

Chapter 4. Laplace Transform

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Laplace Transform

性質9:

Initial Value Theorem 初值定理

Final Value Theorem 終值定理

$$F(s) = L\{f(t)\} \Rightarrow f(t) = L^{-1}\{F(s)\}$$

能不能只看 $F(s)$ 而不看 $f(t)$ 就知道 $f(0)$, $f(\infty)$ 之值

If $F(s)$ is given $f(0) = \lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s)$

初值定理

$$f(\infty) = \lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

終值定理

通常為週期函數，無法 inverse

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Laplace Transform

EX : $F(s) = \frac{2s}{(s+1)(s+2)}$

Verify : $f(t) = ?$

$$F(0) = \lim_{s \rightarrow \infty} sF(s) = \lim_{s \rightarrow \infty} \frac{2s^2}{s^2 + 3s + 2} = 2$$

$$F(\infty) = \lim_{s \rightarrow 0} sF(s) = \lim_{s \rightarrow 0} \frac{2s^2}{s^2 + 3s + 2} = 0$$

$$F(s) = \frac{2s}{(s+1)(s+2)} = \frac{a}{s+1} + \frac{b}{s+2}$$

$$\Rightarrow a = \frac{2s}{s+2} \Big|_{s=-1} = \frac{-2}{1} = -2$$

$$b = \frac{2s}{s+1} \Big|_{s=-2} = \frac{-4}{-1} = 4$$

$$\Rightarrow f(t) = -2e^{-t} + 4e^{-2t}, f(0) = 2$$

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Laplace Transform

pf : 聯想 $sF(s)$ 在 $\mathcal{L}\{f'\}$ 出現過(性質6)

$$\mathcal{L}\{f'(t)\} = \int_0^{\infty} f'(t)e^{-st} dt = sF(s) - f(0)$$

$$\lim_{s \rightarrow \infty} \left[\int_0^{\infty} f'(t)e^{-st} dt \right] = \lim_{s \rightarrow \infty} (sF(s) - f(0))$$

$$\int_0^{\infty} f'(t) \lim_{s \rightarrow \infty} [e^{-st}] dt = \lim_{s \rightarrow \infty} sF(s) - f(0)$$

$$\Rightarrow 0 = \lim_{s \rightarrow \infty} sF(s) - f(0)$$

$$\Rightarrow \lim_{s \rightarrow \infty} sF(s) = f(0)$$

$f'(t) \rightarrow \infty$ 不適用 <終值定理>
(不連續，斷掉處為 $\rightarrow \infty$)

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Laplace Transform

$$\mathcal{L}\{f'(t)\} = \int_0^{\infty} f'(t)e^{-st} dt = sF(s) - f(0)$$

$$\lim_{s \rightarrow 0} \left[\int_0^{\infty} f'(t)e^{-st} dt \right] = \lim_{s \rightarrow 0} (sF(s) - f(0))$$

$$\int_0^{\infty} f'(t) dt = \lim_{s \rightarrow 0} sF(s) - f(0)$$

$$f(t) \Big|_0^{\infty} = \lim_{s \rightarrow 0} sF(s) - f(0)$$

$$f(\infty) - f(0) = \lim_{s \rightarrow 0} sF(s) - f(0)$$

$$\lim_{s \rightarrow 0} sF(s) = f(\infty)$$

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Laplace Transform

EX : $F(s) = \frac{2s}{(s-1)(s+2)}$ 當分母根 ≥ 0 , $f(t) \rightarrow \infty$, 終值定理不成立

