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Chapter 8
Sorting in Linear Time

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Lower bounds for sorting

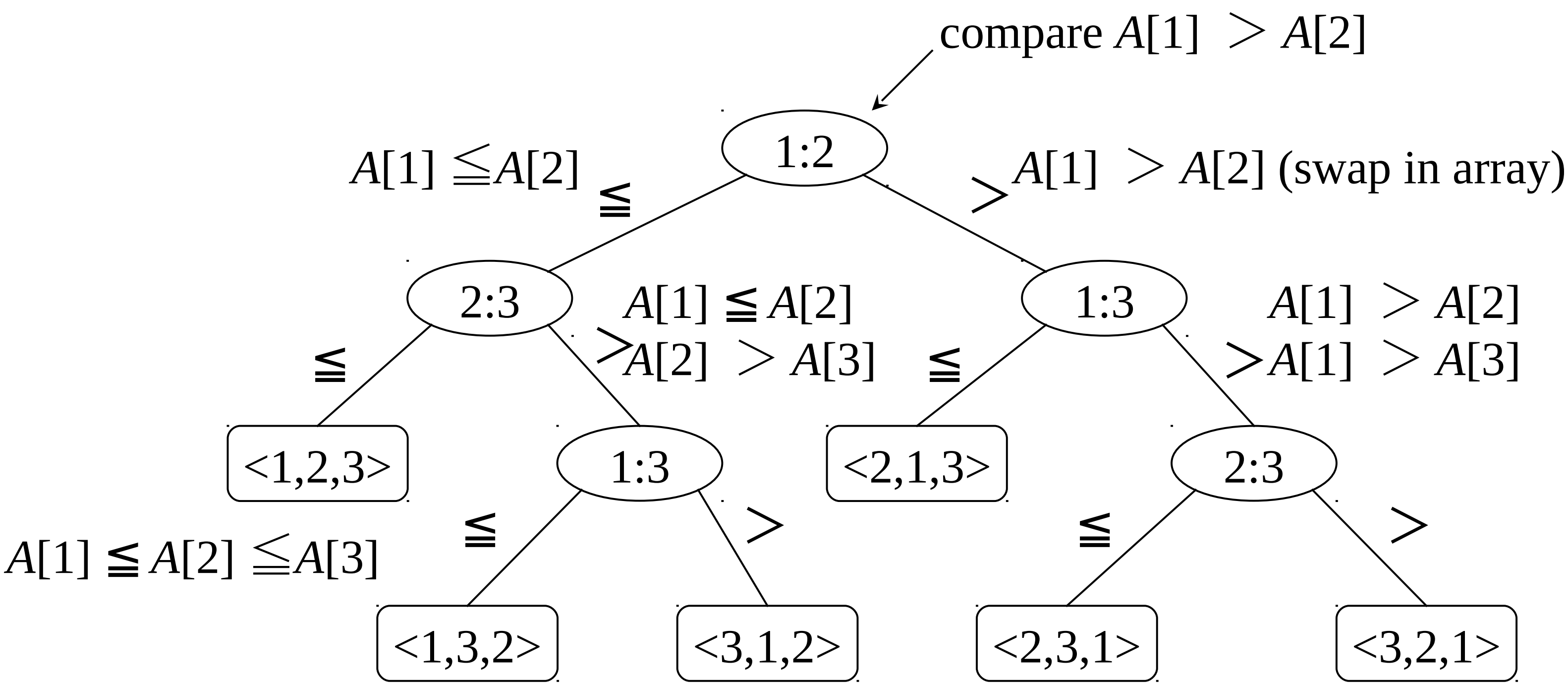
Lower bounds

- $\Omega(n)$ to examine all the input.
- All sorts seen so far are $\Omega(n \lg n)$.
- We'll show that $\Omega(n \lg n)$ is a lower bound for comparison sorts.

Decision tree

- Abstraction of any comparison sort.
- Represents comparisons made by
  - a specific sorting algorithm
  - on inputs of a given size.
- Abstracts away everything else: control and data movement.
- We're counting *only* comparisons.

