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Theoretical and Numerical Schemes for Pricing Exotics

Author:
Koula Stavri

Supervisor:
Dr. Riaz Ahmad

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Abstract

Exotic options have become particularly popular over the last decades due to the increased needs of customers for more flexibility in the contracts and for more broaden choices. However, when it comes to pricing of these options, Black-Scholes equation is sometimes not efficient, as a closed form solution for their price can not be found. Thus, it is of huge interest to study the numerical methods to price those options.
In this paper, we will mainly focus on Asian and Loockback options, which are two of the most popular exotic options. The fact that both belong to the same subcategory of exotics, makes the simultaneous work on both logical. In particular, we will make use of the Monte Carlo technique and explore how the "Updating Rule" works.

Contents

1 Introduction

A derivative is a contract whose value is based on the behaviour of an underlying financial asset. Options are particular type of derivatives. These are contracts which give the right (not the obligation) to the holder either to buy (in case of call option) or sell (if it is a put option) an underlying asset during a certain period date, for a predetermined price, called strike.

Options are not a novelty. The first original description of option contract counts thousands of years ago (around 486 B.C) and was given by Aristotelis. According to his narration, Thales of Miletus, a poor philosopher, wanted to prove that philosophy was not useless and could, indeed, offer him the wealth that he did not have. Using his perceptive abilities, he predicted that olive harvest would be high the coming season. Therefore, he bought the rights of the local mills with his little money in low prices. During harvest time, Thales could bestow mills in high prices, thus making a lot of money. It was his decision to exercise the option, he was not obliged to do so.

A plain vanilla option,as evidenced by its name, is a normal call or put option that has standardized terms and no special or unusual features. The simplicity of these contracts means there are limited opportunities. Since the early 1980s, the increasing needs of customers have led to both international banks and other financial institutions in designing and inventing new and often complex contracts. These contracts are called exotic options and are considered as a sophisticated extension to Vanilla options.

1.1 Overview of exotic options

In finance, an exotic option is a financial instrument which has features making it more complex than commonly traded vanilla options. These features make exotics more difficult to price and hedge than vanillas and they are sometimes very model dependent. Exotic options are not traded on an exchange, but they are designed by the relevant counterparties and are sold privately from one counterparty to another. In other words, they are traded over the counter.

In some cases, they constitute hedges from various investment options. Several times they are used for tax, accounting or legal purposes and are particularly attractive because they generate greater profit from the simple rights. Sometimes, they are issued by financial institutions to look attractive to prospective buyers.
The fact that not many financial institutions trade exotics means that the competition is lower compared to vanilla option market. Therefore, somebody can take advantage on it to obtain better transaction prices and make larger profit by writing exotic options.

Although exotics offer structured protection when vanilla options can not be successfully employed, they, sometimes, present some disadvantages. One of them is that it can be difficult or even impossible to buy or sell sufficient amount of exotic options to hedge investor's portfolio, because of the low liquidity on some exotic option markets. In addition, because exotics offer more flexibility, they are more expensive compared to plain vanilla options. Another disadvantage of exotic options is that the writer might try to exercise control or influence the underlying market especially when approaching maturity, in case the option is about to expire worthless. Finally, the risks inherent in the contracts are usually more obscure and can lead to unexpected losses.

The other fundamental difference observed regarding exotic options, is the way they can be priced. Black-Scholes equation (BSE) is the most important formula for pricing vanilla options. The model appeals to the no arbitrage principle and assumes that the price of the underlying asset follows Geometric Brownian motion. Because of its simplicity, BSE is used in a wide range of financial institutions. This method is commonly used for pricing European options as there is an analytic solution for their price. However, when it comes to pricing exotic options, they can not be priced as easily. Generally, many exotic options are initially priced via a binomial model, and then at some point traders figure out a closed-form pricing model. Sometimes, it turns out that is almost impossible to express the BS PDE in an analytic formula that would calculate the prices of exotic options and so no closed form solution is ever found.
In addition, while binomial tree can also be used for pricing options, once again, it is not applied when pricing exotic options, especially when it is about highly path-dependent options. This technique is mainly used for pricing European and American options.
Monte Carlo simulation is a numerical method to calculate integrals/expectations using random numbers and can be used for pricing options of almost any type, including exotics. It was invented in the Manhattan project in Los Alamos and named after the city of Monaco which is famous for its casinos and gambling games.

2 Classification of Exotic Options

Generally, it is impossible to classify all the options. However, taking into account the most important features of derivative products, we will move on a logical categorization of them to help characterize and analyse these contracts.

1. Time dependence: This refers to contracts where there conditions vary with time, i.e there are key dates when something happens. In this case, we say that the contract is time-inhomogeneous, otherwise it is time-homogeneous (like plain vanilla options). Time dependence implies that more caution is required when applying numerical methods, as we need to take into consideration the time interval when something happens and make it consistent with our numerical discretization.
   Examples of time dependent options are:
   - Cliquet option, where at the end of each year the return is calculated.
   - Bermudan options: this is characterized by intermittent early exercise, as the holder can exercise the option on certain dates, eg every sencond Friday of every month.

2. Cashflows: When during the life of an option, the holder collects some money, then we say that there is a jump condition and the value of the contract jumps by the amount of the cashflow. Thus, if $V(t_0^-)$ and $V(t_0^+)$ are the values of the contract before and after cashflow date ,respectively and this contract pays an amount q at time t_0 (cashflow date), then, by appealing to the no-arbitrage principle, it holds that $V(t_0^-) = V(t_0^+) + q$, otherwise we are led to arbitrage opportunity.
   If the cashflow depends on the underlying asset, S, then we will have q(S). Furthermore, an important point is that cashflow has to be deterministic , otherwise, the jump condition does not necessarily apply. In case it is not deterministic, then we can not appeal to the no-arbitrage argument and the holder's preferences might define the result. Then, we will have $V(t_0^-) = V(t_0^+) + \mathbb{E}[g]$.
   Cashflow might be:
   - discrete: On certain day, the holder receives £10 if the stock is below £150, or
   - continuous: The holder receives £1 every day that the stock is below £80.

   In the latter case, there is no jump condition anymore. What we have to do, is to modify the BSE to add source term.

3. Path dependence: The payoff now is a function of the underlying stock. Once again, we have two subcategories: the weak path dependence and the strong path dependence.
   - Weak path dependence: The option price remains the same,i.e no extra dimensions are added and the option depends only on asset and time.
     - The most popular weak path dependent option is Barrier (knock-in or knock-out). Barrier options come to existence or expire worthless depending on whether a predetermined underlying level (barrier) is triggered.

- Strong path dependence: In this case, an extra variable is introduced. Their payoff does not depend only on the value of the underlying at the present time, but also on, at least one new variable. Thus, the option price becomes V(S,I,t).
  - A well known and interesting option is Asian. In this case, the payoff is a function of the underlying average. By averaging, volatility is lower than the volatility of the value of the underlier itself, thus Asians are cheaper than plain vanilla options.
  - Another example of strong path dependent options are Lookback Options. Now the payoff is a function of the observed maximum/minimum of the asset price during the life of the option. In contrast with Asians, these are very expensive options as they give the opportunity to the owner to buy a call at the lowest and sell put at the highest.

With Asians and Lookbacks we will deal extensively during the next chapters.

4. Dimensionality: This feature refers to the number of underlying independent variables.
- e.g vanilla option is two dimensional, as it has two independent variable: S and t.
The weakly path dependent contracts have the same number of dimensions as their non-dependent cousins., i.e barrier call has the same two dimensions as a vanilla call.
The number of dimensions is an indication for the numerical method that we should use. If the dimension is less or equal to 3, then we should use Finite Difference Method (FDM), whereas if it larger than 4, Monte Carlo performs better. In case the dimension is 4, FDM and Monte Carlo give similar results.

5. Order of an option: Options whose payoff depends only on the underlying asset, are said to be first order options. The term 'Higher order options' refers to options whose payoff depends on the value of another option. Thus, an option on an option is second order. These are called compound options.
- e.g a call to buy a put. This means that at time T_1 the holder has the chance to decide whether he wants to exercise his right to buy a put. This put option expires at T_2. So, when T_2 arrives, he has the chance to sell the asset.

6. Embedded decisions: This is whether the holder or writer of the option, have to make any decisions during the life of the contract.

   - early exercise features: the important point is to decide when is the optimal time to exercise the option. In the PDE framework, we try to maximize the value of the option, choosing S^* that makes the option value V(S) and its delta $\frac{\partial V}{\partial S}$ continuous. This condition is called the Smooth-Pasting Condition and it is what guarantees optimality. Thus, we exercise the option as soon as the asset price reaches the optimal exercise point, i.e the value at which the position price and the payoff meet (S^*).

3 Lookbacks-Asians

3.1 Lookbacks

Lookback Options are options whose payoffs depend on the realized maximum and/or the realised minimum of the asset price over the life of the option. This option allows the investor to "look back" over the history of the asset price over the life of the option and find when it was the most advantageous to exercise it. As mentioned previously, loockbacks are very expensive options, as they offer the opportunity to the owner of the option to buy a call at the lowest and sell -in case of put- at the highest. These options are widely used when the marketplace is more volatile, as in this way, the investors have more chances to gain higher profits.

The payoff in a lookback contract, can have two forms: the fixed strike and the floating strike.

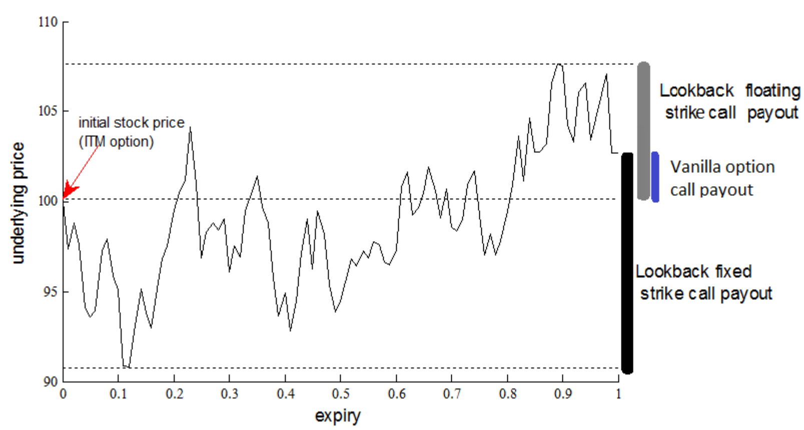
- In case of fixed strike, the strike is predetermined since the time the customer enters the contract. Then the payoff is the same as for vanilla options, but the asset value is replaced by the maximum or minimum stock value.

$$\boxed{H(S_T, J_T) = \begin{cases} (M_T - K)^+, & \text{fixed call} \\ (K - m_T)^+, & \text{fixed put} \end{cases}}$$

- In the second case of floating strike, the strike is not known until maturity. In the payoff, strike is what is replaced by the maximum or minimum.

$$\boxed{H(S_T, J_T) = \begin{cases} S_T - m_T, & \text{floating call} \\ M_T - S_T, & \text{floating put} \end{cases}}$$

where J_T denotes M_T or m_T and $M_T = \max\limits_{0 \le t \le T} S_t$, $m_t = \min\limits_{0 \le t \le T} S_t$



3.1.1 Continuous vs Discrete monitoring

Depending on the way the maximum or minimum we keep track of maximum or minimum, we can have the continuous sampling or discrete sampling.

* *continuous sampling:* the value of the underlying asset is observed continuously over the life of the option. Continuously monitored lookback options can be very expensive

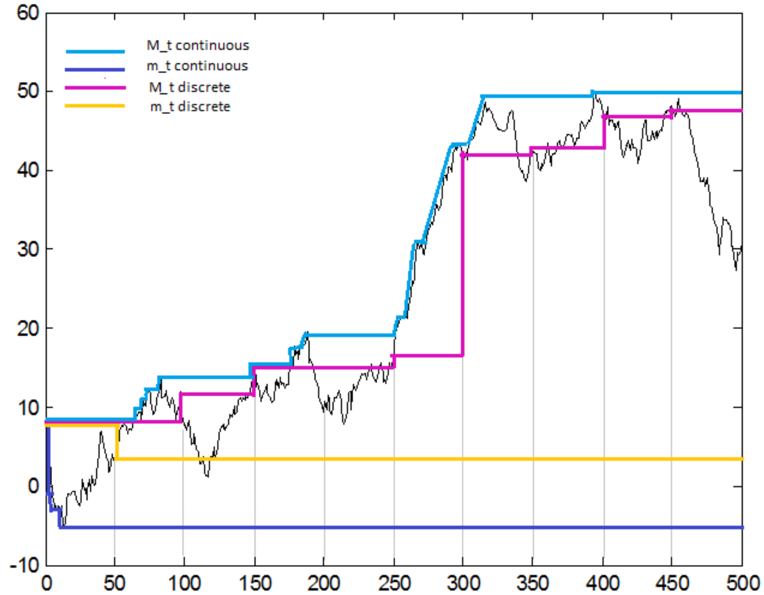
* *discrete sampling:* the value of the underlying asset is monitored is specific moments, e.g at the end of each week. Then, for sampling moments $\{0, t_1, t_2, ..., t_n\}$, we have $M_i = max\{S_{t_1}, S_{t_1}, ..., S_{t_n}\}$ and $m_i = min\{S_{t_1}, S_{t_1}, ..., S_{t_n}\}$.
  This form of sampling is less manipulative, thus cheaper than the continuous sampling.

  Therefore, at each sampling date t_i, $M_{t_i} = max(M_{t_{i-1}}, S_{t_i})$ and $m_{t_i} = min(m_{t_{i-1}}, S_{t_i})$ and between them, the observed maximum or minimum remains constant.

  For the maximum and minimum in continuous and discrete sampling, the following relations hold:

$$\boxed{\begin{array}{l} M_{T_{discrete}} \leq M_{T_{continuous}} \\ m_{T_{discrete}} \geq m_{T_{continuous}} \end{array}} \circledast$$

  By definition, when we have continuous sampling, the running asset value can not be greater than the maximum, i.e $S_t \leq M_t$.
  On the other hand, when the monitoring is discrete, it is possible to have $S_T > M_{t_n}$, and, thus, the holder might not exercise the option.
  This can be observed from the following figure.



We notice that the relations ⊛ hold.

3.2 Asians

These options owe their name to the fact that they were originally used in Tokyo, in 1987 for pricing options linked to the average price of crude oil.
Asians are also strongly path dependent options whose payoff is a function of the average price of the underlying during the option's life. These rights can be used to offset the risk that may expose a company that buys or sells at regular intervals any kind of asset market. With Asians, an increase in spot price will not affect the average price particularly and, on the other hand, if the price increases, profit margins of the buying company will not be affected, because its position is protected by Asian options. As mentioned earlier, the fact that we average the stock price, decreases volatility and, so they are cheaper compared to the plain vanillas - even though this is not strictly accurate. The fast calculation of their price serves to ensure fair negotiations.

The payoff once again comes in two forms: the average price Asian and the average strike Asian.

- In case of average price, the payoff becomes: $\boxed{\text{Payoff}=\begin{cases} max[A-K,0], & \text{call} \\ max[K-A,0], & \text{put} \end{cases}}$

- In the second case of average strike, payoff is formed as follows:

$$\boxed{\text{Payoff}=\begin{cases} max[S_T-A,0], & \text{call} \\ max[A-S_T,0], & \text{put} \end{cases}}$$

,where A is the average value of the underlying asset. It is defined depending on whether the sampling is discrete or continuous, as well as if it is arithmetic or geometric.
Average price is more common than average strike, therefore we will focus on this.

Asian rights first appeared in Asian markets in order for vendors not to have the opportunity to manipulate the share price at the time of exercise, as it is more difficult to do so over an extended period of time than just at the expiration of an option.

3.2.1 Discrete vs Continuous sampling

For Discrete Sampling we have: $\begin{cases} A = \dfrac{1}{N}\sum\limits_{i=1}^{N} S(t_i), & \text{disretely sampled arithmetic average} \\ A = \left(\prod\limits_{i=1}^{N} S(t_i)\right)^{\dfrac{1}{N}}, & \text{disretely sampled geometric average} \end{cases}$

The latter case can be developed as follows:

$$A = \left(\prod_{i=1}^{N} S(t_i)\right)^{\dfrac{1}{N}} \Leftrightarrow lnA = ln\left(\prod_{i=1}^{N} S(t_i)\right)^{\dfrac{1}{N}} \Leftrightarrow lnA = \frac{1}{N}ln\left(\prod_{i=1}^{N} S(t_i)\right) \Leftrightarrow lnA = \frac{1}{N}\sum_{i=1}^{N} lnS(t_i) \Leftrightarrow$$
$$A = \mathrm{e}^{\dfrac{1}{N}\sum\limits_{i=1}^{N} lnS(t_i)}$$

Similarly, for Continuous Sampling: $\begin{cases} A = \dfrac{1}{T}\int_0^T S(t)\,dt, & \text{continuously sampled arithmetic average} \\ A = \mathrm{e}^{\dfrac{1}{T}\int_0^T lnS(t)\,dt}, & \text{continuously sampled geometric average} \end{cases}$

(In practise, Asian options must obviously always be monitored discretely, but with enough observations they can be considered virtually continuous. The discrete solution empirically appears to converge to the continuous one with 1/M, where M is the number of observations.)

