Introduction to Numerical Methods

In this lecture. . .

I

- The justification for pricing by Monte Carlo simulation

II

- Grids and discretization of derivatives

- The explicit finite-difference method

By the end of this lecture you will be able to

- implement the Monte Carlo method for simulating asset paths and pricing options

- implement the explicit finite-difference method for pricing options

60% M.C. ✓

30% FDM ✓

10% trees and transform methods

Introduction

More often than not we must solve option-pricing problems by numerical means.

It is rare to be able to find closed-form solutions for prices unless both the contract and the model are very simple.

probabilistic approach

The most useful numerical techniques are Monte Carlo simulations and finite-difference methods.

grid based

Computational Finance – Numerical methods applied to problems in finance

analysis

Relationship between derivative values and simulations

Theory says:

- The fair value of an option is the present value of the expected payoff at expiry under a *risk-neutral* random walk for the underlying. measure $\mathbb{Q}$

The risk-neutral random walk for S is

$dS = \mu S\,dt + \sigma S\,dX^{\mathbb{P}}$

- $$\boxed{dS = rS\,dt + \sigma S\,dX.}$$

This is simply our usual lognormal random walk but with the risk-free rate instead of the real growth rate.

Justification:

- Binomial method Paul

Riaz

- Black–Scholes Equation similar to backward Kolmogorov equation

Seb.

- Martingale theory

We can therefore write

$r \in \mathbb{R}$

$$\text{option value } = e^{-r(T-t)} E\left[\text{payoff}(S)\right]$$

provided that the expectation is with respect to the risk-neutral random walk, not the *real* one.

$$\mathbb{E}\left[e^{-\int_t^T r_u \, du} \text{payoff}(S)\right]$$

Monte Carlo recipe

The algorithm:

underlying state variable

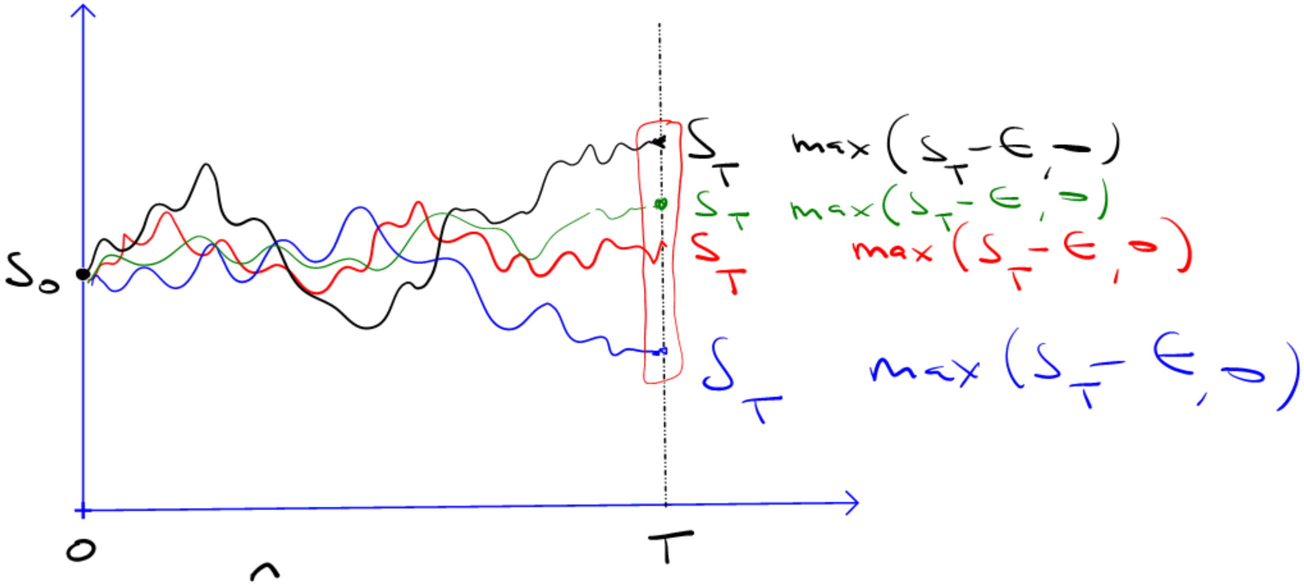
1. Simulate the risk-neutral random walk starting at today's value of the asset S_0 over the required time horizon. This gives one realization of the underlying price path.

2. For this realization calculate the option payoff.

3. Perform many more such realizations over the time horizon.

4. Calculate the average payoff over all realizations.

5. Take the present value of this average, this is the option value.



$$\text{Average} = \frac{1}{n} \sum_{i=1}^{n} \max\left(S_T^{(i)} - E, 0\right)$$

$$e^{-r(T-t)} \times \frac{1}{n} \sum_{i=1}^{n} \max\left(S_T^{(i)} - E, 0\right)$$

$$= \text{Option Value}$$

$$dS = (r - D)S\,dt + \sigma S\,dX_1$$

$$dr = (\eta - \gamma r)\,dt + \sigma\,dX_2$$

How do we simulate the asset?

Two ways:

1. **If** the s.d.e. for the asset path is integrable **and** the contract is not path dependent (or American) **then** simulate in 'one giant leap'

2. **Otherwise** you will have to simulate time step by time step, the entire path

One giant leap: A method that works in special cases

For the lognormal random walk we are lucky that we can find a simple, and exact, time stepping algorithm.

We can write the risk-neutral stochastic differential equation for S in the form

Itô III on log S

$$d(\log S) = \left(r - \tfrac{1}{2}\sigma^2\right) dt + \sigma\, dX.$$

This can be integrated exactly to give

$$S(t) = S(0) \exp\left(\left(r - \tfrac{1}{2}\sigma^2\right) t + \sigma \int_0^t dX\right).$$

i.e.

time step

NORMSINV(RAND())

expiry today

- $$S(T) = S(0) \exp\left(\left(r - \tfrac{1}{2}\sigma^2\right) T + \sigma \sqrt{T}\, \phi\right).$$

sqrt time step

$\phi \sim N(0,1)$

Because this expression is exact and simple it is the best time stepping algorithm to use. . . but only if we have a payoff that only depends on the final asset value, i.e. is European and path independent.

We can then simulate the final asset price in one giant leap, using a time step of T if both of these are true

- the s.d.e. is integrable and $r, \sigma \in \mathbb{R}$ $\exists$ closed form soln
- the contract is European and not path dependent

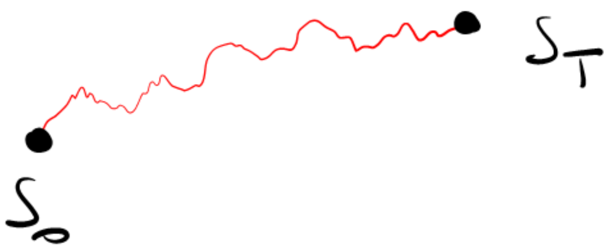
path indep.

Recap from mod I

Simulating the entire path: A method that always works

Price paths are simulated using a discrete version of the stochastic differential equation for S.

An obvious choice is to use



$$\delta S = r S \, \delta t + \sigma S \sqrt{\delta t} \, \phi,$$

$$S_{i+1} = S_i \left(1 + r \, dt + \sigma \, \phi \sqrt{dt}\right)$$

where ϕ is from a standardized Normal distribution.

- This way of simulating the time series is called the **Euler** – Maruyama **method**. This method has an error of $O(\delta t^{1/2})$.

Errors $\times 2$

There are two (at least) sources of error in the Monte Carlo method:

- If the size of the time step is δt then we may introduce errors of $O(\delta t)$ by virtue of the discrete approximation to continuous events   time-step

- Because we are only simulating a finite number of an infinite number of possible paths, the error due to using N realizations of the asset price paths is $O(N^{-1/2})$. — CLT
  no. of simulations

Generating Normal variables — Recap.

fast but inaccurate

- **Quick 'n' dirty**: A useful distribution that is easy to implement on a spreadsheet, and is fast, is the following *approximation* to the Normal distribution:

MILS

$$\frac{12}{n}\left(\sum_{i=1}^{n}\psi_i - \frac{n}{2}\right) \longrightarrow \left(\sum_{i=1}^{12}\psi_i\right) - 6,$$

$n=12$

where the ψ_i are independent random variables, drawn from a uniform distribution over zero to one.

$\psi_i \sim U(0,1)$

There are other methods such as **Box–Muller**, more later.

Polar Marsaglia

acceptance/rejection

Other issues

$$\Delta = \frac{V(S+\delta S) - V(S)}{\boxed{\delta S}}$$

need to do MC twice

- Greeks ; method has to be repeated

- Early exercise (and other decisions)

  → free boundary problem

MC done 3 times

$$\Gamma = \frac{V(S-\delta S) - 2V(S) + V(S+\delta S)}{\delta S^2}$$

Vega $\dfrac{V(\sigma+\delta\sigma) - V(\sigma)}{\boxed{\delta\sigma}}$

$\boxed{\delta t}$

$$\Theta = \frac{V(S, t+\delta t) - V(S,t)}{\delta t}$$

Advantages of Monte Carlo simulations

- The mathematics that you need to perform a Monte Carlo simulation can be very basic

$V(S_1, \ldots, S_n, t)$ Peter J.

- Correlations can be easily modeled, and it is easy to price options on many assets (high-dimensional contracts) R.D.

- It is computationally quite efficient in high dimensions

- There is plenty of software available, at the very least there are spreadsheet functions that will suffice for most of the time

- To get a better accuracy, just run more simulations $\varepsilon \sim \frac{1}{\sqrt{n}}$

- The effort in getting *some* answer is very low

- The models can often be changed without much work

S_0 S_T

Wait till Thurs.

- Complex path dependency can often be easily incorporated

- Many contracts can be priced at the same time

- People accept the technique, and will believe your answers

Less scope for error.

Disadvantages of Monte Carlo simulations

error $= \frac{\sigma}{\sqrt{n}}$

christian chooses

- The method is very slow, you need a lot of simulations to get an accurate answer

$r = r(t)$ $\int r(t) - \frac{1}{2}\sigma^2$

- Finding the greeks can be hard time consuming.

- The method does not cope well with early exercise, due to the 'free boundary'.

Numerical Integration technique $\begin{cases} \text{Simpson's rule} \\ \text{Trapezoidal} \end{cases}$

Milstein Scheme

Finite Difference Methods

Runge Kutta

Monte Carlo simulations can be very slow to converge to the answer, and they do not give us the greeks without further effort.

There is a method that is very similar to the binomial tree method which is the method of choice for certain types of problem.

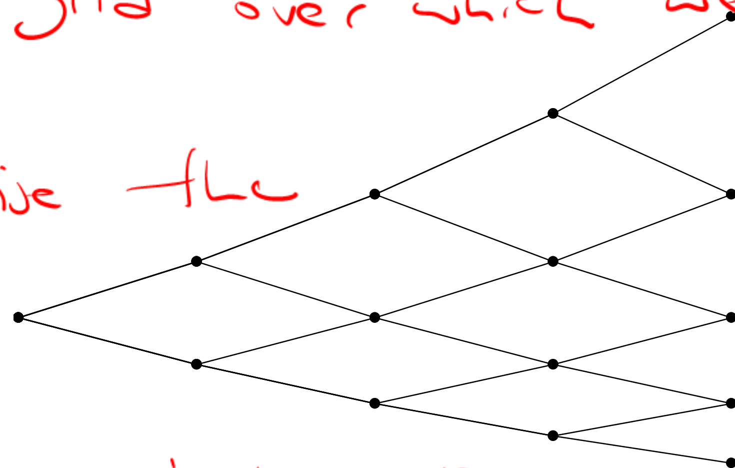
Grids

FDM came to finance from C.F.D.

Recall the shape of the binomial tree. . .

I Create a grid over which we will to integrate

II to discretize the PDE.

