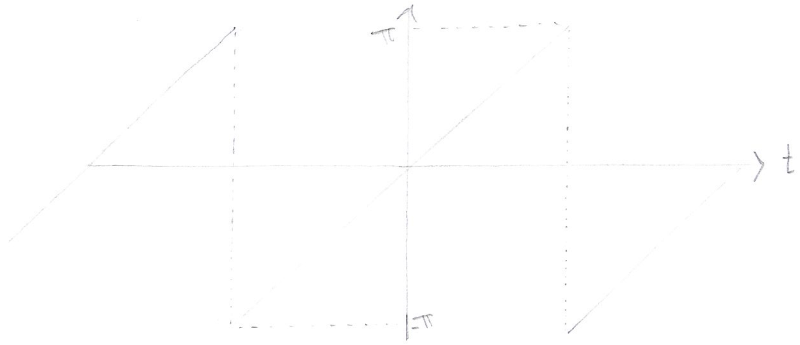
TD_5

ex 1 Fonction ζ de Riemann : $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$, $s \in]1,+\infty[$.

$\boxed{\zeta(2) = \frac{\pi^2}{6}}$ $\boxed{\zeta(4) = \frac{\pi^4}{90}}$ $\boxed{\zeta(6) = \frac{\pi^6}{945}}$ $\boxed{\zeta(8) = \frac{\pi^8}{9450}}$

$\zeta(2p) \longleftrightarrow$ série de Fourier de t^p (prolongée sur $\mathbb{R}$ par 2π périodicité).

$*$ $\zeta(2)$:



$f \in L^2_{2\pi} := \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ } 2\pi \text{ périodique} \}$ et $\int_{-\pi}^{\pi} |f(t)|^2 dt < \infty$

$\mathbb{R}$

$L^2([-\pi,\pi])$

$$\boxed{(f|g) = \int_{-\pi}^{\pi} f(t) g(t)\, dt} \longrightarrow (L^2_{2\pi}, (\cdot|\cdot))\ \text{e.h.}$$

Base hilbertienne $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos nt}{\sqrt{\pi}}, \frac{\sin nt}{\sqrt{\pi}} \right\}_{n\geq 1}$.

- $\|1\|^2_{L^2_{2\pi}} = (1|1) = \int_{-\pi}^{\pi} 1^2 dt = [t]_{-\pi}^{\pi} = 2\pi$.
- $\|\cos(nt)\|^2 = \int_{-\pi}^{\pi} \cos^2(nt)\, dt = \int_{-\pi}^{\pi} \frac{1+\cos(2nt)}{2} dt = \frac{1}{2}\left[t + \frac{\sin(2nt)}{2n}\right]_{-\pi}^{\pi} = \frac{1}{2} \cdot 2\pi = \pi$
- $\|\sin(nt)\|^2 = \int_{-\pi}^{\pi} \sin^2(nt)\, dt = \int_{-\pi}^{\pi} \frac{1-\cos(2nt)}{2} dt = \frac{1}{2}\left[t - \frac{\sin(2nt)}{2n}\right]_{-\pi}^{\pi} = \frac{1}{2} \cdot 2\pi = \pi$

$*$ $L^2_{2\pi}$: $\exists f \underset{\substack{cvds \\ L^2_{2\pi}}}{=} a_0 \frac{1}{\sqrt{2\pi}} + \sum_{n\geq 1} \left(a_n \cdot \frac{\cos(nt)}{\sqrt{\pi}} + b_n \frac{\sin(nt)}{\sqrt{\pi}} \right)$ $(*)$

où $\begin{cases} a_0 = \left(f \mid \frac{1}{\sqrt{2\pi}}\right) \\ a_n = \left(f \mid \frac{\cos(nt)}{\sqrt{\pi}}\right), \quad b_n = \left(f \mid \frac{\sin(nt)}{\sqrt{\pi}}\right) \end{cases}$, $n \geq 1$.

