CQF: Certificate in Quantitative Finance

Credit Derivatives and Structural Models

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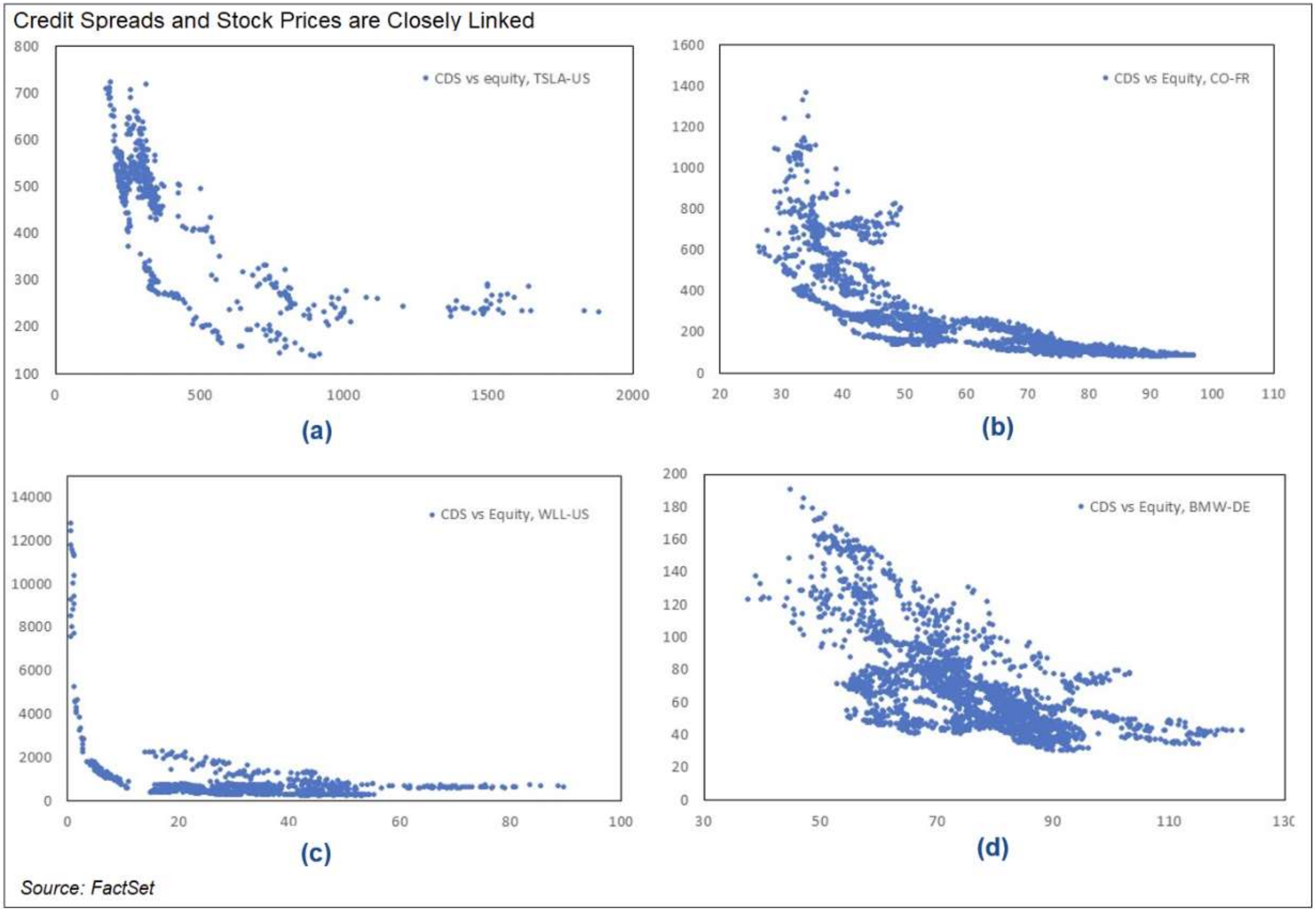
Content

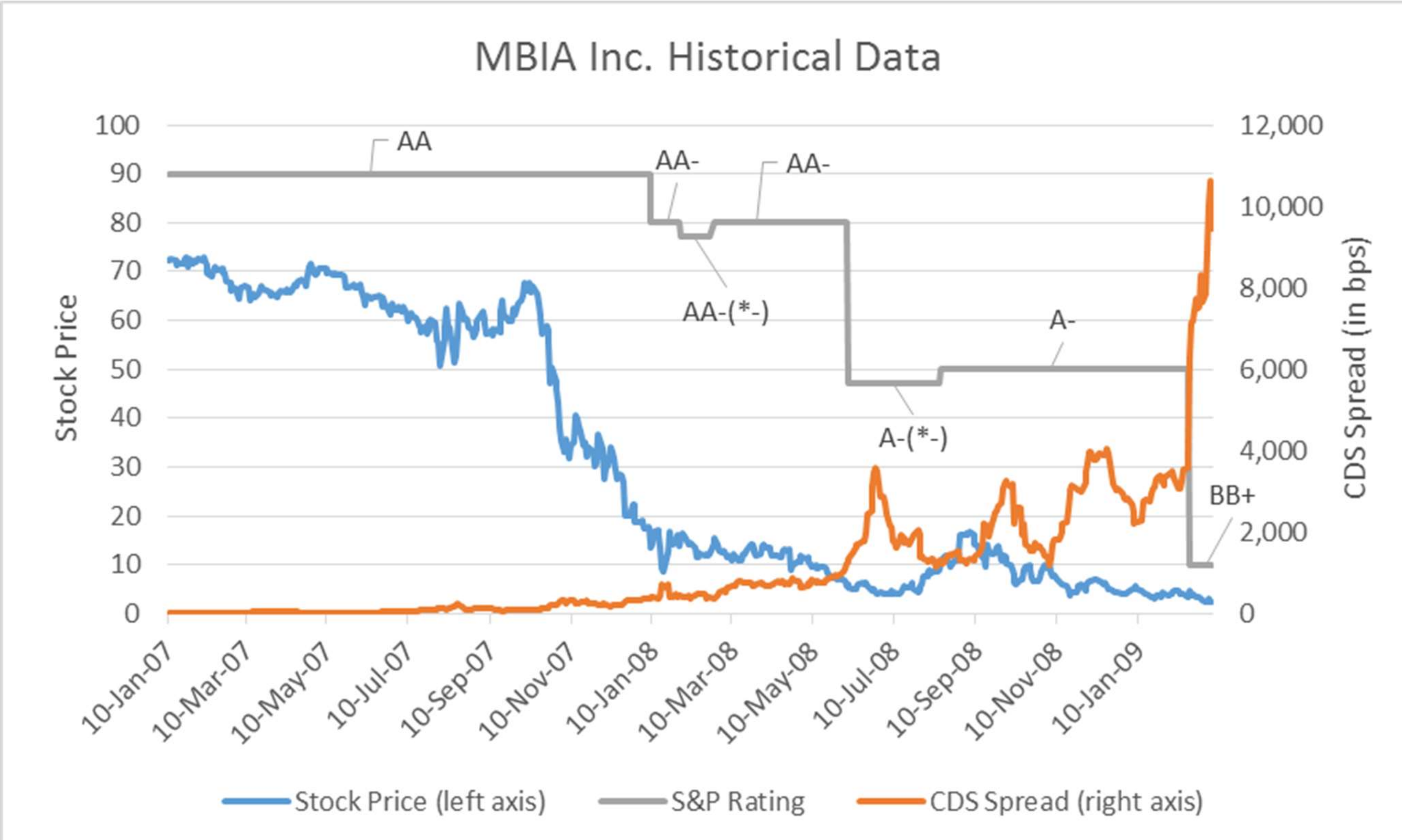
- The Merton Model
- Structural Models and Default Prediction
- Agency Problems
- The First-to-Default problem
- Portfolio Credit Derivatives
- CDOs and the Financial Crisis
- Does Securitisation have any Economic Value?

The Merton Model

Structural Models – Empirical Data and Motivation

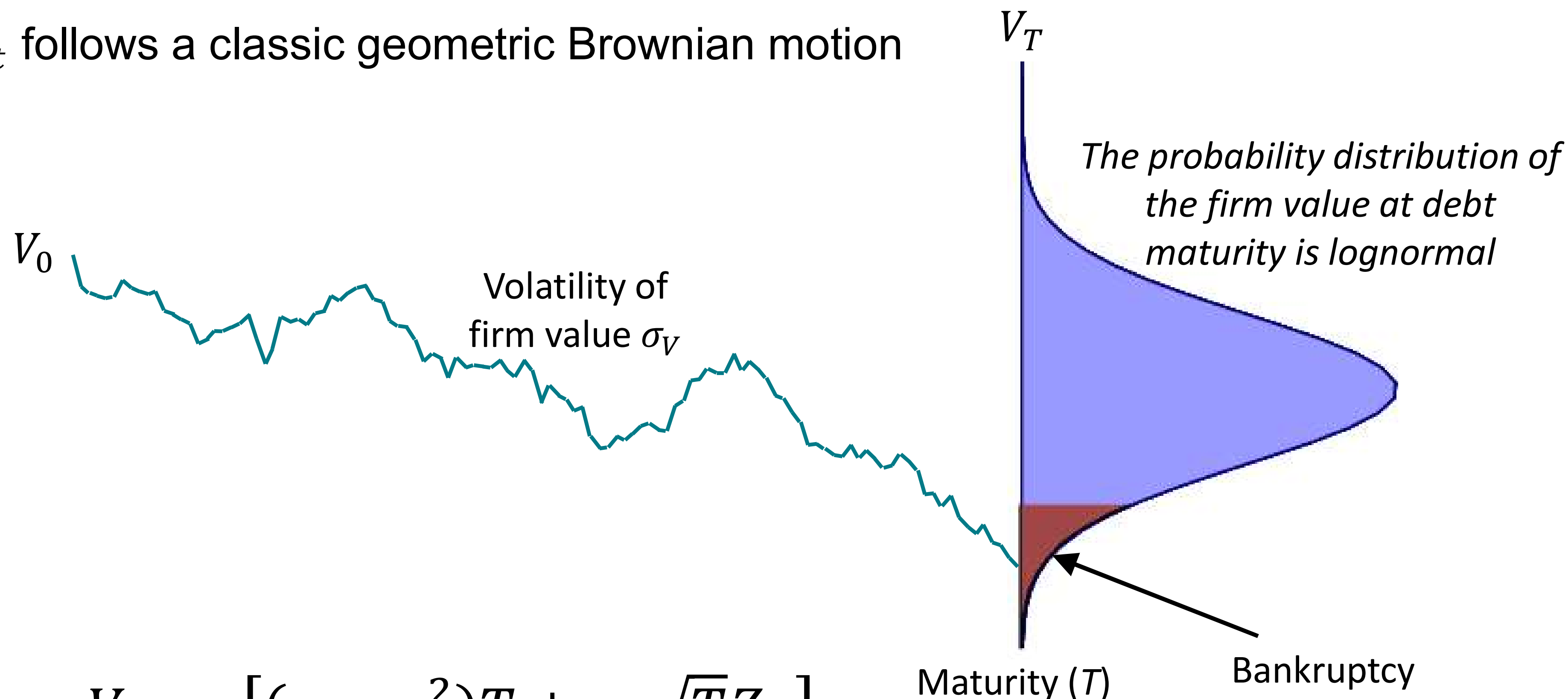


Example – Stock Price, Rating and Credit Spread



The Merton Model

- Value of the firm (V_t) is the sum of the equity (E_t) and debt (B_t)
  - Maturity of (zero coupon) debt is T and outstanding amount is B
- V_t follows a classic geometric Brownian motion



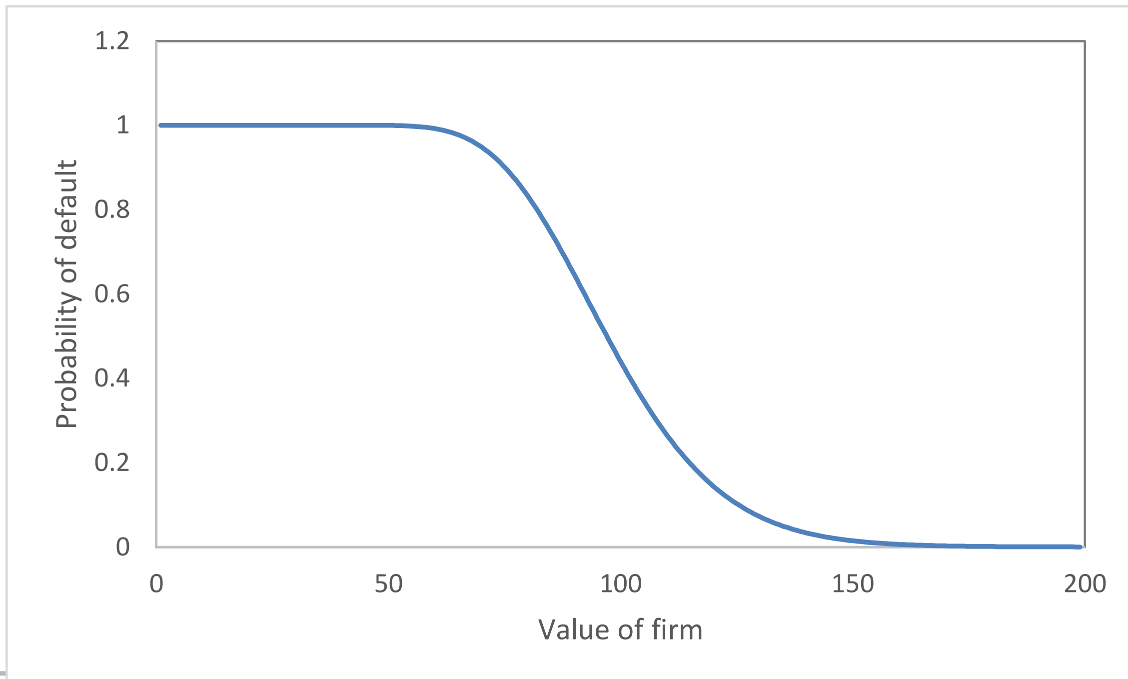
$$V_T = V_0 exp\left[(r - \sigma_V^2)T + \sigma_V\sqrt{T}Z_T\right]$$

Default Probability

- Probability of default (PD)
  - The firm defaults at maturity if the value of the assets is below the face value of the debt

$$PD = \Pr(V_T < B) = N(-d_2)$$

$$d_1 = \frac{\ln(V_0/B) + (r + \sigma_V^2/2)T}{\sigma_A\sqrt{T}} = d_2 + \sigma_V\sqrt{T}$$



Value of Equity and Debt

- Value of equity is call option on value of firm
  - Equity is residual value after bondholders have been paid (European call option)

$$E_T = \max(V_T - B, 0)$$

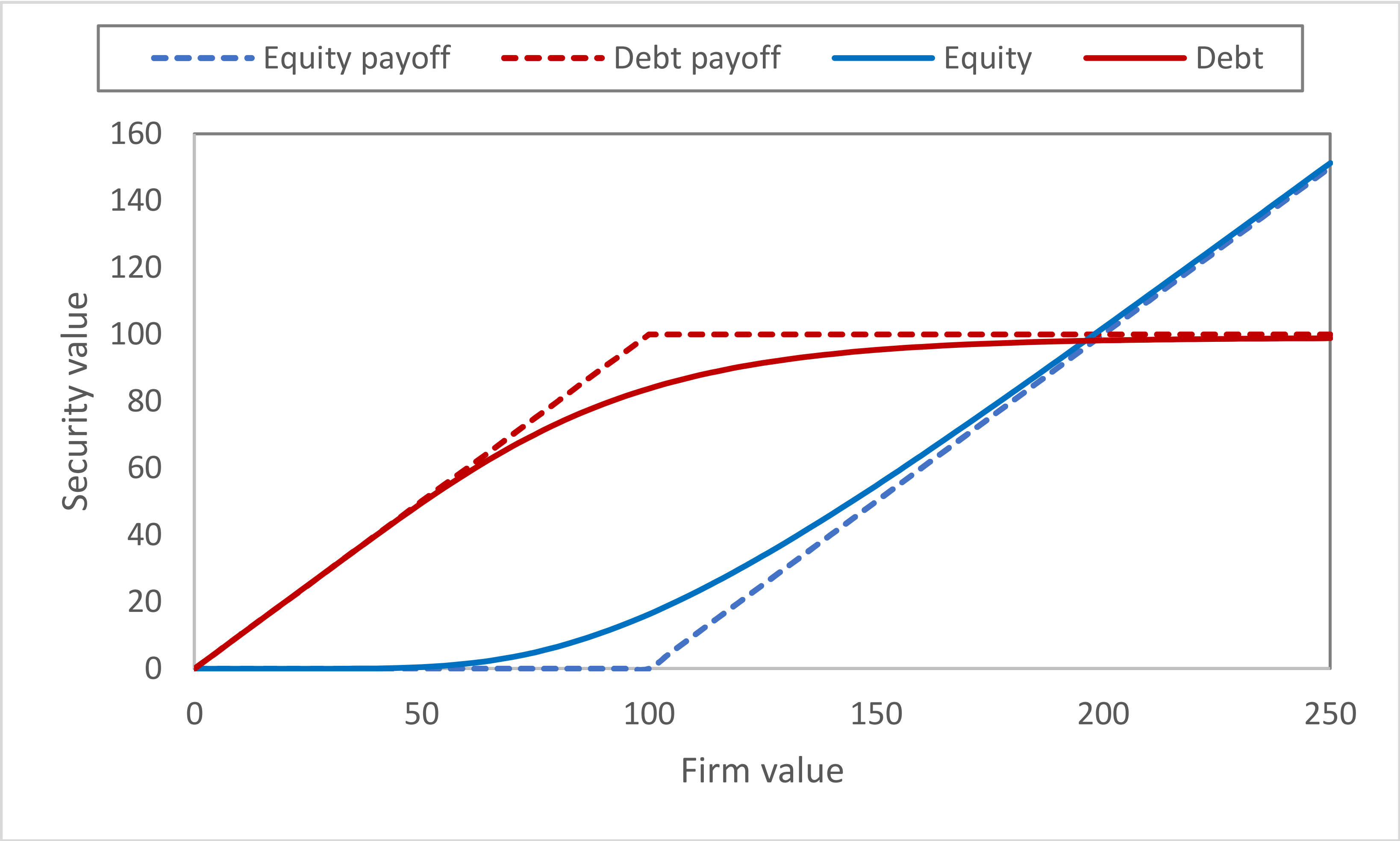
$$E_0 = V_0\Phi(d_1) - e^{-rT}B\Phi(d_2)$$