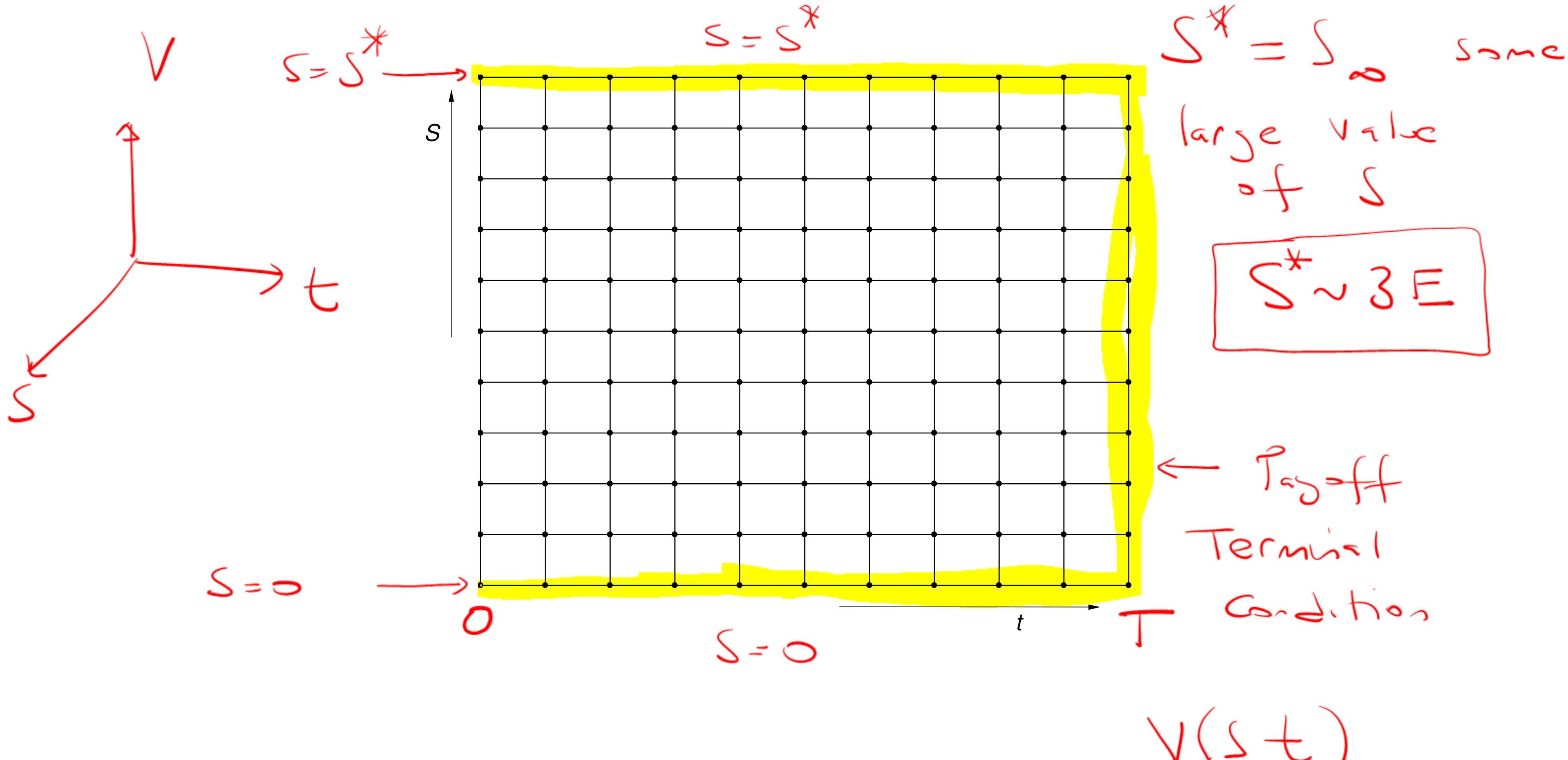


replace each term in the B.S.E with approx^{n}s

PDE $\longrightarrow$ difference eq^{n} (discrete time)

continuous time

The shape of the tree is determined by the asset volatility.



The finite-difference grid.

The finite-difference grid usually has equal time steps and equal S steps.

δt , δS fixed

Discretising time, stock.

Differentiation using the grid : $\frac{\partial V}{\partial t}$, $\frac{\partial V}{\partial S}$, $\frac{\partial^2 V}{\partial S^2}$ approx's

Notation: time step δt and asset step δS. The grid is made up of the points at asset values

I, K total no. of intervals in S and time respectively

$$S := i\,\delta S$$

$$\delta S = \frac{S_\infty}{I} \triangleq$$

and times

defined time as Time to expiry

$$\boxed{t := T - k\,\delta t}$$

$$\tau = T - t$$

$$\delta t = \frac{T}{K}$$

where $0 \le i \le I$ and $0 \le k \le K$.

We will be solving for the asset value going from zero up to the asset value $I\,\delta S$.

The Black–Scholes equation is to be solved for $0 \le S < \infty$ so that $I\,\delta S$ is our approximation to infinity.

step 2 in B.S lecture $\tau = T - t$

Write the option value at each of these grid points as $\frac{\partial}{\partial t} = -\frac{\partial}{\partial \tau}$

time always a superscript.

time

$$V(S, T-t) \longrightarrow V_i^k = V(i\,\delta S, T - k\,\delta t).$$

stock

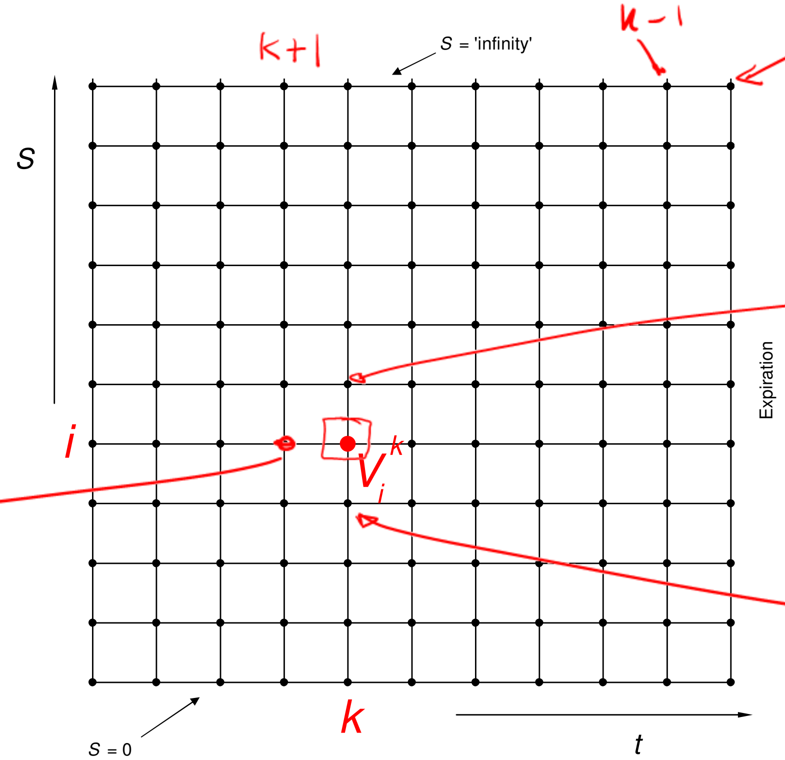
Forward Marching Scheme

- The superscript is the time variable and the subscript the asset variable.

$k = 0, \ldots, K$

time is increasing.

V_i^{k+1}



backward marching.

V_{i+1}^k

V_{i-1}^k

Let us now approx. the derivative terms in the B.S.E.

Approximating θ

The definition of the first time derivative of V is simply

backward diff.

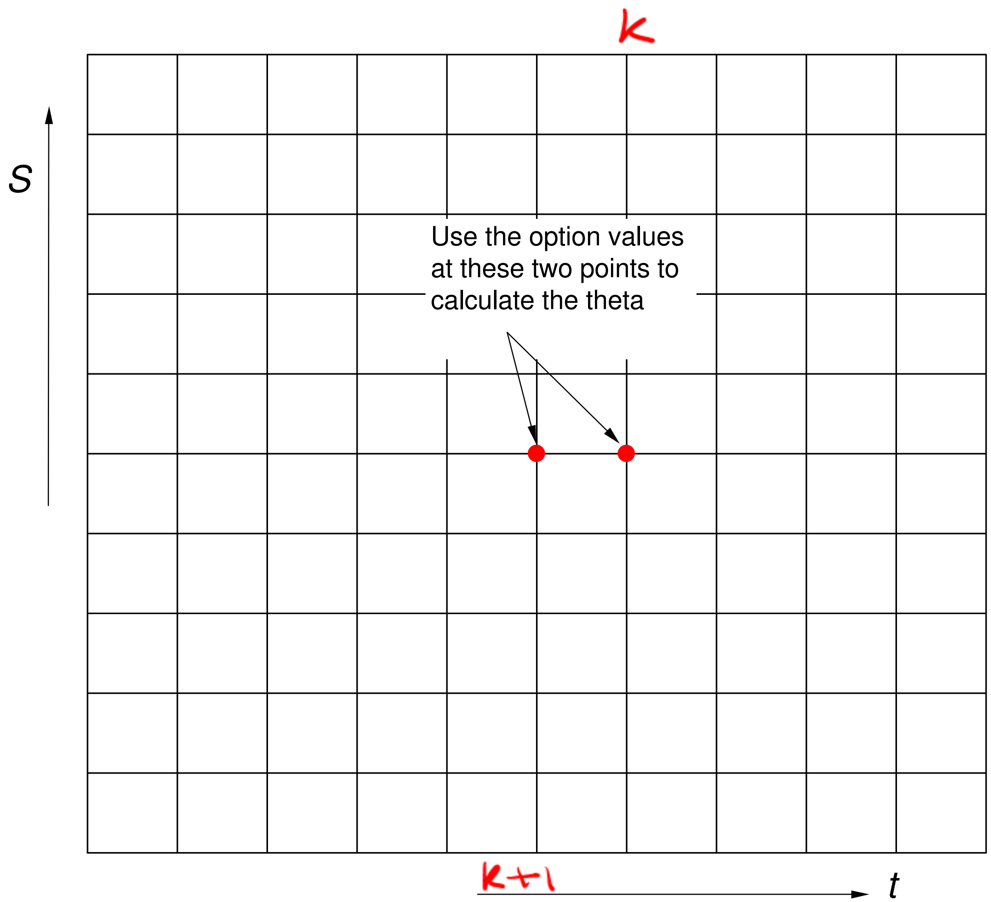
$$\boxed{\frac{\partial V}{\partial t} = \lim_{h \to 0} \frac{V(S,t) - V(S,t-h)}{h}.} \quad h \equiv \delta t$$

It follows naturally that we can approximate the time derivative from our grid of values using

$$\boxed{\frac{\partial V}{\partial t}(S,t) \approx \frac{V_i^k - V_i^{k+1}}{\delta t}.}$$

$\therefore$ of minus coming from $T - k\delta t$

This is our approximation to the option's theta.



Approximating the theta.

How accurate is this approximation?

We can expand the option value at asset value S and time $t - \delta t$ in a Taylor series about the point S, t as follows.

T.S.E $(t-\delta t, t)$

$$V(S, t - \delta t) = V(S, t) - \delta t \frac{\partial V}{\partial t}(S, t) + O(\delta t^2).$$

In terms of values at grid points this is just

$$V_i^k = V_i^{k+1} + \delta t \frac{\partial V}{\partial t}(S, t) + O(\delta t^2).$$

$$V(S, t+\delta t) = V(S, t) + \frac{\partial V}{\partial t}\delta t + O(\delta t^2)$$

Which, upon rearranging, is

$V(S,t)$ $\quad V(S,t+\delta t)$

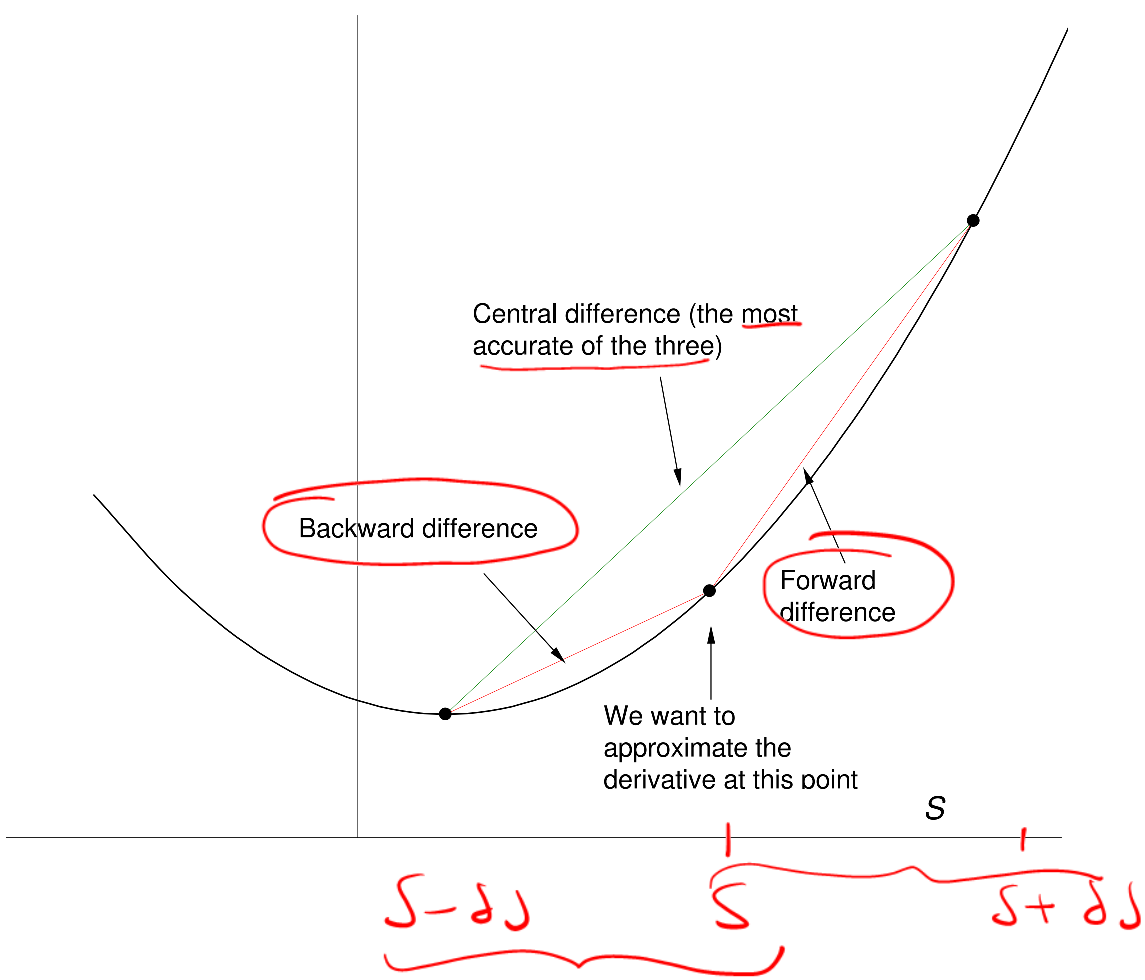
$$\frac{\partial V}{\partial t}(S, t) = \frac{V_i^k - V_i^{k+1}}{\delta t} + O(\delta t).$$

$$\frac{V(S, t+\delta t) - V(S, t)}{\delta t} = \frac{\partial V}{\partial t} + O(\delta t)$$

- The error is $O(\delta t)$.