The following relations are useful to examine if our results are correct:

$$\boxed{\begin{array}{l} C^{fixed}_{Arithm} \geq C^{fixed}_{Geom} \\ P^{fixed}_{Arithm} \leq P^{fixed}_{Geom} \end{array}} \qquad \boxed{\begin{array}{l} C^{float}_{Arithm} \leq C^{fixed}_{Geom} \\ C^{float}_{Arithm} \geq C^{fixed}_{Geom} \end{array}}$$

Proof:

Since lnx is concave function, then from Jensen's inequality [1] it holds:

$$ln\left(\frac{1}{N}\sum_{i=1}^{N} S(t_i)\right) \geq \frac{1}{N}\sum_{i=1}^{N} lnS(t_i) = \frac{1}{N}ln\prod_{i=1}^{N} S(t_i) = ln\left(\prod_{i=1}^{N} S(t_i)\right)^{\frac{1}{N}} \overset{ln\uparrow}{\Longrightarrow} \frac{1}{N}\sum_{i=1}^{N} S(t_i) \geq \left(\prod_{i=1}^{N} S(t_i)\right)^{\frac{1}{N}}$$

$$\Rightarrow \left(\frac{1}{N}\sum_{i=1}^{N} S(t_i) - K\right)^{+} \geq \left(\left(\prod_{i=1}^{N} S(t_i)\right)^{\frac{1}{N}} - K\right)^{+} \text{ and } \left(K - \frac{1}{N}\sum_{i=1}^{N} S(t_i)\right)^{+} \leq \left(K - \left(\prod_{i=1}^{N} S(t_i)\right)^{\frac{1}{N}}\right)^{+}$$

$$\Rightarrow C^{fixed}_{Arithm} = \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-r(T-t)}\left(\frac{1}{N}\sum_{i=1}^{N} S(t_i) - K\right)^{+}|\mathcal{F}_t] \geq \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-r(T-t)}\left(\left(\prod_{i=1}^{N} S(t_i)\right)^{\frac{1}{N}} - K\right)^{+}|\mathcal{F}_t] = C^{fixed}_{Geom}$$

$$\Rightarrow P^{fixed}_{Arithm} = \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-r(T-t)}\left(K - \frac{1}{N}\sum_{i=1}^{N} S(t_i)\right)^{+}|\mathcal{F}_t] \leq \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-r(T-t)}\left(K - \left(\prod_{i=1}^{N} S(t_i)\right)^{\frac{1}{N}}\right)^{+}|\mathcal{F}_t] = P^{fixed}_{Geom}$$

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[1] For concave function ϕ, numbers $x_1, ..., x_n$ and positive weights ω_i: $\phi\left(\frac{\sum_{i=1}^{n}\omega_i x_i}{\sum_{i=1}^{n}\omega_i}\right) \geq \frac{\sum_{i=1}^{n}\omega_i \phi(x_i)}{\sum_{i=1}^{n}\omega_i}$

If the weights ω_i are all equal, then $\phi\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) \geq \frac{\sum_{i=1}^{n}\phi(x_i)}{n}$. Here $\phi = lnx$

Similarly, since $\frac{1}{N}\sum_{i=1}^{N} S(t_i) \geq \left(\prod_{i=1}^{N} S(t_i)\right)^{\frac{1}{N}} \Rightarrow \begin{cases} \left(S_T - \frac{1}{N}\sum_{i=1}^{N} S(t_i)\right)^+ \leq \left(S_T - \left(\prod_{i=1}^{N} S(t_i)\right)^{\frac{1}{N}}\right)^+ \; \textit{and} \\ \left(\frac{1}{N}\sum_{i=1}^{N} S(t_i) - S_T\right)^+ \geq \left(\left(\prod_{i=1}^{N} S(t_i)\right)^{\frac{1}{N}-S_T}\right)^+ \end{cases}$

$$\Rightarrow C^{float}_{Arithm} = \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-r(T-t)}(S_T - \frac{1}{N}\sum_{i=1}^{N} S(t_i))^+|\mathcal{F}_t] \leq \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-r(T-t)}(S_T - \left(\prod_{i=1}^{N} S(t_i)\right)^{\frac{1}{N}})^+|\mathcal{F}_t] = C^{float}_{Geom}$$

$$\Rightarrow P^{float}_{Arithm} = \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-r(T-t)}(\frac{1}{N}\sum_{i=1}^{N} S(t_i) - S_T)^+|\mathcal{F}_t] \geq \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-r(T-t)}(S_T - \left(\prod_{i=1}^{N} S(t_i)\right)^{\frac{1}{N}})^+|\mathcal{F}_t] = P^{float}_{Geom}$$

4 PDE for pricing strongly path dependent equations

In order to find the pde for pricing path dependent options, we will extend the BSE, assuming there are no dividends. Firstly, we will derive the general model for the valuation of all path-dependent options.

Value of path-dependent options depend on an extra variable. Thus, the price function V(S(t),t) becomes V(S(t),I(t),t), where I(t) is the newly introduced path-dependent variable, with I(t) = $\int_0^t f(S(\tau),\tau)d\tau$ and f(.) is a function determined relying on which path-dependent option we consider. Since the variable I does not depend on the current asset price S, the option price V is a function of three independent variables.

Furthermore, we need to obtain the stochastic differential equation, satisfies by I. This can be done easily, since

$I + dI = I(t+dt) = \int_0^{t+dt} f(S(\tau),\tau)d\tau = \int_0^t f(S(\tau),\tau)d\tau + f(S(t),t)dt$
$\therefore dI = f(S,t)dt$

We assume that the underlying asset follows the lognormal random walk $dS = rSdt + \sigma SdX$ (*) , where r represents the risk free interest rate, σ the volatility and dX is the standard Brownina motion.

We now construct a portfolio by longing a path-dependent option and shorting an amount of stock, i.e. Π=V(S,t)-ΔS.
The change in portfolio over one time step is the following:
We assume that at the beginning of each period, Δ can change, but across the time step dt,is kept fixed.

Therefore, over a time step we have

$$d\Pi = dV - \Delta dS \tag{1}$$

From Itô, we know: $dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial I}dI + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial^2 S}d[S,S]$

$$\Rightarrow dV = (\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial^2 S})dt + \frac{\partial V}{\partial I}dI + \sigma S\frac{\partial V}{\partial S}dX \tag{2}$$

From (**??**) and (**??**)

$$d\Pi = (\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial^2 S})dt + \frac{\partial V}{\partial I}dI + \sigma S\frac{\partial V}{\partial S}dX - \Delta dS \tag{3}$$

$$\overset{(*)}{\Longrightarrow} d\Pi = (\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial^2 S})dt + \frac{\partial V}{\partial I}dI + \sigma S\frac{\partial V}{\partial S}dX - \Delta(\mu Sdt + \sigma SdX) \tag{4}$$

The change in portfolio is not riskless as it involves the risky term dX. Thus, in order to eliminate risk, we need to vanish the dX term. We achieve this by setting $\Delta = \frac{\partial V}{\partial S}$.

$\overset{(4)}{\Longrightarrow} d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial^2 S})dt + \frac{\partial V}{\partial I}dI$

dI=f(S,t)dt (**)

$$\overset{(**)}{\Longrightarrow} d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial^2 S})dt + f(S,t)\frac{\partial V}{\partial I}dt \tag{5}$$

Having eliminated the risk, we now appeal to the No-arbitrage principle. Suppose we have an amount Π, put this in the bank, receiving an amount r across a time-step dt. Then

$$d\Pi = r\Pi dt \tag{6}$$

$$\underset{(6)}{\overset{(5)}{\Longrightarrow}} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial^2 S} + f(S,t)\frac{\partial V}{\partial I} = r\Pi$$

$$\Rightarrow \boxed{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial^2 S} + f(S,t)\frac{\partial V}{\partial I} + rS\frac{\partial V}{\partial S} - rV = 0} \boxtimes$$

4.1 The Black-Scholes Equation for Continuous Asians

Generally, Asian options can be valued using a PDE in two space-like dimensions (Wilmott,Dewynne and Howison(1992)), which for the case of a floating strike can be reduced to a one-dimension PDE. For instance, Rogers and Shi (1995) have developed a one-dimensional PDE that can be used for both fixed and floating strike Asian options. This PDE, however, values only European-style options. Moreover, the solution is complicated, due to the small size of the diffusion term. The achievement of one-dimensional PDE can be also done through a change of variables but only for floating strike options. This change also permits the pricing of American style floating strike options in one-dimension, even though this is out of the scope of this thesis. To value fixed strike options with early exercise opportunities, we must solve a two-dimensional PDE.

As mentioned, depending on which path-dependent option we consider, the function f(.) in ☒ is chosen analogously. For the case of Asian options, a pricing PDE can be obtained in two ways.

The first one is based on the running sum $I(t) = \int_0^t S(\tau)d\tau$. Then the first derivative with respect to time is $\dfrac{dI}{dt} = S(t) = f(S(t), t)$.
Thus, the BSE for arithmetic Asian options in terms of variable I(t) is:

$$\boxed{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial^2 S} + S\frac{\partial V}{\partial I} + rS\frac{\partial V}{\partial S} - rV = 0} \diamondsuit$$

There exist a unique known analytic solution, which is for the fixed strike case when K=0. By making a change of variables Ingersoll (1987) and Wilmott, Dewynne and Howison (1993) reduce the two-dimensional PDE satisfied by the price of a floating strike Asian option into one-dimensional one.

For discrete Asian option, theses equations transform to one-dimension PDE $\dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial^2 S} + +rS\dfrac{\partial V}{\partial S} - rV = 0$***, with the condition at the observation date*** $V(t_i^-, S, A_{t_i}) = V(t_i^+, S, A_{t_i} + \alpha_{t_i} S_{t_i})$ ***Forsyth et al***

For geometric Asian options, the same things hold, but now f(S,t)=logS. Hence, the BSE takes the form:

$$\boxed{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial^2 S} + logS\frac{\partial V}{\partial I} + rS\frac{\partial V}{\partial S} - rV = 0}.$$

The second way of representation is given from an equivalent formula in terms of the average $A(t) = \dfrac{I(t)}{t}$, instead of running sum I (Barraquand, Pudet, 1996)

$$\therefore A_t = \frac{1}{t}\int_0^t S(\tau)\,d\tau \Rightarrow dA_t = \frac{tS_t - \int_0^t S(\tau)\,d\tau}{t^2}dt \Rightarrow dA_t = \frac{S_t - A_t}{t}dt$$

From Itô, we know: $dV = \dfrac{\partial V}{\partial t}dt + \dfrac{\partial V}{\partial A}dA + \dfrac{\partial V}{\partial S}dS + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial^2 S}d[S,S]$

$$\Rightarrow dV = (\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial^2 S})dt + \frac{\partial V}{\partial A}dA + \sigma S\frac{\partial V}{\partial S}dX$$
$$d\Pi = (\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial^2 S})dt + \frac{\partial V}{\partial A}dA + \sigma S\frac{\partial V}{\partial S}dX - \Delta dS$$

$$\Rightarrow d\Pi = (\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial^2 S})dt + \frac{\partial V}{\partial A}dA + \sigma S\frac{\partial V}{\partial S}dX - \Delta(\mu Sdt + \sigma SdX)$$

$$\Rightarrow d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial^2 S})dt + \frac{\partial V}{\partial A}dA$$

$$\Rightarrow d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial^2 S})dt + \frac{\partial V}{\partial A}\frac{S-A}{t}dt$$

$$\Rightarrow (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial^2 S})dt + \frac{\partial V}{\partial A}\frac{S-A}{t}dt = r\Pi dt$$

$$\Rightarrow (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial^2 S})dt + \frac{\partial V}{\partial A}\frac{S-A}{t}dt = r(V - S\frac{\partial V}{\partial S})dt$$

$$\Rightarrow \boxed{\Rightarrow \frac{\partial V}{\partial t} + \frac{S-A}{t}\frac{\partial V}{\partial A} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial^2 S} + rS\frac{\partial V}{\partial S} - rV = 0}$$

In the case of geometric Asians, working in the same manner and taking $A = \mathrm{e}^{\frac{1}{t}\int_0^t lnS(\tau)\,d\tau}$, we get $dA = \mathrm{e}^{\frac{1}{t}\int_0^t lnS(\tau)\,d\tau}\left(\frac{t\;lnS - \int_0^t lnS(\tau)\,d\tau}{t^2}dt\right) \Rightarrow dA = \frac{A\;ln\frac{S}{A}}{t}$ and we end up with the following PDE:

$$\boxed{\Rightarrow \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial^2 S} + \frac{A\;ln\frac{S}{A}}{t}\frac{\partial V}{\partial A} + rS\frac{\partial V}{\partial S} - rV = 0}$$

4.1.1 Reduction of dimensionality

The above partial differential equations are two-dimensional PDEs, i.e they consist of partial derivatives with respect to time and two other variables. However, none of them includes a second spatial derivative with respect to the new state variable. Because of this feature, these equations are prone to oscillatory solutions while solving them numerically with standard finite difference methods. Thus, is complicated to value the Asian options using the BS PDEs.

Despite of this fact, several transformations exist in order to reduce those PDEs in one-dimensional PDE. *This is not true only for American type average rate options where each time full two-dimensional PDE must be solved*

■ Rogers and Shi (1995):

We introduce the new state variable $x = \dfrac{K - \int_0^t S(\tau)\mu(d\tau)}{S_t}$, where μ is the probability measure with density $\rho(t)$.
The density for a fixed strike option is $\rho(t) = \dfrac{1}{T}$ and for a floating strike option and K=0 we have $\rho(t) = \dfrac{1}{T} - \delta(T-t)$, where δ is a function with $\delta(x)$ equals 1 if x and 0 otherwise.

Thus, the value of an Asian option, according to Rogers and Shi, is described by the one-dimensional equation :

$\boxed{\dfrac{\partial W}{\partial t} + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 W}{\partial^2 x} - (\rho(t) + rx)\dfrac{\partial W}{\partial x} = 0}$ with final conditions:

$W(x,T) = max(-x, 0)$ for fixed strike call and
$W(x,T) = max(-x-1, 0)$ for floating put option.

The price of a fixed strike Asian call with stock price S_0 and strike price K is $S_0 W\left(\dfrac{K}{S_0}, 0\right)$ and for floating strike put is $S_0 W(0,0)$.

Therefore, we have an one-dimension PDE for both fixed and floating strike options.

The same does not apply with the American style option because of the early exercise nature.

■ Ingersoll (1987)

For pricing equation $\diamondsuit$, appropriate boundary conditions are:

$V(0, I, t) = 0$
$\lim_{S\to\infty} \dfrac{\partial V}{\partial S}(S, I, t) = 1$
$\lim_{I\to\infty} V(S, I, t) = 0$
$V(S_T, I, T, T) = \left(S_T - \dfrac{I_T}{T}\right)^+$.

After we introduce the variable $R = \dfrac{S}{I}$, the option price transforms into V(S,I,t)=IW(R,t).
Then, the new pricing equation is $\boxed{\dfrac{\partial W}{\partial t} + \dfrac{1}{2}\sigma^2 R^2 \dfrac{\partial^2 W}{\partial^2 R} + R(r-R)\dfrac{\partial W}{\partial R} - (r-R)W = 0}$ with boundary conditions
$W(0,t) = 0$

$\lim_{R\to\infty} \dfrac{\partial W}{\partial R}(R,t) = 1$
$W(R_T,T) = \left(R_T - \dfrac{1}{T}\right)^+$

Details on this PDE are given on the next section.

This can not be applicable for fixed strike Asian options, as the terminal condition is not homogeneous in S and I, whereas for floating strike, the pricing equation along with the boundary conditions are homogeneous in those variables.

■ Wilmott, Dewinne and Howison

We consider the following change of variable: $R_t = \dfrac{I_t}{S_t} = \dfrac{1}{S_t}\int_0^t S(\tau)d\tau$. Thus the price function becomes V(S,I,t)=S Z(R,t), with terminal payoff $V(S,I,T) = \left(S_T - \dfrac{I_T}{T}\right)^+ = S_T\left(1 - \dfrac{I_T}{TS_T}\right)^+ = S_T\left(1 - \dfrac{R_T}{T}\right)^+ = S_T Z(R_T,T)$

The SDE for R can be obtained as follows:

$$dR_t = d\left(\frac{I_t}{S_t}\right) = \frac{1}{S_t}dI_t + I_t d\left(\frac{1}{S_t}\right) + d\left\langle I, \frac{1}{S}\right\rangle_t =$$
$$\frac{1}{S_t}S_t dt + I_t\left[\frac{-1}{S_t^2}dS_t + \frac{1}{S_t^3}d < S >_t\right] + d\left\langle I, \frac{1}{S}\right\rangle_t = dt + I_t\left[\frac{-1}{S_t}\left((r-\sigma^2)dt + \sigma dX\right)\right]$$
$$\Rightarrow dR_t = (1 + (\sigma^2 - r)R_t)dt - \sigma R_t dX.$$

$$\frac{\partial V}{\partial t} = S\frac{\partial Z}{\partial t}$$
$$\frac{\partial V}{\partial S} = Z + S\frac{\partial Z}{\partial R}\frac{\partial R}{\partial S} = Z + S\frac{\partial Z}{\partial R}\left(\frac{-I}{S^2}\right) = Z - R\frac{\partial Z}{\partial R}$$
$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial Z}{\partial R}\left(\frac{-I}{S^2}\right) - \frac{\partial^2 Z}{\partial R^2}\left(\frac{-I}{S^2}\right)R - \frac{\partial Z}{\partial R}\frac{\partial R}{\partial S} = \frac{-R}{S}\frac{\partial Z}{\partial R} + \frac{R^2}{S}\frac{\partial^2 Z}{\partial R^2} + \frac{R}{S}\frac{\partial Z}{\partial R}$$
$$\frac{\partial Z}{\partial I} = S\frac{\partial Z}{\partial R}\frac{1}{S} = \frac{\partial Z}{\partial R}$$

Thus the equation ♢ becomes $\boxed{\dfrac{\partial Z}{\partial t} + \dfrac{1}{2}\sigma^2 R^2 \dfrac{\partial^2 Z}{\partial R^2} + (1 - rR)\dfrac{\partial Z}{\partial R} = 0}$ which is not two-dimensional anymore. The boundary conditions become:

$\lim_{R\to\infty} Z(T,t) = 0$
$\frac{\partial Z}{\partial t}(0,t) + \frac{\partial Z}{\partial R}(0,t) = 0$
$Z(R_T,T) = \left(1 - \frac{R_T}{T}\right)^+.$

Again, this reduction to one-dimensional form does not hold for fixed strike options.

4.2 Lookbacks

We will price Lookback options, by extending the BS model.

We define $I_n(t) = \left(\int_0^t (S_\tau)^n \, d\tau\right)^{1/n}$ and $J_n(t) = \left(\int_0^t (S_\tau)^{-n} \, d\tau)\right)^{-1/n}$.

