Fixed Income and Credit – Lecture 1

Fixed Income Products and Analysis

May 2023

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Fixed Income Products and Analysis

Introduction

This lecture is an introduction to basic instruments and concepts in the world of fixed-income. In this context *fixed income* means cash-flows that are not related to stocks or commodities.

We will review some simple fixed-income instruments, in particular *bonds* and *swaps*, and their conventions.

In this lecture we *ignore the possibility of default* by the counterparties.

In this lecture

- names and features the most important vanilla fixed-income products: bond and swaps
- the relationship between the different product's types
- construction of yield curves and forward rates
- conventional ways to analyze bonds: yield, duration and convexity
- short introduction to market infrastructure
- an overview of fixed-income modeling

By the end of this lecture you will

- be able to decompose a simple instruments in cash-flows
- be able to construct yield curves
- understand the concepts of price, yield, duration and convexity for bonds

Fixed Income Products and Analysis

Zero-coupon bond

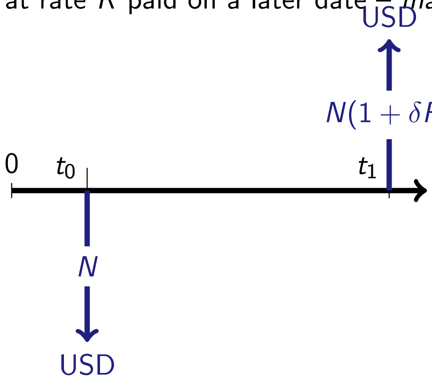
A zero-coupon bond is a contract paying a known fixed amount (N) – the *principal* or *notional* – in a given currency at some known date in the future – the *maturity date* T.

No-arbitrage: The promise of future (positive) wealth is worth something now: it cannot have zero or negative value.

Theoretical instrument, building block.

Deposit

Over-the-counter (OTC) instruments. Payment of an amount on a given date – *settlement date* or *effective date*, typically $T + 2$ – to receive the same amount augmented with interest at rate R paid on a later date – *maturity date*.



The maturity can be as short as one day (overnight deposit).

Day count conventions

Interest amount are computed in a day count convention. The conventions are (arbitrary) mechanisms with an agreed meaning (see ISDA definitions). The multiplier δ is called accrual factor or *year fraction*

Some of those conventions are:

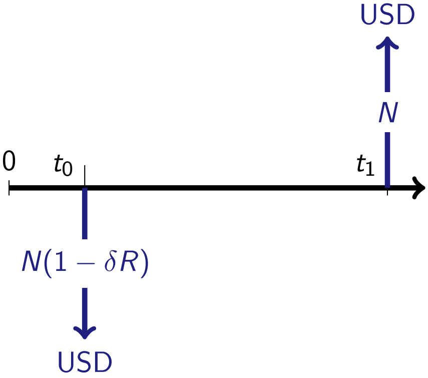
- 1/1, One: *no convention*, the stated interest is paid irrespective of the period.
- Act/360, Actual/360: The number of (calendar) days divided by 360. Standard for money market in USD, EUR.
- Act/365F, Actual/365 Fixed: The number of (calendar) days divided by 365. Standard for money market in GBP.
- 30/360: Each month is 30 days and a year is 360 days.

Guide (Creative Commons): `https://quant.opengamma.io/Interest-Rate-Instruments-and-Market-Conventions.pdf`

Code (Apache v2.0 license):
`https://github.com/OpenGamma/Strata/blob/main/modules/basics/src/main/java/com/opengamma/strata/basics/date/StandardDayCounts.java`

Bills / Certificate of Deposit

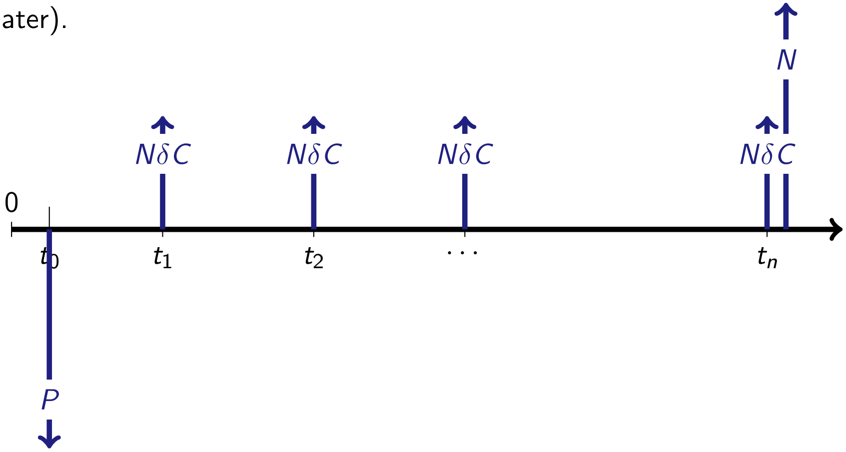
Security version of a deposit.
Traded on a price or discounted basis.



Note: The meaning of "*rate*" in deposits and in bills is not the same.

Coupon-bearing bond / fixed coupon bonds

Security, paying fixed coupons on a regular basis. Traded on a price or yield basis (see later).



Typically the price P is paid in $T+1$, $T+2$ or $T+3$. Also called Note.

STRIPS

STRIPS stands for *Separate Trading of Registered Interest and Principal of Securities*.

The coupons and principal of fixed coupon bonds are *split up*, creating artificial zero-coupon bonds. Those zero-coupon bonds have longer maturity than the bills.

Artificial zero-coupon bonds created from coupon bonds.

Bank account – Cash account

Bank account: accumulates interest at a rate c set on a daily basis.
Theoretical bank account: **Composition**

$$\prod_{i=1}^{n}(1+\delta_i r(s_{i-1},s_i)) = (N^r_{s_0})^{-1} N^r_{s_n} = \exp\left(\int_{s_0}^{s_n} r_\tau d\tau\right)$$

Natural mechanism for interest rate, interest on notional plus interest.

In practice, bank accounts are often based on arithmetic average paid on short periods (monthly): **Arithmetic average**

$$1+\sum_{i=1}^{n}\delta_i I^O(s_{i-1},s_i)$$

Fixed Income Products and Analysis

Benchmarks

A benchmark is an interest rate published by an administrator. The rate is based on trades or market participants survey and published with a clear gouvernance.
In the main jurisdictions, the use of benchmarks is regulated. See EU regulation 2016/1011 *on indices used as benchmarks in financial instruments and financial contracts[...]*

Overnight benchmark

Reference rate based on Overnight deposit, secured or unsecured. t_0 is today and t_1 is the next good business day (in a given calendar).

