$i=\sqrt{-1}\in\mathbb{C}$ $\qquad$ $(i,k)\to(n,m)$

Stability Analysis

By expressing $S := n\delta S$ and $t := T - m\delta t$, we will obtain a difference equation for the Black-Scholes equation.

$$V(S,\, T-t) = V(n\delta S,\, T - m\delta t) = V_n^m.$$

$\delta S = \dfrac{S^*}{N}$ where $S^* \gg E$ is a suitably large value of S ; $\delta t = \dfrac{T}{M}$. Taking N and M steps for S and t respectively, so

$$\begin{aligned} S &:= n\delta S & \quad 0 \le n \le N \\ t &:= T - m\delta t & \quad 0 \le m \le M. \end{aligned}$$

$$\frac{V_n^m - V_n^{m+1}}{\delta t} + \tfrac{1}{2}n^2\sigma^2\left(V_{n-1}^m - 2V_n^m + V_{n+1}^m\right) +$$

$$\tfrac{1}{2}(r-D)\,n\left(V_{n+1}^m - V_{n-1}^m\right) - rV_n^m = 0$$

and rearrange to obtain a *forward marching* scheme in time

$$\begin{aligned} V_n^{m+1} &= V_n^m + \delta t\left(\frac{1}{2}n^2\sigma^2\left(V_{n-1}^m - 2V_n^m + V_{n+1}^m\right)\right) \\ &\quad + \delta t\left(\frac{1}{2}(r-D)\,n\left(V_{n+1}^m - V_{n-1}^m\right) - rV_n^m\right) \\ &\equiv F\left(V_{n-1}^m,\ V_n^m,\ V_{n+1}^m\right) \end{aligned}$$

Is δS and δt related?

Now for the RHS collect coefficients of each variable term V, to get

$$V_n^{m+1} = \alpha_n V_{n-1}^m + \beta_n V_n^m + \gamma_n V_{n+1}^m \quad (1)$$

where

$$\begin{aligned}
\alpha_n &= \frac{1}{2}\left(n^2\sigma^2 - n\left(r-D\right)\right)\delta t, \\
\beta_n &= 1-\left(r+n^2\sigma^2\right)\delta t, \\
\gamma_n &= \frac{1}{2}\left(n^2\sigma^2 + n\left(r-D\right)\right)\delta t
\end{aligned} \quad (2)$$

$e^{i\theta} = \cos\theta + i\sin\theta$

Euler's identity

Fourier Stability (Von Neumann's) Method

A method is called step-wise unstable if for a fixed grid (i.e. δt, δS constant) there exists an initial perturbation which "blows up" as $t \to \infty$, i.e. as we march in time. Here in a forward marching scheme. The question we wish to answer is "do small errors propagate along the grid and grow exponentially?".

error: $E_n^{\boxed{m}} \propto e^{in\omega}$

Assume an initial disturbance which is proportional to $\exp\left(in\omega\right)$. We therefore study the propagation of perturbations created at any given point in time.

G.D. Smith (Oxford) Numerical Methods for PDEs

If $\widehat{V}_n^m$ is an approximation to the exact solution ${V_n}^m$ then

approx. exact error

$\frac{\partial u}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2}$

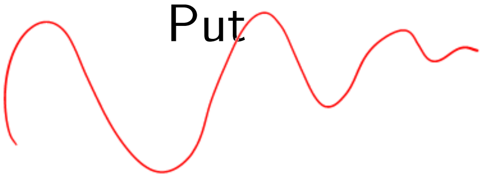
$$\widehat{V}_n^m = {V_n}^m + {E_n}^m$$

where ${E_n}^m$ is the associated error. Then ${E_n}^m$ also satisfies the difference equation (4) to give

$$E_n^{m+1} = \alpha_n E_{n-1}^m + \beta_n E_n^m + \gamma_n E_{n+1}^m$$

Put

freq. ω (cycles per second)



$$\boxed{E_n^m = \overline{a}^m \exp\left(in\omega\right)} \qquad (3)$$

which is oscillatory of amplitude $\overline{a}$ and frequency ω. Substituting (3) into (1) gives

$$\overline{a}^{\cancel{m}+1}\cancel{e^{(in\omega)}} = \alpha_n \cancel{\overline{a}^m} e^{i(\cancel{n}-1)\omega} + \beta_n \cancel{\overline{a}^m} \cancel{e^{in\omega}} + \gamma_n \cancel{\overline{a}^m} e^{i(\cancel{n}+1)\omega}$$

which becomes

$$\overline{a} = \alpha_n e^{-i\omega} + \beta_n + \gamma_n e^{i\omega}.$$

Now stability criteria arises from the balancing of the time dependency and diffusion terms, so that

$\frac{\partial V}{\partial t}$ $\frac{\partial^2 V}{\partial S^2}$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = 0$$

From (2) we take the following contributions $\frac{\partial V}{\partial t}$ & $\frac{\partial^2 V}{\partial S^2}$

$$\alpha_n = \frac{1}{2}n^2\sigma^2\delta t, \;\; \beta_n = 1 - n^2\sigma^2\delta t, \;\; \gamma_n = \frac{1}{2}n^2\sigma^2\delta t$$

$\cos x = \frac{e^{ix} + e^{-ix}}{2}$

$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

$\cos 2x = \begin{cases} \cos^2 x - \sin^2 x \\ 2\cos^2 x - 1 \\ 1 - 2\sin^2 x \end{cases}$

Important Trig identities

$$
\begin{aligned}
\overline{a} &= \frac{1}{2} n^2 \sigma^2 \delta t \left(e^{i\omega} + e^{-i\omega} \right) + 1 - n^2 \sigma^2 \delta t \\
&= n^2 \sigma^2 \delta t \left(\cos \omega - 1 \right) + 1.
\end{aligned}
$$

$\cos\omega$ comes from $\frac{1}{2}\left(e^{i\omega} + e^{-i\omega}\right)$

we have

$$
\overline{a} = 1 - 2n^2 \sigma^2 \sin^2 \frac{\omega}{2} \delta t
$$

For stability $\overline{a}$ must be bounded, i.e. $|\overline{a}| < 1$

$$
\left| 1 - 2n^2 \sigma^2 \sin^2 \frac{\omega}{2} \delta t \right| < 1
$$

$\overline{a}^n$

$|\overline{a}| < 1$

which upon simplifying we find is

stability condition

$$
\boxed{\delta t < \frac{1}{\sigma^2 N^2}} \tag{4}
$$

$-1 < 1 - 2N^2\sigma^2 \delta t < 1$

so $\delta t \sim O\left(N^{-2}\right)$.

explicit scheme

$\cos 2x = 1 - 2\sin^2 x$

$\cos 2x - 1 = -2\sin^2 x$ $\quad x = \frac{\omega}{2}$

$\cos\omega - 1 = -2\sin^2 \frac{\omega}{2}$

$4\,\delta t < \frac{1}{4\sigma^2 N^2}$ $\quad N \longrightarrow 2N$

$\delta S = \frac{S^*}{N}$