Then $\lim\limits_{n\to\infty} = I_n(t) = M_t = \max\limits_{0\le\tau\le t} S_\tau$ (=the running maximum) and
$\lim\limits_{n\to\infty} = J_n(t) = m_t = \min\limits_{0\le\tau\le t} S_\tau$ (=the running minimum)

Proof: Appendix

Having proven this and using the differential

$$dI_n(t) = d\left(\int_0^t (S_\tau)^n \, d\tau\right)^{1/n} = \frac{1}{n}\left(\int_0^t (S_\tau)^n \, d\tau\right)^{1/n-1} S_t^n dt =$$

$$= \frac{1}{n}\left(\int_0^t (S_\tau)^n \, d\tau\right)^{\frac{1-n}{n}} S_t^n dt = \frac{1}{n}\left(\int_0^t (S_\tau)^n \, d\tau\right)^{\frac{-(n-1)}{n}} S_t^n dt = \frac{1}{n}\left(I_n(t)\right)^{-(n-1)} S_t^n dt = \frac{1}{n}\frac{1}{\left(I_n(t)\right)^{n-1}} S_t^n dt$$

, we will follow the same procedure as in the Asian option subsection. Therefore, we have

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial^2 S} + \frac{\partial V}{\partial I_n} dI_n + rS\frac{\partial V}{\partial S} - rV = 0 \Rightarrow \boxed{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial^2 S} + \frac{S^n}{nI_n^{n-1}}\frac{\partial V}{\partial I_n} dt + rS\frac{\partial V}{\partial S} - rV = 0}$$

Since $S_t \le M_t = \max\limits_{0\le\tau\le t} S_\tau \Rightarrow \lim\limits_{n\to\infty} \frac{S^n}{nI_n^{n-1}} \overset{\cdot I^{n+1}}{=} \lim\limits_{n\to\infty} \frac{I_n}{n}\left(\frac{S}{I_n}\right)^n = 0$

However, as we observed and is also obvious from the figure 1, in case of discrete sampling, it is possible to have $S \ge M$. In this case, $\lim\limits_{n\to\infty} \frac{I_n}{n}\left(\frac{S}{I_n}\right)^n = \infty$. So, in this situation, we must have $\lim\limits_{n\to\infty} \frac{\partial V}{\partial I_n} = 0$.

Finally, at $S_t = M_t$ we have $\frac{\partial V}{\partial M} = 0$, i.e the option value is independent of the running maximum.

For a floating strike put, the PDE we need to solve, the boundary and final conditions are as follows:

$$\boxed{\begin{array}{l} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial^2 S} + rS\frac{\partial V}{\partial S} - rV = 0, \text{ for } 0 < S < M \\ \text{with final condition } V(S_T, M_T, T) = M_T - S_T \\ \text{and } \frac{\partial V}{\partial M}(M, M, t) = 0 \end{array}}$$

Similarly, for a floating strike call, we have the differential

$$dJ_n(t) = d\left(\int_0^t (S_\tau)^{-n}\,d\tau\right)^{-1/n} = \frac{-1}{n}\left(\int_0^t (S_\tau)^{-n}\,d\tau\right)^{-1/n-1} S_t^{-n}dt =$$

$$= \frac{-1}{n}\left(\int_0^t (S_\tau)^{-n}\,d\tau\right)^{\dfrac{-1-n}{n}} S_t^{-n}dt = \frac{-1}{n}\left(\int_0^t (S_\tau)^{-n}\,d\tau\right)^{\dfrac{-(n+1)}{n}} S_t^{-n}dt = \frac{-1}{n}\left(J_n(t)\right)^{(n-1)} S_t^{-n}dt$$
$$=$$

$$= \frac{-1}{n}\frac{\left(J_n(t)\right)^{n+1}}{S_t^n}dt$$

Thus, the observed PDE is $\boxed{\dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial^2 S} - \dfrac{1}{n}\dfrac{\left(J_n(t)\right)^{n+1}}{S_t^n}\dfrac{\partial V}{\partial J_n}dt + rS\dfrac{\partial V}{\partial S} - rV = 0}$

For $m_t < S_t$: $\dfrac{\left(J_n(t)\right)^{n+1}}{nS_t^n} \overset{\cdot J^{-n+1}}{=} \dfrac{J_n}{n}\left(\dfrac{J_n}{S_t}\right)^n \xrightarrow{n\to\infty} 0$

As previously, in discrete case, we may have $\dfrac{J_n}{n}\left(\dfrac{J_n}{S_t}\right)^n \xrightarrow{n\to\infty} \infty$. Thus, it must hold $\dfrac{\partial V}{\partial J_n} = 0$

Finally, if $m_t = S_t$, then because the running minimum can not be the final minimum, we need $\dfrac{\partial V}{\partial m} = 0$

For a floating strike call, the PDE we need to solve, the boundary and final conditions are as follows:

$$\boxed{\begin{array}{l} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial^2 S} + rS\dfrac{\partial V}{\partial S} - rV = 0, \text{ for m} \leq \text{S} \\ \text{with final condition } V(S_T, m_T, T) = S_T - m_T \\ \text{and } \dfrac{\partial V}{\partial m}(m, m, t) = 0 \end{array}}$$

5 Value of Asian options

In general, Asian options are difficult to value, because the traditional methods such as the binomial lattice, the PDE and Monte Carlo simulation, are inaccurate and impractical. Zvan, Forsyth and Vetzal have introduced a modified FDM, which seems to be more efficient.

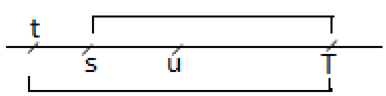
In addition, when we price Asian options at various future dates, the price of the underlying asset is modelled using log-normal distribution function. At expiration, the payoff measured by the arithmetic average of log-normal random variables, is not log-normally distributed anymore. Therefore, pricing European Asian Options, is not trivial at all. Particularly, for continuously monitored European Asian options, an analytic solution exists only if the average is taken geometrically. However, geometric average tends to under-price the value of Asian call option.
Even thought analytic solutions do exist for European Asians using arithmetic average, various obstacles make the procedure complicated. For instance, German and Yor(1993) derived the Laplace transform of the European option price, however, its inversion is quite difficult. Also, there are expressions involving an infinite sum over recursively defined integrals (Dufrense ,2000). Thus, approximations are used (eg Turnbull and Wakeman (1991), Levy and Turnbull (1992), Rogers and Shi (1995) which we will present below). For exact results, or to incorporate with features such as early exercise, numerical methods must be used.

5.1 Asian fixed strike call and put (geometric - cont)

$$S_T = S_t \mathrm{e}^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma\int_t^T dW_u} \Rightarrow$$

$$lnS_T = lnS_t + \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma \int_t^T dW_u \Rightarrow$$

$$\int_t^T lnS_u\,du = \int_t^T lnS_t\,du + \int_t^T \left(r - \frac{\sigma^2}{2}\right)(u-t)\,du + \int_t^T \sigma \int_t^u dW_s\,du \xRightarrow{fubini}$$



$$\int_t^T lnS_u\,du = (T-t)lnS_t + \int_t^T \left(r - \frac{\sigma^2}{2}\right)(u-t)\,du + \sigma \int_t^T \int_s^T du\,dW_s \Rightarrow$$

$$\int_t^T lnS_u\,du = (T-t)lnS_t + \left(r - \frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2} + \sigma \int_t^T (T-s)\,dW_s \Rightarrow$$

$$\frac{1}{T-t}\int_t^T lnS_u\,du = lnS_t + \left(r - \frac{\sigma^2}{2}\right)\frac{(T-t)}{2} + \frac{\sigma}{T-t}\int_t^T (T-s)\,dW_s \otimes$$

$$\overset{\otimes}{\Rightarrow} \int_t^0 lnS_u\,du + \int_0^T lnS_u\,du = (T-t)lnS_t + \left(r - \frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2} + \sigma\int_t^T (T-s)\,dW_s \Rightarrow$$

$$\int_0^T lnS_u\,du = \int_0^t lnS_u\,du + (T-t)lnS_t + \left(r - \frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2} + \sigma\int_t^T (T-s)\,dW_s \Rightarrow$$

$$\frac{1}{T}\int_0^T lnS_u\,du = \frac{1}{T}\int_0^t lnS_u\,du + \frac{(T-t)}{T}lnS_t + \left(r - \frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2T} + \frac{\sigma}{T}\int_t^T (T-s)\,dW_s$$

$$where, \frac{\sigma}{T}\int_t^T (T-s)\,dW_s \sim N\left(0, \frac{\sigma^2}{T^2}\frac{(T-t)^3}{3}\right) \Rightarrow$$

$$\frac{1}{T}\int_0^T lnS_u\,du = \frac{1}{T}\int_0^t lnS_u\,du + \frac{(T-t)}{T}lnS_t + \left(r - \frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2T} + \frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}}z, z \sim N(0,1) \Rightarrow$$

$$e^{\frac{1}{T}\int_0^T lnS_u\,du} = e^{\frac{1}{T}\int_0^t lnS_u\,du} e^{\frac{(T-t)}{T}lnS_t} e^{\left(r-\frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2T}} e^{\frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}}z} \Rightarrow$$

$$e^{\frac{1}{T}\int_0^T lnS_u\,du} = e^{\frac{1}{T}\int_0^t lnS_u\,du} S_t^{\frac{(T-t)}{T}} e^{\left(r-\frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2T}} e^{\frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}}z} \Rightarrow **$$

$$\text{Payoff}= \left[e^{\frac{1}{T}\int_0^T lnS_u\,du} - K\right]^+ = \begin{cases} e^{\frac{1}{T}\int_0^T lnS_u\,du} - K, & e^{\frac{1}{T}\int_0^T lnS_u\,du} - K \geq 0 \\ 0, & e^{\frac{1}{T}\int_0^T lnS_u\,du} - K < 0 \end{cases} =(**)=$$

$$\begin{cases} e^{\frac{1}{T}\int_0^T lnS_u\,du} - K, & e^{\frac{1}{T}\int_0^t lnS_u\,du} S_t^{\frac{(T-t)}{T}} e^{\left(r-\frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2T}} e^{\frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}}z} - K \geq 0 \\ 0, & e^{\frac{1}{T}\int_0^t lnS_u\,du} S_t^{\frac{(T-t)}{T}} e^{\left(r-\frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2T}} e^{\frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}}z} - K < 0 \end{cases} =$$

$$\begin{cases} e^{\frac{1}{T}\int_0^T lnS_u\,du} - K, & \frac{1}{T}\int_0^t lnS_u\,du + lnS_t^{\frac{(T-t)}{T}} \left(r - \frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2T}\frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}}z \geq lnK \\ 0, & \frac{1}{T}\int_0^t lnS_u\,du + lnS_t^{\frac{(T-t)}{T}} \left(r - \frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2T}\frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}}z < lnK \end{cases} =$$

$$
\begin{cases}
e^{\frac{1}{T}\int_0^T lnS_u\,du} - K, & z \geq \dfrac{lnK - \frac{1}{T}\int_0^t lnS_u\,du - \left(r - \frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2T} - lnS_t{}^{\frac{(T-t)}{T}}}{\frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}}} \\[2em]
0, & z < \dfrac{lnK - \frac{1}{T}\int_0^t lnS_u\,du - \left(r - \frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2T} - lnS_t{}^{\frac{(T-t)}{T}}}{\frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}}}
\end{cases}
=
$$

$$
\begin{cases}
e^{\frac{1}{T}\int_0^T lnS_u\,du} - K, & z \geq \dfrac{TlnK - \int_0^t lnS_u\,du - \left(r - \frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2} - (T-t)lnS_t}{\sigma\sqrt{\frac{(T-t)^3}{3}}} \\[2em]
0, & z < \dfrac{TlnK - \int_0^t lnS_u\,du - \left(r - \frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2} - (T-t)lnS_t}{\sigma\sqrt{\frac{(T-t)^3}{3}}}
\end{cases}
=
$$

$$
\begin{cases}
e^{\frac{1}{T}\int_0^T lnS_u\,du} - K, & z \geq \dfrac{\frac{T}{T-t}lnK - \frac{1}{T-t}\int_0^t lnS_u\,du - \left(r - \frac{\sigma^2}{2}\right)\frac{(T-t)}{2} - lnS_t}{\sigma\sqrt{\frac{(T-t)}{3}}} \\[2em]
0, & z < \dfrac{\frac{T}{T-t}lnK - \frac{1}{T-t}\int_0^t lnS_u\,du - \left(r - \frac{\sigma^2}{2}\right)\frac{(T-t)}{2} - lnS_t}{\sigma\sqrt{\frac{(T-t)}{3}}}
\end{cases}
=
$$

$$
\begin{cases}
e^{\frac{1}{T}\int_0^T lnS_u\,du} - K, & z \geq \tilde{d}_1 \\
0, & z < \tilde{d}_1
\end{cases}
$$

$$\tilde{d_1} = \frac{\frac{-T}{T-t} lnK + \frac{1}{T-t}\int_0^t lnS_u\,du + \left(r - \frac{\sigma^2}{2}\right)\frac{(T-t)}{2} + lnS_t}{\sigma\sqrt{\frac{(T-t)}{3}}}$$

$$C_{fix}(t) = \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-r(T-t)Payoff|\mathcal{F}_t}] = \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-r(T-t)}(\mathrm{e}^{\frac{1}{T}\int_0^T lnS_u\,du} - K)^+|\mathcal{F}_t] =$$

$$\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-r(T-t)}(\mathrm{e}^{\frac{1}{T}\int_0^t lnS_u\,du} S_t^{\frac{(T-t)}{T}} \mathrm{e}^{\left(r-\frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2T}} \mathrm{e}^{\frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}}z} - K)^+|\mathcal{F}_t] =$$

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-r(T-t)}}{\sqrt{2\pi}} [\mathrm{e}^{\frac{1}{T}\int_0^t lnS_u\,du} S_t^{\frac{(T-t)}{T}} \mathrm{e}^{\left(r-\frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2T}} \mathrm{e}^{\frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}}z} - K]^+ f(z)\,dz =$$

$$\left(\int_{-\infty}^{\infty} = \cancelto{0}{\int_{-\infty}^{-\tilde{d_1}}} + \int_{-\tilde{d_1}}^{\infty}\right)$$

$$\int_{-\tilde{d_1}}^{\infty} \frac{\mathrm{e}^{-r(T-t)}}{\sqrt{2\pi}} [\mathrm{e}^{\frac{1}{T}\int_0^t lnS_u\,du} S_t^{\frac{(T-t)}{T}} \mathrm{e}^{\left(r-\frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2T}} \mathrm{e}^{\frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}}z} - K] f(z)\,dz =$$

$$\frac{\mathrm{e}^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\tilde{d_1}}^{\infty} \mathrm{e}^{\frac{1}{T}\int_0^t lnS_u\,du} S_t^{\frac{(T-t)}{T}} \mathrm{e}^{\left(r-\frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2T}} \mathrm{e}^{\frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}}z} f(z)\,dz - \frac{\mathrm{e}^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\tilde{d_1}}^{\infty} Kf(z)\,dz =$$

$$\frac{\mathrm{e}^{-r(T-t)}}{\sqrt{2\pi}} \mathrm{e}^{\frac{1}{T}\int_0^t lnS_u\,du} S_t^{\frac{(T-t)}{T}} \mathrm{e}^{\left(r-\frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2T}} \int_{-\tilde{d_1}}^{\infty} \mathrm{e}^{\frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}}z} f(z)\,dz - K\mathrm{e}^{-r(T-t)} N(-(-\tilde{d_1})) =$$

$$\frac{\mathrm{e}^{-r(T-t)}}{\sqrt{2\pi}} \mathrm{e}^{\frac{1}{T}\int_0^t lnS_u\,du} S_t^{\frac{(T-t)}{T}} \mathrm{e}^{\left(r-\frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2T}} \int_{-\tilde{d_1}}^{\infty} \mathrm{e}^{\frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}}z} \mathrm{e}^{\frac{-z^2}{2}}\,dz - K\mathrm{e}^{-r(T-t)} N(\tilde{d_1}) =$$

$$\frac{\mathrm{e}^{-r(T-t)}}{\sqrt{2\pi}} \mathrm{e}^{\frac{1}{T}\int_0^t lnS_u\,du} S_t^{\frac{(T-t)}{T}} \mathrm{e}^{\left(r-\frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2T}} \int_{-\tilde{d_1}}^{\infty} \mathrm{e}^{\frac{-1}{2}\left(z^2 - 2\frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}}z\right)}\,dz - K\mathrm{e}^{-r(T-t)} N(\tilde{d_1}) =$$

$$\frac{\mathrm{e}^{-r(T-t)}}{\sqrt{2\pi}} \mathrm{e}^{\frac{1}{T}\int_0^t lnS_u\,du} S_t^{\frac{(T-t)}{T}} \mathrm{e}^{\left(r-\frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2T}} \int_{-\tilde{d_1}}^{\infty} \mathrm{e}^{\frac{-1}{2}\left(z - \frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}}\right)^2} \mathrm{e}^{\frac{1}{2}\frac{\sigma^2}{T^2}\frac{(T-t)^3}{3}}\,dz -$$

$$K\mathrm{e}^{-r(T-t)} N(\tilde{d_1}) =$$

$$\text{set } z - \frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}} = u$$

$$\frac{e^{-r(T-t)}}{\sqrt{2\pi}} e^{\frac{1}{T}\int_0^t lnS_u\,du} S_t^{\frac{(T-t)}{T}} e^{\left(r-\frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2T}} e^{\frac{\sigma^2}{6}\frac{(T-t)^3}{T^2}} \int_{-\tilde{d}_1-\frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}}}^{\infty} e^{\frac{-u^2}{2}} du - K e^{-r(T-t)} N(\tilde{d}_1) =$$

$$e^{-r(T-t)} e^{\frac{1}{T}\int_0^t lnS_u\,du} S_t^{\frac{(T-t)}{T}} e^{\left(r-\frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2T}} e^{\frac{\sigma^2}{6}\frac{(T-t)^3}{T^2}} N\left(\tilde{d}_1 + \frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}}\right) - K e^{-r(T-t)} N(\tilde{d}_1)$$

$\Rightarrow$

$$\boxed{C_{fix}(t) = e^{-r(T-t)}[e^{\frac{1}{T}\int_0^t lnS_u\,du} S_t^{\frac{(T-t)}{T}} e^{\left(r-\frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2T}} e^{\frac{\sigma^2}{6}\frac{(T-t)^3}{T^2}} N\left(\tilde{d}_1 + \frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}}\right) - K e^{-r(T-t)} N\left(\tilde{d}_1\right)]}$$

