

## CQF Exercises 2.2 (with solutions)

$dW$  is the usual increment of Brownian motion

1. Consider a two-factor model comprising a stochastic differential equation for the stock  $S$  and another for interest rate  $r$ , respectively

$$\begin{aligned} dS &= rSdt + \sigma S dW_t^{(1)}, \sigma \in \mathbb{R} \\ dr &= u(r, t)dt + w(r, t)dW_t^{(2)}. \end{aligned} \quad (1.1)$$

Here both drift  $u(r, t)$  and diffusion  $w(r, t)$  are arbitrary functions. The two Brownian increments  $dW_t^{(1)}$  and  $dW_t^{(2)}$  have a constant correlation of  $\rho$ , such that

$$\mathbb{E} \left[ dW_t^{(1)} dW_t^{(2)} \right] = \rho dt.$$

Consider setting up a delta-hedged portfolio

$$\Pi = V(S, r, t) - \Delta S - \Delta_1 V_1(S, r, t).$$

Derive the pricing partial differential equation for  $V(S, r, t)$ .

**Solution:**

$$\Pi = V - \Delta S - \Delta_1 V_1$$

The change in this portfolio in a time  $dt$  is given by

Step 1. Express the S.D.E. for the portfolio, by expanding  $V(r, t; T)$

$$\begin{aligned} d\Pi &= \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma w S \frac{\partial^2 V}{\partial S \partial r} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} \right) dt \\ &\quad - \Delta_1 \left( \frac{\partial V_1}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho \sigma w S \frac{\partial^2 V_1}{\partial S \partial r} + \frac{1}{2} w^2 \frac{\partial^2 V_1}{\partial r^2} \right) dt \\ &\quad + \left( \frac{\partial V}{\partial S} - \Delta_1 \frac{\partial V_1}{\partial S} - \Delta \right) dS + \left( \frac{\partial V}{\partial r} - \Delta_1 \frac{\partial V_1}{\partial r} \right) dr. \end{aligned}$$

To eliminate all randomness from the portfolio we must choose

Step 2: Eliminate the Randomness by setting Diffusion term to be zero

$$\frac{\partial V}{\partial S} - \Delta_1 \frac{\partial V_1}{\partial S} - \Delta = 0,$$

to eliminate the  $dS$  terms, which are the sources of randomness, and

$$\frac{\partial V}{\partial r} - \Delta_1 \frac{\partial V_1}{\partial r} = 0,$$

to get rid off  $dr$  terms. Therefore our choice of delta terms to make the portfolio risk free become

$$\Delta_1 = \frac{\frac{\partial V}{\partial r}}{\frac{\partial V_1}{\partial r}}$$

and

$$\Delta = \frac{\partial V}{\partial S} - \frac{\frac{\partial V}{\partial r}}{\frac{\partial V_1}{\partial r}} \frac{\partial V_1}{\partial S}.$$

This leaves us with

Step 3: After eliminating the randomness, the origin portfolio becomes risk-free, so we let it equal to  $r\Pi dt$

$$\begin{aligned} d\Pi &= \left( \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma w S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2}w^2 \frac{\partial^2 V}{\partial r^2} \right) dt \\ &\quad - \Delta_1 \left( \frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho\sigma w S \frac{\partial^2 V_1}{\partial S \partial r} + \frac{1}{2}w^2 \frac{\partial^2 V_1}{\partial r^2} \right) dt \\ &= r\Pi dt = r(V - \Delta S - \Delta_1 V_1) dt, \end{aligned}$$

where we have used arbitrage arguments to set the return on the portfolio equal to the risk-free rate. Collecting all  $V$  terms on the left-hand side and all  $V_1$  terms on the right-hand side we find that

$$\begin{aligned} &\frac{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma w S \frac{\partial^2 V}{\partial S \partial r} + \frac{1}{2}w^2 \frac{\partial^2 V}{\partial r^2} + rS \frac{\partial V}{\partial S} - rV}{\frac{\partial V}{\partial r}} \\ &= \frac{\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho\sigma w S \frac{\partial^2 V_1}{\partial S \partial r} + \frac{1}{2}w^2 \frac{\partial^2 V_1}{\partial r^2} + rS \frac{\partial V_1}{\partial S} - rV_1}{\frac{\partial V_1}{\partial r}} \end{aligned}$$

Step 4. Rearrange those two non-Arbitrage equation, move  $V \setminus V_1$  on the same side, and find a B.P.E.