Approximating $\Delta = \frac{\partial V}{\partial S}$

Examine a cross section of the grid at one of the time steps.



These three approximations are

choices for $\frac{\partial V}{\partial S}$

$$\frac{V_{i+1}^k - V_i^k}{\delta S}, \quad \text{fwd}$$

$$\frac{V_i^k - V_{i-1}^k}{\delta S} \quad \text{bwd}$$

$$\text{and} \quad \boxed{\frac{V_{i+1}^k - V_{i-1}^k}{2\,\delta S}.} \quad \text{centred}$$

These are called a **forward difference**, a **backward difference** and a **central difference** respectively.

One of these approximations is better than the others. (last one)

From a Taylor series expansion of the option value about the point $S+\delta S$, t we have

$$① \quad V(S+\delta S,t) = V(S,t) + \delta S\frac{\partial V}{\partial S}(S,t) + \tfrac{1}{2}\delta S^2\frac{\partial^2 V}{\partial S^2}(S,t) + \frac{1}{3!}\delta S^3\frac{\partial^3 V}{\partial S^3} + \cancel{O(\delta S^3)} + O(\delta S^4)$$

Similarly,

$$② \quad V(S-\delta S,t) = V(S,t) - \delta S\frac{\partial V}{\partial S}(S,t) + \tfrac{1}{2}\delta S^2\frac{\partial^2 V}{\partial S^2}(S,t) - \frac{1}{3!}\delta S^3\frac{\partial^3 V}{\partial S^3} + \cancel{O(\delta S^3)} + O(\delta S^4)$$

$① - ②$: $V(S+\delta S) - V(S-\delta S) = 2\,\delta S\frac{\partial V}{\partial S} + O(\delta S^3)$

$$\frac{\partial V}{\partial S} = \frac{V(S+\delta S) - V(S-\delta S)}{2\,\delta S} + O(\delta S^2) \approx \frac{V_{i+1}^k - V_{i-1}^k}{2\,\delta S}$$

$① + ②$: $V(S+\delta S) + V(S-\delta S) = 2V(S) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\delta S^2 + O(\delta S^4)$

$$\frac{V(S-\delta S) - 2V(S) + V(S+\delta S)}{\delta S^2} = \frac{\partial^2 V}{\partial S^2}\cancel{\delta S^2} + O(\delta S^{\cancel{4}\,2})$$

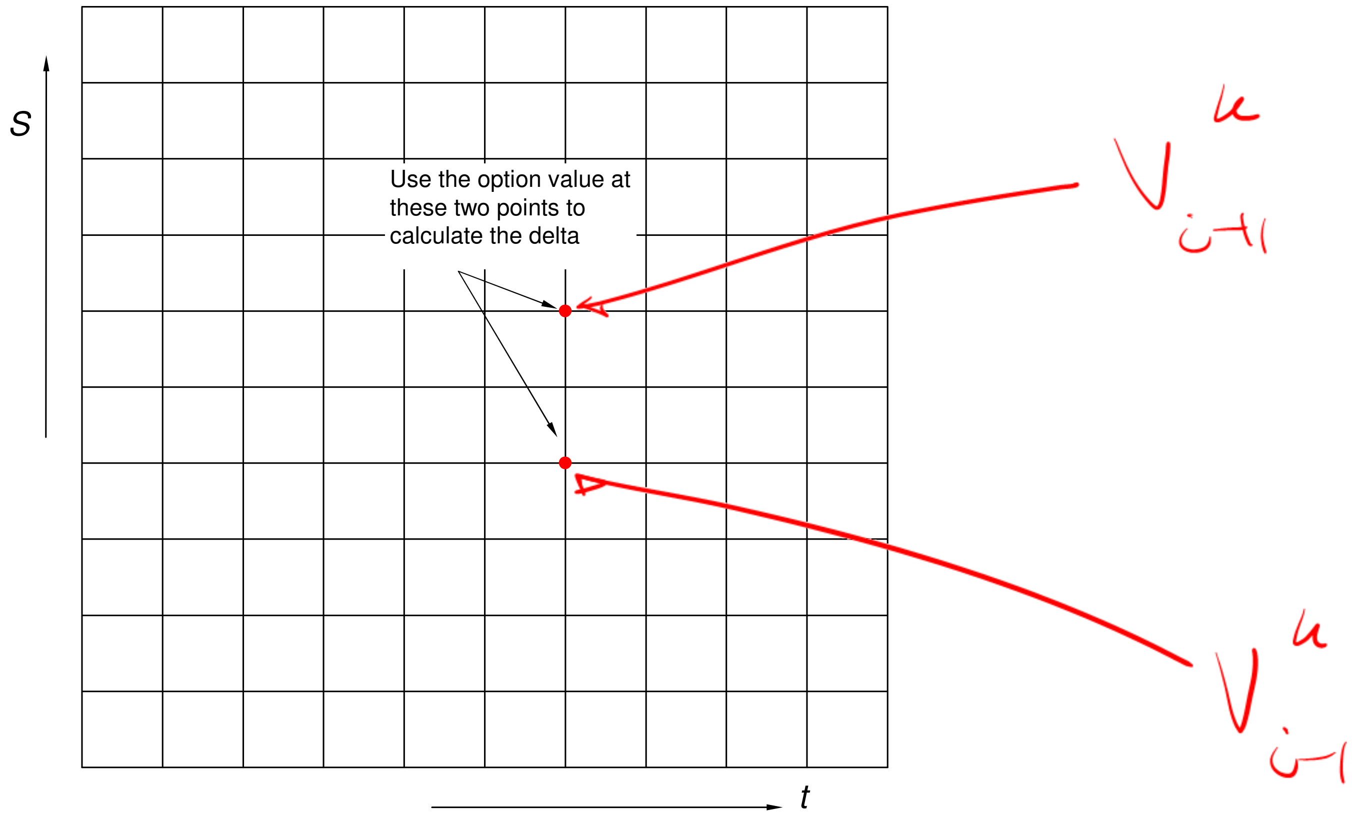
$$\frac{\partial^2 V}{\partial S^2} \approx \frac{V_{i-1}^k - 2V_i^k + V_{i+1}^k}{\delta S^2}$$

From these we get

$$\frac{\partial V}{\partial S}(S,t) = \frac{V_{i+1}^k - V_{i-1}^k}{2\,\delta S} + O(\delta S^2).$$

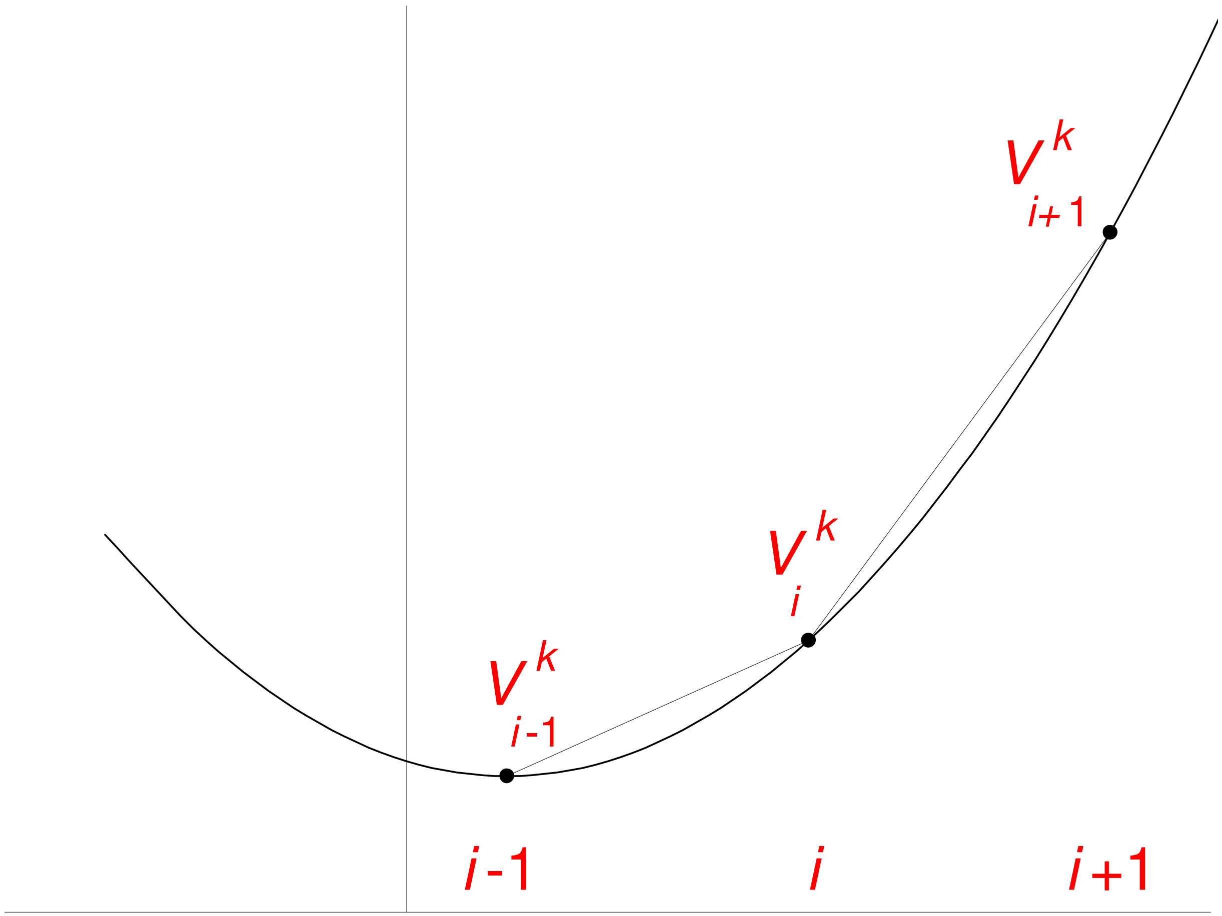
- The central difference has an error of $O(\delta S^2)$, the error in the forward and backward differences are both much larger, $O(\delta S)$.

The central difference calculated at S requires knowledge of the option value at $S + \delta S$ and $S - \delta S$.


S
Use the option value at these two points to calculate the delta
t
V_{i+1}^{k}
V_{i-1}^{k}

Approximating Γ

Gamma is the sensitivity of the delta to the underlying.



