CQF Exercises 2.2 (with solutions)

dW is the usual increment of Brownian motion

1. Consider a two-factor model comprising a stochastic differential equation for the stock S and another for interest rate r, respectively

$$\begin{aligned} dS &= rSdt + \sigma S dW_t^{(1)},\ \sigma \in \mathbb{R} \\ dr &= u(r,t)dt + w(r,t)dW_t^{(2)}. \end{aligned} \tag{1.1}$$

Here both drift $u(r,t)$ and diffusion $w(r,t)$ are arbitrary functions. The two Brownian increments $dW_t^{(1)}$ and $dW_t^{(2)}$ have a constant correlation of ρ, such that

$$\mathbb{E}\left[dW_t^{(1)} dW_t^{(2)}\right] = \rho dt.$$

Consider setting up a delta-hedged portfolio

$$\Pi = V\left(S,r,t\right) - \Delta S - \Delta_1 V_1\left(S,r,t\right).$$

Derive the pricing partial differential equation for $V(S,r,t)$.

Solution:

$$\Pi = V - \Delta S - \Delta_1 V_1$$

The change in this portfolio in a time dt is given by

$$\begin{aligned} d\Pi &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma w S \frac{\partial^2 V}{\partial S \partial r} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2}\right) dt \\ & \quad -\Delta_1 \left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho\sigma w S \frac{\partial^2 V_1}{\partial S \partial r} + \frac{1}{2} w^2 \frac{\partial^2 V_1}{\partial r^2}\right) dt \\ & \quad + \left(\frac{\partial V}{\partial S} - \Delta_1 \frac{\partial V_1}{\partial S} - \Delta\right) dS + \left(\frac{\partial V}{\partial r} - \Delta_1 \frac{\partial V_1}{\partial r}\right) dr. \end{aligned}$$

To eliminate all randomness from the portfolio we must choose

$$\frac{\partial V}{\partial S} - \Delta_1 \frac{\partial V_1}{\partial S} - \Delta = 0,$$

to eliminate the dS terms, which are the sources of randomness, and

$$\frac{\partial V}{\partial \sigma} - \Delta_1 \frac{\partial V_1}{\partial r} = 0,$$

to get rid off dr terms. Therefore our choice of delta terms to make the portfolio risk free become

$$\Delta_1 = \frac{\frac{\partial V}{\partial r}}{\frac{\partial V_1}{\partial r}}$$

and

$$\Delta = \frac{\partial V}{\partial S} - \frac{\frac{\partial V}{\partial r}}{\frac{\partial V_1}{\partial r}} \frac{\partial V_1}{\partial S}.$$

This leaves us with

$$\begin{aligned}
d\Pi &= \left(\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+\rho\sigma wS\frac{\partial^2 V}{\partial S\partial\sigma}+\frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}\right)dt \\
&\quad -\Delta_1\left(\frac{\partial V_1}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V_1}{\partial S^2}+\rho\sigma wS\frac{\partial^2 V_1}{\partial S\partial r}+\frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2}\right)dt \\
&= r\Pi dt = r\left(V-\Delta S-\Delta_1 V_1\right)dt,
\end{aligned}$$

where we have used arbitrage arguments to set the return on the portfolio equal to the risk-free rate. Collecting all V terms on the left-hand side and all V_1 terms on the right-hand side we find that

$$\begin{aligned}
&\frac{\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+\rho\sigma wS\frac{\partial^2 V}{\partial S\partial r}+\frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}+rS\frac{\partial V}{\partial S}-rV}{\frac{\partial V}{\partial r}} \\
=\ &\frac{\frac{\partial V_1}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V_1}{\partial S^2}+\rho\sigma wS\frac{\partial^2 V_1}{\partial S\partial r}+\frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2}+rS\frac{\partial V_1}{\partial S}-rV_1}{\frac{\partial V_1}{\partial r}}
\end{aligned}$$

Therefore both sides can only be functions of the independent variables, S,σ and t. Thus we have

