Course: Cryptography and Network Security
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Lecture – 7: Primes and Congruence

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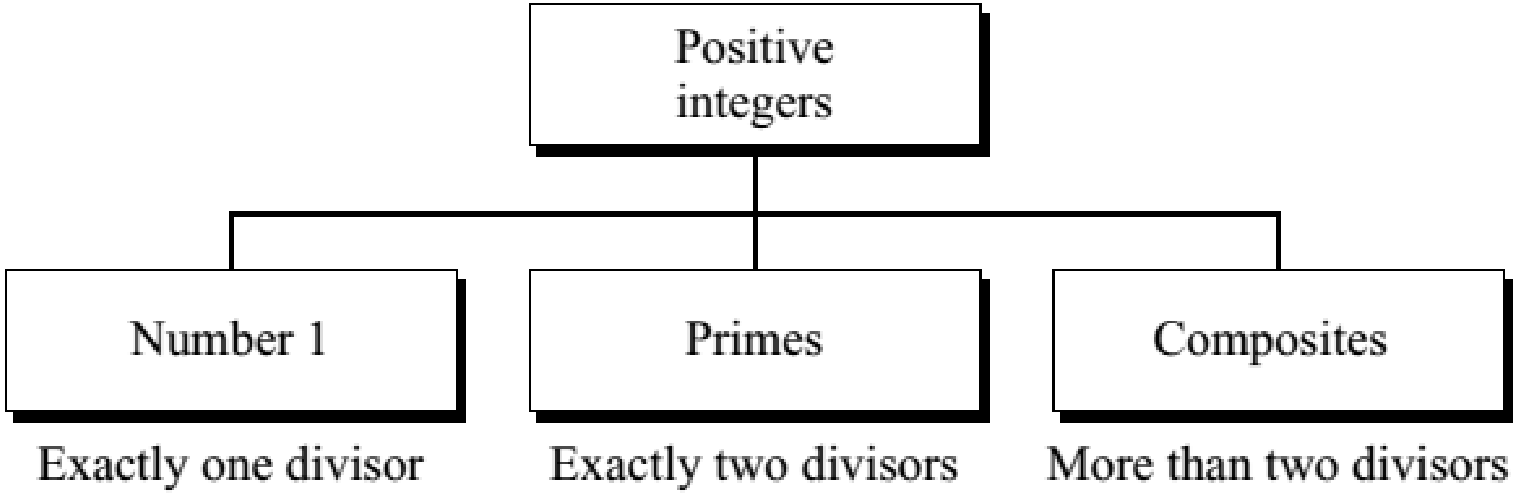
PRIMES

- Asymmetric-key cryptography uses primes extensively.
- The topic of primes is a large part of any book on number theory.
- The positive integers can be divided into three groups: the number 1, primes, and composites.



PRIMES

- A positive integer is a prime if and only if it is exactly divisible by two integers, 1 and itself.
- A composite is a positive integer with more than two divisors.
- Two positive integers, a and b, are relatively prime, or coprime, if gcd (a, b) = 1.
- If p is a prime, then all integers 1 to p − 1 are relatively prime to p.
- Is there a finite number of primes or is the list infinite?
  - There is an infinite number of primes.
- A function called $\pi(n)$ is defined that finds the number of primes smaller than or equal to n.

$$\pi(1) = 0 \quad \pi(2) = 1 \quad \pi(3) = 2 \quad \pi(10) = 4 \quad \pi(20) = 8 \quad \pi(50) = 15 \quad \pi(100) = 25$$

PRIMES

- But if n is very large, how can we calculate π(n)?
- The answer is that we can only use approximation.

$$[n / (\ln n)] \quad < \quad \pi(n) \quad < \quad [n/(\ln n - 1.08366)]$$

- Gauss discovered the upper limit; Lagrange discovered the lower limit.
- Find the number of primes less than 1,000,000.
- The approximation gives the range 72,383 to 78,543. The actual number of primes is 78,498.

Checking for Primeness

- Given a number n, how can we determine if n is a prime?
- The answer is that we need to see if the number is divisible by all primes less than √n.
- Is 97 a prime?
  - The floor of √97= 9.
  - The primes less than 9 are 2, 3, 5, and 7.
  - We need to see if 97 is divisible by any of these numbers.
  - It is not, so 97 is a prime.
- Is 301 a prime?
  - The floor of √301= 17.
  - We need to check 2, 3, 5, 7, 11, 13, and 17.
  - The numbers 2, 3, and 5 do not divide 301, but 7 does.
  - Therefore 301 is not a prime.

