Fixed Income and Credit – Lecture 4

Martingales and Fixed Income Valuation

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Martingales and Fixed Income Valuation

Introduction

Introduction to martingale and fixed income!

Financial mathematics useful.

Short-term rates and risk free zero-coupon bonds do not exists in practice.

The lecture is useful for financial mathematics modelling, presents the building block of the technical tool, but is not a faithful description of the actual market.

This lecture

Apply probabilistic and martingale methods to the pricing of bonds and interest rate derivatives using short-term rate and bond models. We will see:

- the pricing of interest rate products in a probabilistic setting; the equivalent martingale measures;
- the fundamental asset pricing formula for bonds;
- the dynamics of bond prices;
- the forward measure;
- the fundamental asset pricing formula for derivatives on bonds (interest rate);

We work in a single currency setting.

Martingales and Fixed Income Valuation

A Model for the Short-Term Rate

In the world of equity, a single class of models (the Geometric Brownian Motion and its extensions) dominates the landscape. The situation is different in the world of interest rates, where several classes of models coexist:

1. Single factor short-term rate models, such as the Merton, Vasicek, Cox-Ingersoll-Ross, Ho-Lee or Hull-White models;
2. Multifactor models such as the Brennan-Schwartz, Longstaff-Schwartz and Fong-Vasicek, G2++ (Hull-White 2-factor) models;
3. Forward instantaneous rate framework such as the Heath-Jarrow-Morton model
4. Forward Market Model like Brace, Gatarek and Musiela;

Modelled quantities can be (theoretical instantaneous) short-term rate, (theoretical instantaneous) forward rates, discount factors or market rates.

A Model for the Short-Term Rate

Underlying probability space: $(\Omega, \mathcal{F}, \mathbb{P})$. The filtration $\mathcal{F}$ is the $\mathbb{P}$-completed version of the filtration generated by the underlying d-dimensional standard Brownian motion X. Time interval: $[0, \bar{T}]$.

This lecture starting point: short-term rate models whose dynamics under the physical measure $\mathbb{P}$ is of the form

$$dr(t) = \mu(t, r(t))\, dt + \sigma(t, r(t)) \cdot dX(t)$$

where $r(t)$ is the short-term rate at time t and $X(t)$ is a Brownian motion.

To simplify the notation, we will write this equation as

$$dr(t) = \mu_t dt + \sigma_t \cdot dX(t)$$

This specification is general enough to cover both

1. Equilibrium models such as the Vasicek or Cox-Ingersoll-Ross models;
2. No arbitrage models such as Ho-Lee and Hull-White;

A Model for the Short-Term Rate

Equipped with a short-term interest rate model, we define a cash account, *money market account* or *money-in-the-bank* process $A(t)$ as

$$A(t) = \exp\left(\int_0^t r(s)ds\right).$$

The starting value is $A(0) = 1$ and the account grows at the instantaneous risk free rate r:

$$dA(t) = r(t)A(t)dt.$$

Martingales and Fixed Income Valuation

The Zero-Coupon Bond Market

A zero-coupon bond is a bond which pays 1 at maturity time T; we denote its value $B(t, T)$. We define the zero-coupon bond market as the set of all the zero-coupon bonds $B(t, T)$ for all $t \leq T \leq \bar{T}$.

Risk-less zero-coupon bonds do not exists in practice. This is an idealised quantity that is useful for intuition but need to be adapted to the market, e.g. OIS (Overnight Indexed Swaps) discounting.

We cannot expect to find a zero-coupon bond maturing for all times $t \leq T \leq \bar{T}$. Only a discrete number of maturities exists, at most one for each good business day T.

This representation is useful for financial mathematics modelling, as the building block of the technical tool, but not as a faithful description of the market.

Arbitrage-free family of bonds

Definition (Arbitrage-free family of bonds)

A family $B(t, T)$ $(t \leq T \leq \bar{T})$ of adapted processes is called an arbitrage-free family of bond prices relative to short-term interest rate process r if

1. $B(T, T) = 1$ *for every* $T \in [0, \bar{T}]$.
2. *there exists a probability measure* $\mathbb{Q}$ *on* $(\Omega, \mathcal{F}_{\bar{T}})$ *equivalent to* $\mathbb{P}$ *and such that for any maturity* $T \in [0, \bar{T}]$ *the relative bond price*

$$Z(t, T) = \frac{B(t, T)}{A(t)}$$

is a martingale under $\mathbb{Q}$.

Any such probability measure $\mathbb{Q}$ is called a (spot) martingale measure for the family $B(t, T)$ relative to r.

Equivalent Measure or Equivalent Measures?

The definition is consistent with the way we defined the equivalent martingale measure for a stock. But the key difference in the zero-coupon bond case is that the equivalent martingale measure does not apply to a single security, but to a continuum of securities.

As shown in a previous lecture, the existence of (at least) one equivalent martingale measure implies an absence of arbitrage opportunity.