Calculate the delta half way between i and $i+1$, and the delta half way between $i-1$ and i ... and difference them!

$$\text{Forward difference } = \frac{V_{i+1}^k - V_i^k}{\delta S}.$$

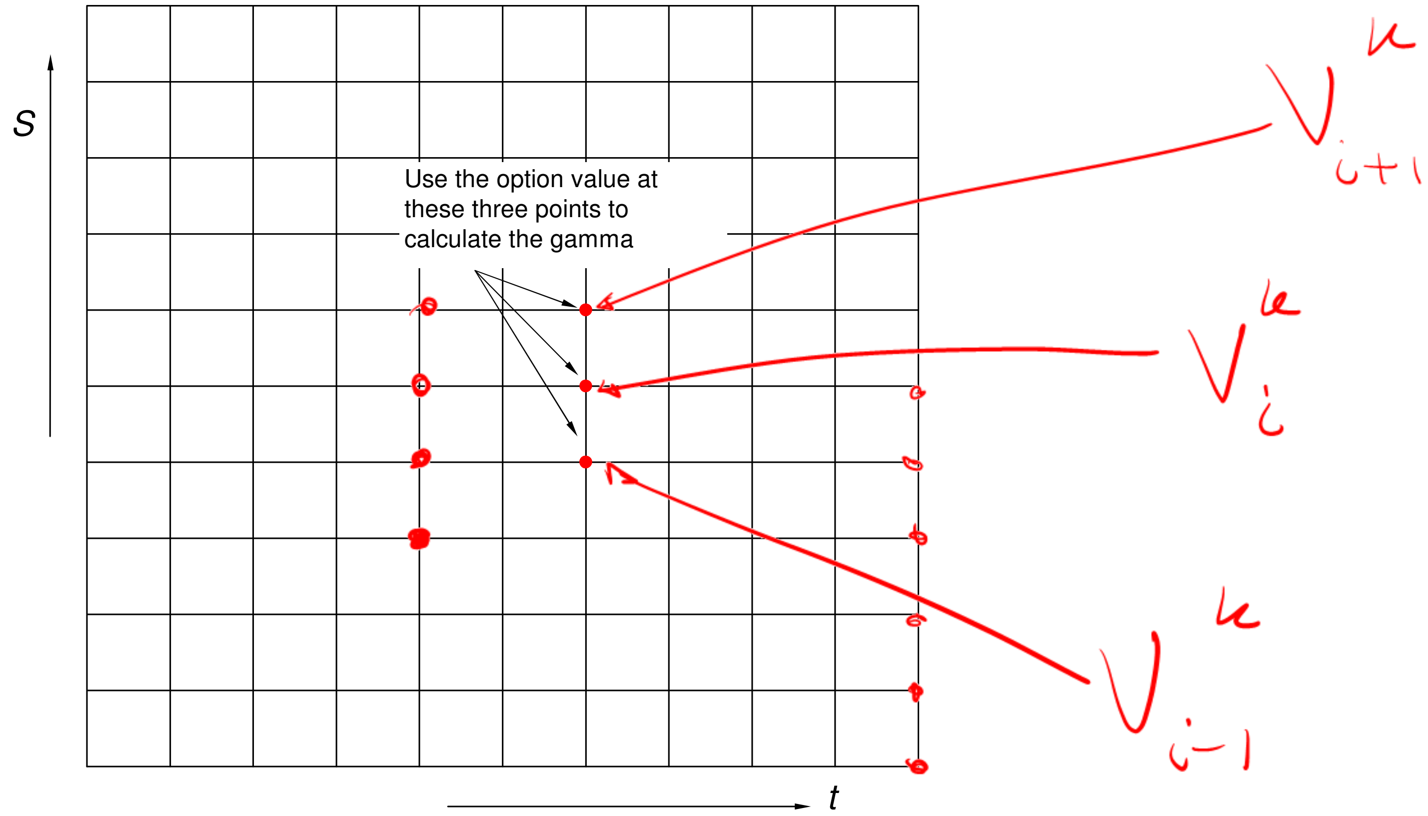
$$\text{Backward difference } = \frac{V_i^k - V_{i-1}^k}{\delta S}.$$

Therefore the natural approximation for the gamma is

$$\frac{\partial^2 V}{\partial S^2}(S,t) \approx \frac{\frac{V_{i+1}^k - V_i^k}{\delta S} - \frac{V_i^k - V_{i-1}^k}{\delta S}}{\delta S}$$

$$= \frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{\delta S^2} + O(\delta S^2)$$

The error in this approximation is also $O(\delta S^2)$.



S
Use the option value at
these three points to
calculate the gamma
t
V_{i+1}^k
V_i^k
V_{i-1}^k

Final conditions and payoffs

We know that at expiry the option value is just the payoff function. At expiry we have

$$V(S,T) = \text{Payoff}(S) \qquad \text{at} \quad t = T$$

or, in our finite-difference notation,

expiry

$k = 0$

$$V_i^0 = \text{Payoff}(i\,\delta S).$$

$t = T - 0 \times \delta t$

The right-hand side is a known function.

For example, if we are pricing a call option we have

$$V_i^0 = \max(\overbrace{i\,\delta S}^{S} - E, 0).$$

This final condition will get our finite-difference scheme started.

Summary:

$$\frac{\partial V}{\partial t} \approx \frac{V_i^k - V_i^{k+1}}{\delta t}; \qquad \frac{\partial V}{\partial S} \approx \frac{V_{i+1}^k - V_{i-1}^k}{2\,\delta S}$$

$$\frac{\partial^2 V}{\partial S^2} \approx \frac{V_{i-1}^k - 2V_i^k + V_{i+1}^k}{\delta S^2}$$

Subst. in B.S.E.

The explicit finite-difference method

Simplest of all F.D.M

The Black–Scholes equation is

y = f(x) explicit fn.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

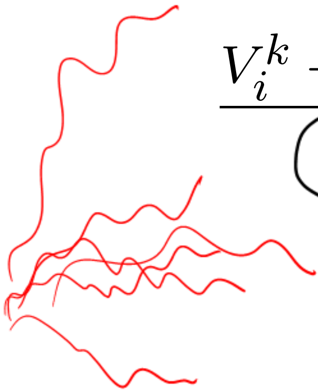
g(x, y) = 0 implicit fn.

Write this as

$$\frac{\partial V}{\partial t} + a(S,t)\frac{\partial^2 V}{\partial S^2} + b(S,t)\frac{\partial V}{\partial S} + c(S,t)V = 0$$

so that we can examine more general problems.

Using the above approximations



unknown $\qquad \frac{1}{2}\sigma^2 i^2 \cancel{\delta S^2} \qquad (r-D)\, i\, \cancel{\delta S}$

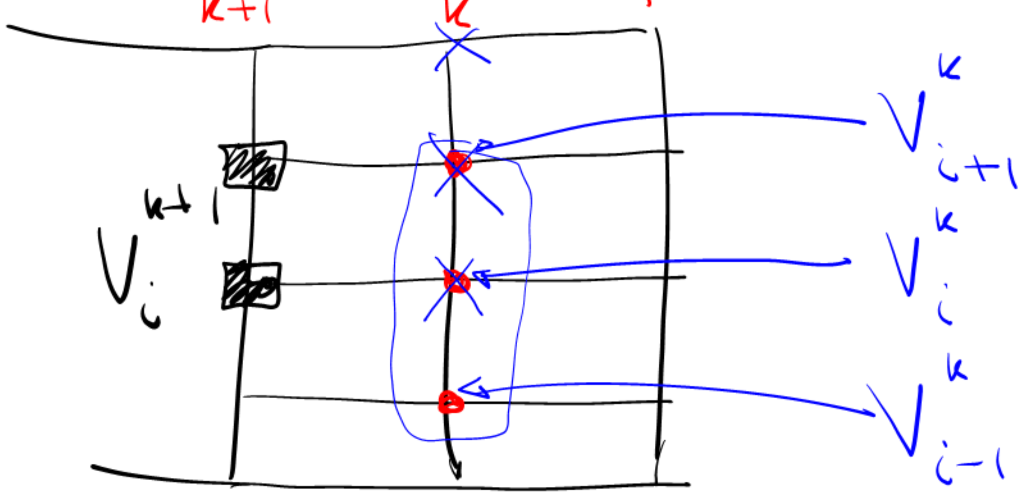
$$\frac{V_i^k - V_i^{k+1}}{\delta t} + \cancel{a_i^k}\left(\frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{\cancel{\delta S^2}}\right) + \cancel{b_i^k}\left(\frac{V_{i+1}^k - V_{i-1}^k}{2\cancel{\delta S}}\right)$$

$-r \qquad 0$

$$\cancel{+c_i^k} V_i^k = \cancel{O(\delta t, \delta S^2)}.$$

This method is $O(\delta t, \delta S^2)$

This can be rearranged. . . In the above all option prices are known except V_i^{k+1}



$$= F\left(V_{i-1}^k,\ V_i^k,\ V_{i+1}^k\right)$$

Difference eq$^{\text{n}}$

$$V_i^{k+1} = \overset{\gamma_i}{\cdots} V_{i+1}^k + \overset{\beta_i}{\cdots} V_i^k + \overset{\alpha_i}{\cdots} V_{i-1}^k.$$

for $i = 1, \ldots, I-1$

$i = I$ does not exist

$i = 0$ does not exist

This is an equation for V_i^{k+1} given three option values at time k.

(That's why this is called the **explicit finite-difference method**.)

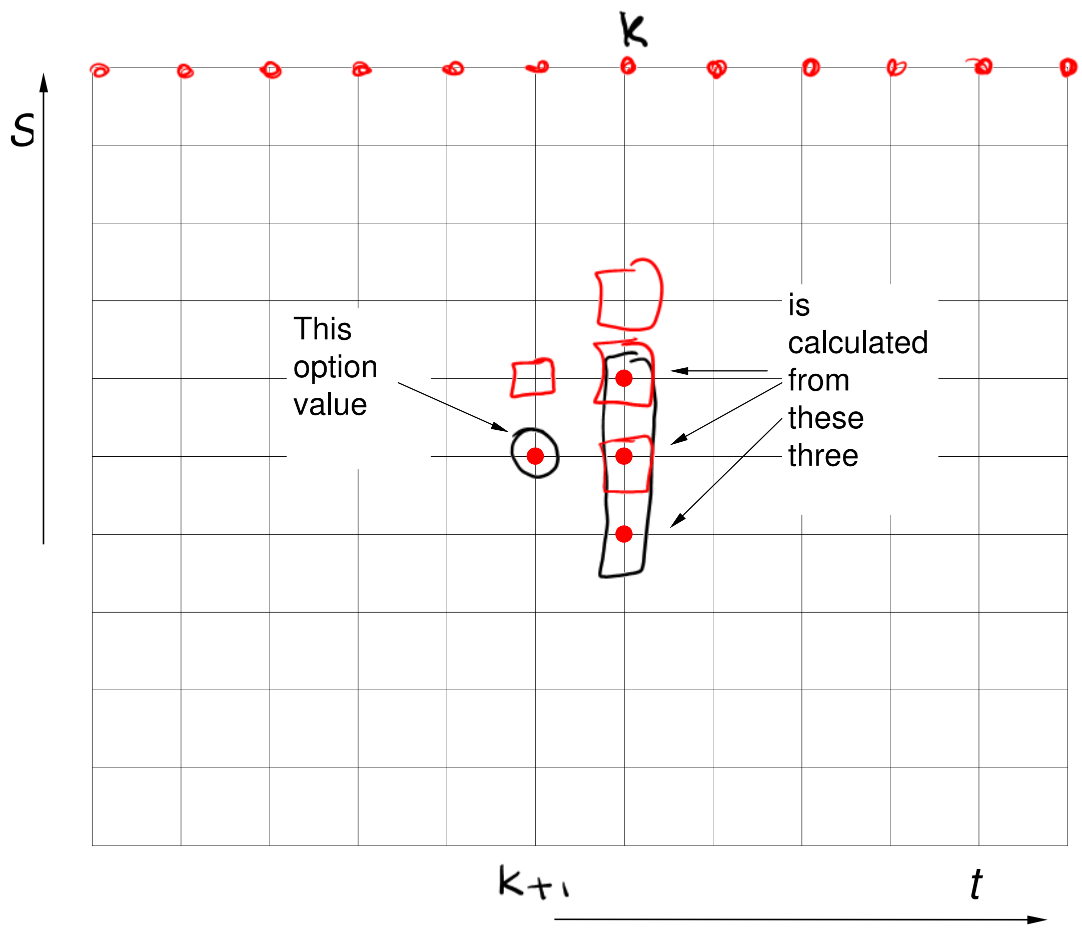
$$V_i^{k+1} = \overline{V_i^k} + \frac{1}{2}\sigma^2 i^2 \delta t \left(V_{i-1}^k - 2V_i^k + V_{i+1}^k\right) + \frac{(r-D)}{2} i\,\delta t \left(V_{i+1}^k - V_{i-1}^k\right) - r V_i^k \delta t$$

V_{i-1}^k : $\frac{1}{2}\sigma^2 i^2 \delta t - \frac{1}{2}(r-D)i\,\delta t$ $\quad \frac{1}{2}\left[\sigma^2 i^2 - (r-D)\right]\delta t$ $\quad \alpha_i$

V_i^k : $1 - \sigma^2 i^2 \delta t - r\delta t$ $\quad 1 - (r + \sigma^2 i^2)\delta t$ $\quad \beta_i$

V_{i+1}^k : $\frac{1}{2}\left(\sigma^2 i^2 + (r-D)i\right)\delta t$ $\quad \gamma_i$

The relationship between the option values in the algorithm is shown in the figure below.



