The Heath, Jarrow and Morton Model

CQF Lecture SOLUTIONS

Three Ways to Derive Instantaneous Forward Rate

1. The price of a zero-coupon bond that matures at time T paying \$1 is given using an integral over *the forward curve*

$$Z(t;T) = e^{-\int_t^T f(t,s)ds} \tag{1}$$

By solving an integral equation, confirm the instantaneous forward rate is defined as

$$f(t,T) = -\frac{\partial}{\partial T}\log Z(t;T) \tag{2}$$

$\boxed{\textbf{Solution:}}$

$$\begin{aligned}
Z(t;T) &= e^{-\int_t^T f(t,s)ds} \\
\log Z(t;T) &= -\int_t^T f(t,s)ds \\
\frac{\partial}{\partial T}\log Z(t;T) &= -\frac{\partial}{\partial T}\left(\int_t^T f(t,s)ds\right) \\
&= \text{due to the fact that derivative is partial } \textit{wrt } T \text{ we have} \\
\frac{\partial}{\partial T}\log Z(t;T) &= -f(t,T) - 0 \\
f(t,T) &= -\frac{\partial}{\partial T}\log Z(t;T)
\end{aligned}$$

Under risk-neutral expectation, forward rate is replaced by the short-term rate $r(t)$:

$$Z(t;T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}|\mathcal{F}_s\right]$$

Let's show the inverse: taking the definition of instantaneous forward rates as given, derive the bond price

$$\begin{aligned}
f(t,T) &= -\frac{\partial}{\partial T}\log Z(t;T) \\
\int_t^T f(t,s)ds &= -\int_t^T \frac{d}{ds}\log Z(t;s)ds = -\int_t^T d\left(\log Z(t;s)\right) \\
&= \text{Leibniz integration gives} \\
\log Z(t;T) - \log Z(t;t) &= -\int_t^T f(t,s)ds \qquad \text{since } Z(t;t) = Z(T;T) = 1 \\
Z(t;T) &= e^{-\int_t^T f(t,s)ds}
\end{aligned}$$

The solutions have been edited by CQF Faculty, Richard Diamond, R.Diamond@fitchlearning.com

2. Consider two bonds $Z(t;T_1)$ and $Z(t;T_2)$ where $T_2 > T_1$, and the forward rate $f(t;T_1,T_2)$ that is locked-in between T_1 and T_2. **By considering present value of 1\$ investment**, back from show that the locked-in forward rates are defined as

$$f(t,T) = -\frac{\partial}{\partial T}\log Z(t;T)$$

$\boxed{\textbf{Solution:}}$

The longer-term bond $Z(t;T_2)$ is a natural discount factor. It is equal to the DF $Z(t;T_1)$ multiplied by $e^{-f_1(T_2-T_1)}$ where f_1 is a forward rate that applies from T_1 to T_2.

$$\begin{aligned}
Z(t;T_2) &= Z(t;T_1)e^{-f(t;T_1,T_2)(T_2-T_1)} \\
\log\frac{Z(t;T_2)}{Z(t;T_1)} &= -f(t;T_1,T_2)(T_2-T_1) \qquad \Rightarrow \\
-f(t;T_1,T_2) &= \frac{\log Z(t;T_2)-\log Z(t;T_1)}{T_2-T_1} \\
&= \text{in continous time, } T_2-T_1=\delta t\to 0 \\
&= \lim_{\delta t\to 0^+} -\frac{\log Z(t;T_1+\delta t)-\log Z(t;T_1)}{\delta t} \\
&= -\frac{\partial}{\partial T}\log Z(t;T)
\end{aligned}$$

3. A forward rate $f(t,T)$ represents the instantaneous continuously compounded rate, that is contracted at time t for a risk-free borrowing at a future time T. Prove the relationship between an instantaneous forward rate and ZCB yield

$$f(t,T) = -\frac{\partial}{\partial T}\log Z(t;T)$$

by considering a self-financing portfolio that is short $Z(t;T)$ and long $Z(t;T+\delta t)$.

$\boxed{\textbf{Solution:}}$ To replicate a forward rate that would apply for a small time period δt, we can take a long position in bond $Z(t;T+\delta t)$ and short position in bond $Z(t;T)$. Then at time $T+\delta t$ we receive $\dfrac{Z(t;T)}{Z(t;T+\delta t)}$ on the notional capital of 1 from the long position.