Euler’s Phi-Function

- Euler’s phi-function, $\phi(n)$, which is sometimes called the Euler’s totient function plays a very important role in cryptography.
- The function finds the number of integers that are both smaller than n and relatively prime to n.
- The following helps to find the value of$\phi(n)$

1. $\phi(1) = 0$.
2. $\phi(p) = p - 1$ if p is a prime.
3. $\phi(m \times n) = \phi(m) \times \phi(n)$ if m and n are relatively prime.
4. $\phi(p^e) = p^e - p^{e-1}$ if p is a prime.

Euler’s Phi-Function

- We can combine the above four rules to find the value of $\phi(n)$. For example, if n can be factored as

$$n = p_1^{e1} \times p_2^{e2} \times \ldots \times p_k^{ek}$$

- Then we combine the third and the fourth rule to find

$$\phi(n) = (p_1^{e_1} - p_1^{e_1 - 1}) \times (p_2^{e_2} - p_2^{e_2 - 1}) \times \cdots \times (p_k^{e_k} - p_k^{e_k - 1})$$

The difficulty of finding ϕ(n) depends on the difficulty of finding the factorization of n.

Euler's Phi-Function

- Example 1
  - What is the value of $\phi(13)$?
- Solution
  - Because 13 is a prime, $\phi(13) = (13 - 1) = 12$.
- Example 2
  - What is the value of $\phi(10)$?
- Solution
  - We can use the third rule: $\phi(10) = \phi(2) \times \phi(5) = 1 \times 4 = 4$, because 2 and 5 are primes.

Euler's Phi-Function

- Example 3
  - What is the value of $\phi(240)$?
- Solution
  - We can write $240 = 2^4 \times 3^1 \times 5^1$. Then

$$\phi\,(240) = (2^4 - 2^3) \times (3^1 - 3^0) \times (5^1 - 5^0) = 64$$

- Example 4
  - Can we say that $\phi\,(49) = \phi\,(7) \times \phi\,(7) = 6 \times 6 = 36$????

Euler's Phi-Function

- Example 3
  - What is the value of $\phi(240)$?
- Solution
  - We can write $240 = 2^4 \times 3^1 \times 5^1$. Then

    $$\phi\,(240) = (2^4 - 2^3) \times (3^1 - 3^0) \times (5^1 - 5^0) = 64$$
- Example 4
  - Can we say that $\phi\,(49) = \phi\,(7) \times \phi\,(7) = 6 \times 6 = 36$????
- Solution
  - No. The third rule applies when *m* and *n* are relatively prime. Here $49 = 7^2$. We need to use the fourth rule: $\phi\,(49) = 7^2 - 7^1 = 42$.

Euler's Phi-Function

- Example 5
  - What is the number of elements in $Z_{14}{}^{*}$?
- Solution
  - The answer is $\phi(14) = \phi(7) \times \phi(2) = 6 \times 1 = 6$. The members are 1, 3, 5, 9, 11, and 13.

Interesting point: If $n > 2$, the value of $\phi(n)$ is even.

Fermat's Little Theorem

- First Version
  - If p is a prime and a is an integer such that p does not divide a,

$$a^{p-1} \equiv 1 \bmod p$$

- Second Version
  - Removes the condition on a
  - If p is prime and a is an integer,

$$a^{p} \equiv a \bmod p$$

Fermat’s Little Theorem

- Example 1
  - Find the result of 6^{10} mod 11.
- Solution
  - We have 6^{10} mod 11 = 1. This is the first version of Fermat’s little theorem where p = 11.
- Example 2
  - Find the result of 3^{12} mod 11.
- Solution

Fermat's Little Theorem

- Example 1
  - Find the result of 6^{10} mod 11.
- Solution
  - We have 6^{10} mod 11 = 1. This is the first version of Fermat's little theorem where p = 11.
- Example 2
  - Find the result of 3^{12} mod 11.
- Solution
  - Here the exponent (12) and the modulus (11) are not the same. With substitution this can be solved using Fermat's little theorem.

