The Heath, Jarrow and Morton Model

Forward Curve. Factorisation with Principal Component Analysis

In this lecture...

- the short rate process, bond price and forward rates
- evolving the forward curve with HJM model – in fact, a system of SDEs for each term forward rate
- derivatives pricing by Monte-Carlo (caplet, spread option)
- HJM is your first multi-factor model. Calibration done with PCA
- yield curve data analysis: revealing the internal structure of factors for yield curve movement

By the end of this lecture you will:

- understand forward rates and their bootstrapping
- get introduced to quant modelling of a yield curve
- be able to analyse the yield curve changes data to calibrate fwd rate volatility
- understand the HJM framework and its calibration issues
- be able to price simple interest rate derivatives by Monte-Carlo

Introduction

The Heath, Jarrow & Morton approach was a major breakthrough that improved risk management in fixed income.

It models the yield curve as a whole.

This is a good starting point to learn about forward rates and their volatility/SDE for curve simulation.

Then you can move onto term forward rates (IBORs), and LIBOR Market Model (LMM).

One-factor models, Vasicek, CIR, Ho & Lee, Hull & White, evolve the short rate $dr(t)$.

- $r(t) = f(t,t)$ represents **one point on the curve**.

  Curve is evolved with $r + dr$, a series of steps. Can price a bond numerically by integrating over the evolution.

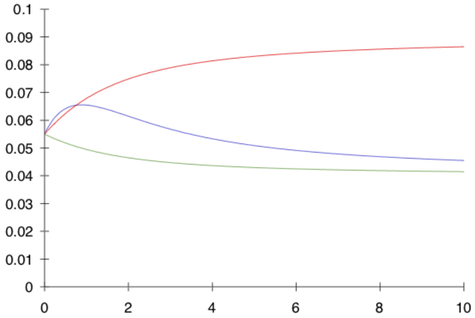
- Calibrated to short-term $Z_M(t^*, T)$ market bonds ONLY.

  There are bond futures, IRS, FRAs written directly on expected LIBOR at tenors t^* $(f_{t^*,6M} - k)$.

  Offer analytical 'bond pricing' $Z(t;T) = e^{A(T) - r\,B(T)}$.

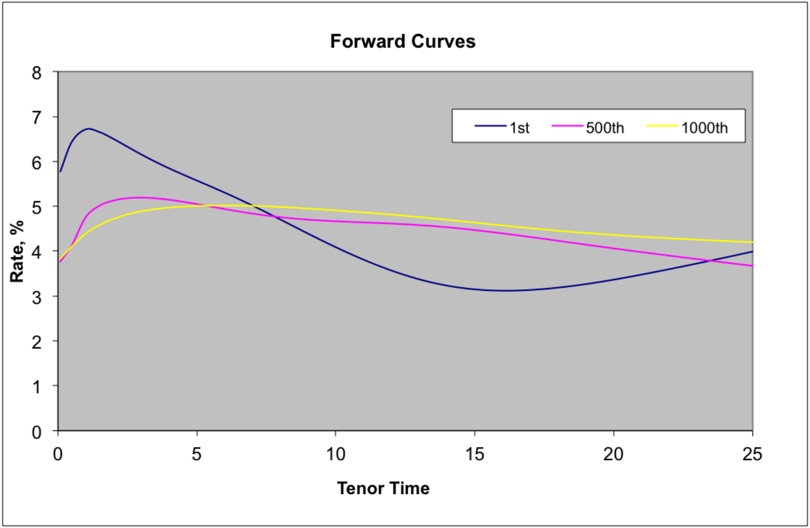
One-factor simulated curves

Types of yield curve given by the Vasicek model $dr = (\eta - \gamma r)\,dt + \beta^{1/2}\,dX$.



From: CQF Lecture on Stochastic Interest Rates, Slide 43.

Market Curves (forward rates, pound sterling)



- 'One factor' translates to one kind of movement: parallel up/down.

$$Corr[\Delta r_t, \Delta f_j] \approx 1$$

- Empirical fact is that **short-term** funding rates move rather independently from long-term rates.

$$Corr[\Delta r_t, \Delta f_j] \approx 0$$

Both cannot be true.

Outcome: One-factor models are affine, easy to simulate but produce simplistic moves for a whole curve. Good as a quick fix for risk calculations: CVA, IRS pricing examples.

It makes sense to model a sequence of forward rates $f(0; t_i, t_{i+1})$,

rather than overlapping discount factors $Z(0, T_1), Z(0, T_2), Z(0, T_3) \ldots$

$$
\begin{aligned}
Z(0, T_1) \times Z(0; T_1, T_2) &= Z(0, T_2) \\
Z(0; T_1, T_2) &= \exp\left[-f(0; t_1, t_2)(t_2 - t_1)\right] \\
f_2 &= -\frac{\ln Z_2 - \ln Z_1}{t_2 - t_1}
\end{aligned}
$$