The time-adjusted rate of return on the portfolio will be

$$\frac{1}{\delta t}\log\frac{Z(t;T)}{Z(t;T+\delta t)} = -\frac{\log Z(t;T+\delta t)-\log Z(t;T)}{\delta t}$$

In continuous time limit, we obtain a derivative of $\log Z$ as a solution for the instantaneous forward rate at time t for investing at time T:

$$f(t,T) = \lim_{\delta t\to 0^+} -\frac{\log Z(t;T+\delta t)-\log Z(t;T)}{\delta t} = -\frac{\partial}{\partial T}\log Z(t;T).$$

Notice the similarity of the solution to discrete expression for the yield to maturity $-\dfrac{\log Z}{T-t}$.

HJM SDE and Musiela Parameterization

Market price of risk. No arbitrage. Tenor time

1. The key parameter that links the real and risk-neutral 'worlds' and explains a global market condition is the market price of (interest rate) risk (MPOR). Mathematically, the market price of risk is a parameter of choice that allows to cancel the drift. By considering a hedged portfolio,
$$\Pi = Z(t;T_1) - \Delta Z(t;T_2)$$
derive the relationship between SDE parameters for $\frac{dZ(t;T)}{Z(t;T)} = \mu(t,T)dt + \sigma(t,T)dX$ and the market price of interest rate risk.
$$\frac{\mu(t,T_1) - r(t)}{\sigma(t,T_1)} = \frac{\mu(t,T_2) - r(t)}{\sigma(t,T_2)}$$
Hint: in the risk-free world, all assets earn the risk-free rate.

Solution: The change in the hedged portfolio is given by
$$\begin{aligned} d\Pi &= dZ(t;T_1) - \Delta dZ(t;T_2) \\ &= Z(t;T_1)\,\left[\mu(t,T_1)dt + \sigma(t,T_1)dX\right] \\ &\qquad -\Delta Z(t;T_2)\,\left[\mu(t,T_2)dt + \sigma(t,T_2)dX\right] \end{aligned}$$
If we choose
$$\Delta = \frac{\sigma(t,T_1)Z(t;T_1)}{\sigma(t,T_2)Z(t;T_2)} \text{ or simply } \frac{\sigma_1 Z_1}{\sigma_2 Z_2}$$
then our portfolio is risk-free – that is, if we substitute Δ then dX terms cancel out:
$$\begin{aligned} d\Pi &= \left(\mu(t,T_1)Z(t;T_1) - \Delta\mu(t,T_2)Z(t;T_2)\right)dt \\ &= \left(\mu_1 Z_1 - \Delta\mu_2 Z_2\right)dt \end{aligned}$$
In the risk-free world, all assets earn the risk-free rate therefore, the portfolio return is also equal $d\Pi = r\Pi dt$. Equating both results for $d\Pi$, we have
$$\begin{aligned} \mu_1 Z_1 - \Delta\mu_2 Z_2 &= r\left(Z_1 - \Delta Z_2\right) \\ Z_1\left(\mu_1 - r\right) &= Z_2\left(\mu_2 - r\right)\frac{\sigma_1 Z_1}{\sigma_2 Z_2} \\ \frac{\mu_1 - r}{\sigma_1} &= \frac{\mu_2 - r}{\sigma_2} \equiv \lambda(r,t) \quad \text{independent of } T_1, T_2 \end{aligned}$$
The named interest rate models operate with the risk-adjusted drift $(u - \lambda\omega)$. One-factor models suggest that the slope of the yield curve *at the short end* is simply $(u - \lambda\omega)/2$.

Further discussion can be found in the original WILMOTT Magazine's technical article on *The Market Price of Interest-rate Risk: Measuring and Modelling Fear and Greed in the Fixed-income Markets* by Riaz Ahmad and Paul Wilmott.

2. Using the definition of the instantaneous forward rate (2)

$$f(t,T) = -\frac{\partial}{\partial T}\log Z(t;T)$$

obtain the corresponding SDE model. Assume the bond price follows a log-Normal model

$$\frac{dZ}{Z} = \mu(t,T)\,dt + \sigma(t,T)\,dX$$

Hint: differentiate with respect to t. The maturity time T is fixed.