$$(*) \Leftrightarrow \int_{-\pi}^{\pi} \left| f(t) - \left(a_0 \cdot \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{N} \left(a_n \frac{\cos(nt)}{\sqrt{\pi}} + b_n \frac{\sin(nt)}{\sqrt{\pi}}\right)\right)\right|^2 dt \xrightarrow[N\to+\infty]{} 0$$ (et Parseval)

$$\boxed{\|f\|^2 = a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)}$$

$\operatorname{Im}\left(\int_0^{\pi} t e^{int}\right)$

- $f_{]-\pi,\pi[}$ impaire $\Rightarrow a_0 = a_n = 0 \quad \forall n \geq 1$.

et $b_n = \int_{-\pi}^{\pi} f(t) \frac{\sin(nt)}{\sqrt{\pi}} dt = \frac{2}{\sqrt{\pi}} \int_0^{\pi} t.\sin(nt)$

• On a $\int_0^{\pi} \underset{u}{t}\, \underset{v'}{e^{int}}\, dt = \left[t \cdot \frac{e^{int}}{in} \right]_0^{\pi} - \int_0^{\pi} 1 \cdot \frac{e^{int}}{in}\, dt$

$$= \frac{\pi e^{in\pi}}{in} - \left[\frac{e^{int}}{(in)^2} \right]_0^{\pi} = \frac{(-1)^n \pi}{in} + \frac{e^{in\pi} - 1}{n^2} = \frac{-i(-1)^n \pi}{n} + \frac{(-1)^n - 1}{n^2}.$$

$$\Rightarrow \boxed{b_n = \frac{2}{\sqrt{\pi}} \cdot \frac{(-1)^{n+1}\pi}{n} = \frac{2\sqrt{\pi}(-1)^{n+1}}{n}}$$

Parseval $\Rightarrow \|f\|^2 = \underset{0}{\cancel{a_0^2}} + \sum_{n\geq 1} (\underset{0}{\cancel{a_n^2}} + b_n^2) = \sum_{n=1}^{\infty} \frac{4\pi}{n^2} = 4\pi \cdot \zeta(2).$

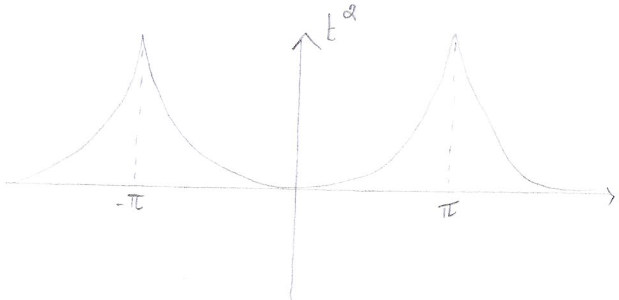
or $\|f\|^2 = \int_{-\pi}^{\pi} t^2\, dt = 2\int_0^{\pi} t^2\, dt = 2\left[\frac{t^3}{3}\right]_0^{\pi} = \frac{2\pi^3}{3}.$

$$\boxed{\zeta(2) = \frac{\pi^2}{6}}$$

so $\zeta(2) = \frac{1}{4\pi} \cdot \frac{2\pi^3}{3} = \frac{\pi^2}{6}.$

Rq : $f = \sum_{n=1}^{\infty} b_n \frac{\sin(nt)}{\sqrt{\pi}} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (\sin(nt)).$

* $\zeta(4)$:



$f_{]-\pi,\pi[}(t) = t^2$ paire.