Any probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ can be obtained by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}_{\bar{T}}\left(\int_0^\cdot -\theta_s \cdot dX_s\right), \quad \mathbb{P} \text{ a.s.}$$

for some predictable $\mathbb{R}^d$-valued process θ.

Equivalent Martingale Measure and No-Arbitrage

The process $\theta(t)$ satisfies the Novikov condition

$$\mathrm{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^{\bar{T}} |\theta(s)|^2\, ds\right)\right] < \infty$$

We suppose the existence of an equivalent martingale measure $\mathbb{Q}$. The conditional expectation of the derivative is

$$\frac{d\mathbb{Q}}{d\mathbb{P}}_{|\mathcal{F}_t} = \eta(t) = \exp\left(-\int_0^t \theta(s)\cdot dX(s) - \frac{1}{2}\int_0^t |\theta(s)|^2 ds\right).$$

Note that η satisfies

$$d\eta(t) = -\eta(t)\,\theta(t)\cdot dX(t).$$

Equivalent Martingale Measure and No-Arbitrage

By Girsanov's theorem, the process X^θ defined as

$$X^\theta(t) = X(t) + \int_0^t \theta(s)\, ds, \quad t \in [0, \bar{T}]$$

is a standard Brownian Motion on $(\Omega, \mathcal{F}, \mathbb{Q})$.

As a result, the dynamics of $r(t)$ under $\mathbb{Q}$ is

$$dr(t) = (\mu_t - \sigma_t \cdot \theta(t))\, dt + \sigma_t \cdot dX^\theta(t).$$

Equivalent Martingale Measure and No-Arbitrage

Applying the martingale property of the rescaled bond, we get

$$B(t, T) = A(t)\, \mathrm{E}^{\mathbb{Q}} \left[\left. \frac{B(T, T)}{A(T)} \right| \mathcal{F}_t \right].$$

Since $B(T, T) = 1$ and $A(T) = \exp\left(\int_0^T r(s)\, ds\right)$ then the formula simplifies to

$$B(t, T) = \mathrm{E}^{\mathbb{Q}} \left[\exp\left(\left. - \int_t^T r(s)\, ds \right) \right| \mathcal{F}_t \right].$$

Note that $B(t, T) > 0$ for all $t \leq T \leq \bar{T}$.

Bond Pricing in Practice: Analytical Solutions

Some form of analytical solutions exist for the most popular models such as the Vacicek model or the CIR model. One way to derive them is to apply Feynman-Kac to the fundamental asset pricing formula to obtain a PDE, and then solve that PDE.

$$\frac{\partial B}{\partial t}(t,r) + (\mu_t - \sigma_t\theta(t))\frac{\partial B}{\partial r}(t,r) + \frac{1}{2}\sigma_t^2\frac{\partial^2 B}{\partial^2 r}(t,r) - r(t)B(t,r) = 0 \quad B(T,r) = 1$$

This lecture is related to martingale approach.

Bond Pricing in Practice: Numerical Solutions

Given a short-rate model with dynamics

$$dr(t) = (\mu_t - \sigma_t \cdot \theta(t))\, dt + \sigma_t \cdot dX^{\theta}(t).$$

under the equivalent martingale measure $\mathbb{Q}$ and the fundamental asset pricing formula applied to zero-coupon bonds

$$B(t, T) = \mathrm{E}^{\mathbb{Q}} \left[\exp\left(- \int_t^T r(s)\, ds \right) \middle| \mathcal{F}_t \right].$$

it is not difficult to obtain a numerical solution using Monte Carlo methods.

Bond Pricing in Practice: Numerical Solutions

A Monte-Carlo algorithm on the short-term rate (dimension 1) looks like

- $r(0) = r$
- Simulations $j = 1, \ldots, N$
  - Time steps t_i, $i = 1, \ldots M$
    - Simulate $N(0,1)$ random variable $Z_{j,i}$
    - Simulate short rate

$$r_j(t_i) = r_j(t_{i-1}) + \left(\mu(t_{i-1}, r_j(t_{i-1})) - \sigma(t_{i-1}, r_j(t_{i-1}))\theta(t_{i-1})\right)(t_i - t_{i-1}) + \sigma(t_{i-1}, r_j(t_{i-1}))Z_{j,i}\sqrt{t_i - t_{i-1}}$$

    - Compute integral $\int_0^T r(s)ds$: $\text{Int} = \text{Int} + 0.5 * (r_j(t_i) + r_j(t_{i-1}))(t_i - t_{i-1})$.
  - Zero-bond under simulation $B_j = \exp(-\text{Int})$
- Value of the bond:

$$B = \frac{1}{N}\sum_{j=1}^{N} B_j.$$

Monte Carlo on r not most efficient numerically. Do it on B! (see end of section on *Pricing bond derivatives*)

Martingales and Fixed Income Valuation

Dynamics of the Zero-Coupon Bond Price

We will consider the pricing of derivatives struck on a zero-coupon bond. To price such derivatives, we will need to understand not only the dynamics of the short-term rate but also the dynamics of the zero-coupon bond prices.