Solution:

$$\begin{aligned}
f(t,T) &= -\frac{\partial}{\partial T}\log Z(t;T) \\
df(t,T) &= d\left(-\frac{\partial}{\partial T}\log Z(t;T)\right) \\
&= -\frac{\partial}{\partial T}\left(d\log Z(t;T)\right) \\
&= -\frac{\partial}{\partial T}\left(\frac{dZ(t;T)}{Z(t;T)} - \frac{1}{2}\frac{dZ(t;T)^2}{Z(t;T)^2}\right) \\
&= \text{we know } \frac{dZ}{Z}\text{. Evaluate } \left(\frac{dZ}{Z}\right)^2 \text{ using } O(dX^2) = dt \text{ dropping other terms} \\
&= -\frac{\partial}{\partial T}\left(\mu(t,T)\,dt + \sigma(t,T)\,dX - \frac{1}{2}\sigma^2(t,T)dt\right) \\
&= -\frac{\partial}{\partial T}\left(\left[\mu(t,T) - \frac{1}{2}\sigma^2(t,T)\right]\,dt + \sigma(t,T)\,dX\right).
\end{aligned}$$

How did we obtain result for $d\left(\log Z(t;T)\right) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dX$?

This is our familiar GBM dynamics for $Z(t;T)$ and Itô application to $F = \log Z$ as follows:

$$dF = \frac{\partial F}{\partial Z}(dZ) + \frac{1}{2}\frac{\partial^2 F}{\partial Z^2}(dZ)^2$$

With $\frac{dF}{dZ} = \frac{1}{Z}$, $\frac{d^2F}{dZ^2} = -\frac{1}{Z^2}$ and $(dZ)^2 = \sigma^2 Z^2 dt$

$$\begin{aligned}
dF &= \frac{1}{Z}\left(\mu Z dt + \sigma Z dX\right) - \frac{1}{2}\frac{1}{Z^2}\sigma^2 Z^2 dt \\
&= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dX.
\end{aligned}$$

3. The raw model for the evolution of (points) $f(t,T_i)$ on the forward curve relates the drift to volatility as

$$df(t,T) = \frac{\partial}{\partial T}\left[\frac{1}{2}\sigma^2(t,T) - \mu(t,T)\right] dt - \frac{\partial}{\partial T}\sigma(t,T)dX \tag{3}$$

Show that, under the risk-neutral measure $\mathbb{Q}$, the model can be expressed as

$$df(t,T) = m(t,T)dt + \nu(t,T)dX$$

where $\nu(t,T) = -\frac{\partial}{\partial T}\sigma(t,T)$ simplifies the diffusion term, and the risk-neutral drift can be expressed solely as a function of volatility (no arbitrage condition)

$$m(t,T) = \nu(t,T)\int_t^T \nu(t,s)ds$$

$\boxed{\textbf{Solution:}}$ Let's consider differentiation $\frac{\partial}{\partial T}$ of the drift of equation (3)

$$m(t,T) = \frac{\partial}{\partial T}\left[\frac{1}{2}\sigma^2(t,T) - \mu(t,T)\right]$$

Under the risk-neutral measure $\mu(t,T) \to r(t)$, where the spot rate does *not* depend on time T, so $\frac{\partial}{\partial T}r(t) = 0$. What we have left is

$$\begin{aligned}
\underline{m(t,T)} &= \frac{\partial}{\partial T}\left[\frac{1}{2}\sigma^2(t,T)\right] \\
&= \frac{1}{2}\frac{\partial}{\partial T}\left[\sigma(t,T)\,\sigma(t,T)\right] \\
&= \frac{1}{2}\left[\sigma(t,T)\sigma(t,T)' + \sigma(t,T)'\sigma(t,T)\right] \\
&= \sigma(t,T)'\sigma(t,T) = \underline{\sigma(t,T)\frac{\partial}{\partial T}\sigma(t,T)} \qquad \text{This is the dirft.}
\end{aligned}$$

Now we have to express the drift in terms of $\nu(t,T) = -\frac{\partial}{\partial T}\sigma(t,T)$. By solving an integral equation we can find the solution for $\sigma(t,T)$

$$\int_t^T \nu(t,s)ds = -\int_t^T d(\sigma(t,s)) = -\sigma(t,T) \qquad \text{where} \quad Z(t;t) = 1 \to \sigma(t,t) = 0.$$

Then, continue by substitution into the underlined expression

$$\begin{aligned}
m(t,T) &= -\int_t^T \nu(t,s)ds \times -\nu(t,T) \\
&= \nu(t,T)\int_t^T \nu(t,s)ds.
\end{aligned}$$

4. Musiela Parametrisation of the HJM model (risk-neutral evolution of the forward curve) provides convenience of operating with fixed tenors $\tau = T - t$ rather than maturity dates.