- Value of debt is face value minus short put option on value of firm
  - Bondholders have first claim on assets of company

$$B_T = \min(V_T, B)$$
$$= B + \min(V_T - B, 0)$$

$$D_0 = V_0\Phi(-d_1) + e^{-rT}B\Phi(d_2)$$

Payoffs and Values



Example

$$B = 100$$

$$r = 5\%$$

$$T = 1 \text{ year}$$

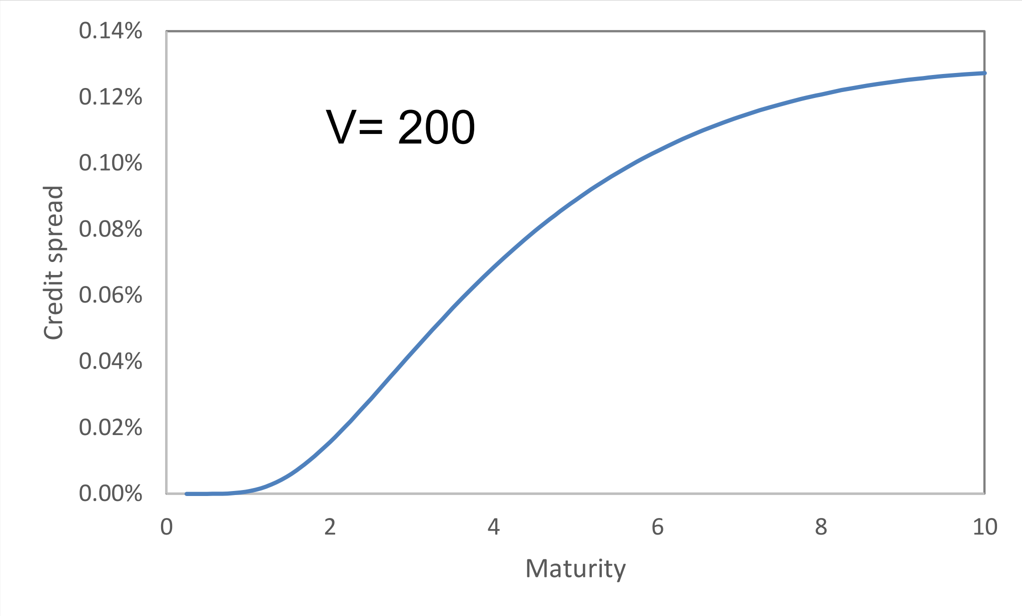
$$\sigma_V = 20\%$$

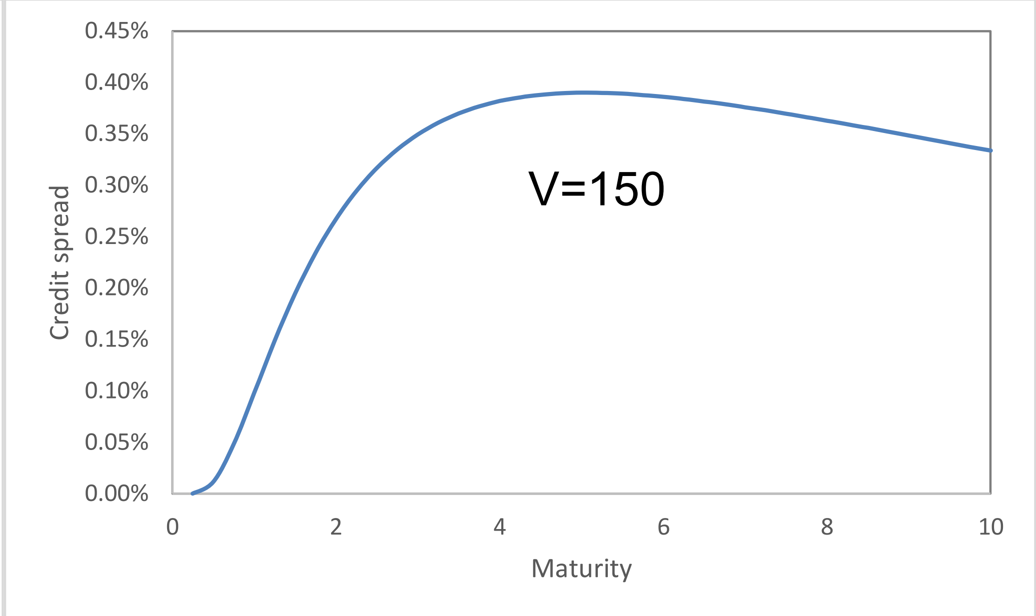
$$V_0 = 100 \quad \Rightarrow B_0 = 89.55, \quad E_0 = 10.45 \quad PD = 44.0\%$$

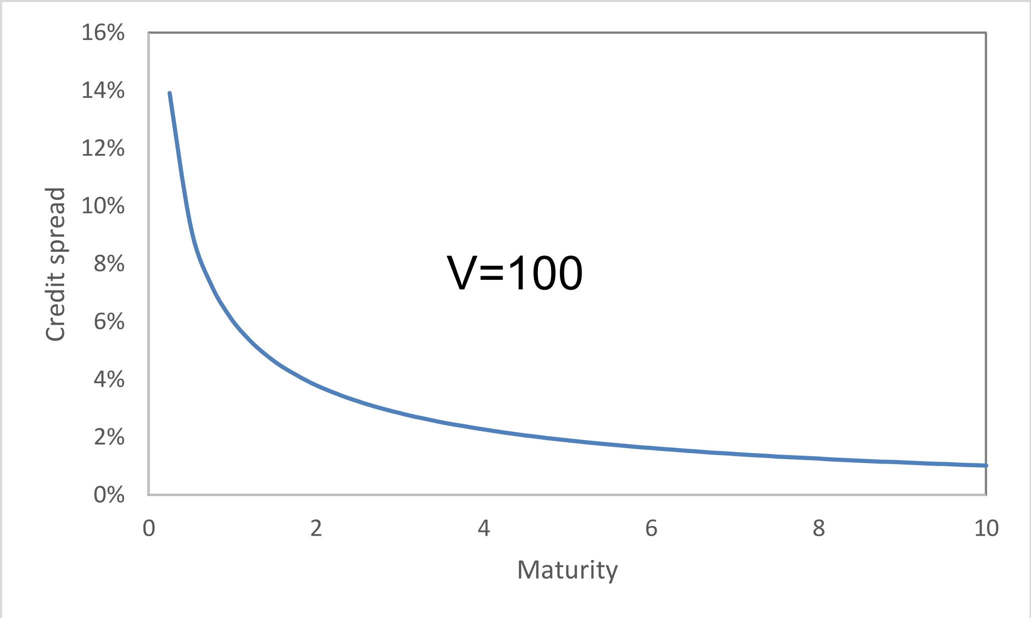
$$V_0 = 150 \quad \Rightarrow B_0 = 95.03, \quad E_0 = 54.97 \quad PD = 1.5\%$$

$$V_0 = 200 \quad \Rightarrow B_0 = 95.12, \quad E_0 = 104.88 \quad PD = 0.015\%$$

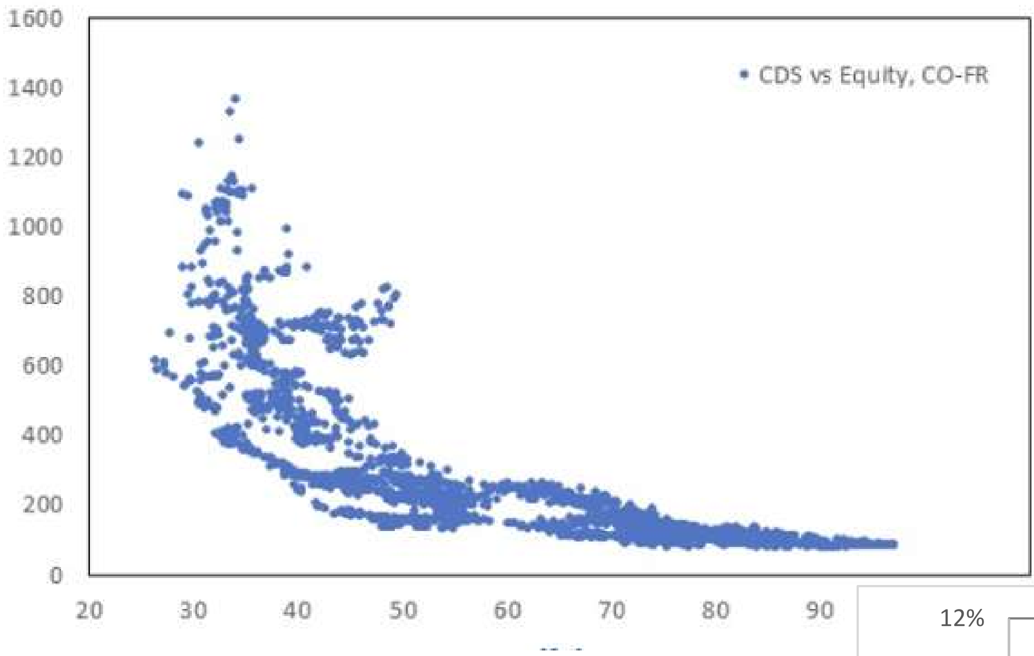
Merton Model Example – Credit Spread Curves

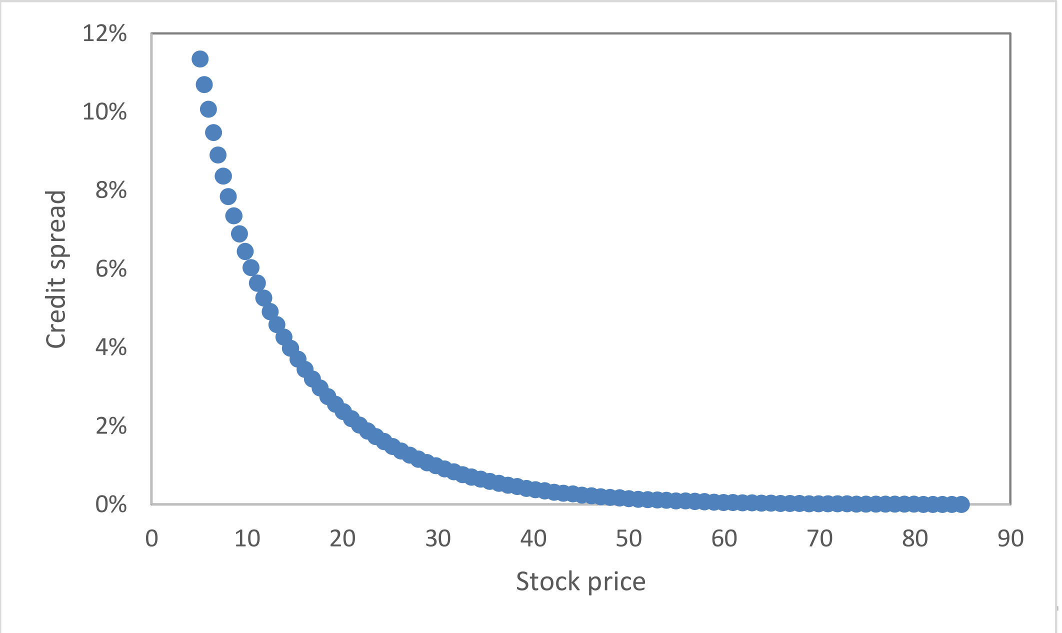






Actual Credit Spread Behaviour vs. Model





Extensions to the Merton Model

- Black and Cox (1976)
  - Default occurs at the first time that the firm's asset value drops below a certain time dependent barrier
- Leland and Toft (1994, 1996)
  - Bankruptcy costs and tax on coupons to consider optimal capital structure
- KMV (1990s)
  - Default prediction with Merton model
  - Acquired by Moody's in 2002
- Capital structure arbitrage (1990s onwards)
  - Proprietary trading desks trading different securities of a company
  - Bonds, CDS, equity and equity puts

Structural Models and Default Prediction

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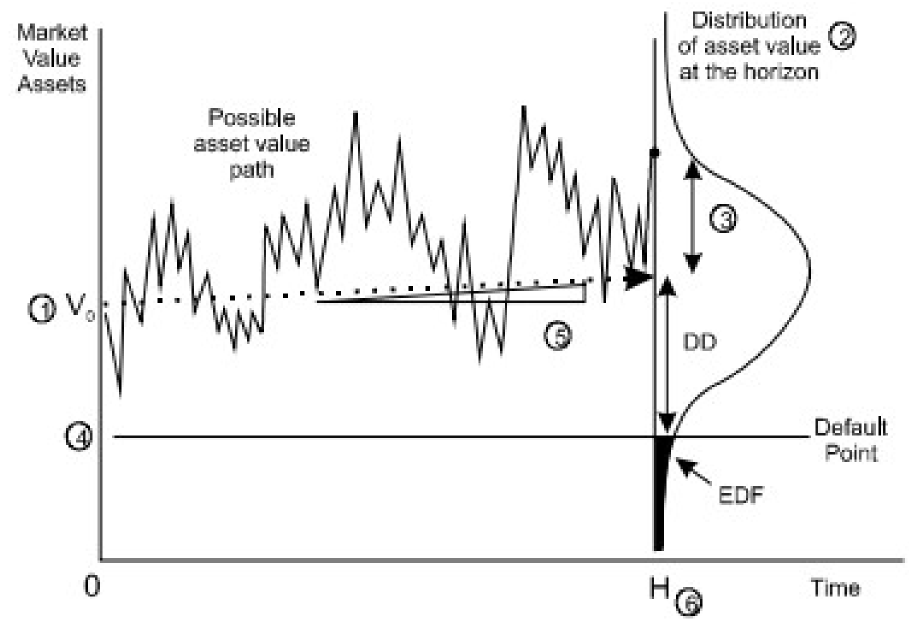
Structural Models in Practice: Moody's KMV

Moody's | K·M·V

MOODY'S KMV RISKCALC™ v3.1 MODEL

NEXT-GENERATION TECHNOLOGY FOR PREDICTING PRIVATE FIRM CREDIT RISK



3.2 Calculate the Distance-to-default

There are six variables that determine the default probability of a firm over some horizon, from now until time H (see Figure 8):

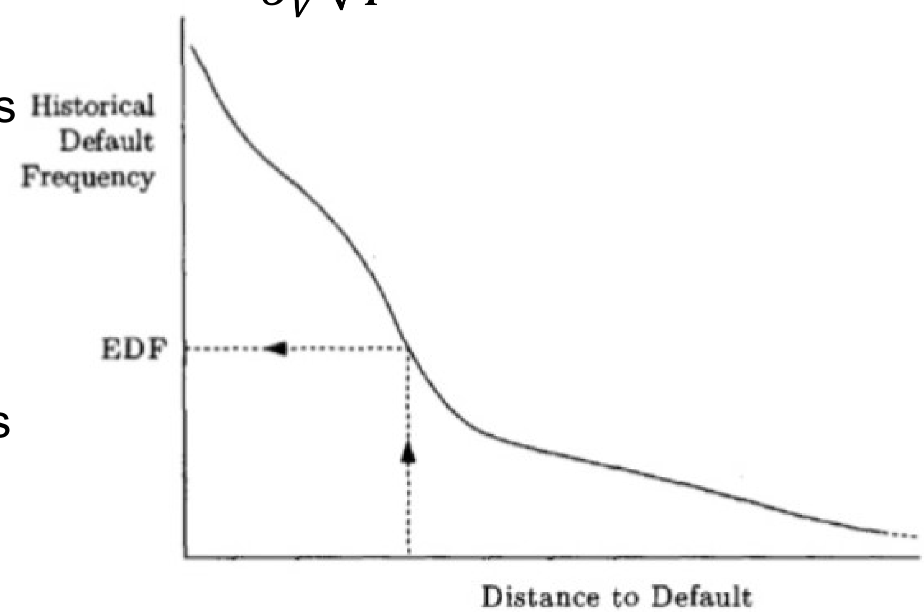
1. The current asset value.
2. The distribution of the asset value at time H.
3. The volatility of the future assets value at time H.
4. The level of the default point, the book value of the liabilities.
5. The expected rate of growth in the asset value over the horizon.
6. The length of the horizon, H.