Our starting point is to look at the process

$$M(t) = Z(t, T)\eta(t) = \frac{B(t, T)}{A(t)}\eta(t)$$

where $\eta(t)$ is the Radon Nikodym derivative.

The reason for considering this process M(t) is that:

If a process $Y(t)$ is a martingale under $\mathbb{Q}$ and $\eta(t) = \frac{d\mathbb{Q}}{d\mathbb{P}}_{|\mathcal{F}_t}$ then the process $M(t) = Y(t)\eta(t)$ is a martingale under $\mathbb{P}$.

Dynamics of the Zero-Coupon Bond Price

Since $Z(t,T)$ is a martingale under $\mathbb{Q}$ (by definition of $\mathbb{Q}$), $M(t) = Z(t,T)\eta(t)$ is a martingale under $\mathbb{P}$. By the *Martingale Representation Theorem*, there exists a stochastic process γ such that $M(t)$ can be represented as

$$\begin{aligned} M(t) &= M(0) + \int_0^t \gamma_s \cdot dX(s) \\ &= Z(0,T) + \int_0^t \gamma_s \cdot dX(s) \end{aligned}$$

or equivalently

$$dM(t) = \gamma_t \cdot dX(t).$$

We express $Z(t,T)$ in term of $M(t)$ as $Z(t,T) = M(t)\eta^{-1}(t)$ and

$$dZ(t,T) = d(M(t)\eta^{-1}(t))$$

Dynamics of the Zero-Coupon Bond Price

The differential of η is

$$d\eta(t) = -\eta(t)\,\theta(t) \cdot dX_t = -\eta(t)\,\theta(t) \cdot (dX^\theta(t) - \theta(t)dt)$$

and thus

$$d\eta^{-1}(t) = \eta^{-1}(t)\left(\theta(t) \cdot dX_t + |\theta(t)|^2 dt\right) = \eta^{-1}(t)\left(\theta(t) \cdot dX^\theta(t)\right)$$

Using the Itô product rule we have

$$\begin{aligned}
dZ(t,T) &= d(M(t)\eta^{-1}(t)) = dM(t)\eta^{-1}(t) + M(t)d\eta^{-1}(t) + \eta^{-1}(t)\gamma(t) \cdot \theta(t)dt \\
&= \eta^{-1}(t)\gamma(t) \cdot (dX^\theta(t) - \theta(t)dt) + M(t)\eta^{-1}(t)\theta(t) \cdot dX^\theta(t) + \eta^{-1}(t)\gamma(t) \cdot \theta(t)dt \\
&= \eta^{-1}(t)(\gamma(t) + M(t)\theta(t)) \cdot dX^\theta(t) = (\eta^{-1}(t)\gamma(t) + Z(t,T)\theta(t)) \cdot dX^\theta(t)
\end{aligned}$$

Dynamics of the Zero-Coupon Bond Price

Recalling that $Z(t,T) = B(t,T)/A(t)$, we get the dynamic of $B(t,T)$ under $\mathbb{Q}$

$$\begin{aligned} dB(t,T) &= d(Z(t,T)A(t)) \\ &= Z(t,T)dA(t) + dZ(t,T)A(t) \\ &= r(t)Z(t,T)A(t)dt + A(t)(\eta^{-1}(t)\gamma(t) + Z(t,T)\theta(t)) \cdot dX^\theta(t). \end{aligned}$$

So

$$\begin{aligned} dB(t,T) &= r(t)B(t,T)dt + (\eta^{-1}(t)A(t)\gamma(t) + B(t,T)\theta(t)) \cdot dX^\theta(t) \\ &= r(t)B(t,T)dt + \left(B(t,T)\frac{\gamma(t)}{Z(t,T)\eta(t)} + B(t,T)\theta(t)\right) \cdot dX^\theta(t) \\ &= r(t)B(t,T)dt + B(t,T)\left(\frac{\gamma(t)}{M(t)} + \theta(t)\right) \cdot dX^\theta(t). \end{aligned}$$

Dynamics of the Zero-Coupon Bond Price

$$
\begin{aligned}
dB(t,T) &= r(t)B(t,T)dt + b^{\theta}(t,T)B(t,T)\cdot dX^{\theta}(t) \\
&= B(t,T)\left(r(t)dt + b^{\theta}(t,T)\cdot dX^{\theta}(t)\right)
\end{aligned}
$$

where

$$
b^{\theta}(t,T) = \frac{\gamma(t)}{M(t)} + \theta(t).
$$

The process $b^{\theta}(t,T)$ is called the *volatility* of the zero-coupon bond of maturity T.

$$
d\left(\frac{B(t,T)}{A(t)}\right) = \frac{B(t,T)}{A(t)} b^{\theta}(t,T)\cdot dX^{\theta}(t)
$$

Dynamics of the Zero-Coupon Bond Price

The dynamics of a zero-coupon bond $B(t, T)$ under the equivalent martingale measure $\mathbb{Q}$ is given by

$$\frac{dB(t,T)}{B(t,T)} = r(t)dt + b^\theta(t,T) \cdot dX^\theta(t) \quad B(T,T) = 1$$

where b^θ is the volatility of the zero-coupon bond. Therefore

$$\begin{aligned} \frac{B(t,T)}{A(t)} &= B(0,T) \exp\left(\int_0^t b^\theta(s,T) \cdot dX^\theta(s) - \frac{1}{2} \int_0^t |b^\theta(s,T)|^2 ds \right) \\ &= B(0,T) \mathcal{E}_t \left(\int_0^{\cdot} b^\theta(s,T) \cdot dX^\theta(s) \right). \end{aligned}$$

An interest rate models family will describe the form of b.