$$3^{12} \bmod 11 = (3^{11} \times 3) \bmod 11 = (3^{11} \bmod 11)\,(3 \bmod 11) = (3 \times 3) \bmod 11 = 9$$

Fermat's Little Theorem

- Multiplicative Inverses

$$a^{-1} \bmod p = a^{p-2} \bmod p$$

- The answers to multiplicative inverses modulo a prime can be found without using the extended Euclidean algorithm:

a. $8^{-1} \bmod 17 = 8^{17-2} \bmod 17 = 8^{15} \bmod 17 = 15 \bmod 17$

b. $5^{-1} \bmod 23 = 5^{23-2} \bmod 23 = 5^{21} \bmod 23 = 14 \bmod 23$

c. $60^{-1} \bmod 101 = 60^{101-2} \bmod 101 = 60^{99} \bmod 101 = 32 \bmod 101$

d. $22^{-1} \bmod 211 = 22^{211-2} \bmod 211 = 22^{209} \bmod 211 = 48 \bmod 211$

Euler's Theorem

- The modulus in the Fermat theorem is a prime, the modulus in Euler's theorem is an integer.
  - First Version
    - If a and n are coprime,

$$a^{\phi(n)} \equiv 1 \pmod{n}$$

  - Second Version
    - Removes the condition that a and n should be coprime

$$a^{k \times \phi(n) + 1} \equiv a \pmod{n}$$

The second version of Euler's theorem is used in the RSA cryptosystem

Euler’s Theorem

- Example 1
  - Find the result of 6^{24} mod 35.
- Solution
  - We have 6^{24} mod 35 = $6^{\phi(35)}$ mod 35 = 1.
- Example 2
  - Find the result of 20^{62} mod 77???

Euler's Theorem

- Example 1
  - Find the result of $6^{24} \bmod 35$.
- Solution
  - We have $6^{24} \bmod 35 = 6^{\phi(35)} \bmod 35 = 1$.
- Example 2
  - Find the result of $20^{62} \bmod 77$.
- Solution

If we let $k = 1$ on the second version, we have

$$20^{62} \bmod 77 = (20 \bmod 77)\,(20^{\phi(77)+1} \bmod 77) \bmod 77$$
$$= (20)(20) \bmod 77 = 15.$$

Euler’s Theorem

- Multiplicative Inverses
  - Euler’s theorem can be used to find multiplicative inverses modulo a composite.

$$a^{-1} \bmod n = a^{\phi(n)-1} \bmod n$$

Euler's Theorem

- Example
  - The answers to multiplicative inverses modulo a composite can be found without using the extended Euclidean algorithm if we know the factorization of the composite:

a. $8^{-1} \bmod 77 = 8^{\phi(77)-1} \bmod 77 = 8^{59} \bmod 77 = 29 \bmod 77$

b. $7^{-1} \bmod 15 = 7^{\phi(15)-1} \bmod 15 = 7^{7} \bmod 15 = 13 \bmod 15$

c. $60^{-1} \bmod 187 = 60^{\phi(187)-1} \bmod 187 = 60^{159} \bmod 187 = 53 \bmod 187$

d. $71^{-1} \bmod 100 = 71^{\phi(100)-1} \bmod 100 = 71^{39} \bmod 100 = 31 \bmod 100$

Generating Primes

Mersenne Primes $\boxed{M_p = 2^p - 1}$

$M_2 = 2^2 - 1 = 3$
$M_3 = 2^3 - 1 = 7$
$M_5 = 2^5 - 1 = 31$
$M_7 = 2^7 - 1 = 127$
$M_{11} = 2^{11} - 1 = 2047$ **Not a prime ($2047 = 23 \times 89$)**
$M_{13} = 2^{13} - 1 = 8191$
$M_{17} = 2^{17} - 1 = 131071$

A number in the form $M_p = 2^p - 1$ is called a Mersenne number and may or may not be a prime.

Generating Primes

Fermat Primes $\boxed{F_n = 2^{2^n} + 1}$

$F_0 = 3$
$F_1 = 5$
$F_2 = 17$
$F_3 = 257$
$F_4 = 65537$
$F_5 = 4294967297 = 641 \times 6700417$ **Not a prime**

No number greater than F4 has been proven to be a prime.
As a matter of fact many numbers up to F24 have been proven to be composite numbers.

