Curve Construction: A Summary

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Practices and meaning of curves

In this review...

- forward rate (as adjusted $r(T)$ process)
- interpolation – introduction (raw)
- bootstrapping where forward is target (eg from swap rates) and ZCBs are known
- bootstrapping where discounting factor is target – Dong Qu 'bootstrap framework'
- interpolation advanced – curve representation (NS, monotone-convex)

- Forward rates are interest rates that apply over given periods *in the future* **for all** instruments.

Bond yields apply only up to maturity (YTM), and each non-ZCB bond has different cashflows leading to its own duration and convexity. That is, own price/yield relationship.

Forward Rate Agreement - 1

- It was custom to introduce forward rates with the concept of a **Forward Rate Agreement** (**FRA**). An agreement between two parties that a prescribed interest rate will apply over period in the future.

The FRA being simple rate opens way to linear maths.

$$
\begin{aligned}
r_1 T_1 + f(T_2 - T_1) &= r_2 T_2 \\
f &= \frac{r_2 T_2 - r_1 T_1}{T_2 - T_1}
\end{aligned}
$$

Forward Rate Agreement - 1

This is already sufficient for a simple computation of forward rates.

Which interest rate is implied by the market price of the **first bond** with shortest maturity? The answer is y_1, the solution of

$$Z_1^M = e^{-y_1(T_1-t)},$$

i.e.

$$y_1 = -\frac{\log(Z_1^M)}{T_1 - t}.$$

This rate will be the rate that we use for discounting between the present and the maturity date T_1 of the first bond.

- It will be applied to *all* instruments whenever we discount over this period.

Now move on to the **second bond**, having maturity date T_2.
We know the rate to apply between now and time T_1, but at what interest rate must we discount between dates T_1 and T_2 to match the theoretical and market prices of the second bond?
The answer is y_2 which solves the equation

$$Z_2^M = e^{-y_1(T_1-t)}e^{-y_2(T_2-T_1)},$$

i.e.

$$y_2 = -\frac{\log\left(Z_2^M/Z_1^M\right)}{T_2 - T_1}.$$

- By this method of **bootstrapping** we build up the forward curve. Note how the forward rates are applied between two dates, for which period we have assumed they are constant.

Overlapping discount factors $Z(0, T_1), Z(0, T_2), Z(0, T_3) \ldots$ encapsulate information about forward rates, also called *hedgeable rates*.

$$
\begin{aligned}
Z(0, T_1) \times Z(0; T_1, T_2) &= Z(0, T_2) \\
Z(0; T_1, T_2) &= \exp\left[-f(0; t_1, t_2)(t_2 - t_1)\right] \\
f_2 &= -\frac{\ln Z_2 - \ln Z_1}{t_2 - t_1}
\end{aligned}
$$

It makes sense to model the sequence of forward rates Overlapping discount factors $Z(0, T_1), Z(0, T_2), Z(0, T_3) \ldots$ encapsulate information about forward rates, also called *hedgeable rates*.

Computing forward rates

A scheme for each discrete tenor j follows from the instantaneous forward rate maths

$$f_0 = -\frac{\ln Z_1 - \ln 1}{T_1 - 0} \qquad f_1 = -\frac{\ln(Z_2/Z_1)}{T_2 - T_1}$$

Tenor, T	**0.08**	**0.17**	**0.25**	**0.33**
Spot	0.5052%	0.5295%	0.5500%	0.5682%
Z(0, T)	0.9996	0.9991	0.9986	0.9981
Forward	0.5063%	0.5552%	0.5927%	0.6244%

Discrete-time forward computation

$$DF(0, T_i) = \frac{1}{\left(1 + \frac{r}{m}\right)^i}$$

where i is an index for period, e.g., $i = T \times m$ for example, $i = 0.25 \times 12 = 3$

$$F(0; T_{i-1}, T_i) = \left[\frac{DF(0, T_i)}{DF(0, T_{i-1})}\right]^{-m} - 1$$

$$F(0; T_{i-1}, T_i) = DF(T_i, T_{i-1})^{-m} - 1$$

Forward Rate Agreement - 2

The cashflows in this agreement are as follows: party A pays party B the principal at time T_1 and B pays A the principal plus agreed interest at time $T_2 > T_1$.

at time t we agree to either

- pay \$1 at T_1 to receive $e^{\int_{T_1}^{T_2} f(t,T)dT}$ at T_2,
- pay $e^{-\int_{T_1}^{T_2} f(t,T)dT}$ at T_1 to receive \$1 at T_2.

