CQF Module 3
Martingales Theory: Application to Option Pricing
Black-Scholes All Over Again

CQF

In this lecture...

... we will apply probabilistic and martingale methods to the pricing of European stock and index options in complete markets. we will see:

- computing the price of a derivative as an expectation;
- Girsanov's theorem and change of measures;
- the fundamental asset pricing formula;
- the Black-Scholes Formula;
- the Feynman-Kăc formula;
- extensions to Black-Scholes: dividends and time-dependent parameters;
- Black's formula for options on futures;

Introduction

In Lecture 3.1. we developed the Black-Scholes derivative pricing approach from a PDE perspective. In this session, we will do the same but from the vantage point of probability theory.

While the concepts used to derive the Black-Scholes PDE are straightforward in their financial and economic interpretation, we will see that the probabilistic approach requires a greater level of abstraction.

However, where the probabilistic approach really shines is in the derivation of the Black-Scholes pricing formula where the fast functional manipulations of the PDE world are replaced by an elegant probabilistic interpretation.

1. The World of Black-Scholes

The setting of the probabilistic approach is not different from the setting seen in Lecture 3.1. It is simply expressed more formally.

We are in a financial market comprised of two traded assets, the risk-free asset B and the underlying security S.

Our objective is to value a financial derivative (a.k.a. contingent claim) χ . The exercise value of this financial derivative is a function $G(S(T))$ of the underlying security S at the expiry time T.

Since we are using a probabilistic framework, we will start by defining a probability space[1] $(\Omega, \mathcal{F}, \mathbb{P})$.

[1] Technical note (everyone, avert your eyes!): we should in fact be defining a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in[0,T]}, \mathbb{P})$.

1.1. Risk-Free Asset

The risk-free asset B evolves according to the dynamics

$$dB_t = rB_t dt, \quad B(0) = B_0.$$

Written as an integral, the value of the risk-free asset at time t is given by:

$$B(t) = B(0)e^{rt}$$

The normal interpretation of the risk-free asset is as a short-term government bond or as a repo trade, r denoting the short-term (in fact instantaneous!) interest rate.

Note that the asset B is effectively risk-free since its dynamics only has a drift but no diffusion.

Without loss of generality, we will take $B(0) = 1$ currency unit (GBP, USD, EUR,...).

1.2. Underlying Asset

The dynamics of the underlying asset S is modelled through a geometric Brownian motion (GBM):

$$dS_t = \mu S_t dt + \sigma S_t dX_t, \quad S(0) = S_0$$

where $X(t)$ is a Brownian motion[2] over $[0, T]$. Written as an integral, the value of the underlying asset at time t is given by:

$$S(t) = S_0 e^{\mu t - \frac{1}{2}\sigma^2 t + \sigma X_t}$$

[2]Technical note (everyone, avert your eyes again!): $X(t)$ is a $\mathcal{F}_t$-Brownian motion, meaning that it is adapted to the filtration $\mathcal{F}_t$.

1.3. Derivative Security

To keep the discussion general but tractable, we will assume that the derivative security χ

1. matures at time T;
2. with payoff at maturity given by a function G of $S(T)$, i.e. $\chi(T, S(T)) = G(S(T))$;
3. is of European style;

1.4. Other Assumptions

We also assume the usual at this stage:

1. short selling is allowed;
2. the market is frictionless, i.e. there is no transaction cost, taxes, lack of liquidity or constraints on shortselling;
3. as a consequence of 1. and 2., the assets can be bought or shorted in unlimited quantity;
4. fractional trading is authorized;
5. trading is conducted continuously;
6. there is no dividend and the coefficients r, μ and σ are constant.

2. The Fundamental Asset Pricing Formula

The most intricate part of this lecture is the derivation of the Fundamental Asset Pricing Formula.

The Fundamental Asset Pricing Formula basically states that

$$\text{Value of Asset} = \mathbb{E}^{\substack{\text{Some equivalent} \\ \text{probability measure}}}\left[\text{PV(Cash Flows)}\right]$$

This fundamental result not only provides a uniform view of asset valuation, but it is also the first step in establishing the Black-Scholes formula.

Admittedly, the Black-Scholes problem is special, because it can be solved analytically. In fact, many valuation problems do not have a closed-form solution. In this case, the Fundamental Asset Pricing Formula can be used as a base for the application of numerical methods, and in particular Monte Carlo simulations.

To derive the Fundamental Asset Pricing Formula, we will need to:

1. Define self-financing trading strategies and arbitrage strategies;
2. Discount the asset price;
3. Find an equivalent probability measure in which this asset price can be expressed as a simple (discounted) expectation;
4. Use the no-arbitrage condition to value the derivative;

2.1 Self-Financing Trading Strategies and Arbitrage Strategies

As in Lecture 3.1, our starting point will be the definition of a portfolio, or trading strategy which replicates the dynamics of the derivative.

Definition (Trading Strategy)

A **trading strategy** $\phi_t = (\phi_t^S, \phi_t^B)$ is a pair of stochastic processes progressively measurable over the time interval $[0, T]$ where

- ϕ_t^S represents the number of units of the underlying asset S held at time $t \in [0, T]$;
- ϕ_t^B represents the number of units of the risk-free asset B held at time $t \in [0, T]$.

This definition of trading strategies is quite wide as it only indicates the number of shares and the amount invested in/borrowed from the bank.

In particular, this definition does not say anything about flow to and from the portfolio such as:

- consumption - cash taken out of the portfolio to finance personal consumption (such as paying for grocery);
- contributions - cash added periodically to the portfolio (such as receiving a salary).

Because cash inflows and outflows are rather difficult to track accurately[3], we will concentrate on portfolios with **NO** cash inflows and **NO** cash outflows.

Such portfolios are called **self-financing**.

[3] Especially since one would need to actually track the *utility* generated by these cash flows rather than the flows themselves.

Formally,

Definition (Self-Financing Trading Strategy)

A trading strategy $\phi_t = (\phi_t^S, \phi_t^B)$ defined over the time interval $[0, T]$ is **self-financing** if its wealth process $V(\phi)$ given by

$$V_t(\phi) = \phi_t^S S_t + \phi_t^B B_t \quad \forall t \in [0, T]$$

satisfies the condition:

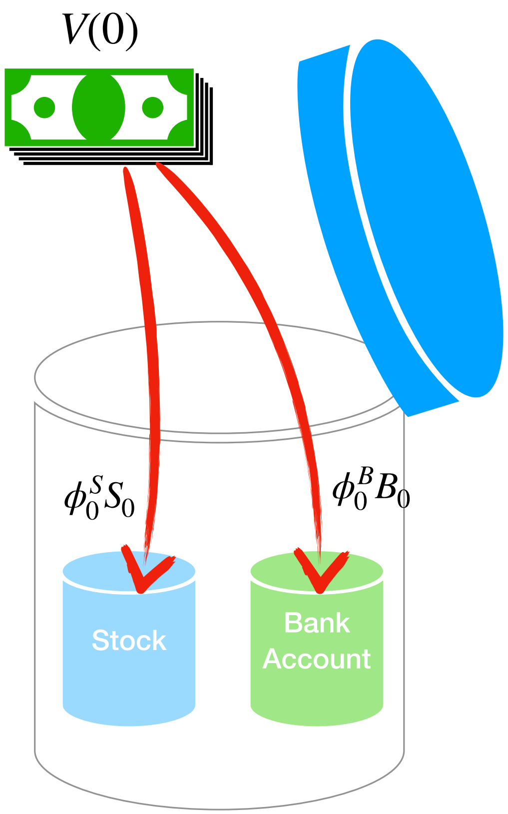
$$V_t(\phi) = V_0(\phi) + \int_0^t \phi_u^S dS_u + \int_0^t \phi_u^B dB_u \quad \forall t \in [0, T] \tag{1}$$

In plain English, what condition (1) is saying is that no cash is ever added or taken out of the portfolio. Hence, any purchase (or sale) of the underlying asset will be financed by (or invested in) the risk-free asset. portfolio is therefore truly **self-financing**.

From a mathematical standpoint, note that since S is stochastic the first integral in condition (1) is understood in the Itô sense. The second integral is a "regular" pathwise Riemann integral.

