TD 5 - Séries de Fourier

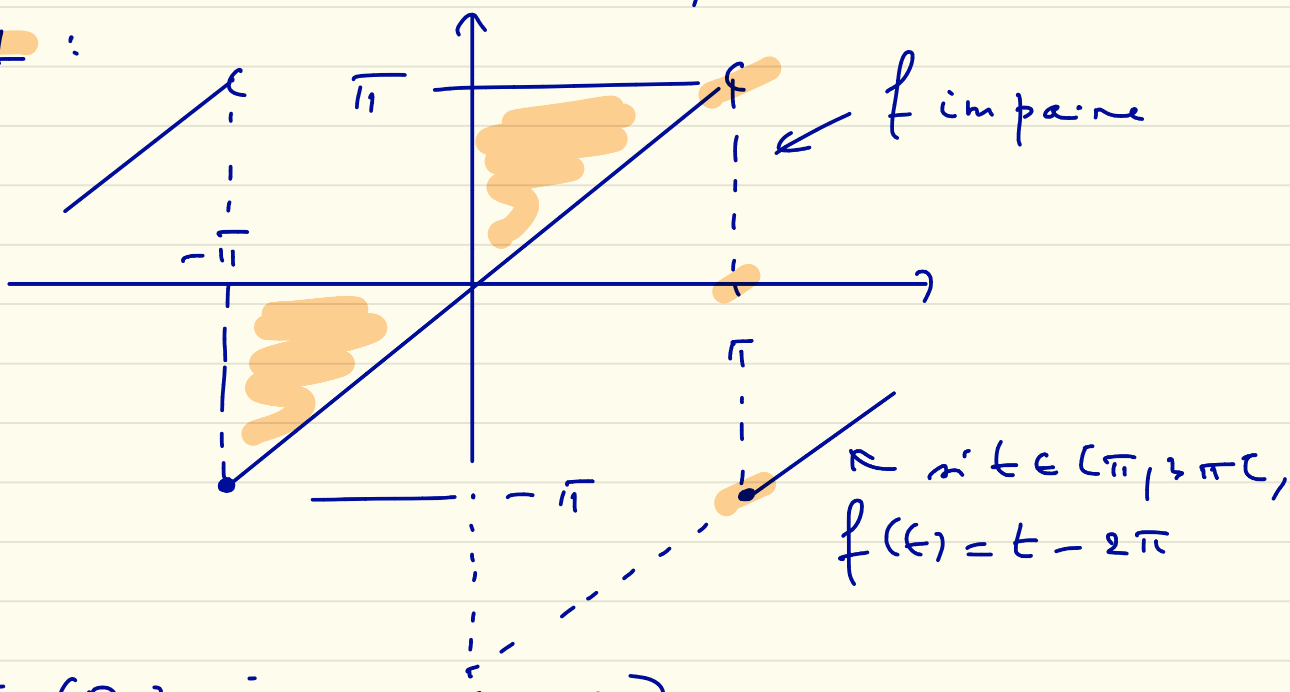
Exo 1. Fonction ζ de Riemann.

$\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$, $\zeta(6) = \frac{\pi^6}{945}$,

$\zeta(8) = \frac{\pi^8}{9450}$, $\zeta(10) = \frac{\pi^{10}}{93555}$ ...

1.1. $\zeta(2)$: pour calculer $\zeta(2p)$, on calcule la série de Fourier de $f : \mathbb{R} \to \mathbb{R}$ 2π-périodique tq $f_{|[-\pi,\pi[}(t) = t^p$;

$p = 1$:



$f \in L^2_{2\pi}(\mathbb{R})$, ie

$$\int_{-\pi}^{\pi} |f(t)|^2 \, dt < \infty \quad (\leq \pi^2)$$

puisque f se prolonge continûment sur $[-\pi, \pi]$, elle est donc bornée sur ce compact, donc de carré intégrable.

On peut donc calculer sa série de Fourier en la décomposant sur la b.h.