Forward Rate Agreement written at par offers a market quantity: the forward rate $\mathrm{FRA}(t, T_1, T_2)$, which a strike.

$$[f - \mathrm{FRA}(t, T_1, T_2)] = 0$$

At its reset time T_1 that FRA rate will coincide with is Term Rate fixing $L(T_1, T_2)$.

The Term Rate was formerly LIBOR, eg 6M LIBOR, reflecting the credit risk as well.

Forward rates

Intuitive interpretation of forward rates as the adjusted $r(T)$ process) – from plots.

- Forward rate process is the actual process of instantaneous quantity, averaging over which gives spot rates $r(T)$

$$
\begin{aligned}
r(t)\,T &= r_{i-1}\,T_{i-1} + \int_{T_{i-1}}^{T} f(s)ds \qquad (1)\\
&\text{as seen before}\\
r_2\,T_2 &= r_1\,T_1 + f(T_2 - T_1)
\end{aligned}
$$

In general case, we use Equation 1

$$\frac{r_i T_i - r_{i-1} T_{i-1}}{T_i - T_{i-1}} = \frac{1}{T_i - T_{i-1}} \int_{T_{i-1}}^{T} f(s)$$

No arbitrage relationship

No arbitrage relationship between a sequence of *simple* forward rates and ZCB yield at time n (**spot rate**) is given by a geometric average

$$(1 + rs_T)^T = (1 + f_0) \times (1 + f_1) \times \ldots \times (1 + f_n).$$

where for the initial period $f_0 = rs_0$.

This no-arb relationship also allows to compute forward rates, given that we know rs_T for a range of maturities T_1, T_2, T_3 and Two outcomes of no-arbitrage: it is possible to bootstrap f You might know this relationship well, and it leads to *a recursive* bootstrapping *from* a set of ZCB prices...

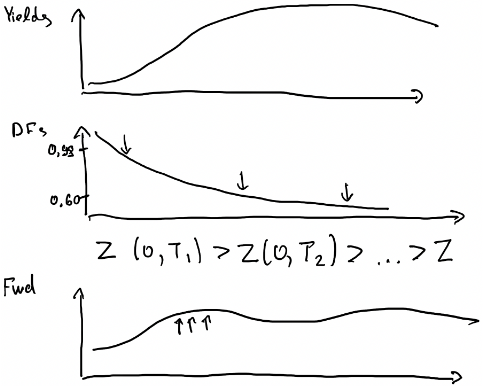
Computing forward rates

1. Let's explore results of boostrapping fwd rates from spot rates, ZCB factors, and compare to BOE inst fwd curve for the same day

2. It is evident that BOE applies additional VRP smoothing on top of this classical fwds bootstrapping.

3. It is recommended to use **raw interpolation**: linear over log discount factors $\ln Z(T_{i-1}, T_i)$.

Please examine *Yield Curve v4.xlsm* spreadsheet and its VBA.



Source: Drawing by Richard Diamond

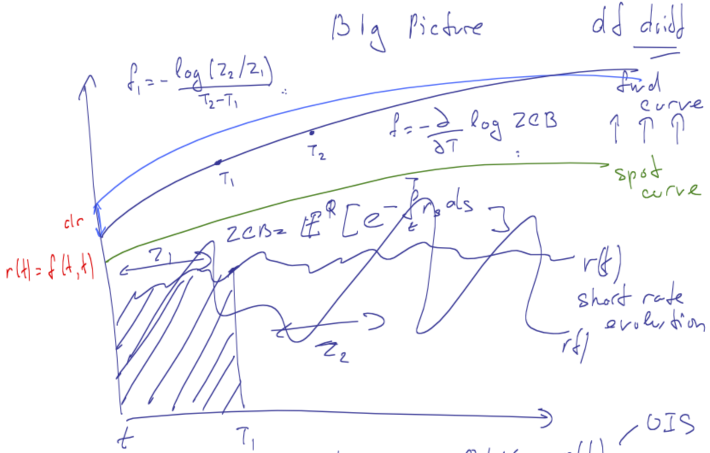
Let's draw a summary of relationship among

a. the short rate process,

b. spot curve, and

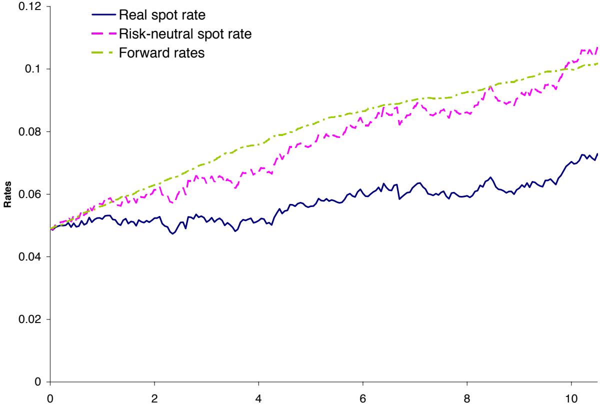
c. forward curve.