Primality Testing

- Finding an algorithm to correctly and efficiently test a very large integer and output a prime or a composite has always been a challenge in number theory.
- Two types
  - Deterministic Algorithms
  - Probabilistic Algorithms

Primality Testing

Divisibility_Test (n) // n is the number to test for primality

{
  $r \leftarrow 2$
  while (r < $\sqrt{n}$)
  {
   if $(r \mid n)$ return ***"a composite"***
   $r \leftarrow r + 1$
  }
  return ***"a prime"***
}

The bit-operation complexity of the divisibility test is $O(2^{n/2})$ (exponential)

Deterministic Algorithms

```
Divisibility_Test (n)                    // n is the number to test for primality
{
  r ← 2
  while (r < √n )
  {
   if (r | n) return "a composite"
   r ← r + 1
  }
  return "a prime"
}
```

The bit-operation complexity of the divisibility test is $O(2^{n_b/2})$ (exponential)

Deterministic Algorithms

- Example
  - Assume n has 200 bits. What is the number of bit operations needed to run the divisibility-test algorithm?
- Solution
  - The bit-operation complexity of this algorithm is $2^{n_b/2}$. This means that the algorithm needs 2^{100} bit operations. On a computer capable of doing 2^{30} bit operations per second, the algorithm needs 2^{70} seconds to do the testing !!!!!

