Solusi PD Linear dengan Transformasi Laplace

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TUJUAN PEMBELAJARAN

Mampu menyelesaikan persamaan diferensial linear homogen / tak homogen dengan Transformasi Laplace

Transformasi Laplace untuk solusi PD

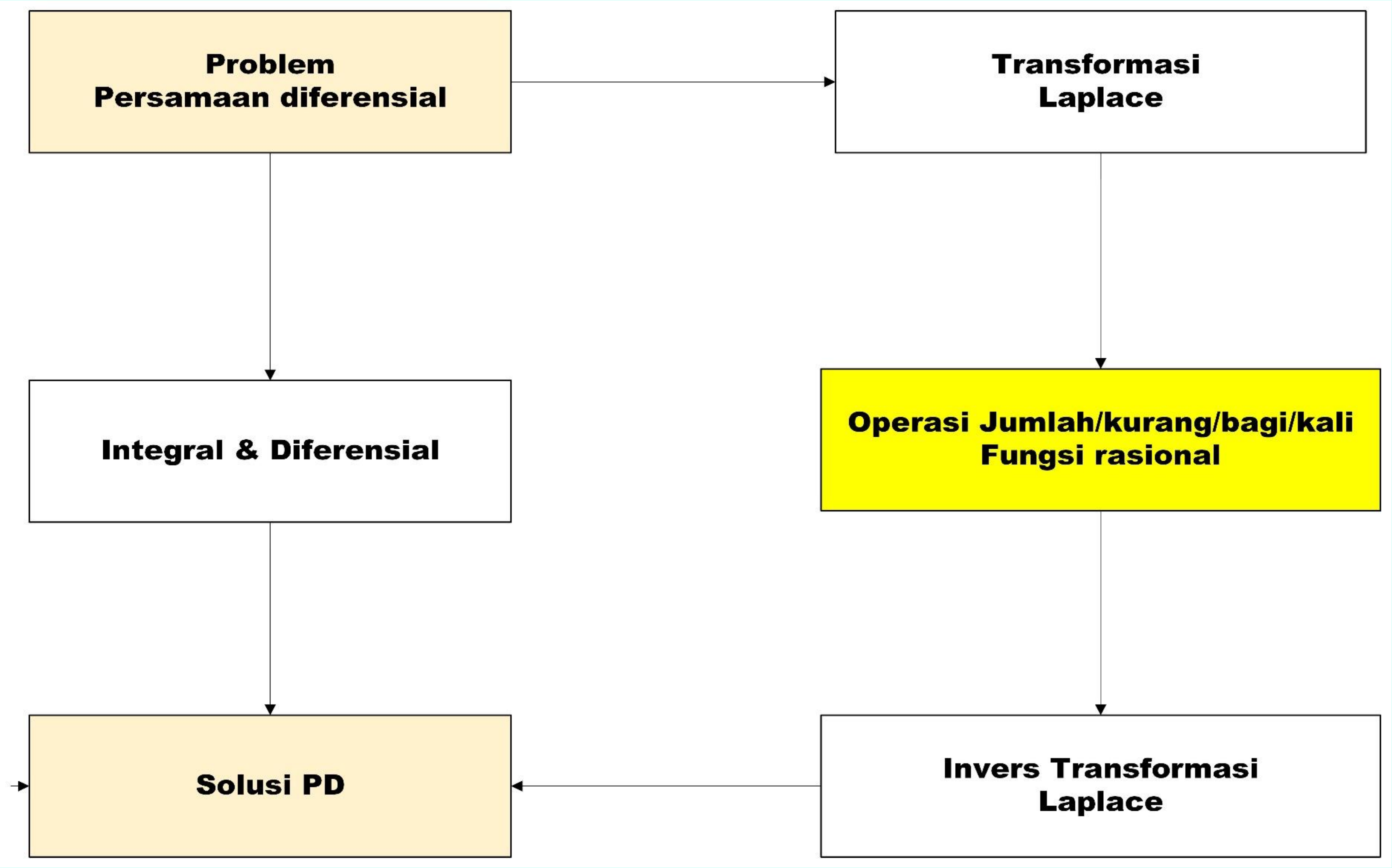

TABLE 7.1	**Brief Table of Laplace Transforms**
$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin bt$	$\frac{b}{s^2 + b^2}, \quad s > 0$
$\cos bt$	$\frac{s}{s^2 + b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2 + b^2}, \quad s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2 + b^2}, \quad s > a$

Invers Transformasi Laplace

$\boldsymbol{F(s)}$	$\boldsymbol{f(t) = \mathcal{L}^{-1}(F(s))}$	**Syarat**
$\frac{1}{s}$	1	s > 0
$\frac{n\,!}{s^{n+1}}$	t^n	s > 0 n > -1
$\frac{1}{s-a}$	e^{at}	s > a
$\frac{a}{s^2+a^2}$	$\sin(at)$	s > 0
$\frac{s}{s^2+a^2}$	$\cos(at)$	s > 0

SOLUSI PD LINEAR DG TRANS. LAPLACE

Langkah penyelesaian :

1) Kerjakan Transformasi Laplace (TL) pada kedua sisi persamaan diferensial dan syarat awalnya.
2) Gunakan sifat-sifat TL untuk mendapatkan fungsi dalam domain Laplace $Y(s)$
3) Selesaikan Invers Transformasi Laplace (ITL) untuk $Y(s)$

Hasil akhir : $\boldsymbol{ITL\ (Y(s)) = y(t)}$ **adl solusinya**

UNTUK DIINGAT BAHWA

1) $\mathcal{L}\left(y^{/}\right) = s\,Y(s) - y(0)$

2) $\mathcal{L}\left(y^{//}\right) = s^2 Y(s) - sy(0) - y^{/}(0)$

CONTOH 1 :