For insertion sort on 3 elements:



[Each internal node is labeled by indices of array elements **from their original positions**. Each leaf is labeled by the permutation of orders that the algorithm determines.]

How many leaves on the decision tree? There are $\geqq n!$ leaves, because every permutation appears at least once.

For any comparison sort,

- 1 tree for each n.
- View the tree as if the algorithm splits in two at each node, based on the information it has determined up to that point.
- The tree models all possible execution traces.

What is the length of the longest path from root to leaf ?

- Depends on the algorithm
- Insertion sort: $\Theta(n^2)$
- Merge sort: $\Theta(n \lg n)$

Lemma

Any binary tree of height h has $\leqq 2^h$ leaves.

In other words:

- l = # of leaves,
- h = height,
- Then $l \leqq 2^h$.

Proof: By induction on h.

Basis: $h = 0$. Tree is just one node, which is a leaf. $2^h = 1$.

Inductive step: Assume true for height $= h - 1$.

Extend tree of height $h - 1$ by making as many new leaves as possible.
Each leaf becomes parent to two new leaves.

$$
\begin{aligned}
\#\text{ of leaves for height } h &= 2 \cdot (\#\text{ of leaves for height } h - 1) \\
&\le 2 \cdot 2^{h-1} && \text{(ind. hypothesis)} \\
&= 2^h. && \text{(lemma)}
\end{aligned}
$$

Theorem. Any decision tree that sorts n elements has height $\Omega(n\lg n)$

Proof

- $l \geqq n!$
- By lemma, $n! \leqq l \leqq 2^h$ or $2^h \geqq n!$
- Take logs: $h \geqq \lg(n!)$
- Use Stirling's approximation: $n! > (n/e)^n$ (by equation (3.16))

(*e* = Euler's number ≈ 2.71828)

$$\begin{aligned} h &\geqq \lg(n/e)^n \\ &= n\lg(n/e) \\ &= n\lg n - n\lg e \\ &= \Omega(n\lg n) \end{aligned}$$

(theorem) ■

Corollary

Heapsort and merge sort are asymptotically optimal comparison sort.

Sorting in linear time

Non-comparison sorts.

Counting sort

Depends on a *key assumption*: numbers to be sorted are integers in $\{0, 1, \ldots, k\}$.

Input: $A[1..n]$, where $A[j] \in \{0, 1, \ldots, k\}$ for $j = 1, 2, \ldots, n$. Array A and values n and k are given as parameters.

Output: $B[1..n]$, sorted. B is assumed to be already allocated and is given as a parameter.

Auxiliary storage: $C[0\ldots k]$

COUNTING-SORT(A, B, n, k)