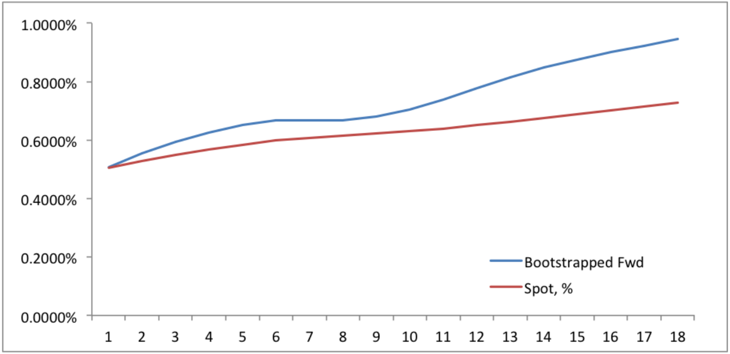
Computing forward rates

A scheme for each discrete tenor j follows from the instantaneous forward rate maths (1)

$$f_0 = -\frac{\ln Z_1 - \ln 1}{T_1 - 0} \qquad f_1 = -\frac{\ln(Z_2/Z_1)}{T_2 - T_1}$$

Tenor, T	0.08	0.17	0.25	0.33
Spot	0.5052%	0.5295%	0.5500%	0.5682%
Z(0, T)	0.9996	0.9991	0.9986	0.9981
Forward	0.5063%	0.5552%	0.5927%	0.6244%

Forward vs. Spot Curve



Data: Bank Liability Curve as of 30 January 2015 (Bank of England)

An **instantaneous forward rate** $F(t, T, T + \Delta t)$, where rate expires at time T and applies over an instant $\Delta t \to 0$.

$$\begin{aligned} f(t,T) &= -\lim_{\Delta t \to 0} \frac{lnZ(0;T+\Delta t) - \ln Z(0;T)}{\Delta t} \\ &= -\frac{\partial}{\partial T} \ln Z(0;T) \end{aligned} \tag{1}$$

Now we use instantaneous notation $f(t, T) \equiv F(t, T, T + \Delta)$.

LIBOR, a simple forward rate

Forward Rate Agreement: at reset, FRA rate and LIBOR fix coincide.

$$\mathrm{FRA}(T_{i-1}, T_i) - L(T_{i-1}, T_i) = 0$$

IRS and caplets/floorlets are market-quoted with simple forward rate.

Notation $L(t, T_{i-1}, T_i)$ or $L(T_{i-1}, T_i)$ or $L_i(t)$ and shortcut is:

$\mathrm{LIBOR} = m\left(e^{f/m} - 1\right)$, where $m = 1/\tau$ is compounding frequency, eg 3M LIBOR compounded $m = 4$ times. LIBOR Model gives:

$$f_i = \frac{1}{T_{i+1} - T_i}\left(\frac{1}{Z(T_i, T_{i+1})} - 1\right)$$

Modelling bond price

The log-normal SDE must be a familiar model:

$$\frac{dZ}{Z} = \mu(t, T)\, dt + \sigma(t, T)\, dX \tag{2}$$

where bond price evolves with t but the maturity date T is fixed.

Empirical observation: if we construct a constant maturity bond price, the changes in the yield give us **the model invariant** (an *iid* process)

$$\begin{aligned} \Delta f &\propto \ln Z(t_{i+1}; T) - \ln Z(t_i; T) \\ \Delta f &\sim Normal(\mu, \sigma^2 \tau) \end{aligned}$$

The model is Normal for forward rates $f(t, T)$ and Log-Normal for bond prices $Z(t; T)$.

Computational Preview

1. Take the data of Δf_j **changes in forward rates** about constant tenors and compute $j \times j$ covariance matrix $\mathbf{\Sigma}$.

$$\mathbf{\Sigma} = \frac{1}{N} \boldsymbol{X}\boldsymbol{X}'$$

2. Conduct PCA on the covariance matrix $\mathbf{\Sigma}$. Each component is a linear contribution to change in the yield curve.

3. Fit volatility functions from PCs (eigenvectors). That gives both, diffusion and drift, for HJM SDE.

4. Pricing of interest rate derivatives is done by Monte-Carlo.

Matrix Decomposition

Symmetric matrix can be decomposed according to *the spectral theorem*:

$$\boldsymbol{\Sigma} = \mathbf{V} \boldsymbol{\Lambda} \mathbf{V}'$$

- $\boldsymbol{\Lambda}$ is a diagonal matrix with eigenvalues $\lambda_1 > \cdots > \lambda_n > 0$ positive and usually ranked in software output (Matlab, R).

$$\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

- $\mathbf{V}$ is a vectorised matrix of eigenvectors *vec*($\mathbf{e}^{(1)}$ $\mathbf{e}^{(2)}$ ... $\mathbf{e}^{(n)}$).

- **PCA Tutorial** end slides cover our Excel demo. **Solutions** offer maths of Jacobi Method.

Start with the forward rate equation $f(t, T) = -\frac{\partial}{\partial T} \ln Z(t, T)$ and use the bond price 'GBM' SDE to derive the evolution of forward rates:

$$df(t, T) = \frac{\partial}{\partial T}\left[\frac{1}{2}\sigma^2(t, T) - \mu(t, T)\right] dt - \frac{\partial}{\partial T}\sigma(t, T)dX \tag{3}$$

The SDE carries the drift of the bond price $\mu(t, T)$.

When we come to pricing, such drift terms are replaced by the risk-free interest rate $r(t)$.

