Chapter 5.
Series Solutions of Linear Differential Equations

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Series Solutions

- 考慮下列方程式

1.試解方程式 $y' + y = 2x^2 + 3x + 1$

$$y_h = Ce^{-x}$$

$$y_p = \frac{1}{D+1}(2x^2 + 3x + 1)$$

$$= (1 - D + D^2 - D^3 + \cdots)(2x^2 + 3x + 1)$$

$$= 2x^2 + 3x + 1 - (4x + 3) + 4$$

$$= 2x^2 - x + 2$$

$$\therefore y = Ce^{-x} + 2x^2 - x + 2$$

$$= C(1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \cdots) + 2x^2 - x + 2$$

Series Solutions

2.利用於x=0的級數解驗證1.的結果

∵x=0為一常點 存在 Taylor 級數解

$$y' + y = 2x^2 + 3x + 1$$

$$y = y_h + y_p$$

$$y_h = Ce^{-x}$$

$$y_p = \frac{1}{D+1}(2x^2 + 3x + 1)$$

$$= (1 - D + D^2 - D^3 + \cdots)(2x^2 + 3x + 1)$$

$$= 2x^2 + 3x + 1 - (4x + 3) + 4$$

$$= 2x^2 - x + 2$$

$$\therefore y = Ce^{-x} + 2x^2 - x + 2$$

$$= C(1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \cdots) + 2x^2 - x + 2$$

Series Solutions

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, |x - 0| < \infty$$

$$y^{'}(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 2x^2 + 3x + 1$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 2x^2 + 3x + 1$$

$$\sum_{n=0}^{\infty} [(n+1) a_{n+1} + a_n] x^n = 2x^2 + 3x + 1$$

$$n = 0, a_1 + a_0 = 1$$

$$n = 1, 2a_2 + a_1 = 3$$

$$n = 2, 3a_3 + a_2 = 2$$

$$n \geq 3, (n+1) a_{n+1} + a_n = 0$$

$$a_2 = 2 - 3a_3$$

$$a_1 = 3 - 2a_2 = 3 - 2(2 - 3a_3) = -1 + 6a_3$$

$$a_0 = 1 - a_1 = 1 - (-1 + 6a_3) = 2 - 6a_3$$

Series Solutions

$$n \ge 3, a_{n+1} = \frac{-1}{n+1} a_n$$

$$n = 3, a_4 = \frac{-1}{4} a_3$$

$$n = 4, a_5 = \frac{-1}{5} a_4 = \frac{-1}{5}\left(\frac{-1}{4} a_3\right) = \frac{(-1)^2}{5*4} a_3$$

$$n = 5, a_6 = \frac{-1}{6} a_5 = \frac{-1}{6}\left(\frac{(-1)^2}{5*4} a_3\right) = \frac{(-1)^3}{6*5*4} a_3$$

$$a_n = \frac{6(-1)^{n-3}}{n!} a_3$$

Series Solutions

$$\therefore y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

$$= (2-6a_3) + (-1+6a_3)x + (2-3a_3)x^2 + \cdots + \frac{6(-1)^{n-3}}{n!} a_3 x^n + \cdots$$

$$= (2 - x + 2x^2) - 6a_3(1 - x + \frac{1}{2}x^2 + \cdots + \frac{(-1)^{n-2}}{n!} x^n + \cdots)$$

$$= (2 - x + 2x^2) - 6a_3 e^{-x}$$

Series Solutions

*上題若改為以x=2作Taylor展開呢?

sol： ∵x=2是常點

$$y(x)=\sum_{n=0}^{\infty}a_n(x-2)^n,\left|x-2\right|<\infty$$

$$y^{'}(x)=\sum_{n=1}^{\infty}na_n(x-2)^{n-1}$$

$$\sum_{n=1}^{\infty}na_n(x-2)^{n-1}+\sum_{n=0}^{\infty}a_n(x-2)^n=2x^2+3x+1$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-2)^n+\sum_{n=0}^{\infty}a_n(x-2)^n=2x^2+3x+1$$