Martingales and Fixed Income Valuation

Market price of risk

Under an equivalent martingale measure $\mathbb{Q}$, the dynamics of a zero-coupon bond is given by

$$\frac{dB(t,T)}{B(t,T)} = r(t)dt + b^{\theta}(t,T) \cdot dX^{\theta}(t) \quad B(T,T) = 1.$$

The dynamic under the original probability measure $\mathbb{P}$ is

$$\frac{dB(t,T)}{B(t,T)} = \left(r(t) + b^{\theta}(t,T) \cdot \theta(t)\right) dt + b^{\theta}(t,T) \cdot dX(t) \quad B(T,T) = 1.$$

Market price of risk

The process $\theta(t)$ through which we defined the relation between the physical/original measure $\mathbb{P}$ and the equivalent martingale measure $\mathbb{Q}$ has an economic meaning: it is the risk premium or market price of interest rate risk.

The market price of risk represents the compensation paid by the market to an investor per unit of risk. In our framework, the risk is represented by the volatility of the zero-coupon bond, $b^\theta(t, T)$, and the total compensation for risk that an investor requires is equal to $b^\theta(t, T) \cdot \theta(t)$.

This observation is consistent with the equity case. In the stock option case, the market price of risk was the process $\theta = (\mu - r)/\sigma$.

The relationship between the physical measure $\mathbb{P}$ and an equivalent martingale $\mathbb{Q}$ measure is established by the market price of risk which acts as the change of measure process. In complete markets the equivalent martingale measure is unique and so is the market price of risk. In incomplete markets, we may have several equivalent martingale measures, each with its own market price of risk.

Martingales and Fixed Income Valuation

Pricing bond derivatives

Fixed income markets present a wide diversity of instruments: bonds, of course, but also forwards, futures, options, caps and floors, numerous swaps and swaptions, structure notes... and this is without mentioning the interconnection between fixed income markets and credit market or between fixed income products and inflation-linked products.

The pricing zero-coupon bond is simply the starting point of any attempt to price fixed income products. In this section, we start to expand our horizons by considering the pricing of derivatives on zero-coupon bonds.

Pricing bond derivatives

In this section, we denote by $X^{\mathbb{Q}}(t)$ a $\mathbb{Q}$-standard Brownian motion and express the bond price dynamics as

$$\frac{dB(t,T)}{B(t,T)} = r(t)dt + b(t,T)\cdot dX^{\mathbb{Q}}(t) \quad B(T,T) = 1.$$

This means

$$\begin{aligned}\frac{B(t,T)}{A(t)} &= B(0,T)\exp\left(\int_0^t b(s,T)\cdot dX^{\mathbb{Q}}(s) - \frac{1}{2}\int_0^t |b(s,T)|^2 ds\right) \\ &= B(0,T)\,\mathcal{E}_t\left(\int_0^{\cdot} b(s,T)\cdot dX^{\mathbb{Q}}(s)\right)\end{aligned}$$

Applying the Fundamental Asset Pricing Formula

The fundamental asset pricing formula tells us that the time t price of a contingent claim paying some (random) amount Y at time T is given by

$$\pi_t(Y) = A(t)\,\mathrm{E}^{\mathbb{Q}}\left[A^{-1}(T)Y \middle| \mathcal{F}_t \right].$$

In particular, the value of a zero-coupon bond maturing at time T is given by

$$B(t,T) = A(t)\,\mathrm{E}^{\mathbb{Q}}\left[A^{-1}(T) \middle| \mathcal{F}_t \right].$$

Now, what would happen if we wanted to price a call option $C(t)$ on a zero-coupon bond maturing at time U? The call option has strike K and expiry $T < U$.
The payoff in T is

$$(B(T,U) - K)^+.$$

Applying the Fundamental Asset Pricing Formula

$$
\begin{aligned}
C(t) &= A(t)\,\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{(B(T,U)-K)^+}{A(T)}\right|\mathcal{F}_t\right]\\
&= A(t)\left(\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{B(T,U)}{A(T)}\mathbb{1}(B(T,U)>K)\right|\mathcal{F}_t\right]-K\,\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{1}{A(T)}\mathbb{1}(B(T,U)>K)\right|\mathcal{F}_t\right]\right)\\
&= A(t)\left(\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{A(T)\,\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{1}{A(U)}\right|\mathcal{F}_T\right]}{A(T)}\mathbb{1}(B(T,U)>K)\right|\mathcal{F}_t\right]-K\,\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{\mathbb{1}(B(T,U)>K)}{A(T)}\right|\mathcal{F}_t\right.\right.\\
&= A(t)\left(\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{\mathbb{1}(B(T,U)>K)}{A(U)}\right|\mathcal{F}_t\right]-K\,\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{\mathbb{1}(B(T,U)>K)}{A(T)}\right|\mathcal{F}_t\right]\right)
\end{aligned}
$$

Applying the Fundamental Asset Pricing Formula

And that's it. We cannot go any further.
To go any further, we would need to know at time t the joint distribution of $B(T, U)$, $A(U)$ and $A(T)$. This is unlikely, unless we make very explicit model.