Examples:

USD-EFFR	Effective Fed Funds Rate or Fed Funds, interbank transactions based.
USD-SOFR	Secured Overnight Financing Rate, repo transactions based.
EUR-ESTR	Euro Short Term Rate, bank clients transactions based.
GBP-SONIA	Sterling OverNight Interbank Average

Benchmarks: IBOR

InterBank Offered Rate.

> *London InterBank Offered Rate: At what rate could you borrow funds, were you to do so by asking for and then accepting interbank offers in a reasonable market size just prior to 11 am London time?*

Term deposit rate: typically 1, 3 and 6 months. Unsecured rate.

LIBOR currencies: CHF, [EUR,] GBP, JPY, USD

Other currencies: EUR-EURIBOR, JPY-TIBOR, AUD-BBSW, PLN-WIBOR, DKK-CIBOR, SEK-STIBOR, CAD-CDOR

LIBOR transition

LIBOR was published for GBP, CHF and JPY (discontinued 31 December 2021)
USD-LIBOR published to June 2023.

The LIBOR discontinuation has lead to important *fallback* activity for the legacy trades.

For the last 40 years, IBOR-like benchmarks have been the most important in fixed-income markets for most currencies. Since a couple of years, they have been partly replaced by overnight benchmarks.

- **GBP and CHF** Transition done
- **USD** Transition in progress. IBOR-like replacements proposed
- **EUR** New overnight benchmark ESTR. EURIBOR still the most used benchmark, no direct plans to discontinue it.
- **JPY** LIBOR discontinued. Partial transition to TONA. TIBOR still exists.
- **Other currencies** Situation varies.

Swaps

A swap is an agreement between two parties to exchange, or swap, future cashflows.

The size of these cashflows is determined by some formula, decided upon at the initiation of the contract. The swaps may be in a single currency or involve the exchange of cashflows in different currencies.

One or more legs are associated to benchmark rates.

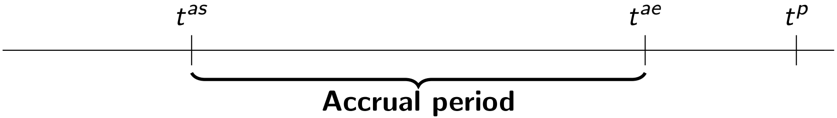
The meaning of the formulas are described in the *ISDA definitions*.

In most cases, the payments take place on regularly spaced dates, e.g. quarterly. Not all days are good business days for payment. The payment days are *adjusted* to fall on a good business day. Most derivatives use a *following*-type adjustment rule. This means that when one of the date is not a good business day, the actual payment take place the next day.

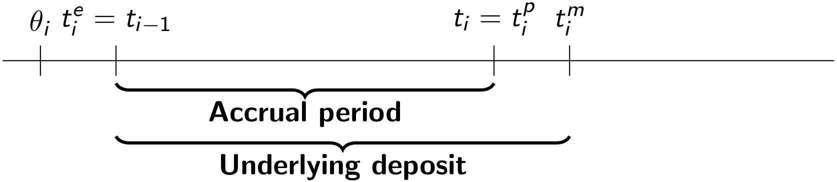
IBOR derivatives – dates

Derivative dates

- Start accrual date (t^{as})
- End accrual date (t^{ae})
- Payment date (t^{p})

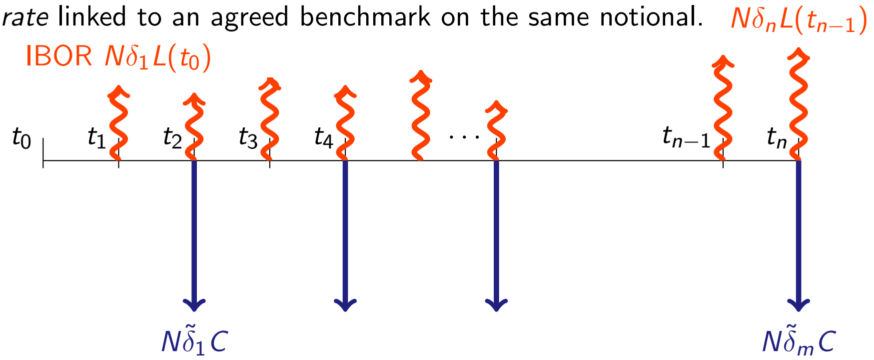


IBOR IRS



Vanilla interest rate swap (IRS)

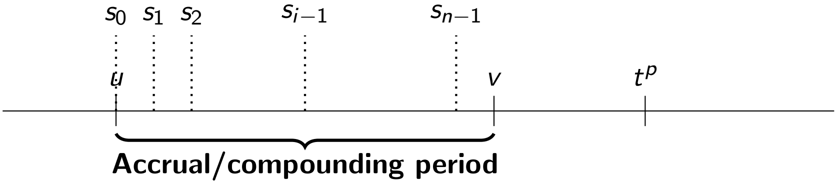
One side pays a *fixed interest rate* on an agreed notional, the other side pays a *floating rate* linked to an agreed benchmark on the same notional.



The frequencies and day counts can be different on each leg.
Standard convention: USD – Fixed 6M 30/360 Vs LIBOR-3M ACT/360
Standard convention: EUR – Fixed 1Y 30/360 Vs EURIBOR-6M ACT/360

Overnight Indexed Swaps (OIS)

Same as IRS but with each floating coupon replace by:



Typically: payment date t^p = last fixing publication + 2 business days

Composition

$$\left(\prod_{i=1}^{n} (1 + \delta_i I^O(s_{i-1}, s_i)) \right) - 1$$

Natural mechanism for interest rate, interest on notional plus interest

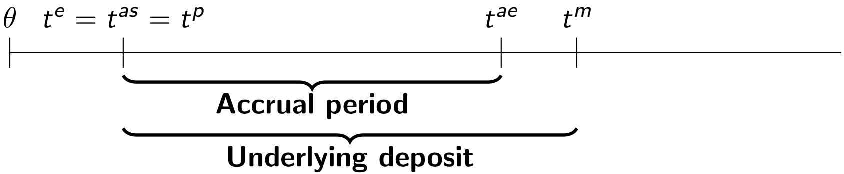
Standard convention: USD – Fixed 1Y ACT/360 Vs SOFR Comp 1Y ACT/360

Standard convention: GBP – Fixed 1Y ACT/365F Vs SONIA Comp 1Y ACT/365F

Forward Rate Agreement

Similar in intuition to a single period swap but with a different settlement mechanism. Forward Rate Agreement (FRA) with *ISDA FRA discounting*: linked to a Ibor benchmark j, a fixing date θ, an accrual factor δ for the period $[t^{as}, t^{ae}]$ and a fixed rate K. At the fixing date θ, the Ibor rate $L^j(\theta)$ is recorded. Payment in $t_p = \text{Spot}(\theta)$

