1.7 Caplet Pricing and Forward LIBOR. (Dr Richard Diamond)

The simple interest rate options are caps and floors. The payoff for a single cap (caplet) is defined as follows:

$$\boxed{Z(0, T_{i+1}) \times \max\left[L(T_i, T_{i+1}) - K, 0\right] \times \tau}$$

where in LIBOR model notation

$$Z(T_i, T_{i+1}) = \frac{1}{1 + \tau_i f_i} \qquad \text{and} \qquad \mathbb{E}^{\mathbb{Q}}_{i+1}\left[L(T_i, T_{i+1})\right] = f_i$$

Notice that discounting factor covers two periods, $[0, T_i]$ and $[T_i, T_{i+1}]$ – they can be separated. Traditional discounting factor is calculated under the same rolling measure $\mathbb{E}^{\mathbb{Q}}_{i+1}$ as the measure of rate $f_i = L(T_i, T_{i+1})$ – from the same simulated curve as one for the rate (row in HJM, diagonal in LMM). That is stochastic discounting. OIS Discounting brings its own challenge: a separate static OIS curve of the day, $\mathrm{DF}_{\mathrm{OIS}}(0, T_{i+1})$.

1.7.1 Forward-Starting Caplets

Unlike in equity options, the long-term cap option (over 1Y, 2Y) *is the sum of cashflows* from the consecutive 6M caplet options – forward-starting caplets written on Forward LIBOR.

Each caplet is **expiring (re-setting) at** T_i and **maturing (paid) at** T_{i+1}. Caplet rate is paid for the time $\tau_i = T_{i+1} - T_i$. The cashflow is equal to the amount of interest accrued $\tau_i \times \max\left[f_i - K, 0\right]$ and is discounted. Here, notional is omitted from formulae.

$$\underbrace{\exp\left(-\int_0^{T_i} \bar{f}(t, \tau) d\tau\right)} \frac{\tau_i}{1 + \tau_i f_i} \max\left[f_i - K, 0\right]$$

Above is valuation of caplet implied under $\mathbb{E}^{\mathbb{Q}}\left[\right]$. The underbrace, the integration over the forward curve (row) gives a stochastic discounting factor $Z[0, T_i]$.

Forward LIBOR: for Caplet and IRS Pricing

We know the following result as the FRA formula, the purpose of which is simply to convert from the continuous-time rate $f_{i,inst}$ to simple LIBOR rate $f_i = L(T_i, T_{i+1})$. The need to convert occurs under the HJM framework. LMM output gives discrete forward rates directly.

$$\begin{aligned} f_i &= \boxed{\frac{1}{T_{i+1} - T_i}\left[\exp\left(\int_{T_i}^{T_{i+1}} \bar{f}_{inst}(T_i, \tau) d\tau\right) - 1\right]} \quad \dagger \\ &= \frac{1}{T_{i+1} - T_i}\left[\frac{1}{Z(T_i, T_{i+1})} - 1\right] \qquad \text{where} \quad Z(T_i, T_{i+1}) = \frac{Z(0, T_{i+1})}{Z(0, T_i)} \end{aligned}$$

Notice that the integration above is really over one-point chunk of the curve, i.e., from T_i to T_{i+1} – we only have one value in our HJM output for that. If you have more than one value (eg, simulated 3M or 1M LIBOR from the outset) the general formula means averaging

$$F(T_i, T_{i+1}) = \frac{1}{T_{i+1} - T_i} \int_{T_i}^{T_{i+1}} \bar{f}(T_i, \tau) d\tau \quad \Rightarrow \quad \boxed{\frac{1}{n} \sum_{j=1}^{n} \bar{f}_j}$$

where n is a number of points between tenors $[T_i, T_{i+1}]$ and integration is over the curve. Forward LIBOR f_i is a simple annualised rate, therefore result F above must be converted using

$$f = m\left(e^{\frac{F}{m}} - 1\right) \quad \dagger$$

where m is compounding frequency per year, for example, 3M LIBOR compounded 4 times a year, 6M LIBOR - 2 times.

HJM Example: notation $\bar{f}(t = T_i, \tau)$ refers to the future curve simulated at times $t = T_i$, for example, $\bar{f}(4.5, \tau)$ means the curve simulated at row 450 with $dt = 0.01$ (remember that HJM output has rows above the simulation so the actual row could be 463, etc). Mathematically, it does not matter which row we pick from the MC HJM output as a sample curve, but it would be a sensible choice to pick up L(t=4.5; 0, 0.5) from the notional row 450, and L(t=0; 4.5, 5) from the notional row 0 (curve today).

