

# Theoretical Computer Science Cheat Sheet

| Definitions  |   | Series  |
|--|---|---|
| $f(n) = O(g(n))$   | iff $\exists$ positive $c, n_0$ such that $0 \leq f(n) \leq cg(n) \forall n \geq n_0$ .   | $\sum_{i=1}^n i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}.$  |
| $f(n) = \Omega(g(n))$  | iff $\exists$ positive $c, n_0$ such that $f(n) \geq cg(n) \geq 0 \forall n \geq n_0$ .   | In general:   |
| $f(n) = \Theta(g(n))$  | iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ .  | $\sum_{i=1}^n i^m = \frac{1}{m+1} \left[ (n+1)^{m+1} - 1 - \sum_{i=1}^n ((i+1)^{m+1} - i^{m+1} - (m+1)i^m) \right]$   |
| $f(n) = o(g(n))$   | iff $\lim_{n \rightarrow \infty} f(n)/g(n) = 0$ .   | $\sum_{i=1}^{n-1} i^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k n^{m+1-k}.$   |
| $\lim_{n \rightarrow \infty} a_n = a$  | iff $\forall \epsilon > 0, \exists n_0$ such that $ a_n - a  < \epsilon, \forall n \geq n_0$ .  | Geometric series:   |
| $\sup S$   | least $b \in \mathbb{R}$ such that $b \geq s, \forall s \in S$ .  | $\sum_{i=0}^n c^i = \frac{c^{n+1} - 1}{c - 1}, \quad c \neq 1, \quad \sum_{i=0}^{\infty} c^i = \frac{1}{1-c}, \quad \sum_{i=1}^{\infty} c^i = \frac{c}{1-c}, \quad  c  < 1,$  |
| $\inf S$   | greatest $b \in \mathbb{R}$ such that $b \leq s, \forall s \in S$ .   | $\sum_{i=0}^n ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \neq 1, \quad \sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad  c  < 1.$   |
| $\liminf_{n \rightarrow \infty} a_n$   | $\lim_{n \rightarrow \infty} \inf \{a_i \mid i \geq n, i \in \mathbb{N}\}.$   | Harmonic series:  |
| $\limsup_{n \rightarrow \infty} a_n$   | $\lim_{n \rightarrow \infty} \sup \{a_i \mid i \geq n, i \in \mathbb{N}\}.$   | $H_n = \sum_{i=1}^n \frac{1}{i}, \quad \sum_{i=1}^n iH_i = \frac{n(n+1)}{2} H_n - \frac{n(n-1)}{4}.$  |
| $\binom{n}{k}$   | Combinations: Size $k$ sub-sets of a size $n$ set.  | $\sum_{i=1}^n H_i = (n+1)H_n - n, \quad \sum_{i=1}^n \binom{i}{m} H_i = \binom{n+1}{m+1} \left( H_{n+1} - \frac{1}{m+1} \right).$   |
| $\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]$  | Stirling numbers (1st kind): Arrangements of an $n$ element set into $k$ cycles.  | 1. $\binom{n}{k} = \frac{n!}{(n-k)!k!}, \quad 2. \sum_{k=0}^n \binom{n}{k} = 2^n, \quad 3. \binom{n}{k} = \binom{n}{n-k},$  |
| $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$  | Stirling numbers (2nd kind): Partitions of an $n$ element set into $k$ non-empty sets.  | 4. $\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}, \quad 5. \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$  |
| $\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle$   | 1st order Eulerian numbers: Permutations $\pi_1 \pi_2 \dots \pi_n$ on $\{1, 2, \dots, n\}$ with $k$ ascents.  | 6. $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}, \quad 7. \sum_{k=0}^n \binom{r+k}{k} = \binom{r+n+1}{n},$  |
| $\langle\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle\rangle$   | 2nd order Eulerian numbers.   | 8. $\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1}, \quad 9. \sum_{k=0}^n \binom{r}{k} \binom{s}{n-k} = \binom{r+s}{n},$  |
| $C_n$  | Catalan Numbers: Binary trees with $n+1$ vertices.  | 10. $\binom{n}{k} = (-1)^k \binom{k-n-1}{k}, \quad 11. \left\{ \begin{smallmatrix} n \\ 1 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\} = 1,$                          |
| 14. $\left[ \begin{smallmatrix} n \\ 1 \end{smallmatrix} \right] = (n-1)!,$  | 15. $\left[ \begin{smallmatrix} n \\ 2 \end{smallmatrix} \right] = (n-1)!H_{n-1},$  | 16. $\left[ \begin{smallmatrix} n \\ n \end{smallmatrix} \right] = 1, \quad 17. \left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right] \geq \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\},$    |
| 18. $\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right] = (n-1) \left[ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right] + \left[ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right],$   | 19. $\left\{ \begin{smallmatrix} n \\ n-1 \end{smallmatrix} \right\} = \left[ \begin{smallmatrix} n \\ n-1 \end{smallmatrix} \right] = \binom{n}{2},$   | 20. $\sum_{k=0}^n \left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right] = n!, \quad 21. C_n = \frac{1}{n+1} \binom{2n}{n},$   |
| 22. $\langle \begin{smallmatrix} n \\ 0 \end{smallmatrix} \rangle = \langle \begin{smallmatrix} n \\ n-1 \end{smallmatrix} \rangle = 1,$   | 23. $\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle = \langle \begin{smallmatrix} n \\ n-1-k \end{smallmatrix} \rangle,$  | 24. $\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle = (k+1) \langle \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \rangle + (n-k) \langle \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \rangle,$ |
| 25. $\langle \begin{smallmatrix} 0 \\ k \end{smallmatrix} \rangle = \begin{cases} 1 & \text{if } k=0, \\ 0 & \text{otherwise} \end{cases}$   | 26. $\langle \begin{smallmatrix} n \\ 1 \end{smallmatrix} \rangle = 2^n - n - 1,$   | 27. $\langle \begin{smallmatrix} n \\ 2 \end{smallmatrix} \rangle = 3^n - (n+1)2^n + \binom{n+1}{2},$   |
| 28. $x^n = \sum_{k=0}^n \langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle \binom{x+k}{n},$  | 29. $\langle \begin{smallmatrix} n \\ m \end{smallmatrix} \rangle = \sum_{k=0}^m \binom{n+1}{k} (m+1-k)^n (-1)^k,$  | 30. $m! \left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\} = \sum_{k=0}^n \langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle \binom{k}{n-m},$  |
| 31. $\langle \begin{smallmatrix} n \\ m \end{smallmatrix} \rangle = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} \binom{n-k}{m} (-1)^{n-k-m} k!,$  | 32. $\langle\langle \begin{smallmatrix} n \\ 0 \end{smallmatrix} \rangle\rangle = 1,$   | 33. $\langle\langle \begin{smallmatrix} n \\ n \end{smallmatrix} \rangle\rangle = 0 \quad \text{for } n \neq 0,$  |
| 34. $\langle\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle\rangle = (k+1) \langle\langle \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \rangle\rangle + (2n-1-k) \langle\langle \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \rangle\rangle,$ | 35. $\sum_{k=0}^n \langle\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle\rangle = \frac{(2n)n}{2^n},$  |   |
| 36. $\left\{ \begin{smallmatrix} x \\ x-n \end{smallmatrix} \right\} = \sum_{k=0}^n \langle\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle\rangle \binom{x+n-1-k}{2n},$   | 37. $\left\{ \begin{smallmatrix} n+1 \\ m+1 \end{smallmatrix} \right\} = \sum_k \binom{n}{k} \left\{ \begin{smallmatrix} k \\ m \end{smallmatrix} \right\} = \sum_{k=0}^n \left\{ \begin{smallmatrix} k \\ m \end{smallmatrix} \right\} (m+1)^{n-k},$ |   |