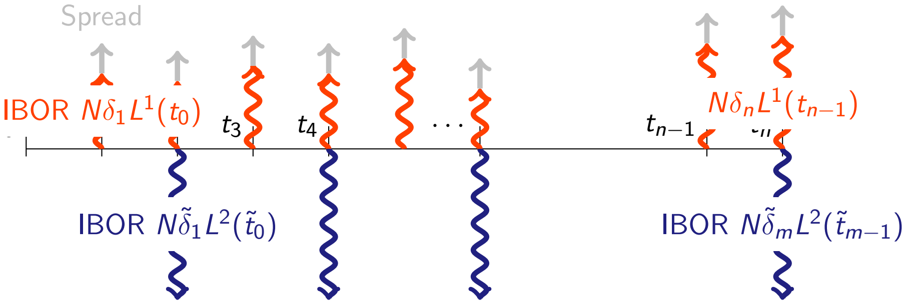
$$\frac{\delta(L^j(\theta) - K)}{1 + \delta L^j(\theta)}.$$



The formula above is not a “quant” valuation formula but a legal term sheet formula as described in ISDA definitions.

Basis swaps IBOR-IBOR

One side pays a *floating rate* linked to a benchmark on an agreed notional, the other side pays a *floating rate* linked to a *different benchmark* on the same notional. The leg with the shorter tenor pays a spread on top of the benchmark.



Vanilla Interest rate derivatives: market size

CCP outstanding amounts (May 2023):

- LCH – USD 429.1 trn – IRS 137.9 / OIS 214.5
  https://www.lch.com/index.php/services/swapclear/volumes
- CME – USD 17.9 trn –
  https://www.cmegroup.com/education/cme-volume-oi-records.html
- EUREX – EUR 33.4 trn –
  https://www.eurex.com/ec-en/clear/clearing-volume

Cleared at LCH SwapClear:

YTD 2023-05-22 : USD 526,902,403,200,540

S&P 500 market: USD 30.1 trn. / US debt: USD 31.3 trn.

Why to use derivatives

Interest rate derivatives, and swap in particular, are widely used in the market for different purposes.

- Hedging
- Speculation
- Comparative advantage
- Hiding value behind complexity

Why to use swaps: Hedging

Swaps can be used to balance cashflows and risk (not the liquidity).

Bank
A saving bank is typically borrowing money through saving accounts (floating rate adjusted on a regular basis) and lending through a mortgage (fixed rate for 20 years). The hedge for that situation would be to enter into a swap paying fix and receiving the floating rate (see below for call provisions and amortisation).

Pension fund.
Pensions are paid on a long term basis. The term can be more than 50 years (between the career start and the worker death). The pension fund may invest in assets with relatively short term horizon (bonds) and needs to adjust the risk to fit the very long term horizon.
The hedge may be a forward starting swap, receiving fixed from the bond maturity to the expected worker longevity.

Why to use swaps: Speculation

At inception, swaps have a present value of 0. The present value of swaps change with the level of rates; the swaps give immediate exposure to interest rates.
Due to the absence of upfront investment and the exposure to rates, the swaps provide a theoretical infinite leverage.

Suppose that you *pay fixed* in a swap and *rates increase* after you entered into the swap. Your payments are agreed at inception and unaffected by the market movements. You now pay less than in a new swap. You are better off than with a new swap, which has present value 0. You have thus a (unrealised) *profit*.

Macro hedge-funds may use IRS or OIS to express their view on rate levels.

Note: in practice, the leverage is limited by Initial Margins (IM).

Why to use swaps: Comparative advantage

Swaps were first created to exploit comparative advantage. This is when two companies who want to borrow money are quoted fixed and floating rates such that by exchanging payments between themselves they benefit, at the same time benefitting the intermediary who puts the deal together.
Two companies A and B want to borrow USD 10MM, to be paid back in two years. They each have a choice of a fixed- or floating-rate loan. They are quoted the interest rates for borrowing at fixed and floating rates shown here.

Co	Fixed	Floating
A	3%	Floating LIBOR-3M + 30bps
B	4.2%	Floating LIBOR-3M + 100bps

Both pay a premium over LIBOR to cover risk of default, which is perceived to be greater for company B.

Why to use swaps: Comparative advantage

Due to the nature of their business, A prefers to borrow at floating and B at fixed. If they each borrow directly then they pay LIBOR + 30bps + 4.2% = LIBOR + 4.5%.

However, if A borrowed at fixed and B at floating they'd be paying 3% + LIBOR-3M + 100bps = LIBOR-3M + 4% which is a potential saving between them of 0.5%.

Suppose A borrows fixed and B floating, even though that's not what they want. Total interest: LIBOR plus 4%. We add a swap in the strategy in which A pays LIBOR and B pays 2.95%.

A pays 3% and LIBOR while receiving 2.95%, a net floating payment of LIBOR plus 5bps. This is floating, as A originally wanted, and it is 25bps better than if they had borrowed directly at the floating rate.

B pays LIBOR plus 100bps and also 2.95% while receiving LIBOR. This nets out at 3.95%, which is fixed, as required, and 25bps better than the original deal.

Why to use swaps: hiding complexity

Some investor, retail in particular, like some type of structured notes. They perceive them as providing a higher return on the long term.

Borrowers prefer to deal with simple structures; like vanilla FRN.

The borrower issue structured notes and swap the structured part with a commercial bank in exchange of simple IBOR/ON + spread coupons.

Frequent borrowers, like International Financial Institutions, often use this mechanism to attract different investors.

Derivative and interbank market

Interbank.

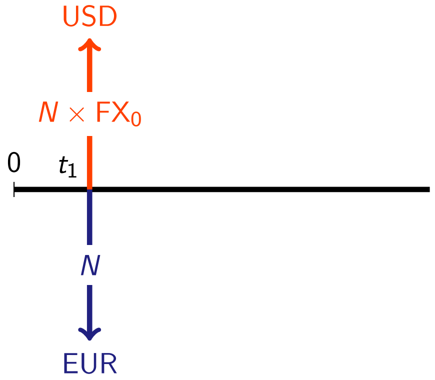
A large part of the volume reported above is *interbank*, i.e. it may take a *couple of intermediaries* before finding the connection between the fixed rate payer and receiver.

A pension fund, client to bank A, is looking to receive on a 50Y swap. The bank itself has not interest in the 50Y risk. It nevertheless offers the swap to its client. In the interbank market, it finds interest with bank B. Bank B has a hedge fund client that expect the long rates to increase and is ready to pay the current rate on the 50Y swap.

