Reduced Form Models

Dr. Siyi Zhou

CQF Fitch Learning

2023

Main Topics

In this lecture we will

- Model default event using Poisson Process.
- Derive risky bond pricing PDE assuming stochastic interest rate and / or stochastic default intensity.
- Review common recovery assumptions can be adopted in risky bond pricing.
- Illustrate Fundamental Pricing Formula for general contingent claims subject to default risk.
- Introduce basic theory of affine intensity models.
- Show an example of two-factor Vasicek intensity model.

Take Away

By the end of this lecture you will be able to

- Understand the concept of default intensity in the context of default risk.
- Explain pros and cons of intensity based model relative to structural model.
- Derive high dimensional risky bond pricing equation employing stochastic interest rate and default intensity.
- Calibrate default probability using bond prices.
- Solve simple affine intensity based models analytically.

Intensity Based (Reduced Form) Model

- In previous lecture we have studied structural approach to modeling default risk. In this lecture, we will introduce a different approach to price default risk in a bond or in a general contingent claim. These models are named as "reduced-form" or "intensity based" models, in which default is treated as an unpredictable event governed by an exogenous intensity process.
- The intensity is linked to (actually determines) the likelihood of default. In addition, the SDEs employed to model intensity are similar to the ones that are used to model short interest rate. Therefore many term structure models that are developed for short interest rate can also be reused to model default risk. The intensity based models, therefore, are one of the most popular credit risk models.

Model Default Risk using Poisson Process

Before embark on valuation of a risky bond, we need to understand the basic concept of intensity and how it can be used to model default events.

The Poisson process is one of the most important stochastic processes in probability theory. It is widely used to model random points in time and space. Furthermore, it is used as a foundation for building a number of more complicated stochastic processes. Like many other discrete and countable events, such as the number of buses will arrive in the next 10 min, default events can be modeled in a Poisson process.

Definition of Poisson Process I

A Poisson Process with intensity λ is a stochastic process

$$N_t : t \geq 0$$

taking values in $S = \{0, 1, 2, \cdots\}$ such that

1. $N_0 = 0$
2. if $s < t$, then $N_s \leq N_t$
3. if $s < t$, then the increment $N_t - N_s$ is independent of what happened during $[0, s]$

Definition of Poisson Process II

4. let $h \to 0^+$

$$Pr(N_{t+h} = n + m | N_t = n) = \begin{cases} \lambda h + o(h), & m = 1 \\ o(h), & m > 1 \\ 1 - \lambda h + o(h), & m = 0 \end{cases}$$

which means within an infinitesimal time interval h, maximum only one event can happen.

Distribution function of N_t

N_t has Poisson distribution with parameter λt

$$Pr(N_t = i) = \frac{(\lambda t)^i}{i!} e^{-\lambda t}$$

where

$$i = 0, 1, 2, \cdots$$

N_t is the number of events occurred by time t. It is also called "counting process". We can use Poisson Process as a starting point to model any default event.

Arrival Time in Poisson Process

Define T_n to be the arrival time of the nth event, i.e.,

$$T_n = \inf\{t : N_t = n\}, \quad T_0 = 0,$$

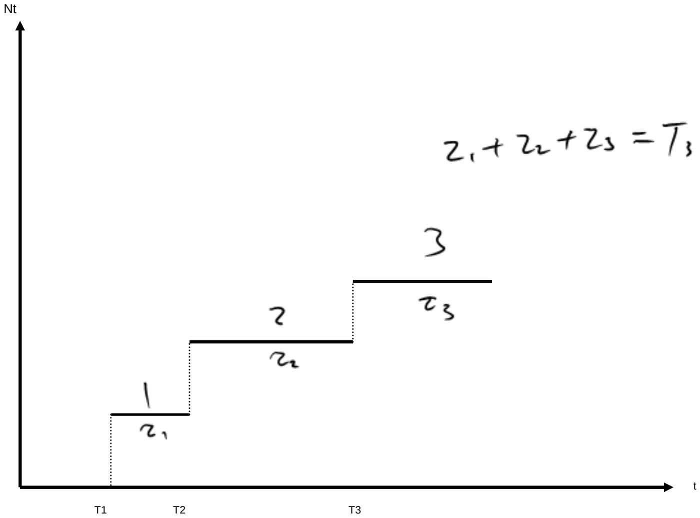
and τ_n to be an inter-arrival time, which is given by:

$$\tau_n = T_n - T_{n-1}.$$

So T_n can be expressed as the sum of all inter-arrive time:

$$T_n = \sum_{i=1}^{n} \tau_i$$

Sample Path of Poisson Process



Distribution of Inter-Arrival Time τ

- Since Poisson Process has independent increments, the associated inter-arrival times are also independent, i.e., $\forall i \neq j$, τ_i and τ_j are independent.
- Further more, every inter-arrival time has an exponential distribution with intensity λ. To summarize τ_i are i.i.d. $\exp(\lambda)$.
- To prove the above, we need to revert back to the 4th property of the Poisson Process.

Definition of Intensity

Denote $F(\tau)$ the Cumulative Distribution Function (CDF) of τ_1(the very first event), and define the survival function as:

$$S(\tau) = 1 - F(\tau).$$

According to the definition of Poisson process (point 4), when $h \to 0^+$

$$Pr(t < \tau \leq t + h | \tau > t) = \lambda h + o(h).$$

So

$$\boxed{\lambda = \lim_{h \to 0^+} \frac{Pr(t < \tau \leq t + h | \tau > t)}{h}}$$

conditional
Instantaneous
default rate

Distribution of τ

$$\lambda = \lim_{h \to 0^+} \frac{Pr(t < \tau \leq t + h)}{hPr(\tau > t)}$$
$$= \lim_{h \to 0^+} \frac{S(t) - S(t+h)}{hS(t)}$$
$$\lambda = -\frac{d \log S(t)}{dt}$$

$\int \lambda dt = -\log S(t) - \log S(0)$

Solve the above ODE for $S(t)$ with initial condition $S(0) = 1$, we get

$$\boxed{S(t) = e^{-\lambda t},}$$

which implies

$$\boxed{S(t) \sim \exp(\lambda).}$$

Simulating Poisson Process

So far we have derived distribution for the 1st inter-arrival time, what about the rest of them? By the third property of Poisson Process we know

$$S_2(t) = Pr(\tau_2 > t|\tau_1) = Pr(\tau_2 > t).$$

Then follow the same argument for τ_1 we have

$$\tau_2 \sim \exp(\lambda),$$

and so on for τ_i.

As a result, simulating a Poisson process is equivalent to simulating consecutive i.i.d. exponential random variables. We will see how to do this in subsequent M5 lecture.

Inhomogeneous Poisson process

What if the intensity isn't constant but deterministic? Then the counting process is called inhomogenous Poisson Process. The analysis are almost the same as before. The the survival function in this case becomes

$$S(t) = \exp\left(-\int_0^t \lambda_s \, ds\right).$$

Cox Process

Later on we will derive pricing PDE in which the intensity is not deterministic but a stochastic process. The pure probabilistic approach is slightly more complex and we don't pursue it in details here. The general idea is to use conditional expectation, i.e., conditional on the filtration to which the path of the intensity is adapted, then the survival function is known. The survival probability in this case is equal to

$$S(t) = \mathbb{E}\left[\exp\left(-\int_0^t \lambda_s \, ds\right)\right].$$

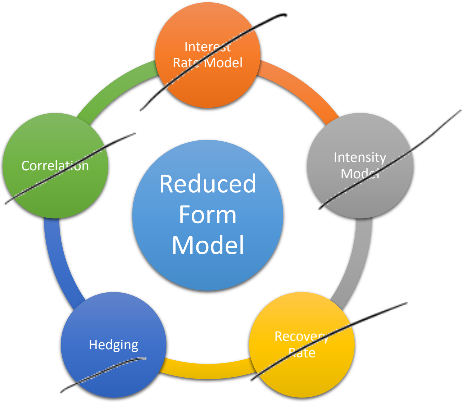
Cox Process: Basic Concept

Later on we will derive risk bond pricing PDEs where the default intensity is stochastic rather than deterministic. **Cox Process** is an inhomogeneous Poisson Process driven by a stochastic intensity. In more rigorous mathematical language we assume

