Exercises 6.3 Calibration and Data Analysis

1. Suppose the spot interest rate r, which is a function of time t, satisfies the stochastic differential equation

$$dr = u(r,t)\,dt + w(r,t)\,dW_t.$$

The bond pricing equation for a security $Z = Z(r,t;T)$ is

$$\frac{\partial Z}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2} + \left(u(r,t) - \lambda(r,t)\,w(r,t)\right)\frac{\partial Z}{\partial r} - rZ = 0, \tag{1.1}$$

together with redemption value $Z(r,T;T) = 1$ where T is the maturity of the bond. The Vasicek model has risk-adjusted drift given by

$$u(r,t) - \lambda(r,t)\,w(r,t) = \eta - \gamma r,$$

for constant η and γ. A zero coupon bond $Z(r,t;T)$ satisfies (1.1) together with redemption value $Z(r,T;T) = 1$. Expand $Z(r,t;T)$ for small times t to maturity T, i.e. in powers of $(T-t)$

$$Z \sim a(r) + b(r)(T-t) + c(r)(T-t)^2 + ..., \tag{1.2}$$

for the unknown coefficients and show that for $t \longrightarrow T$ we have

$$Z \sim 1 - r(T-t) + \tfrac{1}{2}\left(r^2 - \eta + \gamma r\right)(T-t)^2 + ...$$

Hint: start by substituting (1.2) into (1.1).
Solution: Firstly, from redemption we have $Z(r,T;T) = 1 \longrightarrow a(r) = 1$

$$\begin{aligned}
Z &\sim 1 + b(r)(T-t) + c(r)(T-t)^2 + ...\\
\frac{\partial Z}{\partial t} &= -b(r) - 2c(r)(T-t)\\
\frac{\partial Z}{\partial r} &= b'(r)(T-t) + c'(r)(T-t)^2;\ \frac{\partial^2 Z}{\partial r^2} = b''(r)(T-t) + c''(r)(T-t)^2
\end{aligned}$$

Substituting these terms into the BPE gives

$$\begin{aligned}
&-b(r) - 2c(r)(T-t) + \frac{1}{2}w^2\left(b''(r)(T-t) + c''(r)(T-t)^2\right) +\\
&(u-\lambda w)\left(b'(r)(T-t) + c'(r)(T-t)^2\right)\\
= \;& r\left(1 + b(r)(T-t) + c(r)(T-t)^2\right).
\end{aligned}$$

Now compare coefficients of powers of $(T-t)$
$O(1)$:

$$-b(r) = r \longrightarrow b(r) = -r \Longrightarrow b'(r) = -1 \Longrightarrow b''(r) = 0$$

$O\left((T-t)\right):$

$$\begin{aligned} -2c(r)+(u-\lambda w)(-1) &= -r^2 \\ c(r) &= \tfrac{1}{2}\left(r^2-(u-\lambda w)\right)=\tfrac{1}{2}\left(r^2-\eta+\gamma r\right) \end{aligned}$$

So for $t \longrightarrow T$ we have

$$Z \sim 1-r(T-t)+\tfrac{1}{2}\left(r^2-\eta+\gamma r\right)(T-t)^2+...$$

2. Substitute the fitted function for $A(t\,;T)$, using the Ho & Lee model, back into the solution of the bond pricing equation for a zero-coupon bond,

$$Z(r,\ t\,;T)=\exp\left(A(t\,;T)-r(T-t)\right).$$

The form for $A(t\,;T)$ can be found in the lecture notes. What do you notice when $t=t^*$?
Solution: With a Ho & Lee model, the form of the fitted function for $A(t\,;T)$ *is*

$$A(t\,;T)=\log\left(\frac{Z_M(t^*;\ T)}{Z_M(t^*;\ t)}\right)-(T-t)\frac{\partial}{\partial t}\log\left(Z_M(t^*;\ t)\right)-\frac{1}{2}c^2(t-t^*)(T-t)^2.$$

Then

$$\begin{aligned} Z(t;T) &= e^{\log\left(\frac{Z_M(t^*;\ T)}{Z_M(t^*;\ t)}\right)-(T-t)\frac{\partial}{\partial t}\log(Z_M(t^*;\ t))-\frac{1}{2}c^2(t-t^*)(T-t)^2-r(T-t)} \\ &= \frac{Z_M(t^*;\ T)}{Z_M(t^*;\ t)}e^{-(T-t)\left(\frac{\partial}{\partial t}\log(Z_M(t^*;\ t))+\frac{1}{2}c^2(t-t^*)(T-t)+r\right)}. \end{aligned}$$

We note that that when $t=t^*$

$$Z(t^*;T)=\frac{Z_M(t^*;\ T)}{Z_M(t^*;\ t)}e^{-(T-t^*)\left(\frac{\partial}{\partial t}\log(Z_M(t^*;\ t))+\frac{1}{2}c^2(t^*-t^*)(T-t^*)+r\right)}=Z_M(t^*;\ T).$$

