ex. $y'' + 3y' + 2y = \underbrace{\cos x + x}_{r(x)}$

特性方程式: $\lambda^2 + 3\lambda + 2 = 0 \Rightarrow y_h = c_1 e^{-x} + c_2 e^{-2x}$

又 $y_p'' + 3y_p' + 2y_p = \underbrace{\cos x}_{r_1(x)} + \underbrace{x}_{r_2(x)}$ (分開找)

$$\left(\begin{array}{l} \quad y_{p_1}'' + 3y_{p_1}' + 2y_{p_1} = r_1(x) \\ +)\ y_{p_2}'' + 3y_{p_2}' + 2y_{p_2} = r_2(x) \\ \hline (y_{p_1}+y_{p_2})'' + 3(y_{p_1}+y_{p_2})' + 2(y_{p_1}+y_{p_2}) = r_1(x) + r_2(x) \\ \\ \Rightarrow y_p = y_{p_1} + y_{p_2} \end{array}\right)$$

先做 $r_1(x)$.

$\Rightarrow (D^2+3D+2)\, y_{p_1}(x) = \cos x$

$\Rightarrow y_{p_1}(x) = \underbrace{\dfrac{1}{D^2+3D+2}}_{L(D^2)} \cos x = \dfrac{1}{-1+3D+2}\cos x$

$= \dfrac{1}{3D+1}\cos x = \underbrace{\dfrac{1-3D}{1-9D^2}\cos x}_{\text{有理化}}$

$= \dfrac{1-3D}{10}\cos x = \dfrac{1}{10}\cos x - \dfrac{3}{10}D\cos x$ → 相等

$= \dfrac{1}{10}\cos x - \dfrac{3}{10}(-\sin x)$

再做 $r_2(x)$.

$\Rightarrow (D^2+3D+2)\, y_{p_2}(x) = x$

$\Rightarrow y_{p_2}(x) = \dfrac{1}{D^2+3D+2} \cdot x$

$= \dfrac{1}{2\left(1+\frac{D^2+3D}{2}\right)} \cdot x$

$\left(\times \frac{1}{1+r} = 1 - r + r^2 - r^3 \cdots \right)$

$= \frac{1}{2}\left(1 - \frac{D^2+3D}{2} + \left(\frac{D^2+3D}{2}\right)^2 - \cdots\right)x$

$= \frac{1}{2}x - \frac{D^2x+3Dx}{2} = \frac{1}{2}x - \frac{3}{4}$

$\Rightarrow y_p = \frac{1}{10}\cos x + \frac{3}{10}\sin x + \frac{1}{2}x - \frac{3}{4}$

$\Rightarrow y = c_1e^{-x} + c_2e^{-2x} + \frac{1}{10}\cos x + \frac{3}{10}\sin x + \frac{1}{2}x - \frac{3}{4}$ #

Method 4. variation of variables / parameter

變數變換法 / 參數變換法

概念：$y_p = y_h \cdot \phi$

先舉例：$y' + 2y = e^x$

$\Rightarrow y_h = ce^{-2x}$

$\Rightarrow y_p = e^{-2x} \cdot \phi(x)$

$y_p' = e^{-2x} \cdot \phi'(x) - 2e^{-2x} \cdot \phi(x)$ } 代回去

$\Rightarrow e^{-2x}\phi'(x) - \cancel{2e^{-2x}\phi(x)} + \cancel{2e^{-2x} \cdot \phi(x)} = e^x$

$\Rightarrow \phi'(x) = e^{3x}$

$\Rightarrow \phi(x) = \frac{1}{3}e^{3x} + k$

$\Rightarrow y_p = e^{-2x}\left(\frac{1}{3}e^{3x} + k\right)$

$= \frac{1}{3}e^x + ke^{-2x}$

$\Rightarrow y = y_h + y_p = \underline{\underline{ce^{-2x}}} + \frac{1}{3}e^x + \underline{\underline{ke^{-2x}}}$

$= ce^{-2x} + \frac{1}{3}e^x \Rightarrow k$ 以後不用寫

推廣 to 二階常微分方程式

$y''(x) + p(x) \cdot y'(x) + q(x) \cdot y(x) = r(x)$

設 $y_1(x)$, $y_2(x)$ 分別為此方程式的齊性解

$\Rightarrow y_h = c_1 y_1(x) + c_2 y_2(x)$.