Moody’s KMV Approach

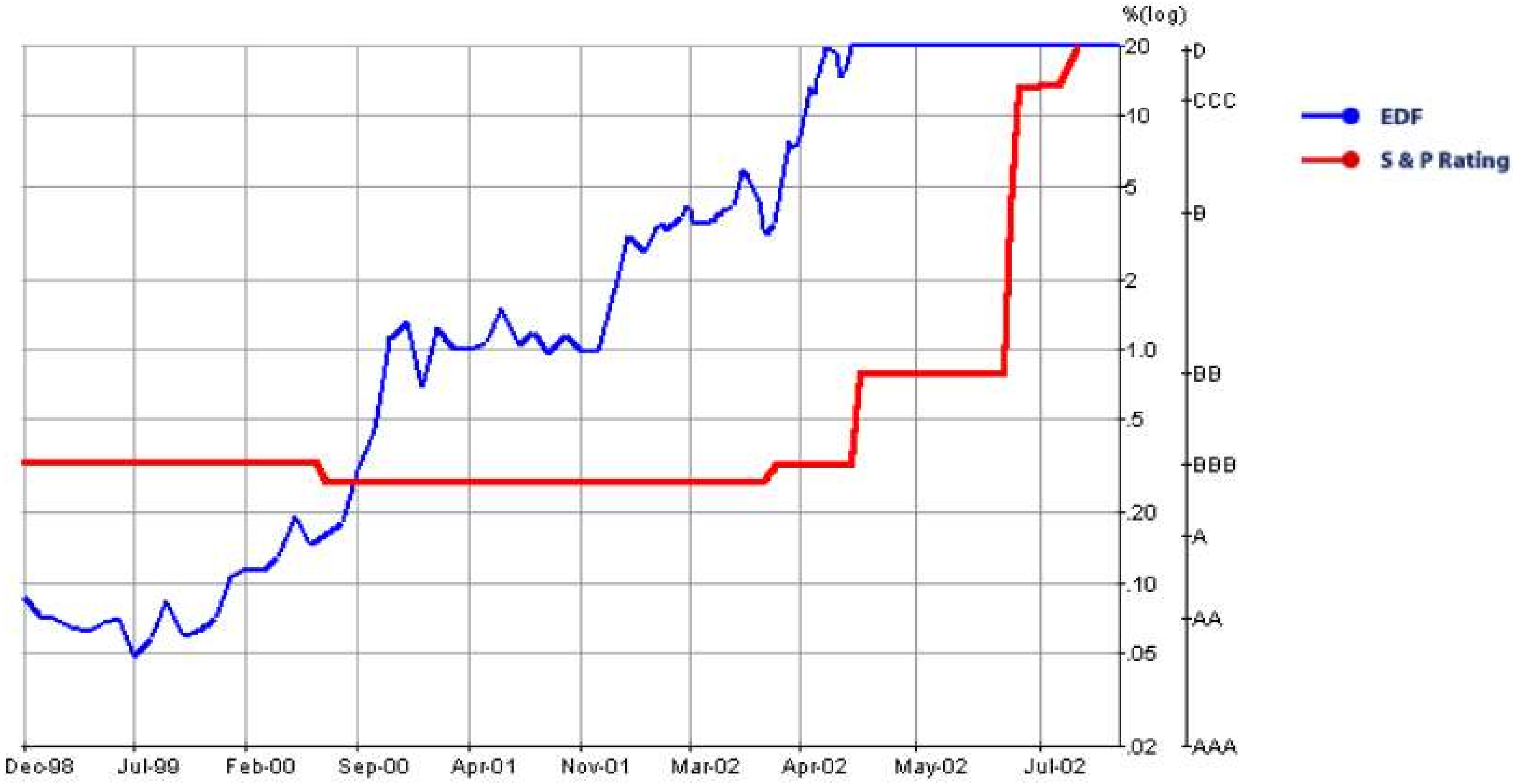
- Some obvious problems in the Merton model
  - Cannot observe firm value (e.g., asset volatility) directly (see Appendix)
  - Default can only occur at maturity of (zero coupon) debt
  - Firm value may not be Gaussian (this is replaced by the empirically estimated distribution)
  - Firms do not default when their value falls just below the face value of their debt (the default point is therefore shifted)

$$DD = \frac{\ln(V_t/B^*) + (r + \sigma_V^2/2)T}{\sigma_V\sqrt{T}}$$

- Distance-to-default
  - B^* is approximated as the short-term debt plus half the long-term debt
  - Gaussian probability of default would be $N(-DD)$
  - The DD is actually compared to previous firms and used to determine the expected default frequency (EDF)

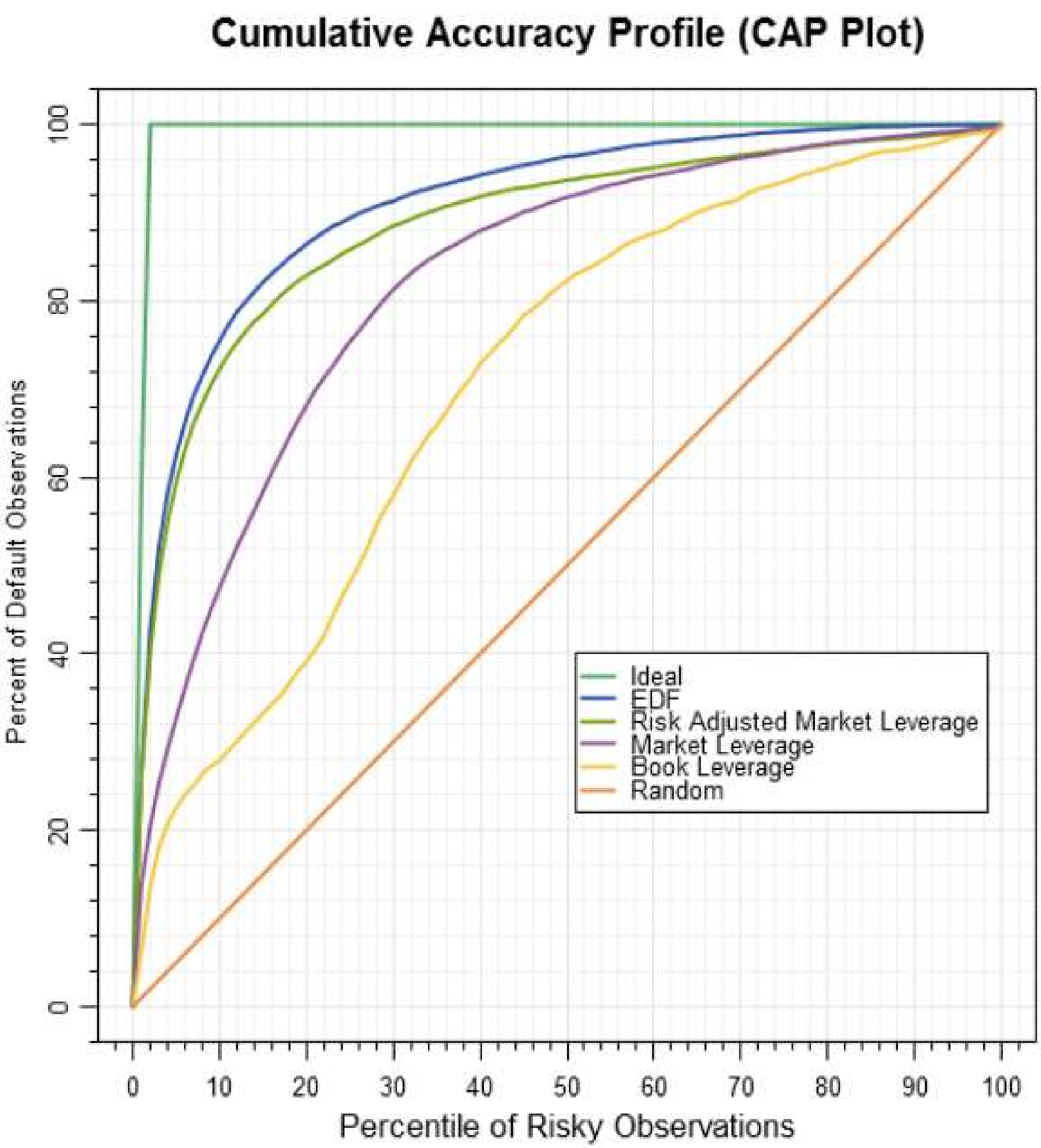


EDF and Credit Ratings



Source: Moody's KMV Case Study: Worldcom

EDF Performance



Source: Moody's KMV

Agency Problems

Firm Value Between Equity and Debt

- Stockholders and bondholders have conflicting interests
- For example
  - Risk shifting
  - Dividends
  - Underinvestment
  - Convertible bonds

Risk Shifting

- The value of a call option is an increasing function of the value of the underlying asset
- By increasing the risk, stockholders benefit at the expense of bondholders
- Example (V = 140, B = 100 T = 2 years, r = 5%)

Volatility	Equity	Debt	Total
30%	53.23	86.77	140
40%	57.38	82.62	140
50%	62.13	77.87	140

Dividends

- Suppose that the shareholders decide to pay themselves a dividend

- Example: B = 100, T = 5 years, r = 5%, σ = 30%

Value of firm	Equity	Debt
100	35.96	64.04

- Dividend = 10

	Value of firm	Equity	Debt
	90	28.60	61.40
Change	-10	-7.36	-2.64
Dividend	-	10	-
Total	100	2.64	-2.64

Underinvestment

- A leveraged company might decide not to undertake projects with positive NPV if financed with equity
- Example: B = 100, T = 5 years, r = 5%, σ = 30%

Value of firm	Equity	Debt
100	35.96	64.04

- Potential project: Required investment 8 & NPV = 2

	Value of firm	Equity	Debt
	110	43.78	66.22
Change	+10	7.84	2.18

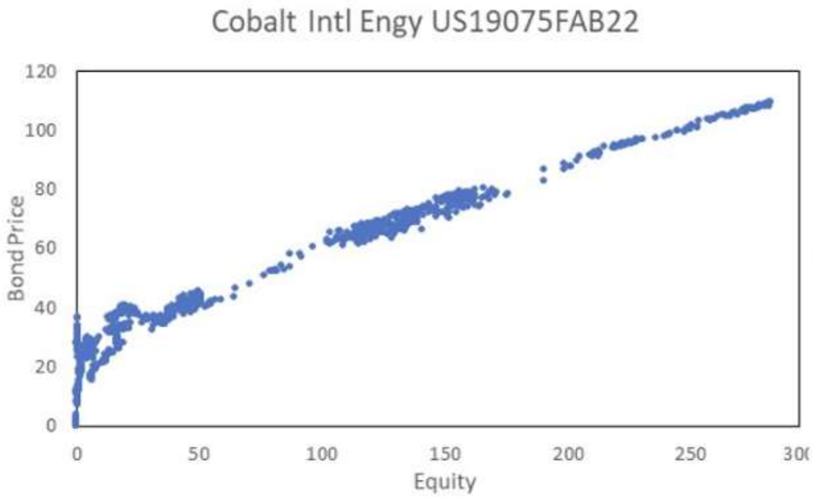
- Shareholders lose if project is financed only by equity
- This can be seen as being related to banks having capital hurdles (KVA)

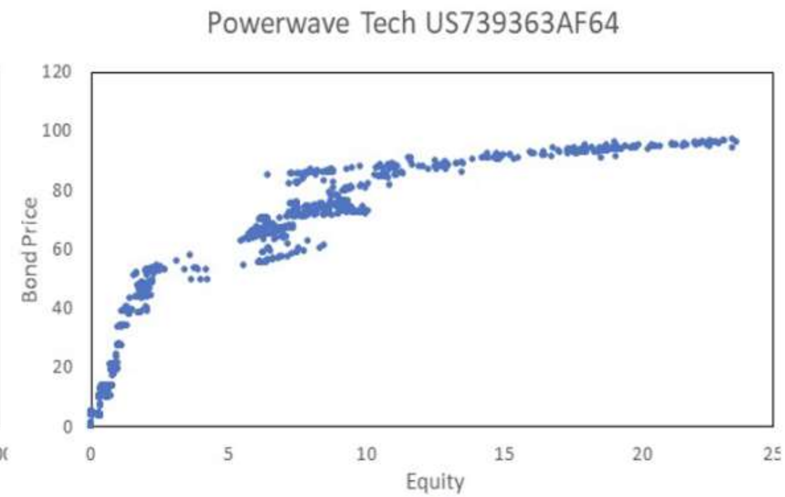
Convertible Bonds

- A convertible bond allows the bondholder to convert into equity at a prespecified price
- This conversion causes a dilution which is at the detriment to the existing shareholders
- However, if it is priced correctly then the shareholders will be compensated for this potential future dilution
  - Suppose firm would issue normal debt at a coupon of 4%
  - Firm issues convertible debt with coupon of 1%
  - Shareholders benefit from cheaper financing in short-term and give up some upside from potential dilution later

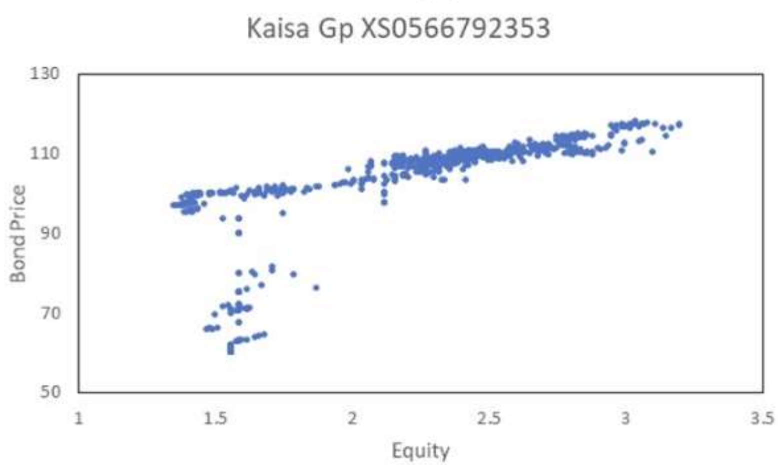
Convertible Bonds – Price Behaviour (I)

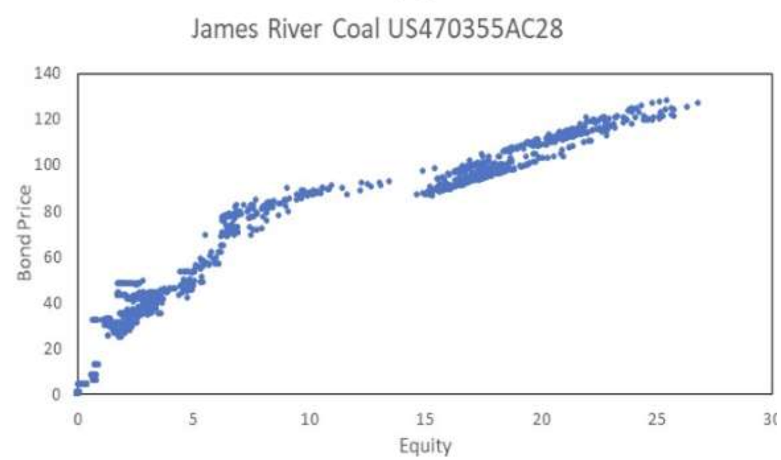
(a) (b)





(c) (d)





(e) (f)

Source: FactSet

Convertible Bonds – Price Behaviour (II)