Risk to risk-neutral

In the real world, our model for he evolution of the forward curve (3) was derived as

$$df(t,T) = \frac{\partial}{\partial T}\left[\frac{1}{2}\sigma^2(t,T) - \underline{\mu(t,T)}\right] dt - \frac{\partial}{\partial T}\sigma(t,T)dX$$

In the risk-neutral world, under measure $\mathbb{Q}$, the model becomes

$$df(t,T) = \frac{\partial}{\partial T}\left[\frac{1}{2}\sigma^2(t,T) - \underline{r(t)}\right] dt - \frac{\partial}{\partial T}\sigma(t,T)dX^{\mathbb{Q}} \tag{4}$$

But $r(t)$ is not a function of T, so $\frac{\partial}{\partial T}r(t) = 0$

Change of measure

Replacing $\mu(t, T)$ with $r(t)$ is a change of measure. Girsanov theorem gives

$$X_t^{\mathbb{Q}} = X_t + \int_t^T \theta_s ds \qquad \frac{d\mathbb{Q}}{d\mathbb{P}} = exp\left\{-\frac{1}{2}\theta^2 T - \theta X_T\right\}$$

The result for market price of risk is familiar $\theta = \frac{\mu - r}{\sigma}$.

$$\frac{dZ}{Z} = \mu(t, T)dt + \sigma dX_t$$

$$\frac{dZ}{Z} = r(t)dt + \sigma dX_t^{\mathbb{Q}}$$

In the risk-neutral economy, the expected return on any traded investment (a bond) is simply $r(t)$.

- By setting up a hedged portfolio $\Pi = Z(t, T_1) - \Delta Z(t, T_2)$ we found that to cancel the drift we require

$$\frac{\mu(t, T_1) - r(t)}{\sigma(t, T_1)} = \frac{\mu(t, T_2) - r(t)}{\sigma(t, T_2)}$$

- Only possible if both sides are equal to some parameter, which is independent of maturity dates T_1, T_2

$$\mu(t, T) = r(t) + \lambda(r, t)\sigma(t, T)$$

where $\lambda(r, t)$ is the market price of risk (MPOR), a unifying global parameter, not a constant!

More in the CQF Extra *The Market Price of Risk: Fear and Greed...*

Drift derivation

We expressed the risk-neutral drift as a function of volatility – of bond prices for now. Chain rule was used for differentiation *wrt* ∂T

$$df(t,T) = \underbrace{\sigma(t,T)\frac{\partial}{\partial T}\sigma(t,T)dt}_{} - \frac{\partial}{\partial T}\sigma(t,T)dX^{\mathbb{Q}} \tag{5}$$

Equivalent to,

$$df(t,T) = -\sigma(t,T)\nu(t,T)dt + \nu(t,T)dX^{\mathbb{Q}}$$

Forward rate volatility

$$\nu(t,T) = -\frac{\partial}{\partial T}\sigma(t,T) \qquad \sigma(t,T) = -\int_t^T \nu(t,s)ds$$

this is a conversion into **volatility of forward rates** $\nu(t,T)$.

$$\begin{aligned} df(t,T) &= \nu(t,T)\underbrace{(-\sigma(t,T))}\,dt + \nu(t,T)dX^{\mathbb{Q}} \\ df(t,T) &= \left[\nu(t,T)\int_t^T \nu(t,s)ds\right]dt + \nu(t,T)dX^{\mathbb{Q}} \end{aligned}$$

From now on, we operate in the risk-neutral world, so we drop $\mathbb{Q}$.

Risk-neutral HJM dynamics

The drift is a function of volatility, **a no arbitrage condition.**

$$m(t, T) = \nu(t, T) \int_t^T \nu(t, s) ds \tag{6}$$

HJM SDE expressed deceiptively simply

$$\boldsymbol{f}(t, T) = m(t, T)dt + \nu(t, T)dX \tag{7}$$

Notation in bold, $\boldsymbol{f}(t, T)$ refers to evolution of the whole curve – that is at any point of tenor time.

We cannot model $\bar{\boldsymbol{f}}(t, \tau)$ at an infinite number of tenors, so we have to discretise into *a system* of SDEs – each is evolved in **a separate column** of *HJM MC.xlsm*.

$$
\begin{aligned}
d\bar{f}(t, \tau_1) &= \bar{m}(t, \tau_1)dt + \sum_{i=1}^{k} \bar{\nu}_i(t, \tau_1)dX_i \\
\vdots \qquad & \qquad \vdots \qquad\qquad \vdots \\
d\bar{f}(t, \tau_j) &= \bar{m}(t, \tau_j)dt + \sum_{i=1}^{k} \bar{\nu}_i(t, \tau_j)dX_i
\end{aligned}
$$

Forward rate at each tenor point $\tau_j = T_j - t$ is its own stochastic variable. j is a counter for tenor 0.5Y, 1Y, 1.5Y, etc.

We have introduced something else: instead of one volatility function $\bar{\nu}_i(t, \tau)$, we have several! That summation means we have **a multi-factor** stochastic model.

$$\sum_{i=1}^{k} \bar{\nu}_i(t, \tau_j)\, dX_i$$

Curve stress-testing: diffusion has several independent sources of randomness dX_i. Each represents uncertainty about curve movement, from a different factor.