Points to note:

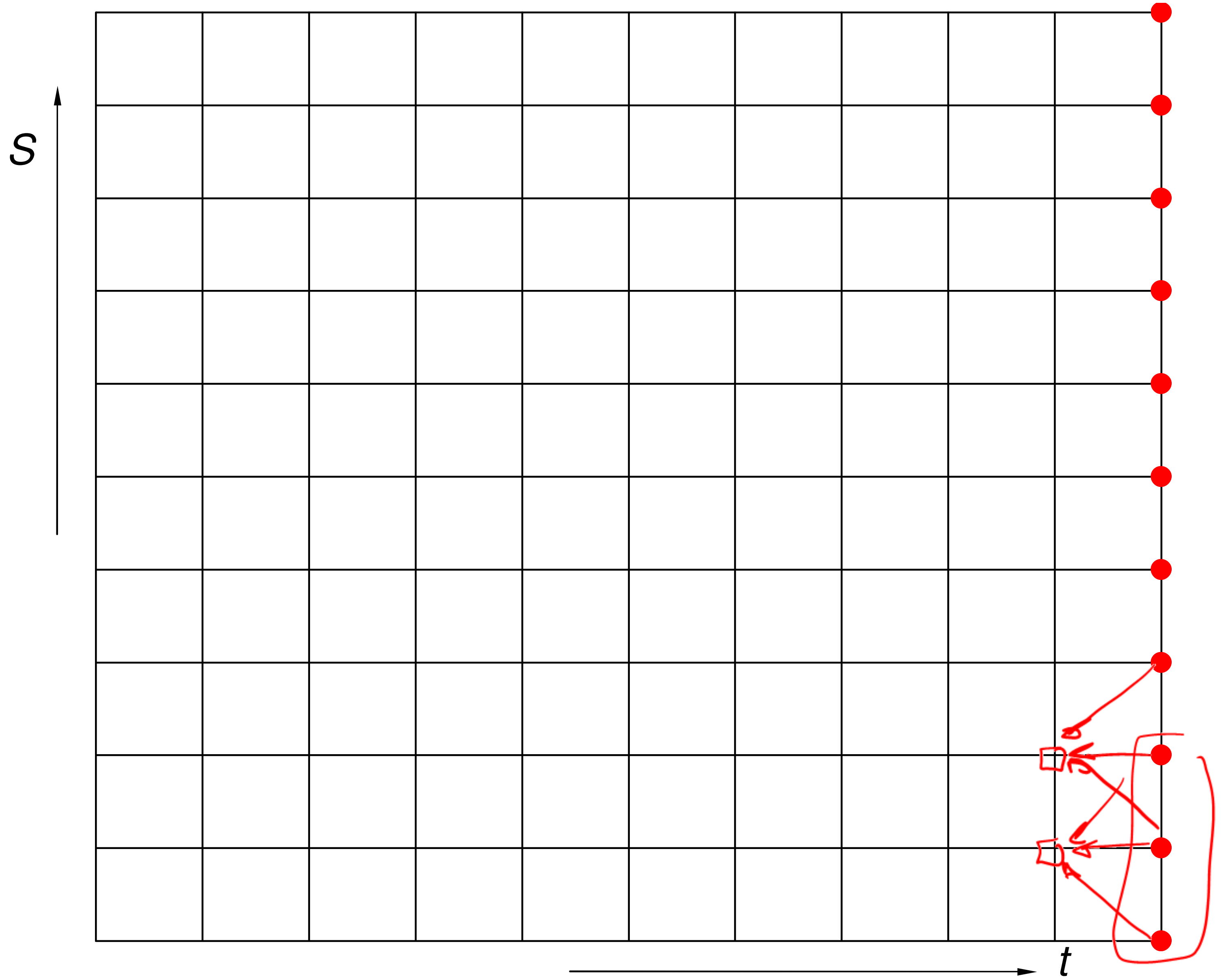
- The time derivative uses the option values at 'times' k and $k+1$, whereas the other terms all use values at k.

- The gamma term is a central difference, in practice one never uses anything else.

- The delta term uses a central difference. There are often times when a one-sided derivative is better. We'll see examples later.

- The asset- and time-dependent functions a, b and c have been valued at $S_i = i\,\delta S$ and $t = T - k\,\delta t$ with the obvious notation.

- The error in the equation is $O(\delta t, \delta S^2)$.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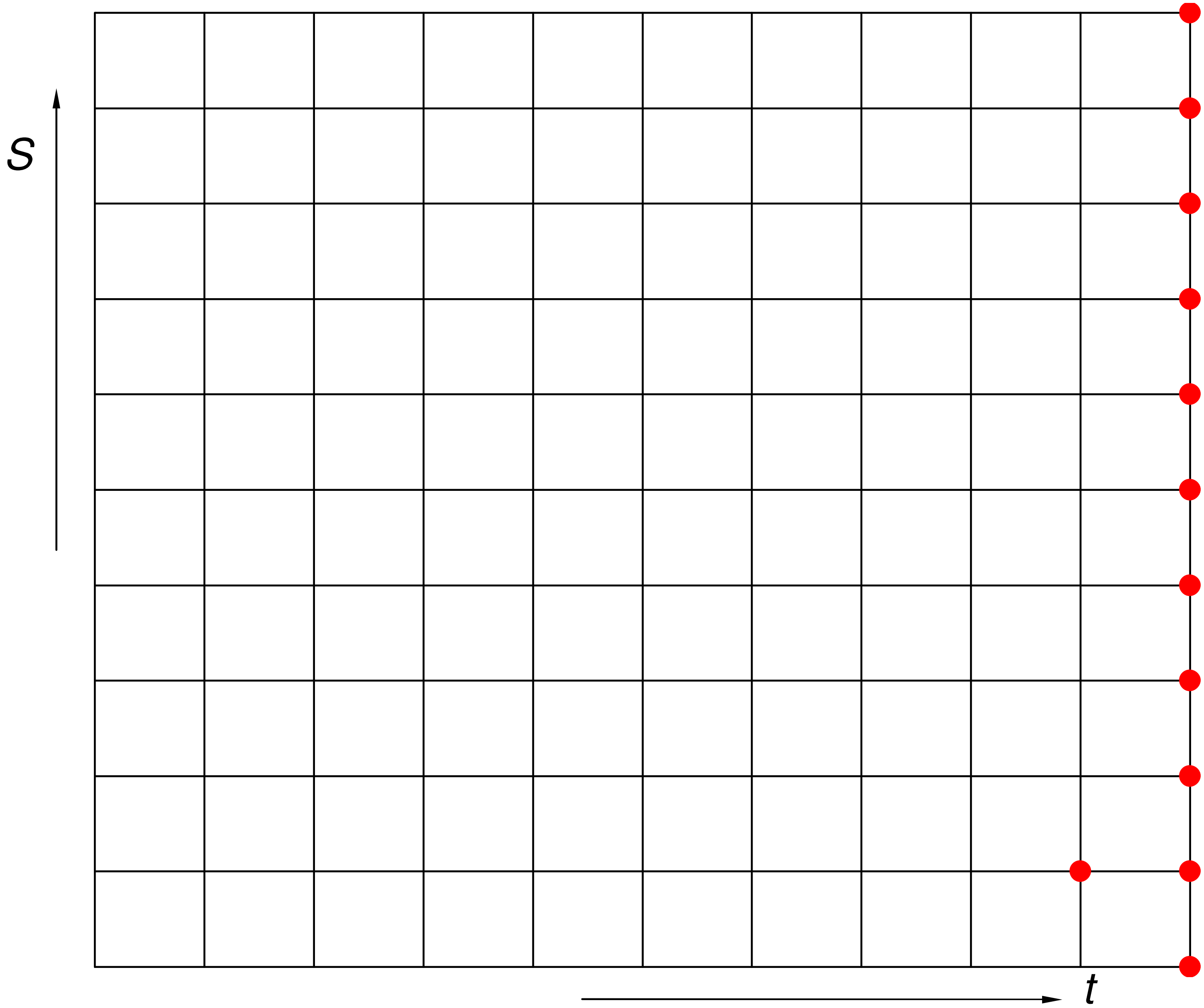
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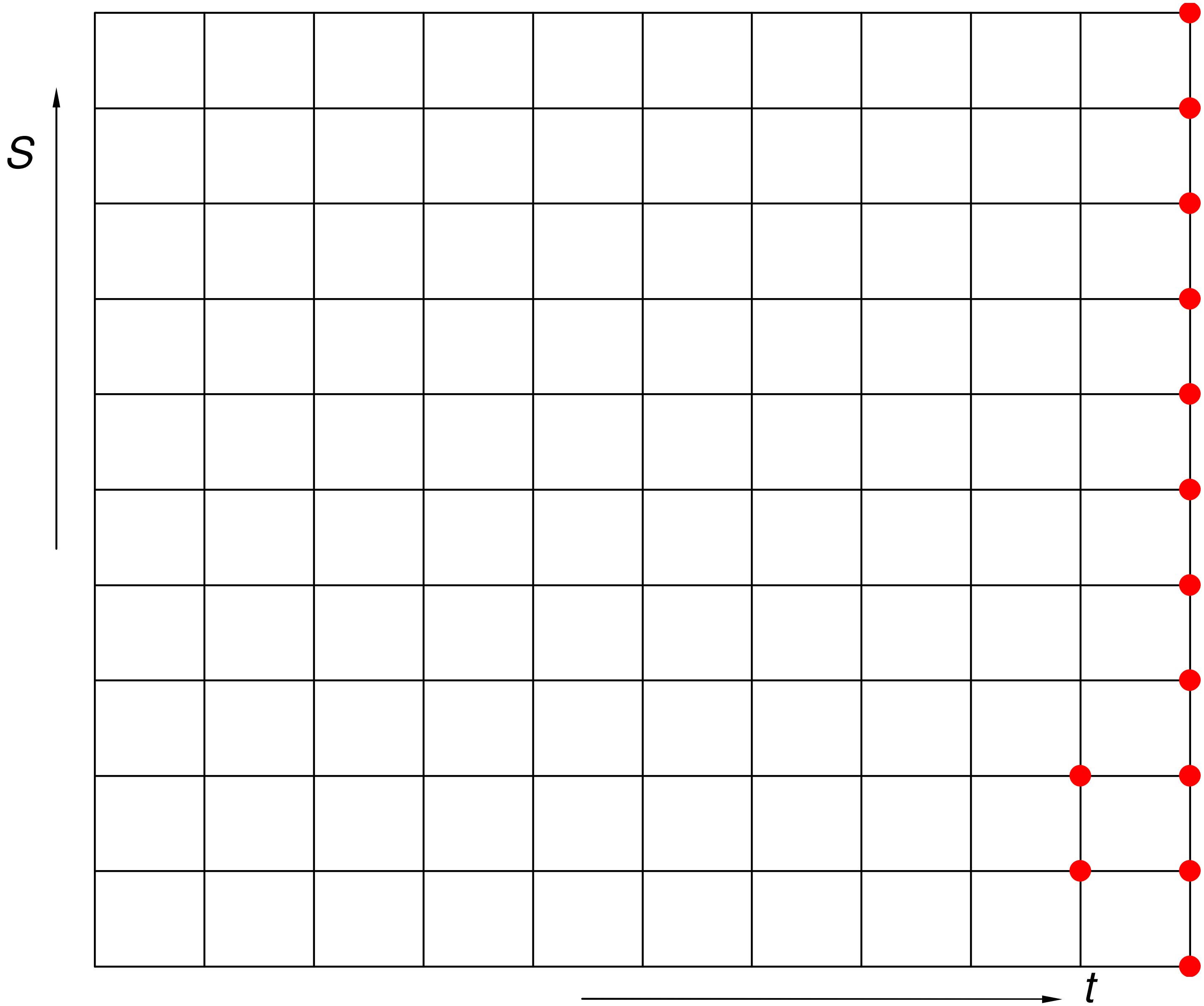