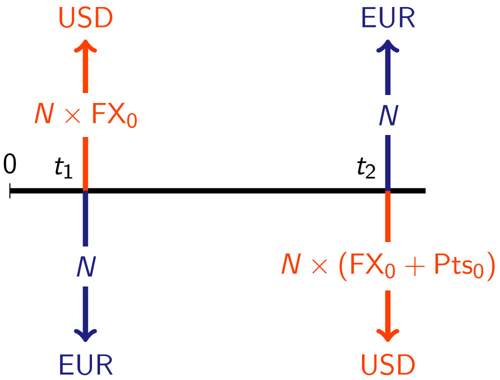
In the interbank game, there may be also some transformation. Bank A offers a exotic swap to a client, offload the vanilla part to bank B and part of the exotic feature to Bank C while dynamically delta hedging other exotic features using vanilla swaps and futures up to maturity.

FX Spot

Exchange of two cash flows, each in a different currency, with agreed notionals. The notional in one currency is typically written as the notional in the other currency multiplied by a *Foreign Exchange Rate*.

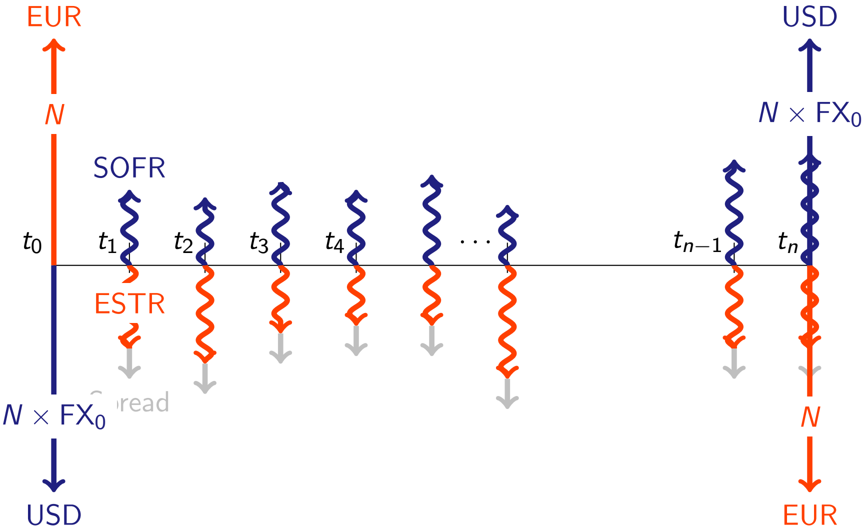


FX Swaps / FX Forwards



FX Swap

Cross-currency swaps



Cross-currency swaps

In the IBOR dominated world (up to a recently), the typical cross-currency swaps were IBOR-3M v IBOR-3M + spread.
The convention is to have the spread on the non-USD leg.

In the overnight world the typical cross currency swaps are ON v ON + spread with quarterly payments.
Note that the payments are quarterly, even if the standard convention for single currency OIS is annual payments in most currencies.

FX reset

For cross-currency swaps, it is common to have a *FX reset* feature – also called *MtM reset*. On a regular basis – e.g. at each coupon payment – the FX rate is reset to the market level. The party that received the converted notional ($N \times F_0$) receives the difference with the new rate ($N \times (F_0 - F_1)$) and the remaining cash flows in that currency (interest and final notional payment) are updated to the new notional $N \times F_1$.

Building blocks

Common feature: vanilla products constructed out of few building blocks

- Fixed cashflows – zero coupon bond
- IBOR coupons
- Overnight compounded coupons

The actual details of the vanilla interest rate instruments is often *more complex* than what is presented in textbooks.

Floating rate note

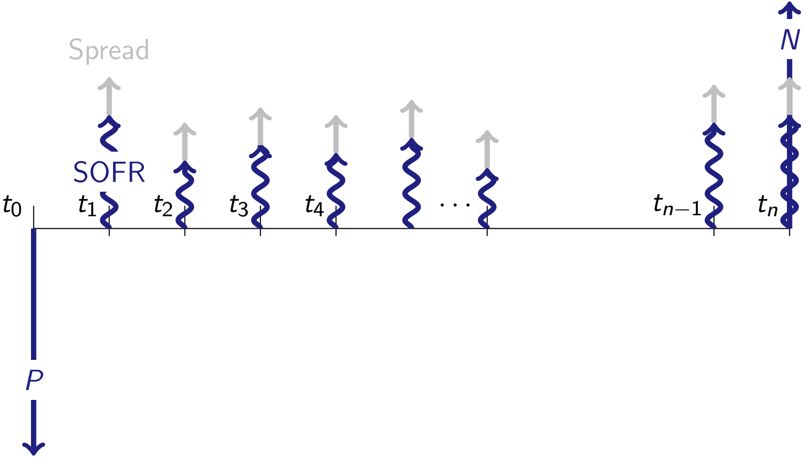
A Floating Rate Note (FRN) is a note/bond for which the coupons paid are linked to a benchmark and a formula.

Up to recently, the most standard one paid a IBOR benchmark plus a spread.

The market has moved in GBP and USD to overnight in-arrears (OIS-like) plus a spread.

It is a security and a cash product, but it is linked to benchmarks like most of the interest rate derivatives.

Floating rate note



Call provision

Some bonds and swaps have call provisions.

For bonds, the issuer can call back the bond on certain dates or at certain periods for a prescribed, possibly time-dependent, amount. Typically the call dates match coupon payment dates.

For swaps the call is written as a cancellation provision. One party has the right to cancel the swap at specific dates.

Callable bonds and swaps are natural hedging products for mortgages. In many countries mortgages have fixed rates over a long tenor (20 to 30 years) and can be reimbursed anticipatively by borrowers.

In particular in the US, GSE issue large amounts of callable bonds.

Amortization

In all of the above products we have assumed that the principal/notional remains fixed at its initial level.

Sometimes this is not the case, the principal can amortize or decrease during the life of the contract.

Such amortization is arranged at the initiation of the contract and may be fixed, so that the rate of decrease of the principal is known beforehand, or can depend on the level of some index, if the index is high the principal amortizes faster for example.

In bonds, the principal is thus paid back gradually and interest is paid on the amount of the principal outstanding.

In single currency swaps, the notional is modified but no notional amount is exchanged.

In cross currency swaps, the notional is modified and the relevant exchange of notional is established.

Fixed Income Products and Analysis

Discounting

We call *discount factor* the present value of a cash flow of 1. We denote by $P(s,t)$ the present value of the cash flow paid in t viewed from s. This means that for a fixed cash flow with amount N paid in t, the value in s is

$$\mathrm{PV}(s) = N \cdot P(s,t).$$

Note that $P(t,t) = 1$.
This is often referred to as risk free discounting.