From Spot Rate to Forward Curve



$r(t)$ is stochastic or $O/N \equiv r(t)$ OIS SONIA

$Z_1(t, T_1)$, $Z_2(t, T_2)$ $T_2 > T_1$



Fwd curve is the expectation of risk-adjusted spot rates. Evident in short maturities, eg Fwd 1M LIBOR is risk-adjusted OIS (Slide 33).

Source: CQF IR Calibration Lecture by Paul Wilmott, Slide 72

Bootstrap Framework

The following process is referred to as **yield curve stripping**

- Setting up par bootstrap equations
- Solving for the unknown curves by stepping through time.

Bootstrap Framework

PV for the fixed leg with fixed rate k – Payer PAYS. $k\,\tau_i$ paid at T_i

$$\text{PV} = k\,N \sum_{i=1}^{n} \tau_i\, D_i$$

PV for the floating leg – Payer RECEIVES. Rate plus **basis spread**

$$\text{PV} = N \sum_{j=1}^{m} \tau_j\,(I_j + s)\;D_j$$

where s is spread, and I_j is Index curve rate, such as 6M LIBOR.

Bootstrap Framework

Equating PV of swap legs we obtain **par bootstrap equation**,

$$k \sum_{i=1}^{n} \tau_i D_i = \sum_{j=1}^{m} \tau_j (l_j + s) D_j$$

$$\tau_i \neq \tau_j$$

In fact, small $\tau_i \neq \tau_{i+1}$, eg ON, 1M, 2M, 3M, 6M, 12M, 2Y, 3Y curve is converted into 6M basis $\tau_j \approx 0.5$.

Instrument	Par Bootstrap Equation	Unknown Curves to Solve
Standard Swap	$k\sum_i^n \tau_i D_i = \sum_j^m \tau_j \left(I_j + s\right) D_j$	Discount curve (D) and index curve (I). As an example, D can be the OIS curve, and I a LIBOR curve.
Tenor Basis Swap	$\sum_i^n \tau_i I_i^x D_i = \sum_j^m \tau_j \left(I_j^y + s\right) D_j$	Discount curve (D) and two index curves. As an example, I_i^x can be the 6-month LIBOR and I_j^y the 3-month LIBOR curve.

From: *Manufacturing and Managing Customer-Driven Derivatives*, Dong Qu, 2016

OIS

$$k\sum_{i}^{n}\tau_i D_i = \sum_{j}^{m}\tau_j R_j D_j$$

$$R_j = \frac{\prod_d (1+\tau_d r_d) - 1}{\sum_d \tau_d}$$

Discount curve (D) is the only unknown, as the daily overnight rate (r_d) is assumed linked to D:

$$r_d = \frac{1}{\tau_d}\left(\frac{D_1}{D_2} - 1\right)$$

R_j is the compound rate of r_d.

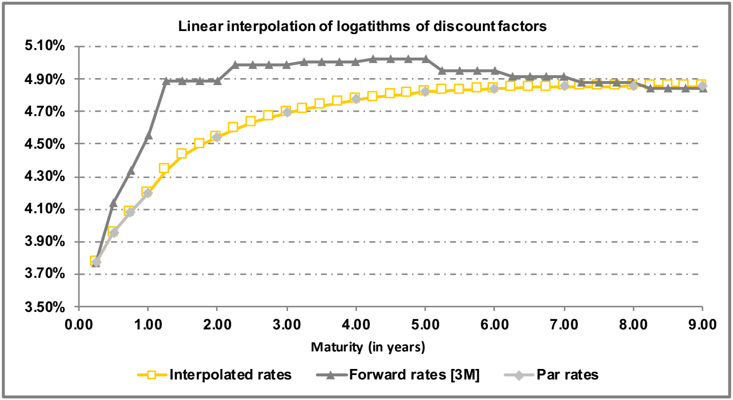
From: *Manufacturing and Managing Customer-Driven Derivatives*, Dong Qu, 2016

Interpolation

Curve stripping and interpolation go together. Instead of boostrapping $\ln Z(t, T) \Rightarrow f(t, T)$, direct interpolation methods are applied to infer Forward LIBOR (simple rates) form par rates $r_s \Rightarrow L(t)_i$

- Linear over zero coupon rates (spot)
- Linear over discount factors
- Linear over **log of discount factors** (raw interpolation)
- Natural cubic spline (also B-splines)
- Monotone-convex spline

Explore **Interpolation and FRA.xlsm** file for the effect of interpolation method on **a.** resulting forward curve and **b.** PV01 of interest rate swap.