$\Rightarrow b_n = 0$

$\Rightarrow a_0 := \left(f \,\Big|\, \frac{1}{\sqrt{2\pi}}\right) = \int_{-\pi}^{\pi} f(t) \frac{1}{\sqrt{2\pi}}\, dt = \frac{2}{\sqrt{2\pi}} \int_0^{\pi} t^2\, dt = \frac{2\pi^3}{3\sqrt{2\pi}}$

$a_n = \left(f \,\Big|\, \frac{\cos(nt)}{\sqrt{\pi}}\right) = \underbrace{2\int_0^{\pi} t^2 \frac{\cos(nt)}{\sqrt{\pi}}\, dt}_{Re\left(\int_0^{\pi} t^2 e^{int}\right)}.$

• $\int_0^{\pi} \underset{u}{t^2}\, \underset{v'}{e^{int}}\, dt = \left[\frac{t^2 e^{int}}{in}\right]_0^{\pi} - \int_0^{\pi} 2t \frac{e^{int}}{in}\, dt = \frac{\pi^2 e^{int}}{in} - \frac{2}{in}\left[\frac{-i(-1)^n\pi}{n} + \frac{(-1)^n - 1}{n^2}\right]$

et $Re\left|\int_0^{\pi} t^2 e^{int}\, dt\right| = \frac{2(-1)^n \pi}{n^2}.$ so $\boxed{a_n = \frac{4(-1)^n \pi}{\sqrt{\pi}\, n^2} = \frac{4(-1)^n \sqrt{\pi}}{n^2}}$

Parseval $\Rightarrow \|f\|^2 = a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + \underset{0}{\cancel{b_n^2}}) = \frac{4\pi^6}{9 \cdot 2\pi} + \sum_{n=1}^{\infty} \frac{16\pi}{n^4} = \frac{2\pi^5}{9} + 16\pi \cdot \zeta(4)$

or $\|f\|^2 = \int_{-\pi}^{\pi} t^4\, dt = 2\int_0^{\pi} t^4\, dt = 2\frac{\pi^5}{5}.$

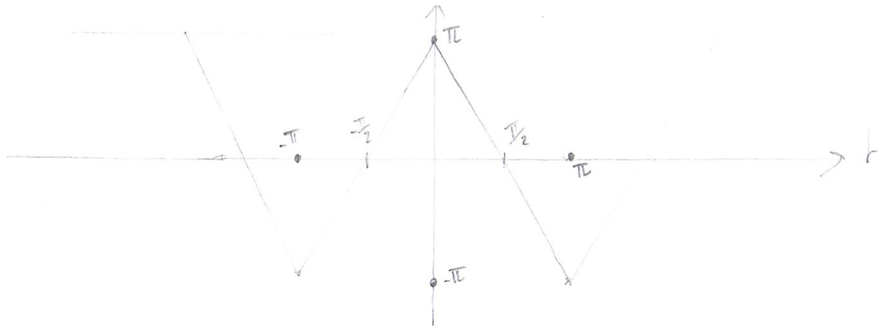
so $\zeta(4) = \frac{1}{16\pi}\left[\frac{2\pi^5}{5} - \frac{2\pi^5}{9}\right]$

$\Rightarrow \zeta(4) = \frac{\pi^4}{16}\left(\frac{2}{5} - \frac{2}{9}\right) = \frac{\pi^4}{8}\left(\frac{4}{45}\right) = \frac{\pi^4}{90}.$

$$\boxed{\zeta(4) = \frac{\pi^4}{90}}$$

ex 2 Soit $f \in \mathbb{R}^{\mathbb{R}}$, paire, 2π périodique def par $f(x) = \pi - 2x$ sur $[0, \pi[$.

1)



$f_{[0,\pi[} = \pi - 2t$.

f paire $\Rightarrow b_n = 0$, $n \geq 1$

de plus, $a_0 = \left(f \mid \frac{1}{\sqrt{2\pi}}\right) = 0$, par symétrie. $a_n = \left(f \mid \frac{\cos(nt)}{\sqrt{\pi}}\right) = 2\int_0^\pi (\pi - 2t)\frac{\cos(nt)}{\sqrt{\pi}}$.

so $a_n = -\frac{4}{\sqrt{\pi}} \underbrace{\int_0^\pi t\cos(nt)\,dt}_{\mathrm{Re}\int_0^\pi t e^{int}dt}$ $\quad$ (cf $\int_0^\pi \pi\cos(nt) = 0$)