- The stochastic intensity process λ_t which drives the probability of default is adapted to a filtration $\mathcal{G}_t$. This filtration can also contain other market observable underlying variables such as rates, FX and so on.
- Jumps(N_t) in the Cox process is adapted to another filtration $\mathcal{H}_t$.
- So the full filtration is obtained by $\boxed{\mathcal{F}_t = \mathcal{G}_t \cup H_t}$

The process Nt is called a Cox Process with stochastic intensity h_t, however, conditional on the background filtration $\mathcal{G}_t$, Nt becomes an inhomogeneous Poisson Process with deterministic intensity.

Input for Reduced Form model



BPE Plan

In the next a few slides, We will derive 3 BPEs for a risky bond based on different assumptions made on default intensity, hedging strategy and recovery rate.

	Int Rate	Intensity	Int Rate Risk	Default Risk	Recovery
1	Stoch	Const	Hedge	Not Hedge	Zero
2	Stoch	Stoch	Hedge	Hedge	Zero
3	Stoch	Stoch	Hedge	Hedge	RMV

We always assume stochastic interest rate and interest rate risk are delta hedged.

Model Assumptions

- Suppose a corporate's default follows a homogenous Poisson process with intensity p, and a ZCB with maturity T is issued by this company, the value of the bond is denoted by $V(t, r; p)$.
- Like usual suppose $Z(t, r)$ is the value of a riskless ZCB with exactly the same maturity where the short interest rate dynamics follows a diffusion process

$$\boxed{dr = u(r, t)\, dt + w(r, t) dX.}$$

- For simplicity we will assume that there is no correlation between the diffusive change in the short interest rate and the Poisson process.

Hedging Portfolio

Now following our convention when pricing fixed income product, we construct a 'hedged' portfolio:

$$\Pi = V(r, t; p) - \Delta Z(r, t).$$

Note here only the interest rate risk is hedged.

$$dZ(r,t) = \left(\frac{\partial Z}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 Z}{\partial w^2}\right) dt + \frac{\partial Z}{\partial r} dr$$
$$= \mathcal{L}(Z)\, dt + \frac{\partial Z}{\partial r} dr$$

$$dV(r,t;p) = \mathcal{L}(V)\, dt + \frac{\partial V}{\partial r} dr$$ No Default

$d\pi$? $t \to t + dt$

Case A: Without default

There is a probability of $(1 - p\,dt)$ that the bond does not default. Then the change in the value of the portfolio during an infinitesimal time step dt is

$$d\Pi = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}\right)dt + \frac{\partial V}{\partial r}dr$$
$$-\Delta\left(\left(\frac{\partial Z}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2}\right)dt + \frac{\partial Z}{\partial r}dr\right).$$

- Choose Δ to eliminate the risky dr term.

Case B: With Default

On the other hand, if the bond defaults, with a probability of $p\,dt$, then the change in the value of the portfolio is

$$d\Pi = -V + O(dt^{1/2}).$$

This is due to the default loss of the risky bond, the second term represents the changes in the riskless bond.

Systematic and Idiosyncratic Risk

- In Robert Merton's Jump Diffusion Model, jump components are assumed to be idiosyncratic risk. And the risk associated with the jumps is diversifiable since the jumps in the individual assets is uncorrelated with the market as a whole.
- Similar assumption can be made here, where default of a bond is similar to a jump in a share price.
- Therefore, the beta of the hedge portfolio is zero, the expected return of the zero-beta portfolio should be equal to the risk-free rate.

