Engineering Mathematics Solution

Final Term, 2012/01/02

1. (15%) If $\mathcal{L}\{\mathrm{f(t)}\} = \frac{k^2}{(s^2+k^2)^2}$, and $\mathrm{f(t)} = \frac{\mathrm{g(t)}}{2k}$, please find g(t).

ANS:

$$\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$$

$$\rightarrow \ \mathrm{f(t)} = \mathcal{L}^{-1}\left\{\frac{k^2}{(s^2+k^2)^2}\right\} = \int_0^t \sin k\tau \sin k(t-\tau)d\tau$$

$$= \frac{1}{2}\int_0^t [\cos k(2\tau - t) - \cos kt]d\tau$$

$$= \frac{1}{2}\left[\frac{1}{2\mathrm{k}}\sin k(2\tau - t) - \tau\cos kt\right]|_0^t$$

$$= \frac{\sin kt - kt\cos kt}{2k}$$

$$\rightarrow \mathrm{g(t)} = 2\mathrm{k}\,\mathrm{f(t)} = \sin kt - kt\cos kt$$

2. (10%) $\mathrm{f(t)} = t^2 + 3\mathrm{t} + 2$, find $\mathcal{L}\{\mathrm{f(t)H(t-2)}\}$

ANS:

$$\mathcal{L}\{\mathrm{f(t)H(t-2)}\}$$
$$= \mathcal{L}\{(t^2 + 3\mathrm{t} + 2)\mathrm{H(t-2)}\}$$
$$= \mathcal{L}\{((t-2)^2 + 7(\mathrm{t}-2) + 12)\mathrm{H(t-2)}\}$$
$$= \frac{2}{s^3}e^{-2s} + \frac{7}{s^2}e^{-2s} + \frac{12}{s}e^{-2s}$$

3. (10%) $\mathcal{L}\left\{e^{2t}\int_0^t t\cdot e^{3t}\mathrm{cost\ dt}\right\} = ?$

ANS:

$$\mathcal{L}\{\cos t\} = \frac{\mathrm{s}}{s^2+1}$$

$$\mathcal{L}\{\mathrm{t}\cos t\} = -\frac{d}{ds}\frac{s}{s^2+1} = \frac{s^2-1}{(s^2+1)^2}$$

$$\mathcal{L}\{e^{3t}\mathrm{t}\cos t\} = \frac{(s-1)^2}{(s^2+1)^2}|_{s=s-3} = \frac{(s-3)^2-1}{((s-3)^2+1)^2} = \frac{(s-3)^2-1}{(s^2-6s+10)^2}$$

$$\mathcal{L}\left\{\int_0^t e^{3t}\mathrm{t}\cos t\,dt\right\} = \frac{1}{s}\frac{(s-3)^2-1}{(s^2-6s+10)^2}$$

$$\mathcal{L}\left\{e^{2t}\int_0^t e^{3t}\mathrm{t}\cos t\,dt\right\} = \frac{(s-3)^2-1}{s(s^2-6s+10)^2}\ |_{s=s-2} = \frac{(s-5)^2-1}{(s-2)(s^2-10s+26)^2}$$

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4. (10%) $\mathrm{f(t)} = ?$ if $\mathrm{F(s)} = \frac{s^2+6s+9}{(s-1)(s-2)(s+4)}$

ANS:

$$\mathrm{F(s)} = \frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{\mathrm{A}}{s-1} + \frac{B}{s-2} + \frac{C}{s+4}$$

$$s^2+6s+9 = A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2)$$

$$\mathrm{s} = 1,2,-4$$

$$\mathrm{A} = -\frac{16}{5}, B = \frac{25}{6}, C = \frac{1}{30}$$

$$\rightarrow f(t) = \mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$$

$$= -\frac{16}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{25}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{30}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$$

5. (15%) $y'' + 2ty' - 4y = 6, y(0) = 0, y'(0) = 0,$ find y=?