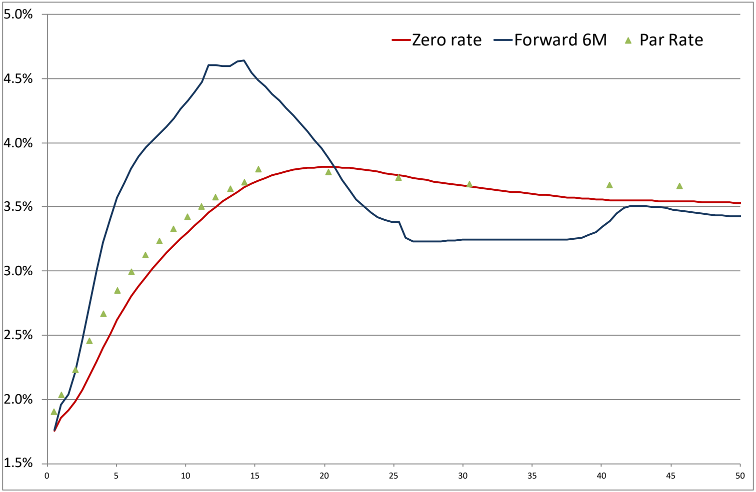
Interpolating over $log\ Z(0, T)$ gives nearly as good curve as advanced methods (spline-based). **Forward curve** (in deep grey) is not zig-zag and 'smooth enough'.

The various representations where the curve is assume to be an object (mathematical function) and represented via such function with parameters.

This can be *seen as* interpolation but the parameters typically require search by optimisaiton. Examples: use of cubic splines and Nelson-Siegel-Svensson on spot curves, new Term SOFR boostrapping.

- The approach ‘disgregards’ the link to market-like quantities – boostrapping of ZCBs... but allows to have smooth, presentable curves.

Let's explore **LIBOR Bootstrap v1.xlsm**. There it is implemented via re-interpolation of spot so $I_j = r(0, T_j)$, $j = [1..100]$.



Market data points (triangles). Bootstrapped spot curve in red – 6M tenor increment (based on exact day count, so assume EONIA

Take **monotone convex splines**, for example.

- Hagan and West [2006] 'ameliorated' the method by making the interpolation less local, ie, using inputs which are two nodes away.

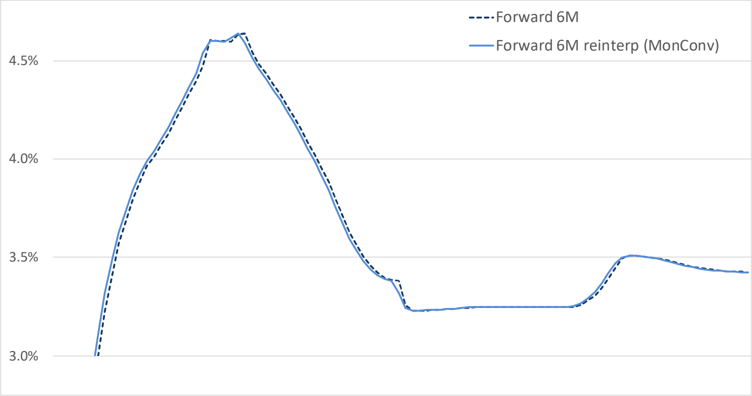
  LIBOR Bootstrap v1.xls has VBA code and use example.

- BIS Paper 25, spot curves (ZCB rates, zero-rates) fitted with

  1) cubic B-spines with knot points, utilised by FRBNY previously

  2) exponential polynomial family (Nelson-Siegel, Svensson) utilised by BOE,

  3) curve fitting with Artificial Neural Nets – see William McGhee (Natwest Markets, 2018) for recipe.



Bootstrapped 6M Fwd re-interpolated using monotone-convex splines. Impact is slight because the rate comes from the already-stripped curve – in regular increment.