BPE

Taking expectation and using the bond-pricing equation of the riskless bond, we find that the value of the risky bond satisfies the following BPE

$$\boxed{\frac{\partial V}{\partial t} + \tfrac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - (r + p)V = 0.}$$

Feynman Kac:1st call

Homo

$$\boxed{V(t,T) = \mathbb{E}\left(e^{-\int_t^T (r_s+p)\,ds} | \mathcal{F}_t\right)}$$

Very similar analysis can be carried out with deterministic default intensity, the result will be almost identical apart from changing p to p_t, that is

In Homo

$$\boxed{V(t,T) = \mathbb{E}\left(e^{-\int_t^T (r_s+p_s)\,ds} | \mathcal{F}_t\right)}$$

$$= e^{-\int_t^T p_s ds} E\left(e^{-\int_t^T r_s ds} / \mathcal{F}_t\right)$$

$$= S_t(T) \; Z(t,T)$$

Yield Spread

The yield to maturity on this bond is now given by

$$y = -\frac{\log(Z(t,T)S_t(T))}{T-t} = y_f + \overbrace{\frac{1}{T-t}\int_t^T p_s ds}^{Sp},$$

where y_f is the yield to maturity of a risk free bond with the same maturity as the risky bond.

Thus the effect of the risk of default on the yield is to add a spread on riskless yield. In this simple model, the spread will be the average of the hazard rate from t to T.

Forward Rate Spread

If one calculates the forward rate implied by the risky bond

$$-\frac{\partial}{\partial T}\log(V(t,T)) = f(t,T) + p_T.$$

The spread is simply equal to default intensity.

Implied Default Probability: Zero Recovery

Given term structure of risk free bond and risky bond, one can extract implied default probability by using

$$S_t(T) = \frac{V(t,T)}{Z(t,T)} = \exp(-(T-t)(\underbrace{y - y_f}_{S})).$$

One can also calculate implied hazard rate by using forward spread.

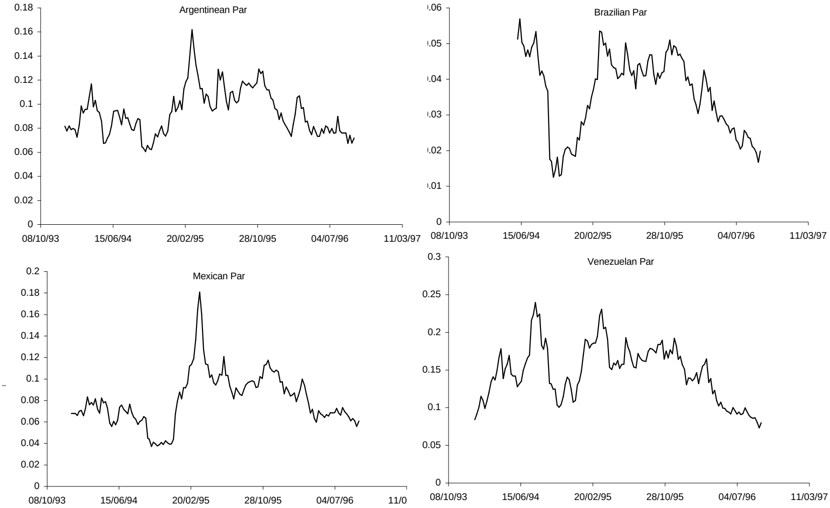
PD Calibration with Zero Recovery Rate

Year	Riskfree zero rate	Risky bond zero rate	Cummulative PD	Marginal PD
1	5%	5.25%	0.2497%	0.2497%
2	5%	5.50%	0.9950%	0.7453%
3	5%	5.70%	2.0781%	1.0831%
4	5%	5.85%	3.3428%	1.2647%
5	5%	5.95%	4.6390%	1.2961%