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| Identities Cont. |  |     | Trees  |  |
|------------------|--|-----|--|--|
| 38.              | $\begin{bmatrix} n+1 \\ m+1 \end{bmatrix} = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ m \end{bmatrix} = \sum_{k=0}^n \begin{bmatrix} k \\ m \end{bmatrix} n^{n-k} = n! \sum_{k=0}^n \frac{1}{k!} \begin{bmatrix} k \\ m \end{bmatrix},$ | 39. | $\begin{bmatrix} x \\ x-n \end{bmatrix} = \sum_{k=0}^n \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle \begin{bmatrix} x+k \\ 2n \end{bmatrix},$         | Every tree with $n$ vertices has $n-1$ edges.<br><br>Kraft inequality: If the depths of the leaves of a binary tree are $d_1, \dots, d_n$ :<br>$\sum_{i=1}^n 2^{-d_i} \leq 1,$<br>and equality holds only if every internal node has 2 sons. |
| 40.              | $\left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \sum_k \binom{n}{k} \left\{ \begin{matrix} k+1 \\ m+1 \end{matrix} \right\} (-1)^{n-k},$  | 41. | $\begin{bmatrix} n \\ m \end{bmatrix} = \sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \begin{bmatrix} k \\ m \end{bmatrix} (-1)^{m-k},$                              |  |
| 42.              | $\left\{ \begin{matrix} m+n+1 \\ m \end{matrix} \right\} = \sum_{k=0}^m k \left\{ \begin{matrix} n+k \\ k \end{matrix} \right\},$  | 43. | $\begin{bmatrix} m+n+1 \\ m \end{bmatrix} = \sum_{k=0}^m k(n+k) \begin{bmatrix} n+k \\ k \end{bmatrix},$   |  |
| 44.              | $\binom{n}{m} = \sum_k \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\} \begin{bmatrix} k \\ m \end{bmatrix} (-1)^{m-k},$   | 45. | $(n-m)! \binom{n}{m} = \sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \left\{ \begin{matrix} k \\ m \end{matrix} \right\} (-1)^{m-k}, \quad \text{for } n \geq m,$    |  |
| 46.              | $\left\{ \begin{matrix} n \\ n-m \end{matrix} \right\} = \sum_k \binom{m-n}{m+k} \binom{m+n}{n+k} \begin{bmatrix} m+k \\ k \end{bmatrix},$   | 47. | $\begin{bmatrix} n \\ n-m \end{bmatrix} = \sum_k \binom{m-n}{m+k} \binom{m+n}{n+k} \left\{ \begin{matrix} m+k \\ k \end{matrix} \right\},$                             |  |
| 48.              | $\left\{ \begin{matrix} n \\ \ell+m \end{matrix} \right\} \binom{\ell+m}{\ell} = \sum_k \left\{ \begin{matrix} k \\ \ell \end{matrix} \right\} \left\{ \begin{matrix} n-k \\ m \end{matrix} \right\} \binom{n}{k},$  | 49. | $\begin{bmatrix} n \\ \ell+m \end{bmatrix} \binom{\ell+m}{\ell} = \sum_k \begin{bmatrix} k \\ \ell \end{bmatrix} \begin{bmatrix} n-k \\ m \end{bmatrix} \binom{n}{k}.$ |  |

| Recurrences  |  |   |
|--|--|---|
| <p>Master method:</p> $T(n) = aT(n/b) + f(n), \quad a \geq 1, b > 1$ <p>If <math>\exists \epsilon &gt; 0</math> such that <math>f(n) = O(n^{\log_b a - \epsilon})</math> then</p> $T(n) = \Theta(n^{\log_b a}).$ <p>If <math>f(n) = \Theta(n^{\log_b a})</math> then</p> $T(n) = \Theta(n^{\log_b a} \log_2 n).$ <p>If <math>\exists \epsilon &gt; 0</math> such that <math>f(n) = \Omega(n^{\log_b a + \epsilon})</math>, and <math>\exists c &lt; 1</math> such that <math>af(n/b) \leq cf(n)</math> for large <math>n</math>, then</p> $T(n) = \Theta(f(n)).$ <p>Substitution (example): Consider the following recurrence</p> $T_{i+1} = 2^{2^i} \cdot T_i^2, \quad T_1 = 2.$ <p>Note that <math>T_i</math> is always a power of two. Let <math>t_i = \log_2 T_i</math>. Then we have</p> $t_{i+1} = 2^i + 2t_i, \quad t_1 = 1.$ <p>Let <math>u_i = t_i/2^i</math>. Dividing both sides of the previous equation by <math>2^{i+1}</math> we get</p> $\frac{t_{i+1}}{2^{i+1}} = \frac{2^i}{2^{i+1}} + \frac{t_i}{2^i}.$ <p>Substituting we find</p> $u_{i+1} = \frac{1}{2} + u_i, \quad u_1 = \frac{1}{2},$ <p>which is simply <math>u_i = i/2</math>. So we find that <math>T_i</math> has the closed form <math>T_i = 2^{i2^{i-1}}</math>.</p> <p>Summing factors (example): Consider the following recurrence</p> $T(n) = 3T(n/2) + n, \quad T(1) = 1.$ <p>Rewrite so that all terms involving <math>T</math> are on the left side</p> $T(n) - 3T(n/2) = n.$ <p>Now expand the recurrence, and choose a factor which makes the left side “telescope”</p> | $1(T(n) - 3T(n/2)) = n$ $3(T(n/2) - 3T(n/4)) = n/2$ $\vdots \quad \vdots \quad \vdots$ $3^{\log_2 n - 1}(T(2) - 3T(1)) = 2$ <p>Let <math>m = \log_2 n</math>. Summing the left side we get <math>T(n) - 3^m T(1) = T(n) - 3^m = T(n) - n^k</math> where <math>k = \log_2 3 \approx 1.58496</math>. Summing the right side we get</p> $\sum_{i=0}^{m-1} \frac{n}{2^i} 3^i = n \sum_{i=0}^{m-1} \left(\frac{3}{2}\right)^i.$ <p>Let <math>c = \frac{3}{2}</math>. Then we have</p> $n \sum_{i=0}^{m-1} c^i = n \left( \frac{c^m - 1}{c - 1} \right)$ $= 2n(c^{\log_2 n} - 1)$ $= 2n(c^{(\log_2 n) \log_2 c} - 1)$ $= 2n^k - 2n,$ <p>and so <math>T(n) = 3n^k - 2n</math>. Full history recurrences can often be changed to limited history ones (example): Consider</p> $T_i = 1 + \sum_{j=0}^{i-1} T_j, \quad T_0 = 1.$ <p>Note that</p> $T_{i+1} = 1 + \sum_{j=0}^i T_j.$ <p>Subtracting we find</p> $T_{i+1} - T_i = 1 + \sum_{j=0}^i T_j - 1 - \sum_{j=0}^{i-1} T_j$ $= T_i.$ <p>And so <math>T_{i+1} = 2T_i = 2^{i+1}</math>.</p> | <p>Generating functions:</p> <ol style="list-style-type: none"> <li>Multiply both sides of the equation by <math>x^i</math>.</li> <li>Sum both sides over all <math>i</math> for which the equation is valid.</li> <li>Choose a generating function <math>G(x)</math>. Usually <math>G(x) = \sum_{i=0}^{\infty} x^i g_i</math>.</li> <li>Rewrite the equation in terms of the generating function <math>G(x)</math>.</li> <li>Solve for <math>G(x)</math>.</li> <li>The coefficient of <math>x^i</math> in <math>G(x)</math> is <math>g_i</math>.</li> </ol> <p>Example:</p> $g_{i+1} = 2g_i + 1, \quad g_0 = 0.$ <p>Multiply and sum:</p> $\sum_{i \geq 0} g_{i+1} x^i = \sum_{i \geq 0} 2g_i x^i + \sum_{i \geq 0} x^i.$ <p>We choose <math>G(x) = \sum_{i \geq 0} x^i g_i</math>. Rewrite in terms of <math>G(x)</math>:</p> $\frac{G(x) - g_0}{x} = 2G(x) + \sum_{i \geq 0} x^i.$ <p>Simplify:</p> $\frac{G(x)}{x} = 2G(x) + \frac{1}{1-x}.$ <p>Solve for <math>G(x)</math>:</p> $G(x) = \frac{x}{(1-x)(1-2x)}.$ <p>Expand this using partial fractions:</p> $G(x) = x \left( \frac{2}{1-2x} - \frac{1}{1-x} \right)$ $= x \left( 2 \sum_{i \geq 0} 2^i x^i - \sum_{i \geq 0} x^i \right)$ $= \sum_{i \geq 0} (2^{i+1} - 1) x^{i+1}.$ <p>So <math>g_i = 2^i - 1</math>.</p> |

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$$\pi \approx 3.14159,$$

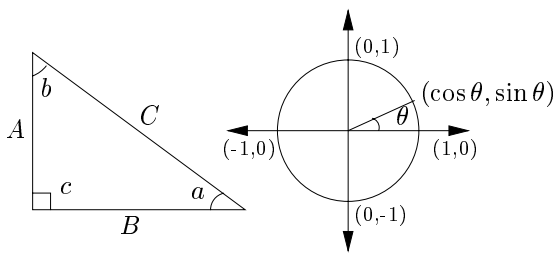
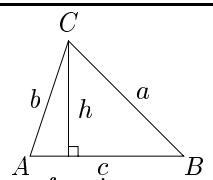
$$e \approx 2.71828,$$

