TRANSFORMASI LAPLACE : Sifat dan Penggunaannya

REFERENSI TAMBAHAN

FUNDAMENTAL OF DIFFERENTIAL EQUATION DAN BOUNDARY VALUE PROBLEM

(NAGLE, SAFF, DAN SNIDER : ADDISON WESLEY 2012)

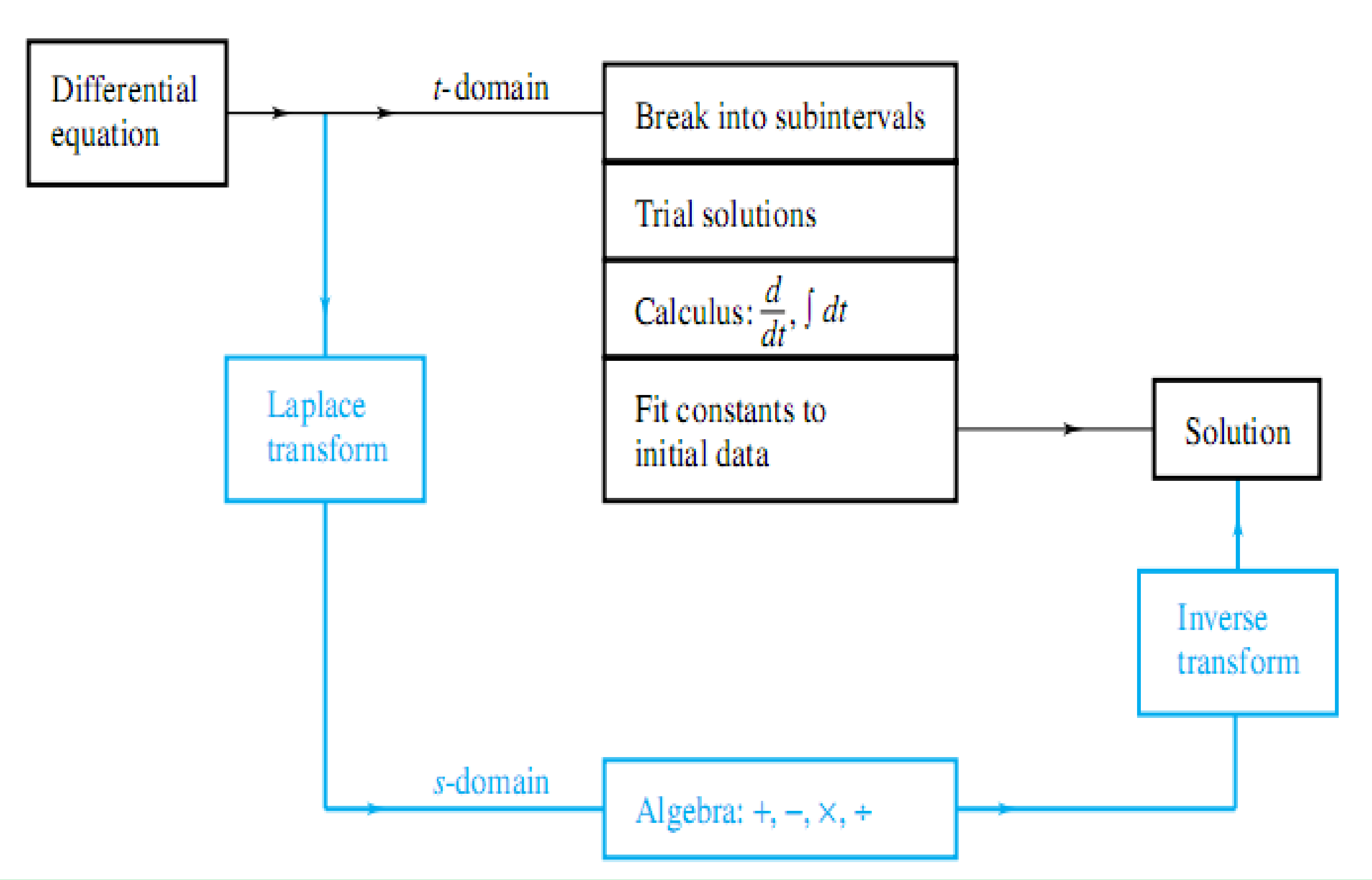
Bagaimana mengubah bentuk persamaan diferensial menjadi persamaan biasa???