3. Differentiate Equation (2) on page 19 of the lecture notes, twice to solve for the value of $\eta^*(t)$. What is the value of a zero-coupon bond with a fitted Vasicek model for the interest rate?
Solution: We have

$$\begin{aligned} &-\int_{t^*}^{T}\eta^*(s)B(s\,;T)\ ds+\frac{c^2}{2\gamma^2}\left((T-t^*)+\frac{2}{\gamma}e^{-\gamma(T-t^*)}-\frac{1}{2\gamma}e^{-2\gamma(T-t^*)}-\frac{3}{2\gamma}\right) \\ = &\ \log\left(Z_M(t^*\,;T)\right)+r^*B(t^*\,;T). \end{aligned}$$

Differentiating with respect to T,

$$\begin{aligned} &-\int_{t^*}^{T}\eta^*(s)\frac{\partial}{\partial T}B(s;T)\ ds-\eta^*(T)B(T\,;T)+\frac{c^2}{2\gamma^2}\left(1-2e^{-\gamma(T-t^*)}+e^{-2\gamma(T-t^*)}\right) \\ = &\ \frac{\partial}{\partial T}\log\left(Z_M(t^*\,;T)\right)+r^*\frac{\partial}{\partial T}B(t^*\,;T). \end{aligned}$$

Now

$$B(t\,;T)=\frac{1}{\gamma}\left(1-e^{-\gamma(T-t)}\right)\quad\text{so}\quad B(T\,;T)=0,$$

and

$$\frac{\partial}{\partial T}B(t\,;T)=e^{-\gamma(T-t)}.$$

Substituting back into the PDE

$$\begin{aligned}&-\int_{t^*}^{T}\eta^*(s)\,e^{-\gamma(T-s)}\;ds+\frac{c^2}{2\gamma^2}\left(1-2e^{-\gamma(T-t^*)}+e^{-2\gamma(T-t^*)}\right)\\ =\;&\frac{\partial}{\partial T}\log\left(Z_M(t^*\,;T)\right)+r^*e^{-\gamma(T-t^*)}.\end{aligned}$$

Differentiating again with respect to T,

$$\begin{aligned}&-\eta^*(T)+\gamma\int_{t^*}^{T}\eta^*(s)\,e^{-\gamma(T-s)}\;ds+\frac{c^2}{2\gamma^2}\left(2\gamma e^{-\gamma(T-t^*)}-2\gamma e^{-2\gamma(T-t^*)}\right)\\ =\;&\frac{\partial^2}{\partial T^2}\log\left(Z_M(t^*\,;T)\right)-\gamma r^*e^{-\gamma(T-t^*)}.\end{aligned}$$

Substituting for the integral from the previous equation, we find

$$\begin{aligned}&-\eta^*(T)+\gamma\left(\frac{c^2}{2\gamma^2}\left(1-2e^{-\gamma(T-t^*)}+e^{-2\gamma(T-t^*)}\right)-\frac{\partial}{\partial T}\log\left(Z_M(t^*\,;T)\right)-r^*e^{-\gamma(T-t^*)}\right)\\ &+\frac{c^2}{2\gamma^2}\left(2\gamma e^{-\gamma(T-t^*)}-2\gamma e^{-2\gamma(T-t^*)}\right)\\ =\;&\frac{\partial^2}{\partial T^2}\log\left(Z_M(t^*\,;T)\right)-\gamma r^*e^{-\gamma(T-t^*)}.\end{aligned}$$

This simplifies to

$$\eta^*(T)=-\frac{\partial^2}{\partial T^2}\log\left(Z_M(t^*\,;T)\right)+\frac{c^2}{2\gamma}-\gamma\frac{\partial}{\partial T}\log\left(Z_M(t^*\,;T)\right)-\frac{c^2}{2\gamma}e^{-2\gamma(T-t^*)},$$

and

$$\eta^*(t)=-\frac{\partial^2}{\partial t^2}\log\left(Z_M(t^*\,;t)\right)-\gamma\frac{\partial}{\partial t}\log\left(Z_M(t^*\,;t)\right)+\frac{c^2}{2\gamma}\left(1-e^{-2\gamma(t-t^*)}\right).$$

We then have

$$A(t\,;T)=-\int_{t}^{T}\eta^*(s)\,B(s\,;T)\;ds+\frac{c^2}{2\gamma^2}\left((T-t)+\frac{2}{\gamma}e^{-\gamma(T-t)}-\frac{1}{2\gamma}e^{-2\gamma(T-t)}-\frac{3}{2\gamma}\right)$$

and substituting for η^* and integrating, we find

$$=\log\left(\frac{Z_M(t^*;\;T)}{Z_M(t^*;\;t)}\right)-B(t\,;T)\frac{\partial}{\partial t}\log\left(Z_M(t^*;\;t)\right)-\frac{c^2}{4\gamma^3}\left(e^{-\gamma(T-t^*)}-e^{-\gamma(t-t^*)}\right)\left(e^{2\gamma(t-t^*)}-1\right).$$

We know the value of a zero-coupon bond is

$$Z(r,\ t\ ;T) = \exp\left(A(t\ ;T) - rB(t\ ;T)\right),$$

with $A(t\ ;T)$ given by the above, and

$$B(t;T) = \frac{1}{\gamma}\left(1 - e^{-\gamma(T-t)}\right).$$