MATH 8230: Homework 1
09/07/2021

1. With the definition of a vector bundle from class, show that the vector space operations define continuous maps:

$$+ : E \times_B E \to E$$
$$\times : \mathbb{R} \times E \to E$$

2. Suppose you are given the following data:

   - Topological spaces B and F
   - A set E and a map of sets $\pi : E \to B$
   - An open cover $\mathcal{U} = \{U_i\}$ of B and for each i a bijection $\phi : \pi^{-1}(U_i) \to U_i \times F$ so that $\pi_1 \circ \phi_i = \pi$.

   Give conditions on the maps ϕ_i so that there is a topology on E making $\phi : E \to B$ into a fiber bundle with $\{(U_i, \phi_i)\}$ as an atlas.

3. An *oriented n-dimensional vector bundle* is a vector bundle $\pi : E \to B$ together with an orientation of each fiber E_b, so that these orientations are continuous in the following sense. For each $b \in B$ there is a chart (U, ϕ) with $b \in U$ and $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ so that for all $b' \in U$, $\phi|_{E_{b'}} : E_{b'} \to \mathbb{R}^n$ is orientation-preserving. Show that given an oriented n-dimensional vector bundle there is an induced principal $GL_+(\mathbb{R}^n)$-bundle (the "bundle of oriented frames"), and conversely given a principal $GL_+(\mathbb{R}^n)$-bundle there is an induced oriented n-plane bundle.

4. A *Riemannian metric* on a vector bundle $\pi : E \to B$ is an inner product $\langle \cdot, \cdot \rangle_b$ on each fiber E_b of E, which is continuous in the sense that the induced map $E \oplus E = E \times_B E \to \mathbb{R}$ is continuous. Show that given a Riemannian metric on a vector bundle, there is an induced principal $O(n)$-bundle (the "bundle of orthonormal frames"), and conversely given a principal $O(n)$-bundle there is an induced vector bundle with Riemannian metric.

5. What operation on principal $O(n)$-bundles corresponds to dualizing a vector bundle? What about the direct sum of vector bundle?

6. For nice spaces X (e.g. CW complexes) and abelian groups G, there is a canonical isomorphism $\check{H}^i(X; G) \cong H^i(X; G)$ between Čech and singular cohomology of X with coefficients in G. A nice, readable proof can be found in Frank Warner's *Foundations of Differential Manifolds and Lie Groups*, Chapter 5. In the rest of this problem, cohomology either means Čech cohomology or singular cohomology after applying this isomorphism.

   (a) Let $\pi : E \to B$ be an n-dimensional vector bundle, or equivalently, a principal $GL(n, \mathbb{R})$-bundle, given by a Čech cocycle $\phi \in \check{H}^1(B; GL(n, \mathbb{R}))$. Show that the sign of the determinant $\circ \det : GL(n, \mathbb{R}) \to \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$ induces a map

   $$\check{H}^1(B; GL(n, \mathbb{R})) \to \check{H}^1(B; \mathbb{Z}/2\mathbb{Z})$$

   and so ϕ induces an element $w_1(E) \in H^1(B; \mathbb{Z}/2\mathbb{Z})$.

   (b) Compute w_1 for the trivial line bundle (1-dimensional vector bundle) over the circle and for the Möbius band.

   (c) Prove that (for nice spaces) a line bundle $\pi : E \to B$ is trivial if and only if $w_1(E) = 0 \in H^1(B; \mathbb{Z}/2\mathbb{Z})$.

7. Show that the exact sequence of abelian topological groups

$$0 \to \mathbb{Z} \to \mathbb{R} \to S^1 = GL(1, \mathbb{C}) \to 0$$

induces an exact sequence in Čech cohomology

$$\check{H}^1(B, \mathbb{Z}) \to \check{H}^1(B, \mathbb{R}) \to \check{H}^1(B; S^1) \xrightarrow{\delta} \check{H}^2(B; \mathbb{Z}).$$

Given a complex line bundle (principal $GL(1, \mathbb{C})$-bundle) $\pi : E \to B$ coming from the cocycle data $\phi \in \check{H}^1(B; GL(1, \mathbb{C}))$, let $c_1(E) = \delta(\phi)$. Compute $c_1(E)$ for some complex line bundle over S^2.