Stability Analysis

stock time T−

By expressing $S = n\delta S$ and $t = m\delta t$, we will obtain a difference equation for the Black-Scholes equation.

T−

$$V(S,t) = V(n\delta S, m\delta t) = V_n^m.$$

V_i^k

$\delta S = \dfrac{S^*}{N}$ where $S^* \gg E$ is a suitably large value of S ;
$\delta t = \dfrac{T}{M}$. Taking N and M steps for S and t respectively, so

$$\begin{aligned} S &= n\delta S \qquad 0 \le n \le N \\ t &= m\delta t \qquad 0 \le m \le M. \end{aligned}$$

$$\frac{V_n^m - V_n^{m+1}}{\delta t} + \frac{1}{2}n^2\sigma^2\left(V_{n-1}^m - 2V_n^m + V_{n+1}^m\right) + \frac{1}{2}(r-D)\,n\left(V_{n+1}^m - V_{n-1}^m\right) - rV_n^m = 0$$

and rearrange to obtain a *forward marching* scheme in time

$$\begin{aligned} \boxed{V_n^{m+1}} &= V_n^m + \delta t\left(\frac{1}{2}n^2\sigma^2\left(V_{n-1}^m - 2V_n^m + V_{n+1}^m\right)\right) \\ &\quad + \delta t\left(\frac{1}{2}(r-D)\,n\left(V_{n+1}^m - V_{n-1}^m\right) - rV_n^m\right) \\ &\equiv F\left(V_{n-1}^m,\ V_n^m,\ V_{n+1}^m\right) \end{aligned}$$

V_{n-1}^m : V_n^m V_{n+1}^m

Now for the RHS collect coefficients of each variable term V, to get

known

$$V_n^{m+1} = \alpha_n V_{n-1}^m + \beta_n V_n^m + \gamma_n V_{n+1}^m \qquad (1)$$

where

$\frac{\partial V}{\partial S}$

$$\begin{aligned} \alpha_n &= \frac{1}{2}\left(n^2\sigma^2 - n\,(r-D)\right)\delta t, \\ \beta_n &= 1-\left(r+n^2\sigma^2\right)\delta t, \\ \gamma_n &= \frac{1}{2}\left(n^2\sigma^2 + n\,(r-D)\right)\delta t \end{aligned} \qquad (2)$$

$\frac{\partial^2 V}{\partial S^2}$

(1) is B.S.E in FD form

(2)

Fourier Stability (Von Neumann's) Method

A method is called step-wise unstable if for a fixed grid (i.e. $\delta t,\ \delta S$ constant) there exists an initial perturbation which "blows up" as $t \longrightarrow \infty$, i.e. as we march in time. Here in a forward marching scheme. The question we wish to answer is "do small errors propagate along the grid and grow exponentially?".

We hope NOT.!!

Assume an initial disturbance which is proportional to $\exp(in\omega)$. We therefore study the propagation of perturbations created at any given point in time.

$i = \sqrt{-1}$

Continuing

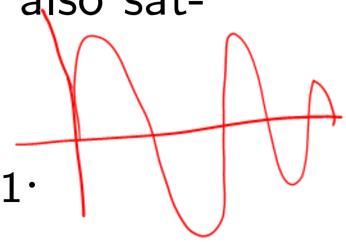
error $\propto e^{in\omega}$

If $\widehat{V}_n^m$ is an approximation to the exact solution ${V_n}^m$ then

approx exact error

$$\widehat{V}_n^m = {V_n}^m + {E_n}^m$$

where ${E_n}^m$ is the associated error. Then ${E_n}^m$ also satisfies the difference equation (4) to give



$$E_n^{m+1} = \alpha_n E_{n-1}^m + \beta_n E_n^m + \gamma_n E_{n+1}^m.$$

Put

amplitude

$$E_n^m = \overline{a}^m \exp(in\omega) \tag{3}$$

which is oscillatory of amplitude $\overline{a}$ and frequency ω. Substituting (3) into (1) gives

$$\overline{a}^{\cancel{m}+1} e^{(in\omega)} = \alpha_n \overline{a}^{\cancel{m}} e^{i(n-1)\omega} + \beta_n \overline{a}^{\cancel{m}} e^{in\omega} + \gamma_n \overline{a}^{\cancel{m}} e^{i(n+1)\omega}$$

which becomes (after cancellation)

$$\overline{a} = \alpha_n e^{-i\omega} + \beta_n + \gamma_n e^{i\omega}.$$

NOTE: Now stability criteria arises from the balancing of the time dependency and diffusion terms, so that

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = 0$$

$\Theta \sim -\frac{1}{2}\sigma^2 S^2 \Gamma$

From (2) we take the following contributions

$$\alpha_n = \frac{1}{2} n^2 \sigma^2 \delta t, \;\; \beta_n = 1 - n^2 \sigma^2 \delta t, \;\; \gamma_n = \frac{1}{2} n^2 \sigma^2 \delta t$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\begin{aligned}
\overline{a} &= \frac{1}{2} n^2\sigma^2\delta t \left(e^{i\omega} + e^{-i\omega}\right) + 1 - n^2\sigma^2\delta t \\
\overline{a} &= n^2\sigma^2\delta t\,(\cos\omega - 1) + 1.
\end{aligned}$$

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

we have

$$\boxed{\overline{a} = 1 - 2n^2\sigma^2 \sin^2\frac{\omega}{2}\delta t}$$

For stability $\overline{a}$ must be bounded, i.e. $|\overline{a}| < 1$

$$\left|1 - 2n^2\sigma^2 \sin^2\frac{\omega}{2}\delta t\right| < 1$$

$n = N$

which upon simplifying we find is

$$\delta t < \frac{1}{\sigma^2 N^2} \tag{4}$$

$\delta t \sim O\left(\frac{1}{N^2}\right)$

stability condition.

so $\delta t \sim O\left(N^{-2}\right)$.

$$-1 < 1 - 2n^2\sigma^2\delta t < 1$$

$$-2 < -2N^2\sigma^2\delta t < 0$$

$$x = \frac{\omega}{2} \qquad \cos\omega = 1 - 2\sin^2\frac{\omega}{2}$$

$$\cos\omega - 1 = -2\sin^2\frac{\omega}{2}$$

$$1 > N^2\sigma^2\delta t > 0$$