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+\rho\sigma Sq\frac{\partial^2 V}{\partial S\partial r}+\frac{1}{2}q^2\frac{\partial^2 V}{\partial r^2}+rS\frac{\partial V}{\partial S}-rV=-\left(u-\lambda w\right)\frac{\partial V}{\partial r},$$

for some function $\lambda\left(S,r,t\right)$. Reordering this equation, we usually write

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+\rho\sigma Sw\frac{\partial^2 V}{\partial S\partial r}+\frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}+rS\frac{\partial V}{\partial S}+\left(u-\lambda w\right)\frac{\partial V}{\partial r}-rV=0.$$

2. a. Suppose the spot interest rate r, which is a function of time t, satisfies the stochastic differential equation

$$dr = dW_t.$$

Using **this** model for the spot rate, by hedging one bond $V\left(r,t;T\right)$ of maturity T, with another of a different maturity, derive the bond pricing equation

$$\frac{\partial V}{\partial t}+\frac{1}{2}\frac{\partial^2 V}{\partial r^2}-\lambda\frac{\partial V}{\partial r}-rV=0, \tag{2.1}$$

where $\lambda=\lambda\left(r,t\right)$ is an arbitrary function.

Solution: Hedge one bond $V\left(r,t;T\right)$ with another bond $V_1\left(r,t;T_1\right)$

$$\Pi = V-\Delta V_1$$

Change in portfolio: $t\to t+dt$

$$d\Pi = dV-\Delta dV_1$$

using $dr^2 = dt$

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}\right) dt + \frac{\partial V}{\partial r} dr - \Delta\left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\frac{\partial^2 V_1}{\partial r^2}\right) dt - \Delta\frac{\partial V_1}{\partial r}$$

To eliminate risk set $\Delta = \frac{\partial V \diagup \partial r}{\partial V_1 \diagup \partial r}$.

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}\right) dt - \frac{\partial V \diagup \partial r}{\partial V_1 \diagup \partial r}\left(\frac{\partial V_1}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2}\right) dt$$

No arbitrage gives

$$\begin{aligned} d\Pi &= r\Pi dt = r\left(V - \frac{\partial V \diagup \partial r}{\partial V_1 \diagup \partial r}V_1\right) dt \\ &\Rightarrow \frac{1}{\partial V \diagup \partial r}\left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial r^2} - rV\right) = \frac{1}{\partial V_1 \diagup \partial r}\left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\frac{\partial^2 V_1}{\partial r^2} - rV_1\right) \\ &= \lambda(r,t) \end{aligned}$$

The LHS is a function of V. The RHS depends only on V_1. The function $\lambda(r,t)$ is a function of only two common variables common to V & $V_1 \Rightarrow$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial r^2} - \lambda(r,t)\frac{\partial V}{\partial r} - rV = 0.$$

b. By considering an unhedged bond and the risk free return, explain how and why λ arises in (2.1).
Solution: In a time step dt the (unhedged) bond changes in value by

$$dV = \frac{\partial V}{\partial r}dW + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}\right) dt. \qquad (1)$$

Rearranging the BPE (2.1)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial r^2} = \lambda\frac{\partial V}{\partial r} + rV. \qquad (2)$$

So substitute (2) in to (1) to give

$$dV = \frac{\partial V}{\partial r}dW + \left(\lambda\frac{\partial V}{\partial r} + rV\right) dt,$$

or

$$dV - rVdt = \frac{\partial V}{\partial r}(dW + \lambda dt). \qquad (3)$$

We can interpret (3) financially as the difference between dV - return on an unhedged bond (contains risk) and $rVdt$ - risk free return, i.e. the money invested from selling a bond which earns at the risk free rate, i.e. $\frac{dV}{dt} = rV$. Since the coefficient of $dW \neq 0$, the portfolio is not riskless and the term λdt is the extra return on the portfolio per unit of risk dW. It is the amount by which the market wishes to be compensated, for hedging with a non-tradeable asset.

