Dis^n of τ

$$\tau = \tau_1 = \inf\{t \mid N_t > 0\}$$

$$\begin{cases} f(t) & \text{pdf} \\ F(t) & \text{CDF} = \Pr(\tau < t) \\ S(t) & \text{survival } f^n = 1 - F(t) \end{cases}$$

Dis^n

Bayer's

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

4th property of PP

$$\Pr\{N_{t+h} = 1 \mid N_t = 0] = \lambda h + o(h)$$

$$\equiv \Pr\{t \le \tau < t+h \mid \tau > t] = \lambda h + o(h)$$

$$\lambda = \lim_{h\to 0^+} \frac{1}{h}\Pr(\tau < t+h \mid \tau > t) + \frac{o(h)}{h} \to 0$$

$$f(t) = F'(t) = \lambda e^{-\lambda t}$$

$$\lambda \approx \lim_{h\to 0^+} \frac{1}{h}\,\frac{S(t) - S(t+h)}{S(t)}$$

$$\boxed{\lambda \approx -\frac{1}{S(t)}\lim_{h\to 0^+}\frac{S(t+h) - S(t)}{h} = -\frac{S'(t)}{S(t)} =} -\frac{d\log S(t)}{dt}$$

$$\Pr(\tau < t) = 1 - S(t) = 1 - e^{-\lambda t}$$

$$\Pr(t < \tau < T) = F(T) - F(t) = e^{-\lambda t} - e^{-\lambda T}$$

$$\to \Pr(t < \tau \le t+dt) = f(t)\,dt = \lambda e^{-\lambda t}\,dt = e^{-\lambda t}\,\lambda\,dt = \begin{cases}\Pr(\tau > t) \\ \times \\ \Pr(\tau < t+dt \mid \tau > t)\end{cases}$$

Case I

① No default   Prob: $1 - p\,dt$

$$
\begin{aligned}
d\Pi &= dV - \Delta\, dZ \\
&= \mathcal{L}(V)\,dt + \frac{\partial V}{\partial r}\,dr - \Delta\left[\mathcal{L}(Z)\,dt + \frac{\partial Z}{\partial r}\,dr\right] \\
&= \left[\mathcal{L}(V) - \Delta\mathcal{L}(Z)\right]dt + \left[\frac{\partial V}{\partial r} - \Delta\frac{\partial Z}{\partial r}\right]dr
\end{aligned}
$$

$$\Delta = \frac{\partial V}{\partial r} \Big/ \frac{\partial Z}{\partial r}$$

$$= \left[\mathcal{L}(V) - \Delta\mathcal{L}(Z)\right]dt$$

③ if defaut $t \to t+dt$ Prob $p\,dt$

$$d\pi = -v + O(\sqrt{dt}) \qquad \text{big "O"}$$

$$E^{Q}(d\pi) = (1-p\,dt)[\mathcal{L}(v) - o\mathcal{L}(z)]\,dt +$$
$$p\,dt[-v + O(\sqrt{dt})]$$

$$= [\mathcal{L}(v) - pv - o\mathcal{L}(z)]\,dt$$

$$= r(v - oz)\,dt$$

$$\mathcal{L}(v) - (p+r)v = \left(\frac{\partial v}{\partial r}\Big/\frac{\partial z}{\partial r}\right)(\mathcal{L}(z) - rz)$$

BPE for Z (risk-free)

$$\boxed{\frac{\partial Z}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2} + (u - \lambda w)\frac{\partial Z}{\partial r} - rZ = 0}$$

$$\mathcal{L}(Z) = \frac{\partial Z}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2}$$

$$\text{RHS} = (\mathcal{L}(Z) - rZ)\left(\frac{\partial V}{\partial r} \Big/ \frac{\partial Z}{\partial r}\right)$$

$$= -(u - \lambda w)\cancel{\frac{\partial Z}{\partial r}}\left(\frac{\partial V}{\partial r} \Big/ \cancel{\frac{\partial Z}{\partial r}}\right)$$

$$= -(u - \lambda w)\frac{\partial V}{\partial r}$$

$$\text{LHS} = \mathcal{L}(V) - (p + r)V = -(u - \lambda w)\frac{\partial V}{\partial r}$$

$$\Longrightarrow \boxed{\mathcal{L}(V) + (u - \lambda w)\frac{\partial V}{\partial r} - (p + r)V = 0}$$

BPE for V

Case I

1) No Default

$$d\pi = dV - \delta dz - \delta_1 dV_1$$

$$= [\mathcal{L}'(V) - \delta\mathcal{L}(z) - \delta_1 \mathcal{L}'(V_1)]\, dt +$$

$$[\frac{\partial V}{\partial r} - \delta \frac{\partial z}{\partial r} - \delta_1 \frac{\partial V_1}{\partial r}]\, dr +$$

$$[\frac{\partial V}{\partial p} - 0 - \delta_1 \frac{\partial V_1}{\partial p}]\, dp$$

$$\Rightarrow \quad \delta_1 = \frac{\partial V}{\partial p} \Big/ \frac{\partial V_1}{\partial p} \qquad \delta = \frac{\frac{\partial V}{\partial r} - \delta_1 \frac{\partial V_1}{\partial r}}{\frac{\partial z}{\partial r}}$$

$$d\pi = [\mathcal{L}'(V) - \delta\mathcal{L}(z) - \mathcal{L}'(V_1)]\, dt$$

② If default

$$d\pi = -V + \delta_1 V_1 + o(\sqrt{dt})$$

$$E(d\pi) = (1-\rho dt)\left[\mathcal{L}'(V) - \delta\mathcal{L}(Z) - \delta_1\mathcal{L}'(V_1)\right]dt + \rho dt\left[-V + \delta_1 V_1 + o(\sqrt{dt})\right]$$

$$E(d\pi) = dt\left[\mathcal{L}'(V) - \rho \quad - \delta\mathcal{L}(Z) - \delta_1\left[\mathcal{L}'(V_1) - \rho \quad V_1\right]\right]$$

$$= r(V - \delta Z - \delta_1 V_1)\,dt$$

$$\mathcal{L}'(V) - (\rho+r)V - \delta_1\left[\mathcal{L}'(V_1) - (\rho+r)V_1\right] = \delta\left[\underline{\mathcal{L}(Z) - rZ}\right]$$

$$RHS = \frac{\frac{\partial v}{\partial r} - \sigma_1 \frac{\partial v_1}{\partial r}}{\cancel{\frac{\partial z}{\partial r}}} \left[-(u - \lambda w) \cancel{\frac{\partial z}{\partial r}} \right]$$

$$= -(u - \lambda w)\left(\frac{\partial v}{\partial r} - \sigma_1 \frac{\partial v_1}{\partial r} \right)$$

$$\mathcal{L}'(v) - (\rho + r)v + (u - \lambda w)\frac{\partial v}{\partial r} = \sigma_1 \left[\mathcal{L}'(v_1) - (\rho + r)v_1 + (u - \lambda w)\frac{\partial v_1}{\partial r} \right]$$

$$\sigma_1 = \frac{\partial v}{\partial \rho} \Big/ \frac{\partial v_1}{\partial \rho}$$

$$\frac{\mathcal{L}'(v) - (\rho + r)v + (u - \lambda w)\frac{\partial v}{\partial r}}{\frac{\partial v}{\partial \rho}} = \frac{\mathcal{L}'(v_1) - (\rho + r)v_1 + (u - \lambda w)\frac{\partial v_1}{\partial r}}{\frac{\partial v_1}{\partial \rho}}$$

$$= a'(r, \rho, t)$$

$$= -(\gamma - \delta\lambda')$$