One way out of this situation would be to look for a measure $\mathbb{P}_T$ such that the expectation in the fundamental asset pricing formula would be a sole function of the derivative payoff $(B(T, U) - K)^+$.

This idea implies that rather than having the "classic" formula

$$C(t) = A(t)\,\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{(B(T, U) - K)^+}{A(T)}\right| \mathcal{F}_t\right]$$

we would start with the "modified" formula

$$C(t) = B(t, T)\,\mathrm{E}^{\mathbb{P}_T}\left[\left.(B(T, U) - K)^+\right| \mathcal{F}_t\right].$$

Applying the Fundamental Asset Pricing Formula

To be in a position to use this "modified" formula, we must answer 2 questions:

1. We do not know what $\mathbb{P}_T$ is. In fact, we do not even know if $\mathbb{P}_T$ exists.
2. Given information up to time t, what would $B(T, U)$ be equal to?

Let's start with the second, and easiest, question. If we are at time t and we would like to know the price at some future time T of a bond maturing at time U, we would use the forward price for that bond.

This leads us to a possible answer to our first question. To define $\mathbb{P}_T$, we could look for a *forward martingale measure*, that is, an equivalent martingale measure defined with respect to forward prices.

Hence, to use the *modified* fundamental asset pricing formula, we need to know a little bit about forwards.

Forward Contracts and Forward Prices

Forward contracts are OTC derivatives securities in which the long party has the obligation to buy an agreed upon quantity of an underlying asset (securities, commodities or others) at an agreed upon time and at an agreed upon price called the forward price.

The outcome does not depend on the decision of the long party. The contract is settled at maturity and typically no cash flow is exchanged in the meantime. As all derivatives, forward contracts are subject to counterparty risk.

Forward Contracts and Forward Prices

Let's say that we want to enter at t into a (long) forward contract on a financial instrument (stock, bond, currency...) whose value at time t is $Y(t)$. The forward matures at time T. Based on our definition of forward contracts, the value of the payoff $G(T, Y(T))$ is

$$G(T, Y(T)) = Y(T) - F_Y(t, T)$$

where $F_Y(t, T)$ is the forward price of Y determined at time t for delivery at time T.

Forward Contracts and Forward Prices

Plugging this into the fundamental asset pricing formula, we see that the time $t \leq \tau \leq T$ value $\pi_\tau(G)$ of a forward contract entered into at time t is equal to

$$\pi_\tau(G) = A(\tau)\,\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{Y(T) - F_Y(t,T)}{A(T)}\right|\mathcal{F}_\tau\right]$$

This formula can be simplified by noting that $F_Y(t,T)$ is a $\mathcal{F}_t$ measurable

$$\pi_\tau(G) = A(\tau)\,\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{Y(T)}{A(T)}\right|\mathcal{F}_\tau\right] - F_Y(t,T)A(\tau)\,\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{1}{A(T)}\right|\mathcal{F}_\tau\right]$$

and now we know the value of a forward contract for any time $t \leq \tau \leq T$.

Forward Contracts and Forward Prices

The forward price $F_Y(t, T)$ was originally set at time t so that the value of the forward contract at time t is 0. Hence,

$$\pi_t(G) = A(t)\,\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{Y(T)}{A(T)}\right|\mathcal{F}_t\right] - F_Y(t, T)A(t)t\,\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{1}{A(T)}\right|\mathcal{F}_t\right] = 0$$

Rearranging,

$$F_Y(t, T) = \frac{A(t)\,\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{Y(T)}{A(T)}\right|\mathcal{F}_t\right]}{A(t)\,\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{1}{A(T)}\right|\mathcal{F}_t\right]} = \frac{\pi_t(Y)}{B(t, T)}.$$

where

$$\pi_t(Y) = Y(t) = A(t)\,\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{Y(T)}{A(T)}\right|\mathcal{F}_t\right].$$

that is $Y(t)$ is the value of a claim paying $Y(T)$ at time T.

The forward price is the number that discounted gives us today's value.

Forward Contracts and Forward Prices

If the underlying asset is a zero-coupon bond of maturity $U > T$, the forward price becomes

$$F_{B(.,U)}(t, T) = \frac{B(t, U)}{B(t, T)}.$$

We use also the notation $F_{B(.,U)}(t, T) = F_B(t, T, U)$.