Example 1 Solve the initial value problem

(1) $$y'' - 2y' + 5y = -8e^{-t} \; ; \qquad y(0) = 2 \, , \qquad y'(0) = 12 \, .$$

Langkah 1 : diperoleh

$$\mathcal{L}\{y'' - 2y' + 5y\} = \mathcal{L}\{-8e^{-t}\}$$

dan

$$\mathcal{L}\{y'\}(s) = sY(s) - y(0) = sY(s) - 2 \, ,$$
$$\mathcal{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s - 12$$

CONTOH 1 :

Langkah 2 :

$$\left[s^2Y(s) - 2s - 12\right] - 2\left[sY(s) - 2\right] + 5Y(s) = \frac{-8}{s+1}$$

$$\left(s^2 - 2s + 5\right)Y(s) = 2s + 8 - \frac{8}{s+1}$$

$$\left(s^2 - 2s + 5\right)Y(s) = \frac{2s^2 + 10s}{s+1}$$

$$Y(s) = \frac{2s^2 + 10s}{\left(s^2 - 2s + 5\right)(s+1)}.$$

CONTOH 1 :

Langkah 3 : selesaikan $\mathcal{L}^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$.

We first observe that the quadratic factor s^2-2s+5 is irreducible (check the sign of the discriminant in the quadratic formula). Next we write the quadratic in the form $(s-\alpha)^2+\beta^2$ by completing the square:

$$s^2-2s+5=(s-1)^2+2^2 .$$

Since s^2-2s+5 and $s+1$ are nonrepeated factors, the partial fraction expansion has the form

$$\frac{2s^2+10s}{(s^2-2s+5)(s+1)}=\frac{A(s-1)+2B}{(s-1)^2+2^2}+\frac{C}{s+1} .$$

CONTOH 1 :

Langkah 3 :

When we multiply both sides by the common denominator, we obtain

$$2s^2 + 10s = \left[A(s-1) + 2B\right](s+1) + C\left(s^2 - 2s + 5\right) . \tag{11}$$

In equation (11), let's put $s = -1$, 1, and 0. With $s = -1$, we find

$$2 - 10 = \left[A(-2) + 2B\right](0) + C(8) ,$$

$$-8 = 8C ,$$

CONTOH 1 :

Langkah 3 :

and, hence, $C = -1$. With $s = 1$ in (11), we obtain

$$2 + 10 = [A(0) + 2B](2) + C(4) ,$$

and since $C = -1$, the last equation becomes $12 = 4B - 4$. Thus $B = 4$. Finally, setting $s = 0$ in (11) and using $C = -1$ and $B = 4$ gives

$$0 = [A(-1) + 2B](1) + C(5) ,$$
$$0 = -A + 8 - 5 ,$$
$$A = 3 .$$

Hence, $A = 3$, $B = 4$, and $C = -1$ so that

$$\frac{2s^2 + 10s}{(s^2 - 2s + 5)(s + 1)} = \frac{3(s - 1) + 2(4)}{(s - 1)^2 + 2^2} - \frac{1}{s + 1} .$$

CONTOH 1 :

Langkah 3 :

With this partial fraction expansion in hand, we can immediately determine the inverse Laplace transform:

$$\mathcal{L}^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{3(s-1)+2(4)}{(s-1)^2+2^2} - \frac{1}{s+1}\right\}(t)$$

$$= 3\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+2^2}\right\}(t)$$

$$+ 4\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+2^2}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t)$$

$$= 3e^t\cos 2t + 4e^t\sin 2t - e^{-t} \,. \quad \blacklozenge$$

CONTOH 1 :

Solusi dari IVP

$$y'' - 2y' + 5y = -8e^{-t}; \quad y(0) = 2, \quad y'(0) = 12$$

adalah

$$y(t) = 3e^{t}\cos 2t + 4e^{t}\sin 2t - e^{-t}$$

CONTOH 2 :

Selesaikan $y^{/} - 5y = 0$ dan $y(0) = 2$

Solusi.

Kenakan transformasi kedua sisi : $\mathcal{L}\left(y^{/} - 5y\right) = \mathcal{L}(0)$ maka

$$[sY(s) - 2] - 5Y(s) = 0$$

Sehingga

$$Y(s) = \frac{2}{s-5}$$

Solusinya, invers $Y(s)$ maka

$$y(t) = \mathcal{L}^{-1}\big(Y(s)\big) = 2\mathcal{L}^{-1}\left(\frac{1}{s-5}\right) = 2e^{5t}$$

CONTOH 3 :

Selesaikan $y^{//} - y^{/} - 2y = 4t^2; y^{/}(0) = 4$ dan $y(0) = 1$

Solusi.

Kenakan transformasi kedua sisi : $\mathcal{L}\left(y^{//} - y^{/} - 2y\right) = \mathcal{L}(4t^2)$

maka

$$[s^2Y(s) - s - 4] - [sY(s) - 1] - 2Y(s) = \frac{8}{s^3}$$

Sehingga

$$Y(s) = \frac{s+3}{s^2 - s - 2} + \frac{8}{s^3(s^2 - s - 2)}$$

Solusinya,

$$y(t) = \mathcal{L}^{-1}\left(Y(s)\right)$$
$$= \left(\tfrac{5}{3}\exp(2t) - \tfrac{2}{3}\exp(-x)\right) + \left((-3 + 2x - 2x^2) + \left(\tfrac{1}{3}\exp(2t) + \tfrac{8}{3}\exp(-x)\right)\right)$$