This is a theoretical concept as in practice, nothing is risk free.

Additive pricing

For a set of cash flows $(c_i)_{i=1,\ldots,n}$ paid in $(t_i)_{i=1,\ldots,n}$, the present value is computed in an additive way

$$\mathrm{PV}(s) = \sum_{i=1}^{n} c_i P(s,t_i).$$

Zero-coupon rates

Instead of looking at the discount factors curves we could look at different quantities, the associated rates. We call zero-coupon rate – or *yield* – the quantities r such that:

- continuously compounded rates: $P(0, u) = \exp(-r(u)u)$.
- periodically compounded rates (frequency m/year): $P(0, u) = \left(1 + \frac{r(u)}{m}\right)^{-mu}$.

The discount factors discuss above are the building block for the pricing of fixed cash flows. What about the other cash flow types we have described in the previous sections, like IBOR or overnight floating payments?

Interpolation

Market data is sparse. Typically the liquid instruments have at best monthly tenor up to one year and yearly tenors above a couple of years.

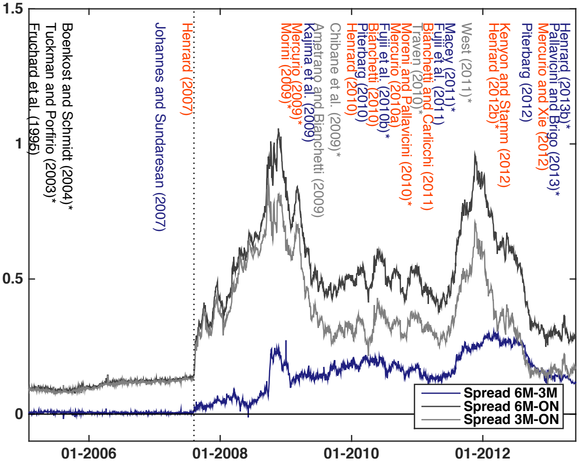
Actual trades in the books have a large variety of cash flow dates. To price existing trades in line with the current market – marked-to-market –, some kind of interpolation is required.

The interpolation can be done on discount factors, continuously compounded rates or periodically compounded rates. The interpolation schemes used in practice are themselves quite diverse. Some typical choices are:

- linear interpolation on zero-rate
- cubic spline on zero-rate
- log-linear on discount factors
- log-cubic spline monotonic on discount factors

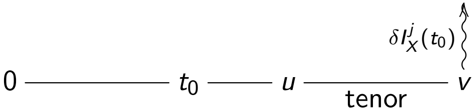
GFC 2007

9 August 2007, the beginning of the Great Financial Crisis in the interest rate world.



Pricing swaps

Ibor replication doesn't work!



Pricing swaps

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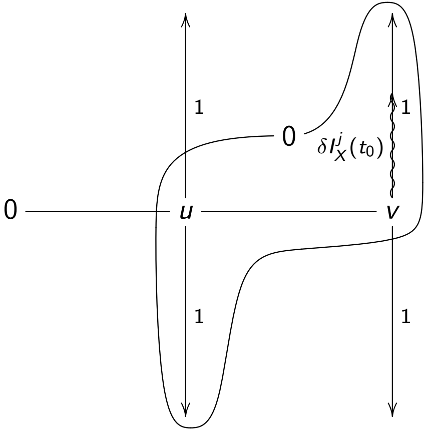
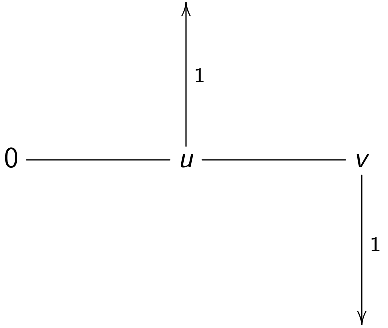


Pricing swaps

Ibor replication doesn't work! $PV = P(0,u) - P(0,v) = P(0,v)\left(\frac{P(0,u)}{P(0,v)} - 1\right)$



If the above reasoning was correct, all basis swaps would have a spread of 0. The issue is coming from credit risk. The IBOR benchmark rate is linked to interbank – credit risky – lending. One cannot simply cut out part of the cash flow and cancel the netting.

Multi-curve framework: IBOR

- The value of a j floating coupon is an asset for each tenor j, each fixing date θ.

Definition (Forward index rate with collateral)

The forward curve $F^j(t, \theta, u, v)$ is the continuous function such that,

$$P(t, v)\delta F^j(t, \theta, u, v)$$

is the present value in t of the j-Ibor coupon with fixing date θ, start date u, maturity date v $(t \leq \theta \leq u = \text{Spot}(t_0) < v)$ and accrual factor δ.

There is a full family of forward rate, one for each index j.

More on IBOR curves

Ibor coupon replication doesn't work but ...

Definition (Coupon pseudo-discount factors curves)

The forward curve P^j is the continuous function such that $P^j(t,t) = 1$, $P^j(t,s)$ is an arbitrary strictly positive function for $t \leq s < \mathrm{Spot}(t) + j$, and for $t_0 \geq t$, $u = \mathrm{Spot}(t_0)$ and $v = u + j$ one has

$$F^j(t, t_0, u, v) = \frac{1}{\delta}\left(\frac{P^j(t,u)}{P^j(t,v)} - 1\right).$$

This is the start of the multi-curve framework.

Swap value and rate – IRS

An IRS is described by a set of fixed coupons or cash flows c_i at dates $\tilde{t}_i$ $(1 \leq i \leq \tilde{n})$. It also contains a set of floating coupons over the periods $[t_{i-1}, t_i]$ with $t_i = t_{i-1} + j$ $(1 \leq i \leq n)$. Accrual factors for the periods $[t_{i-1}, t_i]$: δ_i.
The present value of a (fixed rate) receiver IRS is

$$\sum_{i=1}^{\tilde{n}} c_i P(0, \tilde{t}_i) - \sum_{i=1}^{n} P(0, t_i)\delta_i F^j(0, t_{i-1}, t_i).$$

The *par rate* – also called *market quote* – is

$$K = \frac{\sum_{i=1}^{n} P(0, t_i)\delta_i F^j(0, \theta_i, t_{i-1}, t_i)}{\sum_{i=1}^{\tilde{n}} \tilde{\delta}_i P(0, \tilde{t}_i)}$$

Curve calibration: What are we looking for?

Curve calibration consists in building the theoretical quantities $P(0,.)$ and $F^j(0,.,.)$ from the market quotes for IRS, OIS, FRA, etc.