c. Assuming that λ is a function of t only and a **zero coupon bond** is to be priced by solving (2.1). Find a solution of the form

$$V(r, T; T) = \exp(A(t; T) + rB(t; T)),$$

with redemption value

$$V(r, T; T) = 1$$

where both $A(t; T)$ and $B(t; T)$ should be given.
Solution:

$$\begin{aligned}
\frac{\partial}{\partial t}\left(e^{A(t)+rB(t)}\right) &= \left(\dot{A} + r\dot{B}\right) V; \ \frac{\partial}{\partial r}\left(e^{A+rB}\right) = BV; \\
\frac{\partial^2}{\partial r^2}\left(e^{A(t)-rB(t)}\right) &= B^2 V.
\end{aligned}$$

$$\dot{A} + r\dot{B} + \frac{1}{2}B^2 - \lambda B - r = 0$$

$$\dot{B}(t) = 1 : B(t; T) = t - T$$

$$\dot{A} + \frac{1}{2}B^2 - \lambda B = 0 : \dot{A} = \lambda(t - T) - \tfrac{1}{2}(t - T)^2$$

$$A(t; T) = -\int_t^T \lambda(s)(s - T)\, ds - \frac{(t - T)^3}{6}$$

$$V(r, t; T) = \exp\left(-\int_t^T \lambda(s)(s - T)\, ds - \tfrac{1}{6}(t - T)^3 + r(t - T)\right)$$

3. What final condition (payoff) should be applied to the bond pricing equation for a swap, cap, floor, zero-coupon bond and a bond option?
Solution:
Final condition for a swap:

$$V(r, T) = (r - r_s) P,$$

where r_s is the fixed rate and P is the principal.
Final condition for a cap:

$$V(r, T) = \max(r - r_c, 0) P,$$

where r_c is the cap rate and P is the principal.
Final condition for a floor:

$$V(r, T) = \max(r_f - r, 0) P,$$

where r_f is the floor rate and P is the principal.
Final condition for a zero-coupon bond:

$$V(r,T) = P,$$

where P is the principal.
Final condition for a coupon bond:

$$V(r,T) = (1+c)\,P,$$

where c is the (discrete) coupon rate and P is the principal.
Final condition for a bond option:

$$V(r,T) = \max\left(Z(r,T) - E, 0\right),$$

where E is the exercise price and $Z(r,t)$ is the value of the underlying bond at time t .

4. Consider the spot rate r, which evolves according to the SDE

$$dr = u(r,t)\,dt + w(r,t)\,dX \ ,$$

The extended Hull and White model has drift and diffusion

$$u(r,t) = \eta(t) - \gamma r, \qquad w(r,t) = c,$$

in turn, where $\eta(t)$ is an arbitrary function of time t and γ and c are constants. Deduce that the value of a zero coupon bond, $Z(r,t\,;T)$ which has

$$Z(r,T\,;T) = 1$$

in the extended Hull and White model is given by

$$Z(r,t\,;T) = \exp\left(A(t\,;T) - rB(t\,;T)\right),$$

where

$$B(t\,;T) = \frac{1}{\gamma}\left(1 - e^{-\gamma(T-t)}\right)$$

and

$$A(t;T) = -\int_t^T \eta(\tau)\,B(\tau;T)\,d\tau + \frac{c^2}{2\gamma^2}\left((T-t) + \frac{2}{\gamma}e^{-\gamma(T-t)} - \frac{1}{2\gamma}e^{-2\gamma(T-t)} - \frac{3}{2\gamma}\right).$$

Solution:
You may assume the pricing equation for Vasicek is

$$\frac{\partial Z}{\partial t} + \frac{1}{2}c^2\frac{\partial^2 Z}{\partial r^2} + (\eta(t) - \gamma r)\frac{\partial Z}{\partial r} - rZ = 0 \ , \ Z(r,T\,;T) = 1$$