ANS:

$$s^2Y(s) + s\left(-\frac{d\big(sY(s)\big)}{ds}\right) - 4\mathrm{Y(s)} = \frac{6}{\mathrm{s}}$$

$$s^2Y(s) - 2Y(s) - 2sY'(s) - 4Y(s) = \frac{6}{s}$$

$$-2sY'(s) + (s^2-6)Y(s) = \frac{6}{s}$$

$$Y'(s) + \frac{s^2-6}{-2s}Y(s) = \frac{6}{s(-2s)}$$

$$\mathrm{I} = e^{\int\frac{s^2-6}{-2s}ds} = e^{\int(-\frac{s}{2}+\frac{3}{s})ds} = e^{-\frac{1}{4}s^2+3\ln s} = e^{-\frac{1}{4}s^2} \times s^3$$

$$Y(s) = Ce^{\frac{1}{4}s^2} \times s^{-3} + e^{\frac{1}{4}s^2} \times s^{-3}\int e^{-\frac{1}{4}s^2} \times s^3 \times \frac{6}{s(-2s)}ds$$

$$= Ce^{\frac{1}{4}s^2} \times s^{-3} + 6s^{-3}$$

利用初值定理解 C

$$\mathrm{y(0)} = \lim_{\mathrm{n}\to\infty} \mathrm{sY(s)} = \lim_{\mathrm{n}\to\infty}\left(Ce^{\frac{1}{4}s^2} \times s^{-2} + 6s^{-2}\right) = 0$$

$$\therefore \mathrm{C} = 0$$

$$Y(s) = 6s^{-3}$$

$y(t) = 3t^2$

6. (10%) Find the Taylor series solution at x=0 for the following equation.

$2x(1-x)y'' + (1+x)y' - y = 0$

ANS:

決定 x=0 的級數解

x=0, x=1 均爲異點

$x\frac{(1+x)}{2x(1-x)}, x^2\frac{-1}{2x(1-x)}$於 x=0 皆可微分

$\therefore x = 0$違規則一點

∴存在一 Frobenius 級數解

$$y(x) = x^r\sum_{n=0}^{\infty} a_n x^n, |x-0| < 1 = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y(x) = a_0x^r + a_1x^{r+1} + \cdots + a_nx^{r+n} + \cdots$$

$$y'(x) = ra_0x^{r-1} + (r+1)a_1x^r + \cdots + (r+n)a_nx^{r+n-1} + \cdots$$

$$= \sum_{n=0}^{\infty}(r+n)a_nx^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty}(r+n)(r+n-1)a_nx^{n+r-2}$$

帶回原式

$$2x(1-x)\sum_{n=0}^{\infty}(r+n)(r+n-1)a_nx^{n+r-2} + (1+x)\sum_{n=0}^{\infty}(r+n)a_nx^{n+r-1}$$

$$-\sum_{n=0}^{\infty}a_nx^{n+r} = 0$$

$$2\sum_{n=0}^{\infty}(r+n)(r+n-1)a_nx^{n+r-1} - 2\sum_{n=0}^{\infty}(r+n)(r+n-1)a_nx^{n+r}$$

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$$+\sum_{n=0}^{\infty}(r+n)a_n x^{n+r-1}+\sum_{n=0}^{\infty}(r+n)a_n x^{n+r}-\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\rightarrow 2\sum_{n=-1}^{\infty}(r+n+1)(r+n)a_{n+1}x^{n+r}-2\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{n+r}$$

$$+\sum_{n=-1}^{\infty}(r+n+1)a_{n+1}x^{n+r}+\sum_{n=0}^{\infty}(r+n)a_n x^{n+r}-\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\rightarrow 2r(r-1)a_0 x^{r-1}+ra_0 x^{r-1}$$

$$+\sum_{n=0}^{\infty}\{[2(n+r+1)(n+r)+(n+r+1)]a_{n+1}$$

$$+[-2(n+r)(n+r-1)+(n+r)-1]a_n\}x^{n+r}=0$$

$$\rightarrow [2r(r-1)+r]a_0 x^{r-1}+\sum_{n=0}^{\infty}\{A(n,r)a_{n+1}+B(n,r)a_n\}x^{n+r}=0$$