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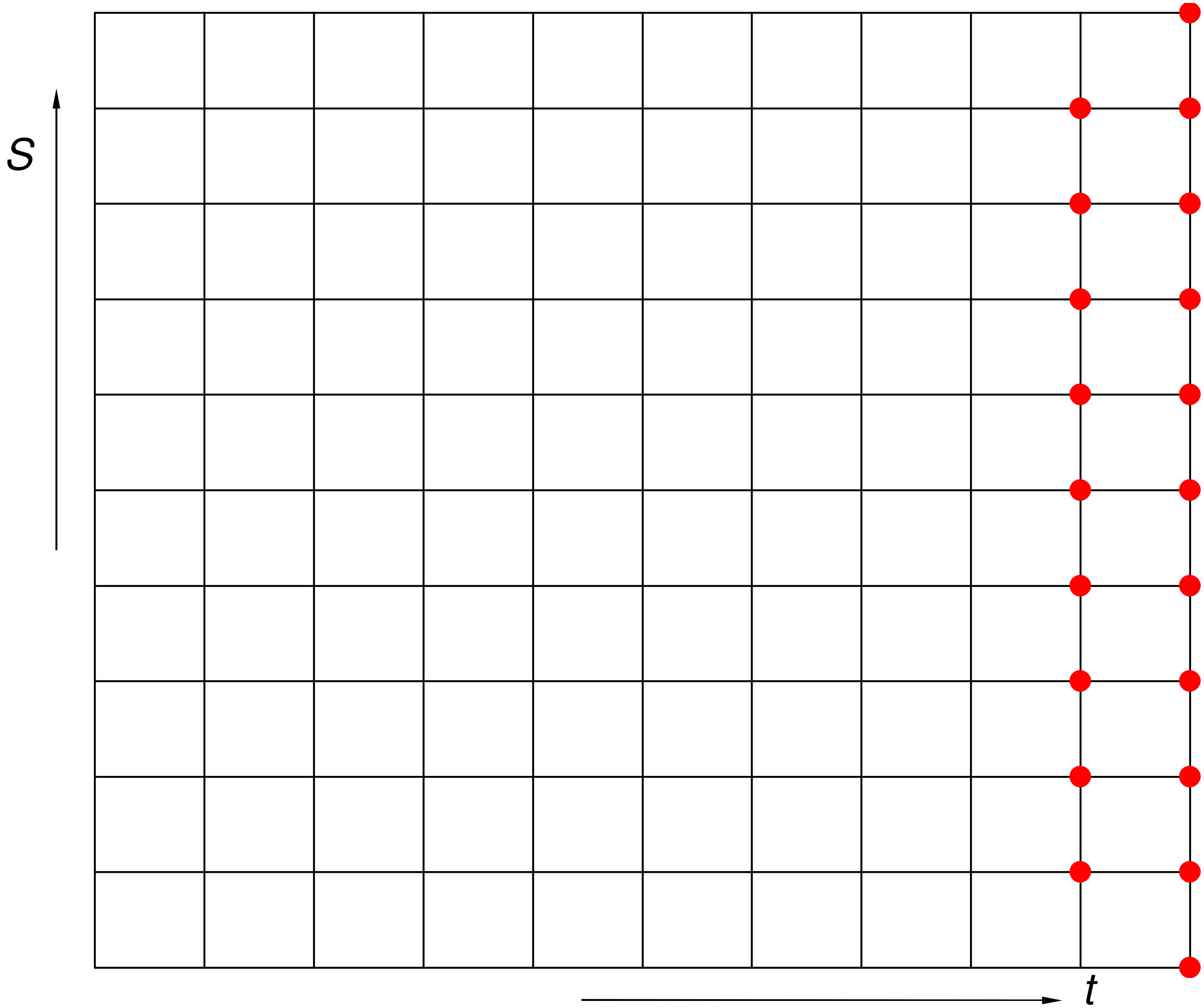
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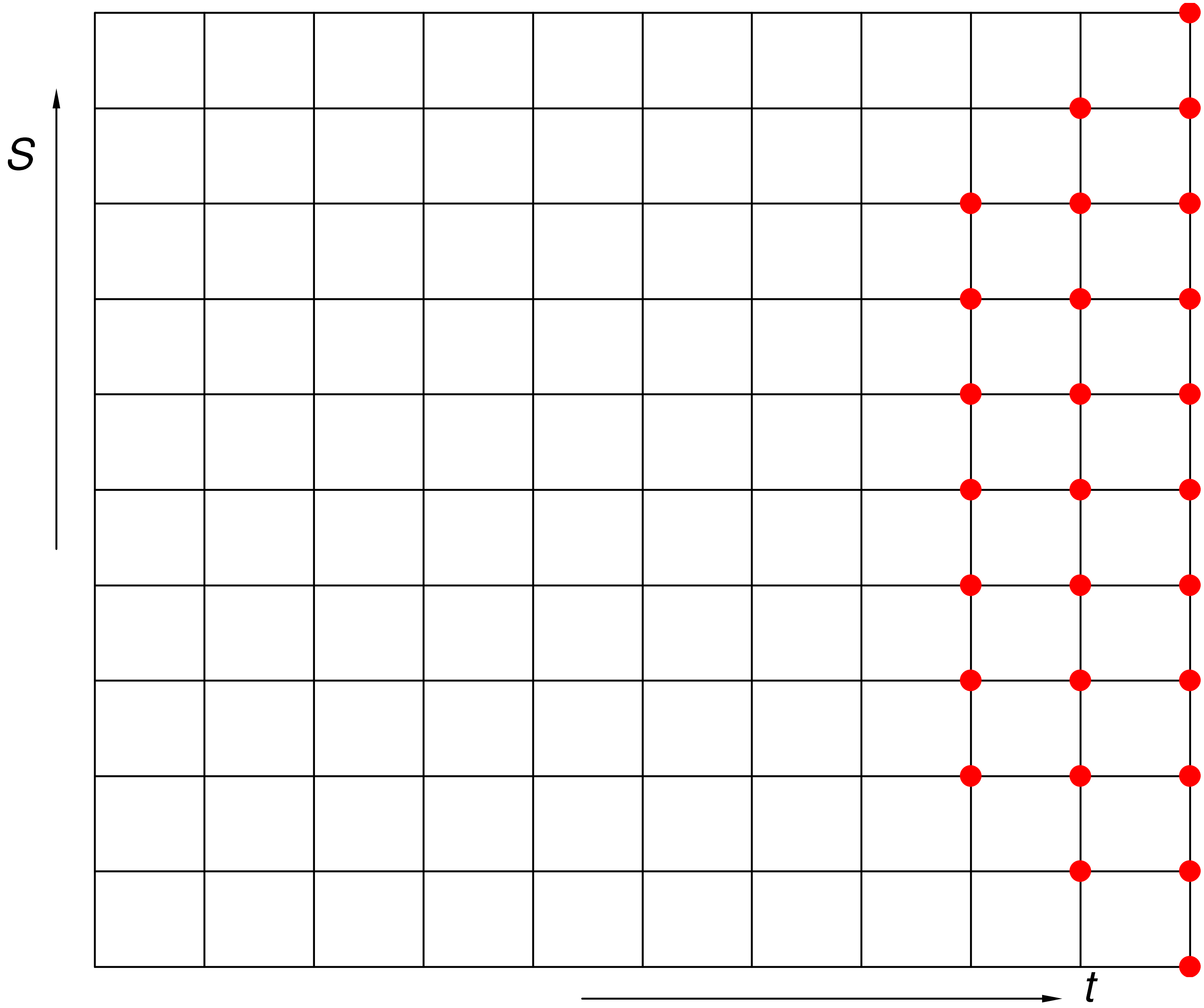
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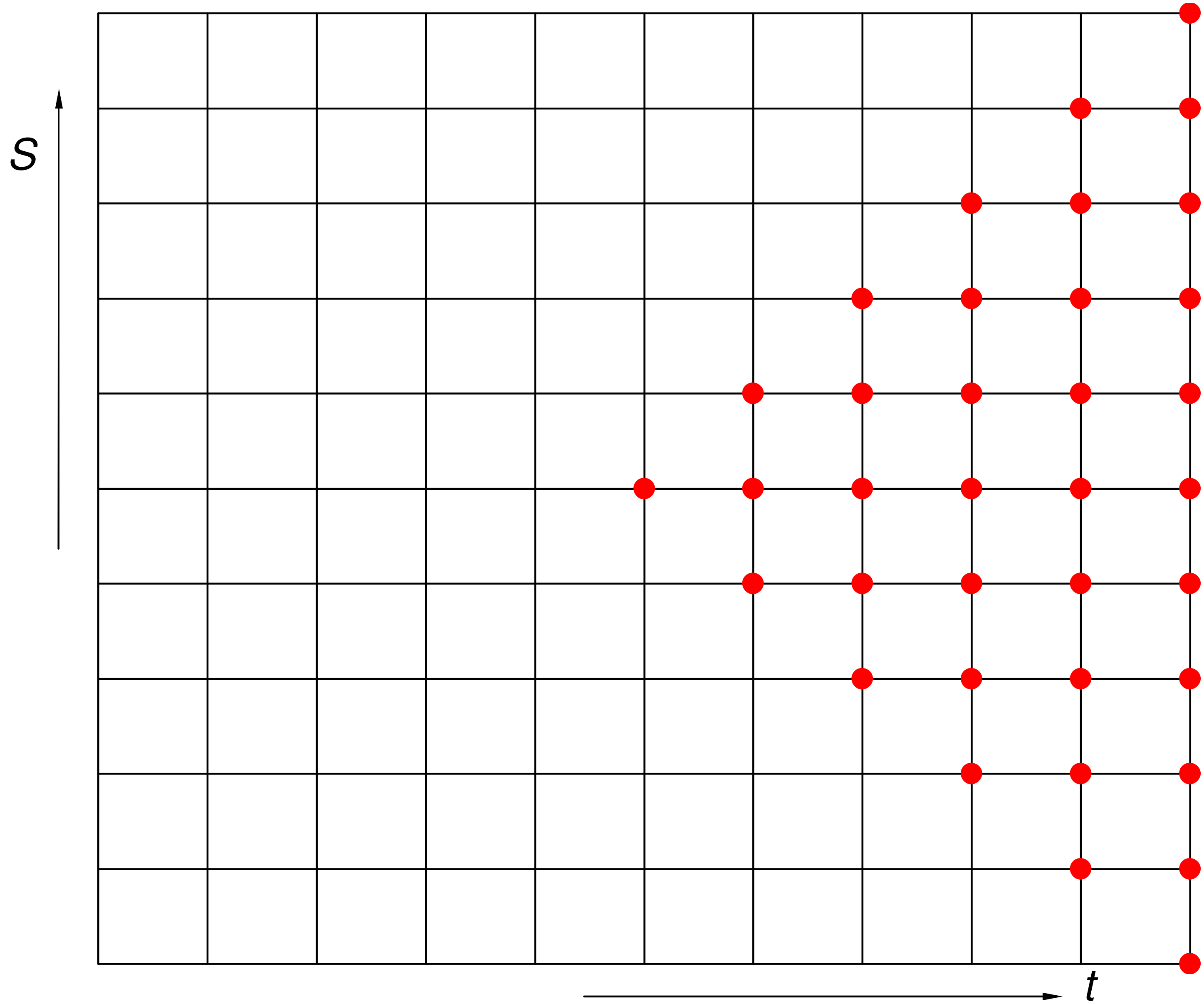

S
t

Boundary conditions

We must specify the option values at the extremes of the region, at $S = 0$ and at $S = I\,\delta S$. They will depend on our option.

Example 1: Call option at $S = 0$

At $S = 0$ we know that the value is always zero, therefore

$$V_0^k = 0.$$

Example 2: Call option for large S

For large S the call value asymptotes to $S - Ee^{-r(T-t)}$. Thus

$$V_I^k = I\,\delta S - Ee^{-rk\,\delta t}.$$

Example 3: Put option at $S = 0$

At $S = 0$ $V = Ee^{-r(T-t)}$. I.e.

$$V_0^k = Ee^{-rk\,\delta t}.$$

Example 4: Put option for large S

The put option becomes worthless for large S and so

$$V_I^k = 0.$$

More sophisticated requiring more thought

Example 5*: General condition at $S = 0$

A useful boundary condition to apply at $S = 0$ is that the diffusion and drift terms 'switch off.'

i.e put $S=0$ in B.S.E $\longrightarrow \frac{\partial V}{\partial t} - rV = 0$

$$\frac{\partial V}{\partial t}(0,t) - rV(0,t) = 0 \qquad V_0^{k+1} = (1 - r\delta t)V_0^{\cancel{k+1}}.$$

at $S=0$ $i=0$ $\quad \frac{V_0^k - V_0^{k+1}}{\delta t} - rV_0^k = 0$ rearrange

In our earlier diff eq^{n} $V_i^{k+1} = \alpha_i V_{i-1}^k + \beta_i V_i^k + \gamma_i V_{i+1}^k$

at $i=0$ $\quad \alpha_0 = \gamma_0 = 0 \quad \beta_0 = 1 - r\delta t$

Example 6*: General condition at infinity

Consider $V_i^{k+1} = \alpha_i V_{i-1}^k + \beta_i V_i^k + \gamma_i V_{i+1}^k$ at $i = I$