	t-Domain	s-Domain
(5)	$x'(t) + \frac{3}{500}x(t) = g(t)$, $x(0) = 30$;	$sX(s) - 30 + \frac{3}{500}X(s) = G(s)$

where $G(s)$ is the Laplace transform of $g(t)$.
solution is simply

(6) $$X(s) = \frac{30}{s + 3/500} + \frac{G(s)}{s + 3/500} .$$

Transformasi Laplace untuk solusi PD



TRANSFORMASI LAPLACE :

Laplace Transform

Definition 1. Let $f(t)$ be a function on $[0, \infty)$. The **Laplace transform** of f is the function F defined by the integral

$$(1) \qquad F(s) := \int_0^{\infty} e^{-st} f(t)\,dt \,.$$

The domain of $F(s)$ is all the values of s for which the integral in (1) exists.† The Laplace transform of f is denoted by both F and $\mathcal{L}\{f\}$.

Example 1 Determine the Laplace transform of the constant function $f(t) = 1, t \geq 0$.

Solution Using the definition of the transform, we compute

$$F(s) = \int_0^{\infty} e^{-st} \cdot 1\, dt = \lim_{N \to \infty} \int_0^{N} e^{-st} dt$$

$$= \lim_{N \to \infty} \frac{-e^{-st}}{s} \Bigg|_{t=0}^{t=N} = \lim_{N \to \infty} \left[\frac{1}{s} - \frac{e^{-sN}}{s} \right] .$$

Since $e^{-sN} \to 0$ when $s > 0$ is fixed and $N \to \infty$, we get

$$F(s) = \frac{1}{s} \quad \text{for} \quad s > 0 .$$

When $s \leq 0$, the integral $\int_0^{\infty} e^{-st} dt$ diverges. (Why?) Hence $F(s) = 1/s$, with the domain of $F(s)$ being all $s > 0$. ♦

Example 2 Determine the Laplace transform of $f(t) = e^{at}$, where a is a constant.

Solution Using the definition of the transform,

$$F(s) = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \lim_{N\to\infty} \int_0^N e^{-(s-a)t}\, dt = \lim_{N\to\infty} \left. \frac{-e^{-(s-a)t}}{s-a} \right|_0^N$$

$$= \lim_{N\to\infty} \left[\frac{1}{s-a} - \frac{e^{-(s-a)N}}{s-a} \right]$$

$$= \frac{1}{s-a} \quad \text{for} \quad s > a \,.$$

Again, if $s \le a$ the integral diverges, and hence the domain of $F(s)$ is all $s > a$. ◆

Example Determine the Laplace transform of

$$f(t)=\begin{cases}2\ , & 0<t<5\ ,\\ 0\ , & 5<t<10\ ,\\ e^{4t}\ , & 10<t\ .\end{cases}$$

Since $f(t)$ is defined by a different formula on different intervals, we begin by breaking up the integral in (1) into three separate parts.† Thus,

$$\begin{aligned}F(s)&=\int_0^{\infty}e^{-st}f(t)\,dt\\&=\int_0^5 e^{-st}\cdot 2\,dt+\int_5^{10}e^{-st}\cdot 0\,dt+\int_{10}^{\infty}e^{-st}e^{4t}\,dt\\&=2\int_0^5 e^{-st}\,dt+\lim_{N\to\infty}\int_{10}^{N}e^{-(s-4)t}\,dt\\&=\frac{2}{s}-\frac{2e^{-5s}}{s}+\lim_{N\to\infty}\left[\frac{e^{-10(s-4)}}{s-4}-\frac{e^{-(s-4)N}}{s-4}\right]\\&=\frac{2}{s}-\frac{2e^{-5s}}{s}+\frac{e^{-10(s-4)}}{s-4}\quad\text{for}\quad s>4\ .\ \blacklozenge\end{aligned}$$

TABLE 7.1 Brief Table of Laplace Transforms

$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$
1	$\dfrac{1}{s}, \quad s > 0$
e^{at}	$\dfrac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
$\sin bt$	$\dfrac{b}{s^2 + b^2}, \quad s > 0$
$\cos bt$	$\dfrac{s}{s^2 + b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at} \sin bt$	$\dfrac{b}{(s-a)^2 + b^2}, \quad s > a$
$e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2 + b^2}, \quad s > a$

Sifat-sifat Transformasi Laplace :

Linearity of the Transform

Theorem 1. Let f, f_1, and f_2 be functions whose Laplace transforms exist for $s > \alpha$ and let c be a constant. Then, for $s > \alpha$,

(2) $$\mathcal{L}\{f_1 + f_2\} = \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\} ,$$

(3) $$\mathcal{L}\{cf\} = c\mathcal{L}\{f\} .$$

Conditions for Existence of the Transform

Theorem 2. If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α, then $\mathcal{L}\{f\}(s)$ exists for $s > \alpha$.