Therefore both sides can only be functions of the independent variables,  $S, \sigma$  and  $t$ . Thus we have

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma S q \frac{\partial^2 V}{\partial S \partial r} + \frac{1}{2}q^2 \frac{\partial^2 V}{\partial r^2} + rS \frac{\partial V}{\partial S} - rV = -(u - \lambda w) \frac{\partial V}{\partial r},$$

for some function  $\lambda(S, r, t)$ . Reordering this equation, we usually write

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma S w \frac{\partial^2 V}{\partial S \partial r} + \frac{1}{2}w^2 \frac{\partial^2 V}{\partial r^2} + rS \frac{\partial V}{\partial S} + (u - \lambda w) \frac{\partial V}{\partial r} - rV = 0.$$

2. a. Suppose the spot interest rate  $r$ , which is a function of time  $t$ , satisfies the stochastic differential equation

$$dr = dW_t.$$

Using **this** model for the spot rate, by hedging one bond  $V(r, t; T)$  of maturity  $T$ , with another of a different maturity, derive the bond pricing equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} - \lambda \frac{\partial V}{\partial r} - rV = 0, \quad (2.1)$$

where  $\lambda = \lambda(r, t)$  is an arbitrary function.

**Solution:** Hedge one bond  $V(r, t; T)$  with another bond  $V_1(r, t; T_1)$

$$\Pi = V - \Delta V_1$$

Change in portfolio:  $t \rightarrow t + dt$

$$d\Pi = dV - \Delta dV_1$$

Step 1. Express the S.D.E. for the portfolio, by expanding  $V(r,t;T)$

using  $dr^2 = dt$

$$d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} \right) dt + \frac{\partial V}{\partial r} dr - \Delta \left( \frac{\partial V_1}{\partial t} + \frac{1}{2} \frac{\partial^2 V_1}{\partial r^2} \right) dt - \Delta \frac{\partial V_1}{\partial r}$$

To eliminate risk set  $\Delta = \frac{\partial V / \partial r}{\partial V_1 / \partial r}$ . Step 2: Eliminate the Randomness by setting Diffusion term to be zero

$$d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} \right) dt - \frac{\partial V / \partial r}{\partial V_1 / \partial r} \left( \frac{\partial V_1}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V_1}{\partial r^2} \right) dt$$

No arbitrage gives

$$d\Pi = r\Pi dt = r \left( V - \frac{\partial V / \partial r}{\partial V_1 / \partial r} V_1 \right) dt$$

Step 3: After eliminating the randomness, the origin portfolio becomes risk-free, so we let it equal to  $r\Pi dt$

$$\Rightarrow \frac{1}{\partial V / \partial r} \left( \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} - rV \right) = \frac{1}{\partial V_1 / \partial r} \left( \frac{\partial V_1}{\partial t} + \frac{1}{2} \frac{\partial^2 V_1}{\partial r^2} - rV_1 \right)$$

$$= \lambda(r, t)$$

Step 4. Rearrange those two non-Arbitrage equation, move  $V \setminus V_1$  on the same side, and find a B.P.E.

The LHS is a function of  $V$ . The RHS depends only on  $V_1$ . The function  $\lambda(r, t)$  is a function of only two common variables common to  $V$  &  $V_1 \Rightarrow$

$$\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} - \lambda(r, t) \frac{\partial V}{\partial r} - rV = 0.$$

- b. By considering an unhedged bond and the risk free return, explain how and why  $\lambda$  arises in (2.1).

**Solution:** In a time step  $dt$  the (unhedged) bond changes in value by

$$dV = \frac{\partial V}{\partial r} dW + \left( \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} \right) dt. \quad (1)$$

Rearranging the BPE (2.1)

$$\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} = \lambda \frac{\partial V}{\partial r} + rV. \quad (2)$$

So substitute (2) in to (1) to give

$$dV = \frac{\partial V}{\partial r} dW + \left( \lambda \frac{\partial V}{\partial r} + rV \right) dt,$$

or

Key Implication!

$$dV - rV dt = \frac{\partial V}{\partial r} (dW + \lambda dt). \quad (3)$$

We can interpret (3) financially as the difference between  $dV$  - return on an unhedged bond (contains risk) and  $rV dt$  - risk free return, i.e. the money invested from selling a bond which earns at the risk free rate, i.e.  $\frac{dV}{dt} = rV$ . Since the coefficient of  $dW \neq 0$ , the portfolio is not riskless and the term  $\lambda dt$  is the extra return on the portfolio per unit of risk  $dW$ . It is the amount by which the market wishes to be compensated, for hedging with a non-tradeable asset.