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos nt}{\sqrt{\pi}}, \frac{\sin nt}{\sqrt{\pi}}, n \geq 1 \right\}$$

<u>Rappel</u> : $(f \mid g)_{L^2_{2\pi}(\mathbb{R})} = \int_{-\pi}^{\pi} f(t) \cdot g(t)\, dt$

$$\Longrightarrow \quad f \underset{\substack{\uparrow \\ \text{cv } L^2_{2\pi}}}{=} \sum_{n=0}^{\infty} (f \mid e_n) \cdot e_n \qquad (\text{cf. } \S$$

i.e : $\left\| f - \sum_{n=0}^{N} (f \mid e_n) \cdot e_n \right\|_2 \xrightarrow[N \to \infty]{} 0$,

et Parseval :
$$\|f\|_2^2 = \sum_{n=0}^{\infty} |(f \mid e_n)|^2 .$$

Ici,
$$f = \underbrace{\left(f \,\middle|\, \frac{1}{\sqrt{2\pi}}\right)}_{a_0} \cdot \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} \overbrace{\left(f \,\middle|\, \frac{\cos nt}{\sqrt{\pi}}\right)}^{a_n} \cdot \frac{\cos nt}{\sqrt{\pi}}$$
$$+ \underbrace{\left(f \,\middle|\, \frac{\sin nt}{\sqrt{\pi}}\right)}_{b_n} \cdot \frac{\sin nt}{\sqrt{\pi}}$$

i.e :
$$\int \left| f(t) - \left(\left(f \,\middle|\, \frac{1}{\sqrt{2\pi}}\right) \cdot \frac{1}{\sqrt{2\pi}} + \sum_{n=0}^{N} \left(f \,\middle|\, \frac{\cos nt}{\sqrt{\pi}}\right) \cdot \frac{\cos nt}{\sqrt{\pi}} + \left(f \,\middle|\, \frac{\sin nt}{\sqrt{\pi}}\right) \cdot \frac{\sin nt}{\sqrt{\pi}} \right) \right|^2 dt \xrightarrow[N \to \infty]{} 0$$

f étant impaire, $a_0 = a_n = 0$, $\forall n \geq 1$, cf.

$$a_0 = \left(f \,\Big|\, \frac{1}{\sqrt{2\pi}}\right) = \underbrace{\int_{-\pi}^{\pi} \underbrace{f(t)}_{\text{imp.}} \cdot \underbrace{\frac{1}{\sqrt{2\pi}}}_{\text{p.}}}_{\text{impair}} dt = 0$$

(idem pour a_n, $n \geq 1$)

$$b_n = \left(f \,\Big|\, \frac{\sin nt}{\sqrt{\pi}}\right)$$

$$= \int_{-\pi}^{\pi} \underbrace{f(t) \cdot \frac{\sin nt}{\sqrt{\pi}}}_{\text{pair}} dt$$

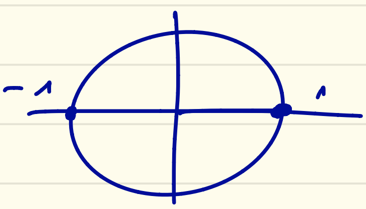
$$= 2 \int_0^{\pi} f(t) \cdot \frac{\sin nt}{\sqrt{\pi}} dt$$

et $\begin{cases} f = \sum_{n=1}^{\infty} b_n \cdot \dfrac{\sin nt}{\sqrt{\pi}} \quad (\text{cv dans } L^2_{2\pi}) \\ \|f\|_2^2 = \sum_{n=1}^{\infty} b_n^2 \end{cases}$

$$b_n = \frac{2}{\sqrt{\pi}} \int_0^{\pi} t \cdot \sin nt \, dt \quad \text{ipp}$$

$$= \frac{2}{\sqrt{\pi}} \left[t \cdot \frac{-\cos nt}{n} \right]_0^{\pi} + \frac{2}{\sqrt{\pi}} \int_0^{\pi} \frac{\cos nt}{n} dt$$

$$= \frac{2}{n\sqrt{\pi}} \left[-\pi \underbrace{\cos n\pi}_{(-1)^n} \right] + \frac{2}{\sqrt{\pi} \cdot n^2} \underbrace{\cancel{\left[\sin nt \right]_0^{\pi}}}_{0}$$



$$\Rightarrow b_n = \frac{(-1)^{n+1} \cdot 2\sqrt{\pi}}{n}$$

Parseval : $\|f\|_2^2 = \sum_{n=1}^{\infty} b_n^2 = \sum_{n=1}^{\infty} \frac{4\pi}{n^2}$

$$\| \quad \int_{-\pi}^{\pi} |f(t)|^2\, dt = 2\int_0^{\pi} t^2 \cdot dt = \frac{2\pi^3}{3}$$

$$\Rightarrow \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4\pi} \cdot \frac{2\pi^3}{3} = \frac{\pi^2}{6}.$$

