Fixed Income and Credit – Lecture 4

Martingales and Fixed Income Valuation - Annexes

CQF

Martingales and Fixed Income Valuation

1. Stochastic calculus results

Reference: Musiela, Rutkowsky, *Martingale Methods in Financial Modelling*.

Ito Formula

$$X_t^i = X_0^i + \int_0^t \alpha_s^i ds + \int_0^t \beta_s^i \cdot dW_s$$

Theorem (Theorem B.1.1)

Suppose that g is a function of class $C^2(\mathbb{R}^k, \mathbb{R})$. Then the following form of Ito's formula is valid

$$dg(X_t) = \sum_{i=1}^{k} g_{x_i}(X_t)\alpha_t^i dt + \sum_{i=1}^{k} g_{x_i}(X_t)\beta_t^i \cdot dW_t + \frac{1}{2}\sum_{i,j=1}^{k} g_{x_i,x_j}(X_t)\beta_t^i \cdot \beta_t^j dt.$$

Doleans exponential

Section B.2

The Doleans exponential is

$$\mathcal{E}_t\left(\int_0^{\cdot} \gamma_u \cdot dW_u\right) = \exp\left(\int_0^t \gamma_u \cdot dW_u - \frac{1}{2}\int_0^t |\gamma_u|^2 du\right).$$

It is the solution of the SDE

$$d\mathcal{E}_t = \mathcal{E}_t \, \gamma_t \cdot dW_t.$$

Equivalent measure – exponential form

Theorem (Proposition B.2.1)

For any probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}_{\bar{T}})$ equivalent to $\mathbb{P}$, there exist a d-dimensional process γ adapted to $\mathcal{F}$ such that the Radon-Nikodym derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$ equals

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}_{|\mathcal{F}_t} = \mathcal{E}_t\left(\int_0^{\cdot} \gamma_u \cdot dW_u\right), \mathbb{P}\text{-as.}$$

Girsanov

Theorem (Theorem B.2.1)

Let W be a standard d-dimensional Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that γ is an adapted $\mathbb{R}^d$-valued process such that

$$\mathrm{E}^{\mathbb{P}}\left[\mathcal{E}_T\left(\int_0^{\cdot} \gamma_u \cdot dW_u\right)\right] = 1.$$

Define a probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}_T)$ equivalent to $\mathbb{P}$ by means of the Radon-Nikodym derivative

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}_T\left(\int_0^{\cdot} \gamma_u \cdot dW_u\right), \mathbb{P}\text{-as.}$$

Then the process $\tilde{W}$, which is given by the formula

$$\tilde{W}_t = W_t - \int_0^t \gamma_u \, du$$

follows a standard d-dimensional Brownian motion on the space $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$.

Bayes's formula

Suppose the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ equals

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \eta$$

Theorem (Lemma A.0.4)

Let $\mathcal{G}$ be a sub-σ-field of the σ-field $\mathcal{F}$, and let ψ be a random variable integrable with respect to $\mathbb{Q}$. Then the following abstract version of the Bayes's formula holds

$$\mathrm{E}^{\mathbb{Q}}\left[\psi \middle| \mathcal{G}\right] = \frac{\mathrm{E}^{\mathbb{P}}\left[\psi\eta \middle| \mathcal{G}\right]}{\mathrm{E}^{\mathbb{P}}\left[\eta \middle| \mathcal{G}\right]}.$$

Martingale representation theorem

Theorem

Let $\mathcal{F}_t$ is the filtration generated by a d-dimensional Brownian motion W_t $(0 \leq t \leq T)$. Let $M(t)$ $(0 \leq t \leq T)$ be a martingale with respect to $\mathcal{F}_t$. Then there exists an adapted d-dimensional process γ_t $(0 \leq t \leq T)$ such that

$$M(t) = M(0) + \int_0^t \gamma_u \cdot dW_u \quad 0 \leq t \leq T.$$

Notation

Radon-Nykodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \eta_{\bar{T}} = \mathcal{E}_{\bar{T}}\left(\int_0^{\cdot} -\theta_s \cdot dX_s\right), \quad \mathbb{P} \text{ a.s.}$$

Bond prices:

$$Z(t,T) = \frac{B(t,T)}{A(t)}$$

and

$$M(t) = Z(t,T)\eta(t)$$

with *Martingale Representation*

$$M(t) = M(0) + \int_0^t \gamma_s \cdot dX(t)$$