If $Z(r,t) = \exp(A(t\,;T) - rB(t\,;T))$ then $Z_r = -BV$ $Z_{rr} = B^2 Z$. Solution with $Z(r,T\,;\,T) = 1$ implies $B(T;T) = A(T;T) = 0$ $Z_t = (A'(t) - rB'(t))V$ so subst. these in to the PDE above

$$\begin{aligned}
A'(t\,;T) - rB'(t\,;T) + \tfrac{1}{2}c^2B^2 - (\eta(t) - \gamma r)B - r &= 0 \quad \forall r \\
\left(A' + \tfrac{1}{2}c^2B^2 - \eta(t)B\right) - r(B' - \gamma B + 1) &= 0 \\
\Rightarrow A' = -\tfrac{1}{2}c^2B^2 + \eta(t)B, \qquad\qquad B' = \gamma B - 1
\end{aligned}$$

From $Z(r,t\,;T) = \exp(A(t) - rB(t))$ we note that as $r \to \infty$, $Z \to 0$. Solving for $B(t\,;T)$

$$\frac{dB}{dt} = \gamma B - 1$$

You can solve this by the variable sep. method:

$$\frac{dB}{\gamma B - 1} = dt$$

Now recall:

$$\int \frac{dx}{ax+1} = \frac{1}{a}\ln|ax+1| + K$$

therefore we have

$$\begin{aligned}
\int_t^T \frac{dB}{\gamma B - 1} &= \int_t^T d\tau = \frac{1}{\gamma}\ln|\gamma B(\tau\,;T) - 1|_t^T = (T-t) \\
\ln\left|\frac{\gamma B(T\,;T) - 1}{\gamma B(t\,;T) - 1}\right| &= \gamma(T-t)
\end{aligned}$$

we know $B(T\,;T) = 0$ so

$$\begin{aligned}
\ln\left|\frac{\gamma B(t\,;T) - 1}{-1}\right|^{-1} &= \gamma(T-t) = -\ln|1 - \gamma B(t\,;T)| = \gamma(T-t) \\
1 - \gamma B(t\,;T) &= \exp[-\gamma(T-t)] = \gamma B(t\,;T) = 1 - \exp[-\gamma(T-t)] \\
B(t\,;T) &= \frac{1}{\gamma}(1 - \exp[-\gamma(T-t)])
\end{aligned}$$

Then

$$A(t\,;T) = \tfrac{1}{2}c^2\int_t^T B^2(\tau\,;T)\,d\tau - \int_t^T B(\tau\,;T)\,\eta(\tau)\,d\tau$$

and

$$\begin{aligned}
\int_t^T B^2(\tau\,;T)\,d\tau &= \frac{1}{\gamma^2}\int_t^T \left(1 - 2e^{-\gamma(T-\tau)} + e^{-2\gamma(T-\tau)}\right)d\tau \\
&= \frac{1}{\gamma^2}\left((T-t) - \frac{2}{\gamma}e^{-\gamma(T-\tau)}\Big|_t^T + \frac{1}{2\gamma}e^{-2\gamma(T-\tau)}\Big|_t^T\right) \\
&= \frac{1}{\gamma^2}\left((T-t) - \frac{2}{\gamma} + \frac{2}{\gamma}e^{-\gamma(T-t)} + \frac{1}{2\gamma} - \frac{1}{2\gamma}e^{-2\gamma(T-t)}\right) \\
&= \frac{1}{\gamma^2}\left((T-t) - \frac{3}{2\gamma} + \frac{2}{\gamma}e^{-\gamma(T-t)} - \frac{1}{2\gamma}e^{-2\gamma(T-t)}\right)
\end{aligned}$$

Hence

$$A\left(t\ ;T\right)=-\int_{t}^{T}B\left(\tau\ ;T\right)\eta\left(\tau\right)d\tau+\frac{c^{2}}{2\gamma^{2}}\left(\left(T-t\right)-\frac{3}{2\gamma}+\frac{2}{\gamma}e^{-\gamma\left(T-t\right)}-\frac{1}{2\gamma}e^{-2\gamma\left(T-t\right)}\right)$$