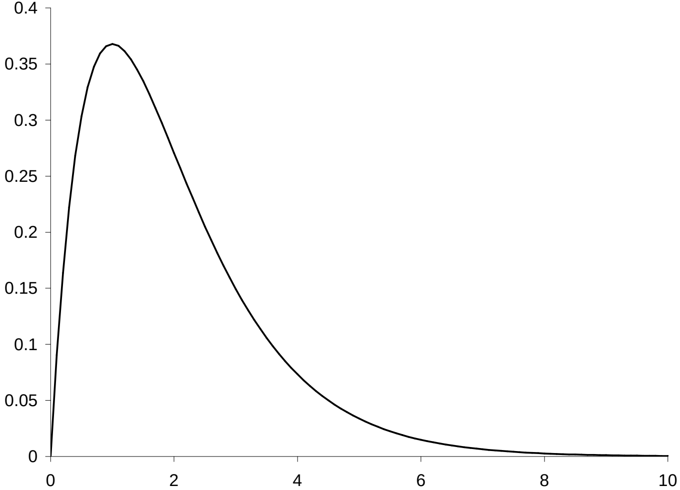
Risky Bond Valuation

1. Find the risk-free yield for the maturity of each cashflow in the risky bond;
2. Add a constant spread, p, to each of these yields;
3. Use this new yield to calculate the present value of each cashflow;
4. Sum all the present values.

Implied PD for 4 Latin American Brady Bonds



Implied Time Dependent Intensity



Stochastic Default Intensity

Now consider a model in which the default intensity is itself random:

$$\boxed{dp = \gamma(r, p, t)dt + \delta(r, p, t)dX_1,}$$

with the interest rate still given by

$$dr = u(r, t)dt + w(r, t)dX_2,$$

where

$$dX_1\, dX_2 = \rho\, dt.$$

Hedging Default Risk

- In the previous model we used riskless bonds to hedge the random movements in the spot interest rate.
- Can we introduce another risky bond or bonds into the portfolio to help with the hedging of the default risk?
- To do this we must assume that default in one bond automatically triggers default in the other bond issued by the same counterparty.

Hedged portfolio

To value our risky zero-coupon bond we construct a portfolio with one of the risky bond, with value $V(r,p,t)$, and delta hedged by shorting Δ unit of a riskless bond, with value $Z(r,t)$, and Δ_1 shorting another risky bond issued by the same company with different maturity, with value $V_1(r,p,t)$:

$$\boxed{\Pi = V(r,p,t) - \Delta Z(r,t) - \Delta_1 V_1(r,p,t).}$$

$$V(r,t;p) \to V(r,p,t)$$

$$dV = \left(\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + \frac{1}{2}\delta^2\frac{\partial^2 V}{\partial p^2} + \rho w \delta \frac{\partial^2 V}{\partial r \partial p}\right)dt + \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial r}dp$$

$$= \mathcal{L}'(V)\,dt + \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial p}dp$$

Case A: Without Default I

Suppose that the bond does not default, the change in the value of the portfolio during an infinitesimal time step is

$$d\Pi = dV - \Delta\, dZ - \Delta_1 dV_1.$$

By using Itô's lemma, above can be written as

$$\begin{aligned} d\Pi &= \left(\mathcal{L}'(V) - \Delta\mathcal{L}(Z) - \Delta_1\mathcal{L}'(V_1)\right) dt \\ &+ \left(\frac{\partial V}{\partial r} - \Delta\frac{\partial Z}{\partial r} - \Delta_1\frac{\partial V_1}{\partial r} r\right) dr \\ &+ \left(\frac{\partial V}{\partial p} - \Delta_1\frac{\partial V_1}{\partial p}\right) dp \end{aligned}$$

Case A: Without Default II

where

$$\mathcal{L}'(V) = \frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + \rho w \delta \frac{\partial^2 V}{\partial r \partial p} + \tfrac{1}{2}\delta^2\frac{\partial^2 V}{\partial p^2}$$

$$\mathcal{L}(Z) = \frac{\partial Z}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2}$$

and ρ is the correlation between dX_1 and dX_2.