$$\gamma \approx 0.57721,$$

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803,$$

$$\hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -.61803$$

| $i$                                 | $2^i$         | $p_i$ | General  | Probability  |
|-------------------------------------|---------------|-------|--|--|
| 1                                   | 2             | 2     | Bernoulli Numbers ( $B_i = 0$ , odd $i \neq 1$ ):  | Continuous distributions: If   |
| 2                                   | 4             | 3     | $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30},$   | $\Pr[a < X < b] = \int_a^b p(x) dx,$   |
| 3                                   | 8             | 5     | $B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}.$  | then $p$ is the probability density function of $X$ . If                                       |
| 4                                   | 16            | 7     | Change of base, quadratic formula:   | $\Pr[X < a] = P(a),$   |
| 5                                   | 32            | 11    | $\log_b x = \frac{\log_a x}{\log_a b}, \quad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$  | then $P$ is the distribution function of $X$ . If $P$ and $p$ both exist then                  |
| 6                                   | 64            | 13    | Euler's number $e$ :   | $P(a) = \int_{-\infty}^a p(x) dx.$   |
| 7                                   | 128           | 17    | $e = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \dots$   | Expectation: If $X$ is discrete  |
| 8                                   | 256           | 19    | $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$  | $\mathbb{E}[g(X)] = \sum_x g(x) \Pr[X = x].$   |
| 9                                   | 512           | 23    | $\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}.$   | If $X$ continuous then   |
| 10                                  | 1,024         | 29    | $\left(1 + \frac{1}{n}\right)^n = e - \frac{e}{2n} + \frac{11e}{24n^2} - O\left(\frac{1}{n^3}\right).$   | $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)p(x) dx = \int_{-\infty}^{\infty} g(x) dP(x).$ |
| 11                                  | 2,048         | 31    | Harmonic numbers:  | Variance, standard deviation:  |
| 12                                  | 4,096         | 37    | $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \frac{49}{20}, \frac{363}{140}, \frac{761}{280}, \frac{7129}{2520}, \dots$   | $\text{VAR}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2,$   |
| 13                                  | 8,192         | 41    | $\ln n < H_n < \ln n + 1,$   | $\sigma = \sqrt{\text{VAR}[X]}.$   |
| 14                                  | 16,384        | 43    | $H_n = \ln n + \gamma + O\left(\frac{1}{n}\right).$  | For events $A$ and $B$ :   |
| 15                                  | 32,768        | 47    | Factorial, Stirling's approximation:   | $\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]$  |
| 16                                  | 65,536        | 53    | $1, 2, 6, 24, 120, 720, 5040, 40320, 362880, \dots$  | $\Pr[A \wedge B] = \Pr[A] \cdot \Pr[B],$   |
| 17                                  | 131,072       | 59    | $n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right).$   | iff $A$ and $B$ are independent.   |
| 18                                  | 262,144       | 61    | Ackermann's function and inverse:  | $\Pr[A B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$  |
| 19                                  | 524,288       | 67    | $a(i, j) = \begin{cases} 2^j & i = 1 \\ a(i-1, 2) & j = 1 \\ a(i-1, a(i, j-1)) & i, j \geq 2 \end{cases}$  | For random variables $X$ and $Y$ :   |
| 20                                  | 1,048,576     | 71    | $\alpha(i) = \min\{j \mid a(j, j) \geq i\}.$   | $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y],$                                   |
| 21                                  | 2,097,152     | 73    |  | if $X$ and $Y$ are independent.  |
| 22                                  | 4,194,304     | 79    |  | $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y],$   |
| 23                                  | 8,388,608     | 83    |  | $\mathbb{E}[cX] = c \mathbb{E}[X].$  |
| 24                                  | 16,777,216    | 89    | Binomial distribution:   | Bayes' theorem:  |
| 25                                  | 33,554,432    | 97    | $\Pr[X = k] = \binom{n}{k} p^k q^{n-k}, \quad q = 1 - p,$  | $\Pr[A_i B] = \frac{\Pr[B A_i] \Pr[A_i]}{\sum_{j=1}^n \Pr[A_j] \Pr[B A_j]}.$                   |
| 26                                  | 67,108,864    | 101   | $\mathbb{E}[X] = \sum_{k=1}^n k \binom{n}{k} p^k q^{n-k} = np.$  | Inclusion-exclusion:   |
| 27                                  | 134,217,728   | 103   | Poisson distribution:  | $\Pr\left[\bigvee_{i=1}^n X_i\right] = \sum_{i=1}^n \Pr[X_i] +$                                |
| 28                                  | 268,435,456   | 107   | $\Pr[X = k] = \frac{e^{-\lambda} \lambda^k}{k!}, \quad \mathbb{E}[X] = \lambda.$   | $\sum_{k=2}^n (-1)^{k+1} \sum_{i_1 < \dots < i_k} \Pr\left[\bigwedge_{j=1}^k X_{i_j}\right].$  |
| 29                                  | 536,870,912   | 109   | Normal (Gaussian) distribution:  | Moment inequalities:   |
| 30                                  | 1,073,741,824 | 113   | $p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad \mathbb{E}[X] = \mu.$  | $\Pr[ X  \geq \lambda \mathbb{E}[X]] \leq \frac{1}{\lambda},$                                  |
| 31                                  | 2,147,483,648 | 127   | The "coupon collector": We are given a random coupon each day, and there are $n$ different types of coupons. The distribution of coupons is uniform. The expected number of days to pass before we to collect all $n$ types is | $\Pr[ X - \mathbb{E}[X]  \geq \lambda \cdot \sigma] \leq \frac{1}{\lambda^2}.$                 |
| 32                                  | 4,294,967,296 | 131   | $nH_n.$  | Geometric distribution:  |
| Pascal's Triangle                   |               |       |  | $\Pr[X = k] = q^{k-1}p, \quad q = 1 - p,$  |
| 1                                   |               |       |  | $\mathbb{E}[X] = \sum_{k=1}^{\infty} k p q^{k-1} = \frac{1}{p}.$                               |
| 1 1                                 |               |       |  |  |
| 1 2 1                               |               |       |  |  |
| 1 3 3 1                             |               |       |  |  |
| 1 4 6 4 1                           |               |       |  |  |
| 1 5 10 10 5 1                       |               |       |  |  |
| 1 6 15 20 15 6 1                    |               |       |  |  |
| 1 7 21 35 35 21 7 1                 |               |       |  |  |
| 1 8 28 56 70 56 28 8 1              |               |       |  |  |
| 1 9 36 84 126 126 84 36 9 1         |               |       |  |  |
| 1 10 45 120 210 252 210 120 45 10 1 |               |       |  |  |