Each volatility function is a Principal Component

$$\bar{\nu}_i(t,\tau) = \sqrt{\lambda_i}\,\boldsymbol{e}_{\boldsymbol{\tau}}^{(\boldsymbol{i})}$$

Through **calibration by PCA** and **fitting by cubic spline** we have the volatility functions $\bar{\nu}_i(t,\tau)$

$$\bar{m}(t,\tau) = \sum_{i=1}^{k} \bar{\nu}_i(t,\tau) \int_0^{\tau} \bar{\nu}_i(t,s)ds + ...$$

Drift computation requires numerical integration $\int_0^{\tau} \bar{\nu}_i(t,s)ds$ implemented by Trapezium Method inside *m()* function in VBA.

Pricing by Monte-Carlo

Simulated Output of HJM

1. Simulation Simulate an evolution of the whole risk-neutral curve for the necessary length of time, from today t^* to T^*.

- For 'a risk-neutral forward curve' we use GLC data (UK Gilts).
- Realizations of the curve $\boldsymbol{f}(t, T)$ over time steps $dt = 0.01$ are **in rows**.
- The paths of forward rates for discretised tenors τ_j are **in columns**.

2. Discounting factors Obtain ZCB values for all required tenors up to T^*. However, discounting factors can come from the outside (e.g., OIS curve) creating a problem of how to match expectations.

Risk-neutral discount factor

Pricing a Zero Coupon Bond with any stochastic short rate $r(t)$

$$Z(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)ds\right)\right]$$

Here is a case when we do not need the entire simulated curve.

Looking at the HJM output, we already know that we only an evolution of the first point on the yield curve $f(t,t) = r(t)$.

$$\begin{aligned} Z(t,T) &= \exp\left(-\sum r_t \, \Delta t\right) \qquad \text{under MC} \\ &= \exp\left(-\text{SUM(COLUMN)} \times 0.01\right) \end{aligned} \tag{8}$$

Monte-Carlo simulation is always discretised over time step dt. Therefore, integration becomes summation.

3. Calculate the value of cashflows of interest and apply discounting. Example: consider $(L - K)^+$ payoff and discount

(Discounting is assumed within the same risk-neutral expectation.)

Return to Step 1 to perform another round of simulation.

- Keep track of the running average of simulated prices, i.e., 1st, 1st + 2nd, 1st + 2nd + 3rd, etc. .
- It must demonstrate convergence/reduction in its variance.

Let's review the pricing.

The analytical solution (PDE approach) is not feasible under the HJM. That leaves us with two choices:

1. **Monte-Carlo method** – estimation of an expectation by simulating the evolution of forward rates.

   Implemented under risk-neutral HJM dynamics and measure $\mathbb{Q}$.

2. The other is to build up **a tree** structure and formalise it into a Finite Difference grid.

EXTRA. HJM and the short rate r(t)

Given by the forward rate for a maturity equal to the current date, i.e.

$$r(t) = f(t, t)$$

If the forward curve today is $f(t^*, T)$ then the short rate for *any* time t in the future is

$$r(t) = f(t^*, t) + \int_{t^*}^{t} df(s, t)$$

In terms of HJM output, $df(s, ...)$ means evolving a rate **in column**.

Check with $f(t^*, t) + f(t, t) - f(t^*, t) = f(t, t)$.

The HJM SDE in terms of spot rate is jammed into

$$dr(t) = \left[\frac{\partial f(t^*, t)}{\partial t} - \left. \frac{\partial \mu(t, s)}{\partial s} \right|_{s=t} \right.$$
$$+ \int_{t^*}^{t} \left(\sigma(s, t) \frac{\partial^2 \sigma(s, t)}{\partial t^2} + \left(\frac{\partial \sigma(s, t)}{\partial t} \right)^2 - \frac{\partial^2 \mu(s, t)}{\partial t^2} \right) ds$$
$$\left. - \underline{\int_{t^*}^{t} \frac{\partial^2 \sigma(s, t)}{\partial t^2} dX(s)} \right] dt - \left. \frac{\partial \sigma(t, s)}{\partial s} \right|_{s=t} dX$$

This SDE is odd because **the drift depends on the history of σ** from the date t^* to the future date t and the stochastic increments dX.

Building a tree

If we attempt to use an evolution path for a forward rate $f(t, t) = r(t)$ to build a tree

- Then we'll find ourselves with an unfortunate result: an up move followed by a down move will **not** end up in the same state.
- Our tree structure becomes 'bushy'. The number of branches grows *exponentially* with the addition of new time steps.

This is a feature of a non-Markov model. Equivalence of paths is what makes the pricing by Binomial Method so efficient.

Non-Markovian nature of HJM

The highly path-dependent drift of dr(t) makes the movement of the short rate **non-Markov**.

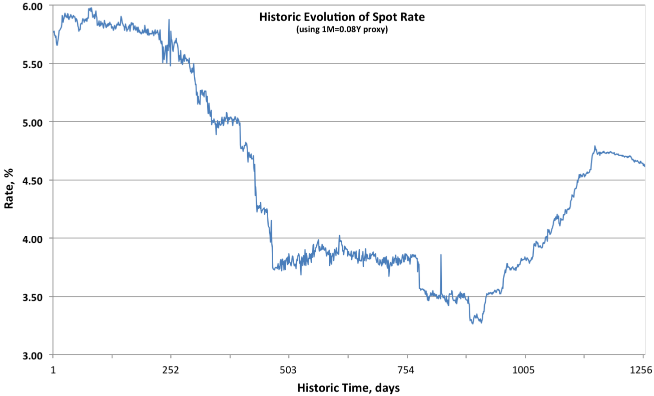
$$r + dr$$

is not recombinable.