1. $[2r(r-1)+r]a_0=0$
2. $A(n,r)a_{n+1}+B(n,r)a_n=0$

$$\because a_0\neq 0 \therefore 2r(r-1)+r=0, r=0\ or\ \frac{1}{2}$$

Case (i) r=0

$$a_{n+1}=-\frac{B(n,r)}{A(n,r)}a_n=-\frac{B(n,0)}{A(n,0)}a_n$$

$$=-\frac{(-2n(n-1)+n-1)}{2(n+1)n+n+1}a_n$$

$$=\frac{(2n-1)(n-1)}{(n+1)(2n+1)}a_n$$

$$n=0, a_1=\frac{(-1)(-1)}{1\times 1}a_0=a_0$$

$$n=1, a_2=\frac{1\times 0}{2\times 3}a_1=0$$

$$y_1(x)=a_0+a_0 x=a_0(1+x)$$

Case (ii) r = $\frac{1}{2}$

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$$\mathrm{a}_{\mathrm{n}+1} = -\frac{B(n,r)}{A(n,r)}\mathrm{a}_{\mathrm{n}} = -\frac{B\left(n,\frac{1}{2}\right)}{A\left(n,\frac{1}{2}\right)}\mathrm{a}_{\mathrm{n}}$$

$$= -\frac{(-2(n+\frac{1}{2})n + n + \frac{1}{2} - 1)}{2\left(n+\frac{1}{2}+1\right)\left(n+\frac{1}{2}\right) + n + \frac{1}{2} + 1}\mathrm{a}_{\mathrm{n}}$$

$$= \frac{(2n-1)n}{(2n+3)\left(n+\frac{3}{2}\right)}\mathrm{a}_{\mathrm{n}}$$

$$n = 0, \mathrm{a}_1 = \frac{0}{3\times\frac{3}{2}}\mathrm{a}_0 = 0$$

$$\mathrm{a}_{\mathrm{n}} = 0$$

$$y_2(x) = \mathrm{a}_0$$

$$y(x) = \mathrm{a}_0 x^{\frac{1}{2}} + \mathrm{a}_0(1+x)$$

7. (20%) $y' + y = 2x^2 + 3x + 1$, (a) Find the Taylor series solution at x=0 (b) Find The Taylor series solution at x=2

ANS:

(a)

$$\mathrm{y(x)} = \sum_{n=0}^{\infty} a_n x^n, |x| < \infty$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 2x^2 + 3x + 1$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 2x^2 + 3x + 1$$

$$\sum_{n=0}^{\infty} ((n+1) a_{n+1} + a_n) x^n = 2x^2 + 3x + 1$$

$$\mathrm{n} = 0, a_1 + a_0 = 1 \rightarrow\ a_0 = 2 - 6a_3$$

$$\mathrm{n} = 1, 2a_2 + a_1 = 3 \rightarrow\ a_1 = 6a_3 - 1$$

$$\mathrm{n} = 2, 3a_3 + a_2 = 2 \rightarrow\ a_2 = 2 - 3a_3$$

$$\mathrm{n} \geq 3, a_{n+1} = -\frac{a_n}{n+1}$$

$a_n = \frac{6a_3(-1)^{n-3}}{n!}$

$\therefore \mathrm{y(x)} = a_0 + a_1 x + a_2 x^2 + \cdots$

$= (2 - 6a_3) + (6a_3 - 1)\mathrm{x} + (2 - 3a_3)x^2 + \cdots + \frac{6a_3(-1)^{n-3}}{n!} + \cdots$

$= 2 - \mathrm{x} + 2x^2 - 6a_3\left(1 - x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots\right)$

$= 2 - \mathrm{x} + 2x^2 - 6a_3 e^{-x}$

(b)