Deterministic Algorithms

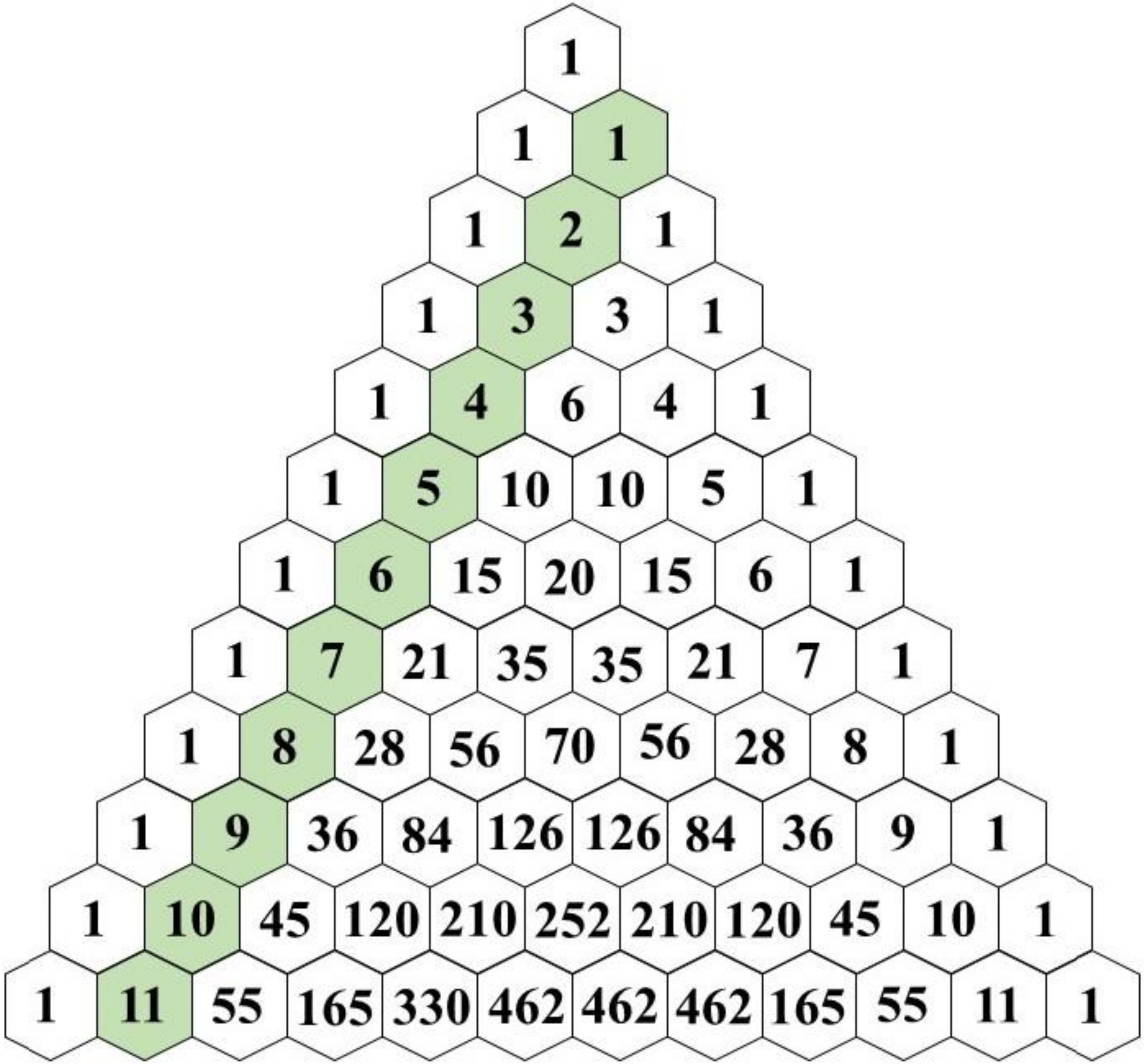
AKS Algorithm

$$(x - a)^p \equiv (x^p - a) \bmod p.$$

$$O((\log_2 n_b)^{12})$$

Tutorial

- Example
  - Assume n has 200 bits. What is the number of bit operations needed to run the AKS algorithm?
- Solution
  - This algorithm needs only $(\log_2 200)^{12}$ = 39,547,615,483 bit operations. On a computer capable of doing 1 billion bit operations per second, the algorithm needs only 40 seconds.



1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 462 165 55 11 1

Probabilistic Algorithms

Fermat Test

If n is a prime, then $a^{n-1} \equiv 1 \bmod n$.

If n is a prime, $a^{n-1} \equiv 1 \bmod n$
If n is a composite, it is possible that $a^{n-1} \equiv 1 \bmod n$

Probabilistic Algorithms

- Example
  - Does the number 561 pass the Fermat test?
- Solution
  - Use base 2

$$2^{561-1} = 1 \bmod 561$$

  - The number passes the Fermat test, but it is not a prime, because 561 = 33 × 17.

Probabilistic Algorithms

Square Root Test

If n is a prime, $\sqrt{1} \bmod n = \pm 1$.
If n is a composite, $\sqrt{1} \bmod n = \pm 1$ and possibly other values.

- Example
  - What are the square roots of 1 mod n if n is 7 (a prime)?
- Solution
  - The only square roots are 1 and −1. We can see that

$1^2 = 1 \bmod 7$	$(-1)^2 = 1 \bmod 7$
$2^2 = 4 \bmod 7$	$(-2)^2 = 4 \bmod 7$
$3^2 = 2 \bmod 7$	$(-3)^2 = 2 \bmod 7$

– Note that we don't have to test 4, 5 and 6 because 4 = –3 mod 7, 5 = –2 mod 7 and 6 = –1 mod 7.

Probabilistic Algorithms

Square Root Test

- Example
  - What are the square roots of 1 mod n if n is 8 (a composite)?
- Solution
  - There are four solutions: 1, 3, 5, and 7 (which is −1). We can see that

$$1^2 = 1 \bmod 8 \qquad (-1)^2 = 1 \bmod 8$$
$$3^2 = 1 \bmod 8 \qquad 5^2 = 1 \bmod 8$$

Probabilistic Algorithms

Square Root Test

- Example
  - What are the square roots of 1 mod n if n is 17 (a prime)?
- Solution
  - There are only two solutions: 1 and −1

$1^2 = 1 \bmod 17$	$(-1)^2 = 1 \bmod 17$
$2^2 = 4 \bmod 17$	$(-2)^2 = 4 \bmod 17$
$3^2 = 9 \bmod 17$	$(-3)^2 = 9 \bmod 17$
$4^2 = 16 \bmod 17$	$(-4)^2 = 16 \bmod 17$
$5^2 = 8 \bmod 17$	$(-5)^2 = 8 \bmod 17$
$6^2 = 2 \bmod 17$	$(-6)^2 = 2 \bmod 17$
$(7)^2 = 15 \bmod 17$	$(-7)^2 = 15 \bmod 17$
$(8)^2 = 13 \bmod 17$	$(-8)^2 = 13 \bmod 17$

Probabilistic Algorithms

Square Root Test

- Example
  - What are the square roots of 1 mod n if n is 22 (a composite)??????