   By applying the change of variable $f(t,T) \to \bar{f}(t,\tau)$ and using the chain rule of differentiation, show that the Musiela Parametrisation of the one-factor HJM model is

$$d\bar{f}(t,\tau) = \left(\bar{\nu}(t,\tau)\int_0^\tau \bar{\nu}(t,s)ds + \frac{\partial \bar{f}(t,\tau)}{\partial \tau}\right)dt + \bar{\nu}(t,\tau)dX$$

   Hint: taking of a derivative of forward rate *wrt* T is equivalent to taking of a derivative of Musiela Parameterisation *wrt* τ, i.e., $\frac{\partial f}{\partial T} \equiv \frac{\partial \bar{f}}{\partial \tau}$.

   Solution:

   There are a number of related results known as 'chain rules' for differentiating the function of multiple variables (that intertwine). We begin with $f(t,T) = f(t, t+T-t)$, have to construct a differential $d\bar{f}(t,\tau)$ using the original variables t, T, done as follows:

$$\frac{d}{dt}\bar{f}(t,T-t) \quad = \quad \left(\frac{\partial}{\partial T}\frac{\partial T}{\partial t} + \frac{\partial}{\partial t}\frac{\partial t}{\partial T}\right) f(t,T) = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial T}\right) f(t,T) \ \dagger$$

   Why? Remember that for $f(t,T)$ our variable of differentiation is t – the SDE evolves with dt, not dT. We assume constant τ and two functions $T(t) = t + \tau$ and $t(T) = T - \tau$. Any differentiation $\frac{\partial}{\partial T}$ would require $\frac{\partial T}{\partial t}$, the same applies to differentiating *wrt* t.

   Making τ a variable, it is straightforward to show (so "τ is the new T")

$$\frac{\partial}{\partial \tau} \quad = \quad \frac{\partial}{\partial T}\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial T} \quad \text{because} \quad \frac{\partial T}{\partial \tau} = 1$$

   To understand Musiela Parameterisation HJM SDE treat the small differential ∂t as 'a real variable' that is moved to the *rhs* 'to multiply' and obtain the boxed expression:

$$\begin{aligned}
\frac{\partial}{\partial t}\bar{f}(t,\tau) \quad &\equiv \quad \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial T}\right) f(t,T) \ \dagger \quad \text{becomes} \quad \boxed{d\bar{f} \equiv df + \frac{\partial f}{\partial T}dt} \\
d\bar{f}(t,\tau) \quad &= \quad \underbrace{df(t,T)} + \frac{\partial f(t,T)}{\partial T}dt \quad \text{then by} \quad \frac{\partial}{\partial \tau} = \frac{\partial}{\partial T}\frac{\cancel{\partial T}}{\cancel{\partial \tau}} \\
&= \quad \underbrace{\left(\nu(t,T)\int_t^T \nu(t,s)ds\right)dt + \nu(t,T)dX} + \frac{\partial \bar{f}(t,\tau)}{\partial \tau}dt \\
&= \quad \left(\bar{\nu}(t,\tau)\int_0^\tau \bar{\nu}(t,s)ds + \frac{\partial \bar{f}(t,\tau)}{\partial \tau}\right)dt + \bar{\nu}(t,\tau)dX
\end{aligned}$$

   The extra forward derivative term $\frac{\partial \bar{f}(t,\tau)}{\partial \tau}$ is a slope of the yield curve. Its role is to maintain constant-tenor points of the yield curve by correcting for 'rolling on the curve' effect. As instruments expire, the curve 'shifts' left in time.

5. Most of the popular models for $r(t)$ have HJM representations. Consider Ho & Lee model for the spot rate $r(t)$,

$$dr(t) = \eta(t)dt + c\,dX, \quad \text{for constant } c.$$

Formulate a bond pricing equation (BPE) and use continuous version of the forward rate bootstrapping formula in order to obtain an SDE for $df(t,T)$. Explain equivalence of terms in this SDE to the HJM SDE (non-Musiela).