The relevant fact is that HJM is calibrated on the Bank Liability Curve, and the yields are approximately equivalent to AA-rated bonds of City of London financial institutions. Implied Spot LIBOR over longer terms than published can be obtained by integration over the curve, for example, $L[0, 2]$ means integration from 0 to $\tau = 2$, however, the pricing of caps and IRS done via a sequence forward-starting LIBORs.

		From HJM Output: Forward LIBOR curve is a composite of **multiple curves** of shortening length. The final from tau=0.5 (column C) applies to [4.5, 5] period. 5Y IRS Example									
t=0	L_6M as mkt	4.5251%	4.2916%	4.2833%	4.3498%	4.4054%	4.4440%	4.4708%	4.4903%	4.5057%	~~**4.5184%**~~
t=0.5Y	L_6M as mkt	3.3450%	3.2939%	3.3454%	3.4124%	3.4630%	3.4980%	3.5219%	3.5381%	**3.5485%**	
t=1Y	L_6M as mkt	3.3800%	3.4826%	3.6168%	3.7398%	3.8416%	3.9252%	3.9942%	**4.0511%**		
t=1.5Y	L_6M as mkt	3.6463%	3.7486%	3.8530%	3.9411%	4.0114%	4.0668%	**4.1098%**			
t=2Y	L_6M as mkt	4.1354%	4.2011%	4.2622%	4.3120%	4.3513%	**4.3823%**				
t=2.5Y	L_6M as mkt	3.7474%	3.7361%	3.7252%	3.7125%	**3.6993%**					

Figure 1.3: Picking Forward LIBOR from HJM Output: curves of shortening length.

1. Full Curve Pricing. Integrate over the simulated curves $\bar{f}(T_i, \tau)$, while future simulated curves get shorter. Consider 5-year interest rate swap and full curve simulated in 6M increments of $d\tau$ in row, and in $dt = 0.01$ timestep columnwise

- Today's curve at time $t = 0$ will have all Forward-starting LIBORs including $L(4.5, 5)$, $L(4, 4.5)$, ...
- Curve simulated in the notional row 50 will be for $t = 0.5$ and therefore, shorter – our swap now has up to 4.5-years of life.
- Curve simulated in the notional row 100 will be for $t = 1$ and therefore, shorter – our swap now has up to 4-years of life. Etc.

The sequence of Forward-starting LIBORs is consistent with the credit risk and the term liquidity risk. Example: even though $L(4.5, 5)$ the rate for only $[4.5, 5]$ period, it is used only as part of the sequence and does reflect the fact the overall swap contract is for 5-year period (and not a breakable contract of constant-maturity Rolling Spot 6M LIBOR). That is, $L(4.5, 5)$ rate does reflect the credit risk of the 5-year borrowing as well as uncertainty about future rates.

Each Forward LIBOR is under its own, rolling forward measure $\mathbb{E}^{\mathbb{Q}}_{i+1}\left[L(T_i, T_{i+1})\right] = f_i$.

2. Constant Maturity. HJM model output offers to us an alternative approach of picking the rate from the same column. For example, instead of $L(t = 0, 4.5, 5)$ from curve at $t = 0$ one can chose the evolved LIBOR $L(t = 4.5, 0, 0.5)$ from curve at $t = 4.5$ – **but those are two rates with different risks!**

Constant-maturity choice would be always from the same column and therefore, will reflect the same credit risk. This is appropriate for the breakable contracts – the industry term is early close-out.

The constant-maturity rate (columnwise) has one measure, therefore, the rate can be renamed as Rolling Spot 6M LIBOR. The notation $\bar{f}(t = T_i, \tau)$ is best suited for the constant maturity case. Remember the heuristic: each HJM output column = own measure.

3. Today's curve in the absence of simulated curves one can compute Forward LIBORs from today's curve. This approach does not allow for modelling of MtM analytics and Potential Future Exposures in CVA computation. *Yield Curve.xlsm* implements Forward LIBOR calculation from the today's curve as demo only.

1.7.2 Market Risk Factors

Market risk factors are always inputs into the pricing formula. The terminology comes from the formal risk management, which regards any input as 'a risk factor'.

1. The standard risk factors are strike, maturity and 'bucket risks', ie, bumping rates at particular tenors and re-pricing. HJM stress-tests the curve by design: each diffusion factor based on principal component and multiplied by $\phi\sqrt{(\delta t)}$.

2. Semi-annual vs. quarterly expiry. BOE forward curve is already smoothed for 6M increments – re-interpolation for 3M not advisable.

3. Discounting: the older choice of DF from the same simulated curve is respects the measure. OIS discounting requires simulated Forward OIS rates which is something not implemented by the industry (as of 2018).