- Solution
  - Surprisingly, there are only two solutions, +1 and −1, although 22 is a composite.

$$1^2 = 1 \bmod 22$$
$$(-1)^2 = 1 \bmod 22$$

Probabilistic Algorithms

- Miller-Rabin Test

$$n - 1 = m \times 2^k$$

$$a^{n-1} = a^{m \times 2^k} = [a^m]^{2^k} = [a^m]^{\overbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}^{k \text{ times}}}$$

The Miller-Rabin test needs from step 0 to step k − 1.

Probabilistic Algorithms

```
Miller_Rabin_Test (n, a)                         // n is the number; a is the base.
{
    Find m and k such that n − 1 = m × 2^k
    T ← a^m mod n
    if (T = ± 1)   return "a prime"
    for (i ← 1 to k − 1)                         // k − 1 is the maximum number of steps.
    {
        T ← T^2 mod n
        if (T = +1) return "a composite"
        if (T = −1) return "a prime"
    }
    return "a composite"
}
```

Probabilistic Algorithms

- Example
  - Does the number 561 pass the Miller-Rabin test?
- Solution
  - Using base 2, let $561 - 1 = 35 \times 2^4$, which means m = 35, k = 4, and a = 2.

Initialization:	$T = 2^{35} \bmod 561 = 263 \bmod 561$	
$k = 1$:	$T = 263^2 \bmod 561 = 166 \bmod 561$	
$k = 2$:	$T = 166^2 \bmod 561 = 67 \bmod 561$	
$k = 3$:	$T = 67^2 \bmod 561 = +1 \bmod 561$	$\rightarrow$ **a composite**

Probabilistic Algorithms

- Example
  - We already know that 27 is not a prime. Let us apply the Miller-Rabin test.
- Solution
  - With base 2, let $27 - 1 = 13 \times 2^1$, which means that $m = 13$, $k = 1$, and $a = 2$. The initialization step: $T = 2^{13} \bmod 27 = 11 \bmod 27$. However, because the algorithm enters the loop only once, it returns a composite.

Probabilistic Algorithms

- Example
  - We know that 61 is a prime, let us see if it passes the Miller-Rabin test.
- Solution
  - We use base 2.

$$61 - 1 = 15 \times 2^2 \quad \rightarrow \quad m = 15 \quad k = 2 \quad a = 2$$

$$\textit{Initialization:} \quad T = 2^{15} \bmod 61 = 11 \bmod 61$$

$$k = 1 \qquad T = 11^2 \bmod 61 = -1 \bmod 61 \quad \rightarrow \quad \textbf{a prime}$$

Recommended Primality test

- Combination of the divisibility test and the Miller-Rabin test.
- Example
  - The number 4033 is a composite (37×109). Does it pass the recommended primality test?
- Solution
  1. Perform the divisibility tests first. The numbers 2, 3, 5, 7, 11, 17, and 23 are not divisors of 4033.
  2. Perform the Miller-Rabin test with a base of 2, $4033 - 1 = 63 \times 64$, which means m is 63 and k is 6.

Initialization:	$T \equiv 2^{63} \pmod{4033} \equiv 3521 \pmod{4033}$	
$k = 1$	$T \equiv T^2 \equiv 3521^2 \pmod{4033} \equiv -1 \pmod{4033}$	$\rightarrow$ **Passes**

Recommended Primality test

But we are not satisfied. We continue with another base, 3.

Initialization: $T \equiv 3^{63} \pmod{4033} \equiv 3551 \pmod{4033}$

$k = 1$	$T \equiv T^2 \equiv 3551^2$ (mod 4033 $\equiv 2443 \pmod{4033}$
$k = 2$	$T \equiv T^2 \equiv 2443^2$ (mod 4033 $\equiv 3442 \pmod{4033}$
$k = 3$	$T \equiv T^2 \equiv 3442^2$ (mod 4033 $\equiv 2443 \pmod{4033}$
$k = 4$	$T \equiv T^2 \equiv 2443^2$ (mod 4033 $\equiv 3442 \pmod{4033}$
$k = 5$	$T \equiv T^2 \equiv 3442^2$ (mod 4033 $\equiv 2443 \pmod{4033}$ $\rightarrow$**Failed (composite)**