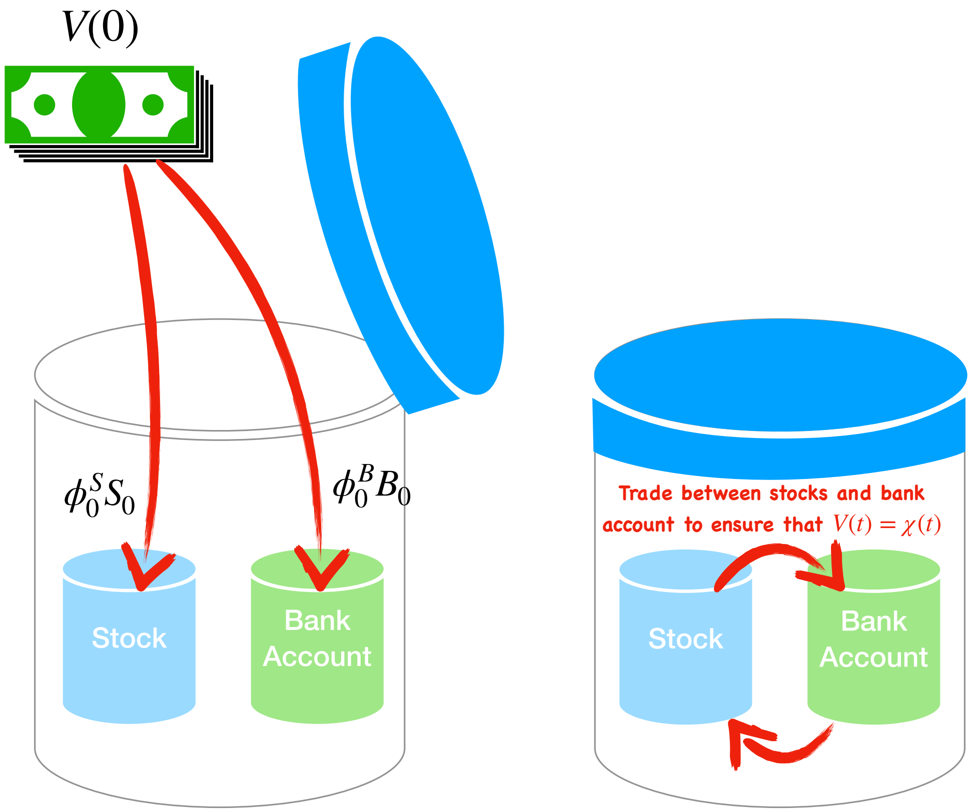
Self-Financing Trading Strategies Illustrated

At starting time $t = 0$

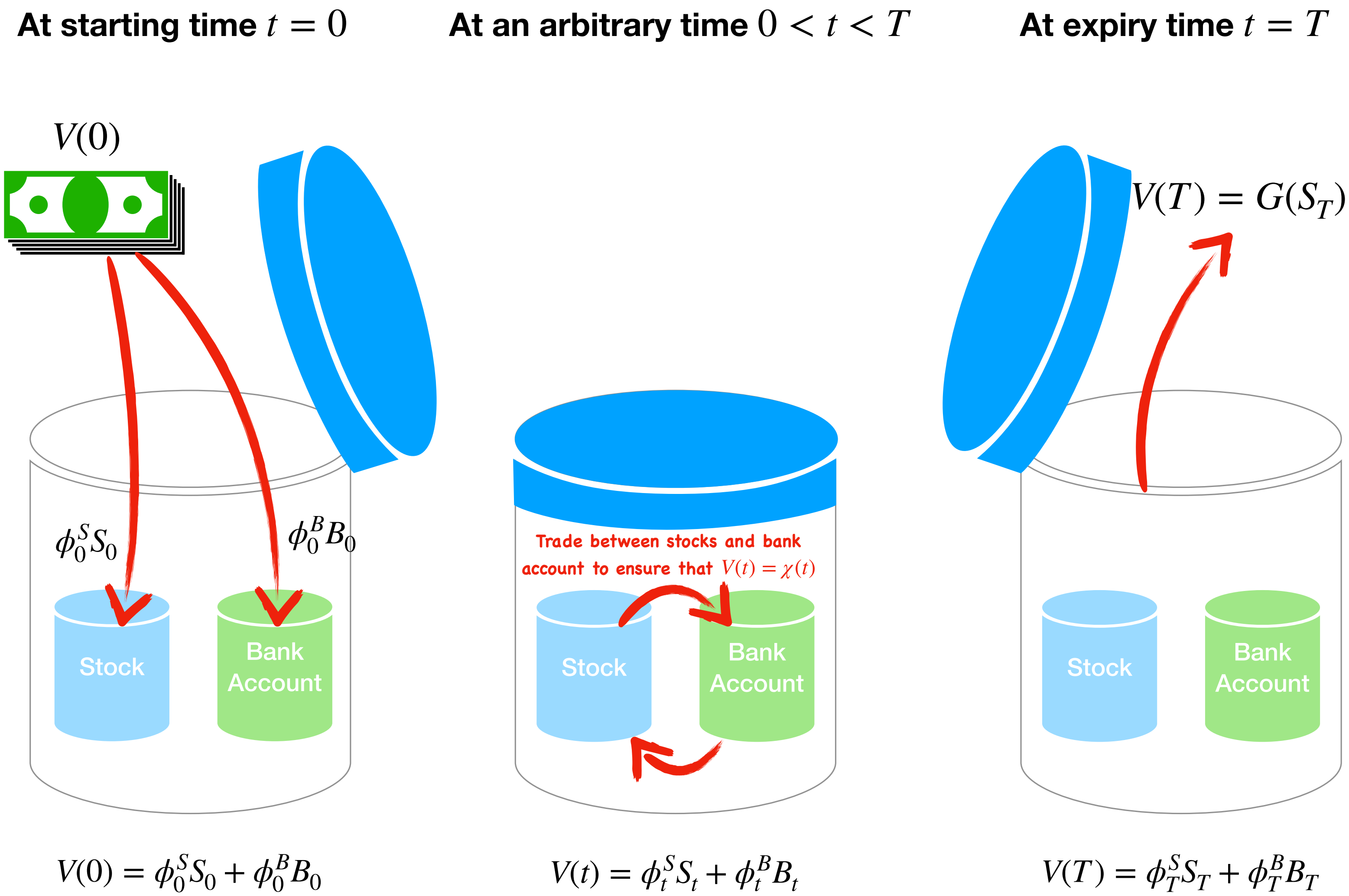


$$V(0) = \phi_0^S S_0 + \phi_0^B B_0$$

At starting time $t = 0$ **At an arbitrary time $0 < t < T$**



$$V(0) = \phi_0^S S_0 + \phi_0^B B_0 \qquad V(t) = \phi_t^S S_t + \phi_t^B B_t$$


At starting time $t = 0$
At an arbitrary time $0 < t < T$
At expiry time $t = T$
$V(0)$
$\phi_0^S S_0$
$\phi_0^B B_0$
Stock
Bank Account
Trade between stocks and bank account to ensure that $V(t) = \chi(t)$
Stock
Bank Account
$V(T) = G(S_T)$
Stock
Bank Account
$V(0) = \phi_0^S S_0 + \phi_0^B B_0$
$V(t) = \phi_t^S S_t + \phi_t^B B_t$
$V(T) = \phi_T^S S_T + \phi_T^B B_T$

Arbitrage

As we have already seen in the binomial model and the PDE approach, the absence of arbitrage opportunities on our market is key to pricing assets.

But what is an arbitrage opportunity?

In essence, an arbitrage opportunity exists if

- we can constitute a portfolio at time 0 at no cost;
- by time T, the portfolio cannot have lost money, but;
- the portfolio has a positive probability of having gained money by time T;

This definition is both wider and more realistic than the traditional academic definition of arbitrage which states that an arbitrage portfolio must deliver some gains by time T.

We now formalize our insight:

Definition (Arbitrage Opportunity)

An **arbitrage opportunity** is a self-financing portfolio ϕ such that

$$V_0(\phi) = 0$$

$$P\left[V_T(\phi) > 0\right] > 0$$

and

$$P\left[V_T(\phi) < 0\right] = 0$$

We say that a market is **arbitrage-free** if no arbitrage opportunity exists.

2.2. Discount the Asset Process

Rather than using the nominal stock price $S(t)$ in our analysis, we will use the **discounted** stock price $S^*(t)$ defined as

$$S^*(t) = \frac{S(t)}{B(t)} = \frac{S(t)}{e^{rt}}$$

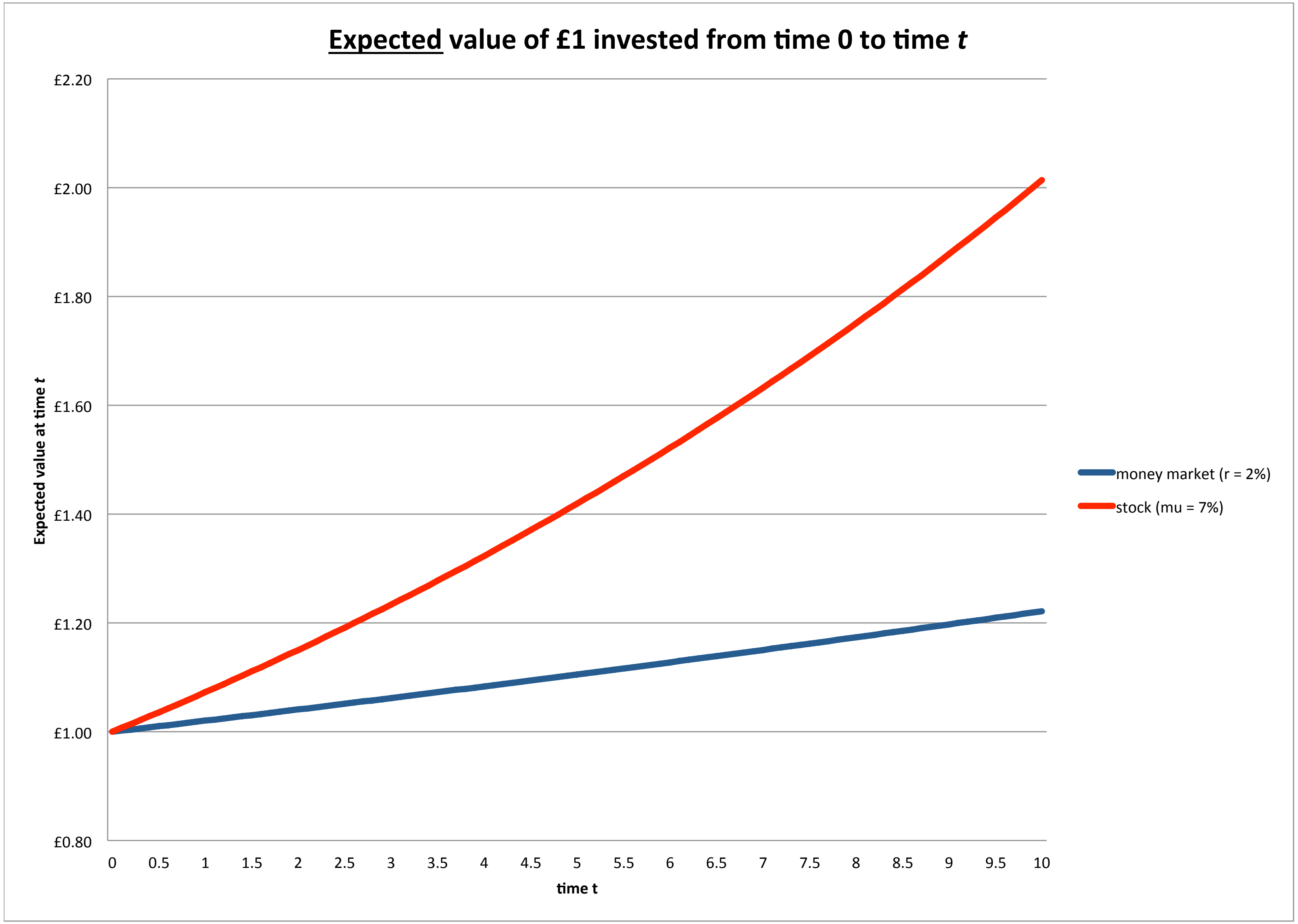
But why consider the discounted price rather than the current price?