5.2 Asian floating strike call and put (Geometric)

$$\boxed{C_{float}(S,A,t) = SN(d_1) - A^{t/T} S^{(T-t)/T} e^{\frac{\left(r+\frac{\sigma^2}{2}\right)(T^2-t^2)}{2T} - \frac{\sigma^2}{6}\frac{T^3-t^3}{T^2}} N(d_2)}$$

$$\boxed{P_{float}(S,A,t) = -SN(-d_1) + A^{t/T} S^{(T-t)/T} e^{\frac{\left(r+\frac{\sigma^2}{2}\right)(T^2-t^2)}{2T} - \frac{\sigma^2}{6}\frac{T^3-t^3}{T^2}} N(-d_2)}$$

$$d_1 = \frac{tln\frac{S}{A} + \left(r+\frac{\sigma^2}{2}\right)\frac{T^2-t^2}{2}}{\sigma\sqrt{\frac{T^3-t^3}{3}}}$$

$$d_2 = \frac{tln\frac{S}{A} + \left(r-\frac{\sigma^2}{2}\right)\frac{T^2-t^2}{2}}{\sigma\sqrt{\frac{T^3-t^3}{3}}} = d_1 - \sigma\sqrt{\frac{T^3-t^3}{3}}$$

put-call parity:(Wilmott) $C_{float}(S,A,t)-P_{float}(S,A,t) = S - A^{t/T}S^{(T-t)/T}\mathrm{e}^{\frac{\sigma^2(T-t)^3}{6T^2}+\frac{\left(r-\frac{\sigma^2}{2}\right)(T-t)^2}{2T}-r(T-t)}$

5.3 Asian fixed strike call and put (arithmetic - cont)

$C_{fix}(S_T, A_T, K) = max(A_T - K, 0),\ A_T = \frac{1}{T}\int_0^T S_u\, du$

$C_{fix}(S_t, A_t, t) = \mathrm{e}^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[max(A_T - K, 0)] = \mathrm{e}^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[max(\frac{1}{T}\int_0^T S_u\, du - K, 0)]$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial^2 S} + rS\frac{\partial V}{\partial S} + S\frac{\partial V}{\partial I} - rV = 0 \quad \boxdot$$

We set $\chi = \frac{1}{S}(I - KT)$ and $f(\chi, t) = \frac{V(S,I,t)}{S}$ $(\Rightarrow V = fS)$, where $I_t = \int_0^t S_u = tA_t$

$\Rightarrow$ final condition: $f(S,I,T) = V(S,I,T)$

$$\begin{aligned} &= max(A_T - K, 0) = max\left(\frac{A_T}{S} - \frac{K}{S}, 0\right) = max\frac{1}{S}(A-K,0) \\ &= \frac{1}{T}max\frac{1}{S}(TA - TK, 0) \Rightarrow f(\chi,T) = \frac{1}{T}max(\chi, 0) \end{aligned}$$

for $I \geq KT$, the final condition becomes $f(\chi, T) = \frac{\chi}{T}$ and a closed form solution can be found. (For, I $\leq$ KT, no closed form solution exists)

Then, $\frac{\partial V}{\partial t} = S\frac{\partial f}{\partial t}$

$$\frac{\partial V}{\partial S} = f - S\frac{\partial f}{\partial \chi}\frac{\partial \chi}{\partial S} = f + S\frac{\partial f}{\partial \chi}\frac{-(I-KT)}{S^2} = f - S\frac{\partial f}{\partial \chi}\frac{\chi}{S} = f - \chi\frac{\partial f}{\partial \chi}$$

$$\frac{\partial^2 V}{\partial^2 S} = \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right) = \cancel{\frac{\partial f}{\partial \chi}\frac{\partial \chi}{\partial S}} - \cancel{\frac{\partial \chi}{\partial S}\frac{\partial f}{\partial \chi}} - \chi\frac{\partial}{\partial \chi}\left(\frac{\partial f}{\partial \chi}\right)\frac{\partial \chi}{\partial S} = \chi\frac{\partial^2 f}{\partial^2 \chi}\frac{(I-KT)}{S^2} = \chi\frac{\partial^2 f}{\partial^2 \chi}\frac{\chi}{S} = \frac{\chi^2}{S}\frac{\partial^2 f}{\partial^2 \chi}$$

$$\frac{\partial V}{\partial I} = \frac{\partial S}{\partial I}f + S\frac{\partial f}{\partial I} = S\frac{\partial f}{\partial \chi}\frac{\partial \chi}{\partial I} = S\frac{1}{S}\frac{\partial f}{\partial \chi}$$

Therefore, ⊡ becomes: $S\dfrac{\partial f}{\partial t}+\dfrac{\sigma^2}{2}S^2\left(\dfrac{\chi^2}{S}\dfrac{\partial^2 f}{\partial^2\chi}\right)+rS\left(f-\chi\dfrac{\partial f}{\partial\chi}\right)+S\left(S\dfrac{1}{S}\dfrac{\partial f}{\partial\chi}\right)-rfS=0$

$$\Rightarrow \frac{\partial f}{\partial t}+\frac{\sigma^2}{2}\chi^2\frac{\partial^2 f}{\partial^2\chi}+r\left(\cancel{f}-\chi\frac{\partial f}{\partial\chi}\right)+\frac{\partial f}{\partial\chi}-\cancel{rf}=0$$

$$\Rightarrow \frac{\partial f}{\partial t}+\frac{\sigma^2}{2}\chi^2\frac{\partial^2 f}{\partial^2\chi}+(1-\chi)\frac{\partial f}{\partial\chi}=0$$

We assume a solution of the form $f(\chi,t)=A(t)\chi+B(t)$, with BCs $A(T)=\dfrac{1}{T}$ and $B(T)=0$

$$\Rightarrow \frac{\partial f}{\partial t}=\dot{A}(t)\chi+\dot{B}(t)$$

$$\frac{\partial f}{\partial\chi}=A(t)$$

$$\frac{\partial^2 f}{\partial^2\chi}=0$$

$$\Rightarrow \dot{A}(t)\chi+\dot{B}(t)+(1-r\chi)A(t)=0$$

$$\Rightarrow [\dot{A}(t)-rA(t)]\chi+[\dot{B}(t)+A(t)]=0\Rightarrow\begin{cases}\dot{A}(t)-rA(t)=0\\ \dot{B}(t)+A(t)=0\end{cases}$$

$$\frac{dA(t)}{dt}-rA(t)=0\Rightarrow\frac{dA(t)}{dA(t)}=rdt$$

By integrating over the horizon t to T, we get: $ln\dfrac{A(T)}{A(t)}=r(T-t)\Rightarrow ln\dfrac{1}{TA(t)}=r(T-t)\Rightarrow$ $\dfrac{1}{TA(t)}=\mathrm{e}^{r(T-t)}\Rightarrow\boxed{A(t)=\dfrac{1}{T}\mathrm{e}^{-r(T-t)}}$

$$\frac{dB(t)}{dt}+A(t)=0\Rightarrow dB(t)=-\frac{1}{T}\mathrm{e}^{-r(T-t)}dt$$

Integrating again from t to T, we end up with: $B(T)-B(t)=-\dfrac{1}{rT}\mathrm{e}^{-r(T-t)}\Big|_t^T\Rightarrow$

$$-B(t)=-\frac{1}{rT}(1-\mathrm{e}^{-r(T-t)})\Rightarrow\boxed{B(t)=\frac{1}{rT}(1-\mathrm{e}^{-r(T-t)})}$$

Hence, $f(\chi,t)=[\dfrac{1}{T}\mathrm{e}^{-r(T-t)}]\chi+\dfrac{1}{rT}(1-\mathrm{e}^{-r(T-t)})$

Since $f(\chi,t)=\dfrac{V(S,I,t)}{S}$ (where, $V(S,I,t)=C_{fix}(S,I,t)$) we have:

$$
\begin{aligned}
C_{fix}(S,I,t) &= [\frac{1}{T}\mathrm{e}^{-r(T-t)}]S\chi + \frac{1}{rT}(1-\mathrm{e}^{-r(T-t)})S \\
&= \frac{1}{T}\mathrm{e}^{-r(T-t)}(I-KT) + \frac{1}{rT}(1-\mathrm{e}^{-r(T-t)})S
\end{aligned}
$$

$$
\Rightarrow \boxed{C_{fix}(S,I,t) = \mathrm{e}^{-r(T-t)}\left(\frac{I}{T}-K\right) + \frac{1}{rT}(1-\mathrm{e}^{-r(T-t)})S}
$$

Since, $P_{fix}(S_T, A_T, K) = max(K - A_T, 0)$, $A_T = \frac{1}{T}\int_0^T S_u\,du$

and $max(A_T - K, 0) - max(K - A_T, 0) = A_T - K$, then

$$
C_{fix}(S_t, A_t, t) - P_{fix}(S_t, A_t, t) = \mathrm{e}^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[max(A_T - K, 0) - max(K - A_T, 0)] =
$$

$$
\mathrm{e}^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[A_T] - K\mathrm{e}^{-r(T-t)} = \mathrm{e}^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\int_0^T S_u\,du\right] - K\mathrm{e}^{-r(T-t)} =
$$

$$
\mathrm{e}^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\int_0^T S_u\,du\right] - K\mathrm{e}^{-r(T-t)} = \mathrm{e}^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\left(\int_0^t S_u\,du + \int_t^T S_u\,du\right)\right] - K\mathrm{e}^{-r(T-t)} =
$$

$$
\mathrm{e}^{-r(T-t)}\frac{1}{T}\int_0^t S_u\,du \;+\; \mathrm{e}^{-r(T-t)}\frac{1}{T}\left[\int_t^T \mathbb{E}^{\mathbb{Q}}[S_u]\,du\right] - K\mathrm{e}^{-r(T-t)} =
$$

$$
\mathrm{e}^{-r(T-t)}\frac{1}{T}\,I_t \;+\; \mathrm{e}^{-r(T-t)}\frac{1}{T}\left[\int_t^T S_t\mathrm{e}^{r(u-t)}\,du\right] - K\mathrm{e}^{-r(T-t)} =
$$

$$
\mathrm{e}^{-r(T-t)}\frac{1}{T}\,I_t + \mathrm{e}^{-r(T-t)}\frac{1}{T}\,S_t\,\frac{1}{r}[e^{r(T-t)} - 1] - K\mathrm{e}^{-r(T-t)} =
$$

$$
\Rightarrow C_{fix}(S_t, A_t, t) - P_{fix}(S_t, A_t, t) = \mathrm{e}^{-r(T-t)}\left[\frac{1}{T}\,I_t - K\right] + \frac{1}{T}\,S_t\,\frac{1}{r}[1 - \mathrm{e}^{-r(T-t)}]
$$

$$
\Rightarrow \boxed{P_{fix}(S_t, A_t, t) = C_{fix}(S_t, A_t, t) - \mathrm{e}^{-r(T-t)}\left[\frac{1}{T}\,I_t - K\right] - \frac{1}{rT}\,[1 - \mathrm{e}^{-r(T-t)}]S_t}
$$

5.4 Asian floating strike call and put (Arithmetic - cont)

It permits a reduction in dimensionality of the problem by the use of a similarity variable. The dimensionality of the continuously-sampled arithmetic floating strike option can be reduced from three to two.

The payoff for the call option is $max\left(S - \frac{1}{T}\int_0^T S(\tau)d\tau, 0\right)$. We write $R = \frac{S}{I}$, where $I_t = \int_0^t S(\tau)d\tau$. R is our new variable. Therefore, at expiry $I_T = \int_0^T S(\tau)d\tau$. Then, the payoff for the call option can be written as $max\left(S - \frac{I}{T}, 0\right) = Imax\left(R - \frac{1}{T}, 0\right)$. The option value takes the form $V(S, R, t) = IW(R, t), R = \frac{S}{I}$, where W(R,t) is some unknown function.

We have already shown that the pricing PDE is $\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial^2 S} + f(S,t)\frac{\partial V}{\partial I} + rS\frac{\partial V}{\partial S} - rV = 0$. Since we are dealing with arithmetic average, f(S,t)=S.

Hence, $\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial^2 S} + S\frac{\partial V}{\partial I} + rS\frac{\partial V}{\partial S} - rV = 0$.

$$\frac{\partial V}{\partial t} = I\frac{\partial W}{\partial t}$$

$$\frac{\partial V}{\partial S} = I\frac{\partial W}{\partial S} = I\frac{\partial W}{\partial R}\frac{\partial R}{\partial S} = I\frac{\partial W}{\partial R}\frac{1}{I} = \frac{\partial W}{\partial R}$$

$$\frac{\partial^2 V}{\partial^2 S} = \frac{\partial}{\partial S}\left(\frac{\partial W}{\partial R}\right) = \frac{\partial}{\partial R}\left(\frac{\partial W}{\partial R}\right)\frac{\partial R}{\partial S} = \frac{\partial^2 W}{\partial^2 R}\frac{1}{I}$$

$$\frac{\partial V}{\partial I} = W + I\frac{\partial W}{\partial I} = W + I\frac{\partial W}{\partial R}\frac{\partial R}{\partial I} = W + I\frac{\partial W}{\partial R}\left(-\tfrac{S}{I^2}\right) = W - \frac{\partial W}{\partial R}\frac{S}{I} = W - \frac{\partial W}{\partial R}R$$

Therefore, the PDE becomes

$$\frac{\partial W}{\partial t}I + \frac{1}{2}\sigma^2 S^2 \frac{1}{I}\frac{\partial^2 W}{\partial^2 R} + rS\frac{\partial W}{\partial R} + S\left(W - R\frac{\partial W}{\partial R}\right) - rIW = 0 \overset{/I}{\Longrightarrow}$$

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{1}{I^2}\frac{\partial^2 W}{\partial^2 R} + r\frac{S}{I}\frac{\partial W}{\partial R} + \frac{S}{I}\left(W - R\frac{\partial W}{\partial R}\right) - rW = 0 \Rightarrow$$

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 W}{\partial^2 R} + R(r - R)\frac{\partial W}{\partial R} - (r - R)W = 0.$$

The final condition becomes $W(R,T) = max\left(R - \frac{1}{T}, 0\right)$,since $V(S,T) = Imax\left(\frac{S}{I} - \frac{1}{T}, 0\right)$ and $V(S,T) = IW(R,T)$

This reduction is not possible for American variants.

For the put case, we will discover the put-call parity that exists. Suppose we have a portfolio consisting of a long position in European floating strike call option and we short one put. The value of this portfolio is identical to one consisting of one asset and a financial product whose payoff is $-\frac{I}{T}$. The payoff of such a portfolio at maturity is $Imax\left(R-\frac{1}{T},0\right)-Imax\left(\frac{1}{T}-R,0\right)=S-\frac{I}{T}$. Then we look for a solution of the form $W(t)=b(t)+a(t)R$, with $a(T)=0, b(T)=\frac{-1}{T}$.

$$\frac{\partial W}{\partial t}=\dot{b(t)}+\dot{a(t)}R$$

$$\frac{\partial W}{\partial R}=a(t)$$

$$\frac{\partial^2 W}{\partial^2 R}=0$$

$$\Rightarrow \dot{b(t)}+\dot{a(t)}R+R(r-R)a(t)-(r-R)(b(t)+a(t)R)=0$$

$$\Rightarrow [\dot{b(t)}-rb(t)]+R[\dot{a(t)}+b(t)]=0 \Rightarrow \begin{cases}\dot{b(t)}-rb(t)=0\\ \dot{a(t)}+b(t)=0\end{cases}$$

$$\frac{db(t)}{dt}-rb(t)=0 \Rightarrow \frac{db(t)}{db(t)}=rdt$$

By integrating over the horizon t to T, we get: $ln\frac{b(T)}{b(t)}=r(T-t) \Rightarrow ln\frac{-1}{Tb(t)}=r(T-t) \Rightarrow$ $\frac{-1}{Tb(t)}=\mathrm{e}^{r(T-t)} \Rightarrow \boxed{b(t)=-\frac{1}{T}\mathrm{e}^{-r(T-t)}}$

$$\frac{da(t)}{dt}+b(t)=0 \Rightarrow da(t)=\frac{1}{T}\mathrm{e}^{-r(T-t)}dt$$

Integrating again from t to T, we end up with: $a(T)-a(t)=\frac{1}{rT}\mathrm{e}^{-r(T-t)}\Big|_t^T \Rightarrow$

$$a(T)-a(t)=\frac{1}{rT}(1-\mathrm{e}^{-r(T-t)}) \Rightarrow \boxed{a(t)=\frac{1}{rT}(-1+\mathrm{e}^{-r(T-t)})+1}$$

Hence, $W(t)=-\frac{1}{T}\mathrm{e}^{-r(T-t)}+\frac{1}{rT}(-1+\mathrm{e}^{-r(T-t)})R+R$

Since $W(t)=\frac{V}{I}$ (where, $V=C_{float}(S,I,t)$) we have:

$$C_{float}(S,I,t)=-I\frac{1}{T}\mathrm{e}^{-r(T-t)}+I\frac{1}{rT}(-1+\mathrm{e}^{-r(T-t)})R+S$$

$$\Rightarrow \boxed{C_{float}(S,I,t) = \frac{-I}{T}\mathrm{e}^{-r(T-t)} + \frac{1}{rT}(-1+\mathrm{e}^{-r(T-t)})S + S}$$

$$C_{float} - P_{float} = S - \frac{S}{rT}(1-\mathrm{e}^{-r(T-t)}) - \frac{1}{T}\mathrm{e}^{-r(T-t)}I_t \text{ (swsto)}$$

5.5 Geometric Closed Form (Kemna and Vorst)fixed cont geomet

$C_{geom} = S\mathrm{e}^{(b-r)(T-t)}N(d_1) - K\mathrm{e}^{-r(T-t)}N(d_2)$

$P_{geom} = -S\mathrm{e}^{(b-r)(t-t)}N(-d_1) + K\mathrm{e}^{-r(T-t)}N(-d_2)$

,where $d_1 = \dfrac{ln\left(\dfrac{S}{K}\right) + (b+\dfrac{\sigma_A^2}{2})T}{\sigma_A\sqrt{T}}$

$d_2 = \dfrac{ln\left(\dfrac{S}{K}\right) + (b-\dfrac{\sigma_A^2}{2})T}{\sigma_A\sqrt{T}} = d_1 - \sigma_A\sqrt{T}$

$$\sigma_A = \frac{\sigma}{\sqrt{3}}$$

$b = \dfrac{1}{2}\left(r - \dfrac{\sigma^2}{6}\right)$

5.6 Arithmetic Rate Approximation (Turnbull and Wakeman) fixed cont arithm

$C_{TW} \approx S\mathrm{e}^{(b-r)T_2}N(d_1) - K\mathrm{e}^{-rT_2}N(d_2)$

$P_{TW} \approx -S\mathrm{e}^{(b-r)T_2}N(-d_1) + K\mathrm{e}^{-rT_2}N(-d_2)$

$$d_1 = \frac{ln\left(\dfrac{S}{K}\right) + (b+\dfrac{\sigma_A^2}{2})T_2}{\sigma_A^2\sqrt{T_2}}$$

$$d_2 = \frac{ln\left(\frac{S}{K}\right) + (b - \frac{\sigma_A^2}{2})T}{\sigma_A^2\sqrt{T_2}} = d_1 - \sigma_A^2\sqrt{T_2}$$

where T_2 is the time remaining until maturity. For averaging options which have already begun their averaging period, then T_2 is simply T (the original time to maturity), if the averaging period has not yet begun, then T_2 is $T_2 - \tau$.

$$\sigma_A = \sqrt{\frac{ln(M_2)}{T} - 2b}$$

$$b = \frac{ln(M_1)}{T}$$

To generalise the equations, we assume that the averaging period has not yet begun and give the first and second moments as:

$$M_1 = \frac{e^{rT} - e^{r\tau}}{r(T - \tau)}$$

$$M_2 = \frac{2e^{(2r+\sigma^2)T}S^2}{(r+\sigma^2)(2r+\sigma^2)T^2} + \frac{2S^2}{rT^2}\left(\frac{1}{2r+\sigma^2} - \frac{e^{rT}}{r+\sigma^2}\right)$$

If the averaging period has already begun, we must adjust the strike price accordingly as:

$K_A = \frac{T}{T_2}K - \frac{(T - T_2)}{T_2}S_{Avg}$, where S_{Avg} is the average asset price.