| Trigonometry  | Matrices   | More Trig.  |                      |               |               |   |   |   |   |                 |               |                      |                      |                 |                      |                      |   |                 |                      |               |            |                 |   |   |          |   |
|---|--|---|----------------------|---------------|---------------|---|---|---|---|-----------------|---------------|----------------------|----------------------|-----------------|----------------------|----------------------|---|-----------------|----------------------|---------------|------------|-----------------|---|---|----------|---|
| <div></div> <p>Pythagorean theorem:<br/><math>C^2 = A^2 + B^2</math>.</p> <p>Definitions:</p> $\sin a = A/C, \quad \cos a = B/C,$ $\csc a = C/A, \quad \sec a = C/B,$ $\tan a = \frac{\sin a}{\cos a} = \frac{A}{B}, \quad \cot a = \frac{\cos a}{\sin a} = \frac{B}{A}.$ <p>Area, radius of inscribed circle:</p> $\frac{1}{2}AB, \quad \frac{AB}{A+B+C}.$ <p>Identities:</p> $\sin x = \frac{1}{\csc x}, \quad \cos x = \frac{1}{\sec x},$ $\tan x = \frac{1}{\cot x}, \quad \sin^2 x + \cos^2 x = 1,$ $1 + \tan^2 x = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x,$ $\sin x = \cos\left(\frac{\pi}{2} - x\right), \quad \sin x = \sin(\pi - x),$ $\cos x = -\cos(\pi - x), \quad \tan x = \cot\left(\frac{\pi}{2} - x\right),$ $\cot x = -\cot(\pi - x), \quad \csc x = \cot \frac{\pi}{2} - \cot x,$ $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$ $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$ $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$ $\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y},$ $\sin 2x = 2 \sin x \cos x, \quad \sin 2x = \frac{2 \tan x}{1 + \tan^2 x},$ $\cos 2x = \cos^2 x - \sin^2 x, \quad \cos 2x = 2 \cos^2 x - 1,$ $\cos 2x = 1 - 2 \sin^2 x, \quad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x},$ $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}, \quad \cot 2x = \frac{\cot^2 x - 1}{2 \cot x},$ $\sin(x+y) \sin(x-y) = \sin^2 x - \sin^2 y,$ $\cos(x+y) \cos(x-y) = \cos^2 x - \sin^2 y.$ <p>Euler's equation:</p> $e^{ix} = \cos x + i \sin x, \quad e^{i\pi} = -1.$ | <p>Multiplication:</p> $C = A \cdot B, \quad c_{i,j} = \sum_{k=1}^n a_{i,k} b_{k,j}.$ <p>Determinants: <math>\det A \neq 0</math> iff <math>A</math> is non-singular.</p> $\det A \cdot B = \det A \cdot \det B,$ $\det A = \sum_{\pi} \prod_{i=1}^n \text{sign}(\pi) a_{i,\pi(i)}.$ <p><math>2 \times 2</math> and <math>3 \times 3</math> determinant:</p> $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$ $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = g \begin{vmatrix} b & c \\ e & f \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix} + i \begin{vmatrix} a & b \\ d & e \end{vmatrix}$ $= aei + bfg + cdh - ceg - fha - ibd.$ <p>Permanents:</p> $\text{perm } A = \sum_{\pi} \prod_{i=1}^n a_{i,\pi(i)}.$  | <div></div> <p>Law of cosines:<br/><math>c^2 = a^2 + b^2 - 2ab \cos C</math>.</p> <p>Area:</p> $A = \frac{1}{2}hc,$ $= \frac{1}{2}ab \sin C,$ $= \frac{c^2 \sin A \sin B}{2 \sin C}.$ <p>Heron's formula:</p> $A = \sqrt{s \cdot s_a \cdot s_b \cdot s_c},$ $s = \frac{1}{2}(a + b + c),$ $s_a = s - a,$ $s_b = s - b,$ $s_c = s - c.$ <p>More identities:</p> $\sin \frac{x}{2} = \sqrt{\frac{1 - \cos x}{2}},$ $\cos \frac{x}{2} = \sqrt{\frac{1 + \cos x}{2}},$ $\tan \frac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}},$ $= \frac{1 - \cos x}{\sin x},$ $= \frac{\sin x}{1 + \cos x},$ $\cot \frac{x}{2} = \sqrt{\frac{1 + \cos x}{1 - \cos x}},$ $= \frac{1 + \cos x}{\sin x},$ $= \frac{\sin x}{1 - \cos x},$ $\sin x = \frac{e^{ix} - e^{-ix}}{2i},$ $\cos x = \frac{e^{ix} + e^{-ix}}{2},$ $\tan x = -i \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}},$ $= -i \frac{e^{2ix} - 1}{e^{2ix} + 1},$ $\sin x = \frac{\sinh ix}{i},$ $\cos x = \cosh ix,$ $\tan x = \frac{\tanh ix}{i}.$ |                      |               |               |   |   |   |   |                 |               |                      |                      |                 |                      |                      |   |                 |                      |               |            |                 |   |   |          |   |
|   | <p align="center"><b>Hyperbolic Functions</b></p> <p>Definitions:</p> $\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2},$ $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \text{csch } x = \frac{1}{\sinh x},$ $\text{sech } x = \frac{1}{\cosh x}, \quad \coth x = \frac{1}{\tanh x}.$ <p>Identities:</p> $\cosh^2 x - \sinh^2 x = 1, \quad \tanh^2 x + \text{sech}^2 x = 1,$ $\coth^2 x - \text{csch}^2 x = 1, \quad \sinh(-x) = -\sinh x,$ $\cosh(-x) = \cosh x, \quad \tanh(-x) = -\tanh x,$ $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$ $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$ $\sinh 2x = 2 \sinh x \cosh x,$ $\cosh 2x = \cosh^2 x + \sinh^2 x,$ $\cosh x + \sinh x = e^x, \quad \cosh x - \sinh x = e^{-x},$ $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx, \quad n \in \mathbb{Z},$ $2 \sinh^2 \frac{x}{2} = \cosh x - 1, \quad 2 \cosh^2 \frac{x}{2} = \cosh x + 1.$ |   |                      |               |               |   |   |   |   |                 |               |                      |                      |                 |                      |                      |   |                 |                      |               |            |                 |   |   |          |   |
|   | <table><tr><th><math>\theta</math></th><th><math>\sin \theta</math></th><th><math>\cos \theta</math></th><th><math>\tan \theta</math></th></tr><tr><td>0</td><td>0</td><td>1</td><td>0</td></tr><tr><td><math>\frac{\pi}{6}</math></td><td><math>\frac{1}{2}</math></td><td><math>\frac{\sqrt{3}}{2}</math></td><td><math>\frac{\sqrt{3}}{3}</math></td></tr><tr><td><math>\frac{\pi}{4}</math></td><td><math>\frac{\sqrt{2}}{2}</math></td><td><math>\frac{\sqrt{2}}{2}</math></td><td>1</td></tr><tr><td><math>\frac{\pi}{3}</math></td><td><math>\frac{\sqrt{3}}{2}</math></td><td><math>\frac{1}{2}</math></td><td><math>\sqrt{3}</math></td></tr><tr><td><math>\frac{\pi}{2}</math></td><td>1</td><td>0</td><td><math>\infty</math></td></tr></table>   | $\theta$  | $\sin \theta$        | $\cos \theta$ | $\tan \theta$ | 0 | 0 | 1 | 0 | $\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{3}$ | $\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 | $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ | $\frac{\pi}{2}$ | 1 | 0 | $\infty$ | <p>... in mathematics you don't understand things, you just get used to them.</p> <p align="right">- J. von Neumann</p> |
| $\theta$  | $\sin \theta$  | $\cos \theta$   | $\tan \theta$        |               |               |   |   |   |   |                 |               |                      |                      |                 |                      |                      |   |                 |                      |               |            |                 |   |   |          |   |
| 0   | 0  | 1   | 0                    |               |               |   |   |   |   |                 |               |                      |                      |                 |                      |                      |   |                 |                      |               |            |                 |   |   |          |   |
| $\frac{\pi}{6}$   | $\frac{1}{2}$  | $\frac{\sqrt{3}}{2}$  | $\frac{\sqrt{3}}{3}$ |               |               |   |   |   |   |                 |               |                      |                      |                 |                      |                      |   |                 |                      |               |            |                 |   |   |          |   |
| $\frac{\pi}{4}$   | $\frac{\sqrt{2}}{2}$   | $\frac{\sqrt{2}}{2}$  | 1                    |               |               |   |   |   |   |                 |               |                      |                      |                 |                      |                      |   |                 |                      |               |            |                 |   |   |          |   |
| $\frac{\pi}{3}$   | $\frac{\sqrt{3}}{2}$   | $\frac{1}{2}$   | $\sqrt{3}$           |               |               |   |   |   |   |                 |               |                      |                      |                 |                      |                      |   |                 |                      |               |            |                 |   |   |          |   |
| $\frac{\pi}{2}$   | 1  | 0   | $\infty$             |               |               |   |   |   |   |                 |               |                      |                      |                 |                      |                      |   |                 |                      |               |            |                 |   |   |          |   |

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# Theoretical Computer Science Cheat Sheet

## Number Theory

The Chinese remainder theorem: There exists a number  $C$  such that:

$$C \equiv r_1 \pmod{m_1}$$

$$\vdots \quad \vdots \quad \vdots$$

$$C \equiv r_n \pmod{m_n}$$

if  $m_i$  and  $m_j$  are relatively prime for  $i \neq j$ .

Euler's function:  $\phi(x)$  is the number of positive integers less than  $x$  relatively prime to  $x$ . If  $\prod_{i=1}^n p_i^{e_i}$  is the prime factorization of  $x$  then

$$\phi(x) = \prod_{i=1}^n p_i^{e_i-1} (p_i - 1).$$

Euler's theorem: If  $a$  and  $b$  are relatively prime then

$$1 \equiv a^{\phi(b)} \pmod{b}.$$

Fermat's theorem:

$$1 \equiv a^{p-1} \pmod{p}.$$

The Euclidean algorithm: if  $a > b$  are integers then

$$\gcd(a, b) = \gcd(a \bmod b, b).$$

If  $\prod_{i=1}^n p_i^{e_i}$  is the prime factorization of  $x$  then

$$S(x) = \sum_{d|x} d = \prod_{i=1}^n \frac{p_i^{e_i+1} - 1}{p_i - 1}.$$

Perfect Numbers:  $x$  is an even perfect number iff  $x = 2^{n-1}(2^n - 1)$  and  $2^n - 1$  is prime.

Wilson's theorem:  $n$  is a prime iff

$$(n-1)! \equiv -1 \pmod{n}.$$

Möbius inversion:

$$\mu(i) = \begin{cases} 1 & \text{if } i = 1. \\ 0 & \text{if } i \text{ is not square-free.} \\ (-1)^r & \text{if } i \text{ is the product of } r \text{ distinct primes.} \end{cases}$$

If

$$G(a) = \sum_{d|a} F(d),$$

then

$$F(a) = \sum_{d|a} \mu(d) G\left(\frac{a}{d}\right).$$

Prime numbers:

$$p_n = n \ln n + n \ln \ln n - n + n \frac{\ln \ln n}{\ln n}$$

$$+ O\left(\frac{n}{\ln n}\right),$$

$$\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!n}{(\ln n)^3}$$

$$+ O\left(\frac{n}{(\ln n)^4}\right).$$

## Graph Theory

Definitions:

*Loop* An edge connecting a vertex to itself.

*Directed* Each edge has a direction.

*Simple* Graph with no loops or multi-edges.

*Walk* A sequence  $v_0 e_1 v_1 \dots e_\ell v_\ell$ .

*Trail* A walk with distinct edges.

*Path* A trail with distinct vertices.

*Connected* A graph where there exists a path between any two vertices.

*Component* A maximal connected subgraph.

*Tree* A connected acyclic graph.

*Free tree* A tree with no root.

*DAG* Directed acyclic graph.

*Eulerian* Graph with a trail visiting each edge exactly once.

*Hamiltonian* Graph with a cycle visiting each vertex exactly once.

*Cut* A set of edges whose removal increases the number of components.

*Cut-set* A minimal cut.

*Cut edge* A size 1 cut.

*k-Connected* A graph connected with the removal of any  $k-1$  vertices.

*k-Tough*  $\forall S \subseteq V, S \neq \emptyset$  we have  $k \cdot c(G-S) \leq |S|$ .

*k-Regular* A graph where all vertices have degree  $k$ .

*k-Factor* A  $k$ -regular spanning subgraph.

*Matching* A set of edges, no two of which are adjacent.

*Clique* A set of vertices, all of which are adjacent.

*Ind. set* A set of vertices, none of which are adjacent.

*Vertex cover* A set of vertices which cover all edges.

*Planar graph* A graph which can be embedded in the plane.

*Plane graph* An embedding of a planar graph.

$$\sum_{v \in V} \deg(v) = 2m.$$

If  $G$  is planar then  $n - m + f = 2$ , so

$$f \leq 2n - 4, \quad m \leq 3n - 6.$$

Any planar graph has a vertex with degree  $\leq 5$ .