In a **Markov chain** only the present state of a variable determines the possible future (albeit random) state.

Markov process is a stochastic process without a **memory**. But let's look at the empirical $r(t)$.

Memory in r(t)



The **long memory** for stationary time series means that decay in autocorrelation is slower than exponential $\text{Corr}[r_t, r_s] = \beta^{t-s}$.

Working in tenor time τ

Modelling in tenor time τ (change of variable) is called **Musiela Parametrisation** of the HJM SDE, which we have derived as $df(t, T)$

- as t changes however, the time distance $\tau = T - t$ changes too. Would be modelling a different tenor rate!!

$$T_{6M-0.00}, T_{6M-0.01}, T_{6M-0.02}, \ldots, T_{6M-0.42}, \ldots$$

The volatility function keeps its form:

$$\nu(t, T) = -\frac{\partial}{\partial T}\sigma(t, T) \quad \Rightarrow \quad -\frac{\partial}{\partial \tau}\bar{\sigma}(t, \tau)\,\frac{\cancel{\partial \tau}}{\cancel{\partial T}} = \bar{\nu}(t, \tau)$$

$$\frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial T} \equiv \frac{\partial}{\partial T}\frac{\partial \tau}{\partial \tau}$$

$$\tau = T - t \quad \text{and} \quad T(t) = t + \tau \quad \text{and} \quad t(T) = T - \tau$$

We want to evolve an SDE in dt (lhs). In order 'to get to τ'

$$\frac{d\bar{f}(t,\tau)}{dt} \equiv \left[\frac{\partial}{\cancel{\partial T}} \frac{\cancel{\partial T}}{\partial t} + \frac{\partial}{\cancel{\partial t}} \frac{\cancel{\partial t}}{\partial T} \right] f(t,\tau)$$

$$\frac{d\bar{f}}{dt} \equiv \frac{df}{dt} + \frac{\partial f}{\partial T} \qquad \text{to put simply}$$

For comparison only,

$$\frac{d}{dt} \underbrace{f(t,\tau)} + \frac{\partial}{\partial T} f(t, T - t)$$

$$d\bar{f} = df + \frac{\partial f}{\partial T} dt \quad \text{or} \quad \boxed{d\bar{f} = df + \frac{\partial \bar{f}}{\partial \tau} dt}$$

$$\begin{aligned}
d\bar{f}(t,\tau) &= \underbrace{df(t,\tau)}+\frac{\partial \bar{f}(t,\tau)}{\partial \tau} dt \\
&= \left(\bar{\nu}(t,\tau)\int_0^{\tau} \bar{\nu}(t,s)ds\right) dt + \bar{\nu}(t,\tau)dX + \frac{\partial \bar{f}(t,\tau)}{\partial \tau} dt \\
&\quad \text{original } f(t,T) = \left(\nu(t,T)\int_t^{T} \nu(t,s)ds\right) dt + \nu(t,T)dX
\end{aligned}$$

The drift gains an extra term $\frac{\partial \bar{f}}{\partial \tau}$, which is a forward slope.

When we come to simulate in *HJM MC.xlsm* the complete multi-factor SDE (in drift and in diffusion) is simulated column-by-column.

$$d\bar{f}(t,\tau) = \left(\sum_{i=1}^{k} \bar{\nu}_i(t,\tau) \int_0^{\tau} \bar{\nu}_i(t,s)ds \right) dt + \sum_{i=1}^{k} \bar{\nu}_i(t,\tau)dX_i + \frac{\partial \bar{f}}{\partial \tau} dt \quad (9)$$

$$d\bar{f}(t,\tau) = \bar{m}(\tau)dt + \sum_{i=1}^{k=3} \bar{\nu}_i(t,\tau)dX_i + \frac{\partial \bar{f}}{\partial \tau} dt \quad (10)$$

We simulate,

$$\bar{f}_{t+dt} = \bar{f}_t + d\bar{f}$$

In cells, computation $d\bar{f} = \bar{m}(\tau)dt + SUM(Vol_i * \phi_i)\sqrt{dt} + \frac{d\bar{f}}{d\tau}dt$

```
B$2*dt +
```

```
(B$3*$BC13+B$4*$BD13+B$5*$BE13)*SQRT(dt)+
```

```
(C12-B12)/(C$11-B$11)*dt
```

$k = 3$ independent factors feature in SDE for each term rate (each column).