Step 5. Find the General Solution by  
 (1) assuming the form of Affine Solution as below,  
 (2) Redemption Value.

- c. Assuming that  $\lambda$  is a function of  $t$  only and a **zero coupon bond** is to be priced by solving (2.1). Find a solution of the form

$$V(r, T; T) = \exp(A(t; T) + rB(t; T)),$$

with redemption value

$$V(r, T; T) = 1$$

where both  $A(t; T)$  and  $B(t; T)$  should be given.

**Solution:**

$$\begin{aligned} \frac{\partial}{\partial t} (e^{A(t)+rB(t)}) &= (\dot{A} + r\dot{B})V; \quad \frac{\partial}{\partial r} (e^{A+rB}) = BV; \\ \frac{\partial^2}{\partial r^2} (e^{A(t)+rB(t)}) &= B^2V. \end{aligned}$$

$$\dot{A} + r\dot{B} + \frac{1}{2}B^2 - \lambda B - r = 0$$

$$\dot{B}(t) = 1 : B(t; T) = t - T$$

$$\dot{A} + \frac{1}{2}B^2 - \lambda B = 0 : \dot{A} = \lambda(t - T) - \frac{1}{2}(t - T)^2$$

$$A(t; T) = -\int_t^T \lambda(s)(s - T)ds - \frac{(t - T)^3}{6}$$

$$V(r, t; T) = \exp\left(-\int_t^T \lambda(s)(s - T)ds - \frac{1}{6}(t - T)^3 + r(t - T)\right)$$

3. What final condition (payoff) should be applied to the bond pricing equation for a swap, cap, floor, zero-coupon bond and a bond option?

**Solution:**

Final condition for a swap:

$$V(r, T) = (r - r_s)P,$$

where  $r_s$  is the fixed rate and  $P$  is the principal.

Final condition for a cap:

$$V(r, T) = \max(r - r_c, 0)P,$$

where  $r_c$  is the cap rate and  $P$  is the principal.

Final condition for a floor:

$$V(r, T) = \max(r_f - r, 0)P,$$

where  $r_f$  is the floor rate and  $P$  is the principal.  
Final condition for a zero-coupon bond:

$$V(r, T) = P,$$

where  $P$  is the principal.  
Final condition for a coupon bond:

$$V(r, T) = (1 + c) P,$$

where  $c$  is the (discrete) coupon rate and  $P$  is the principal.  
Final condition for a bond option:

$$V(r, T) = \max(Z(r, T) - E, 0),$$

where  $E$  is the exercise price and  $Z(r, t)$  is the value of the underlying bond at time  $t$ . **Step 6. Example of How to Solve for the A(.) and B(.) in the Affine Solution**

4. Consider the spot rate  $r$ , which evolves according to the SDE

$$dr = u(r, t) dt + w(r, t) dX,$$

The extended Hull and White model has drift and diffusion

$$u(r, t) = \eta(t) - \gamma r, \quad w(r, t) = c,$$

in turn, where  $\eta(t)$  is an arbitrary function of time  $t$  and  $\gamma$  and  $c$  are constants. Deduce that the value of a zero coupon bond,  $Z(r, t; T)$  which has

$$Z(r, T; T) = 1$$

in the extended Hull and White model is given by

$$Z(r, t; T) = \exp(A(t; T) - rB(t; T)),$$

where

$$B(t; T) = \frac{1}{\gamma} \left( 1 - e^{-\gamma(T-t)} \right)$$

and

$$A(t; T) = - \int_t^T \eta(\tau) B(\tau; T) d\tau + \frac{c^2}{2\gamma^2} \left( (T-t) + \frac{2}{\gamma} e^{-\gamma(T-t)} - \frac{1}{2\gamma} e^{-2\gamma(T-t)} - \frac{3}{2\gamma} \right).$$