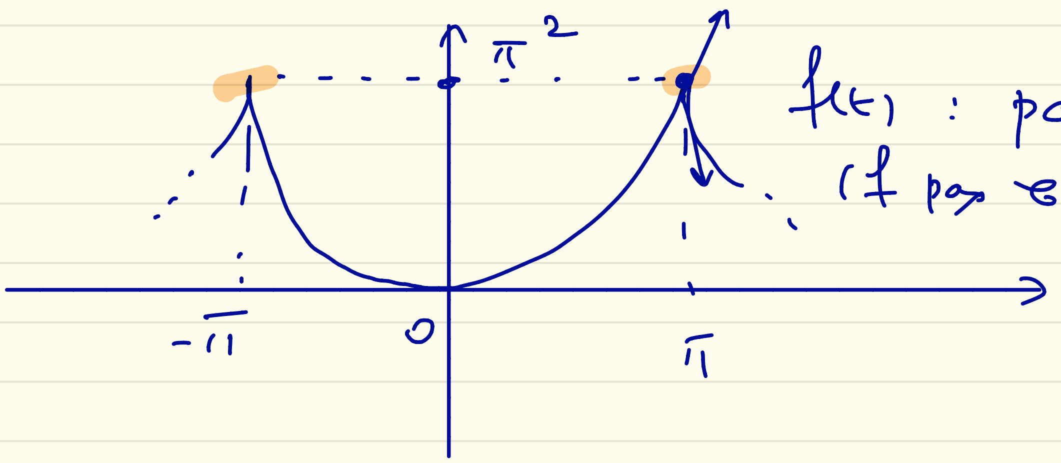
Remarque : f $\mathcal{C}^1$ par morceaux donc

$$S_N(t) \xrightarrow[N\to\infty]{} \frac{f(t^+)+f(t^-)}{2}$$

Si $t \neq (2k+1)\pi$ ($k \in \mathbb{Z}$), f continue en t, donc $S_N(t) \xrightarrow[N\to\infty]{} f(t)$;

Si $t = (2k+1)\pi$, $S_N(t) \xrightarrow[N\to\infty]{} 0$

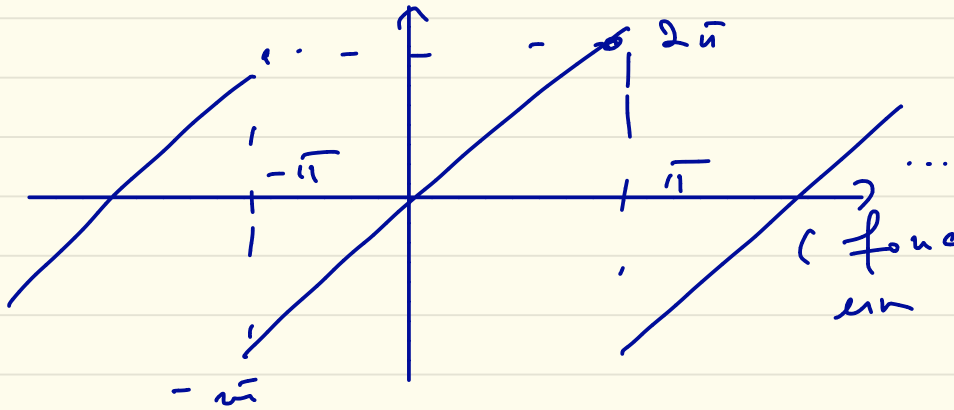
1.2. $\zeta(4)$: $p = 2$, $f(t) = t^2$ si $t \in [-\pi, \pi]$, prolongée par 2π — périodicité sur $\mathbb{R}$:



$f(t)$: paire (f pas $\mathcal{C}^1$!)

Remarque : $f \in H^1_{2\pi}$ car $f(t) = f(0) + \int_0^t g(s)\,ds$, $g \in L^2_{2\pi}$

où g est la fonction tq $g(t) = 2t$ si $t \in [-\pi, \pi[$ (et prolongée à $\mathbb{R}$ par 2π-périodicité) :



(fonction du 1.1 à un coeff. 2 près)