Case A: Without Default III

Choose Δ to eliminate the risky terms.

$$\Delta_1 = \frac{\partial V}{\partial p} \Big/ \frac{\partial V_1}{\partial p}$$

and

$$\Delta = \frac{\frac{\partial V}{\partial r} - \Delta_1 \frac{\partial V_1}{\partial r}}{\frac{\partial Z}{\partial r}}$$

Case B: With Default

If the bond defaults then the change in the value of the portfolio is

$$d\Pi = -V + \Delta_1 V_1 + O(dt^{1/2}).$$

Taking expectations and using the bond-pricing equation for the riskless bond, we find that the value of the risky bond satisfies

$\mathbb{E} \to$ $\mathcal{L}'(V)$

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + \rho w \delta \frac{\partial^2 V}{\partial r \partial p} + \frac{1}{2}\delta^2 \frac{\partial^2 V}{\partial p^2} +$$

$$(u - \lambda w)\frac{\partial V}{\partial r} + (\gamma - \lambda' \delta)\frac{\partial V}{\partial p} - (r + p)V = 0.$$

Feynman Kac: 2nd call

Similar to the interest rate risk, λ' is called the market price of default risk. So the fundamental pricing formula for the risky bond under risk neutral measure is

$$\boxed{V(t, T) = \mathbb{E}\left(e^{-\int_t^T (r_s + p_s) ds} | \mathcal{F}_t\right).}$$

Recovery Rate Historic Analysis

In default there is usually *some* payment, not all of the money is lost. In the table are shown the mean and standard deviations for recovery according to the seniority of the debt. This emphasizes the fact that the rate of recovery is itself very uncertain.

Class	**Mean (%)**	**Std Dev. (%)**
Senior secured	53.80	26.86
Senior unsecured	51.13	25.45
Senior subordinated	38.52	23.81
Subordinated	32.74	20.18
Junior subordinated	17.09	10.90

There is also a statistical relationship between rate of recovery and default rates. (Years with low default rates have higher recovery when there is default.)

Recovery of Market Value

Suppose that on default we know that we will loss l percent of pre-default value. This will change the partial differential equation.

$\theta = 1 - l \leftarrow$ LGD
φ
recovery rate

Upon default we have

$$d\Pi = -lV + l\Delta_1 V_1 + O(dt^{1/2});$$

we suffer a loss from the first bond but gain $1 - l$ from the second bond. The pricing equation becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + \rho w\delta\frac{\partial^2 V}{\partial r\partial p} + \frac{1}{2}\delta^2\frac{\partial^2 V}{\partial p^2}$$

$$+(u - \lambda w)\frac{\partial V}{\partial r} + (\gamma - \lambda'\delta)\frac{\partial V}{\partial p} - (r + lp)V = 0.$$

Recovery of Treasury Value

- Although the assumption of recovery on market value is convenient for the purpose of mathematical modeling and makes economic sense since it measures the loss in value associated with default, it is impossible to give immediate expression for implied default probability.
- Recovery on treasury assumes that, if a corporate bond defaults, its value will be replaced by a treasury bond with the same maturity. Under this assumption and with the independence of interest rate and hazard rate, the bond price will be

$$\boxed{V(0,t) = (1 - F(t))\, Z(0,t) + F(t)\theta Z(0,t).}$$

- Implied default probability can be easily extracted from the above relationship.

$$F(t) = \frac{1 - e^{-st}}{1-\theta}$$

PD Calibration with recovery of Treasury Value

Year	Riskfree zero rate	Risky bond zero rate	Cummulative PD	Marginal PD
1	5%	5.25%	0.4161%	0.4161%
2	5%	5.50%	1.6584%	1.2422%
3	5%	5.70%	3.4635%	1.8051%
4	5%	5.85%	5.5714%	2.1079%
5	5%	5.95%	7.7316%	2.1602%
	Recovery	40%		

Feynman Kac: 3rd Call

payoff $g(\cdot)$

$$\boxed{V(t,T) = \mathbb{E}\left(e^{-\int_t^T (r_s + lp_s)ds} \middle| \mathcal{F}_t\right).}$$

It can be rewritten as

$$V(t,T) = \mathbb{E}\left(e^{-\int_t^T R_s ds} \middle| \mathcal{F}_t\right).$$

Where

$$\boxed{R_t = r_t + l\, p_t,}$$

is called the risk adjusted discount rate.

