Homework 1

Section **1.4**. 3, 14 ,**17**, 21, 24, 25, **37**
Section **1.5**. 13, **24**, 41, 57, 68,**71**,**72**
Section **2.1**. 5, 8
Section **2.2**. **6**, 9, 12, **13**, 34, **37**, 52
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Section **2.4**. 3, 17, 25, **29**,**43**

Section 1.4 Exponential Functions

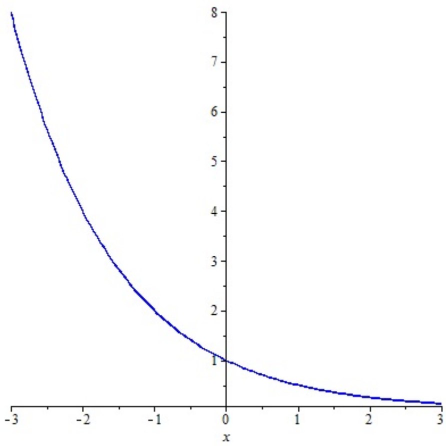
EX.3

(a) $b^8(2b)^4 = b^8 \cdot 16b^4 = 16b^{12}$

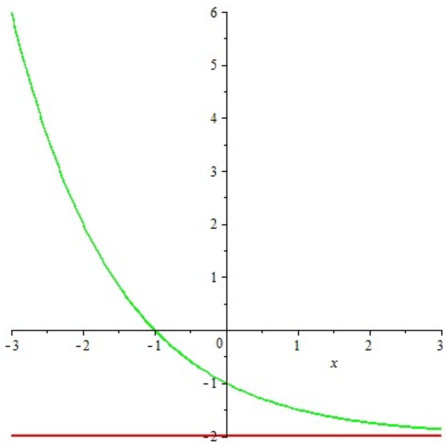
(b) $\frac{(6y^3)^4}{2y^5} = \frac{1296y^{12}}{2y^5} = 648y^7$

EX.14

We start with the graph of $y = (0.5)^x$ and shift it 2 units downward to obtain the graph of $y = (0.5)^x - 2$. The horizontal asymptote of the final graph is $y = -2$.



(a) $y = (0.5)^x$



(b) $y = (0.5)^x - 2$

EX.17

(a) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units downward, we substract 2 from the oringinal function to get $y = e^x - 2$.

(b) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units downward, we replace x with $x - 2$ in the oringinal function to get $y = e^{x-2}$.

(c) To find the equation of the graph that results from shifting the graph of $y = e^x$ about the x-axis, we multiply the oringinal function by -1 to get $y = -e^x$.

(d) To find the equation of the graph that results from shifting the graph of $y = e^x$ about the y-axis, we replace x with -x in the oringinal function to get $y = e^{-x}$.

(e) To find the equation of the graph that results from shifting the graph of $y = e^x$ about the x-axis and then about the y-axis, we first multiply the oringinal function by -1 to get (to get $y = -e^{-x}$) and then replace x with $-x$ in this equation to get $y = -e^{-x}$.

EX.21

Let $y = Cb^x$, then substitute $(1, 6)$ and $(3, 24)$ into y

$$\begin{cases} f(1) = Cb = 6 \\ f(3) = Cb^3 = 24 \end{cases}$$

We know $b^2 = 4$, $b = 2$ (since $b > 0$) and $C = 3$. So

$$f(x) = 3 \cdot 2^x$$

EX.24

Suppose the month is February. Your payment on the 28th day would be $2^{28-1} = 2^{27} = 134,217,728$ cents. Clearly, the second method of payment results in a larger amount for any month.

EX.25

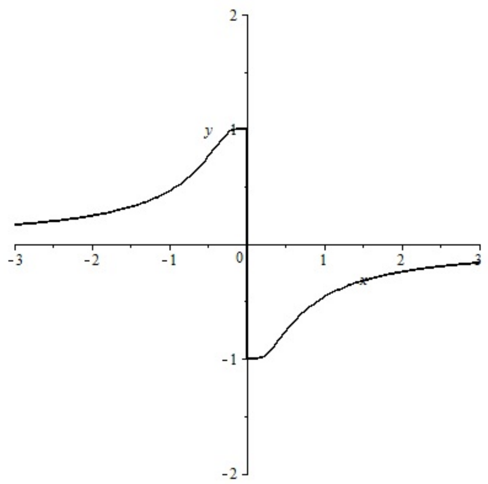
$1m$=$100cm$, $f(100) = 100^2 cm = 10000cm = 100m$.
$g(100) = 2^{100}cm = \frac{2^{100}}{100 \cdot 1000} km \approx 1.27 \times 10^{25} km > 10^{25} km$.