$\mathrm{y(x)} = \sum_{n=0}^{\infty} a_n (x-2)^n, |x-2| < \infty$

$$y'(x) = \sum_{n=1}^{\infty} n a_n (x-2)^{n-1}$$

$$\sum_{n=1}^{\infty} n a_n (x-2)^{n-1} + \sum_{n=0}^{\infty} a_n (x-2)^n = 2x^2 + 3x + 1$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} (x-2)^n + \sum_{n=0}^{\infty} a_n (x-2)^n = 2x^2 + 3x + 1$$

$$\sum_{n=0}^{\infty} [(n+1) a_{n+1} + a_n](x-2)^n = 2x^2 + 3x + 1$$

$= \mathrm{m}(x-2)^2 + n(x-2) + \mathrm{k}$

$= 2(x-2)^2 + 11(x-2) + 15$

$\mathrm{n} = 0, a_1 + a_0 = 15$

$\mathrm{n} = 1, 2a_2 + a_1 = 11$

$\mathrm{n} = 2, 3a_3 + a_2 = 2$

$\mathrm{n} \geq 3, a_{n+1} = \frac{-1}{n+1} a_n$

$a_2 = 2 - 3a_3$

$a_1 = 11 - 2a_2 = 11 - 2(2 - 3a_3) = 7 + 6a_3$

$a_0 = 15 - a_1 = 1 - (7 + 6a_3) = 8 - 6a_3$

$a_4 = \frac{-1}{4} a_3$

$a_5 = \frac{-1}{5 \times 4} a_3$

$$a_n = \frac{6(-1)^{n-3}}{n!}a_3$$

$$\therefore \mathrm{y(x)} = a_0 + a_1(x-2) + a_2(x-2)^2 + \cdots$$

$$= 8 - 6a_3 + (7+6a)(x-2) + (2-3a_3)(x-2)^2 + \cdots$$

$$+ \frac{6(-1)^{n-3}}{n!}a_3(x-2)^n + \cdots$$

$$= 8 + 7(\mathrm{x}-2) + 2(x-2)^2$$

$$- 6a_3\left[1-(x-2)+\frac{1}{2}(x-2)^2+\cdots+\frac{(-1)^n}{n!}(x-2)^n+\cdots\right]$$

$$= 8 + 7(\mathrm{x}-2) + 2(x-2)^2 - 6a_3e^{-(x-2)}$$

8. (10%) Please summarize the possible cases in the series solution at x=a (a is a constant) when we solve the differential equation $\mathrm{p(x)}y'' + q(x)y' + r(x)y = 0$, where p(x), q(x), r(x) have no common terms.

ANS:

1. $\mathrm{p}(a) \neq 0, a$爲常點，則存在 Taylar 級數解

$$\mathrm{y}(x) = \sum_{n=0}^{\infty} a_n(x-a)^n, |x-a| \leq L, L爲\ \mathrm{x} = \mathrm{a}\ 到最近異點的距離$$

2. $\mathrm{p}(a) = 0, a$爲異點，若$(x-a)\frac{q(x)}{p(x)}, (x-a)^2\frac{r(x)}{p(x)}$這二項於$x=a$均可微分，

$x = a$爲規則異點，則存在 Frobenius 級數解

$$y(x) = \sum_{n=0}^{\infty} a_n(x-a)^{n+r}, |x-a| < L, L爲\ x = a到最近異點的距離$$

將y項用級數代入，化成同次方後相同項整理合併，由$(x-a)$之最低次方之係數令$a_0 \neq 0$，可得 r 的指標方程式

$r = r_1 \rightarrow y_1(x)$

$r = r_2 \rightarrow y_2(x)$

[I] $r_1 \neq r_2, |r_1 - r_2| \notin \mathcal{N}$

y_1, y_2一定線性獨立，構成一組基底解$\mathrm{y}(x) = c_1y_1(x) + c_2y_2(x)$

[II] $r_1 \neq r_2, |r_1 - r_2| \in \mathcal{N}$

(A)若$y1, y2$ 獨立，$\mathrm{y}(x) = c_1y_1(x) + c_2y_2(x)$

(B)若$y1, y2$ 相依，則用參數變異法求解$\overline{y_2}$，$\mathrm{y}(x) = c_1y_1(x) + c_2\overline{y_2}(x)$

[III] $r_1 = r_2 = r$有獨立解y_1，另一獨立解可由參數變異法求得

3. $x = a$爲不規則異點，則方程式於$x = a$無級數解