Because of the **time value of money**.

Time value of money encodes a time dependence into the cash flow structure or the price dynamics of assets. It is helpful in determining time-dependent preferences between

1. receiving a cash flow C_0 at time 0; and,
2. receiving a cash flow C_T at time T.

In our model, time value of money is represented by the risk-free rate r. So an investor should be indifferent between the two cashflows if $C_0 = e^{-rt} C_T$.


Expected value of £1 invested from time 0 to time t
Expected value at time t
£2.20
£2.00
£1.80
£1.60
£1.40
£1.20
£1.00
£0.80
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
6.5
7
7.5
8
8.5
9
9.5
10
time t
money market (r = 2%)
stock (mu = 7%)

Because of time value of money, the risk-free rate is already embedded inside the drift of all financial assets. So we want to remove it and consider the underlying asset's dynamics freed from the time-dependent relationship encoded in the time value of money.

For any time $t \in [0, T]$, the discounted asset process is given by

$$S^*(t) = \frac{S(t)}{B(t)} = S_0^* e^{\left(\mu - r - \frac{1}{2}\sigma^2\right)t + \sigma X(t)}$$

with an equivalent SDE given by

$$dS^*(t) = (\mu - r)S^*(t)dt + \sigma S^*(t)dX(t), \quad S^*(0) = S_0^*$$

Looking at the SDE, we see that the effect of discounting is actually to remove the risk-free rate from the drift, or said otherwise, to remove the portion of the stock returns “guaranteed” by the time value of money.

2.3. Change the Measure

We have already seen in the CQF martingales are very nice processes to work with since they have many enviable properties:

- they are driftless, so all we have to think about is the randomness of the driving Brownian motion;
- their conditional expectation is easy to compute;
- Itô integrals are martingales;
- martingales can be represented as Itô integrals;

To ease our derivation we would like our main stochastic process, the discounted asset price $S^*(t)$, to be a martingale.

However, the discounted asset price $S^*(t)$ is not generally a martingale under the measure $\mathbb{P}$.

Indeed, for the discounted asset price $S^*(t)$ to be a martingale under the measure $\mathbb{P}$, we would need $S^*(t)$ to be driftless, i.e.

$$(\mu - r)S^*(t)dt = 0 \Leftrightarrow \mu = r$$

By extension,

- if $\mu > r$ (as should be the case in financial markets) then $S^*(t)$ is a *submartingale*; and
- if $\mu < r$, $S^*(t)$ is a *supermartingale*.


Expected value of £1 invested from time 0 to time t and discounted back at the risk-free rate: the stock is not a martingale!
£1.70
£1.60
£1.50
£1.40
£1.30
£1.20
£1.10
£1.00
£0.90
£0.80
Expected value at time t
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5 5.5 6 6.5 7 7.5 8 8.5 9 9.5 10
time t
discounted stock price (mu - r= 5%)
martingale: E[M_t |F_0] = M_0 = £1

If we want the process $S^*(t)$ to benefit from all the nice properties of martingales, we will need to move away from the measure $\mathbb{P}$ and into a new measure under which $S^*(t)$ is always a martingale.

Our broad objective is therefore to

- find a measure $\mathbb{Q}$ under which S^* is a martingale; and then,
- use the Radon Nikodym theorem to perform the change of measure.

The measure $\mathbb{Q}$ we are seeking is called a **martingale measure**.

Definition (Martingale Measure)

A probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ and equivalent to $\mathbb{P}$ is called a martingale measure for $S^*(t)$ if $S^*(t)$ is a martingale under $\mathbb{Q}$.

The trouble we encounter now is that while Radon Nikodym theorem we saw in the Martingale I lecture can be used to help us change measure once we know the measure we want to change into, it cannot help us to identify or define the martingale measure $\mathbb{Q}$ we are looking for.

To identify it, we need to use an additional result called **Girsanov's theorem**.

Key Fact (Girsanov's Theorem)

Given a process θ such that

$$M_t^{\theta} = \exp\left(-\int_0^t \theta_s dX_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right), \quad t \in [0, T]$$

is a martingale.
Then we can define the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ equivalent to $\mathbb{P}$ through the Radon Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^t \theta_s dX_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right), \quad t \in [0, T]$$

In this case, the process $X^{\mathbb{Q}}$ defined as

$$X_t^{\mathbb{Q}} = X_t + \int_0^t \theta(s) ds, \quad t \in [0, T] \tag{2}$$

is a standard Brownian Motion on $(\Omega, \mathcal{F}, \mathbb{Q})$.

Aside: Spotlight on the Process M_t^θ

The process

$$M_t^\theta = \exp\left(-\int_0^t \theta_s dX_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right), \quad t \in [0, T]$$

is exponential.

When it is also a martingale, we call it an exponential martingale.

Let's look at a special case...

For simplicity, we focus on the case where $\theta(t)$ is either constant or a deterministic function of time.

Taking the logarithm and applying Itô, we see that M_t^θ satisfies the SDE:

$$dM_t^\theta = -\theta_t M_t^\theta dM_t,$$

with the initial condition $M_0^\theta = 1$.

Therefore, M_t^θ is an exponential martingale in this case.

So, in the general case, how do we know if M_t^θ is an exponential martingale?

In general, it is enough to check that $E[M_0^\theta] = 1$ to conclude that M_t^θ is an exponential martingale.

However, proving that $E[M_0^\theta] = 1$ might be difficult when θ_t is itself stochastic.

Fortunately, we can check either of two sufficient conditions to ascertain whether a process M_t^θ is an exponential martingale:

- the Novikov condition;
- the Kazamaki condition.

Note that **we only need to check one** to be sure that M_t^θ is an exponential martingale!

Key Fact (Novikov condition)

A process θ_t satisfies the **Novikov condition** *if*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \theta_s^2 ds\right)\right] < \infty$$

Key Fact (Kazamaki condition)

A process θ_t satisfies the **Kazamaki condition** *if*

$$\mathbb{E}\left[\exp\left(-\frac{1}{2}\int_0^T \theta_s dX(s)\right)\right] < \infty$$

So, What Does Girsanov Actually Do?

Girsanov effectively extends the Radon Nikodym result by

- giving an expression for the Radon Nikodym derivative;
- expliciting a correspondence between the $\mathbb{P}$ measure and the $\mathbb{Q}$ measure in terms of their respective Brownian motions.

The process θ is vital because it acts as the "key" enabling us to define the measure $\mathbb{Q}$ via the Radon Nikodym derivative and to travel in between the $\mathbb{P}$ measure and the $\mathbb{Q}$ measure via the Brownian motion correspondence.

The difficulty with Girsanov is that the theorem stops short of identifying the process θ. We therefore need to have an idea of what θ is like and make sure that it satisfies the Novikov condition.

So, how can we guess what θ is?

In our case, we want to find θ such that S^* is a $\mathbb{Q}$-martingale. Applying Girsanov for an arbitrary process θ, we see that under $\mathbb{Q}$, the dynamics of S^* is given by

$$\frac{dS_t^*}{S_t^*} = (\mu - r)dt + \sigma\left(-\theta dt + dX_t^{\mathbb{Q}}\right)$$

Then we have

$$\frac{dS_t^*}{S_t^*} = (\mu - r - \sigma\theta(t))dt + \sigma dX_t^{\mathbb{Q}}$$

under $\mathbb{Q}$.

For S_t^* to be a martingale under $\mathbb{Q}$, its dynamics must be driftless, which implies

$$\mu - r - \sigma\theta(t) = 0$$

i.e.

$$\theta(t) = \frac{\mu - r}{\sigma} =: \theta$$

So θ is a constant.