When the option has a payoff that is linear in the underlying for large S then

V_{I+1}^k is not defined

S^* is $\lim_{S\to\infty}$

for large S $\Delta = \Delta(t)$

$$\frac{\partial^2 V}{\partial S^2}(S,t) \to 0 \quad \text{as } S \to \infty.$$

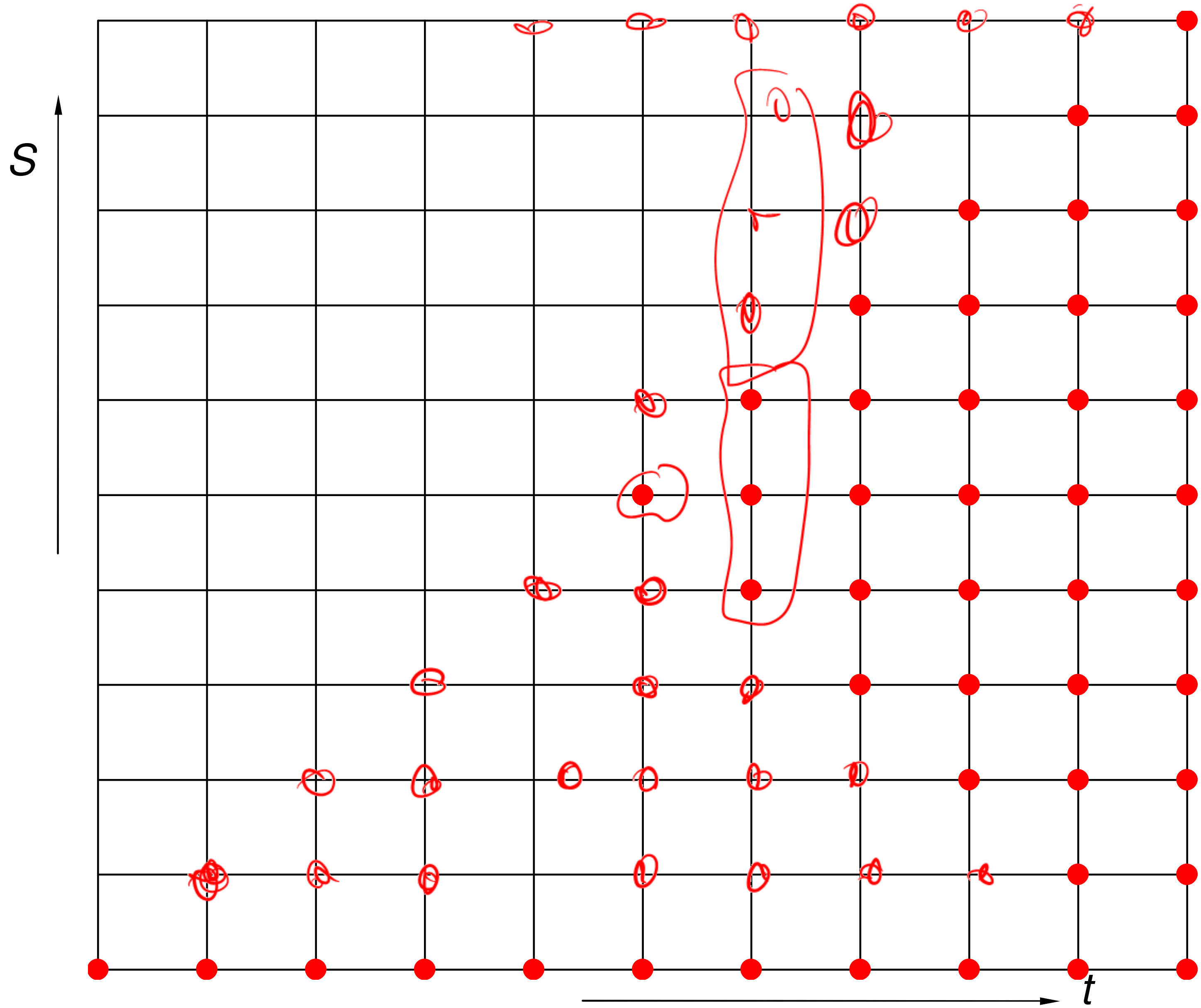
$\Gamma = \frac{\partial}{\partial S}\Delta = 0$ for large S

The finite-difference representation is

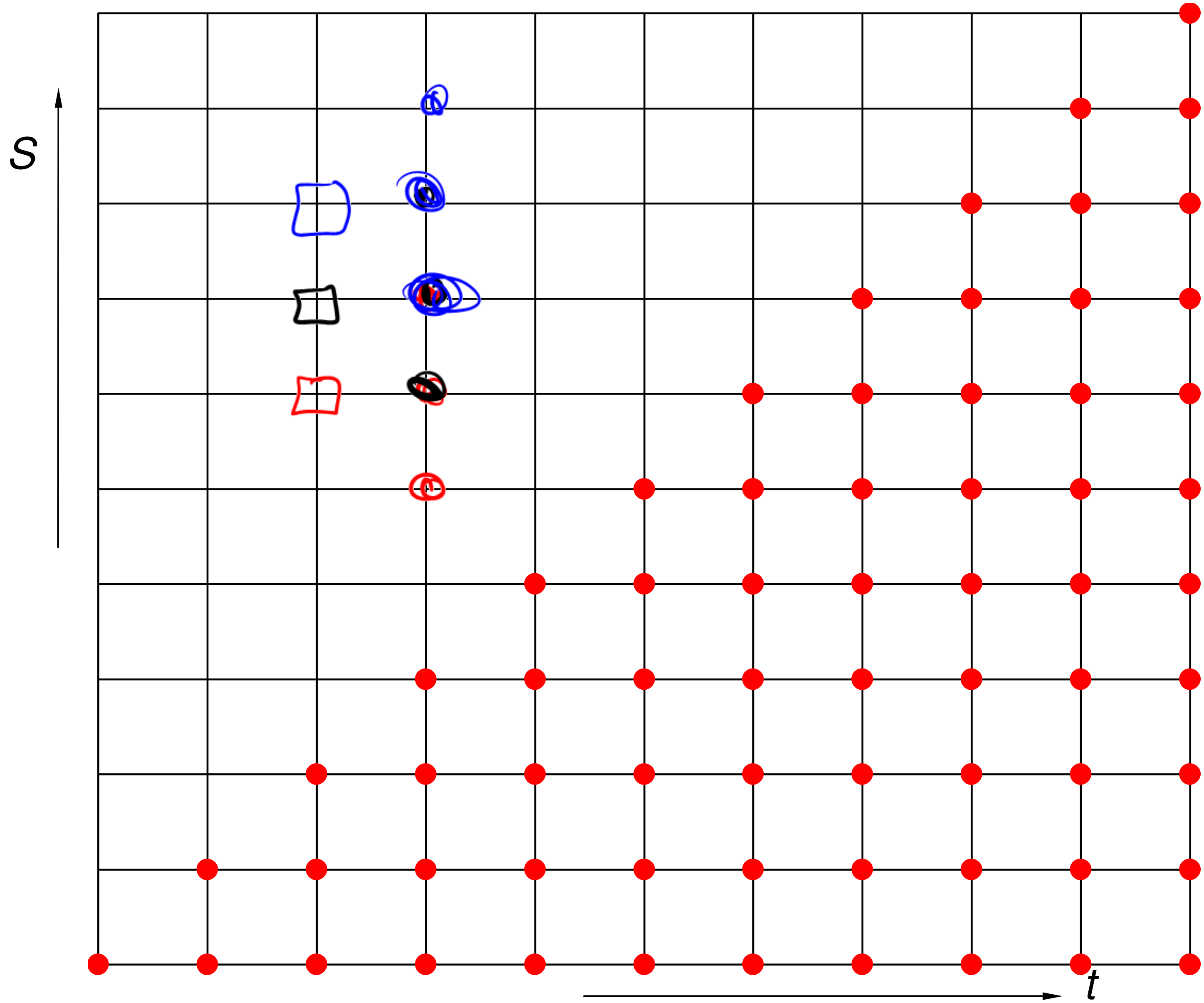
$$\boxed{V_I^k = 2V_{I-1}^k - V_{I-2}^k.}$$

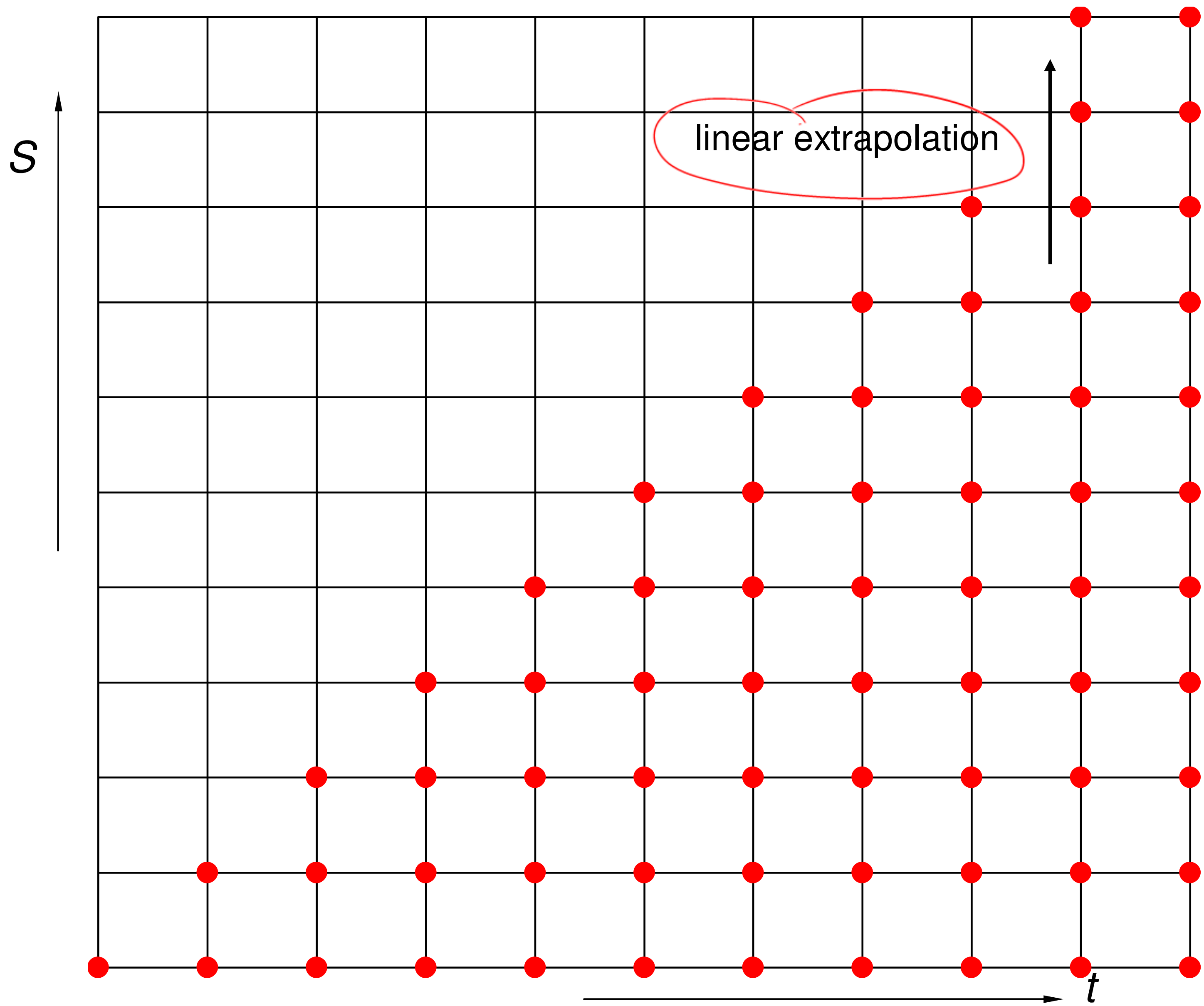
at $i = I$ $\Gamma \approx \frac{V_{I-1}^k - 2V_I^k + V_{I+1}^k}{\delta S^2} \approx 0$