FACTORIZATION

Fundamental Theorem of Arithmetic

$$n = p_1^{e1} \times p_2^{e2} \times \cdots \times p_k^{ek}$$

- Greatest Common Divisor

$$a = p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_k^{a_k} \qquad b = p_1^{b_1} \times p_2^{b_2} \times \cdots \times p_k^{bk}$$

$$\gcd(a, b) = p_1^{\min(a_1, b_1)} \times p_2^{\min(a_2, b_2)} \times \cdots \times p_k^{\min(a_k, b_k)}$$

- Least Common Multiplier

$$a = p_1^{a1} \times p_2^{a2} \times \cdots \times p_k^{ak} \qquad b = p_1^{b1} \times p_2^{b2} \times \cdots \times p_k^{bk}$$

$$\mathrm{lcm}(a, b) = p_1^{\max(a_1, b_1)} \times p_2^{\max(a_2, b_2)} \times \cdots \times p_k^{\max(a_k, b_k)}$$

$$\mathrm{lcm}(a, b) \times \gcd(a, b) = a \times b$$

Factorization methods

- Trial Division Method

```
Trial_Division_Factorization (n)                    // n is the number to be factored
{
    a ← 2
    while (a ≤ √n )
    {
        while (n mod a = 0)
        {
            output a                                 // Factors are output one by one
            n = n / a
        }
        a ← a + 1
    }
    if (n > 1) output n                             // n has no more factors
}
```

FACTORIZATION

- Example
  - Use the trial division algorithm to find the factors of 1233.
- Solution
  - We run a program based on the algorithm and get the following result.

$$1233 = 3^2 \times 137$$

FACTORIZATION

- Example
  - Use the trial division algorithm to find the factors of 1523357784
- Solution
  - We run a program based on the algorithm and get the following result.

$$1523357784 = 2^3 \times 3^2 \times 13 \times 37 \times 43987$$

Fermat Method

$$n = x^2 - y^2 = a \times b \qquad \text{with } a = (x + y) \text{ and } b = (x - y)$$

```
Feramat_Factorization (n)                // n is the number to be factored
{
    x ← √n                               // smallest integer greater than √n
    while (x < n)
    {
    w ← x² − n
    if(w is perfect square)  y ← √w;  a ← x+y;  b ← x−y;   return a and b
    x ← x + 1
    }

}
```

FACTORIZATION

- More methods
  - Pollard p-1
  - Pollard rho
  - Number Field Sieve
  - Quadratic Sieve

Tutorial

CHINESE REMAINDER THEOREM

Used to solve a set of congruent equations with one variable but different moduli, which are relatively prime

$$
\begin{aligned}
x &\equiv a_1 \pmod{m_1} \\
x &\equiv a_2 \pmod{m_2} \\
&\cdots \\
x &\equiv a_k \pmod{m_k}
\end{aligned}
$$

CHINESE REMAINDER THEOREM

Used to solve a set of congruent equations with one variable but different moduli, which are relatively prime

$$
\begin{gathered}
x \equiv a_1 \pmod{m_1} \\
x \equiv a_2 \pmod{m_2} \\
\cdots \\
x \equiv a_k \pmod{m_k}
\end{gathered}
$$

CHINESE REMAINDER THEOREM

- Example
  - The following is an example of a set of equations with different moduli:

$$x \equiv 2 \pmod{3}$$
$$x \equiv 3 \pmod{5}$$
$$x \equiv 2 \pmod{7}$$

  - The solution to this set of equations is given in the next section; for the moment, note that the answer to this set of equations is $x = 23$. This value satisfies all equations: $23 \equiv 2 \pmod{3}$, $23 \equiv 3 \pmod{5}$, and $23 \equiv 2 \pmod{7}$.

CHINESE REMAINDER THEOREM

- Solution To Chinese Remainder Theorem
  - Find $M = m_1 \times m_2 \times \ldots \times m_k$. This is the common modulus.
  - Find $M_1 = M/m_1$, $M_2 = M/m_2$, …, $M_k = M/m_k$.
  - Find the multiplicative inverse of M_1, M_2, …, M_k using the corresponding moduli (m_1, m_2, …, m_k). Call the inverses M_1^{-1}, M_2^{-1}, …, M_k^{-1}.
  - The solution to the simultaneous equations is

$$\boxed{x = (a_1 \times M_1 \times M_1^{-1} + a_2 \times M_2 \times M_2^{-1} + \cdots + a_k \times M_k \times M_k^{-1}) \bmod M}$$

CHINESE REMAINDER THEOREM

- Example
  - Find the solution to the simultaneous equations:

$$
\begin{aligned}
x &\equiv 2 \pmod{3} \\
x &\equiv 3 \pmod{5} \\
x &\equiv 2 \pmod{7}
\end{aligned}
$$

- Solution: We follow the four steps.