We observe that θ satisfies the Novikov condition. Invoking **Girsanov's theorem**, we can finally define the equivalent martingale measure $\mathbb{Q}$ via the Radon Nikodym derivative:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{\mu - r}{\sigma}X_t - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}t\right), \quad t \in [0, T]$$

Moreover, the $\mathbb{Q}$-Brownian Motion, $X^{\mathbb{Q}}$, is defined as

$$X_t^{\mathbb{Q}} = X_t + \frac{\mu - r}{\sigma}t, \quad t \in [0, T]$$

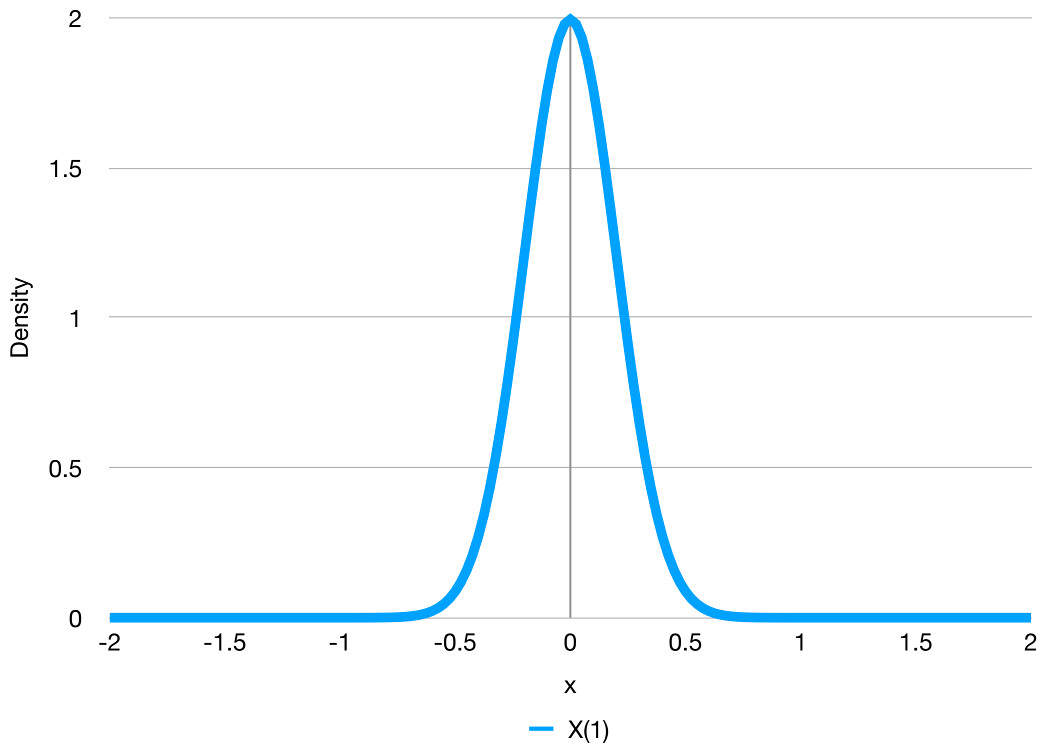
and under $\mathbb{Q}$ the discounted asset process is indeed a martingale since

$$\frac{dS_t^*}{S_t^*} = \sigma dX_t^{\mathbb{Q}}$$

So, concretely, what happens when we change measure?

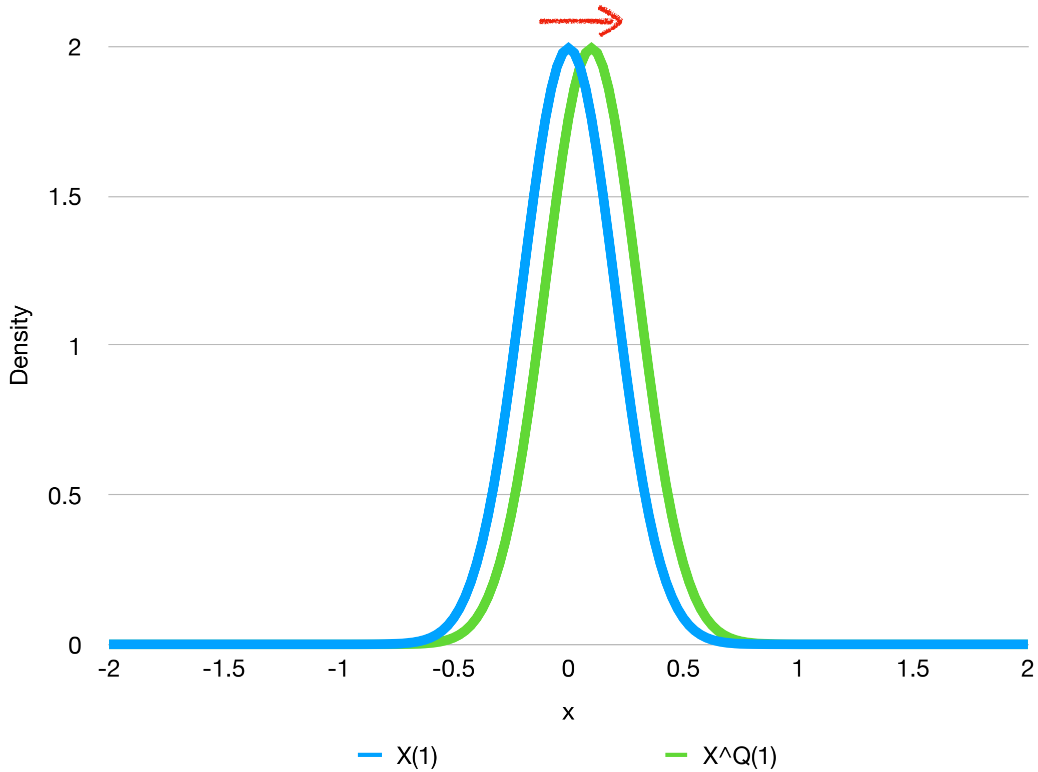
Let's start by taking the perspective of the $\mathbb{P}$-measure.

In the $\mathbb{P}$-measure, $X(t)$ is a standard Brownian motion, so $X(t) \sim N(0, t)$.



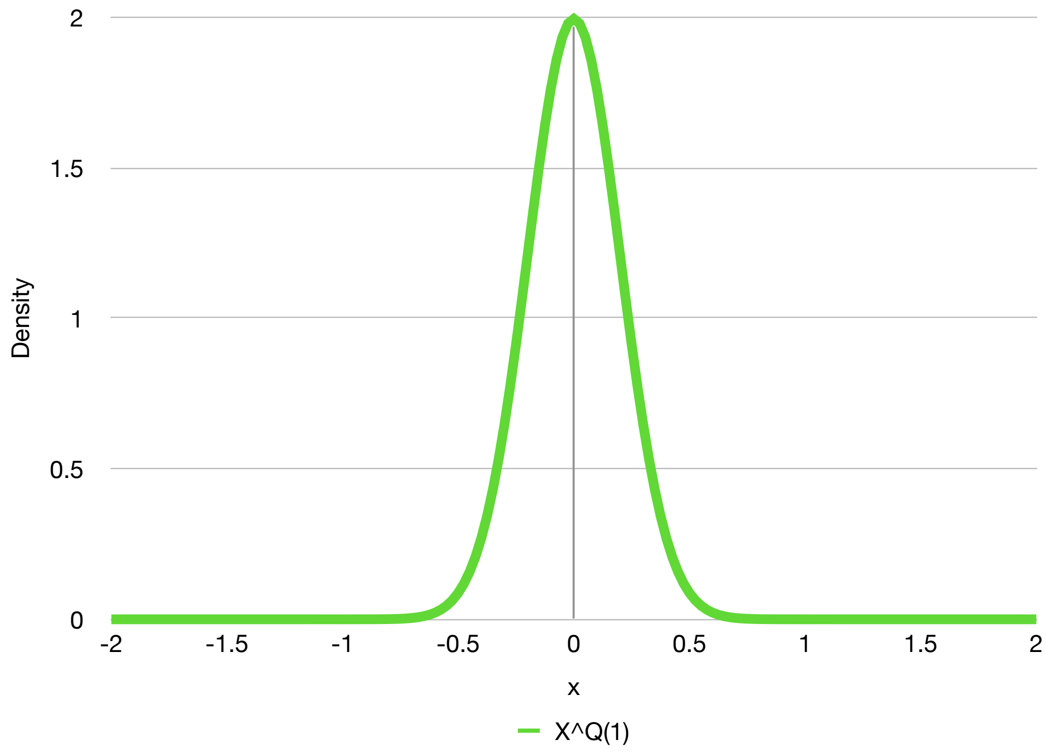
Now, let's look at $X^{\mathbb{Q}}$, still from the $\mathbb{P}$-measure's perspective.

In the $\mathbb{P}$-measure, $X^{\mathbb{Q}}(t) \sim N\left(\frac{\mu - r}{\sigma} t, t\right)$.