Now rearrange $V_{I+1}^k = 2V_I^k - V_{I-1}^k$

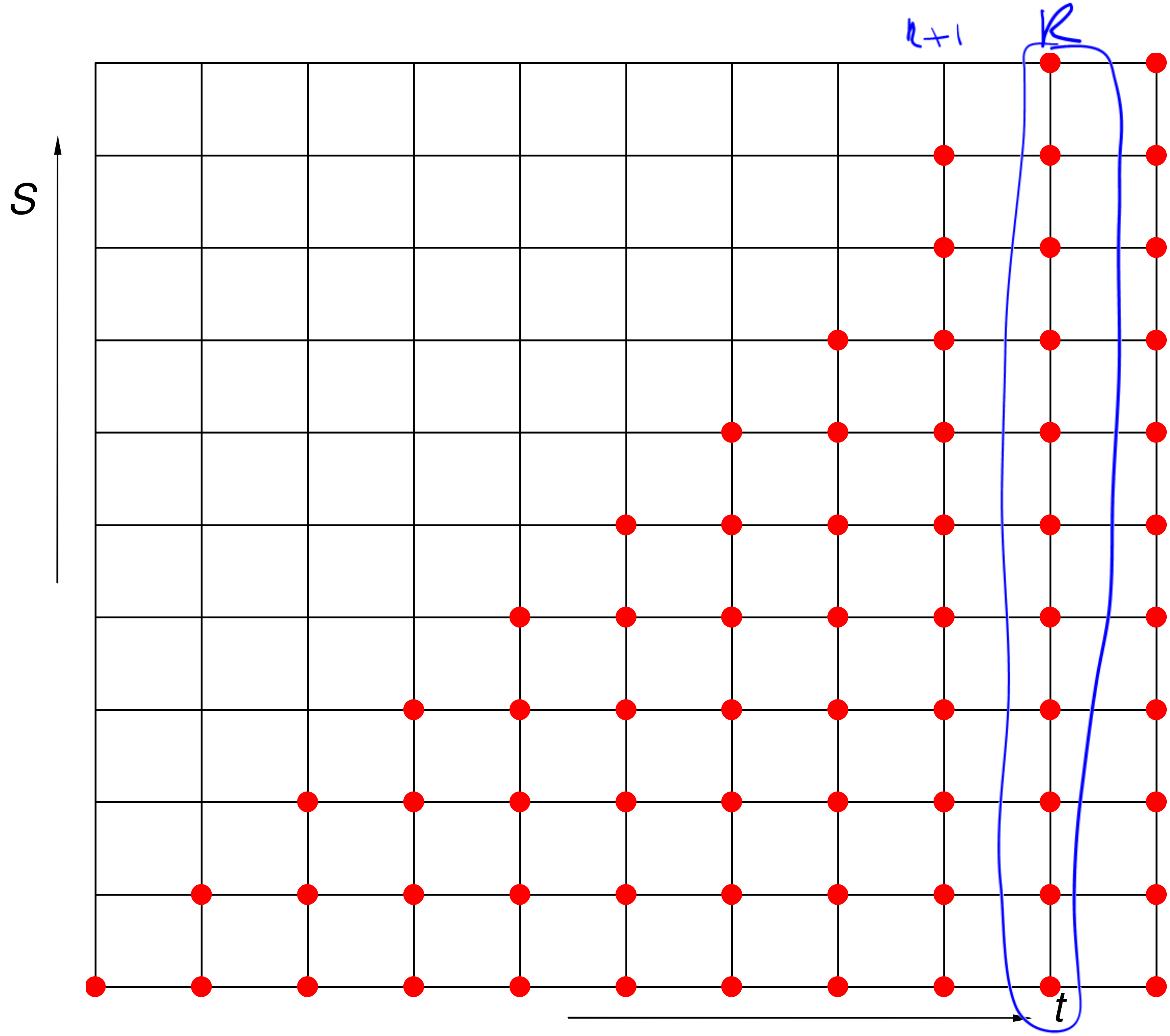


S
t

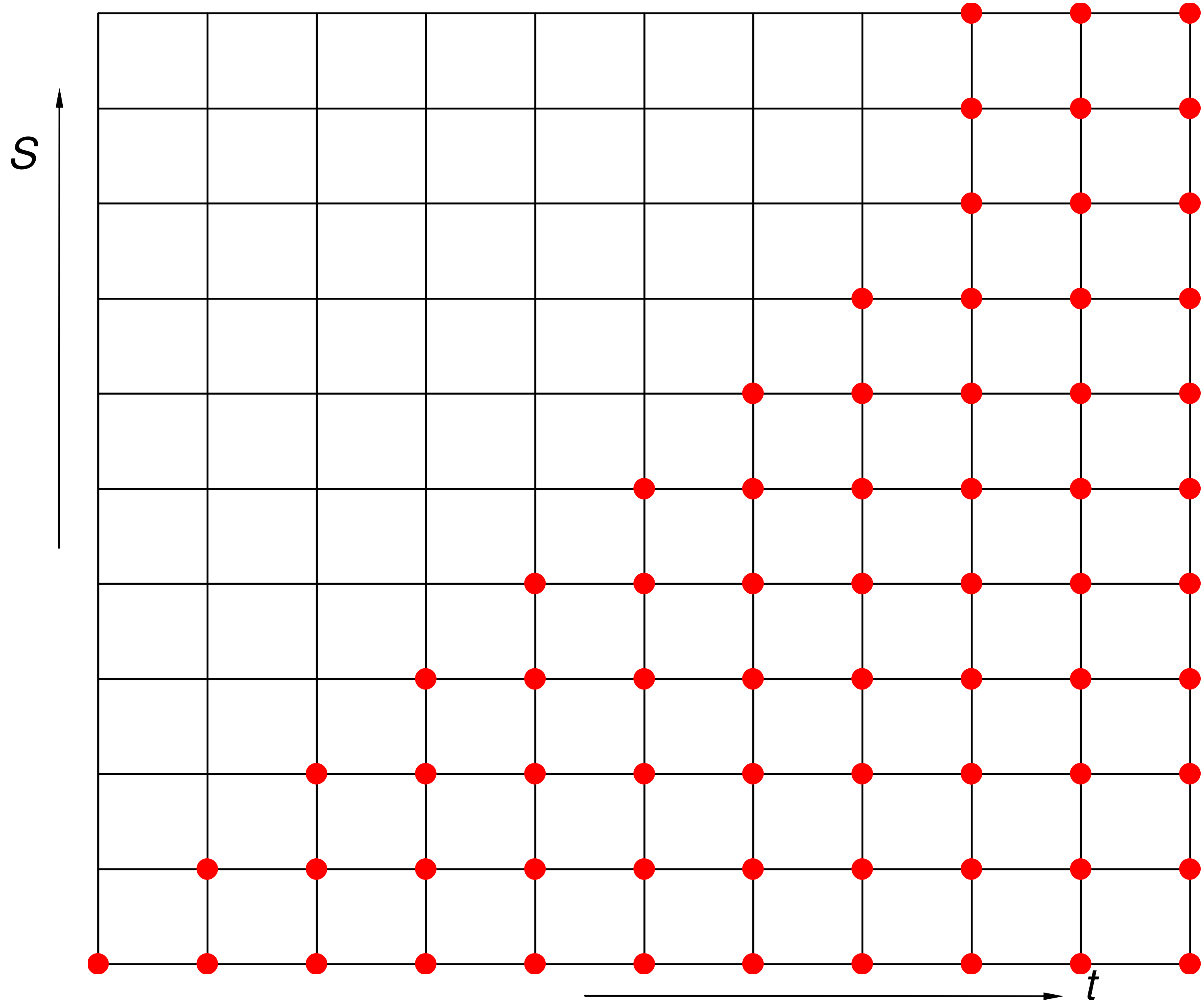

S
t



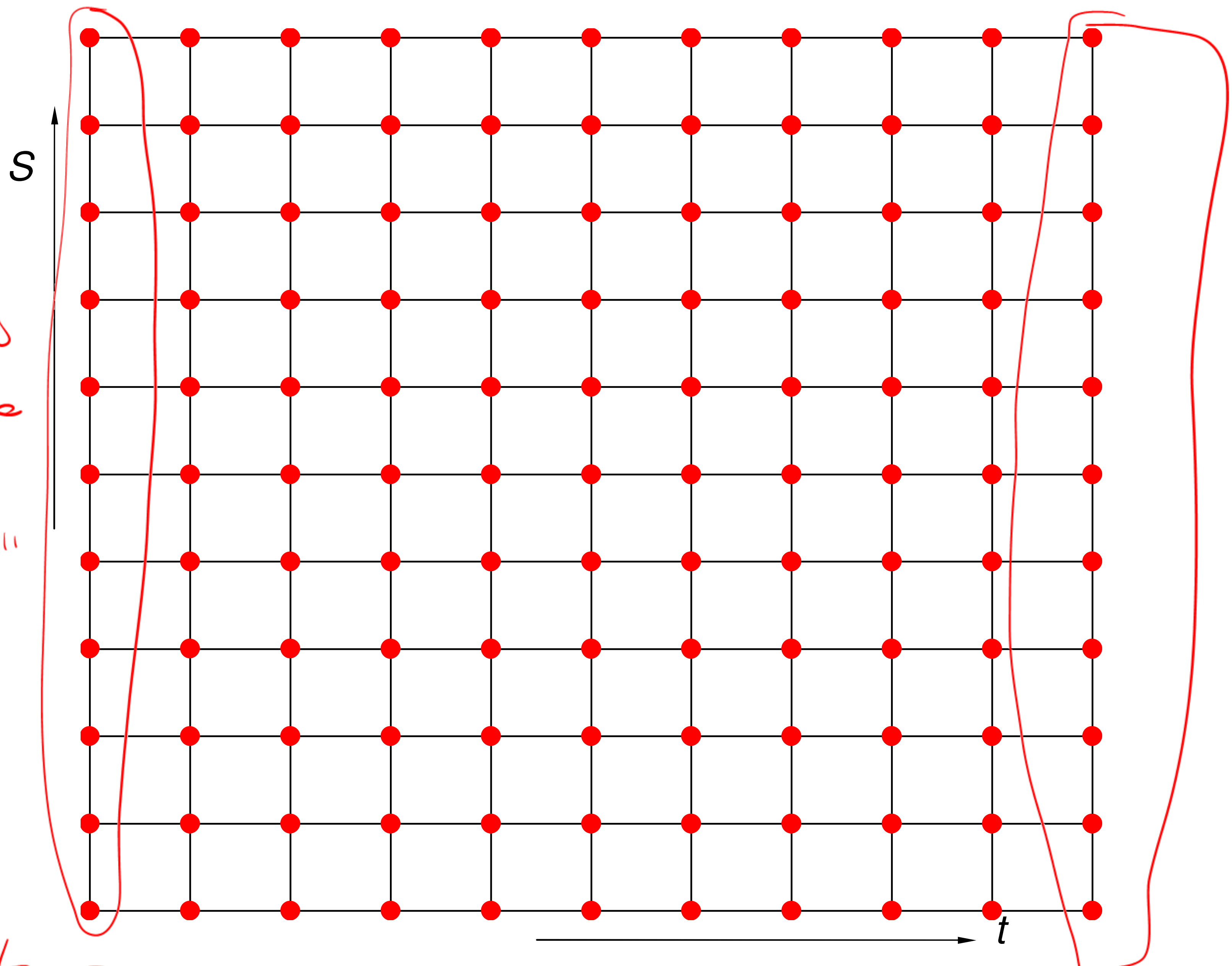
S
linear extrapolation
t



k+1
k
S
t


S
t

which is what we wanted "hooray!"



t = 0

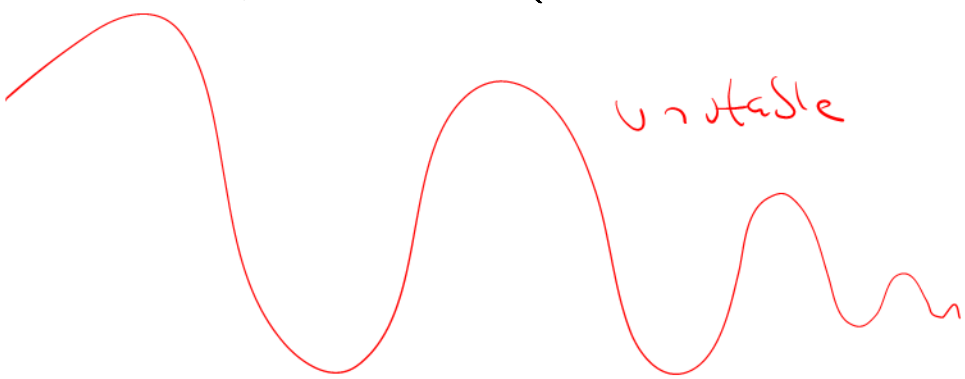
Other issues

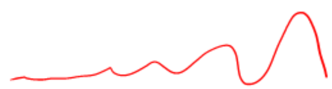
- Greeks

- Early exercise (and other decisions)

very good for early exercise in FDM

29 March.





The advantages of the explicit method

- It is very easy to program and hard to make mistakes
- When it does go unstable it is usually obvious
- It copes well with coefficients that are asset and/or time dependent
- it copes very well with early exercise
- It can be used for modern complex option-pricing models

The disadvantages of the explicit method

- There are restrictions on the time step   Big problem

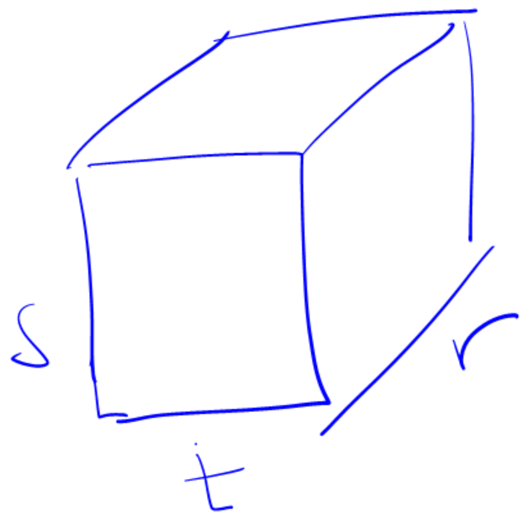
- It is slower than Monte Carlo in high dimensions,

terrible

$V(\underbrace{S, r, t})$

exponent

vol.



Summary

Please take away the following important ideas

- There are two main numerical methods for pricing derivatives

Simulation / Expectation

- Monte Carlo methods exploit the relationship between option prices and expectations

pos.

- The finite-difference method solved a discretized version of the $\boxed{\text{Black–Scholes equation}}$

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