Ici, f paire $\Longrightarrow b_n = 0$, $n \geqslant 1$;

$$a_0 = \left(f \,\middle|\, \frac{1}{\sqrt{2\pi}}\right) = \int_{-\pi}^{\pi} f(t) \cdot \frac{1}{\sqrt{2\pi}}\, dt$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\pi} f(t)\, dt$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{t^3}{3} \right]_0^{\pi} = \frac{2\pi^3}{3\sqrt{2\pi}}$$

$$a_n = \left(f \,\middle|\, \frac{\cos nt}{\sqrt{\pi}}\right) = \int_{-\pi}^{\pi} f(t) \cdot \frac{\cos nt}{\sqrt{\pi}}\, dt$$

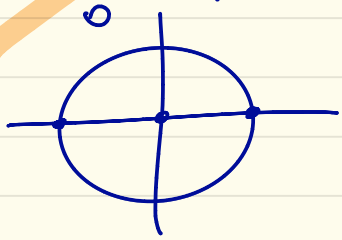
$$= 2 \int_0^{\pi} f(t) \cdot \frac{\cos nt}{\sqrt{\pi}}\, dt$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\pi} \underbrace{t^2 \cdot \cos nt}_{\text{i.p.p.}}\, dt$$

$$\int_0^\pi \underbrace{t^2}_{u} \cdot \underbrace{\cos nt}_{v'}\, dt$$

$$= \cancel{\left[t^2 \cdot \frac{\sin nt}{n}\right]_0^\pi} - \int_0^\pi 2t \cdot \frac{\sin nt}{n} = -\frac{2}{n} \underbrace{\int_0^\pi t \cdot \sin nt}_{\frac{-\pi(-1)^n}{n}}$$

0 (cf. 1.1)



$\sin n\pi = 0$

$$\Rightarrow a_n = \frac{2}{\sqrt{\pi}} \cdot \frac{2}{n} \cdot \frac{\pi(-1)^n}{n}$$

$$= \frac{4\sqrt{\pi}}{n^2} \cdot (-1)^n \quad \text{et } b_n = 0$$

Parseval : $\|f\|_2^2 = a_0^2 + \sum_{n=1}^{\infty} a_n^2 \quad \left(= \sum_{n=0}^{\infty} a_n^2\right)$

$$\|$$

$$2\int_0^\pi (t^2)^2\, dt = \left.2\frac{t^5}{5}\right]_0^\pi = \frac{2\pi^5}{5}$$

$$\Rightarrow \frac{2\pi^5}{5} = \frac{2\pi^5}{9} + \sum_{n=1}^{\infty} \frac{16 \cdot \pi}{n^4}$$

$$\Rightarrow \zeta(4) = \frac{\pi^5}{16 \cdot \pi}\left(\frac{2}{5} - \frac{2}{5}\right)$$

$$= \frac{\pi^4}{8}\left(\frac{4}{45}\right) = \frac{\pi^4}{90}.$$

Remarque : $(S_N)_N$ cvu vers $f \Rightarrow$ cvs en tout t :

$(\forall t \in \mathbb{R})$: $S_N(t) \xrightarrow[N \to \infty]{} f(t)$;