1. $M = 3 \times 5 \times 7 = 105$

2. $M_1 = 105 / 3 = 35$, $M_2 = 105 / 5 = 21$, $M_3 = 105 / 7 = 15$

3. The inverses are $M_1^{-1} = 2$, $M_2^{-1} = 1$, $M_3^{-1} = 1$

4. $x = (2 \times 35 \times 2 + 3 \times 21 \times 1 + 2 \times 15 \times 1) \bmod 105 = 23 \bmod 105$

CHINESE REMAINDER THEOREM

- Example
  - Find an integer that has a remainder of 3 when divided by 7 and 13, but is divisible by 12.
- Solution ????

CHINESE REMAINDER THEOREM

- Example
  - Find an integer that has a remainder of 3 when divided by 7 and 13, but is divisible by 12.
- Solution
  - This is a CRT problem. We can form three equations and solve them to find the value of x.

$$x = 3 \bmod 7$$
$$x = 3 \bmod 13$$
$$x = 0 \bmod 12$$

  - If we follow the four steps, we find x = 276. We can check that
    276 = 3 mod 7, 276 = 3 mod 13 and 276 is divisible by 12 (the quotient is 23 and the remainder is zero).

CHINESE REMAINDER THEOREM

- Assume we need to calculate z = x + y where x = 123 and y = 334, but our system accepts only numbers less than 100. These numbers can be represented as follows:

$$
\begin{array}{ll}
x \equiv 24 \pmod{99} & y \equiv 37 \pmod{99} \\
x \equiv 25 \pmod{98} & y \equiv 40 \pmod{98} \\
x \equiv 26 \pmod{97} & y \equiv 43 \pmod{97}
\end{array}
$$

- Adding each congruence in x with the corresponding congruence in y gives

$$
\begin{array}{lcl}
x + y \equiv 61 \pmod{99} & \rightarrow & z \equiv 61 \pmod{99} \\
x + y \equiv 65 \pmod{98} & \rightarrow & z \equiv 65 \pmod{98} \\
x + y \equiv 69 \pmod{97} & \rightarrow & z \equiv 69 \pmod{97}
\end{array}
$$

- Now three equations can be solved using the Chinese remainder theorem to find z. One of the acceptable answers is z = 457.

CHINESE REMAINDER THEOREM

Secret Sharing scheme in cryptography aims to distribute and later recover secret S among n parties. Secret S is distributed in form of shares which are generated from secret. Without cooperation of k no. of parties, the secret cannot be reconstructed from shares directly. Consider the following example:

Say our secret is S. The shares for n=4 no. of parties are generated taking modulus 11,13,17 and 19. They are respectively 1,12,2 and 3 and given by following equations:

$S \equiv 1 \bmod 11,$

$S \equiv 12 \bmod 13,$

$S \equiv 2 \bmod 17,$

$S \equiv 3 \bmod 19.$

Now, from four possible sets of k=3 shares (as k shares are necessary to reconstruct the secret), consider one possible set {1, 12, 2} and recover the secret S from it.

CHINESE REMAINDER THEOREM

Solution: The problem can be solved by Chinese remainder theorem.

For the set {1,12,2}, the equations available are,

$S \equiv 1 \bmod 11,$

$S \equiv 12 \bmod 13,$

$S \equiv 2 \bmod 17,$

Now solving this equation using CRT, M=11 *13*17 = 2431,

M1 = 2431/11=221,

M2 = 2431/13=187,

M3=2431/17=143

$M1^{-1}$, $M2^{-1}$ and $M3^{-1}$ can be calculated using Extended Euclidean Algorithm.

$M1^{-1} = 1$

$M2^{-1} = 8$

$M3^{-1} = 5$

Now, secret S= ((1*221*1) + (12*187*8) + (2*143*5)) mod 2431

S = 155 mod 2431