Solution:

$Z(r,t;T)$ in the Ho & Lee model satisfies the following BPE and corresponding solution:

$$\frac{\partial Z}{\partial t} + \frac{1}{2}c^2\frac{\partial^2 Z}{\partial r^2} + \eta(t)\frac{\partial Z}{\partial r} - rZ = 0$$

$$Z(r,T;T) = 1$$

$$Z(r,t;T) = \exp\left(\frac{1}{6}c^2(T-t)^3 - \int_t^T \eta(s)(T-s)ds - (T-t)r\right)$$

where $\eta(t)$ is chosen to fit the yield curve at time t^*.

In forward rate terms $f(t,T) = -\frac{\partial}{\partial T}\log Z(t;T)$ means that

$$f(t^*,T) = r(t^*) - \frac{1}{2}c^2(T-t^*)^2 + \int_{t^*}^T \eta(s)ds$$

and so, at any time $t > t^*$

$$\eta(t) = \frac{\partial f(t^*;t)}{\partial t} + c^2(t-t^*)$$

$$f(t,T) = r(t) - \frac{1}{2}c^2(T-t)^2 + \int_t^T \eta(s)ds$$

Use this Ho & Lee solution for the forward rate to obtain the SDE for $df(t,T)$,

$$\begin{aligned}
\frac{\partial}{\partial t}f(t,T) &= \frac{\partial}{\partial t}r(t) + \frac{\partial}{\partial t}\left[-\frac{1}{2}c^2(T-t)^2 + \int_t^T \eta(s)ds\right] \\
&= \text{(differentiate } \textit{wrt } t \text{ and move differential to the } \textit{rhs}\text{)} \\
df(t,T) &= dr(t) + c^2(T-t)dt - \eta(t)dt \\
&= \eta(t)dt + c\,dX + c^2(T-t)dt - \eta(t)dt \\
&= c^2(T-t)dt + c\,dX
\end{aligned}$$

One of the interim results for the HJM SDE (below) makes it straightforward to identify $\nu(t,T) = c$ and $\sigma(t,T) = -c(T-t)$,

$$df(t,T) = -\sigma(t,T)\nu(t,T)dt + \nu(t,T)\,dX$$

Ho & Lee model for the spot rate $r(t)$ offers a simple yield curve fitting, making it the most suitable to draw a first comparison to the HJM framework.

Numerical Methods for PCA: Jacobi Transformation

Jacobi Transformation is a *tractable* numerical method of matrix diagonalization (e.g., obtaining a diagonal matrix of eigenvalues). The method is based on eliminating the largest off-diagonal element by rotating the matrix. 'Rotation' is implemented by pre-multiplying matrix $\mathbf{A}$, which we ultimately want to decompose, by matrix $\mathbf{P_{p,q}}$ that is specially constructed in order to cancel an off-diagonal element $a_{p,q}$ so that $a'_{p,q} = 0$.

$$P_{p,q} = \begin{bmatrix} 1 & & & & & & 0 \\ & \ddots 1 & & \vdots & & \iddots & \\ & & \cos\phi & \cdots & 0 & \cdots & \sin\phi & & \\ & \cdots & 0 & \cdots & 1 & \cdots & 0 & \cdots & \\ & & -\sin\phi & \cdots & 0 & \cdots & \cos\phi & & \\ & \iddots & & \vdots & & 1 \ddots & \\ 0 & & & & & & 1 \end{bmatrix}$$

For each rotation, we multiply

$$\mathbf{A}' = \mathbf{P_{p,q}}^{\mathbf{T}} \mathbf{A} \mathbf{P_{p,q}}$$

For a covariance matrix, the rotation occurs within the unit circle, and therefore, properties of trigonometric functions can be efficiently used. Key to implementation is calculation of the angle of rotation ϕ.

1. Describe the purpose of applying Jacobi Transformation to a covariance matrix.
   $\boxed{\textbf{Solution:}}$ The method is part of the specific class of *spectral decomposition* that factorizes a matrix into eigenvalues and corresponding eigenvectors. Spectral decomposition is used to identify main uncorrelated (orthogonal) factors that determine the most variance of a system, usually expressed with a co-variance matrix.