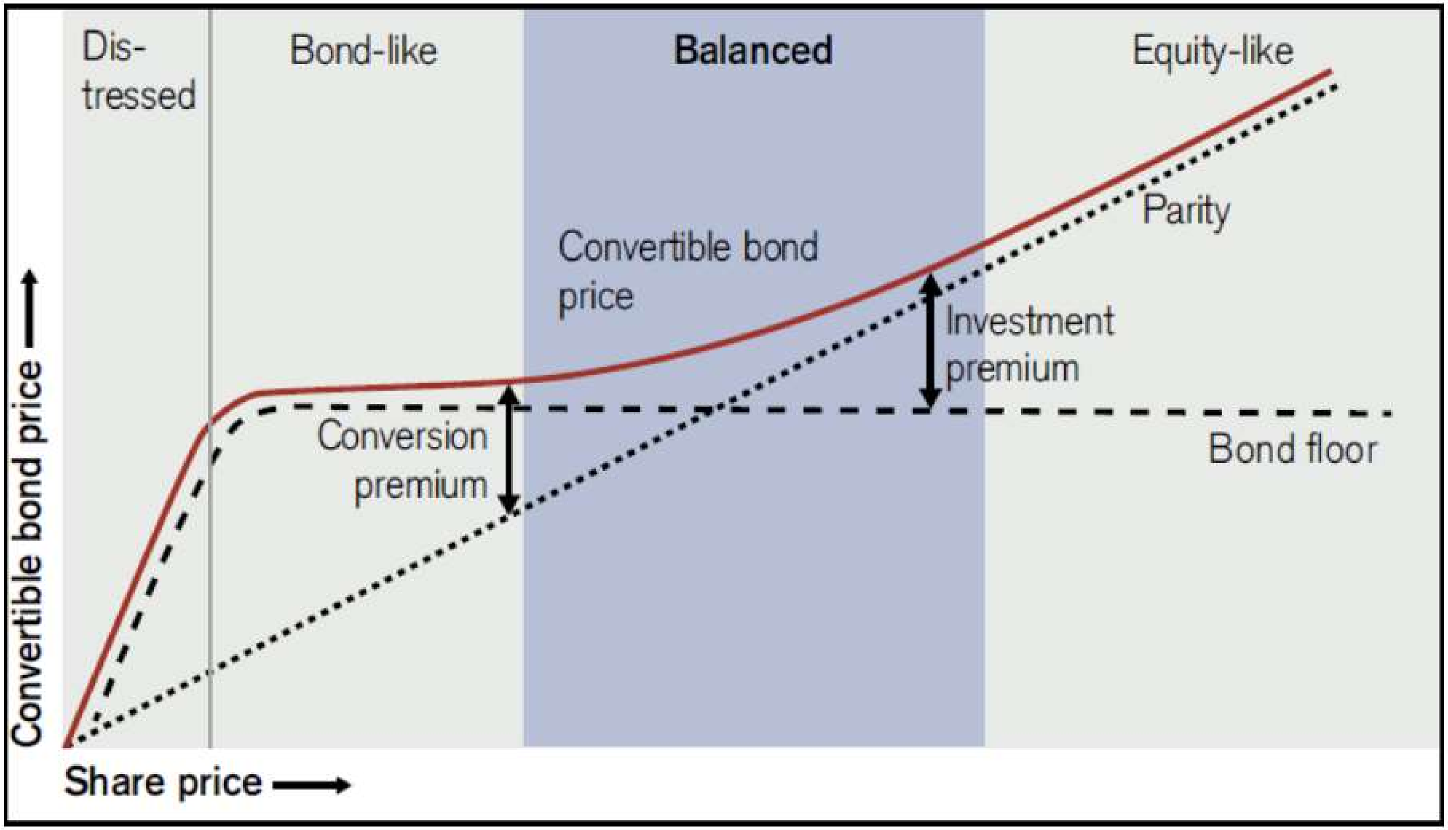
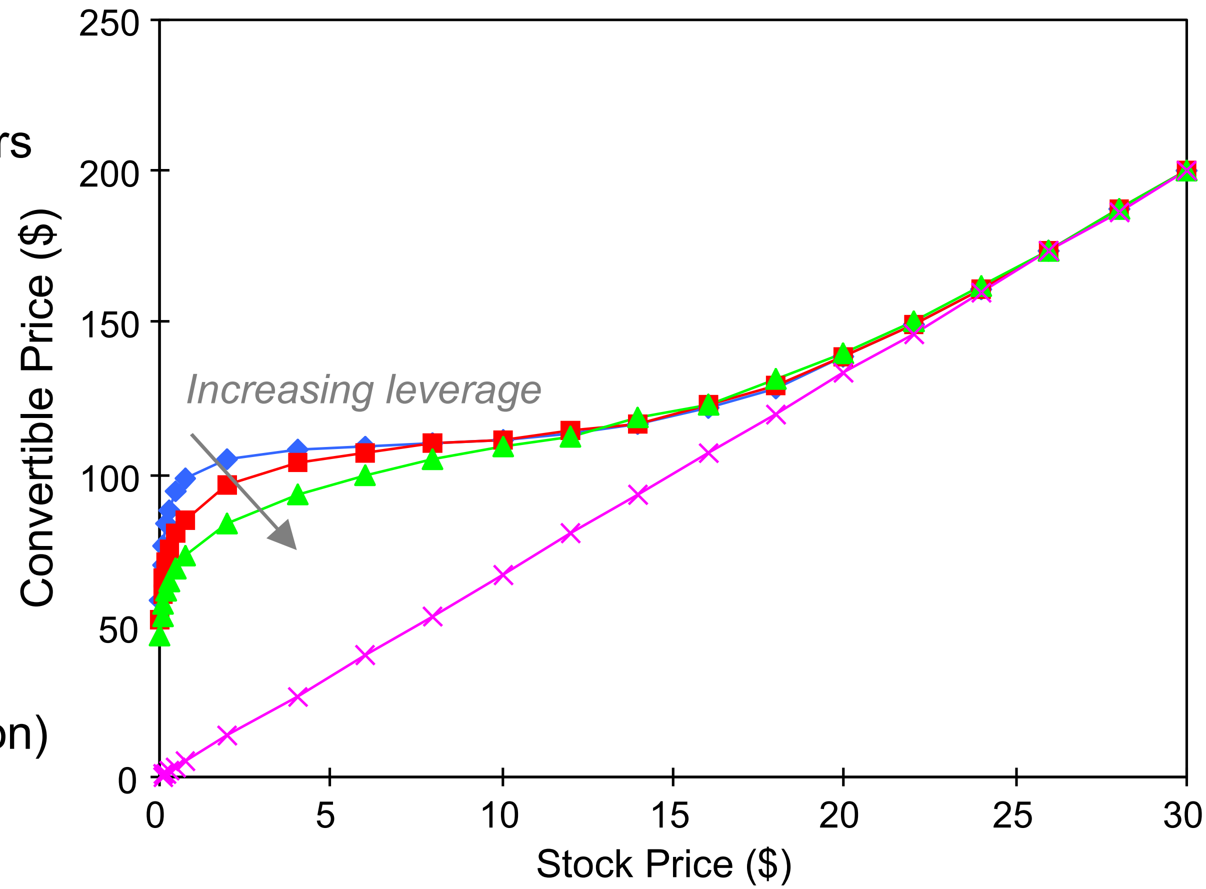


Figure 1: Convertible bond price, parity and bond floor - Source: Credit Suisse (2014)

Convertible Bonds and Structural Models

- Convertible bond valuation is a complex interplay between equity and debt and optionality on the capital structure of an issuer
  - Structural models are a potential choice
- As firm value changes:
  - High – convertible bond holders convert into equity
  - Low – company defaults and bondholders receive some recovery value
  - Medium – convertible bond holders keep NPV of their cashflows (coupons/redemption)



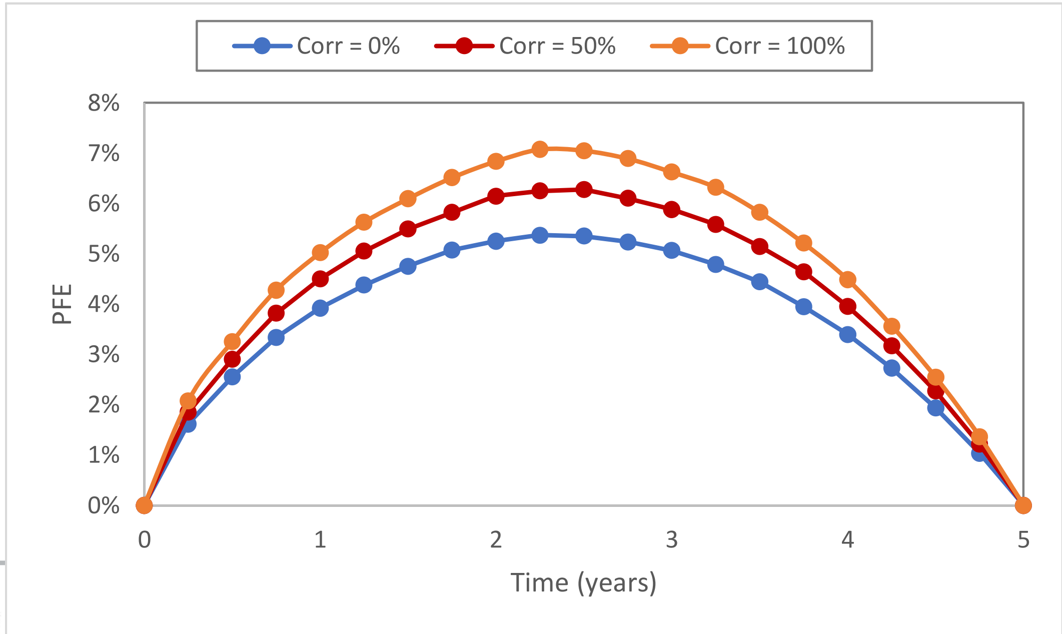
Source: SolvencyAnalytics.com

The First to Default Problem

CDS Counterparty Risk

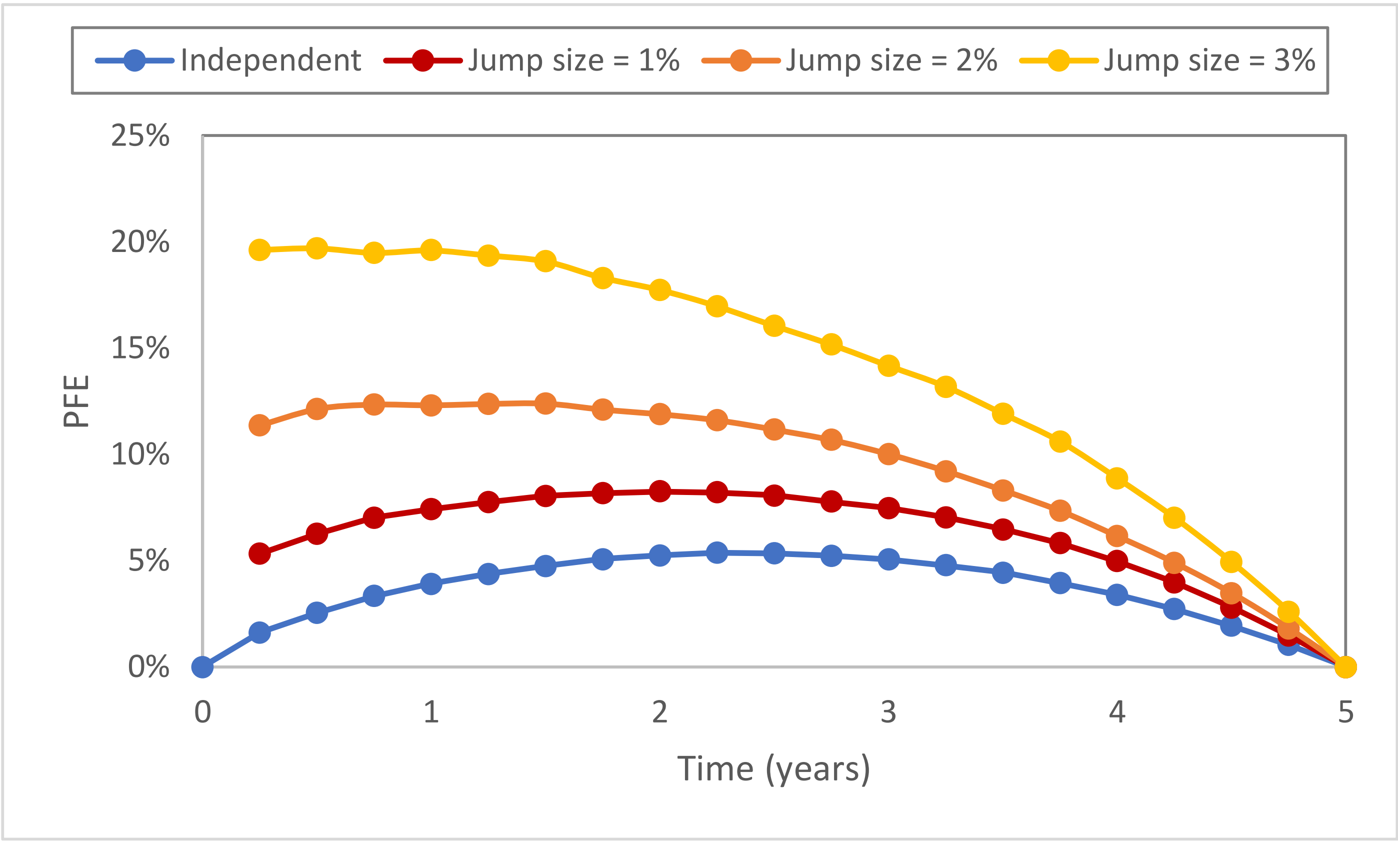
- Suppose we buy protection on a Sovereign from a bank
  - What is our exposure to the bank? Technically, the maximum exposure is 100% which occurs if both the sovereign and bank default and there is no recovery
  - The key component is the correlation between the default events
  - Model counterparty and reference entity by CIR processes with dW_t^c and dW_t^r correlated
  - Calculate potential future exposure (PFE) as a quantile of exposure distribution (97.5%)

$$d\lambda_t^c = k(\theta - \lambda_t^c)dt + \sigma\sqrt{\lambda_t^c}dW_t^c \qquad d\lambda_t^r = k(\theta - \lambda_t^r)dt + \sigma\sqrt{\lambda_t^r}dW_t^r$$



Jump Diffusion Model - Examples

- Can include both intensity correlation and jumps
  - Examples below all have correlation of 50% and various (common) jump sizes
  - More reasonable behaviour but still some way off the maximum of 100%!
  - Simultaneous defaults are still unlikely



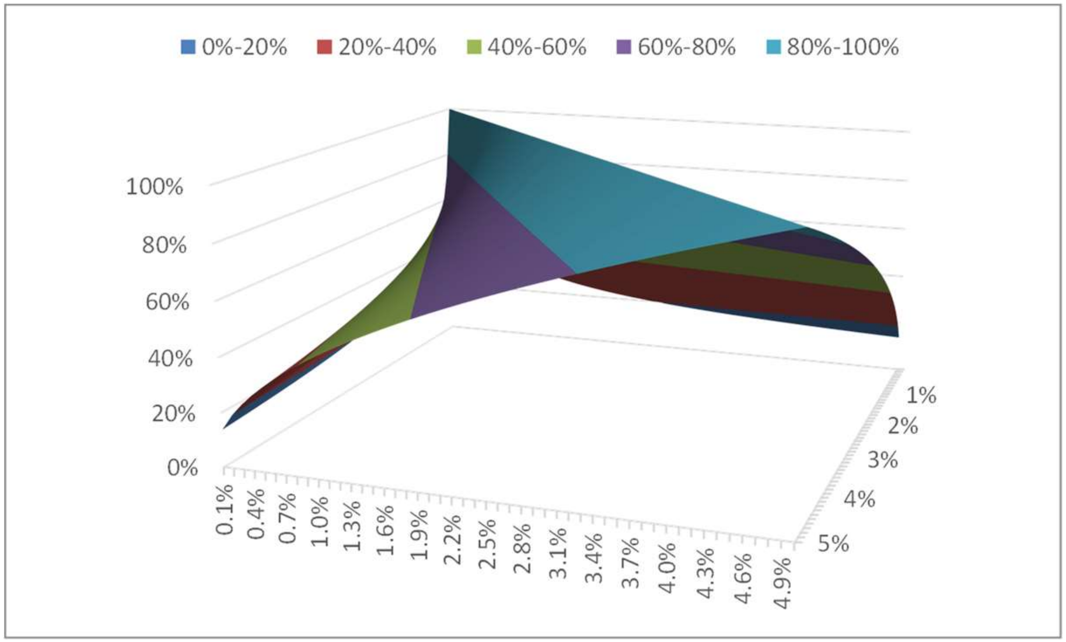
Default Correlation

- Modelling the joint probability of two events, knowing the individual probabilities
- Joint default probability p_{12}
- Clearly need to model default correlation (counterparty and reference entity)
- Default correlation definition

$$\rho_{12} = \frac{p_{12} - p_1 p_2}{\sqrt{p_1(1-p_1)p_2(1-p_2)}}$$

- Since $p_{12} \leq p_1$

$$\rho_{12} \leq \sqrt{\frac{p_1(1-p_2)}{p_2(1-p_1)}}$$



Gaussian Approach to Default Modelling

- Modelling two firms via Merton model
  - Joint default driven by joint returns
  - Which are each Gaussian
  - Correlation between asset returns will drive joint default (equity correlation?)