The Forward Martingale Measure

We will define the T-forward martingale measure, or simply *forward measure*, via the Radon-Nikodym derivative λ_T defined as

$$\lambda_T = \frac{d\mathbb{P}_T}{d\mathbb{Q}} = \frac{A(0)}{A(T)} \frac{B(T,T)}{B(0,T)}$$

Thus

$$\begin{aligned}
\lambda_t = \frac{d\mathbb{P}_T}{d\mathbb{Q}}_{|\mathcal{F}_t} &= \frac{A(0)}{B(0,T)} \mathrm{E}^{\mathbb{Q}}\left[\left. \frac{1}{A(T)} \right| \mathcal{F}_t \right] = \frac{A(0)}{A(t)B(0,T)} \left(A(t)\, \mathrm{E}^{\mathbb{Q}}\left[\left. \frac{1}{A(T)} \right| \mathcal{F}_t \right] \right) \\
&= \frac{A(0)}{A(t)} \frac{B(t,T)}{B(0,T)}
\end{aligned}$$

Note that λ_t defined through a conditional expectation is a $\mathbb{Q}$-martingale and $\mathrm{E}^{\mathbb{Q}}\left[\lambda_T\right] = \lambda_0 = 1$.

The Forward Martingale Measure

We know that the time t value of a zero-coupon bond is given by

$$\frac{B(t,T)}{A(t)} = B(0,T)\exp\left(\int_0^t b(s,T)\cdot dX^{\mathbb{Q}}(s) - \frac{1}{2}\int_0^t |b(s,T)|^2 ds\right)$$

with

$$A(t) = \exp\left(\int_0^t r(s)ds\right).$$

The Forward Martingale Measure

The derivative is such that

$$\begin{aligned}\lambda_t &= \frac{1}{A(t)}\frac{B(t,T)}{B(0,T)}\\ &= \exp\left(\int_0^t b(s,T)\cdot dX^{\mathbb{Q}}(s) - \frac{1}{2}\int_0^t |b(s,T)|^2 ds\right)\end{aligned}$$

It is an exponential martingale with a known exponent.
By Girsanov's theorem, the process X^T defined as

$$X_t^T = X_t^{\mathbb{Q}} - \int_0^t b(s,T)ds$$

is a standard Brownian Motion under the forward measure $\mathbb{P}_T$.
The random process X^T is called the *T-forward Brownian motion*.

Pricing a Derivative Under the Forward Measure

We now have a measure $\mathbb{P}_T$. But before we can use the forward asset pricing formula

$$V(t) = B(t,T)\,\mathrm{E}^{\mathbb{P}_T}\left[\left.\frac{Y}{B(T,T)}\right|\mathcal{F}_t\right]$$

we need to make sure that it will give the same result as the "classic" fundamental asset pricing formula

$$V(t) = A(t)\,\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{Y}{A(T)}\right|\mathcal{F}_t\right].$$

Pricing a Derivative Under the Forward Measure

The price of a contingent claim on Y is

$$\begin{aligned}
\pi_t(Y) &= A(t)\,\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{Y}{A(T)}\right|\mathcal{F}_t\right] \\
&= A(t)\,\mathrm{E}^{\mathbb{Q}}\left[\left.A(T)B(0,T)\lambda_T\frac{Y}{A(T)}\right|\mathcal{F}_t\right] \\
&= A(t)B(0,T)\,\mathrm{E}^{\mathbb{Q}}\left[Y\lambda_T\middle|\mathcal{F}_t\right]
\end{aligned}$$

Pricing a Derivative Under the Forward Measure

By extension of Bayes' formula (see for example Musiela, Rutkowsky, *Martingale Methods in Financial Modelling*, Lemma A.0.4)

$$\mathrm{E}^{\mathbb{P}_T}\left[Y \middle| \mathcal{F}_t \right] = \frac{\mathrm{E}^{\mathbb{Q}}\left[Y\lambda_T \middle| \mathcal{F}_t \right]}{\mathrm{E}^{\mathbb{Q}}\left[\lambda_T \middle| \mathcal{F}_t \right]}.$$

Therefore

$$\begin{aligned}
\pi_t(Y) &= A(t)B(0,T)\,\mathrm{E}^{\mathbb{Q}}\left[\lambda_T \middle| \mathcal{F}_t \right] \mathrm{E}^{\mathbb{P}_T}\left[Y \middle| \mathcal{F}_t \right] \\
&= A(t)B(0,T)\lambda_t\,\mathrm{E}^{\mathbb{P}_T}\left[Y \middle| \mathcal{F}_t \right] \\
&= A(t)B(0,T)\frac{A(0)B(t,T)}{A(t)B(0,T)}\,\mathrm{E}^{\mathbb{P}_T}\left[Y \middle| \mathcal{F}_t \right] \\
&= B(t,T)\,\mathrm{E}^{\mathbb{P}_T}\left[Y \middle| \mathcal{F}_t \right]
\end{aligned}$$