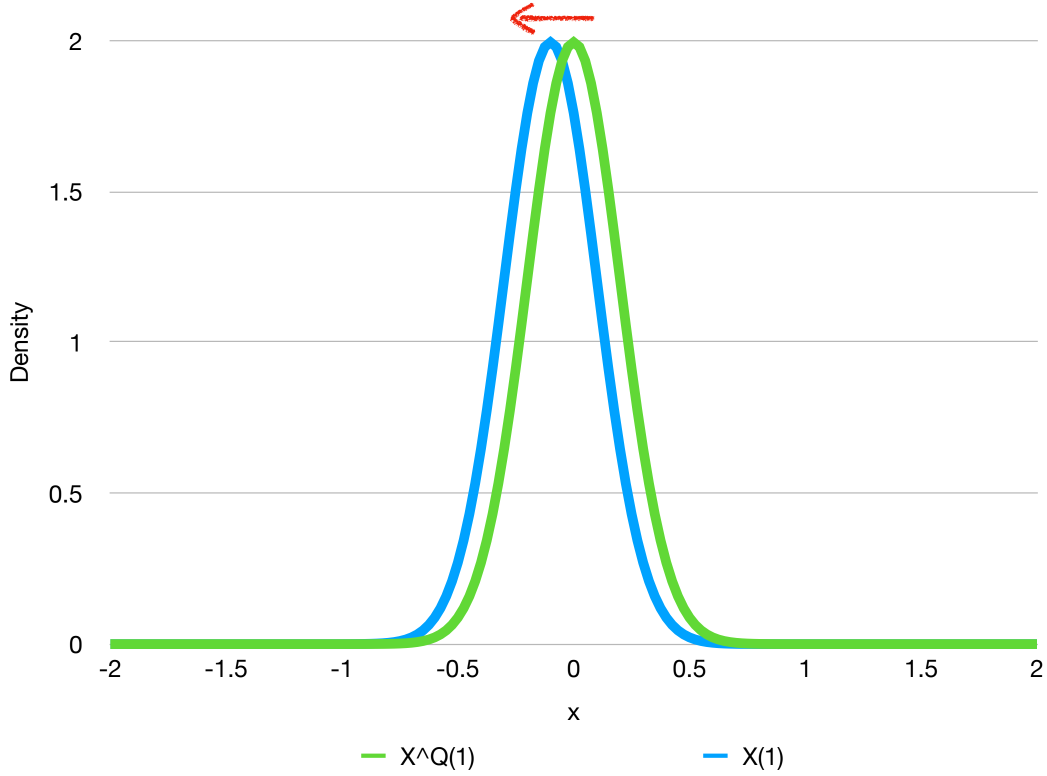


Now, let's change perspective. Let's look at the change of measure from the $\mathbb{Q}$-measure's perspective.

In the $\mathbb{Q}$-measure, $X^{\mathbb{Q}}(t)$ is a standard Brownian motion, so $X^{\mathbb{Q}}(t) \sim N(0, t)$...



… and in the $\mathbb{Q}$-measure, $X(t) \sim N\left(-\frac{\mu - r}{\sigma} t, t\right)$.



So a change of measure is really just a **translation** of the Brownian motion.

2.4. Derivative Valuation

We will denote by $\chi(t, S_t)$ the time t arbitrage-free value of the derivative we are attempting to price.

By analogy with what we have done earlier with the share price, we will consider the discounted value of the replicating portfolio. We define the time t discounted portfolio value V_t^* by

$$V_t^*(\phi) \quad = \quad \frac{V_t(\phi)}{B_t}, \quad t \in [0, T]$$

To prevent arbitrage, the value of the replicating portfolio must be equal to the value of the derivative:

$$\chi(t, S_t) = V_t, \quad t \in [0, T]$$

Once discounted, this equation becomes

$$\frac{\chi(t, S_t)}{B_t} = V_t^*(\phi), \quad t \in [0, T] \tag{3}$$

In particular, for $t = T$, we have

$$\frac{\chi(T, S_T)}{B_T} = \frac{G(S_T)}{B_T} = V_T^*$$

Taking the conditional expectation under $\mathbb{Q}$, we get

$$\mathbb{E}^{\mathbb{Q}}\left[V_T^*(\phi)|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[B_T^{-1}G(S_T)|\mathcal{F}_t\right], \quad t \in [0, T] \tag{4}$$

Now, how can we link expressions (3) and (4) in order to find the time t value of the derivative maturing at time T?

Answer: through our self-financing trading strategy!

First note that

$$dB_t^{-1} = -rB_t^{-1}dt$$

By the **Itô Product Rule**,

$$\begin{aligned}
dV_t^* &= d(V_t B_t^{-1}) \\
&= V_t dB_t^{-1} + B_t^{-1} dV_t \\
&= \left(\phi_t^S S_t + \phi_t^B B_t\right) dB_t^{-1} + B_t^{-1}\left(\phi_t^S dS_t + \phi_t^B dB_t\right) \\
&= \phi_t^S \left(B_t^{-1} dS_t + S_t dB_t^{-1}\right) \\
&= \phi_t^S dS_t^*
\end{aligned}$$

Integrating, we see that

$$\begin{aligned} V_t^* &= V_0^* + \int_0^t \phi_u^S dS_u^* \\ &= V_0^* + \int_0^t \phi_u^S \sigma S_u^* dX_u^{\mathbb{Q}} \end{aligned} \tag{5}$$

because under $\mathbb{Q}$,

$$dS_u^* = \sigma S_u^* dX_u^{\mathbb{Q}}$$

Now, note that

$$\int_0^t \phi_u^S \sigma S_u^* dX_u^{\mathbb{Q}}$$

is an Itô integral and therefore a martingale. Then, the discounted portfolio value V^* is a martingale.

Hence, under $\mathbb{Q}$, not only is S^* a martingale, but so is V^*!

Since V^* is a martingale, then by definition

$$V_t^* = \mathbb{E}^{\mathbb{Q}}\left[V_T^*(\phi)|\mathcal{F}_t\right], \quad t \in [0, T] \tag{6}$$

Considering in addition relationships (3) and (4), we obtain

$$\begin{aligned}\frac{\chi(t, S_t)}{B_t} &= V_t^*(\phi) \\ &= \mathbb{E}^{\mathbb{Q}}\left[V_T^*(\phi)|\mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[B_T^{-1}G(S_T)|\mathcal{F}_t\right], \quad t \in [0, T]\end{aligned} \tag{7}$$

Equation (7) is the cornerstone of asset valuation[4].

Key Fact

The value at time t of a derivative maturing at time T is the expected value under the $\mathbb{Q}$ measure of the discounted terminal cash flow of the contract.

$$\chi(t, S_t) \quad = \quad B_t \mathbb{E}^{\mathbb{Q}} \left[B_T^{-1} G(S_T) | \mathcal{F}_t \right], \quad t \in [0, T] \tag{8}$$

[4] and of the application of Monte-Carlo methods.

Aside: Is The Trading Strategy Considered Truly Self-Financing?

We have seen that under $\mathbb{Q}$, $V^*(t)$ is a martingale. Hence, by the **Martingale Representation Theorem** (see the Martingale I Lecture), there exists a process $\psi(t)$ satisfying technical condition such that

$$\begin{aligned} V_t^* &= V_0^* + \int_0^t \psi_u dX_u^{\mathbb{Q}} \\ &= V_0^* + \int_0^t h_u dS_u^*, \quad t \in [0, T] \end{aligned}$$

where $h_t = \frac{\psi_t}{\sigma S_t^*}$.

Consider a trading strategy ϕ defined as:

$$\begin{aligned}\phi_t^S &= h_t \\ \phi_t^B &= V_t^* - h_t S_t^* = B_t^{-1}(V_t - h_t S_t)\end{aligned}$$

We already know that $V_T(\phi) = G(S_T)$. We will now check that the strategy ϕ is self-financing. To do so, we need to go back to the current, or undiscounted, value of the replicating portfolio.

By the Itô Product Rule:

$$
\begin{aligned}
dV_t(\phi) &= d(B_t V_t^*) \\
&= B_t dV_t^* + V_t^* dB_t \\
&= B_t h_t dS_t^* + rV_t dt \\
&= B_t h_t \left(B_t^{-1} dS_t - rB_t^{-1} S_t dt \right) + rV_t dt \\
&= h_t dS_t + r(V_t - h_t S_t) dt
\end{aligned}
$$

which confirms the fact that the portfolio is indeed self financing.

Note that we do not know the specifics of the trading strategy, namely how much of the underlying asset to hold. All we know is that a strategy exists and that it is self-financing.

After seeing the importance of the Delta-hedging strategy in both the PDE approach and the Binomial model, it may seem quite strange to be dealing with an approach that do not require a specific knowledge of the strategy.

But this is precisely what the probabilistic approach does. It only checks that some technical conditions are fulfilled in order to guarantee that there exists an "appropriate" replicating strategy, without actually defining it.

End of aside.

The Black-Scholes Call Option Problem

We will now use the valuation formula (8) to solve the Black-Scholes European call option problem. A similar derivation could be done for European put options and for binary options.

Since r is constant, we can simplify (8) by writing

$$\chi(t, S_t) \quad = \quad e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[F(S_T)|\mathcal{F}_t\right], \quad t \in [0, T]$$

Going back to our *undiscounted* asset price process, S, we observe that under the measure $\mathbb{Q}$

$$\frac{dS_t}{S_t} = rdt + \sigma dX_t^{\mathbb{Q}}$$

Seems familiar? It is indeed what we saw during the previous lecture on Black-Scholes through PDE as the "Risk-Neutral" GBM.