5.7 Arithmetic Rate Approximation (Levy)fixed cont arithm

$C_{Levy} \approx S_Z N(d_1) - K_Z e^{-rT_2} N(d_2)$ and through put-call parity, we get the price of a put as: $P_{Levy} \approx C_{Levy} - S_Z + K_Z e^{-rT_2}$

$$d_1 = \frac{1}{\sqrt{\nu}}\left[\frac{ln(L)}{2} - ln(K_Z)\right]$$

$$d_2 = d_1 - \sqrt{\nu}$$

$$S_Z = \frac{S}{rT}(1 - e^{-rT_2})$$

$$K_Z = K - S_{Avg}\frac{T - T_2}{T}$$

$\nu = ln(L) - 2[rT_2 + ln(S_Z)]$

$L = \dfrac{M}{T^2}$

$M = \dfrac{2S^2}{r+\sigma^2}\left[\dfrac{\mathrm{e}^{(2r+\sigma^2)T_2-1}}{2r+\sigma^2}\right] - \dfrac{\mathrm{e}^{rT_2}-1}{r}$

Furthermore, transposing the 2 call values as a function of the strike price illustrates the similarity between the two methods.

S_{Avg} the average asset price.

6 Value of European lookback options

The price of the underlying asset follows Geometric Brownian motion.

$$\Rightarrow dS_t = \left(r - \frac{\sigma^2}{2}\right) S_t dt + \sigma S_t dW_t \Rightarrow S_T = S_t \mathrm{e}^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma(T-t)} \Rightarrow \frac{S_T}{S_t} = \mathrm{e}^{\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma(T-t)}$$

$$\Rightarrow ln\frac{S_T}{S_t} = \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma(T-t) \Rightarrow ln\frac{S_T}{S_t} \sim N\left(\left(r - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right)$$

$$\Rightarrow pdf_{ln\frac{S_T}{S_t}} = \frac{1}{\sigma\sqrt{T-t}\sqrt{2\pi}} \mathrm{e}^{\frac{\left(ln\frac{S_T}{S_t} - \mu(T-t)\right)^2}{2\sigma^2(T-t)}}$$

We define $\psi_T = ln\frac{m_t^T}{S_t}$ and $\Psi_T = ln\frac{M_t^T}{S_t}$

Then, the joint distribution function of $ln\frac{S_T}{S_t}$ and $ln\frac{m_t^T}{S_t}$ is:

$$\mathcal{P}\left(ln\frac{S_T^T}{S_t} \geq \chi, \psi_T \geq \psi\right) = N\left(\frac{-\chi + \mu(T-t)}{\sigma\sqrt{T-t}}\right) - \mathrm{e}^{\frac{2\mu\psi}{\sigma^2}} N\left(\frac{-\chi + 2\psi + \mu(T-t)}{\sigma\sqrt{T-t}}\right)$$

and the joint distribution function of $ln\frac{S_T}{S_t}$ and $ln\frac{M_t^T}{S_t}$ is:

$$\mathcal{P}\left(ln\frac{S_T^T}{S_t} \geq \chi, \Psi_T \geq \psi\right) = N\left(\frac{\chi - \mu(T-t)}{\sigma\sqrt{T-t}}\right) - \mathrm{e}^{\frac{2\mu\psi}{\sigma^2}} N\left(\frac{\chi - 2\psi - \mu(T-t)}{\sigma\sqrt{T-t}}\right)$$

For $\psi = \chi$,we can derive the probability distribution functions for ψ_T and Ψ_T :

$$\boxed{\begin{aligned} &\mathcal{P}\left(\psi_T \geq \psi\right) = N\left(\frac{-\psi + \mu(T-t)}{\sigma\sqrt{T-t}}\right) - \mathrm{e}^{\frac{2\mu\psi}{\sigma^2}} N\left(\frac{\psi + \mu(T-t)}{\sigma\sqrt{T-t}}\right) \\ &\mathcal{P}\left(\Psi_T \geq \psi\right) = N\left(\frac{\Psi - \mu(T-t)}{\sigma\sqrt{T-t}}\right) - \mathrm{e}^{\frac{2\mu\psi}{\sigma^2}} N\left(\frac{-2\psi - \mu(T-t)}{\sigma\sqrt{T-t}}\right) \end{aligned}} \triangleleft$$

6.1 European fixed strike call and put (cont)

terminal payoff = max(M-K,0)
value= $C_{fix}(S, M, t) = \mathrm{e}^{-r(T-t)}\mathbb{E}\left[max(max(M, M_t^T) - K, 0)\right]$

We distinguish the cases:
i)$M \leq K \Rightarrow$ payoff $= max[M_t^T - K, 0]$

ii) $M > K \Rightarrow$ payoff $= max[M_t^T - K, 0] + (M - K)$

$$\boxed{C_{fix}(S,M,T-t) = \begin{cases} e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}} max(M_t^T - K, 0), & \text{if M} \leq \text{K} \\ e^{-r(T-t)}(M-K) + e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}} max(M_t^T - K, 0), & \text{if } M > K \end{cases}}$$

- if $M \leq K$: $C_{fix}(S,M,t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}} max(M_t^T - K, 0) = e^{-r(T-t)} \int_0^{\infty} \mathcal{P}\left(M_t^T - K \geq \chi\right) d\chi =$

$$e^{-r(T-t)} \int_0^{\infty} \mathcal{P}\left(M_t^T \geq \chi + K\right) d\chi \overset{z=\chi+K}{=\!=\!=} e^{-r(T-t)} \int_K^{\infty} \mathcal{P}\left(M_t^T \geq z\right) dz =$$

$$e^{-r(T-t)} \int_K^{\infty} P[ln\frac{M_t^T}{S} \geq ln\frac{z}{S}]\, dz \underset{d\psi = \frac{1}{z}dz = \frac{1}{Se^{\psi}}dz}{\overset{\psi = ln\frac{z}{S}}{\overline{\overline{\ \Downarrow\ }}}} e^{-r(T-t)} \int_{ln\frac{K}{S}}^{\infty} Se^{\psi} P[ln\frac{M_t^T}{S} \geq \psi]\, d\psi$$

$$\Rightarrow C_{fix}(S,M,t) = e^{-r(T-t)} \int_{ln\frac{K}{S}}^{\infty} Se^{\psi} P[\Psi_T \geq \psi]\, d\psi \overset{\triangleleft}{=\!=\!=}$$

$$e^{-r(T-t)} \int_{ln\frac{K}{S}}^{\infty} Se^{\psi} \left[\underbrace{N\left(\frac{-\psi + \mu(T-t)}{\sigma\sqrt{T-t}}\right)}_{\text{A}} + \underbrace{N\left(\frac{-\psi - \mu(T-t)}{\sigma\sqrt{T-t}}\right)}_{\text{B}} \right] d\psi =^* \text{(appendix)}$$

$$\overset{*}{\Rightarrow} \boxed{C_{fix}(S,M,t) = SN(\hat{d}1) \text{ -}Ke^{-r(T-t)}N(\hat{d}2) + Se^{-r(T-t)}\frac{\sigma^2}{2r}\left[e^{r(T-t)}N(\hat{d}1) - \left(\frac{S}{K}\right)^{\frac{-2r}{\sigma^2}} N(\hat{d}3)\right]}$$

$$\hat{d}1 = \frac{ln\frac{S}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$\hat{d}2 = \frac{ln\frac{S}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = \hat{d}1 - \sigma\sqrt{T-t}$$

$$\hat{d}3 = \frac{ln\frac{S}{K} - (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = \hat{d}1 - \frac{2r\sqrt{T-t}}{\sigma}$$

- if $M > K$:

$$\overset{*}{\Rightarrow} \boxed{C_{fix}(S,M,t) = (M-K)\mathrm{e}^{-r(T-t)} + SN(\hat{d}1) \text{ -}M\mathrm{e}^{-r(T-t)}N(\hat{d}2) + S\mathrm{e}^{-r(T-t)}\frac{\sigma^2}{2r}\left[\mathrm{e}^{r(T-t)}N(\hat{d}1) - \left(\frac{S}{M}\right)^{\frac{-2r}{\sigma^2}} N(\hat{d}3)\right]}$$

$$\hat{d}1 = \frac{ln\frac{S}{M} + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$\hat{d}2 = \frac{ln\frac{S}{M} + \left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = \hat{d}1 - \sigma\sqrt{T-t}$$

$$\hat{d}3 = \frac{ln\frac{S}{M} - (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = \hat{d}1 - \frac{2r\sqrt{T-t}}{\sigma}$$

European fixed strike put

terminal payoff = max(K-m,0)
value= $P_{fix}(S,m,t) = \mathrm{e}^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}max(max(m,m_t^T) - K, 0)$

i)$m \geq K \Rightarrow$ payoff $= max[K - m_t^T, 0]$

ii) $m < K \Rightarrow$ payoff $= max[K - m_t^T, 0] + (K - m)$

$$\boxed{P_{fix}(S,m,T-t)= \begin{cases} \mathrm{e}^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}max(K-m_t^T,0), & \text{if m} \geq \text{K} \\ \mathrm{e}^{-r(T-t)}(K-m) + \mathrm{e}^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}max(K-m_t^T), & \text{if } m < K \end{cases}}$$

- if $m \geq K$: $\quad P_{fix}(S,m,t) = \mathrm{e}^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}max(K - m_t^T, 0)$

$$\Rightarrow \boxed{P_{fix}(S,m,t) = K\mathrm{e}^{-r(T-t)}N(\check{\mathrm{d}}1) - SN(\check{\mathrm{d}}2) + S\mathrm{e}^{-r(T-t)}\frac{\sigma^2}{2r}[\left(\frac{S}{K}\right)^{\frac{-2r}{\sigma^2}} N(\check{\mathrm{d}}3) - \mathrm{e}^{r(T-t)}N(\check{\mathrm{d}}2)]}$$

$$\check{\mathrm{d}}1 = \frac{-ln\frac{S}{K} - \left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$\check{\mathrm{d}}2 = \frac{-ln\frac{S}{K} - (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = \check{\mathrm{d}}1 - \sigma\sqrt{T-t}$$

$$\breve{d}3 = \frac{-ln\dfrac{S}{K} + \left(r - \dfrac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = \breve{d}2 + \frac{2r\sqrt{T-t}}{\sigma}$$

- if $m < K$: $P_{fix}(S, m, t) =$

$$\boxed{(K-m)e^{-r(T-t)} + me^{-r(T-t)}N(\breve{d}1) - SN(\breve{d}2) + Se^{-r(T-t)}\frac{\sigma^2}{2r}[\left(\frac{S}{m}\right)^{\frac{-2r}{\sigma^2}} N(\breve{d}3) - e^{r(T-t)}N(\breve{d}2)]}$$

$$\breve{d}1 = \frac{-ln\dfrac{S}{m} - \left(r - \dfrac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$\breve{d}2 = \frac{-ln\dfrac{S}{m} - (r + \dfrac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = \breve{d}1 - \sigma\sqrt{T-t}$$

$$\breve{d}3 = \frac{-ln\dfrac{S}{m} + \left(r - \dfrac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = \breve{d}2 + \frac{2r\sqrt{T-t}}{\sigma}$$

6.2 European floating strike call and put (cont.)

Floating strike cases for continuously sampled version can be obtained by using the results in fixed strike.

$$\begin{aligned} C_{fl}(S, m, T) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[S_T - min(m, m_t^T)] \overset{-min=max}{=\!=\!=} e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[S_T + max(m, m_t^T)] = \\ &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[(S_T - m) + max(m - m_t^T, 0)] = \\ &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[S_T - m] + e^{-r(T-t)}[max(m - m_t^T, 0)] = \\ &= e^{-r(T-t)}(S_T - m) + e^{-r(T-t)}[max(m - m_t^T, 0)] = \\ &= e^{-r(T-t)}(S_T - m) + P_{fix}(S, m, t) = \end{aligned}$$

$$= e^{-r(T-t)}(S_T - m) + me^{-r(T-t)}N\left(\frac{-ln\dfrac{S}{m} - \left(r - \dfrac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) - SN\left(\frac{-ln\dfrac{S}{m} - (r + \dfrac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right)$$

$$+ Se^{-r(T-t)}\frac{\sigma^2}{2r}\left[\left(\frac{S}{m}\right)^{\frac{-2r}{\sigma^2}} N\left(\frac{-ln\dfrac{S}{m} + \left(r - \dfrac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) - e^{r(T-t)}N\left(\frac{-ln\dfrac{S}{m} - \left(r + \dfrac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right] =$$

$$= S\left(1 - N\left(\frac{-ln\frac{S}{m} - (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right)\right) + me^{-r(T-t)}\left(-1 + N\left(\frac{-ln\frac{S}{m} - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right)$$

$$+ Se^{-r(T-t)}\frac{\sigma^2}{2r}\left[\left(\frac{S}{m}\right)^{\frac{-2r}{\sigma^2}} N\left(\frac{-ln\frac{S}{m} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) - e^{r(T-t)}N\left(\frac{-ln\frac{S}{m} - \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right] =$$

$$\stackrel{N(\alpha)+N(-\alpha)=1}{=\!=} S\left(N\left(\frac{ln\frac{S}{m} + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right)\right) - me^{-r(T-t)}\left(N\left(\frac{ln\frac{S}{m} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right)$$

$$+ Se^{-r(T-t)}\frac{\sigma^2}{2r}\left[\left(\frac{S}{m}\right)^{\frac{-2r}{\sigma^2}} N\left(\frac{-ln\frac{S}{m} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) - e^{r(T-t)}N\left(\frac{-ln\frac{S}{m} - \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right] \Rightarrow$$

$$\boxed{C_{fl}(S,m,T) = SN(\bar{d}1) - me^{-r(T-t)}N(\bar{d}2) + Se^{-r(T-t)}\frac{\sigma^2}{2r}\left[\left(\frac{S}{m}\right)^{\frac{-2r}{\sigma^2}} N(\bar{d}3) - e^{r(T-t)}N(-\bar{d}1)\right]},$$

for 0 < m ≤ S

$$\bar{d}1 = \frac{ln\frac{S}{m} + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$\bar{d}2 = \frac{ln\frac{S}{m} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = \bar{d}1 - \sigma\sqrt{T-t}$$

$$\bar{d}3 = \frac{-ln\frac{S}{m} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = -\bar{d}1 + \frac{2r\sqrt{T-t}}{\sigma}$$

Similarly we work for the case of the put floating strike option:

$$P_{fl}(S,M,T) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[max(M, M_t^T) - S_T] = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[max(M_t^T - M, 0) - (S_T - M)] =$$

$$= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[max(M_t^T - M, 0)] - (S - Me^{-r(T-t)}) =$$

$$= C_{fix}(S,t,M) - (S - M\mathrm{e}^{-r(T-t)}) =$$

$$= SN\left(\frac{ln\frac{S}{M} + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \text{-}M\mathrm{e}^{-r(T-t)}N\left(\frac{ln\frac{S}{M} + \left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$
$$+ S\mathrm{e}^{-r(T-t)}\frac{\sigma^2}{2r}\left[\mathrm{e}^{r(T-t)}N\left(\frac{ln\frac{S}{M} + \left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) - \left(\frac{S}{M}\right)^{\frac{-2r}{\sigma^2}}N\left(\frac{ln\frac{S}{M} - \left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right] -$$
$$(S - M\mathrm{e}^{-r(T-t)}) =$$

$$= S\left(N\left(\frac{ln\frac{S}{M} + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) - 1\right) + M\mathrm{e}^{-r(T-t)}\left(-N\left(\frac{ln\frac{S}{M} + \left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) + 1\right)$$
$$+ S\mathrm{e}^{-r(T-t)}\frac{\sigma^2}{2r}\left[\mathrm{e}^{r(T-t)}N\left(\frac{ln\frac{S}{M} + \left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) - \left(\frac{S}{M}\right)^{\frac{-2r}{\sigma^2}}N\left(\frac{ln\frac{S}{M} - \left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right] =$$

$$= S\left(-N\left(\frac{-ln\frac{S}{M} - (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right)\right) + M\mathrm{e}^{-r(T-t)}\left(N\left(\frac{-ln\frac{S}{M} - \left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right)$$
$$+ S\mathrm{e}^{-r(T-t)}\frac{\sigma^2}{2r}\left[\mathrm{e}^{r(T-t)}N\left(\frac{ln\frac{S}{M} + \left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) - \left(\frac{S}{M}\right)^{\frac{-2r}{\sigma^2}}N\left(\frac{ln\frac{S}{M} - \left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right] \Rightarrow$$

$$\boxed{P_fl(S,M,T) = M\mathrm{e}^{-r(T-t)}N(\bar{d1}) - SN(\bar{d2}) + S\mathrm{e}^{-r(T-t)}\frac{\sigma^2}{2r}\left[\mathrm{e}^{r(T-t)}N(-\bar{d2}) - \left(\frac{S}{M}\right)^{\frac{-2r}{\sigma^2}}N\left(\bar{d3}\right)\right]}$$

, for S $\leq$ M

$$\bar{d1} = \frac{-ln\frac{S}{M} - \left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$\bar{d2} = \frac{-ln\frac{S}{M} - (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = \bar{d1} - \sigma\sqrt{T-t}$$

$$\bar{d}3 = \frac{ln\frac{S}{M} - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = -\bar{d}2 - \frac{2r\sqrt{T-t}}{\sigma}$$

7 Updating Rule

In the continuous case, the continuously sampled average is modelled as an integral. In the discrete case, one averages a finite number of option values picked up during the life of the option. For example, we may use a weekly fixing. In practice (due to legal and practical issues), Asian options are generally monitored at discrete time.