for $i \leftarrow 0$ **to** k

    do $C[i] \leftarrow 0$

for $j \leftarrow 1$ **to** n

    do $C[A[j]] \leftarrow C[A[j]] + 1$

for $i \leftarrow 1$ **to** k

    do $C[i] \leftarrow C[i] + C[i - 1]$

for $j \leftarrow n$ **downto** 1

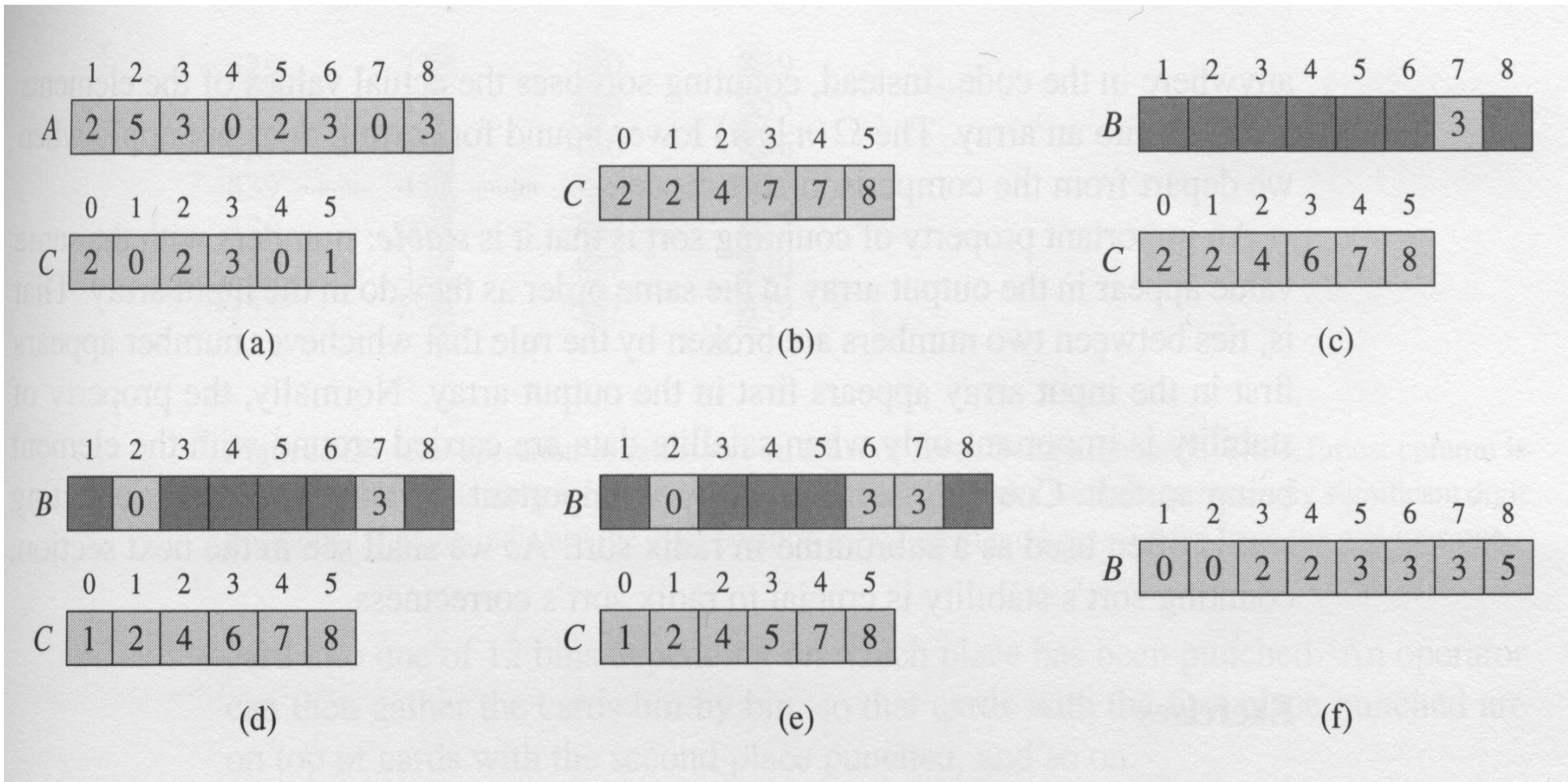
    do $B[C[A[j]]] \leftarrow A[j]$

        $C[A[j]] \leftarrow C[A[j]] - 1$

Do an example for $A = 2_1, 5_1, 3_1, 0_1, 2_2, 3_2, 0_2, 3_3$

Counting sort is ***stable*** (keys with same value appear in same order in output as they did in input) because of how the last loop works.

The operation of Counting-sort on an input array *A*[1…8]



Analysis: $\Theta(n + k)$, which is $\Theta(n)$ if $k = O(n)$.

How big a k is practical?

- Good for sorting 32-bit values? No.
- 16-bit? Probably not.
- 8-bit? Maybe, depending on n.
- 4-bit? Probably (unless n is really small).

Counting sort will be used in radix sort.

Radix sort

Key idea: Sort *least* significant digits first.

