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Homework 2

Question 1 + 3

$\boxed{1A}$ Fundamental Theorem of Calculus:

$$F(y) - F(x) = \int_0^1 \nabla F(x + t(y-x))^T (y-x)\, dt$$

Prove form of Lipschitz gradient upperbound:

$$F(y) \leq F(x) + [\nabla F(x)]^T (y-x) + \frac{L}{2} \|y-x\|_2^2$$

$$F(y) - F(x) - [\nabla F(x)]^T (y-x) \leq \frac{L}{2} \|y-x\|_2^2$$

$$F(y) - F(x) - [\nabla F(x)]^T (y-x) = \int_0^1 \nabla F(x+t(y-x))^T (y-x)\, dt - [\nabla F(x)]^T (y-x)$$

$$= \int_0^1 \left(\nabla F(x+t(y-x)) - \nabla F(x)\right)^T (y-x)\, dt$$

(Cauchy-Schwarz) $\leq \int_0^1 \|\nabla F(x+t(y-x)) - \nabla F(x)\| \cdot \|y-x\|\, dt$

(by definition of Lipschitz-continuous objective gradients) $\leq \int_0^1 L \|y-x\|_2^2\, t\, dt$

$$= L \|y-x\|_2^2 \int_0^1 t\, dt$$

$$= \frac{L}{2} \|y-x\|_2^2$$

$\blacksquare$

Supplemental

Cauchy - Schwarz : $|\langle u, v\rangle| \leq \|u\| \cdot \|v\|$

Lipschitz continuous
objective gradient definition : $\|\nabla F(y) - F(x)\|_2 \leq L\|y-x\|_2$

$\boxed{1\ B}$ Variance bounded by M. i.e. $Var(\|g\|_2) \leq M$

Upper bound for SGD

$$\mathbb{E}\left[F(w_{K+1})\right] - F(w_K) \leq \left(\frac{\alpha_K^2 L}{2} - \alpha_K\right)\|g_K\|_2^2 + \frac{\alpha_K LM}{2}$$

$$\frac{2}{2\alpha_K}\left(\left(\frac{\alpha_K^2 L}{2} - \alpha_K\right)\|g_K\|_2^2 + \frac{\alpha_K^2 LM}{2}\right)$$

$$= \alpha_K L \|g\|_2^2 - \|g\|_2^2 + \alpha_K LM$$

$$= \alpha_K L\left(\|g_K\|_2^2 + M\right) - \|g_K\|_2^2$$

<u>When $\alpha_K = \frac{1}{L}$</u>

$$\frac{1}{L}\cdot L\left(\|g\|_2^2 + M\right) - \|g\|_2^2 = 0$$

$$\boxed{M = 0}$$

When $\alpha_K = \frac{1}{2L}$

$$\frac{1}{2L} \cdot L\left(\|g\|_2^2 + M\right) - \|g\|_2^2 = 0$$

$$\frac{1}{2}\|g\|_2^2 + \frac{1}{2}M = \|g\|_2^2$$

$$\boxed{M = \|g_K\|_2^2}$$

When $\alpha_K = \frac{1}{10L}$

$$\frac{1}{10L} \cdot L\left(\|g\|_2^2 + M\right) - \|g\|_2^2 = 0$$

$$\frac{1}{10}\|g\|_2^2 + \frac{1}{10}M = \|g\|_2^2$$

$$\boxed{M = 9\|g_K\|_2^2}$$

$\boxed{3}$

a) This homework took 8-12 hours

b) I adhered to the Duke Community Standard in the completion of this assignment

NAI