen (lihat Teo. 1), maka

$$\int_2^{\infty} \frac{dx}{\ln(x)}$$

divergen

CONTOH 2

Buktikan

$$\int_0^\infty e^{-x^2}\, dx$$

Konveregen. Gunakan teorema 4 untuk p = 2 >1 , maka

$$\lim_{x\to\infty} x^2 e^{-x^2} = 0$$

Sehingga improper tersebut konvergen.

LATIHAN 1 : TIPE 1

- **Selidiki konvergensi**

(a) $\displaystyle\int_0^1 \frac{dx}{(x+1)\sqrt{1-x^2}}$

(b) $\displaystyle\int_0^1 \frac{\cos x}{x^2}\,dx$

(c) $\displaystyle\int_0^{\pi/2} \frac{e^{-x}\cos x}{x}\,dx$

(d) $\displaystyle\int_1^2 \frac{\ln x}{\sqrt[3]{8-x^3}}\,dx$

LATIHAN 2 : TIPE 2

- Selidiki konvergensi

(a) $\int_0^\infty \frac{x^2+1}{x^4+1}\,dx$

(d) $\int_0^\infty \frac{dx}{x^4+4}$

(b) $\int_2^\infty \frac{x\,dx}{\sqrt{x^3-1}}$

(c) $\int_t^\infty \frac{dx}{x\sqrt{3x+2}}$