To sort *d* digits:

RADIX-SORT(A, d)

for $i \leftarrow 1$ **to** d

  do use a stable sort to sort array A on digit i

Example:

sorted

326		690		704		326
453		751		608		435
608	→	453	→	326	→	453
835		704		835		608
751		835		435		690
435		435		751		704
704		326		453		751
690		608		690		835

Correctness:

- Induction on number of passes (*i* in pseudocode).
- Assume digits 1, 2, …, $i - 1$ are sorted.
- Show that a stable sort on digit *i* leaves digits 1,…, *i* sorted:
  - If 2 digits in position *i* are different, ordering by position *i* is correct, and positions 1, …, $i - 1$ are irrelevant.
  - If 2 digits in position *i* are equal, numbers are already in the right order (by inductive hypothesis). The stable sort on digit *i* leaves them in the right order.

This argument shows why it's so important to use a stable sort for intermediate sort.

Analysis: Assume that we use counting sort as the intermediate sort.

- $\Theta(n + k)$ per pass (digits in range $0, \ldots, k$)
- d passes
- $\Theta(d(n + k))$ total
- If $k = O(n)$, time $= \Theta(dn)$.

How to break each key into digits?

- n words
- b bits/word
- Break into r-bit digits. Have $d = \lceil b/r \rceil$
- Use counting sort, $k = 2^r - 1$

  Example: 32-bit words, 8-bit digits. b=32, r=8, d= $\lceil 32/8 \rceil$ =4, k=$2^8 - 1$= 255
- Time = $\Theta\left(\frac{b}{r}\left(n + 2^r\right)\right)$

How to choose r? Balance b/r and $n + 2^r$. Choosing $r \approx \lg n$ gives us

$$\Theta\left(\frac{b}{\lg n}(n+n)\right) = \Theta(bn/\lg n)$$

- If we choose $r < \lg n$, then $b/r > b/\lg n$, and $n + 2^r$ term doesn't improve.
- If we choose $r > \lg n$, then $n + 2^r$ term gets big. Example:

$$r = 2\lg n \Rightarrow 2^r = 2^{2\lg n} = (2^{\lg n})^2 = n^2.$$

So, to sort 2^{16} 32-bit numbers, use $r = \lg 2^{16} = 16$ bit. $\lceil b/r \rceil = 2$ passes.

Compare radix sort to merge sort and quicksort:

- 1 million (2^{20}) 32-bit integers.
- Radix sort: $\lceil 32/20 \rceil = 2$ passes.
- Merge sort/quicksort: lg n = 20 passes.
- Remember, though, that each radix sort "pass" is really 2 passes– one to take census, and one to move data.

How does radix sort violate the ground rules for a comparison sort?

- Using counting sort allows us to gain information about keys by means other than directly comparing 2 keys.
- Used keys as array indices.

Bucket sort

Assumes the input is generated by a random process that distributes elements uniformly over [0, 1).

Idea:

- Divide [0, 1) into *n* equal-sized *buckets*.
- Distribute the *n* input values into the buckets.
- Sort each bucket.
- Then go through buckets in order, listing elements in each one.

Input: $A[1..n]$, where $0 \leqq A[i] < 1$ for all i.

Auxiliary array: $B[0..n-1]$ of linked lists, each list initially empty.

BUCKET-SORT(A, n)

for $i \leftarrow 1$ **to** n

do insert $A[i]$ into list $B\left[\lfloor n \cdot A[i] \rfloor\right]$

for $i \leftarrow 0$ **to** $n-1$

do sort list $B[i]$ with insertion sort

concatenate lists $B[0], B[1], \ldots, B[n-1]$ together in order

return the concatenated lists

Correctness: Consider $A[i]$, $A[j]$. Assume without loss of generality that $A[i] \leqq A[j]$. Then $\lfloor n \cdot A[i] \rfloor \leq \lfloor n \cdot A[j] \rfloor$. So $A[i]$ is placed into the same bucket as $A[j]$ or into a bucket with a lower index.

- If same bucket, insertion sort fixes up.
- If earlier bucket, concatenation of lists fixes up.