Volatility functions $\sqrt{\lambda_i}\,\boldsymbol{e}_{\boldsymbol{\tau}}^{(\boldsymbol{i})}$ were fitted to cubic spline by $LINEST()$

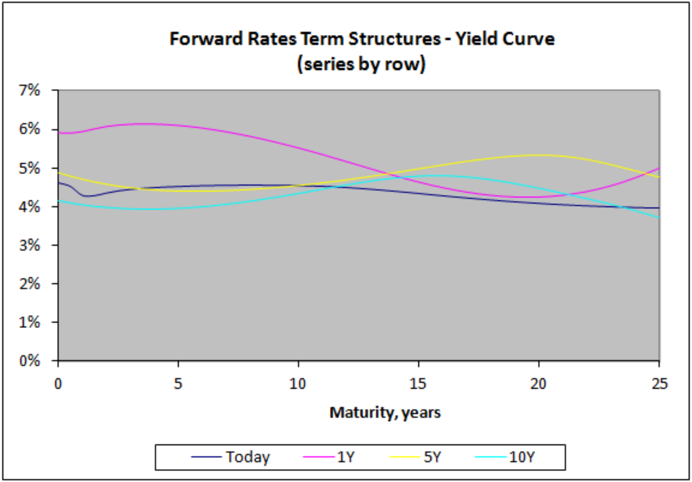
```
Vol_i=b0+b1*tau+b2*tau^2+b3*tau^3
```

Numerical Methods (reference)

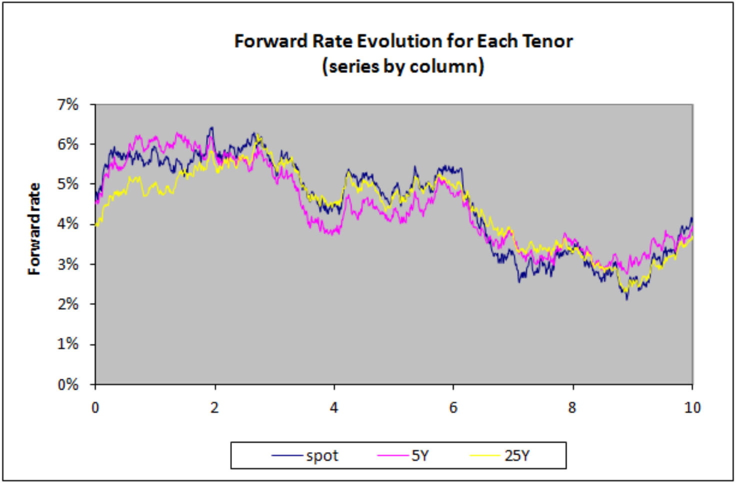
- Drift $\bar{m}(t,\tau)$ computed using *numerical integration* over the fitted volatility functions. **Trapezium rule** is used.
- Eigenvectors and eigenvalues obtained by rotation $\boldsymbol{\Sigma}' = \boldsymbol{P_{p,q}}^{\boldsymbol{T}} \boldsymbol{\Sigma} \boldsymbol{P_{p,q}}$ to eliminate the largest off-diagonal element p, q. Rotation repeated. [See Solutions]
- To initialise, the first row is a static curve from data.
- The forward derivative $\frac{\partial \bar{f}}{\partial \tau}$ is calculated using the row above.

We simulate a realisation of the entire curve at each time step. The simulation is Gaussian HJM and <u>negative rates are possible</u>.

Simulated Forward Curves



Simulated Instantaneous Forward Rates



ZCB Pricing

There are **two approaches** for pricing a bond under HJM framework.

- Integrating over a current forward curve $\bar{f}(t^*, \tau_j)$ – in a row

$$Z(t, T) = \exp\left(-\int_0^{T=\tau} \bar{f}(t^*, \tau) d\tau \right)$$

  This requires a no-arbitrage interpolation of the curve, followed by numerical integration. Things can get very technical!

- Using a simulated path of $r(t) = f(t, t)$ - -from the first column

$$Z(t, T) = \mathbb{E}^{\mathbb{Q}} \left[\exp\left(-\int_t^T r(s) ds \right) \right]$$

Risk-neutral discount factor

Using $r(t)$, the calculation becomes a summation over the first column of simulated data (HJM Model - MC Excel)

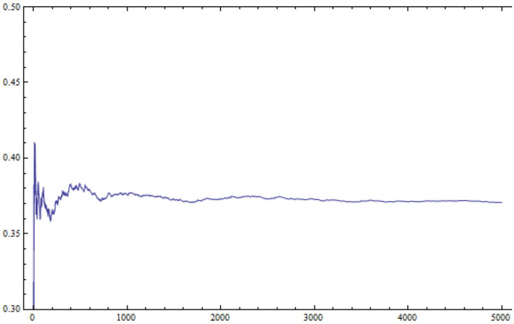
$$\begin{aligned} Z(t,T) &= \exp\left(-\sum r_t \, \Delta t\right) \qquad \text{under MC} \qquad (11) \\ &= \exp\left(-\mathrm{SUM(COLUMN)} \times 0.01\right) \end{aligned}$$

Solution: to price a half-year bond starting today Z(0; 0.5), we will carry out summation over 50 rows of the first column, $\Delta t = 0.01$.

Convenient!

To satisfy the risk-neutral expectation $\mathbb{E}^{\mathbb{Q}}$ we have to conduct the Monte-Carlo.

ZCB price convergence, $T > 10Y$



Monte-Carlo pricing means that we produce **a running average** of simulated prices, e.g., 1st, 1st + 2nd, 1st + 2nd + 3rd. The running average must demonstrate convergence/reduction in its variance.

Please take away the following important ideas:

- with the HJM model, we evolve the entire forward curve
- calibration means linear factorisation of forward rate volatility
- Principal Component Analysis reveals the internal data structure:

  the key factors of curve movement are level, steepness/flatness and curvature
- pricing under the HJM is done using the Monte-Carlo

Slides next offer **PCA Tutorial**

for **Data Analytics on Yield Curve**.

