The Heath, Jarrow and Morton Model

CQF Lecture EXERCISES

## Three Ways to Derive Instantaneous Forward Rate

1. The price of a zero-coupon bond that matures at time $T$ paying \$1 is given using an integral over *the forward curve*

$$Z(t;T) = e^{-\int_t^T f(t;s)ds} \tag{1}$$

**By solving an integral equation**, confirm the instantaneous forward rate is defined as

$$f(t;T) = -\frac{\partial}{\partial T} \log Z(t;T) \tag{2}$$

2. Consider two bonds $Z(t;T_1)$ and $Z(t;T_2)$ where $T_2 > T_1$, and the forward rate $f(t;T_1,T_2)$ that is locked-in between $T_1$ and $T_2$. **By considering present value of 1\$ investment**, back from show that the locked-in forward rates are defined as

$$f(t;T) = -\frac{\partial}{\partial T} \log Z(t;T)$$

3. A forward rate $f(t;T)$ represents the instantaneous continuously compounded rate, that is contracted at time t for a riskless borrowing at future time $T$. Prove the relationship between an instantaneous forward rate and ZCB yield

$$f(t;T) = -\frac{\partial}{\partial T} \log Z(t;T)$$

**by considering a self-financing portfolio** that is short $Z(t;T)$ and long $Z(t;T+\delta t)$.

## HJM SDE and Musiela Parameterization

### Market price of risk. No arbitrage. Tenor time

1. The key parameter that links the real and risk-neutral 'worlds' and explains a global market condition is the market price of (interest rate) risk (MPOR). Mathematically, the market price of risk is a parameter of choice that allows to cancel the drift. By considering a hedged portfolio,

$$\Pi = Z(t;T_1) - \Delta Z(t;T_2)$$

derive the relationship between SDE parameters for $\frac{dZ(t;T)}{Z(t;T)} = \mu(t,T)dt + \sigma(t,T)dX$ and the market price of interest rate risk.

$$\frac{\mu(t,T_1) - r(t)}{\sigma(t,T_1)} = \frac{\mu(t,T_2) - r(t)}{\sigma(t,T_2)}$$

**Hint:** in the risk-free world, all assets earn the risk-free rate.

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2. Using the definition of the instantaneous forward rate (2)

$$f(t;T) = -\frac{\partial}{\partial T} \log Z(t;T)$$

obtain the corresponding SDE model. Assume the bond price follows a log-Normal model

$$\frac{dZ}{Z} = \mu(t,T)\, dt + \sigma(t,T)\, dX$$

**Hint:** differentiate with respect to $t$. The maturity time $T$ is fixed.

3. The raw model for the evolution of (points) $f(t,T_i)$ on the forward curve relates the drift to volatility as

$$df(t,T) = \frac{\partial}{\partial T}\left[\frac{1}{2}\sigma^2(t,T) - \mu(t,T)\right] dt - \frac{\partial}{\partial T}\sigma(t,T) dX^{\mathbb{Q}} \tag{3}$$

Show that, under the risk-neutral measure $\mathbb{Q}$, the model can be expressed as

$$df(t,T) = m(t,T)dt + \nu(t,T)dX$$

where $\nu(t,T) = -\frac{\partial}{\partial T}\sigma(t,T)$ simplifies the diffusion term, and the risk-neutral drift can be expressed solely as a function of volatility (no arbitrage condition)

$$m(t,T) = \nu(t,T) \int_t^T \nu(t,s)ds$$

4. Musiela Parametrisation of the HJM model (risk-neutral evolution of the forward curve) provides convenience of operating with fixed tenors $\tau = T - t$ rather than maturity dates.

   By applying the change of variable $f(t,T) \to \bar{f}(t,\tau)$ and using the chain rule of differentiation, show that the Musiela Parametrisation of the one-factor HJM model is

$$d\bar{f}(t,\tau) = \left(\bar{\nu}(t,\tau) \int_0^\tau \bar{\nu}(t,s)ds + \frac{\partial \bar{f}(t,\tau)}{\partial \tau}\right) dt + \bar{\nu}(t,\tau)dX$$

   **Hint:** taking of a derivative of forward rate *wrt* $T$ is equivalent to taking of a derivative of Musiela Parameterisation *wrt* $\tau$, i.e., $\frac{\partial f}{\partial T} \equiv \frac{\partial \bar{f}}{\partial \tau}$.

5. Most of the popular models for $r(t)$ have HJM representations. Consider Ho & Lee model for the spot rate $r(t)$,

$$dr(t) = \eta(t)dt + c\, dX, \quad \text{for constant } c.$$

   Formulate a bond pricing equation (BPE) and use continuous version of the forward rate bootstrapping formula in order to obtain an SDE for $df(t,T)$. Explain equivalence of terms in this SDE to the HJM SDE (non-Musiela).

## Numerical Methods for PCA: Jacobi Transformation

Jacobi Transformation is a *tractable* numerical method of matrix diagonalization (e.g., obtaining a diagonal matrix). The method is based on eliminating the largest off-diagonal element by rotating the matrix. 'Rotation' is implemented by pre-multiplying matrix $\mathbf{A}$, which we ultimately want to decompose, by matrix $\mathbf{P_{p,q}}$ that is specially constructed in order to cancel an off-diagonal element $a_{p,q}$ so that $a'_{p,q} = 0$.

$$P_{p,q} = \begin{bmatrix} 1 & & & & & & & 0 \\ & \ddots 1 & & & \vdots & & \iddots & \\ & & \cos\phi & \cdots & 0 & \cdots & \sin\phi & \\ & \cdots & 0 & \cdots & 1 & \cdots & 0 & \cdots \\ & & -\sin\phi & \cdots & 0 & \cdots & \cos\phi & \\ & \iddots & & & \vdots & & 1\ddots & \\ 0 & & & & & & & 1 \end{bmatrix}$$

For each rotation, we multiply

$$\mathbf{A}' = \mathbf{P_{p,q}}^{\mathbf{T}}\mathbf{A}\mathbf{P_{p,q}}$$

For a covariance matrix, the rotation occurs within the unit circle, and therefore, properties of trigonometric functions can be efficiently used. Key to implementation is calculation of the angle of rotation $\phi$.

1. Describe the purpose of applying Jacobi Transformation to a covariance matrix.
2. Deduce why in order to eliminate the matrix element $a'_{p,q} = 0$ it is necessary that $\tan(2\phi) = \dfrac{2a_{p,q}}{a_{q,q} - a_{p,p}}$. **Hint:** consider multiplication of individual matrix elements.
3. Jacobi method is not the most computationally efficient because each new rotation destroys zero result obtained on the previous step. Nonetheless, *convergence* of the sum of the off-diagonal elements to zero occurs. Given that Jacobi method chooses $a_{p,q}$ to be greater than other off-diagonal elements on average

$$a^2_{p,q} \geq \frac{\sum_{i \neq j} a^2_{i,j}}{n^2 - n}, \tag{4}$$

   show that for a matrix $n \times n$ convergence occurs with the factor of $1 - \dfrac{2}{n^2 - n}$.
4. Explore VBA code that implements Jacobi Transformation in Excel PCA file. Names of variables are self-explanatory and linked to the mathematical model, for example, *Athis(i,j)* for $\mathbf{A}$ and *Awork(i,j)* for $\mathbf{A}'$.