In particular, at time T,

$$S_T = S_t \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T - t) + \sigma\left(X_T^{\mathbb{Q}} - X_t^{\mathbb{Q}}\right)\right\}$$

3.1. Direct Derivation

Define a new random variable $Y_T := \ln \frac{S_T}{S_t}, \quad \forall t \in [0, T]$. Since S_t is Lognormally distributed, then the log return of the asset over the period $[0, T]$, Y_T, is normally distributed with mean

$$\left(r - \frac{1}{2}\sigma^2\right)(T - t)$$

and variance

$$\sigma^2(T - t)$$

i.e.

$$Y_T \sim \mathcal{N}\left(\left(r - \frac{1}{2}\sigma^2\right)(T - t), \sigma^2(T - t)\right)$$

Using the basic definition of expectations, the expectation in our pricing formula (8) can be rewritten in terms of the random variable Y_T as

$$\chi(t, S_t) = e^{-r(T-t)} \int_{-\infty}^{\infty} G(S_t e^y) p(y) dy$$

where p is the PDF of Y.

To simplify our notation, we define

$$\tilde{r} = r - \frac{1}{2}\sigma^2$$
$$\tau = T - t$$

With this new notation, we have

$$Y_T \sim \mathcal{N}\left(\tilde{r}\tau, \sigma^2\tau\right)$$

To further simplify our calculations, define the *standardized Normal random variable*[5] Z

$$Z := \frac{Y - \tilde{r}\tau}{\sigma\sqrt{\tau}} \Leftrightarrow Y = \tilde{r}\tau + Z\sigma\sqrt{\tau}$$

After a change of variable, the expectation (9) can be expressed in terms of Z as

$$\chi(t, S_t) \quad = \quad e^{-r\tau} \int_{-\infty}^{\infty} G\left(S_t e^{\tilde{r}\tau + \sigma\sqrt{\tau} z}\right) \varphi(z) dz$$

where φ is the standard normal PDF:

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

[5] A standard or standardized Normal random variable is a Normally distributed random variable with mean 0 and variance 1

The payoff function for a call is given by

$$G(S_T) = \max\left[S_T - E, 0\right]$$

Substituting into the pricing equation (9), we get

$$\chi(t, S_t) = e^{-r\tau} \int_{-\infty}^{\infty} \max\left[S_t e^{\tilde{r}\tau + \sigma\sqrt{\tau}z} - E, 0\right] \varphi(z) dz$$

We can get rid of the max function by noticing that the integral vanishes when

$$S_t e^{\tilde{r}\tau + \sigma\sqrt{\tau}z} < E$$

i.e. when

$$z < z_0 := \frac{\ln\left(\frac{E}{S_t}\right) - \tilde{r}\tau}{\sigma\sqrt{\tau}}$$

The pricing formula now becomes

$$\begin{aligned}
\chi(t, S_t) &= e^{-r\tau}\int_{z_0}^{\infty}\left(S_t e^{\tilde{r}\tau + \sigma\sqrt{\tau}z} - E\right)\varphi(z)dz \\
&= e^{-r\tau}\int_{z_0}^{\infty} S_t e^{\tilde{r}\tau + \sigma\sqrt{\tau}z}\varphi(z)dz - e^{-r\tau}\int_{z_0}^{\infty} E\varphi(z)dz
\end{aligned} \tag{9}$$

The second term on the right-hand side of (9) seems easier to evaluate, so let's start with this one

$$
\begin{aligned}
-e^{-r\tau}\int_{z_0}^{\infty} E\varphi(z)dz &= -Ee^{-r\tau}\int_{z_0}^{\infty}\varphi(z)dz \\
&= -Ee^{-r\tau}P[Z \geq z_0]
\end{aligned}
$$

By symmetry of the normal distribution, this can also be written as

$$
-Ee^{-r\tau}P[Z \leq -z_0] = -Ee^{-r\tau}N(-z_0)
$$

where N is the standard normal CDF.

To evaluate the first term on the right side of (9), we need to complete the square in the exponent:

$$
\begin{aligned}
e^{-r\tau}\int_{z_0}^{\infty} S_t e^{\tilde{r}\tau+\sigma\sqrt{\tau}z}\varphi(z)dz &= \frac{e^{(\tilde{r}-r)\tau}S_t}{\sqrt{2\pi}}\int_{z_0}^{\infty} e^{\sigma\sqrt{\tau}z-\frac{1}{2}z^2}dz \\
&= \frac{e^{(\tilde{r}-r)\tau}S_t}{\sqrt{2\pi}}\int_{z_0}^{\infty} e^{-\frac{1}{2}(z-\sigma\sqrt{\tau})^2+\frac{1}{2}\sigma^2\tau}dz \\
&= \frac{S_t}{\sqrt{2\pi}}\int_{z_0}^{\infty} e^{-\frac{1}{2}(z-\sigma\sqrt{\tau})^2}dz \\
&= S_t P[U \geq z_0 - \sigma\sqrt{\tau}] \\
&= S_t P[U \leq -z_0 + \sigma\sqrt{\tau}]
\end{aligned}
$$

where $U = Z - \sigma\sqrt{\tau}$ so that $U \sim \mathcal{N}(-\sigma\sqrt{\tau}, 1)$. Standardizing, we see that the first term is actually equal to

$$S_t N(-z_0 + \sigma\sqrt{\tau})$$

In order to write this in the more familiar form, all we need to do is to define d_1 and d_2 as

$$
\begin{aligned}
d_1 &= -z_0 + \sigma\sqrt{\tau} = \frac{\ln\left(\frac{S_t}{E}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\
d_2 &= -z_0 = \frac{\ln\left(\frac{S_t}{E}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}
\end{aligned}
$$

and to substitute the first and the second term into the pricing equation:

$$\chi(t, S_t) = S_t N(d_1) - Ee^{-r(T-t)} N(d_2)$$

3.2. Alternative Derivation Through Change of Measure

In the case of a call, the valuation equation (8) can be expressed as

$$\begin{aligned} & B_t\mathbb{E}^{\mathbb{Q}}\left[B_T^{-1}\left[S_T-K\right]^+|\mathcal{F}_t\right] \\ = \quad & B_t\mathbb{E}^{\mathbb{Q}}\left[B_T^{-1}\left[S_t e^{\sigma(X_T^{\mathbb{Q}}-X_t)+(r-\frac{1}{2}\sigma^2)(T-t)}-K\right]^+|\mathcal{F}_t\right]\end{aligned}$$

where the strike price is now denoted by K to avoid any confusion with $\mathbb{E}$, the expectation.

Focusing on the case $t=0$, we can drop the conditional expectation and deal with an unconditional expectation:

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}}\left[B_T^{-1}\left[S_T-K\right]^+\right] \\ = \quad & \mathbb{E}^{\mathbb{Q}}\left[B_T^{-1}S_T\mathbf{1}_{\{S_T>K\}}\right]-\mathbb{E}^{\mathbb{Q}}\left[B_T^{-1}K\mathbf{1}_{\{S_T>K\}}\right]\end{aligned}$$

Tackling the second expectation on the RHS,

$$
\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}\left[B_T^{-1} K \mathbf{1}_{\{S_T > K\}}\right] \\
= \quad & e^{-rT} K P^{\mathbb{Q}}\left[S_T > K\right] \\
= \quad & e^{-rT} K P^{\mathbb{Q}}\left[S_0 e^{\sigma X_T^{\mathbb{Q}} + (r - \frac{1}{2}\sigma^2)T} > K\right] \\
= \quad & e^{-rT} K P^{\mathbb{Q}}\left[\ln\left(\frac{S_0}{K}\right) + (r - \frac{1}{2}\sigma^2)T > -\sigma X_T^{\mathbb{Q}}\right] \\
= \quad & e^{-rT} K P^{\mathbb{Q}}\left[\frac{\ln\left(\frac{S_0}{K}\right) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} > \xi\right] \\
= \quad & e^{-rT} K N(d_2)
\end{aligned}
$$

where we have emphasized the fact that the probability P is taken with respect to the measure $\mathbb{Q}$ and have defined $\xi = -X_T^{\mathbb{Q}}/\sqrt{T}$, which is standard Normal random variable: $\xi \sim \mathcal{N}(0, 1)$.

As for the first expectation on the RHS,

$$
\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}\left[B_T^{-1} S_T \mathbf{1}_{\{S_T > K\}}\right] \\
= \; & \mathbb{E}^{\mathbb{Q}}\left[S_T^* \mathbf{1}_{\{S_T > K\}}\right] \\
= \; & \mathbb{E}^{\mathbb{Q}}\left[S_0 \exp\left\{\sigma X_T^{\mathbb{Q}} - \frac{1}{2}\sigma^2 T\right\} \mathbf{1}_{\{S_T > K\}}\right]
\end{aligned}
$$

Unless we can find a trick, it does not seem that this expectation can be computed analytically...

But, as happens frequently in mathematics, we can find a simple trick!

Aside: The Stochastic Exponential

Sometimes, in the literature, you will encounter the notation

$$\mathcal{E}\left(\int_0^t \theta_s dX_s\right)$$

This "curly E" denotes the **stochastic** (or **Doléans**) **exponential**, which is defined as

$$\mathcal{E}\left(\int_0^t \theta_s dX_s\right) = \exp\left(\int_0^t \theta_s dX_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right)$$

The Girsanov Theorem can also be defined in terms of the stochastic exponential.

Key Fact (Girsanov's Theorem in terms of stochastic exponential)

Given a process θ satisfying the condition

$$\mathbb{E}\left[\mathcal{E}\left(\int_0^T \theta_s dX_s\right)\right] = 1$$

we can define the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ equivalent to $\mathbb{P}$ through the Radon Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}\left(\int_0^t \theta_s dX_s\right), \quad t \in [0, T]$$

In this case, the process $X^{\mathbb{Q}}$ defined as

$$X_t^{\mathbb{Q}} = X_t - \int_0^t \theta(s) ds, \quad t \in [0, T] \tag{10}$$

is a standard Brownian Motion on $(\Omega, \mathcal{F}, \mathbb{Q})$.