Notation:

$E(G)$  Edge set

$V(G)$  Vertex set

$c(G)$  Number of components

$G[S]$  Induced subgraph

$\deg(v)$  Degree of  $v$

$\Delta(G)$  Maximum degree

$\delta(G)$  Minimum degree

$\chi(G)$  Chromatic number

$\chi_E(G)$  Edge chromatic number

$G^c$  Complement graph

$K_n$  Complete graph

$K_{n_1, n_2}$  Complete bipartite graph

$r(k, \ell)$  Ramsey number

## Geometry

Projective coordinates: triples  $(x, y, z)$ , not all  $x, y$  and  $z$  zero.

$$(x, y, z) = (cx, cy, cz) \quad \forall c \neq 0.$$

Cartesian Projective

$$(x, y) \quad (x, y, 1)$$

$$y = mx + b \quad (m, -1, b)$$

$$x = c \quad (1, 0, -c)$$

Distance formula,  $L_p$  and  $L_\infty$  metric:

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$$

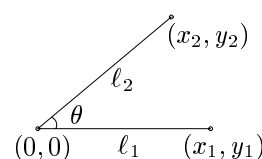
$$[|x_1 - x_0|^p + |y_1 - y_0|^p]^{1/p},$$

$$\lim_{p \rightarrow \infty} [|x_1 - x_0|^p + |y_1 - y_0|^p]^{1/p}.$$

Area of triangle  $(x_0, y_0)$ ,  $(x_1, y_1)$  and  $(x_2, y_2)$ :

$$\frac{1}{2} \text{abs} \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}.$$

Angle formed by three points:



$$\cos \theta = \frac{(x_1, y_1) \cdot (x_2, y_2)}{l_1 l_2}.$$

Line through two points  $(x_0, y_0)$  and  $(x_1, y_1)$ :

$$\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$$

Area of circle, volume of sphere:

$$A = \pi r^2, \quad V = \frac{4}{3} \pi r^3.$$

If I have seen farther than others, it is because I have stood on the shoulders of giants.

– Issac Newton

$\pi$

Wallis' identity:

$$\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

Brouncker's continued fraction expansion:

$$\frac{\pi}{4} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{2 + \cdots}}}}$$

Gregory's series:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

Newton's series:

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots$$

Sharp's series:

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} \left( 1 - \frac{1}{3^1 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \cdots \right)$$

Euler's series:

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots$$

### Partial Fractions

Let  $N(x)$  and  $D(x)$  be polynomial functions of  $x$ . We can break down  $N(x)/D(x)$  using partial fraction expansion. First, if the degree of  $N$  is greater than or equal to the degree of  $D$ , divide  $N$  by  $D$ , obtaining

$$\frac{N(x)}{D(x)} = Q(x) + \frac{N'(x)}{D(x)},$$

where the degree of  $N'$  is less than that of  $D$ . Second, factor  $D(x)$ . Use the following rules: For a non-repeated factor:

$$\frac{N(x)}{(x-a)D(x)} = \frac{A}{x-a} + \frac{N'(x)}{D(x)},$$

where

$$A = \left[ \frac{N(x)}{D(x)} \right]_{x=a}.$$

For a repeated factor:

$$\frac{N(x)}{(x-a)^m D(x)} = \sum_{k=0}^{m-1} \frac{A_k}{(x-a)^{m-k}} + \frac{N'(x)}{D(x)},$$

where

$$A_k = \frac{1}{k!} \left[ \frac{d^k}{dx^k} \left( \frac{N(x)}{D(x)} \right) \right]_{x=a}.$$

The reasonable man adapts himself to the world; the unreasonable persists in trying to adapt the world to himself. Therefore all progress depends on the unreasonable.  
– George Bernard Shaw

### Calculus

Derivatives:

$$1. \frac{d(cu)}{dx} = c \frac{du}{dx}, \quad 2. \frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}, \quad 3. \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx},$$

$$4. \frac{d(u^n)}{dx} = nu^{n-1} \frac{du}{dx}, \quad 5. \frac{d(u/v)}{dx} = \frac{v \left( \frac{du}{dx} \right) - u \left( \frac{dv}{dx} \right)}{v^2}, \quad 6. \frac{d(e^{cu})}{dx} = ce^{cu} \frac{du}{dx},$$

$$7. \frac{d(c^u)}{dx} = (\ln c) c^u \frac{du}{dx}, \quad 8. \frac{d(\ln u)}{dx} = \frac{1}{u} \frac{du}{dx},$$

$$9. \frac{d(\sin u)}{dx} = \cos u \frac{du}{dx}, \quad 10. \frac{d(\cos u)}{dx} = -\sin u \frac{du}{dx},$$

$$11. \frac{d(\tan u)}{dx} = \sec^2 u \frac{du}{dx}, \quad 12. \frac{d(\cot u)}{dx} = -\csc^2 u \frac{du}{dx},$$

$$13. \frac{d(\sec u)}{dx} = \tan u \sec u \frac{du}{dx}, \quad 14. \frac{d(\csc u)}{dx} = -\cot u \csc u \frac{du}{dx},$$

$$15. \frac{d(\arcsin u)}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad 16. \frac{d(\arccos u)}{dx} = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx},$$

$$17. \frac{d(\arctan u)}{dx} = \frac{1}{1+u^2} \frac{du}{dx}, \quad 18. \frac{d(\operatorname{arccot} u)}{dx} = \frac{-1}{1+u^2} \frac{du}{dx},$$

$$19. \frac{d(\operatorname{arcsec} u)}{dx} = \frac{1}{u\sqrt{1-u^2}} \frac{du}{dx}, \quad 20. \frac{d(\operatorname{arccsc} u)}{dx} = \frac{-1}{u\sqrt{1-u^2}} \frac{du}{dx},$$

$$21. \frac{d(\sinh u)}{dx} = \cosh u \frac{du}{dx}, \quad 22. \frac{d(\cosh u)}{dx} = \sinh u \frac{du}{dx},$$

$$23. \frac{d(\tanh u)}{dx} = \operatorname{sech}^2 u \frac{du}{dx}, \quad 24. \frac{d(\coth u)}{dx} = -\operatorname{csch}^2 u \frac{du}{dx},$$

$$25. \frac{d(\operatorname{sech} u)}{dx} = -\operatorname{sech} u \tanh u \frac{du}{dx}, \quad 26. \frac{d(\operatorname{csch} u)}{dx} = -\operatorname{csch} u \coth u \frac{du}{dx},$$

$$27. \frac{d(\operatorname{arcsinh} u)}{dx} = \frac{1}{\sqrt{1+u^2}} \frac{du}{dx}, \quad 28. \frac{d(\operatorname{arccosh} u)}{dx} = \frac{1}{\sqrt{u^2-1}} \frac{du}{dx},$$

$$29. \frac{d(\operatorname{arctanh} u)}{dx} = \frac{1}{1-u^2} \frac{du}{dx}, \quad 30. \frac{d(\operatorname{arccoth} u)}{dx} = \frac{1}{u^2-1} \frac{du}{dx},$$

$$31. \frac{d(\operatorname{arcsech} u)}{dx} = \frac{-1}{u\sqrt{1-u^2}} \frac{du}{dx}, \quad 32. \frac{d(\operatorname{arccsch} u)}{dx} = \frac{-1}{|u|\sqrt{1+u^2}} \frac{du}{dx}.$$

Integrals:

$$1. \int cu \, dx = c \int u \, dx, \quad 2. \int (u+v) \, dx = \int u \, dx + \int v \, dx,$$

$$3. \int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1, \quad 4. \int \frac{1}{x} \, dx = \ln x, \quad 5. \int e^x \, dx = e^x,$$

$$6. \int \frac{dx}{1+x^2} = \arctan x, \quad 7. \int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx,$$

$$8. \int \sin x \, dx = -\cos x, \quad 9. \int \cos x \, dx = \sin x,$$

$$10. \int \tan x \, dx = -\ln |\cos x|, \quad 11. \int \cot x \, dx = \ln |\cos x|,$$

$$12. \int \sec x \, dx = \ln |\sec x + \tan x|, \quad 13. \int \csc x \, dx = \ln |\csc x + \cot x|,$$

$$14. \int \arcsin \frac{x}{a} \, dx = \arcsin \frac{x}{a} + \sqrt{a^2 - x^2}, \quad a > 0,$$