Daniel Duffie and Kenneth J. Singleton (1999)
Modelling of Term Structure of defaultable bond.

Introduction

- In M4, we have seem affine short rate models lead to explicit solutions for bond prices, e.g. Vasicek, CIR and Ho& Lee etc.
- Similarly, affine intensity based models lead to analytical solutions for risk bond prices.

Pricing for a General Contingent Claim

For a general contingent claim $g(X_T)$ at time T, where

$$X_t = (x_{1t}, x_{2t}, \cdots, x_{nt})$$

is a vector of state variable, The fundamental pricing formula is

$$V(t,T) = \mathbb{E}\left(e^{-\int_t^T R(X_s)ds} g(X_T) | \mathcal{F}_t\right).$$

risk adj discnt

Conditions
$$\begin{cases} X: \text{ Affine process} \\ R: \text{ is liner in } X \Rightarrow \text{Affine} \\ g: \text{ is linear in } X \end{cases}$$

Solution of General Affine Model

If the model is affine, i.e.,

- X_t: affine process
- $R(X_s)$: affine in X_t
- $g(X_s)$: affine is X_t

then the general solution is

General Affine Model Solution

$$\boxed{V(t,T) = e^{\alpha(t,T)+\beta(t,T)X_t}.}$$

Affine Conditions

In previous section while we deriving risky bond pricing equation, we have the stochastic interest rate

$$dr = u(r,t)dt + w(r,t)dX_1,$$

and the stochastic intensity

$$dp = \gamma(p,t)dt + \delta(p,t)dX_2.$$

To be affine intensity model

- we must choose the functions $u - \lambda w$, w, $\gamma - \lambda'\delta$, δ and ρ carefully.
- We must choose $u - \lambda w$ and w^2 to be linear in state variables, same for $\gamma - \lambda'\delta$ and δ^2.
- The form of the correlation coefficient is assumed to be constant, i.e., $dX_1 dX_2 = \rho\, dt$.

Solution for Risky ZCB

Suppose recovery rate is a constant $\theta = 1 - l$, and $s_t = (1-\theta)p_t$.
With appropriate choices of the functions in the two stochastic differential equations we find that the solution with final condition $V(r, s, T) = 1$ is

$$V = \exp\{A(t, T) - B(t, T)r - C(t, T)s\}$$

where A, B and C satisfy non-linear first-order ordinary differential equations.

Calibration

- In fixed income modelling, if we allow the spot interest rate model to have some simple time dependence then we have the freedom to fit the initial yield curve.
- Similarly, if there is time dependence in the model for the intensity, and the model is sufficiently tractable, then you can also fit risky bond term structure.

Standard Two-Factor Vasicek Model

Suppose there are two state variables X and Y whose dynamics can be written as

$$
\begin{aligned}
dX &= (a_1 - b_{11}X - b_{12}Y)dt + \sigma_1 dW_1 \\
dY &= (a_2 - b_{21}X - b_{22}Y)dt + \sigma_2 dW_2
\end{aligned}
$$

where

$$dW_1 dW_2 = \rho dt$$

The risk adjusted discount rate is

$$R = g_0 + g_1 X + g_2 Y.$$

Modified Canonical Form

To simplify the parameterizations(standard form doesn't usually have unique solution) we work with demeaned canonical form

$$
\begin{aligned}
dX &= -a\,Xdt + \sigma dW_1 \\
dY &= -b\,Ydt + \eta dW_2
\end{aligned}
$$

and

$$R(t) = \phi(t) + X(t) + Y(t).$$

Note in order to calibrate on the risky bond yield we employ a time dependent parameter $\phi(t)$ into risk adjusted rate. Here we correspond the state variable X to the short rate r and Y to the spread $s = (1-\theta)p$.