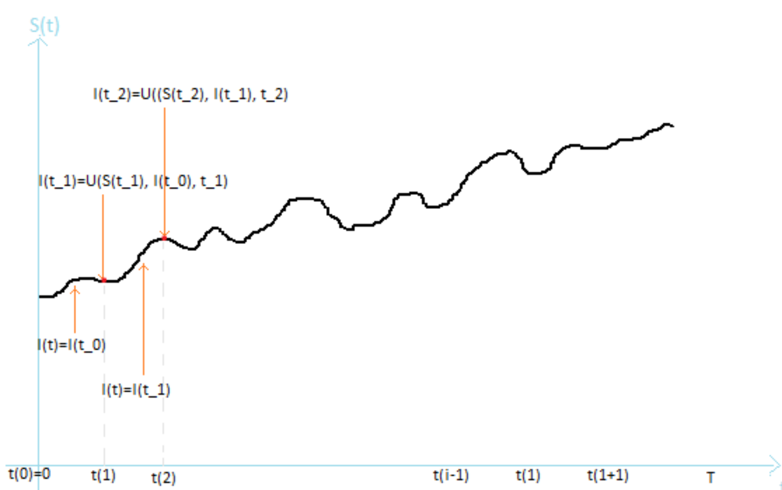
Sometimes, to value path-dependent quantities, instead of doing it continuously, we do this only on key dates, i.e discretely. In reality, path-dependent quantities are never measured continuously. If the time between sampling dates is small, then we can use a continuously sampled model, with the error being very small. In a different case, we must apply a different technique, the so-called updating rule.

Suppose we have the sampling dates t_i and again I_T is defined as $I_T = \int_0^T f(S,\tau)\,d\tau = TA_T \Rightarrow I_t = \int_0^t f(S,\tau)\,d\tau \Rightarrow dI = f(S,t)dt$.

In case of discrete sampling, the value of I changes only at sampling dates, whereas between them remains stable.

Hence, between sampling dates $dI = 0 \Rightarrow f(S,t) = 0 \Rightarrow \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial^2 S} + rS\dfrac{\partial V}{\partial S} - rV = 0$ which is independent of I.
On the other hand, on sampling dates $dI \neq 0$. The new value of I is determined by the old value of I, the value of the underlying on the sampling date and the sampling date. As we get closer and closer to the sampling date, we become more confident about the value of I according to the updating rule.

The updating rule is applied as follows: If $t_i \leq t < t_{i+1}$, then $I(t) = I(t_i)$ (at not sampling points, I does not change) If $t = t_i + 1$, the the quantity $I(t_{i-1})$ is updated: $I(t_{i+1}) = U(S(t_{i+1}), I(t_i), t_{i+1})$



Across a sampling date, the option value is continuous, i.e $V(S, I(t_{i-1}), t_i^-) = V(S, I(t_{i-1}), t_i^+)$, where t_i^- and t_i^+ denote the moment just before and just after t_i, respectively.
$\Rightarrow V(S, I, t_i^-) = V(S, U(S, I, t_i), t_i^+)$ This is called jump condition.
jumb condition: $V(S, A, t_i^-) = V(S, \frac{i-1}{i}A + \frac{1}{i}S, t_i^+)$
$V(S, A, t_i^-) = V(S, \mathrm{e}^{\frac{i-1}{i}log(A) + \frac{1}{i}logS}, t_i^+)$ (2)

The algorithm for updating rule for lookback options is as follows:

Define $I(t_j) = max(S(t_1), ..., S(t_j))$

Analytically: $I(t_1) = S(t_1)$

$$\begin{aligned}
&I(t_2) = max(S(t_2), S(t_1)) = max(S(t_2), I(t_1)) \\
&I(t_3) = max(S(t_3), S(t_2), S(t_1)) = max(S(t_3), I(t_2)) \\
&. \\
&. \\
&. \\
&I(t_{j+1}) = max(S(t_{j+1}), I(t_j))
\end{aligned}$$

The algorithm for updating rule for arithmetic options is as follows:

Define $A_j = \frac{1}{j}\sum_{i=1}^{j} S(t_i)$

Analytically: $A_1 = S(t_1)$

$$\begin{aligned}
&A_2 = \frac{S(t_1) + S(t_2)}{2} = \frac{A_1}{2} + \frac{S(t_2)}{2} \\
&A_3 = \frac{S(t_1) + S(t_2) + S(t_3)}{3} = \frac{S(t_1) + S(t_2)}{3} + \frac{S(t_3)}{3} = \frac{2}{3}A_2 + \frac{S(t_3)}{3} \\
&. \\
&. \\
&. \\
&A_j = \frac{j-1}{j}A_{j-1} + \frac{S(t_j)}{j}
\end{aligned}$$

Lookbacks: continuous $\Rightarrow$ analytic discrete $\Rightarrow$ numerical Asian: geometric $\Rightarrow$ analytic arithmetic $\Rightarrow$ numerical

8 Implementation

8.1 Why Matlab

For the implementation of the above methods, we used Matlab (Matrix Laboratory). The software Matlab is a modern integrated mathematical package used extensively in universities and industries. It is a high performance language for technical computing, popular for its user-friendliness. It was created by the MathWorks and it allows interfacing with programs written in other languages including C,C++,Java and Fortran. As suggested by the name, the basic data element in Matlab is the matrix. Even a simple integer is considered as 1x1 matrix. This enables user to write programs which involve computations of matrices and vectors.
The high level environment implies that user does not need to be aware of specifics of the hardware or the system. Thus, it is more difficult to cause permanent damage to our computer while working on it, making it suitable for amateurs. This partially justifies the fact that it is widely used. In addition, it provides numerous predefined functions, such as integration, cumulative distribution etc, which in other programs would require lines of code to define them. Furthermore, enables the user to present graphs easily, thus offering huge performance advantages.
On the other hand, Matlab presents a major disadvantage. It is an interpreted language. This makes it slower, since, when we run the program, each line is executed by another program. However, nowadays, high level languages are not strictly interpreted. They are compiled into pseudocode that is not machine code, but is faster to interpret. Furthermore, it might be very expensive and it demands huge amount of memory, thus it is very hard to use on slow computers. Finally, procedures in CPU take as much time as Windows allows them to. Thus, real-time applications may be very complicated.

Our code ran on a 1.80GHz Intel® Pentium® dual core Processor with 1GB RAM running 32-bit Windows 8.

8.2 Monte Carlo

Monte Carlo is a method to calculate integrals or expectations using random numbers and probabilities. This technique is the most popular and most commonly used. It is massively used in many fields of applied mathematics. Most trading systems use Monte Carlo to price and risk manage derivatives positions.

It was invented in the Manhattan project[2] in Los Alamos. The name "Monte Carlo" was coined by Nicholas Metropolis[3] when cooperating with John von Neumann[4] and Stanslaw Ulam[5] in finding a solution for the neutron transport in fission material.
In "The beginning of the Monte Carlo Method"[6] is referred by Nicholas Metropolis:

[2] The Manhattan Project was a research and development project that produced the first atomic bombs during World War II. The Manhattan Project created the first nuclear bombs.
[3] 1915-1999, Greek-American mathematician, physicist and computer scientist
[4] 1903-1957, Hungarian American mathematician and scientist
[5] 1909-1984, Polish mathematician, participated in the Manhattan Project
[6] N. Metropolis, "The Beginning of Monte Carlo Method," Los Alamos Science, Special Issue dedicated to Stanislaw Ulam: 125-130, 1987.

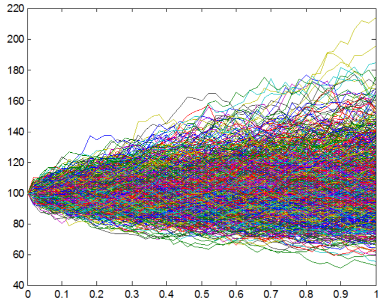
"The spirit of this method was consistent with Stan's interest in random processes [...] he would cite the times he drove into a filled parking lot at the same moment someone was accommodatingly leaving. More seriously, he created the concept of "lucky numbers", whose distribution was much like that of prime numbers; he was intrigued by the theory of branching processes and contributed much to its development, including its application during the war to neutron multiplication in fission devices [...] John von Neumann saw the relevance of Ulam's suggestion and, on March 11, 1947, sent a handwritten letter to Robert Richtmyer, the Theoretical Division leader(see "Stan Ulam,John von Neumann, and the Monte Carlo Method")[...]The spirit of Monte Carlo is best conveyed by the example discussed in von Neumann's letter to Richtmyer. Consider a spherical core of fissionable material surrounded by a shell of tamper material. Assume some initial distribution of neutrons in space and in velocity but ignore radiative and hydrodynamic effects. The idea is to now follow the development of a large number of individual neutron chains as a consequence of scattering, absorption, fission, and escape".

Generally, the calculations involved in the method are based on replacing an expectation E(f(X)) ,where X is a random variables, by a sample average $\frac{1}{n}\sum_{i=1}^{n} f(X_i)$, where $X_1, ..., X_n$ are samples of the random variable X. The mathematical justification for this is called Law of Large Numbers[7].
Since Monte Carlo Method is just an approximation to the precise value, there exists an error between the actual and approximated price. Bounding errors is generally impossible in Monte Carlo, since the error is random and thus it can be of any size. However the errors will follow some kind of distribution centred around a mean and with some variance. Particularly, The Central Limit Theorem[8] states that the error can be approximated by a $N\left(0, \frac{\sigma^2}{n}\right)$ random variable. It is more common to describe the error by its standard deviation: $\frac{\sigma}{\sqrt{n}}$. The LLN states that the average approaches expectation as n increases. In other words, we expect the variance to be small if n is large.

The advantage of this method is that the accuracy of the result can be increased by simply increasing the number of simulations. By the CLT quadrupling the number of simulation runs, approximately halves the error in the simulated price.
On the other hand, Monte Carlo Method consists of repeated calculation of random numbers. Thus slow computers will difficulty respond to the demands of the method with large number of simulations, since increasing the number of simulation runs, increases the computation time significantly. Hence, it is of great importance to find the right balance between the accuracy of the result and the time required for the approximation, i.e the suitable number of simulation runs. The fact that this technique is such time-consuming, implies that this method is applied when deterministic algorithms for the calculations are not available and it is not recommended when closed form formulas do exist. Furthermore, it is not very effective for derivatives that expire before maturity.

[7]Strong Law of Large Numbers: If x_i are identically distributed random variables with mean μ, then there exist a set of measure zero such that out of this set: $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_i = \mu$.

[8]If X_i are independently and identically distributed random variables with mean μ and standard deviation σ, then $\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i - N\mu}{\sqrt{n}\sigma} \Longrightarrow N(0,1)$, where $\Longrightarrow$ indicates convergence in law. This means basically that the cdf of the LHS approaches the cdf of the RHS as n increases.



8.2.1 The Crude Monte Carlo Method

The main benefit of Monte Carlo simulation is that it is easily implemented and can efficiently be used to value a large spectrum of European style exotic options. In addition, it provides accurate enough results as long as sufficient number of sample paths are simulated. Therefore, it can be used for valuation of various complex European style exotic options, especially in case that explicit formulas are not available. Despite the fact that it works very well for pricing European-style path dependent options, it presents the drawback that it is difficult to be applied for early exercise (American-style) options.

When pricing options using the Monte Carlo method, the valuation should be under the risk neutral measure. The random variables or the underlying stock price are assumed to follow Geometric Brownian Motion, i.e $dS_t = \mu S_t dt + \sigma S_t dX_t$, where X_t is the Wiener process. Then, to calculate the expected payoff, we simulate the underlying state variable under the risk neutral measure and discount the payoff depending on the type of the derivative security. In other words, we have $\frac{dS_t}{S_t} = rdt + \sigma dX$ and we can write the value of the option in the form $V(S,t) = \mathrm{e}^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[P(S)]$, which is the present value of the expected payoff with respect to the risk-neutral probabilistic density Q and $\mathbb{E}^{\mathbb{Q}}[P(S)] = \int_0^{\infty} \tilde{p}(S,t;S',T)P(S')dS'$ with $\tilde{p}(S,t;S',T)$ representing the transition density. Finally, we take the average of the discounted payoffs: $V(S,t) = \mathrm{e}^{-r(T-t)}\frac{1}{N}\sum_{i=1}^{N} Payoff(S_i)$.

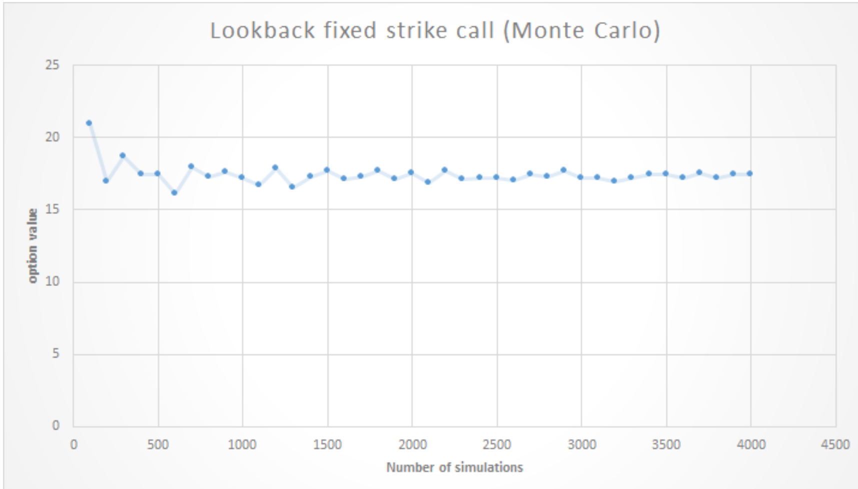
The method tends to provide fairly accurate results and is very flexible with different types of Asian options. The method, however, lacks efficiency because it is very time-consuming to run the simulations.

Asian arithmetic option value				
N	Fixed strike call	Fixed strike put	Floating strike call	Floating strike put
1000	5.6504	3.1439	6.2391	3.5126
2000	5.7197	3.3069	5.5958	3.4400
3000	5.7695	3.4818	5.8365	3.2974
4000	5.5383	3.5004	5.8568	3.3291
5000	5.8192	3.3496	5.8618	3.4989
6000	5.7148	3.3376	5.7057	3.4432
7000	5.7185	3.1908	5.7201	3.3689
8000	5.6784	3.3401	5.8571	3.3322
9000	5.8014	3.4504	5.7885	3.3550
10000	5.7328	3.3834	5.8743	3.3729

Asian geometric option value				
N	Fixed strike call	Fixed strike put	Floating strike call	Floating strike put
1000	5.4352	3.6214	6.2584	3.2087
2000	5.1410	3.3773	5.9600	3.2393
3000	5.4337	3.4606	5.9814	3.3588
4000	5.6634	3.3791	6.0810	3.3634
5000	5.7337	3.5006	5.8635	3.3693
6000	5.4786	3.5122	5.9130	3.3729
7000	5.4607	3.4543	6.2343	3.2667
8000	5.5760	3.5439	6.1003	3.2778
9000	5.5404	3.3736	6.0009	3.3752
10000	5.7043	3.3717	5.9531	3.3316

These methods find applicability in pricing exotic options such as asian and lookback options.

Lookback option value				
N	Fixed strike call	Fixed strike put	Floating strike call	Floating strike put
1000	17.2437	11.1790	16.8438	12.1716
2000	17.5310	10.8207	15.9717	12.6212
3000	17.6535	10.8936	16.3269	12.3942
4000	17.3328	11.2316	16.0166	12.6159
5000	16.9818	10.9377	15.6718	12.3710
6000	17.3504	11.0351	15.7626	12.3447
7000	17.2023	11.1074	15.8924	12.7080
8000	17.5333	10.9108	15.7639	12.5182
9000	17.5532	10.9155	15.8528	12.2795
10000	17.4201	11.1238	15.7216	12.6911



(By the assumption of the Black Scholes model, any MC simulation must converge to the Black Scholes option value in the limit as the number of paths in the sample n $\to \infty$.)

8.3 Improving the efficiency of simulation

Monte Carlo Method is generally applicable to all European-style options, but some form of variance reduction is crucial to ensure precision of price estimates. Since the error is proportional to the variance, by reducing the variance, the error becomes smaller.

8.3.1 The antithetic variates method

One of the most popular methods in order to achieve this, is the antithetics method, which we applied here.

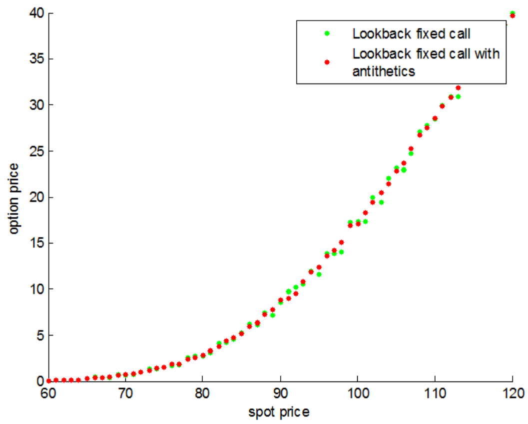
The antithetics method is based on the fact that the normal density is symmetric and ,therefore, if X is a normally distributed variable, -X is also normally distributed. Therefore we could recycle a random number X by using -X in our simulation.
Mathematically this is written as $\mathbb{E}[f(X)] = \mathbb{E}\left[\frac{f(X)+f(-X)}{2}\right]$.