15.  $\int \arccos \frac{x}{a} dx = \arccos \frac{x}{a} - \sqrt{a^2 - x^2}, \quad a > 0,$
16.  $\int \arctan \frac{x}{a} dx = x \arctan \frac{x}{a} - \frac{a}{2} \ln(a^2 + x^2), \quad a > 0,$
17.  $\int \sin^2(ax) dx = \frac{1}{2a} (ax - \sin(ax) \cos(ax)),$
18.  $\int \cos^2(ax) dx = \frac{1}{2a} (ax + \sin(ax) \cos(ax)),$
19.  $\int \sec^2 x dx = \tan x,$
20.  $\int \csc^2 x dx = -\cot x,$
21.  $\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx,$
22.  $\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx,$
23.  $\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx, \quad n \neq 1,$
24.  $\int \cot^n x dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x dx, \quad n \neq 1,$
25.  $\int \sec^n x dx = \frac{\tan x \sec^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx, \quad n \neq 1,$
26.  $\int \csc^n x dx = -\frac{\cot x \csc^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \csc^{n-2} x dx, \quad n \neq 1,$
27.  $\int \sinh x dx = \cosh x,$
28.  $\int \cosh x dx = \sinh x,$
29.  $\int \tanh x dx = \ln |\cosh x|,$
30.  $\int \coth x dx = \ln |\sinh x|,$
31.  $\int \operatorname{sech} x dx = \arctan \sinh x,$
32.  $\int \operatorname{csch} x dx = \ln \left| \tanh \frac{x}{2} \right|,$
33.  $\int \sinh^2 x dx = \frac{1}{4} \sinh(2x) - \frac{1}{2} x,$
34.  $\int \cosh^2 x dx = \frac{1}{4} \sinh(2x) + \frac{1}{2} x,$
35.  $\int \operatorname{sech}^2 x dx = \tanh x,$
36.  $\int \operatorname{arcsinh} \frac{x}{a} dx = x \operatorname{arcsinh} \frac{x}{a} - \sqrt{x^2 + a^2}, \quad a > 0,$
37.  $\int \operatorname{arctanh} \frac{x}{a} dx = x \operatorname{arctanh} \frac{x}{a} + \frac{a}{2} \ln |a^2 - x^2|,$
38.  $\int \operatorname{arccosh} \frac{x}{a} dx = \begin{cases} x \operatorname{arccosh} \frac{x}{a} - \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} > 0 \text{ and } a > 0, \\ x \operatorname{arccosh} \frac{x}{a} + \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} < 0 \text{ and } a > 0, \end{cases}$
39.  $\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln \left( x + \sqrt{a^2 + x^2} \right), \quad a > 0,$
40.  $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a}, \quad a > 0,$
41.  $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad a > 0,$
42.  $\int (a^2 - x^2)^{3/2} dx = \frac{x}{8} (5a^2 - 2x^2) \sqrt{a^2 - x^2} + \frac{3a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$
43.  $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a}, \quad a > 0,$
44.  $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right|,$
45.  $\int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2 \sqrt{a^2 - x^2}},$
46.  $\int \sqrt{a^2 \pm x^2} dx = \frac{x}{2} \sqrt{a^2 \pm x^2} \pm \frac{a^2}{2} \ln \left| x + \sqrt{a^2 \pm x^2} \right|,$
47.  $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right|, \quad a > 0,$
48.  $\int \frac{dx}{ax^2 + bx} = \frac{1}{a} \ln \left| \frac{x}{a+bx} \right|,$
49.  $\int x \sqrt{a+bx} dx = \frac{2(3bx-2a)(a+bx)^{3/2}}{15b^2},$
50.  $\int \frac{\sqrt{a+bx}}{x} dx = 2\sqrt{a+bx} + a \int \frac{1}{x\sqrt{a+bx}} dx,$
51.  $\int \frac{x}{\sqrt{a+bx}} dx = \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}} \right|, \quad a > 0,$
52.  $\int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} - a \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right|,$
53.  $\int x \sqrt{a^2 - x^2} dx = -\frac{1}{3} (a^2 - x^2)^{3/2},$
54.  $\int x^2 \sqrt{a^2 - x^2} dx = \frac{x}{8} (2x^2 - a^2) \sqrt{a^2 - x^2} + \frac{a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$
55.  $\int \frac{dx}{\sqrt{a^2 - x^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right|,$
56.  $\int \frac{x dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2},$
57.  $\int \frac{x^2 dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad a > 0,$
58.  $\int \frac{\sqrt{a^2 + x^2}}{x} dx = \sqrt{a^2 + x^2} - a \ln \left| \frac{a + \sqrt{a^2 + x^2}}{x} \right|,$
59.  $\int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \arccos \frac{a}{|x|}, \quad a > 0,$
60.  $\int x \sqrt{x^2 \pm a^2} dx = \frac{1}{3} (x^2 \pm a^2)^{3/2},$
61.  $\int \frac{dx}{x \sqrt{x^2 + a^2}} = \frac{1}{a} \ln \left| \frac{x}{a + \sqrt{a^2 + x^2}} \right|,$

Calculus Cont.

$$\begin{aligned}
 \mathbf{62.} \quad & \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \arccos \left| \frac{a}{x} \right|, \quad a > 0, & \mathbf{63.} \quad & \int \frac{dx}{x^2\sqrt{x^2 \pm a^2}} = \mp \frac{\sqrt{x^2 \pm a^2}}{a^2 x}, \\
 \mathbf{64.} \quad & \int \frac{x \, dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}, & \mathbf{65.} \quad & \int \frac{\sqrt{x^2 \pm a^2}}{x^4} dx = \mp \frac{(x^2 + a^2)^{3/2}}{3a^2 x^3}, \\
 \mathbf{66.} \quad & \int \frac{dx}{ax^2 + bx + c} = \begin{cases} \frac{1}{\sqrt{b^2 - 4ac}} \ln \left| \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}} \right|, & \text{if } b^2 > 4ac, \\ \frac{2}{\sqrt{4ac - b^2}} \arctan \frac{2ax + b}{\sqrt{4ac - b^2}}, & \text{if } b^2 < 4ac, \end{cases} \\
 \mathbf{67.} \quad & \int \frac{dx}{\sqrt{ax^2 + bx + c}} = \begin{cases} \frac{1}{\sqrt{a}} \ln \left| 2ax + b + 2\sqrt{a}\sqrt{ax^2 + bx + c} \right|, & \text{if } a > 0, \\ \frac{1}{\sqrt{-a}} \arcsin \frac{-2ax - b}{\sqrt{b^2 - 4ac}}, & \text{if } a < 0, \end{cases} \\
 \mathbf{68.} \quad & \int \sqrt{ax^2 + bx + c} \, dx = \frac{2ax + b}{4a} \sqrt{ax^2 + bx + c} + \frac{4ac - b^2}{8a} \int \frac{dx}{\sqrt{ax^2 + bx + c}}, \\
 \mathbf{69.} \quad & \int \frac{x \, dx}{\sqrt{ax^2 + bx + c}} = \frac{\sqrt{ax^2 + bx + c}}{a} - \frac{b}{2a} \int \frac{dx}{\sqrt{ax^2 + bx + c}}, \\
 \mathbf{70.} \quad & \int \frac{dx}{x\sqrt{ax^2 + bx + c}} = \begin{cases} \frac{-1}{\sqrt{c}} \ln \left| \frac{2\sqrt{c}\sqrt{ax^2 + bx + c} + bx + 2c}{x} \right|, & \text{if } c > 0, \\ \frac{1}{\sqrt{-c}} \arcsin \frac{bx + 2c}{|x|\sqrt{b^2 - 4ac}}, & \text{if } c < 0, \end{cases} \\
 \mathbf{71.} \quad & \int x^3 \sqrt{x^2 + a^2} \, dx = \left(\frac{1}{3}x^2 - \frac{2}{15}a^2\right)(x^2 + a^2)^{3/2}, \\
 \mathbf{72.} \quad & \int x^n \sin(ax) \, dx = -\frac{1}{a}x^n \cos(ax) + \frac{n}{a} \int x^{n-1} \cos(ax) \, dx, \\
 \mathbf{73.} \quad & \int x^n \cos(ax) \, dx = \frac{1}{a}x^n \sin(ax) - \frac{n}{a} \int x^{n-1} \sin(ax) \, dx, \\
 \mathbf{74.} \quad & \int x^n e^{ax} \, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx, \\
 \mathbf{75.} \quad & \int x^n \ln(ax) \, dx = x^{n+1} \left( \frac{\ln(ax)}{n+1} - \frac{1}{(n+1)^2} \right), \\
 \mathbf{76.} \quad & \int x^n (\ln ax)^m \, dx = \frac{x^{n+1}}{n+1} (\ln ax)^m - \frac{m}{n+1} \int x^n (\ln ax)^{m-1} \, dx.
 \end{aligned}$$

Finite Calculus

Difference, shift operators:

$$\Delta f(x) = f(x+1) - f(x),$$

$$\mathbf{E} f(x) = f(x+1).$$

Fundamental Theorem:

$$f(x) = \Delta F(x) \Leftrightarrow \sum f(x) \delta x = F(x) + C.$$

$$\sum_a^b f(x) \delta x = \sum_{i=a}^{b-1} f(i).$$

Differences:

$$\Delta(cu) = c\Delta u, \quad \Delta(u+v) = \Delta u + \Delta v,$$

$$\Delta(uv) = u\Delta v + \mathbf{E} v \Delta u,$$

$$\Delta(x^n) = nx^{n-1},$$

$$\Delta(H_x) = x^{-1}, \quad \Delta(2^x) = 2^x,$$

$$\Delta(c^x) = (c-1)c^x, \quad \Delta \binom{x}{m} = \binom{x}{m-1}.$$

Sums:

$$\sum cu \, \delta x = c \sum u \, \delta x,$$

$$\sum (u+v) \, \delta x = \sum u \, \delta x + \sum v \, \delta x,$$

$$\sum u \Delta v \, \delta x = uv - \sum \mathbf{E} v \Delta u \, \delta x,$$

$$\sum x^n \delta x = \frac{x^{n+1}}{n+1}, \quad \sum x^{-1} \delta x = H_x,$$

$$\sum c^x \delta x = \frac{c^x}{c-1}, \quad \sum \binom{x}{m} \delta x = \binom{x}{m+1}.$$