Two-factor Vasicek risky BPE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} + \frac{1}{2}\eta^2\frac{\partial^2 V}{\partial y^2} + \sigma\eta\rho\frac{\partial^2 V}{\partial x \partial y} - ax\frac{\partial V}{\partial x} - by\frac{\partial V}{\partial y} - RV = 0$$

$$\frac{\partial v}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 v}{\partial r^2} + \frac{1}{2}\delta\frac{\partial^2 v}{\partial p^2} + \rho\sigma\delta\frac{\partial^2 v}{\partial r\partial p} - \cdots\cdots - (r+p\ell)v = 0$$

$$V = \exp\{A(t,T) - B(t,T)x - C(t,T)Y\}$$

$$\frac{\partial v}{\partial t} = \qquad \frac{\partial v}{\partial x} \qquad \frac{\partial^2 v}{\partial x^2} \qquad \frac{\partial^2 v}{\partial x \partial y}. \qquad \frac{\partial v}{\partial y} \qquad \frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial v}{\partial t} = V(\dot{A} - \dot{B}x - \dot{C}Y)$$

$$\frac{\partial v}{\partial x} = -BV \qquad \frac{\partial^2 v}{\partial x^2} = B^2 V$$

$$\frac{\partial v}{\partial y} = -CV \qquad \frac{\partial^2 v}{\partial y^2} = C^2 V$$

$$\frac{\partial^2 v}{\partial x \partial y} = CBV$$

Risky Bond Solution I

Plug the general affine solution into the BPE, come up with 3 ODEs w.r.t A, B and C respectively, solve them one by one to obtain the price of risky bond.

B

$$V(t,T) = \exp\left\{ -\int_t^T \phi(s)ds - \frac{1-e^{-a(T-t)}}{a}X(t) - \overset{C}{\frac{1-e^{-b(T-t)}}{b}}Y(t) + \frac{1}{2}M(t,T) \right\}$$

$$\begin{cases} \dot{A} + \frac{1}{2}\sigma^2 B^2 + \frac{1}{2}\eta^2 C^2 + \rho\sigma\eta BC - \phi(t) = 0 \\ \dot{B}x - axB + x = 0 \Rightarrow B = \dfrac{1-e^{-a(T-t)}}{a} \\ \dot{C}y - byC + y = 0 \Rightarrow C = \dfrac{1-e^{-b(T-t)}}{a} \end{cases}$$

Risky Bond Solution II

where

$$
\begin{aligned}
M(t,T) &= \frac{\sigma^2}{a^2}\left[T-t+\frac{2}{a}e^{-a(T-t)}-\frac{1}{2a}e^{-2a(T-t)}-\frac{3}{2a}\right] \\
&+ \frac{\eta^2}{b^2}\left[T-t+\frac{2}{b}e^{-b(T-t)}-\frac{1}{2b}e^{-2b(T-t)}-\frac{3}{2b}\right] \\
&+ 2\rho\frac{\sigma\eta}{ab}\left[T-t-\frac{1-e^{-a(T-t)}}{a}-\frac{1-e^{-b(T-t)}}{b}\right. \\
&- \left.\frac{e^{-(a+b)(T-t)-1}}{a+b}\right]
\end{aligned}
$$

Structural Model Vs Reduced Form Model

1. Structural models assume that the modeler has the same information set as the firm's manager-complete knowledge of all the firm's assets and liabilities. In contrast, reduced form models assume that the modeler has the same information set as the market-incomplete knowledge of the firm's condition.
2. Structural models use a firm's asset and debt values to determine the time of default, thus defaults are endogenously generated within the model. In contrast, the time of default in intensity models is determined by the first jump of an jump process whose hazard rate is given by exogenous stochastic process.

Take Away

Please Take Away the following important ideas

- Poisson process assumes constant intensity and it can be used to model default.
- Risky bond is discounted by risk-adjusted interest rate.
- stochastic default intensity increases dimension of bond pricing partial differential equation.
- Risky bond pricing partial differential equationis consistent with fundamental pricing formula through Feynman Kac.
- Reduced-form models are tractable when it satisfies affine structure.

SiyiZhou@hotmail.com