When applying this method, we intend to reduce the error is smaller and thus in order for Monte Carlo to converge faster (using the fact that error $\sim N\left(0, \frac{\sigma^2}{n}\right)$. So if we calculate the standard deviation of the error, calculating option price in the usual way, and calculating it using the antithetics method, the second standard deviation should be significantly smaller.

The variance in the antithetic method is:

$\mathbb{V}_{antithetics} = \mathbb{V}\left[\frac{f(X)+f(-X)}{2}\right] = \frac{1}{4}[\mathbb{V}(f(X))+\mathbb{V}(f(-X))+2Cov(f(X),f(-X))] = \frac{\mathbb{V}Payoff(1+\rho)}{2}$
where ρ is the correlation between f(X) and f(-X), with $\rho \in [-1,1]$.

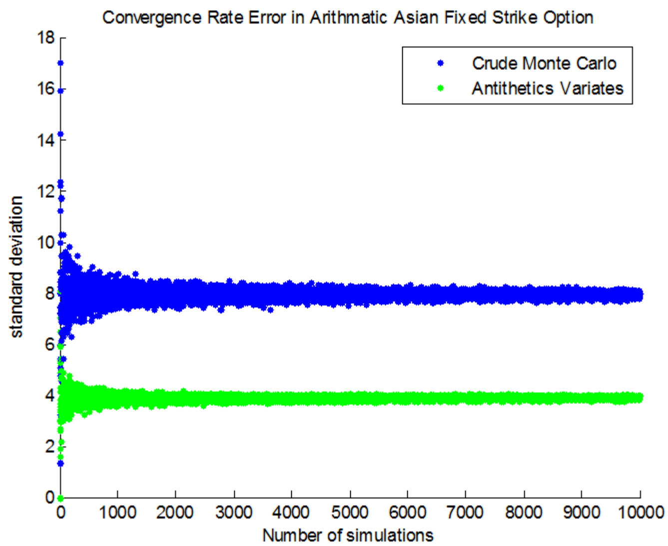
Therefore, $\sigma_{antithetics} = \sigma_{Payoff}\sqrt{\frac{1+\rho}{2}}$. From this we conclude that antithetics method is more effective when ρ is negative.

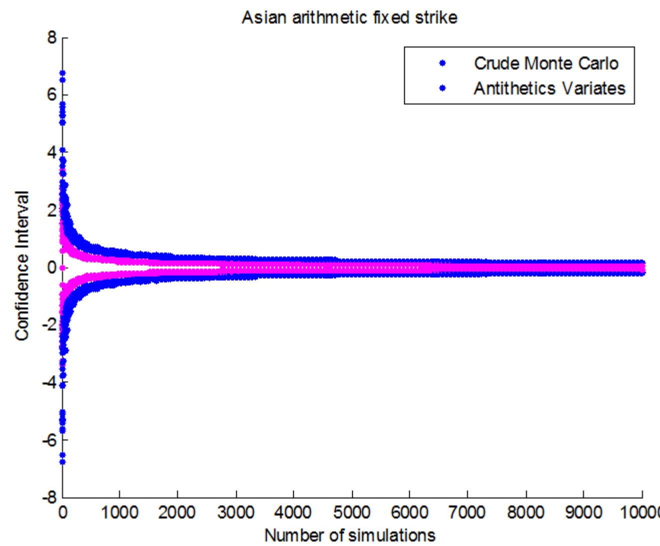


Standard deviation of Lookback option payoff							
Fixed strike call	Fixed strike call (antithetics)	Fixed strike put	Fixed strike put (antithetics)	Floating strike call	Floating strike call (antithetics)	Floating strike put	Floating strike put (antithetics)
15.7741	6.2046	8.7444	3.4645	14.9552	5.8369	9.8299	4.1432

Standard deviation of Asian option payoff (arithmetic)							
Fixed strike call	Fixed strike call (antithetics)	Fixed strike put	Fixed strike put (antithetics)	Floating strike call	Floating strike call (antithetics)	Floating strike put	Floating strike put (antithetics)
8.0260	5.7427	5.2812	3.7102	8.6521	6.1302	5.0568	3.6037

Standard deviation of Asian option payoff (geometric)							
Fixed strike call	Fixed strike call (antithetics)	Fixed strike put	Fixed strike put (antithetics)	Floating strike call	Floating strike call (antithetics)	Floating strike put	Floating strike put (antithetics)
7.7415	5.5358	5.4187	3.9712	8.9392	6.3368	4.9064	3.4931





MC is a probabilistic method. Since the normally distributed samples that are used can take any value from -∞ to ∞, the values of the averaged payout can in principle vary a lot. Thus, the error in MC cannot be bound exactly. What can be done, is to bound it given probability. To achieve this, we use the fact that the error is approximately normal with mean zero and variance equal to $\frac{Variance\ of\ payout}{Number of Simulations}$. The number 'Variance of payout' can be estimated by running a few samples and calculating the standard deviation. Since $\mathbb{P}(-1.96 \leq N(0,1) \leq 1.96) = 95\%$, then $\mathbb{P}\left(-1.96\sqrt{\frac{Variance\ of\ payout}{Number\ of\ Simulations}} \leq N\left(0, \frac{Variance\ of\ payout}{Number\ of\ Simulations}\right) \leq 1.96\sqrt{\frac{Variance\ of\ payout}{Number\ of\ Simulations}}\right) = 95\%$.

$\Rightarrow \mathbb{P}\left(-1.96\sqrt{\frac{Variance\ of\ payout}{Number\ of\ Simulations}} \leq Error \leq 1.96\sqrt{\frac{Variance\ of\ payout}{Number\ of\ Simulations}}\right) = 95\%$

8.3.2 Milstein correction

According to earlier assumption, the underlying asset follows geometric Brownian motion under the risk neutral measure, i.e $dS_t = rS_t dt + \sigma S_t dX_t$, where risk-free rate r and volatility σ are constants and X_t is the Wiener process. By applying Ito's lemma to the function f=lnS, we conclude that the solutions is $S_t = S_0 exp(\left(r - \frac{\sigma^2}{2}\right) t + \sigma X_t)$. Since $X_t \sim N(0,t) \Rightarrow \phi = \frac{X_t}{\sqrt{t}} \sim N(0,1)$. Hence, $S_t = S_0 exp(\left(r - \frac{\sigma^2}{2}\right) t + \sigma\phi\sqrt{t})$, This more generally is written as $S_{t+\delta t} = S_t exp(\left(r - \frac{\sigma^2}{2}\right) \delta t + \sigma\phi\sqrt{\delta t})$ ■, which is the exact solution.

The Forward Euler-Maruyama mathod for GBM gives: $\frac{dS_t}{S_t} = \frac{S_{t+\delta t} - S_t}{S_t} \sim r\delta t + \sigma\phi\sqrt{\delta t} \Rightarrow S_{t+\delta t} \sim S_t(1 + r\delta t + \sigma\phi\sqrt{\delta t})$.

Now, if we do Taylor Series Expansion of the exact solution ■, we get:

$$e^{\left(r-\frac{\sigma^2}{2}\right)\delta t+\sigma\phi\sqrt{\delta t}} \sim 1+\left(r-\frac{\sigma^2}{2}\right)\delta t+\sigma\phi\sqrt{\delta t}+\frac{1}{2}\sigma^2(\phi^2-1)\delta t+... = \underbrace{1+r\delta t+\sigma\phi\sqrt{\delta t}}_{\text{Forward Euler}}+\underbrace{\frac{1}{2}\sigma^2(\phi^2-1)\delta t}_{\text{extra term o(}\delta\text{t)}}+...$$

$$\Rightarrow \boxed{S_{t+\delta t} = S_t(1+r\delta t+\sigma\phi\sqrt{\delta t}+\frac{1}{2}\sigma^2(\phi^2-1)\delta t)+...)}.$$

The term $\frac{1}{2}(\phi^2-1)\delta t$ is called Milstein correction.

Forward Euler-Maruyamma vs Milstein (Asian options)			
	Euler	Milstein	Correction (abs)
fixed call arithmetic	5.7773	5.7824	0.0052
fixed put arithmetic	3.4636	3.4563	0.0073
fixed call geometric	5.5594	5.5640	0.0045
fixed put geometric	3.5860	3.5781	0.0079
floating call arithmetic	5.8604	5.8642	0.0039
floating put arithmetic	3.4544	3.4500	0.0044
floating call geometric	6.0689	6.0735	0.0046
floating put geometric	3.3225	3.3189	0.0037

Forward Euler-Maruyamma vs Milstein (Lookback options)			
	Euler	Milstein	Correction (abs)
fixed call	17.7593	17.7401	0.0192
fixed put	10.6913	10.6803	0.0110
floating call	16.1044	16.0791	0.0253
floating put	12.3462	12.3413	0.0049

8.3.3 Quasi Monte Carlo

A fundamental part of the accuracy of Monte Carlo approximations is the generation of random numbers. Typically, a random number generator generates a "random" integer n in a given interval 1,...,N for large N. This is converted to a $\mathcal{U}[0,1]$ by $X = \frac{n}{N}$.
There are three types of Monte Carlo generators:

i True random number generators: These are random numbers, generated by hardware devices typically relying on the probabilistic nature of subatomic processes (quantum mechanics).

ii Pseudo random number generators: They are not true random numbers. They are algorithms that attempt to simulate real randomness and are often based on modular arithmetic.

iii Quasi random number generator generators: Quasi random numbers do not intend to be random, but, instead, they attempt to sample space as uniformly as possible. The main method to do this is called Sobol method. This method relies on non-trivial algebra over finite fields(primitive polynomials in $F_2[t]$,...). It often improves quasi-random numbers by orders of magnitude.

Such points depend explicitly on the problem dimension d. In the QMC method the vector xi cannot be built-up taking sets of d consecutive elements from a scalar sequence as in standard MC

approach.
The MC has the advantage to be independent on the problem dimnesion, whereas in QMC the dimension must be identified explicitly before the generation of points. Lower dimensional problems generally display smaller errors.

σ	K	Call (K&V)
0.1	85	16.5956
	90	11.8527
	95	7.2608
	100	3.3847
0.2	85	16.5245
	90	12.1618
	95	8.2966
	100	5.1635
0.3	85	16.9088
	90	13.0747
	95	9.7400
	100	6.9723
0.4	85	17.6688
	90	14.2592
	95	11.2815
	100	8.7496
0.5	85	18.6253
	90	15.5439
	95	12.8291
	100	10.4757

T-t	K	Call (K&V)
0.25 (t=3/4)	85	15.3523
	90	10.4630
	95	5.9256
	100	2.5047
0.5 (t=2/4)	85	15.7271
	90	11.0406
	95	6.8478
	100	3.6057
0.75 (t=1/4)	85	16.1247
	90	11.6179
	95	7.6232
	100	4.4532
1 (t=0)	85	16.5245
	90	12.1618
	95	8.2966
	100	5.1635

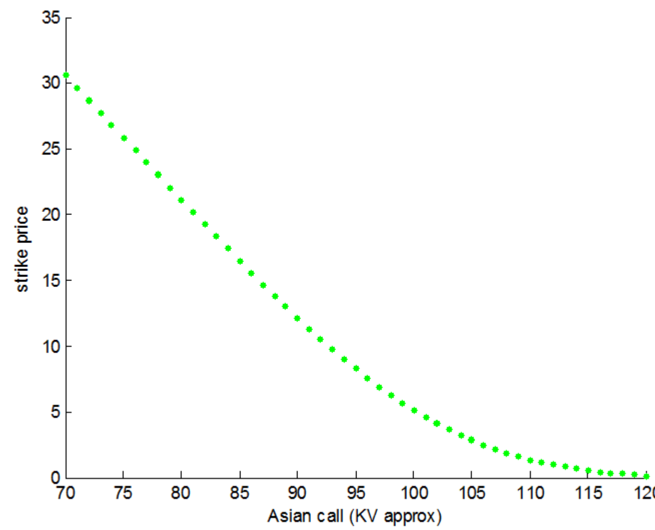
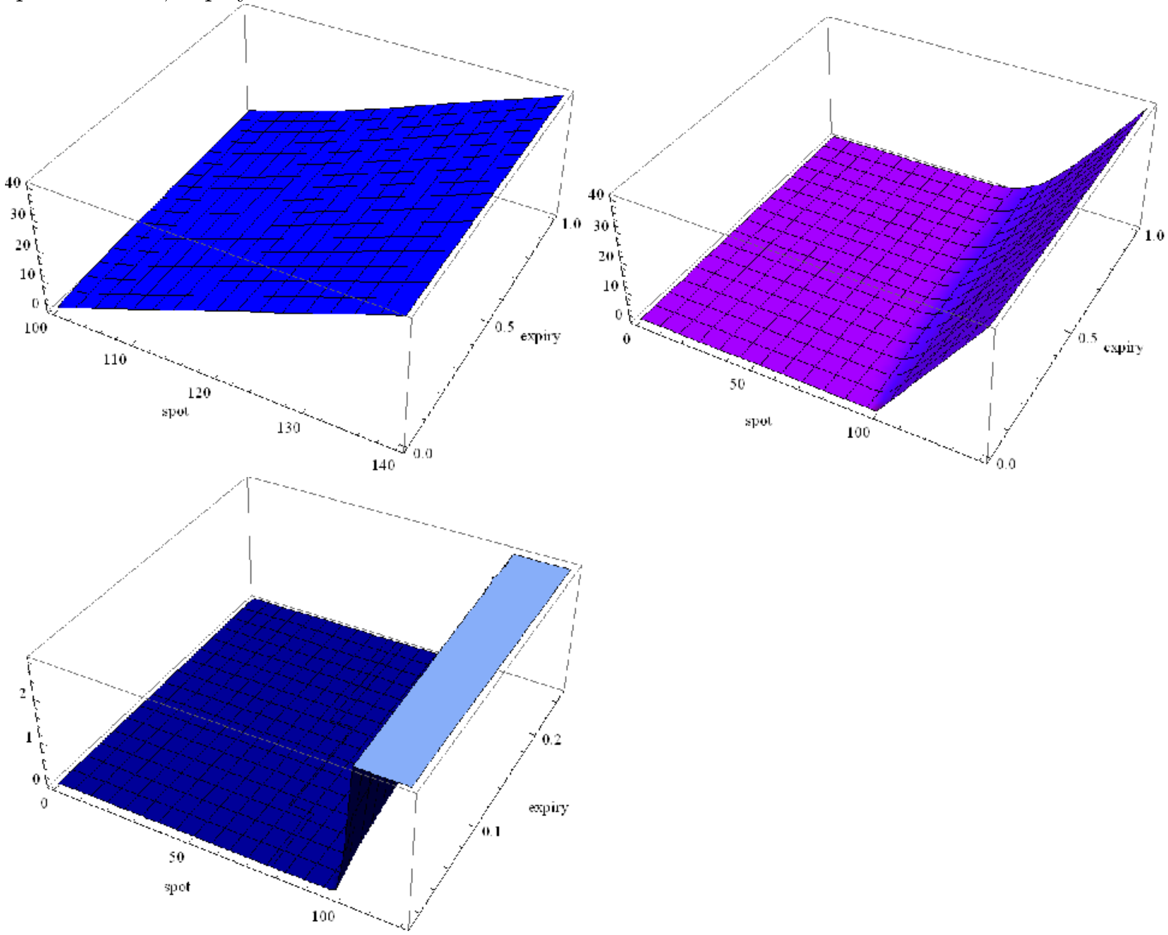
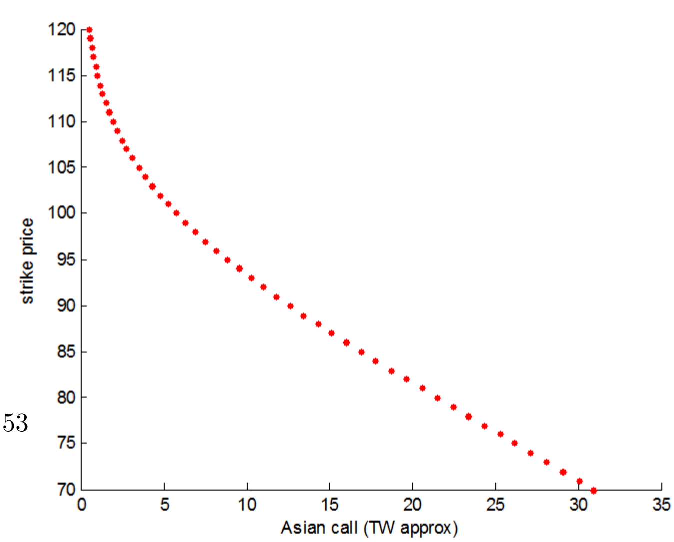


Figure 1: plot 1: spot=0:120, expiry=0.01:0.25, plot 2: spot=0:140, expiry=0:1 , plot 3: spot=100:140, expiry=0:1



σ	K	Call (T&W)
0.1	85	16.6875
	90	11.9533
	95	7.4152
	100	3.6475
0.2	85	16.9198
	90	12.6298
	95	8.8525
	100	5.7828
0.3	85	17.8092
	90	14.0381
	95	10.7488
	100	7.9925
0.4	85	19.1650
	90	15.7768
	95	12.7956
	100	10.2303
0.5	85	20.7852
	90	17.6792
	95	14.9157
	100	12.4895



```
Lookback_fixed_call_exact =

   19.1676


Lookback_fixed_put_exact =

   12.3397


Lookback_float_call_exact =

   17.2168


Lookback_float_put_exact =

   14.2906
```

```
Asian_call_fixed_geometric_exact =

    5.5304


Asian_put_fixed_geometric_exact =

    3.4469


Asian_call_float_geometric_exact =

    5.1563


Asian_put_float_geometric_exact =

    2.3628
```

A Appendix A

Proof (section 4.3)

- $[0,t]$ is compact, thus closed and bounded, with $\tau \in [0,t]$ and S_t is continuous on it. Therefore, integral is well defined and maximum and minimum exist.