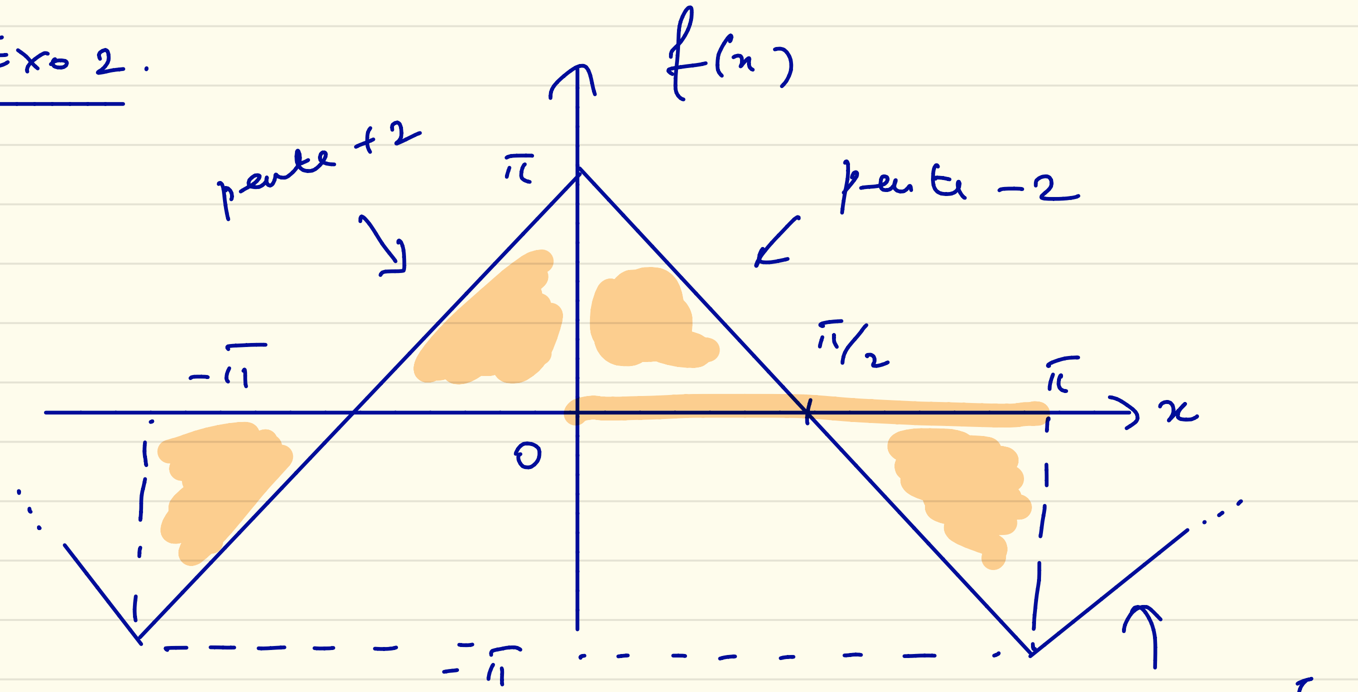
en particulier, $f(0) = 0 = a_0 \cdot \underbrace{\frac{1}{\sqrt{2\pi}}} + \sum_{n=1}^{\infty} a_n \cdot \frac{\overbrace{\cos nt}^{1}}{\sqrt{\pi}}\Big|_{t=0}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{\pi}} = -a_0$

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n 4 \cancel{\sqrt{\pi}}}{n^2} \cdot \frac{1}{\cancel{\sqrt{\pi}}} = -\frac{2}{3} \cdot \frac{\pi^3}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$ (cf. $\zeta(2) = \frac{\pi^2}{6}$)

Exo 2.



$t \in [\pi, 2\pi[$, $f(x) = 2x - 3\pi$

f $C^0 \Rightarrow f \in L^2_{2\pi}(\mathbb{R})$

2.1. f paire $\Rightarrow b_n = 0$

et $f \underset{\text{CV } L^2_{2\pi}}{=} a_0 \cdot \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} a_n \cdot \frac{\cos nt}{\sqrt{\pi}}$

$$a_0 = \left(f \,\middle|\, \frac{1}{\sqrt{2\pi}}\right) = \int_{-\pi}^{\pi} f(x) \cdot \frac{1}{\sqrt{2\pi}}\, dx$$

$$= 2 \int_0^{\pi} f(x) \cdot \frac{1}{\sqrt{2\pi}}\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\pi} (\pi - 2x) \cdot 1 \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \left[\pi x - x^2\right]_0^{\pi}$$

$$= 0$$

$$a_n = \left(f \,\middle|\, \frac{\cos nx}{\sqrt{\pi}}\right) = \int_{-\pi}^{\pi} f(x) \cdot \frac{\cos nx}{\sqrt{\pi}}\, dx$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\pi} \underbrace{(\pi - 2x) \cos nx}_{ipp} \cdot dx$$