We start with a case for multi-factor modelling of the yield curve, and review the PCA side of implementation.

Case for a multi-factor model

A single-factor model for the short rate $r(t)$ can't hope to capture the richness of yield curve movements.

Consider *a spread option*. Its payoff is the difference between rates at two different tenors, e.g., $(L_{6M} - L_{3M})$.

- **If movements of two rates are not correlated (going up/down in sync), there is an extra source of risk**, ie, another factor.

  Instrument is sensitive to more than one factor.

  Cannot hedge with a single bond.

Bucket risks vs. Curve movement

Consider two 'natural' bucket risks, some short rate $f_{0.08Y}$ and a long-term rate f_{7Y},

the common risk methodology will study the CVA or derivative price *wrt* change at a single bucket.

However, if rates at $0.08, 7$ tenors move in the opposite direction, they represent another kind of curve movement, one systematic factor:

- steepening or flattening of the curve.

$$Corr[\Delta f_{3Y}, \Delta f_{5Y}], Corr[\Delta f_{3Y}, \Delta f_{10Y}], ... \gg 0$$

Interest rate *changes* are well-correlated across distant tenors (except *wrt* the short end).

But the covariance of changes $\boldsymbol{\Sigma(\Delta f_j, \Delta f_{j+h})}$ can be explained with a few independent factors.

There follows a possibility to represent the change about tenor τ_j as a linear decomposition of 'orthogonal' (independent) changes:

$$\Delta f_j = \mathrm{PC}_1 + \mathrm{PC}_2 + ... + \mathrm{PC}_k$$

The linear components come from the **Principal Component Analysis**.

- Systematic factors that describe movement of a curve as a whole.
- Factor attribution is well-established for yield curve analysis.

HJM SDE (9), each **volatility function** is equal to the scaled principal component *i*.

$$\bar{\nu}(\tau) = \text{Std Dev} \times \text{Eigenvector}_\tau \qquad \text{or} \qquad \sqrt{\lambda_i}\,\boldsymbol{e}_{\boldsymbol{\tau}}^{(i)}$$

volatility structure matches the data and calibration is fully numerical.

Data Preparation

If we have time series of each rate going back a few years, we can calculate covariances between **changes** in the rates.

- Inst. forward rates from BOE Yield Curve Statistics (use BLC)
- $\tau = 0.5$ increment for $0.08Y$ ... $25Y$ gives 50 columns.
- Jan 2002 – Jan 2007 regime. Consider regimes since then.

Government Liability Curve (GLC) bootstrapped from repo agreements, spot bonds (Gilts) and bond futures.

Bank Liability Curve built from short sterling futures, and FRAs. It is more suitable for pricing IR derivatives.

Covariance Matrix – Changes Percentage Rates

To estimate the covariance matrix $\mathbf{\Sigma}$,

1. Compute **daily differences** in fwd rate at each tenor, columnwise. Subtract the mean, if not a small quantity.

```
B3-B2 - AVERAGE(B), B4-B3 - AVERAGE(B), ...
5.7680 - 5.7733, 5.7757 - 5.7680
```

2. $\mathbf{\Sigma} = \frac{1}{N}\boldsymbol{XX'}$ where $\boldsymbol{X}$ relates to the dataset of Differences, see the tab in *HJM PCA.xls*.

3. We annualise covariance, and correct for the fact that we used percentages, eg, 5.768 not 0.05768 when computing differences,

$$\times \frac{252}{100 \times 100}.$$

Matrix Decomposition

$\boldsymbol{\Sigma}$ is the covariance matrix of fwd rate **changes**. Such symmetric matrix can be decomposed according to *the spectral theorem*:

$$\boldsymbol{\Sigma} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}'$$

- $\boldsymbol{\Lambda}$ is a diagonal matrix with eigenvalues $\lambda_1 > \cdots > \lambda_n > 0$ positive and usually ranked in software output (Matlab, R).

$$\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

- $\mathbf{V}$ is a vectorised matrix of eigenvectors $vec(\mathbf{e}^{(1)}\ \mathbf{e}^{(2)}\ \ldots\ \mathbf{e}^{(n)})$.

$$\Delta f(\tau_j) = \sqrt{\lambda_1}\boldsymbol{e}^{(1)}_{\tau_j} + \sqrt{\lambda_2}\boldsymbol{e}^{(2)}_{\tau_j} + \sqrt{\lambda_3}\boldsymbol{e}^{(3)}_{\tau_j} + \ldots \quad \text{in rows}$$

Using PCA output, we express a system of HJM SDEs in matrix form,

$$d\boldsymbol{f}(t,T) = \boldsymbol{M}(t,T)dt + \boldsymbol{V}\boldsymbol{\Lambda}^{\frac{1}{2}}\, d\boldsymbol{X} \tag{12}$$

where $d\boldsymbol{X}$ is a multi-dimensional Brownian Motion representing k independent factors.