$$f(0) = \lim_{s \rightarrow \infty} \frac{2s^2}{(s-1)(s+2)} = 2$$

$$f(\infty) = \lim_{s \rightarrow 0} \frac{2s^2}{(s-1)(s+2)} = 0$$

$$\text{但 } F(s) = \frac{2}{s-1} + \frac{4}{s+2}$$

$$f(t) = \frac{2}{3}e^{\infty} + \frac{4}{3}e^{-2t}$$

$$f(0) = \frac{2}{3} + \frac{4}{3} = 2$$

$$f(\infty) = \infty \quad \text{矛盾}$$

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Laplace Transform

終值定理要成立，必須滿足某些條件才行

$$F(s) = \frac{N(s)}{D(s)}$$

$D(s)$ 的根不能在S-平面的右平面 i.e. $\text{Re}(s) > 0$

若在虛軸上不可有重根不能有共軛虛根

Laplace Transform

EX : $F(s) = \frac{2s}{(s^2 + 4)^2}, s^2 + 4 = 0, s = \pm 2i$

$$f(\infty) = \lim_{s \rightarrow 0} sF(s) = 0$$

$$f(t) = \frac{1}{2} t \sin 2t \text{ (如何得到?)}$$

$$\Rightarrow \int_s^\infty \frac{2s}{(s^2 + 4)^2} ds, \text{ 令 } u = s^2 + 4$$

$$= \int_{s^2+4}^\infty \frac{1}{u^2} du$$

$$= \frac{-1}{u} \Big|_{s^2+4}^\infty = \frac{1}{s^2 + 4}$$

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 4} \right\} = \frac{1}{2} t \sin 2t$$

$$f(t) = \frac{1}{2} t \sin 2t$$

Laplace Transform

1. 基本function 1~6

2. 基本性質 1~9

3. 週期函數

性質11. 週期函數的 Laplace transform

考慮函數 $f(t)$ 若滿足 $f(t+nT)=f(t)$

則稱 T 為 $f(t)$ 的週期，而 $f(t)$ 為週期函數 $h \in \mathbb{Z}$

$$\cos x \xrightarrow{T} 2\pi$$

$$\cos(x + 2n\pi) = \cos x$$

$$\cos 2x \xrightarrow{T} \pi$$

$$\cos(2x + n\pi) = \cos 2x$$

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Laplace Transform

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st} dt$$

$$= \int_0^T f(t)e^{-st} dt + \int_T^{2T} f(t)e^{-st} dt + \int_{2T}^{3T} f(t)e^{-st} dt + \dots$$

$$= \int_0^T f(t)e^{-st} dt + \int_T^{2T} f(t)e^{-st} dt \quad \text{令 } t = x + T \quad dt = dx$$

$$\text{其中所以} \Rightarrow \int_0^T f(t)e^{-s(x+T)} dx$$

$$= e^{-sT} \int_0^T f(t)e^{-sx} dx$$

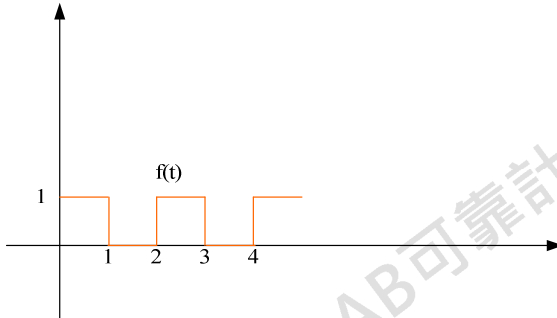
$$\Rightarrow \mathcal{L}\{f(t)\} = \int_0^T f(t)e^{-st} dt (1 + e^{-sT} + e^{-2sT} + \dots)$$

$$= \int_0^T f(t)e^{-st} dt \frac{1}{1 - e^{-sT}}$$

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Laplace Transform

EX : $f(t), T=2$



$$L\{f(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 f(t)e^{-st} dt$$

$$= \frac{1}{1-e^{-2s}} \int_0^2 1e^{-st} dt$$

$$= \frac{1}{1-e^{-2s}} \left(-\frac{1}{s} e^{-st} \right) \Big|_0^2$$

$$= \frac{1}{1-e^{-2s}} \left(-\frac{1}{s} e^{-2s} - \left(-\frac{1}{s} \right) \right)$$

$$= \frac{1}{1-e^{-2s}} \frac{1}{s} (1-e^{-2s})$$

$$= \frac{1}{s(1+e^{-s})} \times (1-e^{-s})$$

反求T，分母變為1-e^(-sT)，分母有理化

$$\text{已知 } \mathcal{L}^{-1} \left\{ \frac{1}{s(1-e^{-s})} \right\} = g(t)$$

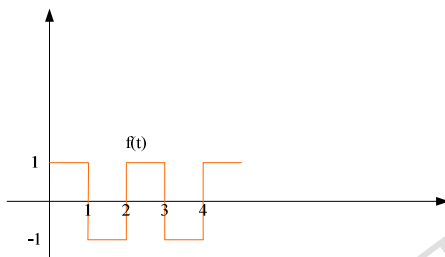
$$\text{求 } g(0) = 1$$

$$g(0) = \lim_{s \rightarrow \infty} sF(s) = \lim_{s \rightarrow \infty} \frac{1}{1-e^{-s}} = 1$$