Theory:

$$? \longrightarrow \text{Curves} \longrightarrow \text{Present Value}$$

Practice:

$$\begin{tikzcd}
\text{Market Quotes} \arrow[r] & \text{Curves} \arrow[r] \arrow[dr] \arrow[ddr, dotted] & \text{Present Value} \\
\text{Hypothesis} \arrow[ur] & & \text{Sensitivities} \\
& & \text{VaR, stress tests, etc.}
\end{tikzcd}$$

Curve calibration: What are we looking for?

Curves:
Discounting: $P(0,.) : [0,T] \to \mathbb{R}^+$.
Forward: $F^j(0,.) : [0,T] \to \mathbb{R}$

The curves $P(0,.)$ can be represented by

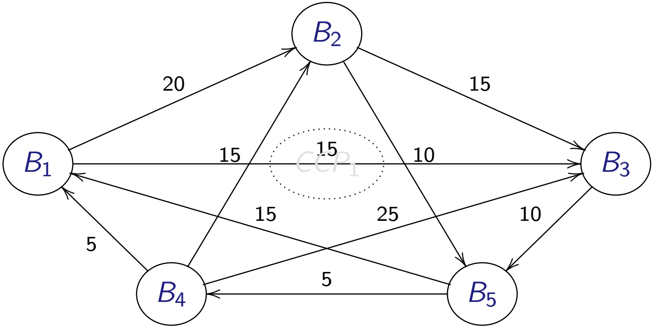
- discount factors
- continuously compounded rates: $P(0,u) = \exp(-r(0,u)u)$.
- periodically compounded rates: $P(0,u) = \left(1 + \frac{y(u)}{m}\right)^{mu}$.

The curves $F^j(0,.)$ can be represented by

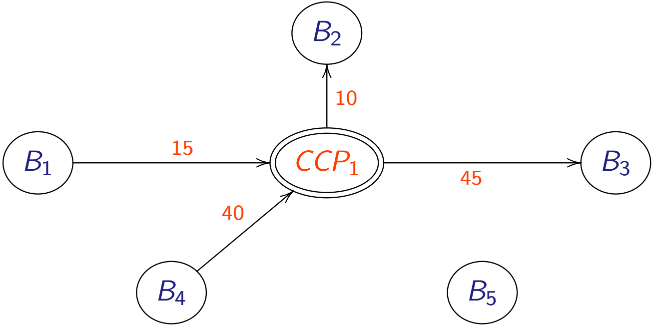
- pseudo-discount factors (see before)
- direct forward curves

Fixed Income Products and Analysis

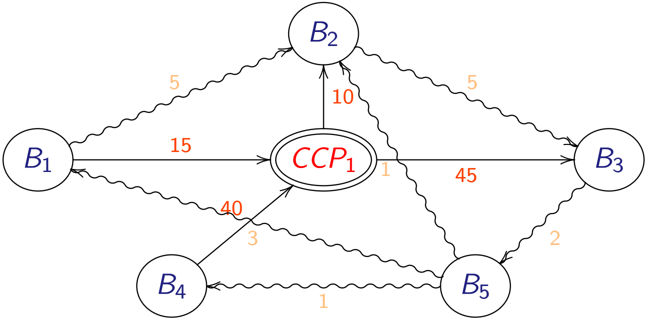
Central counterparty



Central counterparty



Central counterparty



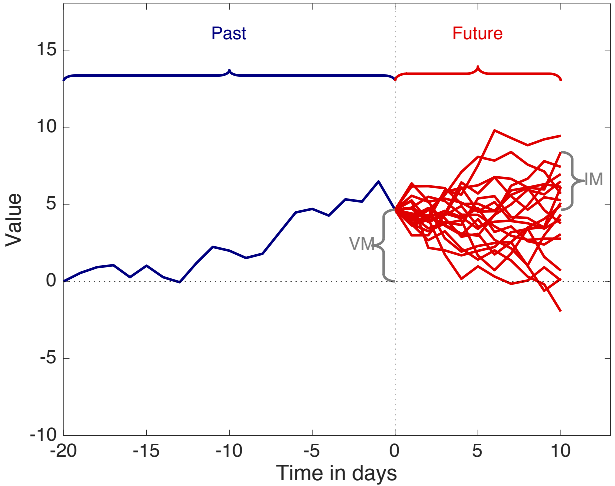
Clearing

Nowadays, the vast majority of vanilla interest rate derivatives are cleared (LCH, CME, EUREX, etc.). Even market participants not using the CCPs/clearing mechanism are impacted by it.

There is some clearing for FX forwards (LCH, CME, Eurex, etc.)

There is no clearing of cross-currency swaps (with minor exceptions).

Collateral framework – Variation Margin



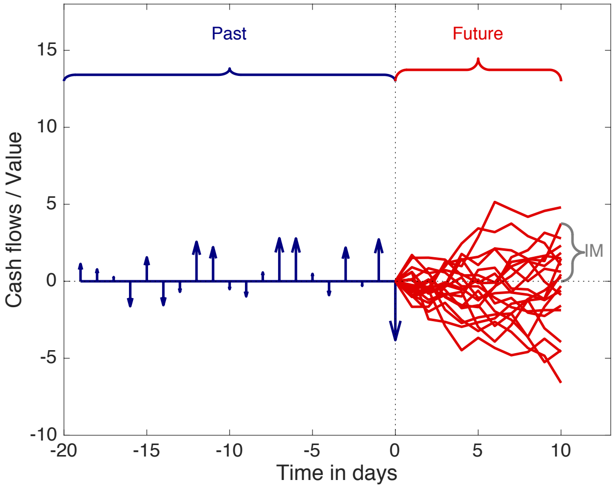
Variation Margin

The *variation margin* (VM) is the exchange of the portfolio present value between (counter)parties on a regular basis – usually daily. The credit risk embedded in the transaction decreases to the extend that today's value is covered by the collateral paid. A default of the counterparty with an instantaneous resolution and replacement of the portfolio at the computed present value would have no financial impact on the surviving party.