Pricing a Derivative Under the Forward Measure

The forward (martingale) measure $\mathbb{P}_T$ is defined in terms of the equivalent martingale measure $\mathbb{Q}$ via the Radon-Nikodym derivative

$$\frac{d\mathbb{P}_T}{d\mathbb{Q}} = \frac{A(0)}{A(T)} \frac{B(T,T)}{B(0,T)}.$$

Key Fact (European derivative)

The price of a European derivative expiring at time T with payoff Y is

$$\pi_t(Y) = B(t,T)\,\mathrm{E}^{\mathbb{P}_T}\left[Y | \mathcal{F}_t \right]$$

Pricing a Derivative Under the Forward Measure

Key Fact (European derivative)

The price of a European derivative expiring at time $u < T$ with payoff Y_u is

$$\pi_t(Y) = B(t,T)\,\mathrm{E}^{\mathbb{P}_T}\left[\left.\frac{Y_u}{B(u,T)}\right|\mathcal{F}_t\right]$$

Strategy: Invest from u into zero-coupon bond to obtain the payoff $Y_u/B(u,T)$ in T.

$$A(t)\,\mathrm{E}^{\mathbb{Q}}\left[\left.\frac{Y_u}{A(u)}\right|\mathcal{F}_t\right] = B(t,T)\,\mathrm{E}^{\mathbb{P}_T}\left[\left.\frac{Y_u}{B(u,T)}\right|\mathcal{F}_t\right]$$

Martingales and Fixed Income Valuation

Pricing a Call on a Zero-Coupon Bond

We express the call payoff at time T, not in terms of the zero-coupon bond, but in terms of a forward on the zero-coupon bond as

$$(B(T,U) - K)^+ = (F_B(T,T,U) - K)^+$$

where $F_B(t,T,U)$ is the forward price at time t for settlement at time T of a zero-coupon bond maturing at time $U > T$. Note that $F_B(T,T,U)$ is the instantaneous forward price at time T, which is equal to the spot price $B(T,U)$.

Pricing a Call on a Zero-Coupon Bond

Applying the forward pricing formula, we deduce that the zero-coupon forward price $F_B(t, T, U)$ is given by

$$F_B(t, T, U) = \frac{B(t, U)}{B(t, T)}$$

Pricing a Call on a Zero-Coupon Bond

As a result, the $\mathbb{Q}$-dynamics of the forward price is given by

$$\begin{aligned} F_B(t,T,U) &= \frac{B(0,U)}{B(0,T)} \exp\left(\int_0^t (b(s,U)-b(s,T)) \cdot dX^{\mathbb{Q}}(s) - \frac{1}{2}\int_0^t |b(s,U)|^2 - |b(s,T)|^2 ds \right. \\ &= F_B(0,T,U) \exp\left(-\int_0^t (b(s,U)-b(s,T)) \cdot b(s,T) ds \right) \\ & \quad \exp\left(\int_0^t (b(s,U)-b(s,T)) \cdot dX^{\mathbb{Q}}(s) - \frac{1}{2}\int_0^t |b(s,U)-b(s,T)|^2 ds \right) \end{aligned}$$

or

$$\frac{dF_B(t,T,U)}{F_B(t,T,U)} = (b(t,U)-b(t,T)) \cdot dX^{\mathbb{Q}}(t) - (b(t,U)-b(t,T)) \cdot b(t,T) dt.$$

Pricing a Call on a Zero-Coupon Bond

Deriving the dynamics of $F_B(t, T, U)$ under the $\mathbb{Q}$-measure is a promising start. As we are going to price the call option using the forward asset pricing formula, we need to know the dynamics of the forward price $F_B(t, T, U)$ under the forward measure $\mathbb{P}_T$. Recalling that

$$X_t^T = X_t^{\mathbb{Q}} - \int_0^t b(s, T) ds$$

is a standard Brownian Motion under the forward measure $\mathbb{P}^T$, we immediately get

$$\frac{dF_B(t, T, U)}{F_B(t, T, U)} = (b(t, U) - b(t, T)) \cdot dX^T(t).$$

and

$$F_B(t, T, U) = F_B(0, T, U) \mathcal{E}_t \left(\int_0^{\cdot} (b(s, U) - b(s, T)) \cdot dX^T(s) \right)$$

which implies that the forward price is a martingale under the forward measure.

Pricing a Call on a Zero-Coupon Bond

We now have all we need to solve the Call option pricing problem using the forward asset pricing formula

$$C(t) = B(t, T)\,\mathrm{E}^{\mathbb{P}_T}\left[(F_B(T, T, U) - K)^+ \middle| \mathcal{F}_t \right].$$

In the case of a call on a zero-coupon bond and deterministic volatility, a Black-Scholes-type formula exists.