- Default of a single name is determined by Gaussian variable V_1

$$V_1 < \Phi^{-1}(p_1)$$

- Default of second correlated name

$$V_2 = \rho_{12} V_1 + \sqrt{1 - \rho_{12}^2} \tilde{V} < \Phi^{-1}(p_2)$$

- Joint default is given by bivariate gaussian distribution

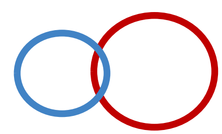
$$\Phi_{2d}(\Phi^{-1}(p_1), \Phi^{-1}(p_2); \rho_{12})$$

Gaussian Copula

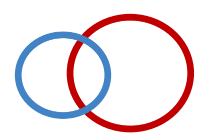
- Two clear advantages
  - Tractability (Gaussian distributions, 100% correlation is meaningful)
  - Calibration via equity correlations (correlation of asset values in Merton model)
- Applications
  - Counterparty risk
  - Structured credit securities with collateral (e.g. AAA CDO collateralised by AAA bond)
  - Portfolio credit products (more names)

$$p_1 = 1\% \qquad\qquad p_2 = 2\%$$

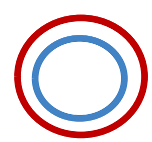
$$\rho_{12} = 0\% \qquad p_{12} = 0.02\%$$



$$\rho_{12} = 50\% \qquad p_{12} = 0.21\%$$

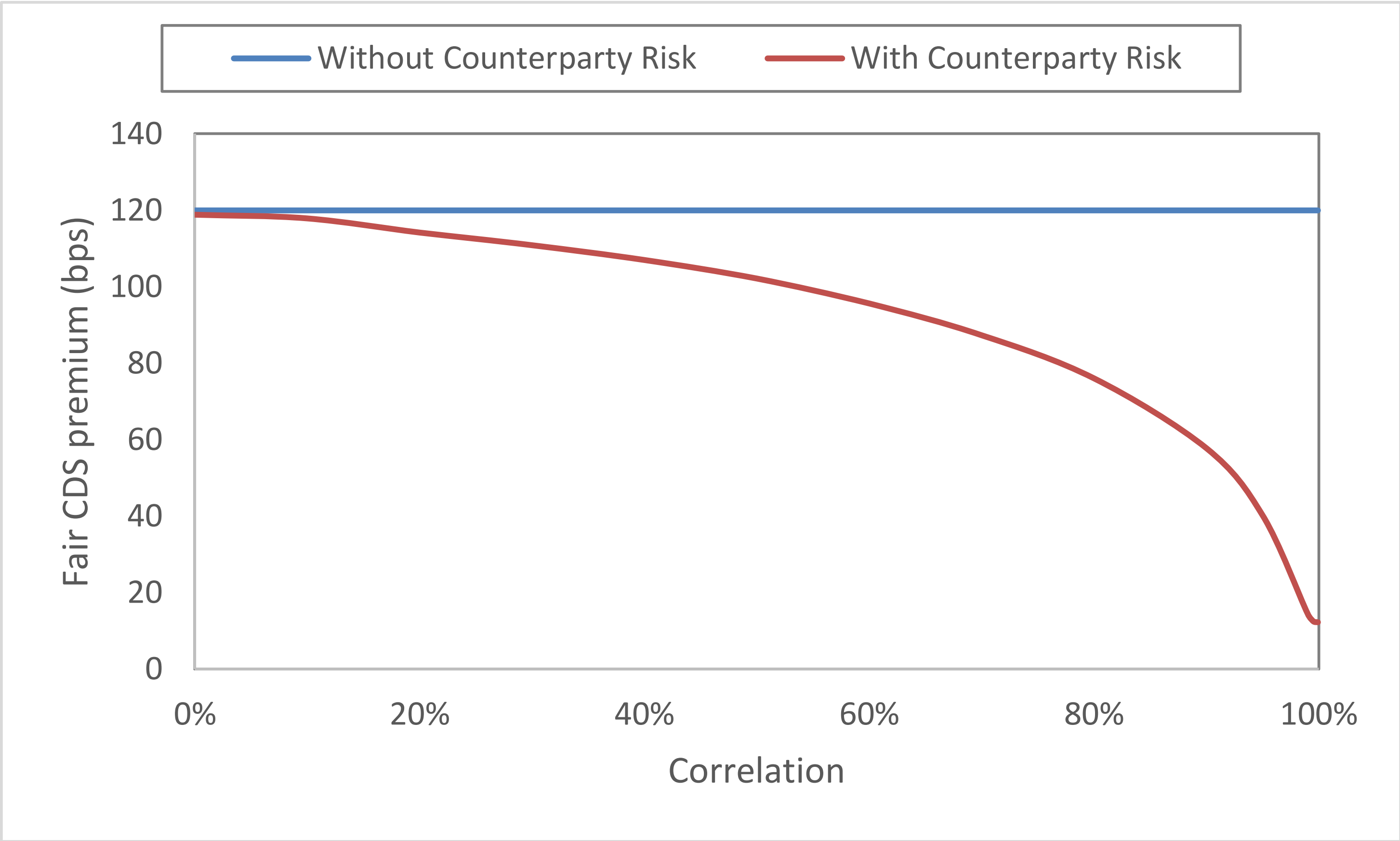


$$\rho_{12} = 100\% \qquad p_{12} = 1\% = p_1$$



CDS Counterparty Risk – Example

- What would you pay to buy CDS on a reference entity with a (risk-free) spread of 1.2% from a counterparty with a spread of 2.4% (recovery rate at 10%)



Gregory J., 2011, “Counterparty risk in credit derivative contracts”, The Oxford Handbook of Credit Derivatives, A. Lipton and A. Rennie (Eds), Oxford University Press.

Portfolio Credit Derivatives

FitchLearning

Merton Model for Portfolio Defaults

- In the Merton model, default of a single name is determined by Gaussian V_1

$$V_1 < \Phi^{-1}(p_1)$$

- Correlation structure (e.g. Gaussian) between variables $V_1, V_2, \dots \dots, V_n$

- A factor model is useful: $V_i = \sqrt{\rho} Z + \sqrt{1-\rho} Y_i$

Idiosyncratic variable

Correlation

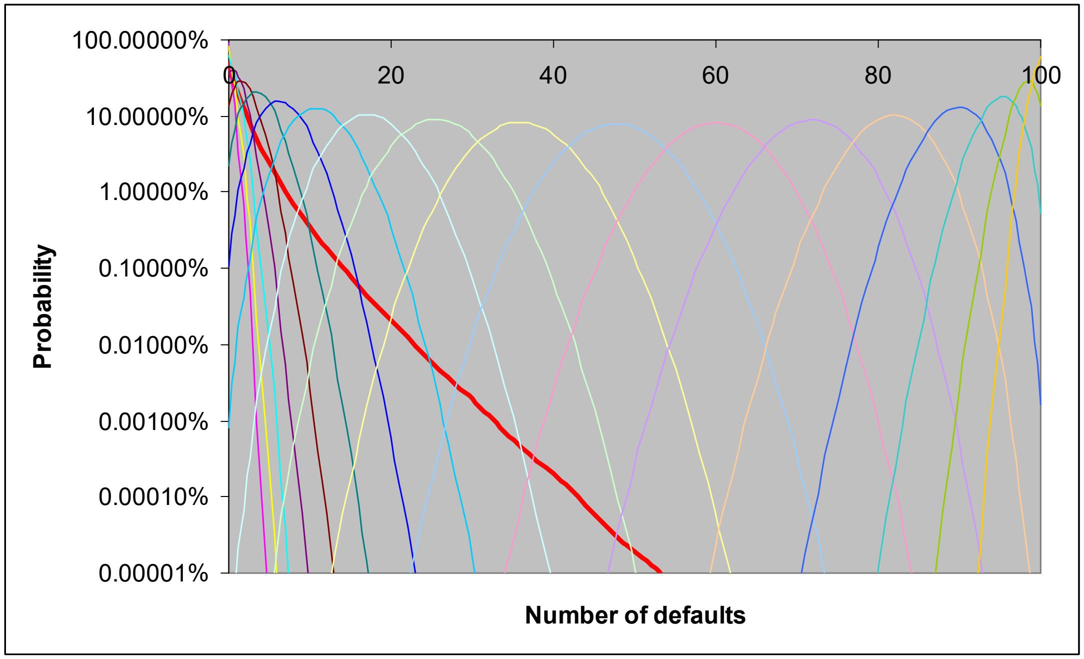
Systematic variable

- Conditional independence:

$$p_i = \Pr\left(V_i < \Phi^{-1}(p_i)\right) = \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho} Z}{\sqrt{1-\rho}}\right)$$

Factor Model Interpretation

- Default probabilities are independent conditional on a latent variable(s)
  - The latent variable (Z) can be thought of as a state of the world, bad states of the world lead to high default probabilities
- The actual world is a weighted average of all of these worlds different underlying default probabilities and independent defaults
- This leads to some tractable implementation procedures
  - For example, for equal default probabilities the actual loss distribution is a linear combination of binomial distributions



Simple Example

- Suppose we have *n* credits each with default probability *p*
- If they are independent then the loss distribution is binomial

$$\Pr(num\ defaults = k) = p^k (1-p)^{n-k} \frac{n!}{(n-k)!\,k!}$$

- What if they are correlated?
  - The conditional default probabilities are:

$$p_i^Z = \Pr\left(V_i < \Phi^{-1}(p)\right) = \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho} Z}{\sqrt{1-\rho}}\right)$$

  - Note that they are independent conditional on the value of *Z*
- So, the loss distribution just requires integrating (e.g., Quadrature) over binomial distributions

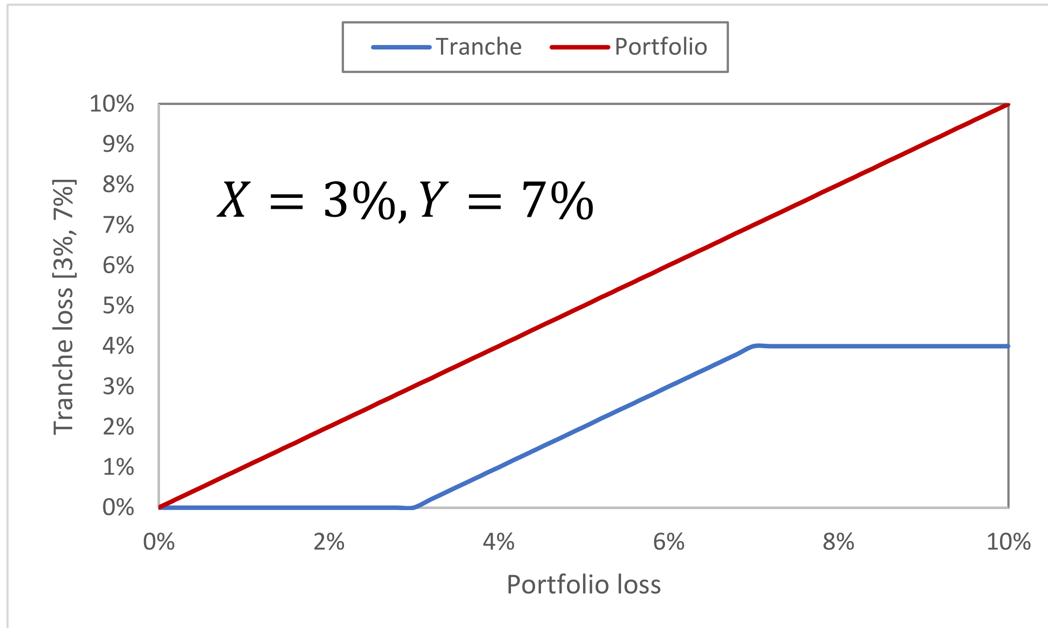
Heterogenous Methods

- When default probabilities, notionals and recoveries are different
- Large portfolio approximation (Vasicek) – see Basel lecture
  - Also granularity adjustment formulas
- Recursion
  - Build loss distribution sequentially using bins (least common multiple)
  - Depending on notionals and recoveries, may have to approximate
- Conditional normal approximation
  - Approximate all conditional distributions as Gaussian (this sounds like a terrible idea!)
  - But as the latent variable becomes more positive, the conditional distribution becomes more Gaussian-like
  - Doesn't work well at 0% correlation by definition (!) Works best for tail probabilities and when correlation is high
  - Resulting distribution is continuous

Pricing a Collateralised Debt Obligation (1)

- A tranche of a CDO is exposed to portfolio losses in the range $[X, Y]$ with X and Y being defined in terms of the total portfolio notional
- Losses on the tranche (M_t) are equivalent to a call spread on the portfolio losses (L_t)

$$M_t = (L_t - X)_+ - (L_t - Y)_+$$

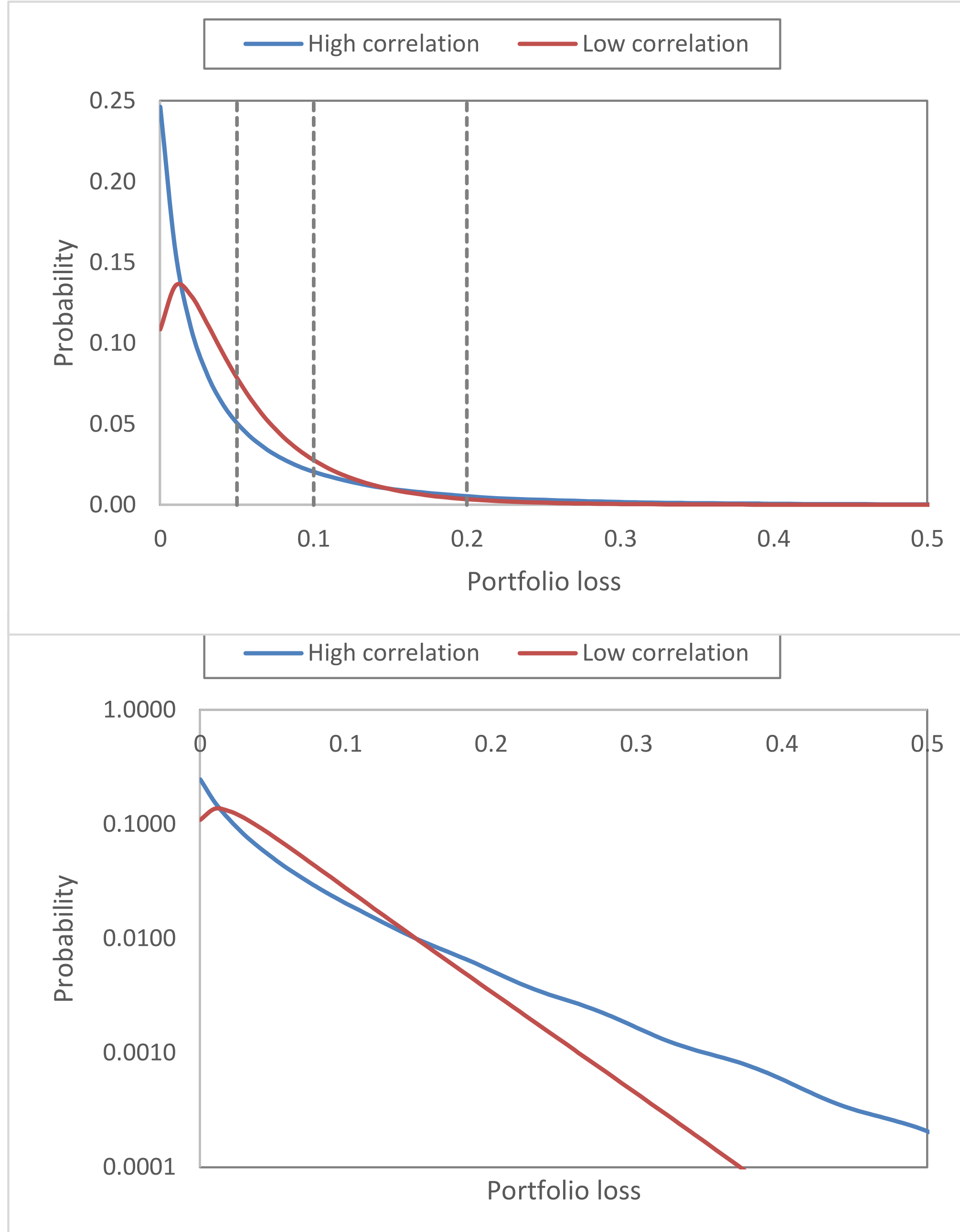


- Need a model for the portfolio loss distribution L_t
- A structural model is the obvious choice (hazard rate models do not produce strong enough dependency)

Tranche Losses - Example

	Low correlation	High correlation
[0-5%]	12.8%	10.4%
[5-10%]	4.7%	4.5%
[10-20%]	1.1%	1.8%
[20-100%]	0.02%	0.10%
[0-100%]	1.0%	1.0%

- This does not consider discounting and timing of defaults



Pricing a Collateralised Debt Obligation (2)

- The CDO tranche can be priced by integrating over the tranche loss process (e.g. Gregory and Laurent [2003])

$$E\left[\int_0^T B(0,s)\,dM_s\right]$$

Maturity of tranche

Risk-free discount factor

Tranche loss process

I will survive

Jon Gregory and Jean-Paul Laurent apply an analytical conditional dependence framework to the valuation of default baskets and synthetic CDO tranches, matching Monte Carlo results for pricing and showing significant improvement in the calculation of deltas

The credit derivatives market has grown exponentially in recent years. Liquidity in the credit default swap (CDS) market for sovereign, investment-grade and high-yield credits is improving to the extent that name-by-name dynamic risk management of portfolio credit derivatives is practical. This allows hedging of credit spread changes and (to an extent) default events, although there remains exposure to correlation risk. The potential to dynamically risk manage collateralised debt obligation (CDO) tranches broadens the range of products that can be offered to investors. On the theoretical side, much effort is being put into the issue of modelling correlated credits. This is relevant for pricing *k*th-to-default baskets and CDO tranches. This article describes a widely applicable and useful technique for these products and offers a powerful framework for pricing and risk management of a credit derivatives correlation book.