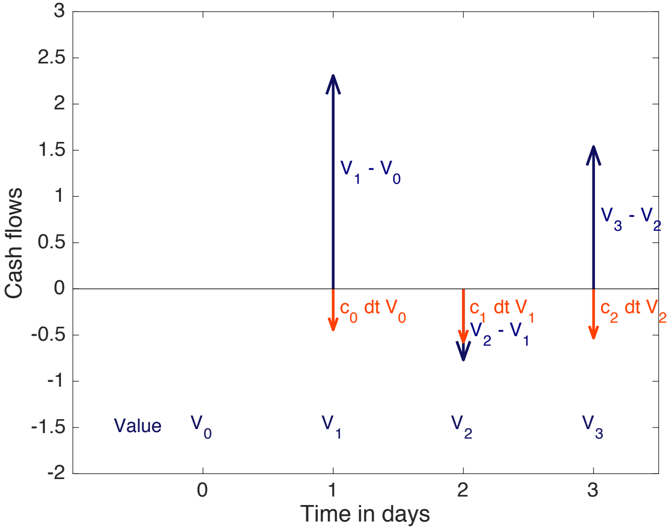
No large payments on coupon dates anymore. It is replace by daily small cash-flows through the life of the product.

The amount paid as variation margin is collateral to guarantee the paying party obligations. It does not come from the term sheet cash flows discussed in the previous sections.
The party receiving the collateral pays interest on the amount received. In standard agreement – e.g. Credit Support Annex (CSA) – the interest paid on the collateral is the main overnight benchmark in the relevant currency.

Collateral framework – Variation Margin



Cash collateral: closer view



Cash collateral pricing formula

Looking for: coherent pricing framework removing risk in presence of VM collateral.

Simplest case: 1 day horizon and fixed payoff with amount 1, i.e. $V(1) = 1$.

What is $V(0)$?

Mechanism:

Date	Description	Amount
Date 0	Receive cash	$V(0)$
Date 1	pay collateral + interest	$-V(0)(1+\delta_0 c(0))$
	receive the payoff	1

The risk disappears if $V(0)(1+\delta_0 c(0)) = 1$, i.e. if

$$V(0) = \frac{1}{1+\delta_0 c(0)}.$$

Whatever the market rate is, the valuation mechanism is using the "legal" rate c as discounting rate.

Cash collateral pricing formula

Those daily payments (variation margin and collateral interest) create a very complex set of cash flows. Fortunately those magically simplify and the discounting can be done on a day by day basis using the cash account associated to the collateral rate c.

Theorem (Collateral with cash price formula)

In presence of collateral with rate c, the value in time 0 of an asset with value V_u^c in time u is

$$V_0^c = \mathrm{E}^{\mathbb{X}}\left[(N_u^c)^{-1} V_u^c\right]$$

for some measure $\mathbb{X}$ (identical for all assets, but potentially currency dependent).

Formula also called collateral account discounting. The discounting is not done with a theoretical risk free rate anymore, but with the rate used in practice in the collateral agreement.

Pricing OIS

The pseudo-discount factor associated to the rate c for discounting from u to s is denoted $P^c(s,u)$:

$$P^c(0,u) = \mathrm{E}^{\mathbb{X}}\left[(N_u^c)^{-1}\right].$$

In the context of collateral discounting, the OIS coupon accumulate the overnight rate over the composition period and the discounting does in the opposite direction with the same mechanism.

Theorem (OIS coupon pricing)

In the collateral framework, the price of the overnight coupon over the period $[t_0, t_n]$ in the overnight collateral framework is given by

$$P^c(0,t_0) - P^c(0,t_n).$$

For OIS and collateral discounting, the floating coupon replication works again!

Pricing OIS

OIS coupon value:

$$V_0^c = \mathrm{E}^{\mathbb{X}}\left[(N_{s_n}^c)^{-1} \left(\left(\prod_{i=1}^{n} (1 + \delta_i I^c(s_{i-1}, s_i)) \right) - 1 \right) \right]$$

The value of a (fixed rate) receiver OIS is

$$\sum_{i=1}^{\tilde{n}} c_i P(0, \tilde{t}_i) - (P(0, s_0) - P(0, s_n)) \,.$$

The *par rate* – also called *market quote* – is

$$K = \frac{P(0, t_0) - P(0, t_n)}{\sum_{i=1}^{\tilde{n}} \tilde{\delta}_i P(0, \tilde{t}_i)}$$

Fixed Income Products and Analysis

Bond Convention – Prices

The most used day count convention for bonds is **ACT/ACT ICMA**
The accrual factor is

$$\frac{1}{\text{Freq}}\text{Adjustment}$$

where Freq is the number of coupon per year and Adjustment depends of the type of stub period.

None The Adjustment is 1. The second expression reduces to 1 and the coupon is $1/\text{Freq}$.

Different adjustments are used for long and short coupons.

The amount paid for a bond is described by a *dirty price* – denoted Dirty below – with the amount paid equal to notional times dirty price: $N \times \text{Dirty}$. The dirty price of a bond is the total price to be paid (at the settlement date) in exchange of the bond.

Prices

Bond trades are typically not settled on a same day basis but T+1, T+2, or T+3 depending on the markets.

The accrued interest at date t in a coupon C with frequency m coupon per year and for period $[t_i^c, t_{i+1}^c]$

$$\text{Accrued}(t) = \frac{\text{DaysBetween}(t_i^c, t)}{\text{DaysBetween}(t_i^c, t_{i+1}^c)} * \frac{C}{m}.$$

The *clean price* of a bond is the dirty price minus the accrued interest. We denote the accrued interest at date t by $\text{Accrued}(t)$. The relation between dirty and clean price for settlement in t is

$$\text{Dirty}(t) = \text{Clean}(t) + \text{Accrued}(t).$$

Bonds are typically traded in clean price.

See spreadsheet for examples.

Internal rate of return

The *Internal Rate of Return* or *Yield* is a conventional bond measure.
Its first use is to be able to compare the (theoretical) return of several instruments.

The most used convention for bonds is: **Street conventions**
Consider a bond with frequency of m coupon / year and coupons C_i $(1 \leq i \leq n)$.

Case 1: settlement on a coupon date
The *Yield* Y when there are n coupons left (including today's coupon) is given by the solution to

$$\text{Dirty} = \sum_{i=2}^{n} \frac{C_i}{(1 + \frac{1}{m}Y)^{i-1}} + \frac{1}{(1 + \frac{1}{m}Y)^{n-1}}$$

Special treatment in the last coupon period.

