MATH 8230: Homework 2
09/22/2021

1. The *Spin* group:

   (a) Show that for each $n \geq 2$, the group $SO(n)$ has a unique connected 2-fold covering space. Call this space $Spin(n)$.

   (b) Use the group structure of $SO(n)$ to induce a group structure on $Spin(n)$.

   (c) What familiar space is $Spin(3)$? Also, show that as topological groups, $Spin(3) \cong SU(2)$.

2. For non-injective (or non-effective) structure groups G, we imposed an extra requirement that the lifts $\tilde{\phi}_{ij} \to G$ of the transition maps $\phi_{ij} : U_i \cap U_j \to Homeo(F, F)$ form a cocycle. Give an example of an oriented n-plane bundle with transition maps ϕ_{ij} which do lift to $Spin(n)$ but where the lifts cannot be chosen to satisfy the cocycle condition. (Hint: decompose the sphere as a union of two hemispheres and consider the Čech cocycle induced by a generator of $\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$, say. This corresponds to an oriented 3-plane bundle over S^2, and does not lift to a Spin-bundle. Further divide the sphere into open sets so that the intersection of each pair is contractible. Observe that you can lift the cocycle on each of these intersections, but that the lifts do not form a cocycle.)

3. More on existence of *Spin*-structures:

   (a) Show that there is an exact sequence of Čech cohomology groups

   $$\check{H}^1(X; \mathbb{Z}/2\mathbb{Z}) \to \check{H}^1(X, Spin(n)) \to \check{H}^1(X; SO(n)) \xrightarrow{\delta} \check{H}^2(X; \mathbb{Z}/2\mathbb{Z})$$

   (b) Given an $SO(n)$-bundle P coming from $\alpha \in \check{H}^1(X; SO(n))$, let $w_2(P) = \delta(\alpha)$. Deduce that P admits a lift to a $Spin(n)$-bundle if and only if $w_2(P) = 0$.

   (c) Compute w_2 for your example in the previous problem.

   (d) Extend the exact sequence from part (a) to an exact sequence

   $$0 \to \check{H}^1(X; \mathbb{Z}/2\mathbb{Z}) \to \check{H}^1(X, Spin(n)) \to \check{H}^1(X; SO(n)) \xrightarrow{\delta} \check{H}^2(X; \mathbb{Z}/2\mathbb{Z})$$

   and deduce that if an $SO(n)$-bundle E admits a $Spin(n)$-lift then the set of lifts form an affine space for $H^1(X, \mathbb{Z}/2\mathbb{Z})$.

4. (Segal, Proposition 2.1.)

   (a) Suppose that $F : \mathscr{C} \to \mathscr{D}$ is a functor. Show that there is an induced continuous map $\mathcal{N}F : \mathcal{N}\mathscr{C} \to \mathcal{N}\mathscr{D}$ of nerves.

   (b) Suppose that $F_0, F_1 : \mathscr{C} \to \mathscr{D}$ are functors and η is a natural transformation from F_0 to F_1. Show that $\mathcal{N}F_0$ and $\mathcal{N}F_1$ are homotopic. (Hint: think of η as a map from $\mathscr{C} \times (\bullet \to \bullet)$ to $\mathscr{D}$.)

   (c) Give another proof that if $\mathscr{C}$ has an initial (or terminal) object then $\mathcal{N}\mathscr{C}$ is contractible.

5. *Spin* structures again.

   (a) Let $f : G \to H$ be a continuous homomorphism of (reasonable) topological groups. Show that for appropriate choices of BG and BH there is an induced fibration $BG \to BH$. (Hint: choose any EG and EH and observe that G acts on $EG \times EH$, using f.)

   (b) Setting $G = Spin(n)$ and $H = SO(n)$, what is the fiber of the fibration $BSpin(n) \to BSO(n)$.

   (c) Show that this fibration is principal (see the section in Hatcher on Postnikov towers).

   (d) Give another construction of the obstruction w_2 to existence of a *Spin*-structure and another proof that the set of *Spin*-structures on an oriented vector bundle E with $w_2(E) = 0$ is in bijection with $H_1(B; \mathbb{Z}/2\mathbb{Z})$.