Independence is achieved by decomposition of covariance matrix

$$\boldsymbol{\Sigma} = \mathbf{V}\boldsymbol{\Lambda}^{\frac{1}{2}}\left(\mathbf{V}\boldsymbol{\Lambda}^{\frac{1}{2}}\right)' = \boldsymbol{A}\boldsymbol{A}' \quad \text{Cholesky decomposition}$$

The covariance matrix is estimated from **changes in forward rates**

$$\boldsymbol{\Sigma} = \mathbb{C}ov[\Delta f(\tau_j), \Delta f(\tau_{j+h})].$$

Eigenvectors

Volatility functions of τ

- For each *column* eigenvector $\mathbf{e}^{(\mathbf{i})}$, the first entry is the movement of one-month rate ($\tau = 0.08$), the second entry is of the six-month rate ($\tau = 0.5$) and so on.

$$\overline{\nu}_i(t^*, \tau) = \sqrt{\lambda_i}\, \boldsymbol{e}_{\boldsymbol{\tau}}^{(\boldsymbol{i})} \tag{13}$$

  To obtain a volatility function, it is naturally convenient to fit a column eigenvector to tenor τ. Eigenvector $\boldsymbol{e}_\tau^{(i)}$ has values at

$$\tau = 0.08Y,\ 0.5Y,\ 1Y,\ \dots,\ 25Y$$

- Instead of picking numbers from the matrix of eigenvectors $\mathbf{V}$, **we use the fitted volatility functions**.

Polynomial Fitting

- The fitting is done by a single **cubic spline** *wrt* tenor τ

  $$\overline{\nu}(t,\tau) = \beta_0 + \beta_1\tau + \beta_2\tau^2 + \beta_3\tau^3 \quad \forall\, \tau_j$$

  A spline is a piecewise-defined smooth polynomial function.

  In general, we can fit exactly using a piecewise polynomial – here, an improvement can be made by enquiring into fitting and methods behind functions like *polyfit()* in Matlab and *nls()* in R.

  Fitting recipes are domain-specific for yield curve, implied vol. under LMM/LVM/SABR. Principal components are polynomials, eg, the best PC4 fit requires τ^4.

Despite using *LINEST()* to calculate β, here we are **not** conducting any regression analysis.

In our PCA application to Pound Sterling curve, three factors explain 93.33% of movement (variation) in the yield curve.

Tenor	λ	Cum. R^2
1Y	0.002027	71.31%
25Y	0.000463	87.58%
6Y	0.000164	93.33%

But how did we choose these $k = 3$ eigenvectors to be our volatility functions?

$$\boldsymbol{e}^{(1Y)},\ \boldsymbol{e}^{(6Y)},\ \boldsymbol{e}^{(25Y)}$$

By the largest corresponding eigenvalue.

Factor Significance

Eigenvalue λ_i **is variance** of the movements of a curve in each eigendirection. For example, the first factor explains

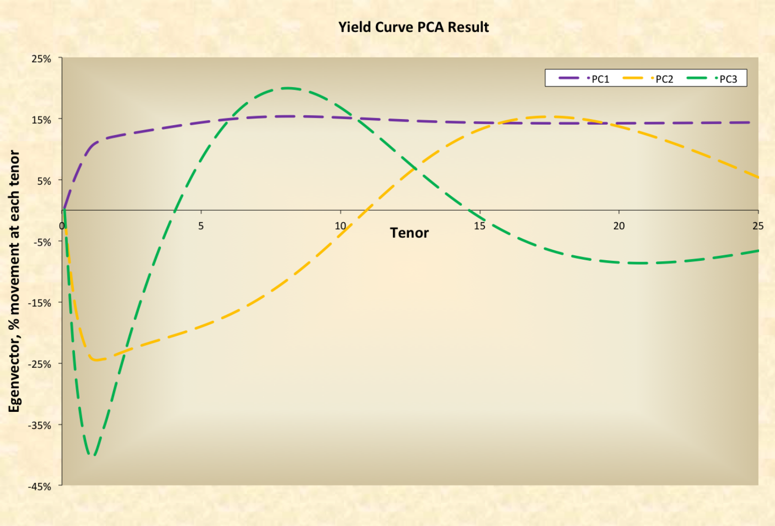
$$\frac{\lambda_1}{\lambda_1 + \lambda_2 + \cdots + \lambda_N}$$

The cumulative goodness of fit statistic for the k-factor model is

$$\text{Cum. } R^2 = \frac{\sum_{i=1}^{k} \lambda_i}{\sum_{i=1}^{N} \lambda_i}$$

By choosing the largest-impact factors we reduce an N-dimensional model to the three-factor model. Each factor represents systemic movement by the curve.

PCA Result: Three Largest Factors



Factor Attribution

- **Parallel shift in overall level of rates** is the largest principal component of forward curve movement, common to all tenors.
- **Steepening/flattening of the curve** is the second important component (i.e., change of *skew* across the term structure)

  Inverted curve (backwardation for commodities term structure) would have a different shape for the PC2.
- **Bending about specific maturity points** is the third component to curve movement that mostly affects *curvature* (convexity).

Disclaimer. These are commonly accepted attributions but please read next.

PCA Factors and the Curve

Changes in the rate at certain tenor 1Y, 6Y, 25Y can be particularly sensitive to a systematic factor **BUT** causality is not proven, and eigenvectors tend to rotate particularly for PC2 vs. PC3 and above.

Short end of the curve has low correlation of changes with the longer tenors. Short end is sensitive to and often represents PC1.

In the current regime of low interest rates (to 2017) and flattened curve, PC2, PC3 components might have no attribution and rotate often. PCA is a limited tool for periods of rapid shifts in interest rates.

END OF TUTORIAL