Analysis:

- Relies on no bucket getting too many values.
- All lines of algorithm except insertion sorting take $\Theta(n)$ altogether.
- Intuitively, if each bucket gets a constant number of elements, it takes $O(1)$ time to sort each bucket $\Rightarrow$ $O(n)$ sort time for all buckets.
- We "expect" each bucket to have few elements, since the average is 1 element per bucket.
- But we need to do a careful analysis.

Define a random variable:

- n_i = the number of elements placed in bucket $B[i]$.

Because insertion sort runs in quadratic time, bucket sort time is $T(n) = \Theta(n) + \sum_{i=0}^{n-1} O(n_i^2)$

Take expectations of both sides:

$$\begin{aligned}
\mathrm{E}[T(n)] &= \mathrm{E}\left[\Theta(n) + \sum_{i=0}^{n-1} O(n_i^2)\right] \\
&= \Theta(n) + \sum_{i=0}^{n-1} \mathrm{E}\left[O(n_i^2)\right] \qquad (\text{linearity of expectation}) \\
&= \Theta(n) + \sum_{i=0}^{n-1} O\left(E\left[n_i^2\right]\right) \qquad \left(E[aX] = aE[X]\right)
\end{aligned}$$

Claim

$$\mathrm{E}\left[n_i^2\right] = 2 - (1/n) \text{ for } i = 0, \ldots, n - 1.$$

Proof of claim

Define indicator random variables:

- $X_{ij} = \mathrm{I}\{A[j] \text{ falls in bucket } i\}$
- $\Pr\{A[j] \text{ falls in bucket } i\} = 1/n$
- $n_i = \sum_{j=1}^{n} X_{ij}$

Then

$$\begin{aligned}
\mathrm{E}\left[n_i^2\right] &= \mathrm{E}\left[\left(\sum_{j=1}^{n} X_{ij}\right)^2\right] \\
&= \mathrm{E}\left[\sum_{j=1}^{n} X_{ij}^2 + 2\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} X_{ij}X_{ik}\right] \\
&= \sum_{j=1}^{n} \mathrm{E}\left[X_{ij}^2\right] + 2\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} \mathrm{E}\left[X_{ij}X_{ik}\right] \quad (\text{linearity of expectation})
\end{aligned}$$

$$E\left[X_{ij}^{2}\right] = 0^2 \cdot \Pr\left\{A\left[j\right] \text{ doesn't fall in bucket } i\right\} + 1^2 \cdot \Pr\left\{A\left[j\right] \text{ falls in bucket } i\right\}$$

$$= 0 \cdot \left(1 - \frac{1}{n}\right) + 1 \cdot \frac{1}{n}$$

$$= \frac{1}{n}$$

$E[X_{ij}X_{ik}]$ for $j \neq k$: Since $j \neq k$, X_{ij} and X_{ik} are independent random variables

$\Rightarrow$ $E[X_{ij}X_{ik}] = E[X_{ij}]E[X_{ik}]$

$$= \frac{1}{n} \cdot \frac{1}{n}$$

$$= \frac{1}{n^2}$$

Therefore:

$$\mathrm{E}\left[n_i^2\right] = \sum_{j=1}^{n} \frac{1}{n} + 2\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} \frac{1}{n^2}$$

$$= n \cdot \frac{1}{n} + 2\binom{n}{2}\frac{1}{n^2}$$

$$= 1 + \frac{n-1}{n}$$

$$= 1 + 1 - \frac{1}{n}$$

$$= 2 - \frac{1}{n}$$

■ (claim)

Therefore:

$$
\begin{aligned}
\mathrm{E}[T(n)] &= \Theta(n) + \sum_{i=0}^{n-1} O(2-1/n) \\
&= \Theta(n) + O(n) \\
&= \Theta(n)
\end{aligned}
$$

- Again, not a comparison sort. Used a function of key values to index into an array.
- This is a ***probabilistic analysis***—we used probability to analyze an algorithm whose running time depends on the distribution of inputs.
- Different from a ***randomized algorithm***, where we use randomization to impose a distribution.
- With bucket sort, if the input isn't drawn from a uniform distribution on [0,1), all bets are off (performance-wise, but the algorithm is still correct).