Observe that condition (10) requires that the stochastic exponential is a martingale. It is therefore equivalent to the Novikov condition in our standard presentation of Girsanov.

Also, note the sign change between (2) and (10), which is due to the difference in sign of the stochastic integral in between the Novikov condition and the stochastic exponential.

End of aside and back to our problem...

Notice that the term

$$\exp\left\{\sigma X_T^{\mathbb{Q}} - \frac{1}{2}\sigma^2 T\right\} \tag{11}$$

inside the expectation looks reminiscent of the Doléans exponential we introduced in our second formulation of Girsanov's theorem:

$$\mathcal{E}\left(\int_0^t \theta_s dX_s\right) = \exp\left(\int_0^t \theta_s dX_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right)$$

In fact, we can reformulate (11) as

$$\exp\left\{\int_0^T \sigma dX_t^{\mathbb{Q}} - \frac{1}{2}\int_0^T \sigma^2 dt\right\}$$

and check that this is indeed the the Doléans exponential with $\theta = \sigma$!

This is an important observation, because it means that we can get rid of this bothersome term in our expectation by defining a new measure via Girsanov's theorem.

All we do do is check that

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left\{\sigma X_T^{\mathbb{Q}} - \frac{1}{2}\sigma^2 T\right\}\right] = 1$$

i.e. that (11) is an exponential martingale.

(check left to the reader since we have done something very similar at the end of Lecture 2.5)

We now define a new probability measure $\bar{\mathbb{Q}}$ via the Radon Nikodym derivative

$$\frac{d\bar{\mathbb{Q}}}{d\mathbb{Q}} = \exp\left\{\sigma X_T^{\mathbb{Q}} - \frac{1}{2}\sigma^2 T\right\}$$

Note that under the $\bar{\mathbb{Q}}$ measure,

$$X_t^{\bar{\mathbb{Q}}} = X_t^{\mathbb{Q}} - \sigma t, \qquad t \in [0, T]$$

is a Brownian motion and

$$S_T^* = S_0 \exp\left\{\sigma X_T^{\bar{\mathbb{Q}}} + \frac{1}{2}\sigma^2 T\right\} \tag{12}$$

Therefore,

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[S_T^* \mathbf{1}_{\{S_T>K\}}\right] &= \mathbb{E}^{\mathbb{Q}}\left[S_0 \exp\left\{\sigma X_T^{\mathbb{Q}} - \frac{1}{2}\sigma^2 T\right\} \mathbf{1}_{\{S_T>K\}}\right] \\
&= S_0 \int \frac{d\bar{\mathbb{Q}}}{d\mathbb{Q}} \mathbf{1}_{\{S_T>K\}} d\mathbb{Q} \\
&= S_0 P^{\bar{\mathbb{Q}}}\left[S_T > K\right] \\
&= S_0 P^{\bar{\mathbb{Q}}}\left[S_T^* > K B_T^{-1}\right] \\
&= S_0 P^{\bar{\mathbb{Q}}}\left[S_0 \exp\left\{\sigma X_T^{\bar{\mathbb{Q}}} + \frac{1}{2}\sigma^2 T\right\} > K e^{-rT}\right] \\
&= S_0 P^{\bar{\mathbb{Q}}}\left[\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right) T > -\sigma X_T^{\bar{\mathbb{Q}}}\right] \\
&= S_0 P^{\bar{\mathbb{Q}}}\left[\frac{\ln\left(\frac{S_0}{K}\right) + (r + \frac{1}{2}\sigma^2) T}{\sigma\sqrt{T}} > \xi\right] \\
&= S_0 N(d_1)
\end{aligned}
$$

where we have emphasized the fact that the probability P is taken with respect to the measure $\bar{\mathbb{Q}}$ and have defined $\xi = -X_T^{\mathbb{Q}}/\sqrt{T}$.

3.3. A few concluding notes on this section...

This derivation provides a wealth of probability related information on the derivative and on the relationship between the underlying price and the exercise price:

- $N(d_1)$ and $N(d_2)$ are probabilities based on a normal distribution;
- the normal distribution itself is a consequence of our assumption that the price of the underlying asset is lognormally distributed and hence that its log returns are normally distributed;
- the **probability under $\mathbb{Q}$ of exercising the option at maturity** is explicitly given: it is N(d2)!
- the **probability under $\bar{\mathbb{Q}}$ of exercising the option at maturity** is explicitly given: it is N(d1)!

While this derivation is quite logical overall, the most counter-intuitive aspect is certainly the fact that you do not need to know the specifics of the replicating strategy. One of the direct implications of this fact is that, since delta-hedging has not been considered and the Greeks have not been defined, the probabilistic approach presented here cannot help you with **local** risk management.

4. The Numéraire Pair

The **Fundamental Asset Pricing Formula** is much more general than what we have seen so far and there are some excellent reasons to call it "fundamental"...

We can extend the Fundamental Asset Pricing Formula very naturally to take advantage of the idea of a **numéraire pair** $(N_t, \mathbb{Q}^N)$.

Key Fact

The numéraire pair $(N_t, \mathbb{Q}^N)$ is comprised of:

- *a numéraire process N_t: any stochastic process $N_t > 0$ that can be viewed as a price can be used as a numéraire;*
- *an equivalent martingale measure $\mathbb{Q}^N$ under which any asset price discounted using the numéraire is a martingale.*

... and here is where the numéraire pair comes in handy:

Key Fact (Fundamental Asset Pricing Formula (Revisited))

The value at time t of a derivative maturing at time T is the expected value under the $\mathbb{Q}^N$ measure of the terminal cash flow of the contract, discounted using the numéraire asset.

$$\chi(t, S_t) \quad = \quad N_t \mathbb{E}^{\mathbb{Q}^N}\left[N_T^{-1} G(S_T) | \mathcal{F}_t\right], \quad t \in [0, T]$$

In fact we have already seen two numéraire pairs:

- $(B_t, \mathbb{Q})$: the risk-free asset and the risk-neutral measure;
- $(S_t, \bar{\mathbb{Q}})$: the stock and the auxiliary measure $\bar{\mathbb{Q}}$ we used to compute $N(d_1)$.

We can also price a derivative under the **real-world $\mathbb{P}$ measure**. In this case, the numéraire asset is the **log-optimal (or Kelly) portfolio**:

- Portfolio maximizing the log return on (or log utility of) wealth;
- In the Black-Scholes universe, the proportion of the log-optimal portfolio invested in the stock is $\frac{\mu - r}{\sigma^2}$;
- More on this portfolio in Module 6!

5. The Feynman-Kač Formula

During our investigation of the martingale methods, we derived the fundamental asset pricing equation (8).

Since in the Black-Scholes problem we have assumed that the interest rate is constant, we can rewrite this formula as:

$$\chi(t, S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[G(S_T)|\mathcal{F}_t\right], \quad t \in [0, T] \tag{13}$$

It turns out that this type of expectation has a PDE representation, thanks to the **Feynman-Kač formula**.

Key Fact (The Feynman-Kač Formula)

Assume that $V(t,s)$ solves the boundary value problem

$$\begin{aligned} \frac{\partial V}{\partial t}(t,s) + \mu(t,s)\frac{\partial V}{\partial s}(t,s) + \frac{1}{2}\sigma^2(t,s)\frac{\partial^2 V}{\partial s^2}(t,s) - rV(t,s) &= 0 \\ V(T,s) = G(s) & \end{aligned} \tag{14}$$

and that the process $S(t)$ follows the dynamics

$$dS_t = \mu(t,S_t)dt + \sigma(t,S_t)dX(t)$$

where $X(t)$ is a Brownian motion. Then, the function V has the representation

$$V(t,S_t) = e^{-r(T-t)}\mathbb{E}\left[G(S_T)|\mathcal{F}_t\right] \tag{15}$$

Application

In the Black-Scholes model, the option value under the risk-neutral measure can be expressed as the expectation:

$$V(t, S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[G(S_T)|\mathcal{F}_t\right], \quad t \in [0, T]$$

where S_t follows the dynamics:

$$dS_t = rS_t dt + \sigma S_t dX^{\mathbb{Q}}(t) \tag{16}$$

in which $X^{\mathbb{Q}}(t)$ is a Brownian motion under $\mathbb{Q}$.

By the **Feynman-Kač formula**, the value $V(t, S_t)$ of the option solves the boundary value problem

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV &= 0 \\ V(T, s) = G(s) \end{aligned}$$

which is the **Black-Scholes PDE**.

A Few Remarks...

1. **the Feynman-Kač formula works both ways:** we can represent a PDE of the form (14) as an expectation, and we can represent an expectation of the form (15). However, since historically Feynman-Kač was established to represent PDEs as expectation, we generally quote the formula as the representation of a PDE;
2. **the Feynman-Kač formula is independent from the measure:** Feynman-Kač works for any measure and does not imply any change of measure. Indeed, in the previous slide, we have used Feynman-Kač in the measure of the expectation, i.e the risk-neutral measure. **Tip:** make sure that you are using the "correct" dynamics for the process $S(t)$ (i.e. the dynamics under the same measure as the expectation), otherwise the $\frac{\partial V}{\partial s}$ coefficient in the PDE will be wrong!

A Few Remarks...

