Nathan Inkawhich

Homework 4

Questions 1 & 5

$\boxed{1A}$ Show $D_{KL}(q(z|x) \| p(z|x)) = 0$ iff $q(z|x) = p(z|x)$

$$D_{KL}(q(z|x) \| p(z|x)) = \int q(z|x) \log \frac{q(z|x)}{p(z|x)} \, dz$$

$$= \mathbb{E}_{z \sim q}\left[\log \frac{q(z|x)}{p(z|x)} \right]$$

$$= \mathbb{E}_{z \sim q}\left[-\log \frac{p(z|x)}{q(z|x)} \right]$$

Jensen's Inequality

$$\geq -\log \mathbb{E}_{z \sim q}\left[\frac{p(z|x)}{q(z|x)} \right]$$

$$= -\log \int q(z|x) \frac{p(z|x)}{q(z|x)} \, dz$$

$$= -\log \int p(z|x) \, dz$$

$$= -\log 1$$

$$= 0$$

May only use Jensen's Inequality (i.e. $f(\mathbb{E}[x]) \leq \mathbb{E}[f(x)]$) if $p(z|x) = c\, q(z|x)$ where c is a constant. And $\int p(z|x)\, dz = c \int q(z|x)\, dz$, so $c = 1$ meaning $p(z|x) = q(z|x)$

This makes $\log \frac{p(z|x)}{q(z|x)}$ convex.

$\boxed{1B}$ Show for $L(q;x) = \int q(z|x) \log \frac{p(x,z)}{q(z|x)} dz$

$$\log p(x) = D_{KL}(q(z|x) \| p(z|x)) + L(q;x)$$

$$= \int q(z|x) \log \frac{q(z|x)}{p(z|x)} dz + \int q(z|x) \log \frac{p(x,z)}{q(z|x)} dz$$

$$= \int q(z|x) \left(\log \frac{q(z|x)}{p(z|x)} + \log \frac{p(x,z)}{q(z|x)} \right) dz$$

$$= \int q(z|x) \left(\log q(z|x) - \log p(z|x) + \log p(x,z) - \log q(z|x) \right) dz$$

$$= \int q(z|x) \left(\log p(x,z) - \log p(z|x) \right) dz$$

$$= \int q(z|x) \left(\log (p(z|x) p(x)) - \log p(z|x) \right) dz$$

$$= \int q(z|x) \left(\log p(z|x) + \log p(x) - \log p(z|x) \right) dz$$

$$= \int q(z|x) \log p(x) \, dz$$

$$= \log p(x) \int q(z|x) \, dz$$

$$= \log p(x)$$

✓

$\boxed{5}$

A) This assignment took 8 - 12 hours

B) I adhered to the Duke Community Standard in the completion of this assignment

NAI