Other conventions: **US Treasury**, **United Kingdom**, **France**, **Germany**, **Australian**

Internal rate of return

Case 2: settlement not on a coupon date

Factor to next coupon w:

$$w = \frac{\text{Accrued}(t^c_{i+1}) - \text{Accrued}(t_0)}{\text{Accrued}(t^c_{i+1})}.$$

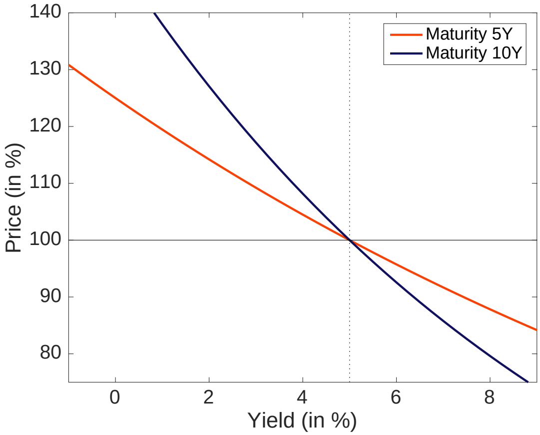
The convention yield Y is given by the solution to

$$\text{Dirty} = \left(1 + \frac{1}{m}Y\right)^{-w} \left(\sum_{i=1}^{n} \frac{C_i}{(1 + \frac{1}{m}Y)^{i-1}} + \frac{1}{(1 + \frac{1}{m}Y)^{n-1}} \right)$$

The conventional yield ignores the exact payment dates (non-good business days).

See spreadsheet for examples.

Price/Yield relationship



Bond: Coupon 5.00%, coupon frequency 2.

Duration

The *modified duration* is a (dirty) price sensitivity measure. It is the relative derivative of the price with respect to the conventional yield, i.e.

$$\text{Duration}_{\text{Modified}} = -\frac{1}{\text{Dirty}} \frac{\partial \text{Dirty}}{\partial Y}.$$

To the first order one has

$$\text{Dirty}(Y) \simeq \text{Dirty}(Y_0) + \frac{\partial \text{Dirty}}{\partial Y}(Y_0)(Y - Y_0) = \text{Dirty}(Y_0)\left(1 - \text{Duration}_{\text{Modified}}(Y - Y_0)\right)$$

For the *US street*,

$$\text{Duration}_{\text{Modified}} = \left(\sum_{i=1}^{n} \frac{C_i}{\left(1 + \frac{Y}{m}\right)^{w+i}} \frac{w+i-1}{m} + \frac{N}{\left(1 + \frac{Y}{m}\right)^{w+n}} \frac{w+n-1}{m} \right) \frac{1}{\text{Dirty}}.$$

Convexity

The *convexity* is the relative second order derivative of the price with respect to the conventional yield, i.e.

$$\frac{\partial^2 \mathrm{Dirty}}{\partial y^2} = \mathrm{Convexity} \cdot \mathrm{Dirty}.$$

The approximation in term of yield can be written as

$$\mathrm{Dirty}(Y_0 + \Delta Y) = \mathrm{Dirty}(Y_0)\left(1 - \mathrm{Duration}_{\mathrm{Modified}} \cdot \Delta Y + \frac{1}{2}\mathrm{Convexity} \cdot (\Delta Y)^2\right).$$

For the *US street* the convexity is

$$\mathrm{Convexity} = \frac{1}{\mathrm{Dirty}}\left(\sum_{i=1}^{n} \frac{C_i}{\left(1+\frac{y}{m}\right)^{w+i+1}} \frac{(w+i-1)}{m} \frac{(w+i)}{m} + \frac{N}{\left(1+\frac{y}{m}\right)^{w+n+1}} \frac{(w+n-1)}{m} \frac{(w+n}{m}\right.$$

Bond price approximation

The quantities defined above can be used to approximate the change of bond price to the second order. The approximation in term of yield can be written as

$$\begin{aligned}\text{Dirty}(Y_0 + \Delta Y) &\simeq \text{Dirty}(Y_0) + \frac{\partial \text{Dirty}}{\partial y}(Y_0) \cdot \Delta Y + \frac{\partial^2 \text{Dirty}}{\partial y^2}(Y_0) \cdot (\Delta Y)^2 \\ &= \text{Dirty}(Y_0)\left(1 - \text{Duration}_{\text{Modified}} \cdot \Delta Y + \frac{1}{2}\text{Convexity} \cdot (\Delta Y)^2\right)\end{aligned}$$

But: reality is multi-factor and cannot be reduced to a single yield.

The above formulas are useful to understand the basic of bond dynamics but are not enough by themselves to manage a bond portfolio. The term structure of interest rates may be as important than the yield to maturity. The yield curves don't always move in a parallel way.

Fixed Income Products and Analysis

Different approaches to interest-rate modelling

The curve calibration provides the information about the current curve $P(0, t)$. One would like also to model what could happen in the future, i.e. $P(s, t)$ or an equivalent quantity like $r(s, t)$.

- Deterministic
- Black - one rate models
- Short rate
- Heath, Jarrow, Morton

The models can be single factor or multi-factor.

Deterministic rates

The future is estimated as a deterministic function of the present. We suppose that the forward rates will be realised.

In term of discount factors, one has

$$P(s,t) = \frac{P(0,t)}{P(0,s)}.$$

In terms of forward rates, one has

$$F^j(t,\theta,t_s,t_e) = F^j(0,\theta,t_s,t_e)$$

Black - one rate models

The idea behind this approach is to treat the different quantities seen in fixed income – e.g. bond price, forward rate, swap rate – like in the equity option formulas. All you need to know is the volatility of the 'underlying' quantity.

Some cleaver bit of quantitative finance may be needed to show that this approach actually makes sense from a quantitative finance perspective and arbitrage free pricing.

The main advantage of this approach is that for simple fixed-income contracts there are simple formulas for their prices.

Only one quantity is modelled. The relation of that quantity with the rest of the market and the yield curve is not modelled.

Stochastic spot rate / short rate

In this approach one models the very short-term rate, i.e. the rate r we have used in described the cash account. This is the theoretical equivalent of the overnight rate.

Typically we suppose that the rate satisfy a stochastic differential equation. The relation between the discount factors and this short rate is obtained through the option pricing formula as

$$P(s,t) = \mathrm{E}^{\mathbb{X}}\left[(N_t^r)^{-1} \middle| \mathcal{F}_s \right].$$

The pricing of derivatives can be sone by many different numerical methods, like finite-difference methods for solving partial differential equations, trees, Monte Carlo simulations, explicit solutions, etc.

Models of this type are Vasicek, Cox, Ingersoll and Ross, Hull and White, etc.

Heath, Jarrow, Morton

When the discount curve is regular enough, there exists $f(t, u)$ such that

$$P^c(t, u) = \exp\left(-\int_t^u f(t, s)ds\right).$$

The short rate associated to the curve is $(r(t))_{0 \leq t \leq T}$ with $r_t = f(t, t)$.

The idea of Heath, Jarrow, and Morton, (1992) was to model f with a stochastic differential equation

$$df(t, u) = \mu(t, u)dt + \sigma(t, u)dW_t$$

for some suitable μ and σ and deducing the behavior of P from there.

Heath, D., Jarrow, R., and Morton, A. (1992). Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation. *Econometrica*, 60(1):77–105.

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