  It holds

  $0 < \int_0^t (S_\tau)^n \, d\tau \leq \int_0^t (M_t)^n \, d\tau = M_t^n t \xrightarrow{(\,)^{1/n}} 0 < I_n(t) \leq t^{1/n} M_t$

  $\overset{\lim}{\Longrightarrow} \boxed{\lim_{n\to\infty} I_n(t) \leq 1 \cdot M_t = M_t}$ (1)

  We now choose $\epsilon > 0$ and let $A_\epsilon(t) \subseteq [0,t]$ be the set of $\tau \in [0,t]$ such that $S_\tau \geq M_t - \epsilon$. Furthermore, we define $L_\epsilon(t)$ to be the measure of this set, i.e $L_\epsilon(t) = \int_{A_\epsilon(t)} d\tau$

  Because $0 < L_\epsilon(t) \leq t$ and S_τ is continuous, we have:

  $\int_0^t (S_\tau)^n \, d\tau \geq \int_{A_\epsilon(t)} (S_\tau)^n \, d\tau \geq \int_{A_\epsilon(t)} (M_t - \epsilon)^n \, d\tau = (M_t - \epsilon)^n \cdot L_\epsilon(t)$

  $\xrightarrow{(\,)^{1/n}} I_n(t) \geq (M_t - \epsilon)(L_\epsilon(t))^{1/n} \overset{\lim}{\Longrightarrow} \boxed{\lim_{n\to\infty} I_n(t) \geq M_t - \epsilon \cdot 1 = M_t - \epsilon}$ (2)

  $\underset{(2)}{\overset{(1)}{\Longrightarrow}} M_t - \epsilon \leq \lim_{n\to\infty} I_n(t) \leq M_t$, for any $\epsilon > 0 \Rightarrow \boxed{\lim_{n\to\infty} I_n(t) = M_t}$

- Making use of the same arguments as in first case, we have: $m_t \leq S_\tau \Rightarrow \dfrac{1}{S_\tau} \leq \dfrac{1}{m_t} \Rightarrow$

  $\int_0^t \dfrac{1}{(S_\tau)^n} \, d\tau \leq \int_0^t \dfrac{1}{m_t^n} \, d\tau \xrightarrow{(\,)^{-1/n}} \left(\int_0^t \dfrac{1}{(S_\tau)^n} \, d\tau \right)^{(-1/n)} \geq \left(\int_0^t \dfrac{1}{m_t^n} \, d\tau \right)^{(-1/n)} \overset{\lim}{\Longrightarrow} \boxed{\lim_{n\to\infty} J_n(t) \geq 1 \cdot m_t = m_t}$ (3)

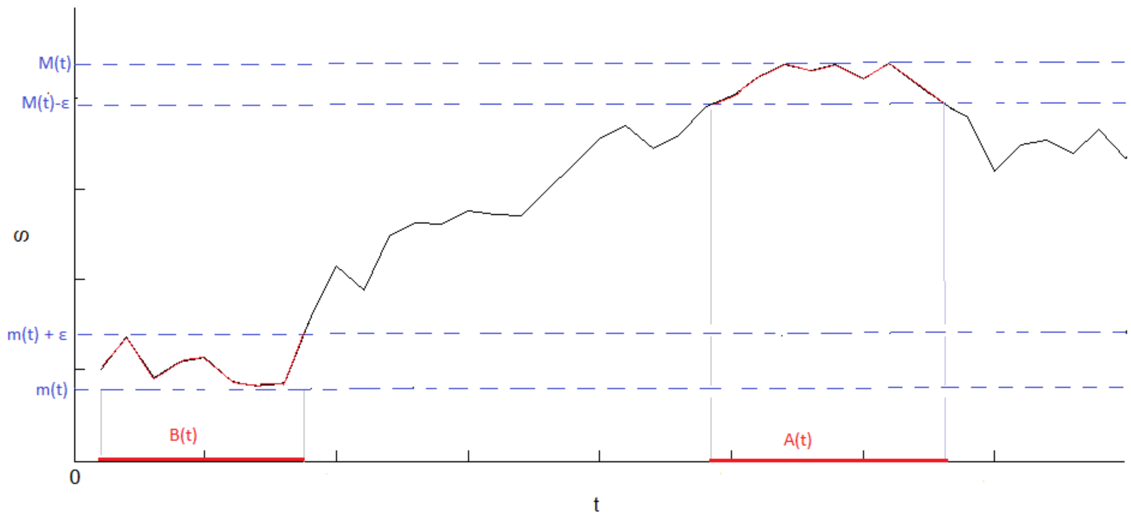
  Let $B_\epsilon \subseteq [0,t]$ be the set of $\tau \in [0,t]$ such that $S_\tau \leq m_t + \epsilon$. Furthermore, we define $Ll_\epsilon(t)$ to be the measure of this set, i.e $l_\epsilon(t) = \int_{B_\epsilon(t)} d\tau$

  Because $0 < l_\epsilon(t) \leq t$ and S_τ is continuous, we have:

  $\int_0^t \left(\dfrac{1}{(S_\tau)} \right)^n \, d\tau \geq \int_{B_\epsilon(t)} \left(\dfrac{1}{(S_\tau)} \right)^n \, d\tau > \int_{B_\epsilon(t)} \left(\dfrac{1}{(m_t + \epsilon)} \right)^n \, d\tau = \dfrac{1}{(m_t + \epsilon)} \cdot L_\epsilon(t) \xrightarrow{(\,)^{-1/n}}$
  $\left(\int_0^t \left(\dfrac{1}{(S_\tau)} \right)^n \, d\tau \right)^{-1/n} < \left(\dfrac{1}{(m_t + \epsilon)} L_\epsilon(t) \right)^{-1/n} \overset{\lim}{\Longrightarrow} \boxed{\lim_{n\to\infty} J_n(t) \leq m_t + \epsilon \cdot 1 = m_t + \epsilon}$ (4)

  $\underset{(4)}{\overset{(3)}{\Longrightarrow}} m_t \leq \lim_{n\to\infty} J_n(t) \leq m_t + \epsilon$, for any $\epsilon > 0 \Rightarrow \boxed{\lim_{n\to\infty} J_n(t) = m_t}$

Note that we could prove the second case just by using the fact that $minS_\tau = max\dfrac{1}{S_\tau}$



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Proof (section 5.1)

$$A= \quad S\int_{ln\frac{K}{S}}^{\infty} e^{\psi} N\left(\frac{-\psi+\mu(T-t)}{\sigma\sqrt{T-t}}\right) d\psi = S\int_{ln\frac{K}{S}}^{\infty} N\left(\frac{-\psi+\mu(T-t)}{\sigma\sqrt{T-t}}\right) d(e^{\psi}) =$$

$$Se^{\psi}N\left(\frac{-\psi+\mu(T-t)}{\sigma\sqrt{T-t}}\right)\Bigg|_{ln\frac{K}{S}}^{\infty} + S\int_{ln\frac{K}{S}}^{\infty} e^{\psi}\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma\sqrt{T-t}} e^{\frac{-1}{2}\left[\frac{-\psi+\mu(T-t)}{\sigma\sqrt{T-t}}\right]^2} d\psi =$$

set $\dfrac{-\psi+\mu(T-t)}{\sigma\sqrt{T-t}} = \kappa \Rightarrow \psi = -\kappa\sigma\sqrt{T-t}+\mu(T-t) \Rightarrow d\psi = -\sigma\sqrt{T-t}d\kappa$

$$= 0 - Se^{ln\frac{K}{S}} N\left(\frac{-ln\frac{K}{S}+\mu(T-t)}{\sigma\sqrt{T-t}}\right) - S\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma\sqrt{T-t}}\int_{\frac{-ln\frac{K}{S}+\mu(T-t)}{\sigma\sqrt{T-t}}}^{-\infty} e^{\mu(T-t)}e^{-\kappa\sigma\sqrt{T-t}}e^{\frac{-\kappa^2}{2}}\sigma\sqrt{T-t}\,d\kappa =$$

$$= -KN\left(\frac{-ln\frac{K}{S}+\mu(T-t)}{\sigma\sqrt{T-t}}\right) - S\frac{1}{\sqrt{2\pi}}e^{\mu(T-t)}\int_{\frac{-ln\frac{K}{S}+\mu(T-t)}{\sigma\sqrt{T-t}}}^{-\infty} e^{\frac{-1}{2}(\kappa^2+2\sigma\kappa\sqrt{T-t})} d\kappa =$$

$$= -KN\left(\frac{-ln\dfrac{K}{S}+\mu(T-t)}{\sigma\sqrt{T-t}}\right) - S\frac{1}{\sqrt{2\pi}}e^{\mu(T-t)}\int_{\frac{-ln\frac{K}{S}+\mu(T-t)}{\sigma\sqrt{T-t}}}^{-\infty} e^{\frac{-1}{2}(\kappa+\sigma\sqrt{T-t})^2} e^{\frac{\sigma^2(T-t)}{2}}\, d\kappa$$

set $\kappa+\sigma\sqrt{T-t}=u$

$$= -KN\left(\frac{-ln\dfrac{K}{S}+\mu(T-t)}{\sigma\sqrt{T-t}}\right) + S\frac{1}{\sqrt{2\pi}}e^{\left(\mu+\frac{\sigma^2}{2}\right)(T-t)}\int_{\frac{-ln\frac{K}{S}+\mu(T-t)}{\sigma\sqrt{T-t}}+\sigma\sqrt{T-t}}^{-\infty} e^{\frac{-u^2}{2}}\, du =$$

$$= -KN\left(\frac{-ln\dfrac{K}{S}+\mu(T-t)}{\sigma\sqrt{T-t}}\right) + Se^{r(T-t)}N\left(\frac{ln\dfrac{S}{K}+\mu(T-t)+\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)$$

set $\quad \mu = r - \dfrac{\sigma^2}{2}$

$$\Rightarrow \boxed{\text{A}= -KN\left(\frac{ln\dfrac{S}{K}+\left(r-\dfrac{\sigma^2}{2}\right)(T-t))}{\sigma\sqrt{T-t}}\right) + Se^{r(T-t)}N\left(\frac{ln\dfrac{S}{K}+(r+\dfrac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right)}$$

$$\text{B}= \ S\int_{ln\frac{K}{S}}^{\infty} e^{\psi}e^{\frac{2\mu\psi}{\sigma^2}} N\left(\frac{-\psi-\mu(T-t)}{\sigma\sqrt{T-t}}\right) d\psi = \ S\int_{ln\frac{K}{S}}^{\infty} N\left(\frac{-\psi-\mu(T-t)}{\sigma\sqrt{T-t}}\right)\frac{1}{1+\dfrac{2\mu}{\sigma^2}}\, de^{\left(1+\frac{2\mu}{\sigma^2}\right)\psi} =$$

$$= \frac{S}{1+\dfrac{2\mu}{\sigma^2}}N\left(\frac{-\psi-\mu(T-t)}{\sigma\sqrt{T-t}}\right)e^{\left(1+\frac{2\mu}{\sigma^2}\right)\psi}\Big|_{ln\frac{K}{S}}^{\infty} + \frac{S}{1+\dfrac{2\mu}{\sigma^2}}\int_{ln\frac{K}{S}}^{\infty} e^{\left(1+\frac{2\mu}{\sigma^2}\right)\psi}\frac{1}{\sqrt{2\pi}}e^{\frac{-1}{2}\left(\frac{-\psi-\mu(T-t)}{\sigma\sqrt{T-t}}\right)^2}\frac{1}{\sigma\sqrt{T-t}}\, d\psi$$

$$= \frac{-S}{1+\dfrac{2\mu}{\sigma^2}}N\left(\frac{-ln\dfrac{K}{S}-\mu(T-t)}{\sigma\sqrt{T-t}}\right)e^{\left(1+\frac{2\mu}{\sigma^2}\right)ln\frac{K}{S}} + \frac{S}{1+\dfrac{2\mu}{\sigma^2}}\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma\sqrt{T-t}}\int_{ln\frac{K}{S}}^{\infty} e^{\left(1+\frac{2\mu}{\sigma^2}\right)\psi}e^{\frac{-1}{2}\left(\frac{-\psi-\mu(T-t)}{\sigma\sqrt{T-t}}\right)^2}\, d\psi$$

set $\quad \dfrac{-\psi-\mu(T-t)}{\sigma\sqrt{T-t}} = \upsilon \Rightarrow \psi = -\upsilon\sigma\sqrt{T-t}-\mu(T-t) \Rightarrow d\psi = -\sigma\sqrt{T-t}d\upsilon$

$$= \frac{-S}{1+\frac{2\mu}{\sigma^2}}\left(\frac{K}{S}\right)^{\left(1+\frac{2\mu}{\sigma^2}\right)} N\left(\frac{ln\frac{S}{K}-\mu(T-t)}{\sigma\sqrt{T-t}}\right)-$$

$$\frac{S}{1+\frac{2\mu}{\sigma^2}}\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma\sqrt{T-t}}\int_{\frac{-ln\frac{K}{S}-\mu(T-t)}{\sigma\sqrt{T-t}}}^{-\infty} \mathrm{e}^{-\left(1+\frac{2\mu}{\sigma^2}\right)\mu(T-t)}\mathrm{e}^{-\upsilon\sigma\sqrt{T-t}\left(1+\frac{2\mu}{\sigma^2}\right)}\mathrm{e}^{\frac{-\upsilon^2}{2}}\sigma\sqrt{T-t}d\upsilon =$$

$$= \frac{-S}{1+\frac{2\mu}{\sigma^2}}\left(\frac{K}{S}\right)^{\left(1+\frac{2\mu}{\sigma^2}\right)} N\left(\frac{ln\frac{S}{K}-\mu(T-t)}{\sigma\sqrt{T-t}}\right)-$$

$$\frac{S}{1+\frac{2\mu}{\sigma^2}}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\left(1+\frac{2\mu}{\sigma^2}\right)\mu(T-t)}\int_{\frac{-ln\frac{K}{S}-\mu(T-t)}{\sigma\sqrt{T-t}}}^{-\infty}\mathrm{e}^{\frac{-1}{2}\left(\upsilon^2+2\left(1+\frac{2\mu}{\sigma^2}\right)\sigma\upsilon\sqrt{T-t}\right)}d\upsilon =$$

$$= \frac{-S}{1+\frac{2\mu}{\sigma^2}}\left(\frac{K}{S}\right)^{\left(1+\frac{2\mu}{\sigma^2}\right)} N\left(\frac{ln\frac{S}{K}-\mu(T-t)}{\sigma\sqrt{T-t}}\right)-$$

$$\frac{S}{1+\frac{2\mu}{\sigma^2}}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\left(1+\frac{2\mu}{\sigma^2}\right)\mu(T-t)}\mathrm{e}^{\frac{1}{2}\left(1+\frac{2\mu}{\sigma^2}\right)^2\sigma^2(T-t)}\int_{\frac{-ln\frac{K}{S}-\mu(T-t)}{\sigma\sqrt{T-t}}}^{-\infty}\mathrm{e}^{\frac{-1}{2}\left(\upsilon+\left(1+\frac{2\mu}{\sigma^2}\right)\sigma\sqrt{T-t}\right)^2}d\upsilon$$

$$\text{set} \quad \upsilon+\left(1+\frac{2\mu}{\sigma^2}\right)\sigma\sqrt{T-t}=u$$

$$\frac{-S}{1+\frac{2\mu}{\sigma^2}}\left(\frac{K}{S}\right)^{\left(1+\frac{2\mu}{\sigma^2}\right)} N\left(\frac{ln\frac{S}{K}-\mu(T-t)}{\sigma\sqrt{T-t}}\right)-$$

$$\frac{S}{1+\frac{2\mu}{\sigma^2}}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\left(1+\frac{2\mu}{\sigma^2}\right)\mu(T-t)}\mathrm{e}^{\frac{1}{2}\left(1+\frac{2\mu}{\sigma^2}\right)^2\sigma^2(T-t)}\int_{\frac{-ln\frac{K}{S}-\mu(T-t)}{\sigma\sqrt{T-t}}+\left(1+\frac{2\mu}{\sigma^2}\right)\sigma\sqrt{T-t}}^{-\infty}\mathrm{e}^{\frac{-u^2}{2}}dz =$$

$$= \frac{-S}{1+\frac{2\mu}{\sigma^2}}\left(\frac{K}{S}\right)^{\left(1+\frac{2\mu}{\sigma^2}\right)} N\left(\frac{ln\frac{S}{K}-\mu(T-t)}{\sigma\sqrt{T-t}}\right)+$$

$$\frac{S}{1+\frac{2\mu}{\sigma^2}}\mathrm{e}^{-\left(1+\frac{2\mu}{\sigma^2}\right)\mu(T-t)}\mathrm{e}^{\frac{1}{2}\left(1+\frac{2\mu}{\sigma^2}\right)^2\sigma^2(T-t)}N\left(\frac{ln\frac{S}{K}-\mu(T-t)\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}+\left(1+\frac{2\mu}{\sigma^2}\right)\sigma\sqrt{T-t}\right)$$

set $\quad \mu = r - \dfrac{\sigma^2}{2}$

$$\frac{-S}{1+\dfrac{2r-\sigma^2}{\sigma^2}}\left(\frac{K}{S}\right)^{\left(1+\dfrac{2r-\sigma^2}{\sigma^2}\right)} N\left(\frac{ln\dfrac{S}{K}-\left(r-\dfrac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)+$$

$$\frac{S}{1+\dfrac{2r-\sigma^2}{\sigma^2}}\mathrm{e}^{\left(1+\dfrac{2\mu}{\sigma^2}\right)(T-t)\left(-\mu+\dfrac{1}{2}\left(1+\dfrac{2\mu}{\sigma^2}\right)\sigma^2\right)} N\left(\frac{ln\dfrac{S}{K}-\left(r-\dfrac{\sigma^2}{2}\right)(T-t)+\left(1+\dfrac{2\mu}{\sigma^2}\right)\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)$$

$$= \frac{-S}{\dfrac{2r}{\sigma^2}}\left(\frac{K}{S}\right)^{\dfrac{2r}{\sigma^2}} N\left(\frac{ln\dfrac{S}{K}-\left(r-\dfrac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)-\frac{S}{\dfrac{2r}{\sigma^2}}\mathrm{e}^{r(T-t)}N\left(\frac{ln\dfrac{S}{K}-\left(r-\dfrac{\sigma^2}{2}\right)(T-t)+\dfrac{2r}{\sigma^2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)$$

$=$

$$\Rightarrow \boxed{\mathrm{B}= -S\frac{\sigma^2}{2r}\left(\frac{S}{K}\right)^{\dfrac{-2r}{\sigma^2}} N\left(\frac{ln\dfrac{S}{K}-\left(r-\dfrac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)+S\frac{\sigma^2}{2r}\mathrm{e}^{r(T-t)}N\left(\frac{ln\dfrac{S}{K}+\left(r+\dfrac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)}$$