Falling Factorial Powers:

$$x^{\underline{n}} = x(x-1) \cdots (x-n+1), \quad n > 0,$$

$$x^{\underline{0}} = 1,$$

$$x^{\underline{n}} = \frac{1}{(x+1) \cdots (x+|n|)}, \quad n < 0,$$

$$x^{\overline{n+m}} = x^{\overline{m}}(x-m)^{\underline{n}}.$$

Rising Factorial Powers:

$$x^{\overline{n}} = x(x+1) \cdots (x+n-1), \quad n > 0,$$

$$x^{\overline{0}} = 1,$$

$$x^{\overline{n}} = \frac{1}{(x-1) \cdots (x-|n|)}, \quad n < 0,$$

$$x^{\overline{n+m}} = x^{\overline{m}}(x+m)^{\overline{n}}.$$

Conversion:

$$x^{\underline{n}} = (-1)^n (-x)^{\overline{n}} = (x-n+1)^{\overline{n}}$$

$$= 1/(x+1)^{\overline{-n}},$$

$$x^{\overline{n}} = (-1)^n (-x)^{\underline{n}} = (x+n-1)^{\underline{n}}$$

$$= 1/(x-1)^{\underline{-n}},$$

$$x^n = \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^{\underline{k}} = \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (-1)^{n-k} x^{\overline{k}},$$

$$x^{\underline{n}} = \sum_{k=1}^n \left[ \begin{matrix} n \\ k \end{matrix} \right] (-1)^{n-k} x^k,$$

$$x^{\overline{n}} = \sum_{k=1}^n \left[ \begin{matrix} n \\ k \end{matrix} \right] x^k.$$

|                      |   |                       |  |
|----------------------|---|-----------------------|--|
| $x^1 =$              | $x^{\underline{1}}$   | $=$                   | $x^{\overline{1}}$   |
| $x^2 =$              | $x^{\underline{2}} + x^{\underline{1}}$   | $=$                   | $x^{\overline{2}} - x^{\overline{1}}$  |
| $x^3 =$              | $x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}}$  | $=$                   | $x^{\overline{3}} - 3x^{\overline{2}} + x^{\overline{1}}$  |
| $x^4 =$              | $x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{2}} + x^{\underline{1}}$                         | $=$                   | $x^{\overline{4}} - 6x^{\overline{3}} + 7x^{\overline{2}} - x^{\overline{1}}$                        |
| $x^5 =$              | $x^{\underline{5}} + 15x^{\underline{4}} + 25x^{\underline{3}} + 10x^{\underline{2}} + x^{\underline{1}}$ | $=$                   | $x^{\overline{5}} - 15x^{\overline{4}} + 25x^{\overline{3}} - 10x^{\overline{2}} + x^{\overline{1}}$ |
| $x^{\overline{1}} =$ | $x^1$   | $x^{\underline{1}} =$ | $x^1$  |
| $x^{\overline{2}} =$ | $x^2 + x^1$   | $x^{\underline{2}} =$ | $x^2 - x^1$  |
| $x^{\overline{3}} =$ | $x^3 + 3x^2 + 2x^1$   | $x^{\underline{3}} =$ | $x^3 - 3x^2 + 2x^1$  |
| $x^{\overline{4}} =$ | $x^4 + 6x^3 + 11x^2 + 6x^1$   | $x^{\underline{4}} =$ | $x^4 - 6x^3 + 11x^2 - 6x^1$  |
| $x^{\overline{5}} =$ | $x^5 + 10x^4 + 35x^3 + 50x^2 + 24x^1$   | $x^{\underline{5}} =$ | $x^5 - 10x^4 + 35x^3 - 50x^2 + 24x^1$  |



**Series**

Taylor's series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \dots = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!} f^{(i)}(a).$$

Expansions:

|   |   |  |
|---|---|--|
| $\frac{1}{1-x}$   | $= 1 + x + x^2 + x^3 + x^4 + \dots$                                 | $= \sum_{i=0}^{\infty} x^i,$                             |
| $\frac{1}{1-cx}$  | $= 1 + cx + c^2x^2 + c^3x^3 + \dots$                                | $= \sum_{i=0}^{\infty} c^i x^i,$                         |
| $\frac{1}{1-x^n}$   | $= 1 + x^n + x^{2n} + x^{3n} + \dots$                               | $= \sum_{i=0}^{\infty} x^{ni},$                          |
| $\frac{x}{(1-x)^2}$   | $= x + 2x^2 + 3x^3 + 4x^4 + \dots$                                  | $= \sum_{i=0}^{\infty} ix^i,$                            |
| $x^k \frac{d^n}{dx^n} \left( \frac{1}{1-x} \right)$                 | $= x + 2^n x^2 + 3^n x^3 + 4^n x^4 + \dots$                         | $= \sum_{i=0}^{\infty} i^n x^i,$                         |
| $e^x$   | $= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$                 | $= \sum_{i=0}^{\infty} \frac{x^i}{i!},$                  |
| $\ln(1+x)$  | $= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 - \dots$    | $= \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^i}{i},$        |
| $\ln \frac{1}{1-x}$   | $= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \dots$    | $= \sum_{i=1}^{\infty} \frac{x^i}{i},$                   |
| $\sin x$  | $= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$ | $= \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!},$ |
| $\cos x$  | $= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$ | $= \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!},$     |
| $\tan^{-1} x$   | $= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$    | $= \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)},$  |
| $(1+x)^n$   | $= 1 + nx + \frac{n(n-1)}{2}x^2 + \dots$                            | $= \sum_{i=0}^{\infty} \binom{n}{i} x^i,$                |
| $\frac{1}{(1-x)^{n+1}}$   | $= 1 + (n+1)x + \binom{n+2}{2}x^2 + \dots$                          | $= \sum_{i=0}^{\infty} \binom{i+n}{i} x^i,$              |
| $\frac{x}{e^x - 1}$   | $= 1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \dots$   | $= \sum_{i=0}^{\infty} \frac{B_i x^i}{i!},$              |
| $\frac{1}{2x}(1 - \sqrt{1-4x})$                                     | $= 1 + x + 2x^2 + 5x^3 + \dots$                                     | $= \sum_{i=0}^{\infty} \frac{1}{i+1} \binom{2i}{i} x^i,$ |
| $\frac{1}{\sqrt{1-4x}}$   | $= 1 + x + 2x^2 + 6x^3 + \dots$                                     | $= \sum_{i=0}^{\infty} \binom{2i}{i} x^i,$               |
| $\frac{1}{\sqrt{1-4x}} \left( \frac{1 - \sqrt{1-4x}}{2x} \right)^n$ | $= 1 + (2+n)x + \binom{4+n}{2}x^2 + \dots$                          | $= \sum_{i=0}^{\infty} \binom{2i+n}{i} x^i,$             |
| $\frac{1}{1-x} \ln \frac{1}{1-x}$                                   | $= x + \frac{3}{2}x^2 + \frac{11}{6}x^3 + \frac{25}{12}x^4 + \dots$ | $= \sum_{i=1}^{\infty} H_i x^i,$                         |
| $\frac{1}{2} \left( \ln \frac{1}{1-x} \right)^2$                    | $= \frac{1}{2}x^2 + \frac{3}{4}x^3 + \frac{11}{24}x^4 + \dots$      | $= \sum_{i=2}^{\infty} \frac{H_{i-1} x^i}{i},$           |
| $\frac{x}{1-x-x^2}$   | $= x + x^2 + 2x^3 + 3x^4 + \dots$                                   | $= \sum_{i=0}^{\infty} F_i x^i,$                         |
| $\frac{F_n x}{1 - (F_{n-1} + F_{n+1})x - (-1)^n x^2}$               | $= F_n x + F_{2n} x^2 + F_{3n} x^3 + \dots$                         | $= \sum_{i=0}^{\infty} F_{ni} x^i.$                      |

Ordinary power series:

$$A(x) = \sum_{i=0}^{\infty} a_i x^i.$$

Exponential power series:

$$A(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}.$$

Dirichlet power series:

$$A(x) = \sum_{i=1}^{\infty} \frac{a_i}{i^x}.$$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k.$$

Difference of like powers:

$$x^n - y^n = (x-y) \sum_{k=0}^{n-1} x^{n-1-k} y^k.$$

For ordinary power series:

$$\alpha A(x) + \beta B(x) = \sum_{i=0}^{\infty} (\alpha a_i + \beta b_i) x^i,$$

$$x^k A(x) = \sum_{i=k}^{\infty} a_{i-k} x^i,$$

$$\frac{A(x) - \sum_{i=0}^{k-1} a_i x^i}{x^k} = \sum_{i=0}^{\infty} a_{i+k} x^i,$$

$$A(cx) = \sum_{i=0}^{\infty} c^i a_i x^i,$$

$$A'(x) = \sum_{i=0}^{\infty} (i+1) a_{i+1} x^i,$$

$$xA'(x) = \sum_{i=1}^{\infty} i a_i x^i,$$

$$\int A(x) dx = \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} x^i,$$

$$\frac{A(x) + A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i} x^{2i},$$

$$\frac{A(x) - A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i+1} x^{2i+1}.$$