Pricing models

☐ **The firm-value approach.** For pricing baskets, the model described independently by Hull & White (2000), Arvanitis & Gregory (2001) and, briefly, by Finger (2000) is a structural approach in the spirit of Merton (1974) with defaults driven by a multi-dimensional diffusion process. This

and Merino & Nyfeler, 2002) and is here extended to consistently account for various time horizons. The factor approach allows us to deal with many names and leads to very tractable pricing results. We will denote by $p_t^{i|V} = Q(\tau_i \le t|V)$ and $q_t^{i|V} = Q(\tau_i > t|V)$ the conditional default and survival probabilities. Conditional on V, the joint survival function is:

$$S(t_1,\dots,t_n|V) = \prod_{1\le i\le n} q_{t_i}^{i|V}$$

We detail the previous framework using some examples:

☐ The one-factor Gaussian copula corresponds to CreditMetrics and was introduced by Vasicek (1987). These are also known as probit models in statistics. The internal ratings-based approach in Basel II is based on a one-factor Gaussian copula. Here, $V, \bar{V}_i$ are independent Gaussian random variables. We define $V_i = \rho_i V + \sqrt{1-\rho_i^2}\bar{V}_i$ and $\tau_i = F_i^{-1}(\Phi(V_i))^2$ for $i = 1, \dots, n$. Here:

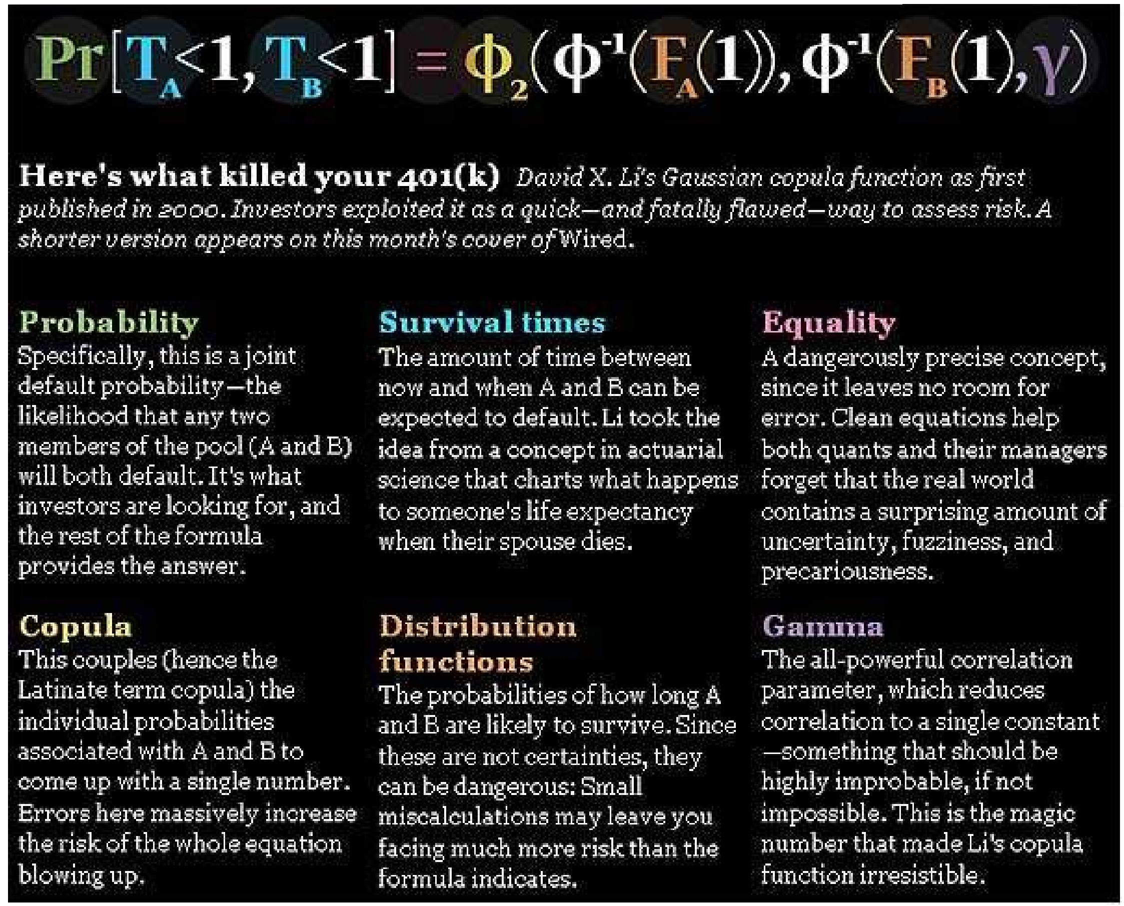
$$p_t^{i|V} = \Phi\left(\frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}}\right)$$

☐ Archimedean copulas are known in statistics, duration analysis and ac-

On Default Correlation: A Copula Function Approach

David X. Li

This draft: April 2000
First draft: September 1999



$$\Pr[T_A<1, T_B<1] = \Phi_2(\Phi^{-1}(F_A(1)), \Phi^{-1}(F_B(1)), \gamma)$$

Here's what killed your 401(k) *David X. Li's Gaussian copula function as first published in 2000. Investors exploited it as a quick—and fatally flawed—way to assess risk. A shorter version appears on this month's cover of* Wired.

Probability
Specifically, this is a joint default probability—the likelihood that any two members of the pool (A and B) will both default. It's what investors are looking for, and the rest of the formula provides the answer.

Survival times
The amount of time between now and when A and B can be expected to default. Li took the idea from a concept in actuarial science that charts what happens to someone's life expectancy when their spouse dies.

Equality
A dangerously precise concept, since it leaves no room for error. Clean equations help both quants and their managers forget that the real world contains a surprising amount of uncertainty, fuzziness, and precariousness.

Copula
This couples (hence the Latinate term copula) the individual probabilities associated with A and B to come up with a single number. Errors here massively increase the risk of the whole equation blowing up.

Distribution functions
The probabilities of how long A and B are likely to survive. Since these are not certainties, they can be dangerous: Small miscalculations may leave you facing much more risk than the formula indicates.

Gamma
The all-powerful correlation parameter, which reduces correlation to a single constant—something that should be highly improbable, if not impossible. This is the magic number that made Li's copula function irresistible.

The Failure of the Gaussian Copula Model

	Market (bp pa)	Gaussian Copula Model
Super Senior 22-100%	14.9	0.03
12-22%	34.3	7.1
9-12%	60.5	44.1
6-9%	84.9	128.6
3-6%	156.3	411.0
Equity 0-3%	25.05%	37.2%

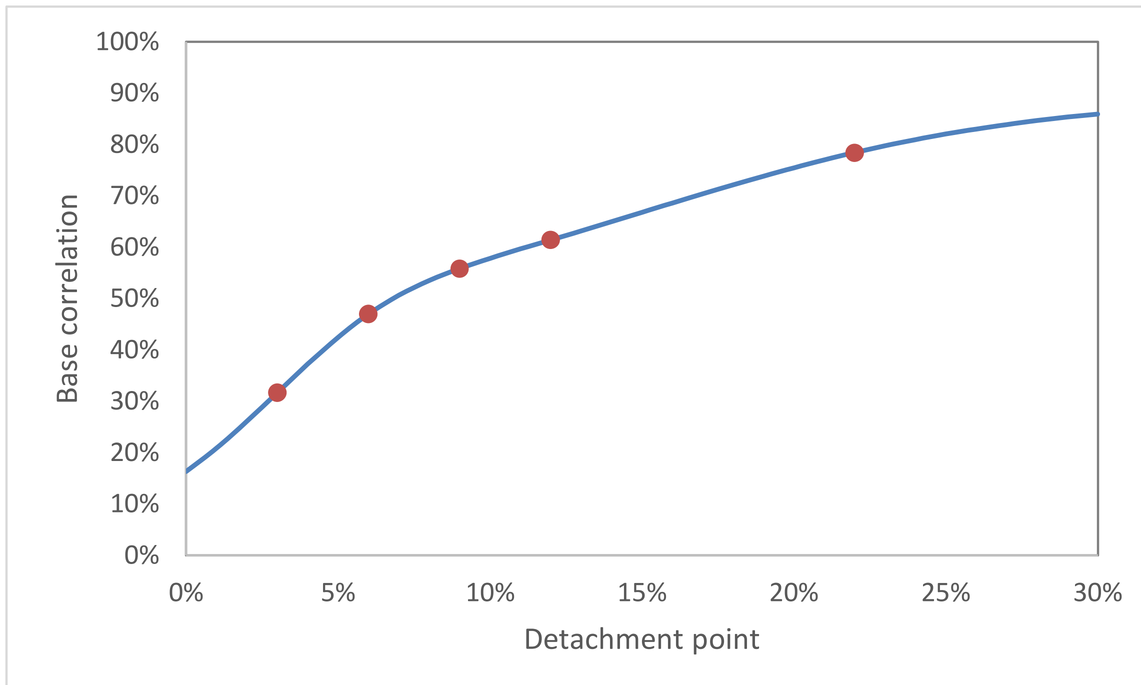
Upfront premium

Base Correlation

- A CDO tranche is a call spread payoff
- Split into two option type 'base tranches'

	Market	Correlation	Model
[0-3%]	25.05%	31.7%	
[3-6%]	156.3	47.0%	
[6-9%]	84.9	55.8%	
[9-12%]	60.5	61.4%	
[12-22%]	34.3	78.4%	
[22-100%]			14.9
[8-15%]			57.3

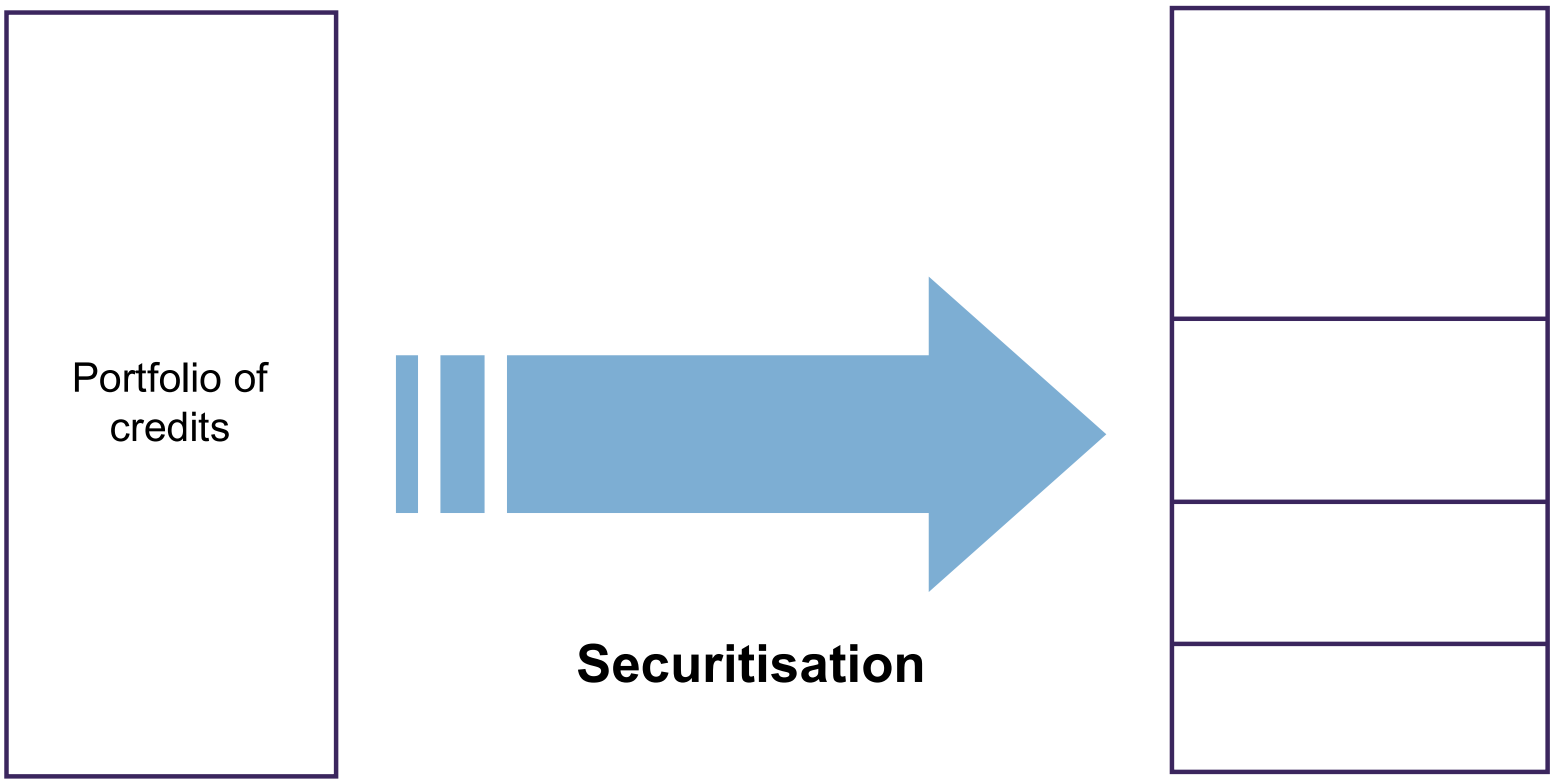
$$CDO[X,Y] \equiv CDO[0,Y] - CDO[0,X]$$



CDOs and the Financial Crisis

FitchLearning

Securitisation



	Size	Tranching	Rating
Super senior	850	[15-100%]	NR
Class A	50	[10-15%]	Aaa/AAA
Class B	30	[7-10%]	Aa2/AA
Class C	30	[4-7%]	Baa2/BBB
Equity	40	[0-4%]	NR

Credit Ratings: Agency Definitions

Description	Fitch and S&P		Moody's		Short term	Explanation
Highest credit quality	AAA		Aaa		F1	Exceptionally strong capacity for timely payment of financial commitments which is highly unlikely to be adversely affected by foreseeable events.
Very high credit quality	AA	AA+ AA AA-	Aa	Aa1 Aa2 Aa3		Very strong capacity for timely payment of financial commitments which is not significantly vulnerable to foreseeable events.
High credit quality	A	A+ A A-	A	A 1 A 2 A 3	F2	Strong capacity for timely payment of financial commitments which may be more vulnerable to changes in circumstances/ economic conditions.
Good credit quality	BBB	BBB+ BBB BBB-	Baa	Baa1 Baa2 Baa3	F3	Adequate capacity for timely payment of financial commitments but adverse changes in circumstances/ economic conditions are more likely to impair this capacity.
Speculative	BB	BB+ BB BB-	Ba	Ba1 Ba2 Ba3		Possibility of credit risk developing, particularly due to adverse economic change over time. Business/financial alternatives may be available to allow financial commitments to be met.
Highly speculative	B	B+ B B-	B	B1 B2 B3	B	Significant credit risk with a limited margin of safety. Financial commitments currently being met; however, continued payment is contingent upon a sustained, favourable business and economic environment.
High default risk	CCC		Caa		C	Default is a real possibility. Capacity for meeting financial commitments is solely reliant upon sustained, favourable business or economic developments.
Probable default	CC		Ca			Default of some kind appears probable.
Likely default	C		C			Default imminent.

The answer to how we got in this mess... if you can do the 'math'

WITH impeccable timing comes Standard & Poor's Handbook of Structured Finance, 785 pages with an incomprehensible equation (like this one below) on nearly all of them.

This tome, it says on the back, "provides a comprehensive overview of quantitative techniques needed to measure and manage risks..."

It also promises to "employ risk-measurement techniques such as ratings, and 'The Greeks' in structured deals". Perhaps they meant to say "Geeks", since neither word is in the index.

S&P, of course, is the rating agency that OK'd all that subprime mortgage junk which is poisoning the world's financial system, and if you wondered how all those clever people could screw things up so royally, this book tells you.

As these instruments got more and more complicated, so ever-more sophisticated models were needed to justify their splendid credit ratings and thus the high prices their creators had demanded from the mugs who were buying.

So arcane was the "math" (it's mostly American, after all) that there was no room for anything like common sense. How the sow's ear of a subprime mortgage can somehow be transmuted into the silk purse of a top-notch credit is swamped under the bell curves, analytical models and (of course) management fees.

$$
\begin{aligned}
C_j(0) &= D(T)\,EL_j(T) + \int_0^T EL_j(t)\,dD(t) \\
&= D(T)\,EL_j(T) + \int_0^T EL_j(t)\,D(t)\,f(t)\,dt
\end{aligned}
$$

Arnaud de Servigny, one of the book's authors, was head of S&P's quantitative analytics, whatever they are. He's now doing something similar at Barclays Wealth, a business that sounds somewhat oxymoronic today.

I wish I could recommend his handbook but you'd need a PhD in higher maths to understand it, and besides, after the summer meltdown you can't afford the 54 quid S&P want for it.

Source : Evening Standard August 2007

Last month, Fitch paid $27m to settle the *Mid-Coast Council v Fitch Ratings* case. The settlement resulted in the recovery of about 95% of the losses incurred by claimants in their purchase of synthetic collateralised debt obligations (SCDOs), which were rated by Fitch.

BUSINESS NEWS AUGUST 10, 2018 / 4:15 AM / 2 YEARS AGO

S&P settles landmark derivatives-rating lawsuit in Australia

3 MIN READ

SYDNEY (Reuters) - Standard & Poor's said on Friday it has settled a lawsuit in Australia over claims by pension funds and local governments that the credit rating firm had overlooked risks when awarding high ratings to opaque investments that imploded in the global financial crisis.

Australia's Federal Court approved the settlement on Thursday, S&P said in an emailed statement which did not disclose the settlement sum or terms.

"S&P Global is pleased to reach a settlement on the class action lawsuit, the last of the significant litigation pertaining to our previous ratings actions on collateralised debt obligations," it said.

The U.S.-based ratings agency was sued for at least A$190 million ($140 million) by two local governments and two pension funds in Australia, which lost money on

Does Securitisation have any Economic Value?

Do CDOs Work?

FitchLearning

Overview

- In the period 1998 to 2007, CDOs increased exponentially in both volume and diversity
  - Prior to 2007, the CDO was seen as a successful financial innovation
- However, the global financial crisis was partly catalysed by an implosion in the CDO market and caused massive losses for:
  - Issuers (banks) through investments held, litigation, failed hedges, reputation
  - Investors, both in terms of default losses and those from forced liquidation
  - Third parties (e.g. rating agencies through loss of fees, reputation issues and litigation)
- An obvious question is therefore:
  - Is there something fundamentally wrong with the concept of a CDO – and more broadly – the concept of securitisation?
  - Does it have economic value?