3. since we have not gone through the Δ-hedging argument of the PDE method, we do not know that $\frac{\partial V}{\partial s}$ represents the number of stocks to be held to hedge/replicate an option.

6. Extensions of the Basic Framework

In this section we consider three extensions to the basic Black-Scholes framework

1. Dividend paying stocks;
2. Stock price process with time-dependent parameters;
3. Valuations of options on futures (Black's formula);

5.1. Constant Dividends

If the underlying asset pays a constant dividend yield D, then its evolution becomes

$$dS_t = (\mu - D)S_t dt + \sigma S_t dX_t, \quad S_0 > 0$$

In this case, self-financing condition needs to be adjusted from

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi_u^S dS_u + \int_0^t \phi_u^B dB_u, \quad \forall t \in [0, T]$$

to

$$\begin{aligned} V_t(\phi) &= V_0(\phi) + \int_0^t \phi_u^S dS_u + \int_0^t D\phi_u^S S_u d_u + \int_0^t \phi_u^B dB_u, \\ & \forall t \in [0, T] \end{aligned}$$

in order to account for the dividend.

In this case, it is generally best to adjust the asset for the dividend paid. Define $\tilde{S}$ as dividend-adjusted asset dynamics, given by

$$\tilde{S}_t = S_t e^{Dt}$$

$\tilde{S}$ is the value of an investment in the stock if all dividends paid are automatically reinvested.

We can also define $\mu_D = \mu + D$ in order to write the evolution of $\tilde{S}$ as

$$d\tilde{S}_t = \mu_D \tilde{S}_t dt + \sigma \tilde{S}_t dX_t$$

The derivation we did for S now fully applies to $\tilde{S}$ and we quickly conclude that, since $\tilde{S}_t = e^{Dt} S_t$, the value of a European call on a dividend-paying asset is given by:

$$\begin{aligned}
\chi(t, S_t) &= \tilde{S}_t N(d_1) - E e^{-r(T-t)} N(d_2) \\
&= S_t e^{-D(T-t)} N(d_1) - E e^{-r(T-t)} N(d_2)
\end{aligned}$$

with d_1 and d_2 given by

$$\begin{aligned}
d_1 &= \frac{\ln\left(\frac{S_t}{E}\right) + \left(r - D + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\
d_2 &= \frac{\ln\left(\frac{S_t}{E}\right) + \left(r - D - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}
\end{aligned}$$

5.2. Time-Dependent Parameters

The Black-Scholes formula is not materially affected if instead of evolving according to a geometric Brownian motion, the underlying asset evolves according to a more general, time-dependent dynamics

$$dS_t = \mu(t, S_t)dt + \sigma(t)S_t dX_t, \quad S_0 > 0$$

and the risk-free rate is itself time-dependent, so that

$$dB_t = r(t)B(t)dt, \quad B_0 = 1$$

The price of the risk-free asset is now given by

$$B_t = e^{\int_0^t r(u)du}$$

However, the martingale measure $\mathbb{Q}$ is still unique, with the process θ used in the change of measure defined as

$$\theta = \frac{\frac{\mu(t,S_t)}{S_t} - r(t)}{\sigma(t)}$$

Note that, as expected, under the martingale measure $\mathbb{Q}$, the dynamics of S_t is given by

$$dS_t = r(t)S_t dt + \sigma(t)S_t dX_t^{\mathbb{Q}}, \quad S_0 > 0$$

In addition, the no-arbitrage pricing equation (7) is still valid, and as a consequence, the value of a derivative is given by:

$$\chi_t = e^{-\int_t^T r(u)du}\mathbb{E}\left[G(S_T)|\mathcal{F}_t\right]$$

Hence, the Black-Scholes formula that we obtained in the constant parameter case is also valid provided that we make two substitutions:

- replace $r(T-t)$ with $\int_t^T r(u)du$;
- replace σ^2 with $v^2 = \int_t^T \sigma^2(u)du$

Note that since the drift of the underlying asset $\mu(t, S_t)$ does not have any impact on the solution of the problem, we could actually choose any functional form we could think of to represent it.

5.3. Black's Model for Options on Futures

In 1976, Black published an option model in which the underlying asset is not traded spot (i.e. bought and sold for value today) but is a futures contract (i.e. a derivative bought and sold today for value at a later date).

What is particularly interesting for us is that by comparing the valuation of options on spot instruments and on futures or forwards, we can see the important role played by the Time Value of Money in holding together instruments and market across time.

This in turn, is an added motivation for looking at better ways to model interest rates instead of keeping them constant or simply time-dependent.

5.3.1. Futures Forwards and Forward Price

Forward contracts are OTC derivatives securities in which the long party has the obligation to buy an agreed upon quantity of an underlying asset (securities, commodities or others) at an agreed upon time and at an agreed upon price called the forward price. Forward contracts are symmetrical contracts. Therefore, the obligations of the short party mirror those of the long party. The contract is settled at maturity and typically no cash flow is exchanged in the meantime. As they are OTC derivatives, forward contracts are subject to counterparty risk.

Futures contracts are an exchange-traded type of forwards. Since they are traded on an exchange, futures are heavily standardized. Counterparty risk is mitigated by the exchange's clearinghouse. In particular the clearinghouse requires that exchange participants posts margin and to settle the their contracts daily. The combination of daily settlement and margins generates a stream of daily cash-flow through the life of the contract.

Forward contracts can be priced easily through the no-arbitrage condition. The time t forward price for a contract maturing at time T on an underlying S is given by:

$$F(t;T) = S_t e^{r(T-t)}$$

where r is the (constant) risk-free rate.

As a result of the cash flows generated by margin and settlement, futures and forward prices only coincides when interest rates are modelled as constant or time dependent, but not when interest rates are stochastic. However, although the forward and futures price do not often coincide, the forward price still plays a role, at least as an approximation, since it enables us to bridge the time gap between time t and maturity T.

For more information on pricing forwards and futures, please refer to Paul Wilmott's book.

5.3.2. Pricing Options on Futures

We will keep the same setting and assumptions as for the Black-Scholes model, including the hypothesis that the asset's dynamics is given by a geometric Brownian motion.

The dynamics of the time t futures price, f_t, evolves according to a Geometric Brownian Motion:

$$df_t = \mu_f f_t dt + \sigma_f f_t dX, \quad f(0) = f_0$$

where μ_f and σ_f represent respectively the drift and diffusion of the futures.

Since interest rates are deterministic, the futures price is equal to the forward price and we can write:

$$f_t = F(t; T) = S_t e^{r(T-t)} \tag{17}$$

and in particular

$$f_0 = S_0 e^{rT}$$

Recall that

$$dS_t = \mu S_t dt + \sigma S_t dX, \quad S(0) = S_0$$

Applying Itô to the relationship (17), we can now express the dynamics of f_t as

$$df_t = (\mu - r) f_t dt + \sigma f_t dX, \quad f(0) = S_0 e^{rT}$$

and thus we can see now that

$$\begin{aligned} \mu_f &= \mu - r \\ \sigma_f &= \sigma \end{aligned}$$

namely, the volatility of the futures is equal to the volatility of the spot and the drift of the futures is the discounted drift of the spot.

As a result we can also see clearly that the dynamics (SDE) for f_t is of the same form as the dynamics (SDE) for $\frac{S_t}{B_t} = S_t^*$. In fact, we even have

$$f(t) = \frac{S(t)}{B(t)} e^{rT} = S^*(t) e^{rT}$$

where T is fixed by the contract. This relationship reveals that the futures price is already a (naturally) discounted process. The immediate conclusion from this observation is that we will not need to consider the discounted futures price process. We can just go ahead with the futures price process as it stands.

We will now proceed as in the Black-Scholes model, first defining a self-financing futures strategy with equation

$$V_t(\phi) \quad = \quad V_0(\phi) + \int_0^t \phi_u^f df_u + \int_0^t \phi_u^B dB_u, \quad \forall t \in [0, T]$$

Please note that now, the trading strategy involves the futures and the risk-free asset. It does not involve the underlying asset S.

The martingale measure $\mathbb{Q}$ is still unique, with the process θ used in the change of measure defined as

$$\theta = \frac{\mu_f}{\sigma}$$

As expected, under the martingale measure $\mathbb{Q}$, the dynamics of f_t is given by

$$df_t \quad = \quad \sigma f_t dX_t^{\mathbb{Q}}, \quad S_0 > 0$$

In addition, the no-arbitrage pricing equation (7) is still valid, and as a consequence, the value of a derivative is given by:

$$\chi_t = e^{-r(T-t)}\mathbb{E}\left[G(S_T)|\mathcal{F}_t\right]$$

Solving this equation leads us to Black's formula for a European call on a futures:

$$\chi(t, f_t) \quad = \quad e^{-r(T-t)}\left[f_t N(d_1) - EN(d_2)\right]$$

with d_1 and d_2 given by

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{f_t}{E}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= \frac{\ln\left(\frac{f_t}{E}\right) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

In this lecture, we have seen...

- the probabilistic approach to solving the Black-Scholes problem;
- Girsanov's theorem and how to use it to change measure;
- what an equivalent martingale measure is;
- the derivation of the fundamental asset pricing formula;
- how to get the Black-Scholes fromula from the fundamental asset pricing formula;
- how to use the Feynman-Kǎc formula to go from the fundamental asset pricing formula to a the Black-Scholes PDE;
- extensions to the original Black-Scholes pricing problem: dividends; time-dependent coefficient and options of futures.