$$\int_0^{\pi} (\pi - 2x) \cdot \cos nx \, dx$$

$$= \int_0^{\pi} \overbrace{-2x}^{u} \cdot \overbrace{\cos nx}^{v'} \, dx$$

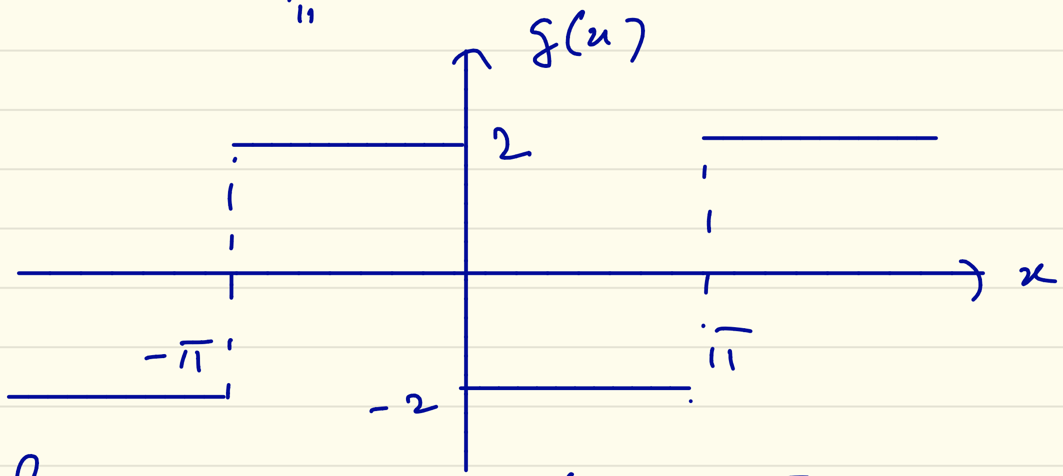
$$= \underbrace{\cancel{\left[-2x \cdot \frac{\sin nx}{x}\right]_0^{\pi}}}_{0} + \int_0^{\pi} 2 \, \frac{\sin nx}{n}\, dx$$

$$= -\frac{2}{n^2} \cos nx \Big]_0^{\pi} = -\frac{2}{n^2} \left((-1)^n - 1\right)$$

$$\begin{cases} = 0 & \text{si } n \text{ pair} \\ = +\dfrac{4}{n^2} & \text{si } n \text{ impair} \end{cases}$$

$$\Rightarrow \begin{cases} a_{2p} = 0, \quad p \geq 0 \ (\text{inclut } a_0 = 0) \\ a_{2p+1} = \dfrac{2}{\sqrt{\pi}}\left(+\dfrac{4}{(2p+1)^2}\right) = +\dfrac{8}{(2p+1)^2\cdot\sqrt{\pi}} \end{cases}$$

2.2. $f(t) = \underbrace{f(0)}_{\pi} + \int_0^t g(s)\,ds, \quad g \in L^2_{2\pi}(\mathbb{R})$



$\Rightarrow$ f est $C^1_{2\pi}$ et sa série de Fourier cvu vers f ; en particulier, cvu $\Rightarrow$ cvs :

$$(\forall t \in \mathbb{R}) : f(t) = a_0 \cdot \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} a_n \cdot \frac{\cos nt}{\sqrt{\pi}} .$$

2.3. En $t=0$, on a :

$$f(0) = \pi = \frac{\cancel{a_0}^{\,0}}{\sqrt{2\pi}} + \sum_{p=0}^{\infty} a_{2p+1} \cdot \frac{1}{\sqrt{\pi}}$$

$$= \sum_{p=0}^{\infty} \frac{8}{(2p+1)^2 \cdot \pi}$$

$$\Rightarrow \sum_{p=0}^{\infty} \frac{1}{(2p+1)^2} = \frac{\pi^2}{8} < \zeta(2) = \frac{\pi^2}{6}$$

__Remarque__ : $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$

$$= \underbrace{\sum_{p=1}^{\infty} \frac{1}{(2p)^2}}_{\frac{1}{4}\cdot\zeta(2)} + \sum_{p=0}^{\infty} \frac{1}{(2p+1)^2}$$

$$\Rightarrow \underbrace{\left(1 - \frac{1}{4}\right)}_{3/4} \underbrace{\zeta(2)}_{\pi^2/6} = \sum_{p=0}^{\infty} \frac{1}{(2p+1)^2} = \frac{\pi^2}{8}.$$

De même : $\underbrace{\left(1 - \frac{1}{2^4}\right)}_{15/16} \underbrace{\zeta(4)}_{\pi^4/90} = \sum_{p=0}^{\infty} \frac{1}{(2p+1)^4} = \frac{\pi^4}{96}$

Ce que l'on trouve avec Parseval :

$$\|f\|_2^2 = \underset{0}{\cancel{a_0^2}} + \sum_{n=1}^{\infty} a_n^2 = \sum_{p=0}^{\infty} a_{2p+1}^2$$

$$= \frac{64}{\pi} \sum_{p=0}^{\infty} \frac{1}{(2p+1)^4}$$

$$2\int_0^{\pi} (\pi - 2x)^2 \, dx = \left[\frac{(2x-\pi)^3}{3}\right]_0^{\pi} = \frac{2\pi^3}{3}$$

$$\Rightarrow \sum_{p=0}^{\infty} \frac{1}{(2p+1)^4} = \frac{\pi}{64} \cdot \frac{2\pi^3}{3} = \frac{\pi^4}{96}.$$