Sifat-sifat Transformasi Laplace :

Translation in s

Theorem 3. If the Laplace transform $\mathcal{L}\{f\}(s) = F(s)$ exists for $s > \alpha$, then

$$(1) \qquad \mathcal{L}\{e^{at}f(t)\}(s) = F(s-a)$$

for $s > \alpha + a$.

Determine the Laplace transform of $e^{at}\sin bt$.

we found that

$$\mathcal{L}\{\sin bt\}(s) = F(s) = \frac{b}{s^2+b^2} .$$

Thus, by the translation property of $F(s)$, we have

$$\mathcal{L}\{e^{at}\sin bt\}(s) = F(s-a) = \frac{b}{(s-a)^2+b^2} . \quad \blacklozenge$$

Sifat-sifat Transformasi Laplace :

Laplace Transform of the Derivative

Theorem 4. Let $f(t)$ be continuous on $[0, \infty)$ and $f'(t)$ be piecewise continuous on $[0, \infty)$, with both of exponential order α. Then, for $s > \alpha$,

(2) $$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0) \ .$$

Laplace Transform of Higher-Order Derivatives

Theorem 5. Let $f(t), f'(t), \ldots, f^{(n-1)}(t)$ be continuous on $[0, \infty)$ and let $f^{(n)}(t)$ be piecewise continuous on $[0, \infty)$, with all these functions of exponential order α. Then, for $s > \alpha$,

(4) $$\mathcal{L}\left\{f^{(n)}\right\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0) \ .$$

Using Theorem 4 and the fact that

$$\mathscr{L}\{\sin bt\}(s) = \frac{b}{s^2 + b^2} ,$$

determine $\mathscr{L}\{\cos bt\}$.

Let $f(t) := \sin bt$. Then $f(0) = 0$ and $f'(t) = b \cos bt$. Substituting into equation (2), we have

$$\mathscr{L}\{f'\}(s) = s\mathscr{L}\{f\}(s) - f(0) ,$$

$$\mathscr{L}\{b \cos bt\}(s) = s\mathscr{L}\{\sin bt\}(s) - 0 ,$$

$$b\mathscr{L}\{\cos bt\}(s) = \frac{sb}{s^2 + b^2} .$$

Dividing by b gives

$$\mathscr{L}\{\cos bt\}(s) = \frac{s}{s^2 + b^2} . \quad \blacklozenge$$

Sifat-sifat Transformasi Laplace :

Derivatives of the Laplace Transform

Theorem 6. Let $F(s) = \mathscr{L}\{f\}(s)$ and assume $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α. Then, for $s > \alpha$,

$$\mathscr{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n F}{ds^n}(s) . \tag{6}$$

Determine $\mathscr{L}\{t \sin bt\}$.

We already know that

$$\mathscr{L}\{\sin bt\}(s) = F(s) = \frac{b}{s^2 + b^2} .$$

Differentiating $F(s)$, we obtain

$$\frac{dF}{ds}(s) = \frac{-2bs}{(s^2 + b^2)^2} .$$

Hence, using formula (6), we have

$$\mathscr{L}\{t \sin bt\}(s) = -\frac{dF}{ds}(s) = \frac{2bs}{(s^2 + b^2)^2} . \quad \blacklozenge$$

Tabel transformasi Laplace

NO	F(t)	L{F(t)}=f(s)	Syarat	Catatan
1.	1	$\frac{1}{s}$	s >0	
2.	t t^n	$\frac{1}{s^2}$ $\frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$	s >0 s > 0 dan n > -1	Fungsi Gamma : $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \, dx$ $\Gamma(n+1) = n!$ $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
3.	e^{at}	$\frac{1}{s-a}$	s > a	a=konstanta sebarang
4.	Sin at	$\frac{a}{s^2+a^2}$	s > 0	
5.	Cos at	$\frac{s}{s^2+a^2}$	s > 0	
6.	Sinh at	$\frac{a}{s^2-a^2}$	s > | a |	$\sinh t = \frac{e^t - e^{-t}}{2}$
7.	Cosh at	$\frac{s}{s^2-a^2}$	s > | a |	$\cosh t = \frac{e^t + e^{-t}}{2}$