Pricing a Call on a Zero-Coupon Bond

Key Fact (European Call)

When the volatility $b(.,V)$ is deterministic *(for all $V \leq \bar{T}$), the time t price of a European Call expiring at time T and with strike K, written on a zero-coupon bond maturing at time $U > T$ is given by the following Black-Scholes type of formula*

$$C(t) = B(t,U)N(d_+(B(t,U),t,T)) - KB(t,T)N(d_-(B(t,U),t,T))$$

where

$$d_\pm(x,t,T) = \left(\ln\left(\frac{x}{KB(t,T)}\right) \pm \frac{1}{2}\sigma_U^2(t,T)\right)\frac{1}{\sigma_U(t,T)}$$

and

$$\sigma_U^2(t,T) = \int_t^T |b(s,U) - b(s,T)|^2 ds.$$

Forward as a martingale

We have mentioned that in our framework with geometric Brownian motion equation for the bond price, the forward price is a martingale in the forward measure. We can show that in general, any forward price is a martingale in the same numeraire. For $s \leq t \leq T$,

$$
\begin{aligned}
\mathrm{E}^{\mathbb{P}_T}\left[F_Y(t,T) \middle| \mathcal{F}_s \right] &= \frac{B(s,T)}{B(s,T)} \mathrm{E}^{\mathbb{P}_T}\left[\left. \frac{Y(t)}{B(t,T)} \right| \mathcal{F}_s \right] \\
&= \frac{Y(s)}{B(s,T)} = F_Y(s,T)
\end{aligned}
$$

Call Pricing in Practice: Numerical Solutions 2

Monte Carlo on the zero-coupon bonds B (one-factor deterministic separable b), e.g. Hull-White one-factor model.

The random variables $\int_0^t b(s, T) dX^{\mathbb{Q}}(s)$ normally distributed. When b is deterministic, its variance is explicitly known: $\alpha^2(t, T) = \int_0^t |b(s, T)|^2 ds$.

When possible, significantly more efficient numerically than working on r. More regularity: $B = \int r$, one level of regularity more!

The quantities $B(0, T)$ provided by the market. The quantity $r(0)$ undefined in practice! It is the limit for $t \to 0$ of a quantity known only discreetly (at best daily maturities).

Call Pricing in Practice: Numerical Solutions 2

A Monte-Carlo algorithm on the bond looks like (with $Z(t, T) = B(t, T)/A(t)$)

- $Z(0, T) = B(0, T)$ and $Z(0, U) = B(0, U)$
- Simulations $j = 1, \ldots, N$
  - Time steps not needed
  - Simulate $N(0, 1)$ random variables W_j
  - Simulate bonds ($X = T, U$)

$$Z_j(T, X) = B(0, X) \exp\left(-\alpha(T, X)W_j - 0.5 * \alpha^2(T, X)\right)$$

- Value of the bond option:

$$C = \frac{1}{N} \sum_{j=1}^{N} (Z_j(T, U) - KZ_j(T, T))^+.$$

Advantage with respect to Monte-Carlo on r: $B(0, T)$ given by the market, no time discretisation, no differentiation, no integration.

Martingales and Fixed Income Valuation

Why this approach?

If we want the bond price to have a lognormal-type behaviour in a short rate model, the bond volatility function $b(t, T)$ may look like a "fudge function". This will however motivate us to turn our models around and specify a bond dynamics first, and then deduce a dynamics of interest rates. This approach forms the base of forward rate models such as the HJM class of models.

Forward measure. The existence of the forward measure and the critical role played by forwards in the pricing of interest rate derivatives also provides a powerful motivation for looking at the term structure of forward (as opposed to spot) rates (see the HJM class of models).

As long as the bond price follows a geometric dynamics, irrespective of the specific interest rate model we chose, the value of a bond derivative will always be of the same form. Natural question: if we assume a geometric dynamics for the bond price, how many interest rate models do we have access to? What is the most general interest rate model we can find such that the bond price follows a geometric dynamics?

What is next? Forward Rate Model

The answer to this question, and next chapter in the development of interest rate models, is the derivation of models of the forward rate dynamics. This critical step was achieved by Heath, Jarrow and Morton (1992) and then further developed by Brace, Gatarek and Musiela (1997).

The key attraction of forward rate models is they start from a (nice) geometric dynamics for the zero-coupon bond price and then deduce the behaviour of the term structure of forward rates; they are a “meta”-model which encompasses all existing interest rate models; as such, you can use them to price or manage the risk of anything, from vanilla derivatives to complex fixed income portfolios (which are heavily dependent on an accurate modelling of the term structure).

Forward rate models are not necessarily theoretically different from short rate models. The same practical model can have several representation: short rate, forward instantaneous rate, market rate, discount factors.

What is next? Forward Rate Model

The key problems related to forward rate models: mathematically sophisticated, sometimes too sophisticated for the applications at hand (such as pricing vanilla derivatives); potentially non-Markov. A good part of the mathematics we have are based on Markov models. Hence, we need to choose our parameters carefully and make assumptions to ensure that the forward rate models we work with are indeed Markov.

Meta-model: no clear indication of which form to use and when to use it.

What is next? Forward Rate Model

In this lecture, we have seen...

- the pricing of interest rate products in a probabilistic setting; the equivalent (spot) martingale measures;
- the fundamental asset pricing formula for bonds;
- the dynamics of bond prices;
- the forward measure;
- the fundamental asset pricing formula for derivatives on bonds;