2. Deduce why in order to eliminate the matrix element $a'_{p,q} = 0$ it is necessary that $\tan(2\phi) = \dfrac{2a_{p,q}}{a_{q,q} - a_{p,p}}$. **Hint:** consider multiplication of individual matrix elements.
   $\boxed{\textbf{Solution:}}$ Consider the result of rotation matrix multiplication on the individual element with row p and column q

$$a'_{p,q} = \frac{1}{2}(a_{p,p} - a_{q,q})\sin(2\phi) + a_{p,q}\cos(2\phi) = 0$$

$$\frac{1}{2}(a_{q,q} - a_{p,p})\sin(2\phi) = a_{p,q}\cos(2\phi)$$

$$\frac{\sin(2\phi)}{\cos(2\phi)} = \frac{2a_{p,q}}{a_{q,q} - a_{p,p}}$$

$$\tan(2\phi) = \frac{2a_{p,q}}{a_{q,q} - a_{p,p}}$$

   Very close eigenvalues $a_{p,p} = a_{q,q}$ will make $\tan(2\phi) \to \infty$ implying that *stability* of the method improves with $\phi \ll \frac{\pi}{4}$.

3. Jacobi method is not the most computationally efficient because each new rotation destroys zero result obtained on the previous step. Nonetheless, *convergence* of the sum of the off-diagonal elements to zero occurs. Given that Jacobi method chooses $a_{p,q}$ to be greater than other off-diagonal elements on average

$$a_{p,q}^2 \geq \frac{\sum_{i \neq j} a_{i,j}^2}{n^2 - n}, \tag{4}$$

show that for a matrix $n \times n$ convergence occurs with the factor of $1 - \dfrac{2}{n^2 - n}$.

Solution: Each rotation reduces the sum of squares of the off-diagonal elements by the amount $2a_{p,q}^2$

$$\sum_{i \neq j} a_{i,j}'^2 = \sum_{i \neq j} a_{i,j}^2 - 2a_{p,q}^2. \tag{5}$$

This is possible to demonstrate with a case of symmetric 2×2 matrix $\mathbf{A} = \begin{bmatrix} a_{p,p} & a_{p,q} \\ a_{p,q} & a_{q,q} \end{bmatrix}$.

Then $\mathbf{A}' = \mathbf{P^T A P}$ implies $a_{p,p}'^2 + a_{q,q}'^2 = a_{p,p}^2 + 2a_{p,q}^2 + a_{q,q}^2$, where the sum of squares of diagonal elements increased by $2a_{p,q}^2$ (remember $2a_{p,q}'^2 = 0$ after a rotation).

The rotation deducts the same amount from off-diagonal elements as it adds to diagonal elements, i.e., the rotation does not change L^2 norms of column vectors constituting the matrix. Substituting (4) into (5) gives

$$\sum_{i \neq j} a_{i,j}'^2 \leq \sum_{i \neq j} a_{i,j}^2 - 2\frac{\sum_{i \neq j} a_{i,j}^2}{n^2 - n}$$

$$\sum_{i \neq j} a_{i,j}'^2 \leq \left(1 - \frac{2}{n^2 - n}\right) \sum_{i \neq j} a_{i,j}^2$$

The closer convergence factor is to 1 the slower is the numerical method because of the small reduction in the sum of squares occurring on each rotation.

4. Explore VBA code that implements Jacobi Transformation in Excel PCA file. Names of variables are self-explanatory and linked to the mathematical model, for example, *Athis(i,j)* for $\mathbf{A}$ and *Awork(i,j)* for $\mathbf{A}'$.

Solution: For the *spectral* decomposition of the covariance matrix

$$\mathbf{\Sigma} = \mathbf{V \Lambda V^T}$$

On convergence, matrix $\mathbf{A}'$ becomes a diagonal matrix with eigenvalues, so $\mathbf{\Lambda} = \mathbf{A}'$. In order to recover eigenvectors, matrices $\mathbf{P_{p,q}}$ from each transformation (rotation) should be multiplied, so $\mathbf{V} = \mathbf{P_0} \times \mathbf{P_1} \times \ldots \times \mathbf{P_m}$.

Note: Jacobi Transformation represents a balance between being tractable and computationally efficient. Power method to calculate eigenvalues by one, starting with the largest, is also simple to present (see Chapter 37.13 in Volume 2 of PWOQF). Other matrix decomposition methods (including non-spectral) can suit the task and work much faster, in particular, see Cholesky decomposition applicable if the matrix is positive definite.