且 $y_1'' + p y_1' + q y_1 = 0$

$y_2'' + p y_2' + q y_2 = 0$

又 $y_p = y_1 \phi_1 + y_2 \phi_2$

$\Rightarrow y_p' = y_1' \phi_1 + y_1 \phi_1' + y_2' \phi_2 + y_2 \phi_2'$

$= (y_1' \phi_1 + y_2' \phi_2) + \underbrace{(y_1 \phi_1' + y_2 \phi_2')}_{令為 0}$;

$\Rightarrow y_p'' = y_1'' \phi_1 + y_1' \phi_1' + y_2'' \phi_2 + y_2' \phi_2'$

代回去

$$\Rightarrow y_1''\phi_1 + y_1'\phi_1' + y_2''\phi_2 + y_2'\phi_2' + p(y_1'\phi_1 + y_2'\phi_2) + q(y_1\phi_1 + y_2\phi_2) = r(x).$$

$$\Rightarrow \phi_1(y_1'' + p y_1' + q y_1) + \phi_2(y_2'' + p y_2' + q y_2) + y_1'\phi_1' + y_2'\phi_2' = r(x)$$

齊性解 $= 0$

$$\Rightarrow \begin{cases} y_1'\phi_1' + y_2'\phi_2' = r(x) \\ y_1\phi_1' + y_2\phi_2' = 0 \end{cases}$$ 要每上式都滿足.

$$\left(\begin{cases} ax + by = c \\ dx + ey = f \end{cases} \Rightarrow x = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} \right)$$

$$\Rightarrow \phi_1' = \frac{\begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}, \quad \phi_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

$$= \frac{-r y_2}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \qquad = \frac{r y_1}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

$$\Rightarrow \phi_1 = \int \frac{-r y_2}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} dx, \quad \phi_2 = \int \frac{r y_1}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} dx$$

$$y_p = y_1\phi_1 + y_2\phi_2.$$

$$= y_1\int \frac{-r y_2}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}dx + y_2\int \frac{r y_1}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}dx.$$

定义. Wronski $(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = w(y_1, y_2)$

$$\Rightarrow y_p = y_1\int \frac{-r y_2}{w(y_1, y_2)}dx + y_2\int \frac{r y_1}{w(y_1, y_2)}dx. \#$$

ex. $y'' + 3y' + 2y = x.$

$y_p = y_1\phi_1 + y_2\phi_2.$

又 $\lambda = -1, -2$

$\Rightarrow y_1 = e^{-2x}, \quad y_2 = e^{-x}.$

$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = e^{-3x}.$

$$y_p = y_1\int \frac{-r y_2}{e^{-3x}}dx + y_2\int \frac{r y_1}{e^{-3x}}dx.$$

$$= -e^{-2x}\left(\tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x}\right) + e^{-x}\left(xe^{x} - e^{x}\right)$$

$$= \tfrac{1}{2}x - \tfrac{3}{4} \#$$

用方法②:

$(D+1)(D+2)y_p = x.$

$I_1 = e^{x}, \quad I_2 = e^{2x}.$

$$\Rightarrow y_p = e^{-2x}\int e^{2x}\left[e^{-x}\int e^{x} r\,dx\right]dx$$

$$= \tfrac{1}{2}x - \tfrac{3}{4} \#$$

ex. $y'' + 8y' + 16y = 3\cdot e^{-4x}.$

$$y_h = c_1 e^{-4x} + c_2 x e^{-4x}.$$

y_p ?

①. $y_p = kx^2 e^{-4x}$

$y_p' = k[x^2(-4e^{-4x}) + (2xe^{-4x})]$

$= k[-4x^2 e^{-4x} + 2xe^{-4x}]$

$y_p'' = k[16x^2e^{-4x} - 16xe^{-4x} + 2e^{-4x}]$ 代回去.

$\Rightarrow k[16x^2e^{-4x} - 16xe^{-4x} + 2e^{-4x}]$

$+k[-32x^2e^{-4x} + 16xe^{-4x}]$ $\Rightarrow k = \frac{3}{2}$.

$+k[16x^2e^{-4x}] = 3\cdot e^{-4x}$.

$\Rightarrow y_p = \frac{3}{2}x^2e^{-4x}$ #.

②. $(D+4)(D+4)y_p = 3\cdot e^{-4x}$.

$I_1 = e^{4x}$, $I_2 = e^{4x}$.

$\Rightarrow y_p = e^{-4x}\int e^{4x}[e^{-4x}\int e^{4x}\cdot r\cdot dx]dx$

$= \frac{3}{2}x^2e^{-4x}$ #.

③. $y_p = \frac{1}{(D+4)^2}\cdot 3e^{-4x}\cdot 1$

$= 3\cdot e^{-4x}\cdot\frac{1}{D^2}\cdot 1$

$= 3\cdot e^{-4x}\iint 1\,dx\,dx = \frac{3}{2}x^2\cdot e^{-4x}$ #.

④. $y_1 = e^{-4x}$, $y_2 = x\cdot e^{-4x}$.

$W(y_1, y_2) = e^{-8x}$.

$y_p = y_1\int\frac{-ry_2}{W}dx + y_2\int\frac{ry_1}{W}dx$.

$= \frac{3}{2}x^2e^{-4x}$ #.

①.③ 受限於 $r(x)$ 的型式.