1.11 Understanding HJM Output. By Dr Richard Diamond

The HJM model outputs a full curve simulated in continuous time (eg, over timestep $dt = 0.01$). Simulated curves data is in the rows. There is certain ambiguity about how to read the forward curve of instantaneous rates (which is the output of HJM model). Each instantaneous rate belongs to the tenor τ so, the *HJM MC file* columns are $\tau = 0.00$, $\tau = 0.5$, $\tau = 1$, $\tau = 1.5$, $\dots$.

The forward rate $\bar{f}(\tau = 0.5)$ belongs only to that tenor point, and can be attributed to both periods $[0, 0.5]$ and $[0.5.1]$. The forward rate $\bar{f}(\tau = 1)$ is inbetween $[0.5, 1]$ and $[1, 1.5]$ period.

Question: for a caplet on 6M LIBOR $L(0.5, 1)$ which column would you use? **Answer:** HJM MC implementation here uses the column for $\tau = 0.5$ – this choice leaves us without the 6M LIBOR rates for the period $[0, 0.5]$ because the first column is equivalent to 1M LIBOR or short rate $r(t) = f(t, t)$. It would be equally acceptable to use the column for $f(\tau = 1)$ for the caplet on 6M LIBOR $L(0.5, 1)$, or things can be resolved by using simple linear interpolation,

$$\frac{\bar{f}(\tau_i) + \bar{f}(\tau_{i+1})}{2}$$

1. For Forward LIBOR, pick values from the relevant simulated curve and cell. For example, you can pick $L(0.5, 1)$ from the notional row 50 as 6M LIBOR expected in at $t = 0.5$. Remember that this is only one simulated value and you have to satisfy the risk-neutral, convergent pricing under **under Monte-Carlo.**

2. Where necessary, for the discounting factor integrate over the curve (in row) from 0 to τ for caplet/floorlet pricing, or from τ_i to τ_{i+1}. Such discounting factor will be under the same rolling forward measure as your Forward LIBOR.

3. Using the first column as $r(t) = f(t, t)$ reduces the HJM to a short rate model. While that is a convenient and quick demo of the stochastic discounting factor please avoid this approach in industry projects.

Evolution of the forward rate at each tenor with the timestep $dt = 0.01$ offers rate paths that seem ready for numerical integration (via summation). However, those are evolution paths for the Rolling Spot LIBOR of given tenor (not rolling forward measure).

It is theoretically consistent to calculate LIBOR-per-period as an average of the instantaneous rate over that period, for example, we average over the column $\tau = 0.5$ from T_i to T_{i+1}. However, even if we average that column from say $T_i = 4$ to $T_{i+1} = 4.5$ that rate will not reflect a credit and term liquidity risk (longer-dated drift) that is required from Forward LIBOR $L(4, 4.5)$. If you attempt caplet pricing from column$\tau = 0.5$ only, that constant maturity approach is likely to produce a relatively flat term structure of implied volatility.

You can plot simulated HJM curves for each timestep $dt = 0.01$ on a single 3D surface to observe the dynamics and make own conclusions.

1.12 Short And Long-Term Relationships Across Yield Curve (with statistical modelling)

The HJM model evolves the full term structure (yield curve) $\bar{\boldsymbol{f}}(\boldsymbol{T};\boldsymbol{\tau})$. As a by-product we have evolutions of the forward rates for each tenor τ. For historical data, we evaluate the relationship between time series $\bar{\boldsymbol{f}}(\tau_j)$, $\bar{\boldsymbol{f}}(\tau_k)$ using either cointegration analysis or correlation coefficients.

Correlation is always estimated between changes (differences) $\Delta\bar{\boldsymbol{f}}$, which themselves are stationary. At the start of the document, there are plots of changes in forward rates at $1Y$ vs. $6Y$ vs. $25Y$ tenors as well as discussion about common types of premium for interest rates. The plots demonstrate independence (in daily movements) of the short rates up to and around $\tau = 1$ year. This is expected because the short rates have no significant credit risk premium component (factor) that can be shared by the rates of various tenors.

[An open study task is to build cross-correlation as well as autocorrelation profile of the yield curve (rates at different tenors) in order to capture dynamics of those correlations.]

The next logical step is **linking rates at different tenors via constant spreads**:

$$\bar{f}(\tau_k) = \bar{f}(\tau_j) + \text{Credit Premium} + \text{Term Premium} \quad \dagger$$

It is market practice for IR Derivatives to strip spreads that capture these premiums.

- One can make empirical estimations of credit risk and term liquidity risk for different tenors, then add to the short rate r_T to construct a full curve. This can be done for the stochastic process of $r(t)$ in order to obtain the full yield curve at each simulation step.
- One can see an actual yield curve as a collection of flat curves for rates of different tenors. This naturally creates a collection of tenor spreads, from which OIS and other relevant spreads (currency) can be subtracted.

  $$\mathrm{L}_{3M} = \mathrm{OIS}_{3M} + \text{Basis Spread} \qquad \text{vs.} \qquad \mathrm{L}_{3M} = \mathrm{OIS}_{ON} + \text{3M to OIS Spread}$$

  '3M to OIS Spread' is an example of basis swap contract traded or stripped from the market. Therefore, you can take 3M USD LIBOR and build up the yield curve in the currency of choice (as farther as the basis swap spread curve allows). You can also take 6M USD LIBOR and build a different yield curve. Use the curve of a matching frequency (ie, 3M vs. 6M) to price and strip the traded instruments.

The simple proposition † 'works' because there is a **cointegrated relationship** between rates at neighbouring tenors (and across the curve): each tenor is stochastic but *on average*, two tenors keep a constant distance and the error correction coefficient $(1-\alpha)$ comes up as significant in

$$\Delta \bar{f}_{k-j} = -(1-\alpha)e_t = -(1-\alpha)(\bar{f}_j - \beta_C \bar{f}_k - \text{Premium})$$

Naively, $\Delta \bar{f}_{k-j}$ = constant but the concept of long run equilibrium suggests that it will be sufficient to have significant term e_t, a stationary residual. This kind of analysis detailing a cointegrated relationship between spot rates of 10Y and 25Y tenors is given in Cointegration Case B (CQF Lecture). The premium can be extracted from the deterministic trend of cointegration.