$$\sum_{n=0}^{\infty}[(n+1)a_{n+1}+a_n](x-2)^n=2x^2+3x+1=m(x-2)^2+n(x-2)+k$$

$$=2(x-2)^2+11(x-2)+15$$

Series Solutions

$$n = 0, a_1 + a_0 = 15$$

$$n = 1, 2a_2 + a_1 = 11$$

$$n = 2, 3a_3 + a_2 = 2$$

$$n \geq 3, a_{n+1} = \frac{-1}{n+1} a_n$$

$$a_2 = 2 - 3a_3$$

$$a_1 = 11 - 2a_2 = 11 - 2(2 - 3a_3) = 7 + 6a_3$$

$$a_0 = 15 - a_1 = 1 - (7 + 6a_3) = 8 - 6a_3$$

$$a_4 = \frac{-1}{4} a_3$$

$$a_5 = \frac{-1}{5 * 4} a_3$$

$$a_n = \frac{6(-1)^{n-3}}{n!} a_3$$

Series Solutions

$$\therefore\ y(x) = a_0 + a_1(x-2) + a_2(x-2)^2 + \cdots$$

$$= 8 - 6a_3 + (7 + 6a_3)(x-2) + (2 - 3a_3)(x-2)^2 + \cdots + \frac{6(-1)^{n-3}}{n!} a_3 (x-2)^n + \cdots$$

$$= 8 + 7(x-2) + 2(x-2)^2 - 6a_3[1 - (x-2) + \frac{1}{2}(x-2)^2 + \cdots + \frac{(-1)^n}{n!}(x-2)^n + \cdots]$$

$$= 8 + 7(x-2) + 2(x-2)^2 - 6a_3 e^{-(x-2)}$$

Series Solutions

*驗證上述級數解為真 by direct solving the D.E.

$$y' + y = 2x^2 + 3x + 1 = 2(x-2)^2 + 11(x-2) + 15$$

令$t = x - 2$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}$$

$$\Rightarrow y'(t) + y(t) = 2t^2 + 11t + 15$$

$$y(t) = y_h + y_p = Ce^{-t} + y_p$$

$$y_p = \frac{1}{D+1}(2t^2 + 11t + 15)$$

$$= (2t^2 + 11t + 15) - (4t + 11) + 4$$

$$= 2t^2 + 7\text{t} + 8$$

Series Solutions

$$y(t) = Ce^{-t} + 2t^2 + 7\text{t} + 8$$

$$y(x) = Ce^{-t} + 2(x-2)^2 + 7(x-2) + 8$$

$$\text{p(x)y}^{''}(x) + g(x)\text{y}^{'}(x) + r(x)\text{y}(x) = 0 \cdots\cdots (1)$$

若$x = a$為(1)式的一個異點$(\text{p(a)} \Rightarrow 0)$

但如果$(\text{x-a})\dfrac{g(x)}{\text{p(x)}}$及$(\text{x-a})^2\dfrac{r(x)}{\text{p(x)}}$於$x = a$均為可微分

則$x = a$稱為(1)的一個規則異點(regular singnlar plint)

否則$x = a$稱為(1)的一個不規則異點(irregular singnlar plint)

Series Solutions

$$\mathrm{p(x)y''}(x) + g(x)\mathrm{y'}(x) + r(x)\mathrm{y}(x) = 0 \cdots\cdots(1)$$

若$x = a$為(1)的一個規則異點,則於$x = a$處

存在一個 *Frobenius*級數解

$$y(x) = (x-a)^r \sum_{n=0}^{\infty} a_n (x-a)^n \text{且} |x-a| < L \text{為收斂區間}$$