### **Solution:**

You may assume the pricing equation for Vasicek is

$$\frac{\partial Z}{\partial t} + \frac{1}{2} c^2 \frac{\partial^2 Z}{\partial r^2} + (\eta(t) - \gamma r) \frac{\partial Z}{\partial r} - rZ = 0, \quad Z(r, T; T) = 1$$

If  $Z(r, t) = \exp(A(t; T) - rB(t; T))$  then  $Z_r = -B$   $Z_{rr} = B^2$ . Solution with  $Z(r, T; T) = 1$  implies  $B(T; T) = A(T; T) = 0$  as  $e^0 = 1 = e^{A - rB}$   
 $Z_t = (A'(t) - rB'(t))V$  so subst. these in to the PDE above

$$\begin{aligned} A'(t; T) - rB'(t; T) + \frac{1}{2}c^2B^2 - (\eta(t) - \gamma r)B - r &= 0 \quad \forall r \\ (A' + \frac{1}{2}c^2B^2 - \eta(t)B) - r(B' - \gamma B + 1) &= 0 \\ \Rightarrow A' = -\frac{1}{2}c^2B^2 + \eta(t)B, & \quad B' = \gamma B - 1 \end{aligned}$$

From  $Z(r, t; T) = \exp(A(t) - rB(t))$  we note that as  $r \rightarrow \infty$ ,  $Z \rightarrow 0$ . Solving for  $B(t; T)$

$$\frac{dB}{dt} = \gamma B - 1$$

You can solve this by the variable sep. method:

$$\frac{dB}{\gamma B - 1} = dt$$

Now recall:

$$\int \frac{dx}{ax + 1} = \frac{1}{a} \ln |ax + 1| + K$$

therefore we have

$$\begin{aligned} \int_t^T \frac{dB}{\gamma B - 1} &= \int_t^T d\tau = \frac{1}{\gamma} \ln |\gamma B(\tau; T) - 1|_t^T = (T - t) \\ \ln \left| \frac{\gamma B(T; T) - 1}{\gamma B(t; T) - 1} \right| &= \gamma(T - t) \end{aligned}$$

we know  $B(T; T) = 0$  so

$$\begin{aligned} \ln \left| \frac{\gamma B(t; T) - 1}{-1} \right|^{-1} &= \gamma(T - t) = -\ln |1 - \gamma B(t; T)| = \gamma(T - t) \\ 1 - \gamma B(t; T) &= \exp[-\gamma(T - t)] = \gamma B(t; T) = 1 - \exp[-\gamma(T - t)] \\ B(t; T) &= \frac{1}{\gamma} (1 - \exp[-\gamma(T - t)]) \end{aligned}$$

Then

$$A(t; T) = \frac{1}{2}c^2 \int_t^T B^2(\tau; T) d\tau - \int_t^T B(\tau; T) \eta(\tau) d\tau$$

and

$$\begin{aligned} \int_t^T B^2(\tau; T) d\tau &= \frac{1}{\gamma^2} \int_t^T \left( 1 - 2e^{-\gamma(T-\tau)} + e^{-2\gamma(T-\tau)} \right) d\tau \\ &= \frac{1}{\gamma^2} \left( (T - t) - \frac{2}{\gamma} e^{-\gamma(T-\tau)} \Big|_t^T + \frac{1}{2\gamma} e^{-2\gamma(T-\tau)} \Big|_t^T \right) \\ &= \frac{1}{\gamma^2} \left( (T - t) - \frac{2}{\gamma} + \frac{2}{\gamma} e^{-\gamma(T-t)} + \frac{1}{2\gamma} - \frac{1}{2\gamma} e^{-2\gamma(T-t)} \right) \\ &= \frac{1}{\gamma^2} \left( (T - t) - \frac{3}{2\gamma} + \frac{2}{\gamma} e^{-\gamma(T-t)} - \frac{1}{2\gamma} e^{-2\gamma(T-t)} \right) \end{aligned}$$

Hence

$$A(t; T) = - \int_t^T B(\tau; T) \eta(\tau) d\tau + \frac{c^2}{2\gamma^2} \left( (T-t) - \frac{3}{2\gamma} + \frac{2}{\gamma} e^{-\gamma(T-t)} - \frac{1}{2\gamma} e^{-2\gamma(T-t)} \right)$$