$\int_0^\pi t e^{int} = \frac{\pi(-1)^n}{in} + \frac{(-1)^n - 1}{n^2}$

$a_n = -\frac{4}{\sqrt{\pi}}\left(\frac{(-1)^n - 1}{n^2}\right)$

$$\begin{cases} a_{2p} = 0 \\ a_{2p+1} = \dfrac{8}{(2p+1)^2\sqrt{\pi}} \quad (*) \end{cases}$$

2) Th de Dirichlet : si $f \in L^2_{2\pi}$ est C^1 par morceaux alors $\forall t \in \mathbb{R}$, la série de Fourier de f évaluée en t converge (ds $\mathbb{R}$) vers $\frac{f(t^+) + f(t^-)}{2}$ $(= f(t)$ si f continue en $t)$

$f(t^+) = \lim_{s \underset{>}{\to} t} f(s)$

$f(t^-) = \lim_{s \underset{<}{\to} t} f(s)$

$(\forall t \in \mathbb{R})\ a_0 \cdot \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{N}\left(a_n \cdot \frac{\cos(nt)}{\sqrt{\pi}} + b_n \frac{\sin(nt)}{\sqrt{\pi}}\right) \underset{\substack{N\to\infty \\ \text{CV ds } \mathbb{R}}}{\longrightarrow} \frac{f(t^+) + f(t^-)}{2}$

ici, la fct est C^1 par morceaux et continue :

$(\forall t \in \mathbb{R})$: $\sum_{p=0}^{\infty} \frac{8}{(2p+1)^2\sqrt{\pi}} \cdot \frac{\cos((2p+1)t)}{\sqrt{\pi}} = f(t) = \sum_{p=0}^{\infty} \frac{8\cos((2p+1)t)}{(2p+1)^2\,\pi}$.

3) en $t = 0$, $\frac{8}{\pi}\sum_{p=0}^{\infty}\frac{1}{(2p+1)^2}\underbrace{\cos(0)}_{1} = f(0) = \pi \Rightarrow \sum_{p=0}^{\infty}\frac{1}{(2p+1)^2} = \frac{\pi^2}{8} < \frac{\pi^2}{6} = \zeta(2)$

Parseval : $\|f\|^2 = \sum_{p=0}^{\infty} a_{p+1}^2$.

. $\|f\|^2 = 2\int_0^\pi (\pi - 2t)^2 dt = 4\int_0^{\frac{\pi}{2}} (\pi - 2t)^2 dt = 4\left[\frac{(\pi - 2t)^3}{-6}\right]_0^{\frac{\pi}{2}} = 2\frac{\pi^3}{3}$.

$\overset{(*)}{=} \sum_{p=0}^{\infty}\frac{64}{\pi}\left(\frac{1}{(2p+1)^4}\right) = \frac{2\pi^3}{3}$ so $\sum_{p=0}^{\infty}\frac{1}{(2p+1)^4} = \frac{\pi^4}{96} < \frac{\pi^4}{90} = \zeta(4)$

verification : • $\zeta(2) = \sum_{n=1} \frac{1}{n^2} = \sum_{p=0} \frac{1}{(2p+1)^2} + \underbrace{\sum_{p=1} \frac{1}{(2p)^2}}_{\frac{1}{4}\zeta(2)}$

$$\Rightarrow \underbrace{\underbrace{\frac{3}{4}\zeta(2)}_{\frac{3}{4}\cdot\frac{\pi^2}{6}}}_{=\frac{\pi^2}{8}} = \sum_{p=0}^{\infty} \frac{1}{(2p+1)^2}$$

• $\zeta(4) = \sum_{p=0}^{\infty} \frac{1}{(2p+1)^4} + \underbrace{\sum_{p=1}^{\infty} \frac{1}{(2p)^4}}_{\frac{1}{6}\zeta(4)}$

$$\Rightarrow \sum_{p=0}^{\infty} \frac{1}{(2p+1)^4} = \underbrace{\left(1 - \frac{1}{16}\right)}_{\frac{15}{16}} \zeta(4) = \frac{\pi^4}{96}$$