L為收斂半徑＝由$x = a$到另外一個最近異點的距離

若$x = a$為一不規則異點,則$x = a$處不存在級數解

Series Solutions

EX： $(x-2)(x+3)^2 y'' + 4(x+1)y' + 5y = 0$

$x = 2$異點

$(x-2)\dfrac{4(x+1)}{(x-2)(x+3)^2}$及$(x-2)^2\dfrac{5}{(x-2)(x+3)^2}$

於$x = 2$皆可微

$$\mathrm{y(x)} = (\mathrm{x-2})^{\mathrm{r}} \sum_{n=0}^{\infty} a_n (x-2)^n$$

$x = -3 \cdots$

Series Solutions

EX： $2x(1-x)y'' + (1+x)y' - y = 0$

決定$x=0$的級數解

$x=0, x=1$均為異點

$x\frac{(1+x)}{2x(1-x)}, x^2\frac{-1}{2x(1-x)}$於$x=0$皆可微

$\therefore x=0$為規則異點

$\therefore$ 存在一*Frobenius*級數解

$$y(x) = x^r\sum_{n=0}^{\infty} a_n x^n, |x-0|<1$$

$$= \sum_{n=0}^{\infty} a_n x^{n+r}$$

Series Solutions

$$y(x) = a_0 x^r + a_1 x^{r+1} + \cdots + a_n x^{r+n} + \cdots$$

$$= \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y'(x) = r a_0 x^{r-1} + (r+1) a_1 x^r + \cdots + (r+n) a_n x^{r+n-1} + \cdots$$

$$= \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x(1-x)\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + (1+x)\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\Rightarrow 2\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} - 2\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r}$$

$$+\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Series Solutions

$$\Rightarrow 2\sum_{n=-1}^{\infty}(n+r+1)(n+r)a_{n+1}x^{n+r} - 2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}$$

$$+\sum_{n=-1}^{\infty}(n+r+1)a_{n+1}x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\Rightarrow 2r(r-1)a_0 x^{r-1} + ra_0 x^{r-1} + \sum_{n=0}^{\infty}\{[2(n+r+1)(n+r)+(n+r+1)]a_{n+1}$$

$$+[-2(n+r)(n+r-1)+(n+r)-1]a_n\}x^{n+r} = 0$$

$$\Rightarrow [2r(r-1)+r]a_0 x^{r-1} +$$

$$\sum_{n=0}^{\infty}\{A(n,r)a_{n+1} + B(n,r)a_n\}x^{n+r} = 0$$

Series Solutions

$1. [2r(r-1)+r]a_0 = 0$

$2. A(n,r)a_{n+1} + B(n,r)a_n = 0$

$\because a_0 \neq 0 \therefore 2r(r-1)+r = 0, r = 0 or \dfrac{1}{2}$

$2r(r-1)+r = 0 \Rightarrow$ 指標方程式(indical equation)

Series Solutions

- Case(i) $r = 0$

$$a_{n+1} = -\frac{B(n,r)}{A(n,r)} a_n$$

$$= -\frac{B(n,0)}{A(n,0)} a_n$$

$$= \frac{-(-2n(n-1)+n-1)}{2(n+1)n+n+1} a_n$$

$$= \frac{(2n-1)(n-1)}{(n+1)(2n+1)} a_n$$

$$n = 0, a_1 = \frac{(-1)(-1)}{1*1} a_0 = a_0$$

$$n = 1, a_2 = \frac{1*0}{2*3} a_1 = 0$$

$$\Rightarrow y_1(x) = a_0 + a_1 x$$

$$= a_0 + a_0 x$$

$$= a_0(1+x)$$

Series Solutions

- Case(ii) $r=\frac{1}{2}$

$$a_{n+1}=\frac{(2n-1)n}{(2n+3)(n+1)}a_n$$

$$n=0, a_1=0$$

$$\vdots$$

$$a_n=0$$

$$y_2(x)=a_0x^{\frac{1}{2}}$$

$$W=\begin{vmatrix}1+x & x^{\frac{1}{2}} \\ 1 & \frac{1}{2}x^{-\frac{1}{2}}\end{vmatrix}=\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{2}x^{\frac{1}{2}}\neq 0,\forall x$$

Series Solutions

$\because y_1, y_2$線性獨立,構成一組基底解

$y(x) = k_1 y_1(x) + k_2 y_2(x)$

$1. r_1 \neq r_2$

$if \ |r_1 - r_2| \notin N$

$y = k_1 y_1 + k_2 y_2$

$2. r_1 \neq r_2, |r_1 - r_2| \in N$

$$\begin{cases} y = k_1 y_1 + k_2 y_2 \\ ?\ y_2 = \varphi y_1 \end{cases}$$

$3. r_1 = r_2$

$$\begin{cases} y_1 \\ ?\ y_2 = \varphi y_1 \end{cases}$$