1.6 The Forward Curve: linking Z, f, $[T_i, T_{i+1}]$ and τ.

This section stems from reviews Lesniewski (2008) and Glasserman (2003), available online, identifying certain critical derivations and linkages between the formulae.

Let's start with a classic **zero-coupon bond** and its pricing under Monte Carlo from simulated evolutions of short rate $r(t)$[4]

$$Z(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}\right]$$

In terms of the forward curve, using the steps $r(t) = f(t,t) = \bar{f}(0,\tau)$ for today's curve $t = 0$

$$\begin{aligned} Z(0,T_i) &= \exp\left(-\int_0^{T_i} \bar{f}(0,\tau)d\tau\right) * \qquad \text{integrating over curve} \\ & \quad \text{discretised as} \\ &= \exp\left(-\sum_{j=0}^{i-1} \bar{f}(0,T_j)(T_{j+1}-T_j)\right) \qquad F(T_j) = f_j \text{ applies to } [T_j, T_{j+1}] \end{aligned}$$

Indexes i, j – both are applied to tenor time from 0 to T_i (curve data in row). When index i is used then the interval $\tau = T_{i+1} - T_i$ is large (eg, $6M$, $1Y$, $2Y$) and so, index j is used for rates inside of that interval $T_i < T_j < \ldots < T_{i+1}$.

LIBOR model gives the following relationship between LIBOR and ZCB. We will be interested in Forward LIBOR, so indexing changed from $Z(0,T_i)$ to $Z(T_i,T_{i+1})$.

$$\begin{aligned} Z(T_i,T_{i+1}) &= \frac{1}{1+\tau_i f_i} \\ (1+\tau_i f_i) &= \frac{1}{Z(T_i,T_{i+1})} \qquad \Longrightarrow \qquad f_i = \frac{1}{T_{i+1}-T_i}\left(\frac{1}{Z(T_i,T_{i+1})} - 1\right) \end{aligned}$$

$$f_i = \frac{1}{T_{i+1}-T_i}\left[\exp\left(\int_{T_i}^{T_{i+1}} \bar{f}(T_i,\tau)d\tau\right) - 1\right] \qquad \text{using } Z(T_i,T_{i+1}) = \exp\left(-\int_{T_i}^{T_{i+1}} \bar{f}(T_i,\tau)d\tau\right) **$$

often presented simply as

$$= \frac{1}{\tau}\left[\exp\int_t^T f(s)ds - 1\right] \qquad \text{where } t = T_i \text{ and } T = T_{i+1}$$

$\mathbb{E}^{\mathbb{Q}}_{i+1}[L(T_i,T_{i+1})] = f_i$ means that the rate taken off the curve is our best filtered expectation of LIBOR rate in the future – except for the first tenor, the LIBOR is **Forward LIBOR**. The numerarie for this rolling expectation is a bond $Z(0,T_{i+1})$.

f_i notation also used for the **Forward Rate Agreement** rate starting at T_i and ending at T_{i+1}, where it is simple annualised rate. FRA contracts and used in LMM calibration. [5]

[4] In the HJM Framework, the first column (forward rate of the shortest tenor) is a proxy for $r(t)$ and can be used to price a bond.

[5] FRAs also used by the Bank of England to construct an instantaneous forward curve $\bar{f}(t,\tau)$. In HJM calibration, we start with a ready instantaneous rates curve and thus need to convert f_{inst} into simple-period LIBOR to be used in caps, floors, IRS, etc.