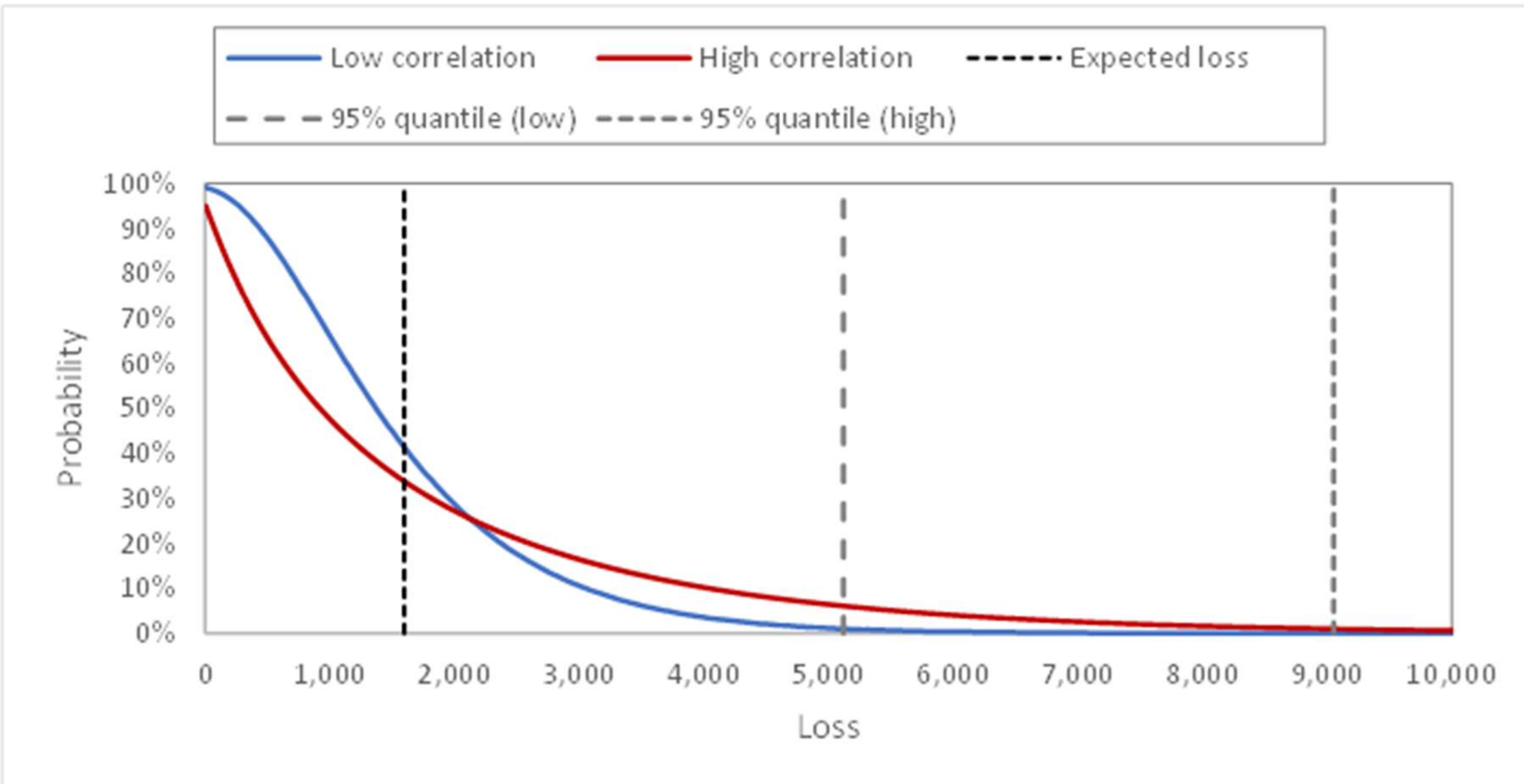
Rating a CDO (I)

- Credit Ratings Agencies (CRAs) from the 1990s starting rating structured credit products based on quantitative modelling
- The starting point would be historical default behaviour
  - For example, this is a Moody's table from around 2006

Rating	Year 1	Year 2	Year 3	Year 4	Year 5	Year 6	Year 7	Year 8	Year 9	Year 10
Aaa	0.000%	0.000%	0.000%	0.035%	0.078%	0.129%	0.185%	0.190%	0.190%	0.190%
Aa1	0.000%	0.000%	0.000%	0.099%	0.149%	0.168%	0.168%	0.168%	0.168%	0.168%
Aa2	0.000%	0.010%	0.044%	0.110%	0.211%	0.258%	0.312%	0.373%	0.443%	0.524%
Aa3	0.018%	0.036%	0.070%	0.123%	0.177%	0.233%	0.268%	0.285%	0.289%	0.343%
A1	0.003%	0.082%	0.218%	0.308%	0.377%	0.447%	0.499%	0.546%	0.615%	0.706%
A2	0.024%	0.076%	0.206%	0.389%	0.557%	0.716%	0.895%	1.082%	1.243%	1.339%
A3	0.034%	0.156%	0.317%	0.429%	0.578%	0.754%	0.884%	1.038%	1.163%	1.218%
Baa1	0.154%	0.425%	0.749%	1.040%	1.308%	1.545%	1.844%	2.040%	2.195%	2.338%
Baa2	0.164%	0.450%	0.818%	1.396%	1.882%	2.380%	2.827%	3.247%	3.808%	4.546%
Baa3	0.329%	0.893%	1.545%	2.280%	3.195%	4.101%	4.920%	5.688%	6.231%	6.703%
Ba1	0.747%	1.958%	3.460%	4.936%	6.477%	8.016%	9.009%	9.874%	10.546%	11.248%
Ba2	0.856%	2.403%	4.287%	6.212%	7.977%	9.187%	10.467%	11.636%	12.762%	13.716%
Ba3	1.929%	5.369%	9.523%	13.671%	17.152%	20.418%	23.610%	26.526%	29.161%	31.678%
B1	3.064%	8.135%	13.408%	18.029%	22.986%	27.485%	31.951%	35.770%	38.872%	41.441%
B2	4.814%	10.905%	16.308%	20.955%	24.864%	28.016%	30.752%	32.732%	35.087%	37.087%
B3	9.525%	17.753%	25.434%	32.257%	38.266%	43.953%	48.097%	51.764%	54.309%	55.904%
Caa1	12.161%	23.751%	35.108%	44.221%	51.517%	56.537%	58.736%	59.286%	59.286%	59.286%
Caa2	20.250%	30.286%	38.358%	45.265%	49.376%	53.825%	57.558%	62.031%	67.122%	73.485%
Caa3	26.482%	38.212%	45.071%	50.421%	55.373%	55.549%	55.549%	55.549%	55.549%	55.549%
Ca-C	33.643%	44.631%	53.222%	58.890%	66.743%	69.954%	74.351%	78.455%	78.455%	78.455%

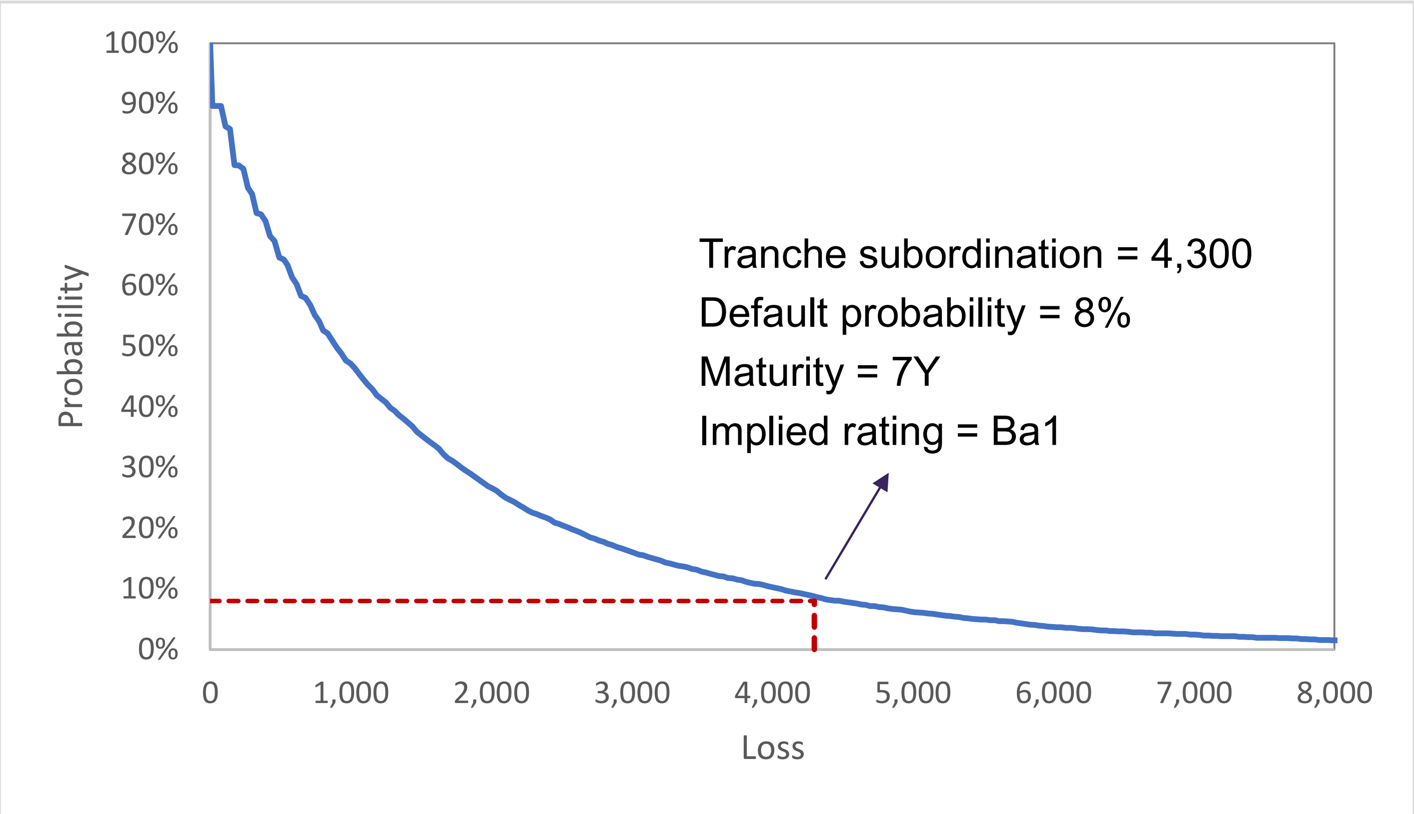
Rating a CDO (II)

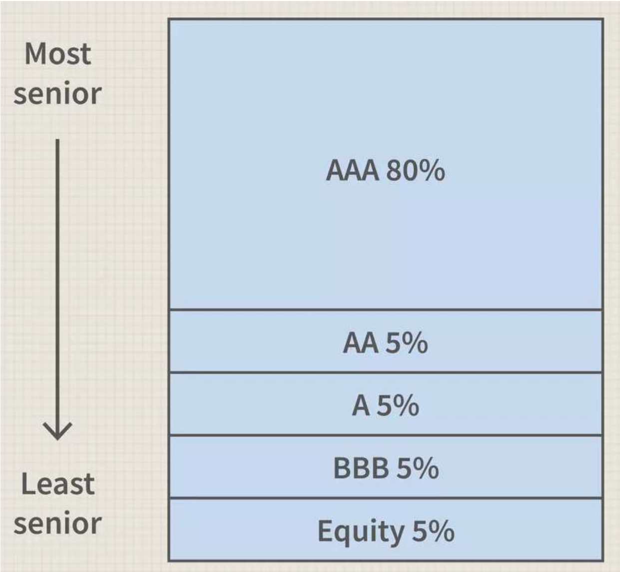
- Using the ratings of each entity in the portfolio to characterize their default behaviour, it is then necessary to make some assumption on default dependence
  - This developed in complexity eventually leading to the use of Monte Carlo simulation and the Gaussian copula as 'advanced'



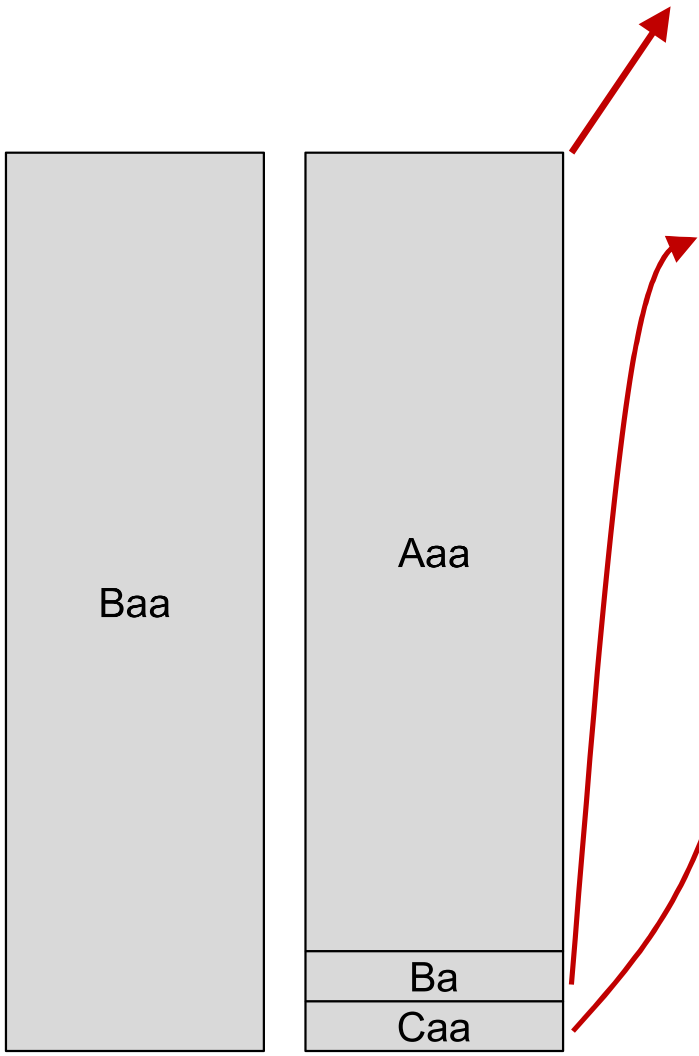
Rating a CDO (III)

- Once the loss distribution has been generated then the rating of a given tranche can be calculated by matching the expected loss (Moody's) or default probability (Standard and Poor's and Fitch)



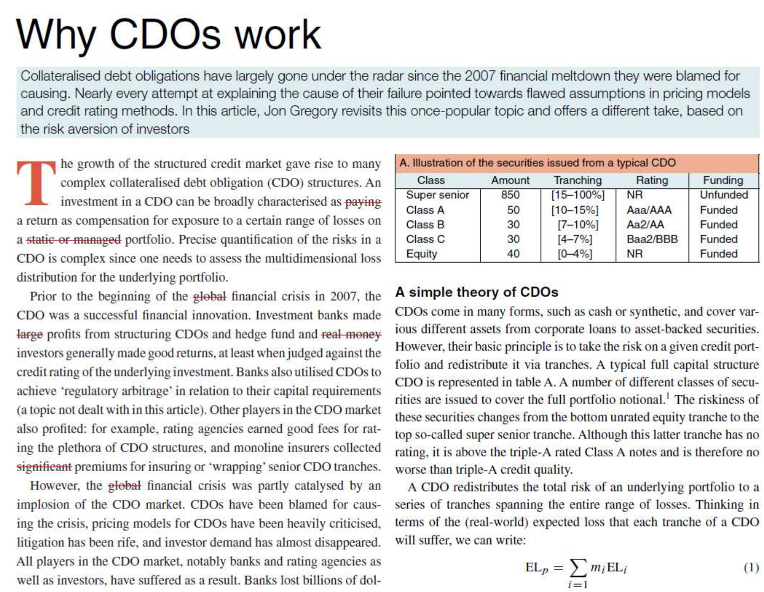


Simple CDO Economics



	Tranching	Rating	Coupon
Baa	[0-100%]	Baa	250
Aaa	[8-100%]	Aaa	100
Ba	[4-8%]	Ba	600
Caa	[0-4%]	Caa	1500

$$
\begin{aligned}
&\mathit{Excess\ Spread} \\
&= (250 \\
&-100 \times 92\% \\
&-600 \times 4\% \\
&-1500 \times 4\%) \\
&\underline{= 74\ bps}
\end{aligned}
$$



Why CDOs work

Collateralised debt obligations have largely gone under the radar since the 2007 financial meltdown they were blamed for causing. Nearly every attempt at explaining the cause of their failure pointed towards flawed assumptions in pricing models and credit rating methods. In this article, Jon Gregory revisits this once-popular topic and offers a different take, based on the risk aversion of investors

The growth of the structured credit market gave rise to many complex collateralised debt obligation (CDO) structures. An investment in a CDO can be broadly characterised as ~~paying~~ a return as compensation for exposure to a certain range of losses on a ~~static or managed~~ portfolio. Precise quantification of the risks in a CDO is complex since one needs to assess the multidimensional loss distribution for the underlying portfolio.