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Laplace Transform

EX : $T=2$



$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 f(t)e^{-st} dt$$

$$= \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} dt - \int_1^2 e^{-st} dt \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{-1}{s} e^{-s} + \frac{1}{s} - \left(-\frac{1}{s} e^{-2s} + \frac{1}{2} e^{-s} \right) \right]$$

$$= \frac{1}{1-e^{-2s}} \frac{1}{s} [-2e^{-s} + 1 + e^{-2s}]$$

$$= \frac{1}{1-e^{-2s}} \frac{1}{s} (1-e^{-s})^2$$

$$= \frac{1-e^{-s}}{1+e^{-s}} \frac{1}{s}$$

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Laplace Transform

- Inverse Laplace transform

$$\text{設 } F(s) = \frac{N(s)}{D(s)}$$

Case I :

$$D(s) = (s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_n)$$

$\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n \in \Re$, 相異(Distinct)實根

$$F(s) = \frac{N(s)}{(s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_n)} = \frac{k_1}{s - \lambda_1} + \frac{k_2}{s - \lambda_2} + \cdots + \frac{k_n}{s - \lambda_n}$$

$$\mathcal{L}^{-1} \Rightarrow f(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} + \cdots + k_n e^{\lambda_n t}$$

$$\text{其中 } k_1 = \left. \frac{N(s)}{(s - \lambda_2) \cdots (s - \lambda_n)} \right|_{s = \lambda_1}$$

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Laplace Transform

EX:

$$F(s) = \frac{2s + 1}{(s + 1)(s + 2)(s + 3)} = \frac{a}{(s + 1)} + \frac{b}{(s + 2)} + \frac{c}{(s + 3)}$$

$$a = \frac{-1}{2}, b = \frac{-3}{-1} = 3, c = \frac{-5}{(-2)(-1)} = \frac{-5}{2}$$

$$f(t) = -\frac{1}{2} e^{-t} + 3e^{-2t} - \frac{5}{2} e^{-3t}$$

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Laplace Transform

EX:

$$F(s) = \frac{2s+3}{(s-1)(s+5)(s-3)} = \frac{a}{(s-1)} + \frac{b}{(s+5)} + \frac{c}{(s-3)}$$

$$a = \frac{-5}{12}, b = \frac{-7}{48}, c = \frac{9}{16}$$

$$f(t) = \frac{-5}{12}e^t + \frac{-7}{48}e^{-5t} + \frac{9}{16}e^{3t}$$

Laplace Transform

Case II

$$F(s) = \frac{N(s)}{D_1(s)(s-\lambda)^n}, \text{其中 } D_1(s) \text{ 為單根}$$

Laplace Transform

EX:

$$F(s) = \frac{s}{(s+1)(s-2)^2} = \frac{N_1(s)}{(s+1)} + \frac{N_2(s)}{(s-2)^2} = \frac{a}{s+1} + \frac{k_1}{(s-2)} + \frac{k_1}{(s-2)^2}$$

$$\Rightarrow a = F(s)(s+1) \Big|_{s=-1} = \frac{-1}{9}$$

$$k_1 = \frac{d}{ds} \left(F(s)(s-2)^2 \right) \Big|_{s=2} = \frac{d}{ds} \left(\frac{s}{s+1} \right) \Big|_{s=2} = \frac{1}{(s+1)^2} \Big|_{s=2} = \frac{1}{9}$$

$$k_2 = F(s)(s-2)^2 \Big|_{s=2} = \frac{2}{3}$$

$$f(t) = \frac{-1}{9} e^{-t} + \frac{1}{9} e^{2t} + \frac{2}{3} t e^{2t}$$

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Laplace Transform

Case II

$$\begin{aligned} F(s) &= \frac{N(s)}{D_1(s)(s-\lambda)^n} = \frac{N_1(s)}{D_1(s)} + \frac{N_2(s)}{(s-\lambda)^n} \\ &= \frac{N_1(s)}{D_1(s)} + \frac{k_1}{s-\lambda} + \frac{k_2}{(s-\lambda)^2} + \cdots + \frac{k_n}{(s-\lambda)^n} \end{aligned}$$

$$\Rightarrow k_n = (s-\lambda)^n F(s) \Big|_{s=\lambda} = \frac{N(s)}{D_1(s)} \Big|_{s=\lambda}$$

\vdots

$$k_1 = \frac{1}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}} ((s-\lambda)^n F(s))$$

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