Summation: If  $b_i = \sum_{j=0}^i a_j$  then

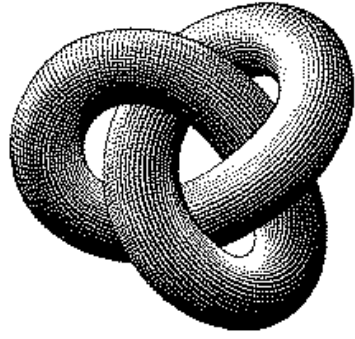
$$B(x) = \frac{1}{1-x} A(x).$$

Convolution:

$$A(x)B(x) = \sum_{i=0}^{\infty} \left( \sum_{j=0}^i a_j b_{i-j} \right) x^i.$$

God made the natural numbers;  
all the rest is the work of man.  
– Leopold Kronecker

# Theoretical Computer Science Cheat Sheet

| Series  |  | Escher's Knot  |
|---|--|--|
| <div>Expansions:</div> <div><math display="block">\frac{1}{(1-x)^{n+1}} \ln \frac{1}{1-x} = \sum_{i=0}^{\infty} (H_{n+i} - H_n) \binom{n+i}{i} x^i,</math><math display="block">x^{\overline{n}} = \sum_{i=0}^{\infty} \left[ \begin{matrix} n \\ i \end{matrix} \right] x^i,</math><math display="block">\left( \ln \frac{1}{1-x} \right)^n = \sum_{i=0}^{\infty} \left[ \begin{matrix} i \\ n \end{matrix} \right] \frac{n! x^i}{i!},</math><math display="block">\tan x = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{2^{2i} (2^{2i} - 1) B_{2i} x^{2i-1}}{(2i)!},</math><math display="block">\frac{1}{\zeta(x)} = \sum_{i=1}^{\infty} \frac{\mu(i)}{i^x},</math><math display="block">\zeta(x) = \prod_p \frac{1}{1 - p^{-x}},</math><math display="block">\zeta^2(x) = \sum_{i=1}^{\infty} \frac{d(i)}{x^i} \quad \text{where } d(n) = \sum_{d n} 1,</math><math display="block">\zeta(x)\zeta(x-1) = \sum_{i=1}^{\infty} \frac{S(i)}{x^i} \quad \text{where } S(n) = \sum_{d n} d,</math><math display="block">\zeta(2n) = \frac{2^{2n-1}  B_{2n} }{(2n)!} \pi^{2n}, \quad n \in \mathbb{N},</math><math display="block">\frac{x}{\sin x} = \sum_{i=0}^{\infty} (-1)^{i-1} \frac{(4^i - 2) B_{2i} x^{2i}}{(2i)!},</math><math display="block">\left( \frac{1 - \sqrt{1-4x}}{2x} \right)^n = \sum_{i=0}^{\infty} \frac{n(2i+n-1)!}{i!(n+i)!} x^i,</math><math display="block">e^x \sin x = \sum_{i=1}^{\infty} \frac{2^{i/2} \sin \frac{i\pi}{4}}{i!} x^i,</math><math display="block">\sqrt{\frac{1 - \sqrt{1-x}}{x}} = \sum_{i=0}^{\infty} \frac{(4i)!}{16^i \sqrt{2} (2i)!(2i+1)!} x^i,</math><math display="block">\left( \frac{\arcsin x}{x} \right)^2 = \sum_{i=0}^{\infty} \frac{4^i i!^2}{(i+1)(2i+1)!} x^{2i}.</math></div> |  |   |
|   |  | Stieltjes Integration  |
|   |  | <div>If <math>G</math> is continuous in the interval <math>[a, b]</math> and <math>F</math> is nondecreasing then</div> <div><math display="block">\int_a^b G(x) dF(x)</math></div> <div>exists. If <math>a \leq b \leq c</math> then</div> <div><math display="block">\int_a^c G(x) dF(x) = \int_a^b G(x) dF(x) + \int_b^c G(x) dF(x).</math></div> <div>If the integrals involved exist</div> <div><math display="block">\int_a^b (G(x) + H(x)) dF(x) = \int_a^b G(x) dF(x) + \int_a^b H(x) dF(x),</math><math display="block">\int_a^b G(x) d(F(x) + H(x)) = \int_a^b G(x) dF(x) + \int_a^b G(x) dH(x),</math><math display="block">\int_a^b c \cdot G(x) dF(x) = \int_a^b G(x) d(c \cdot F(x)) = c \int_a^b G(x) dF(x),</math><math display="block">\int_a^b G(x) dF(x) = G(b)F(b) - G(a)F(a) - \int_a^b F(x) dG(x).</math></div> <div>If the integrals involved exist, and <math>F</math> possesses a derivative <math>F'</math> at every point in <math>[a, b]</math> then</div> <div><math display="block">\int_a^b G(x) dF(x) = \int_a^b G(x) F'(x) dx.</math></div> |
| Cramer's Rule   |  | Fibonacci Numbers  |
| <div>If we have equations:</div> <div><math display="block">a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1</math><math display="block">a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2</math><math display="block">\vdots \qquad \qquad \qquad \vdots</math><math display="block">a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n = b_n</math></div> <div>Let <math>A = (a_{i,j})</math> and <math>B</math> be the column matrix <math>(b_i)</math>. Then there is a unique solution iff <math>\det A \neq 0</math>. Let <math>A_i</math> be <math>A</math> with column <math>i</math> replaced by <math>B</math>. Then</div> <div><math display="block">x_i = \frac{\det A_i}{\det A}.</math></div>   |  | <div>00 47 18 76 29 93 85 34 61 52</div> <div>86 11 57 28 70 39 94 45 02 63</div> <div>95 80 22 67 38 71 49 56 13 04</div> <div>59 96 81 33 07 48 72 60 24 15</div> <div>73 69 90 82 44 17 58 01 35 26</div> <div>68 74 09 91 83 55 27 12 46 30</div> <div>37 08 75 19 92 84 66 23 50 41</div> <div>14 25 36 40 51 62 03 77 88 99</div> <div>21 32 43 54 65 06 10 89 97 78</div> <div>42 53 64 05 16 20 31 98 79 87</div>  |
|   |  | <div>The Fibonacci number system:</div> <div>Every integer <math>n</math> has a unique representation</div> <div><math display="block">n = F_{k_1} + F_{k_2} + \cdots + F_{k_m},</math></div> <div>where <math>k_i \geq k_{i+1} + 2</math> for all <math>i</math>,<br/><math>1 \leq i &lt; m</math> and <math>k_m \geq 2</math>.</div>   |
| Improvement makes strait roads, but the crooked roads without Improvement, are roads of Genius.<br>– William Blake (The Marriage of Heaven and Hell)  |  | <div>1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...</div> <div>Definitions:</div> <div><math display="block">F_i = F_{i-1} + F_{i-2}, \quad F_0 = F_1 = 1,</math><math display="block">F_{-i} = (-1)^{i-1} F_i,</math><math display="block">F_i = \frac{1}{\sqrt{5}} \left( \phi^i - \hat{\phi}^i \right),</math></div> <div>Cassini's identity: for <math>i &gt; 0</math>:</div> <div><math display="block">F_{i+1}F_{i-1} - F_i^2 = (-1)^i.</math></div> <div>Additive rule:</div> <div><math display="block">F_{n+k} = F_k F_{n+1} + F_{k-1} F_n,</math><math display="block">F_{2n} = F_n F_{n+1} + F_{n-1} F_n.</math></div> <div>Calculation by matrices:</div> <div><math display="block">\begin{pmatrix} F_{n-2} &amp; F_{n-1} \\ F_{n-1} &amp; F_n \end{pmatrix} = \begin{pmatrix} 0 &amp; 1 \\ 1 &amp; 1 \end{pmatrix}^n.</math></div>  |

|    |    |    |    |    |    |    |    |    |    |
|----|----|----|----|----|----|----|----|----|----|
| 00 | 47 | 18 | 76 | 29 | 93 | 85 | 34 | 61 | 52 |
| 86 | 11 | 57 | 28 | 70 | 39 | 94 | 45 | 02 | 63 |
| 95 | 80 | 22 | 67 | 38 | 71 | 49 | 56 | 13 | 04 |
| 59 | 96 | 81 | 33 | 07 | 48 | 72 | 60 | 24 | 15 |
| 73 | 69 | 90 | 82 | 44 | 17 | 58 | 01 | 35 | 26 |
| 68 | 74 | 09 | 91 | 83 | 55 | 27 | 12 | 46 | 30 |
| 37 | 08 | 75 | 19 | 92 | 84 | 66 | 23 | 50 | 41 |
| 14 | 25 | 36 | 40 | 51 | 62 | 03 | 77 | 88 | 99 |
| 21 | 32 | 43 | 54 | 65 | 06 | 10 | 89 | 97 | 78 |
| 42 | 53 | 64 | 05 | 16 | 20 | 31 | 98 | 79 | 87 |

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 Every integer  $n$  has a unique representation

$$n = F_{k_1} + F_{k_2} + \dots + F_{k_m},$$

where  $k_i \geq k_{i+1} + 2$  for all  $i$ ,  
 $1 \leq i < m$  and  $k_m \geq 2$ .