Prior to the beginning of the ~~global~~ financial crisis in 2007, the CDO was a successful financial innovation. Investment banks made ~~large~~ profits from structuring CDOs and hedge fund and ~~real money~~ investors generally made good returns, at least when judged against the credit rating of the underlying investment. Banks also utilised CDOs to achieve 'regulatory arbitrage' in relation to their capital requirements (a topic not dealt with in this article). Other players in the CDO market also profited: for example, rating agencies earned good fees for rating the plethora of CDO structures, and monoline insurers collected ~~significant~~ premiums for insuring or 'wrapping' senior CDO tranches.

However, the ~~global~~ financial crisis was partly catalysed by an implosion of the CDO market. CDOs have been blamed for causing the crisis, pricing models for CDOs have been heavily criticised, litigation has been rife, and investor demand has almost disappeared. All players in the CDO market, notably banks and rating agencies as well as investors, have suffered as a result. Banks lost billions of dol-

A. Illustration of the securities issued from a typical CDO

Class	Amount	Tranching	Rating	Funding
Super senior	850	[15–100%]	NR	Unfunded
Class A	50	[10–15%]	Aaa/AAA	Funded
Class B	30	[7–10%]	Aa2/AA	Funded
Class C	30	[4–7%]	Baa2/BBB	Funded
Equity	40	[0–4%]	NR	Funded

A simple theory of CDOs

CDOs come in many forms, such as cash or synthetic, and cover various different assets from corporate loans to asset-backed securities. However, their basic principle is to take the risk on a given credit portfolio and redistribute it via tranches. A typical full capital structure CDO is represented in table A. A number of different classes of securities are issued to cover the full portfolio notional.[1] The riskiness of these securities changes from the bottom unrated equity tranche to the top so-called super senior tranche. Although this latter tranche has no rating, it is above the triple-A rated Class A notes and is therefore no worse than triple-A credit quality.

A CDO redistributes the total risk of an underlying portfolio to a series of tranches spanning the entire range of losses. Thinking in terms of the (real-world) expected loss that each tranche of a CDO will suffer, we can write:

$$\mathrm{EL}_p = \sum_{i=1} m_i \mathrm{EL}_i \quad (1)$$

Due to the short duration of the equity tranche, the excess spread is a conservative measure of the profitability of a CDO

Do CDOs Work?

- Yes
- A CDO works due to
  - The risk preferences of investors
  - The expected loss methodology used in the ratings process
- What did go wrong then?
- Lack of proper assessment of counterparty risk in the structuring process
  - The more senior the tranche, the more counterparty risk (relatively) and monoline insurers (Gregory 2008)
- Lack of appreciation of the systemic risk in senior tranches
  - Were investors sufficiently compensated for this?
  - Gibson, M., 2004, “Understanding the risk of synthetic CDOs”, Finance and Economics Discussion Paper, 2004–36, Federal Reserve Board, Washington DC / Coval et al, 2009, “Economic catastrophe bonds,” American Economic Review, 99(3), 628—66.

Conclusions

- Structural models are used for
  - Individual default prediction (capital structure $\rightarrow$ default probability)
  - Portfolio default behaviour (equity correlation $\rightarrow$ joint default behaviour)
- Default prediction approaches are useful
  - Compliment other measures such as ratings and credit spreads
- Portfolio credit risk modelling in particular remains important
  - Economic capital calculations (see Basel lecture)
  - Counterparty risk in credit default swaps
  - Structured credit (CLOs are still quite popular)
- Securitization ‘works’ due to changing risk profile of a portfolio
- But beware some of the known problems
  - Gaussian perhaps not the best copula choice
  - Parameterisation difficult

Selected References

- Structural models and default prediction
  - R. C. Merton, 1974, “On the pricing of corporate debt: The risk structure of interest rates”. Journal of Finance, vol.29, pp.449-470.
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Appendix

Calibration of Merton Model

- In practice, we cannot observe the value of the firm or its volatility
- However, we can show that:

$$S_0 = \frac{\sigma_V}{\sigma_S} N(d_1) V_0$$

  - Where σ_S is the (observable) equity volatility

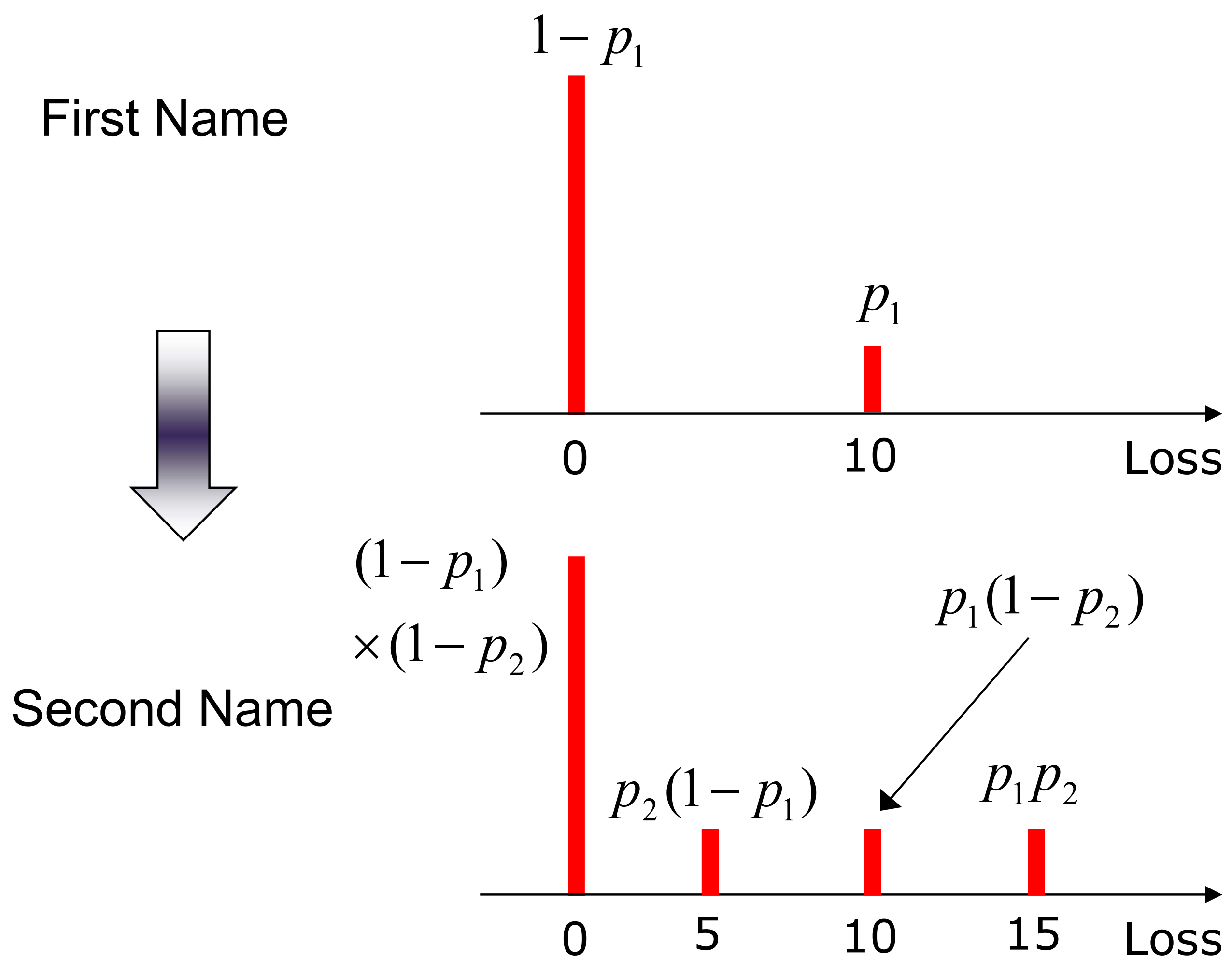
- We also have the value of the equity in terms of the value of the firm

$$E_0 = V_0 \Phi(d_1) - e^{-rT} B \Phi(d_2)$$

$$d_1 = \frac{\ln(V_0/B) + (r + \sigma_V^2/2)T}{\sigma_A \sqrt{T}}$$
$$= d_2 + \sigma_V \sqrt{T}$$

- These two equations can be solved together to find σ_V and V_0
- Example
  - Suppose we have $E_0 = 3, B = 10, \sigma_S = 0.5, T = 1, r = 5\%$
  - We can solve to give $V_0 = 12.5, \sigma_V = 0.12$

Recursion


First Name
$1-p_1$
p_1
0
10
Loss
Second Name
$(1-p_1)$
$\times(1-p_2)$
$p_2(1-p_1)$
$p_1(1-p_2)$
$p_1 p_2$
0
5
10
15
Loss

Conditional Normal Approximation

- Conditional default probabilities p_i^Z, notionals N_i and LGDs LGD_i
- Mean

$$\mu^Z = \sum_i^n p_i^Z N_i LGD_i$$

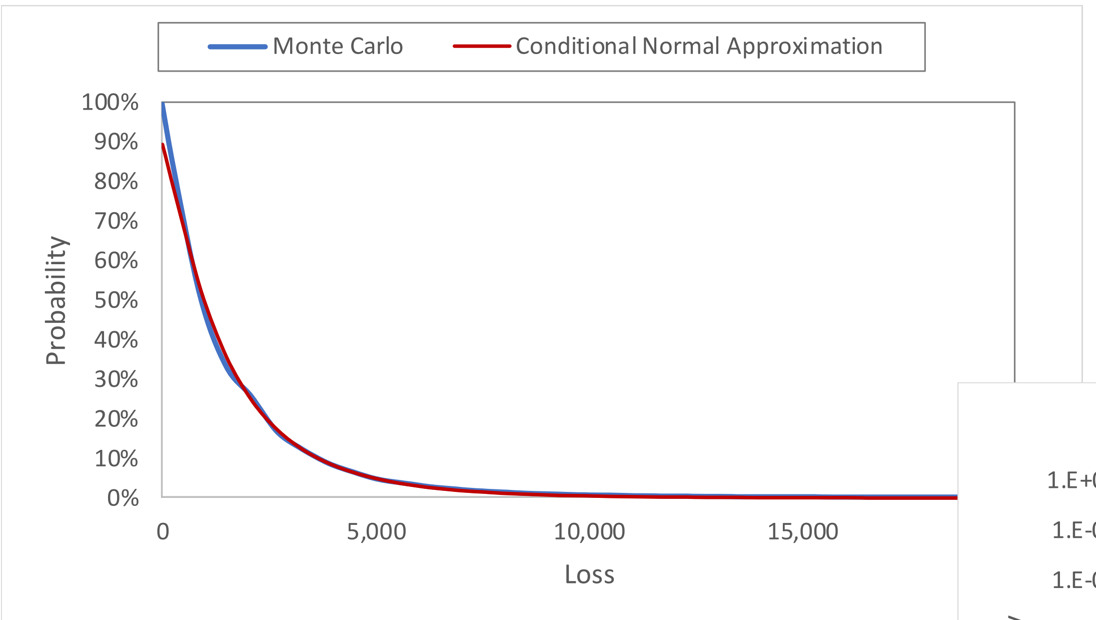
- Standard deviation

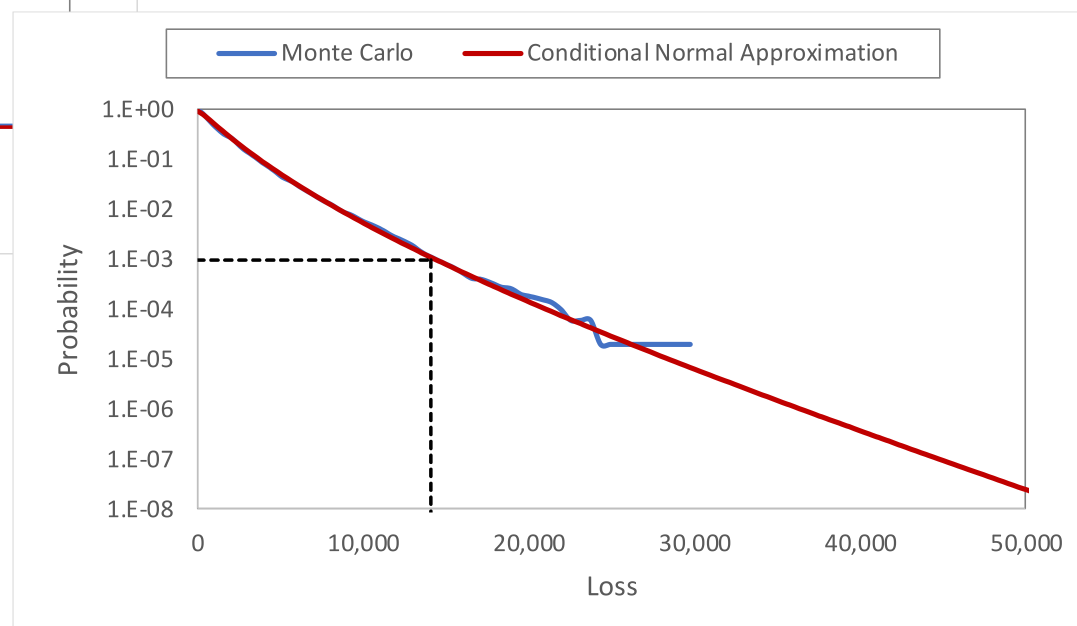
$$\sigma^Z = \sqrt{\sum_i^n p_i^Z \left(1 - p_i^Z\right) N_i^2 LGD_i^2}$$

- Loss distribution is then a weighted average (Quadrature) of normal distributions

Analytical Factor Models

- With conditional independence can calculate loss distribution accurately without Monte Carlo simulation (see also lecture on Basel)





Economics of a CDO

- Suppose there is a continuum of underlying tranches (full capital structure)
- Consider expected loss (EL) as the main quantitative characteristic of the tranche
  - Expected loss must be conserved across the structure

$$EL_P = \sum_i m_i EL_i \qquad \sum_i m_i = 1$$

Expected loss for unit tranche (under physical measure)

Tranche size

- Investors will demand a premium for the losses they take
  - Represented via a multiplier α which represents the risk aversion for a given seniority and will be implicitly determined by the coupon demanded by investors
  - The CDO will "work" if

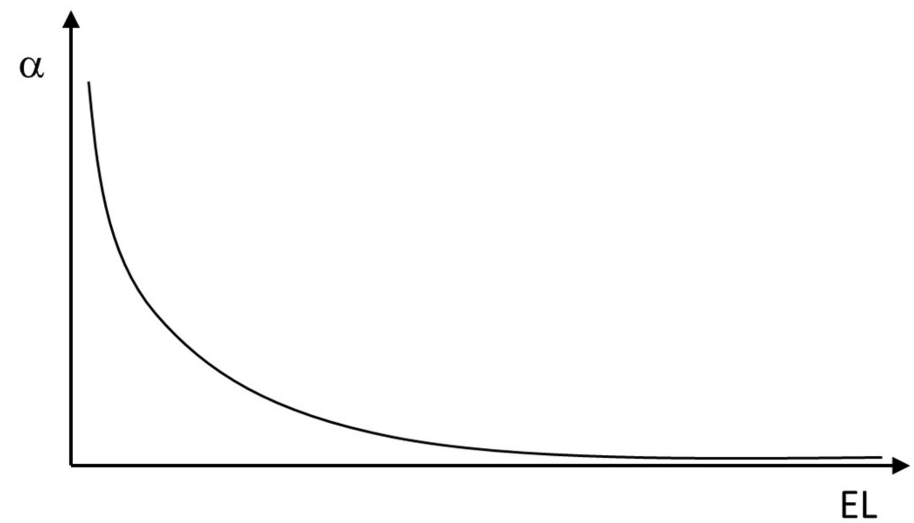
$$\alpha_p EL_P > \sum_i \alpha_i m_i EL_i$$

  - This basically requires that it is possible to buy protection cheaper via the CDO tranches than it is on the underlying portfolio

Risk Aversion and Seniority

- How do we represent α?
  - The primary consideration of investors is the rating of the underlying tranche
  - In turn, the fundamental driver of ratings would be the expected loss of a tranche
  - Hence we assume

$$\alpha_j = \left(\frac{a}{EL_j}\right)^b$$



- Properties
  - Risk-neutral investors, $b = 0$
  - Risk aversion for $a, b > 0$
  - More relative risk aversion for small expected losses

Requirement for CDO to Work

- What parameters are required for a CDO to work?
  - We require:

$$\alpha_p EL_P > \sum_i \alpha_i m_i EL_i \qquad\qquad \alpha_j = \left(\frac{a}{EL_j}\right)^b$$

  - Which becomes:

$$\left(\frac{a}{EL_p}\right)^b EL_P > \sum_i \left(\frac{a}{EL_i}\right)^b m_i EL_i$$

  - Simplifying to:

$$EL_P^{1-b} > \sum_i m_i EL_i^{1-b}$$

  - Which is satisfied when $b < 1$

Example Calibration

- Hull, Predescu and White (2005)
  - Time period, December 1996 to July 2004
  - Merrill Lynch bond indices and Moody's data

	Real world loss (bps)	Risk neutral loss (bps)	Ratio
Aaa	4	67	16.8
Aa	6	78	13.0
A	13	128	9.8
Baa	47	238	5.1
Ba	240	507	2.1
B	749	902	1.2
Caa	1690	2130	1.3