EX.37

From the graph, it appears that f is an odd function (f is undefined for $x = 0$). To prove this, we must show that $f(-x) = -f(x)$.

$$f(-x) = \frac{1-e^{1/(-x)}}{1+e^{1/(-x)}} = \frac{1-e^{(-1/x)}}{1+e^{(-1/x)}} = \frac{1-\frac{1}{e^{1/x}}}{1+\frac{1}{e^{1/x}}} \cdot \frac{e^{1/x}}{e^{1/x}} = \frac{e^{1/x}-1}{e^{1/x}+1}$$
$$= -\frac{1-e^{1/(-x)}}{1+e^{1/(-x)}} = -f(x)$$

so f is an odd function.



$f(x) = \frac{1-e^{1/(-x)}}{1+e^{1/(-x)}}$

Section 1.5 Inverse Functions and Logarithms

EX.13

An arrow will attain every height h up to its maximum height twice: once on the way up, and again on the way down. Thus, even if t_1 does not equal t_2, $f(t_1)$ may equal $f(t_2)$, so f is not $1-1$.

EX.17

First, we must determine x such that $g(x) = 4$. By inspection, we see that if x=0, then $g(x) = 4$. Since g is 1-1 (g is an increasing function), it has an inverse, and $g^{-1}(4) = 0$.

EX.24

$y = f(x) = \frac{e^x}{1+2e^x} \;\Rightarrow\; y = 2ye^x = e^x \;\Rightarrow\; y = e^x - 2ye^x \;\Rightarrow\; y = e^x(1-2y)$ $\Rightarrow\; e^x = \frac{y}{1-2y} \;\Rightarrow\; x = \ln\left(\frac{y}{1-2y}\right)$. Interchange x and y: $y = \ln\left(\frac{x}{1-2x}\right)$. So $f^{-1}(x) = \ln\left(\frac{x}{1-2x}\right)$. Note that the range of f and the domain of f^{-1} is $(0,\frac{1}{2})$.

EX.41

$\frac{1}{3}\ln(x+2)^3 + \frac{1}{2}[\ln x - \ln(x^2+3x+2)^2]$

$= \ln[(x+2)^3]^{(\frac{1}{3})} + \frac{1}{2}\ln\frac{x}{(x^2+3x+2)^2}$ [by laws3,2]

$= \ln(x+2) + \ln(\frac{\sqrt{x}}{x^2+3x+2})$ [by law 3]

$= \ln\frac{(x+2)\sqrt{x}}{(x+1)(x+2)}$ [by law 1]

$= \ln\frac{\sqrt{x}}{x+1}$.

EX.57

(a) We must have $e^x - 3 > 0 \iff e^x > 3 \iff x > \ln 3$. Thus, the domain of $f(x) = \ln(e^x - 3)$ is $(\ln 3, \infty)$.

(b) $y = \ln(e^x - 3) \Rightarrow e^y = e^x - 3 \Rightarrow e^x = e^y + 3 \Rightarrow x = \ln(e^y + 3)$, so $f^{-1}(x) = \ln(e^x + 3)$. Now $e^x + 3 > 0 \Rightarrow e^x > -3$, which is true for any real x, so the domain of f^{-1} is $\mathbf{R}$.

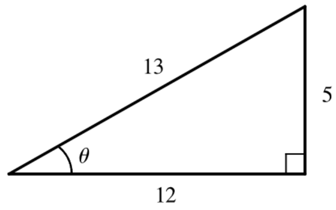
EX.68

(a) $\sin^{-1}(\sin(\frac{5\pi}{4})) = \sin^{-1}(\frac{-1}{\sqrt{2}}) = -\frac{\pi}{4}$ because $\sin(-\frac{\pi}{4}) = -1/\sqrt{2}$ and $-\frac{\pi}{4}$ is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(b) Let $\theta = \sin^{-1}(\frac{5}{13})$ [see the figure].

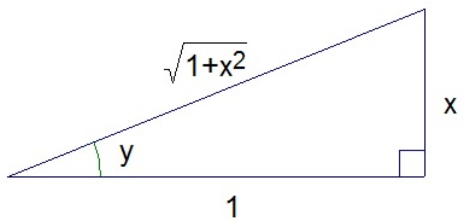
$\cos(2\sin^{-1}(\frac{5}{13})) = \cos 2\theta = \cos^2\theta - \sin^2\theta$

$= (\frac{12}{13})^2 - (\frac{5}{13})^2 = \frac{144}{169} - \frac{25}{169} = \frac{119}{169}$



EX.71

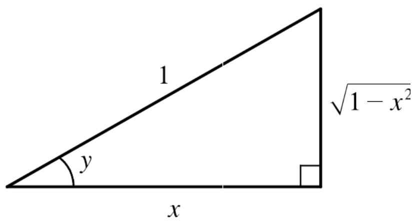
Let $y = \tan^{-1} x$. Then $\tan y = x$,so from the triangle (which illustrates the case $y > 0$), we see that $\tan(\sin^{-1} x) = \sin y = \frac{x}{\sqrt{1+x^2}}$.



EX.72

Let $y = \arccos x$. Then $\cos y = x$, so from the triangle (which illustrates the case $y > 0$), we see that

$$\sin(2\arccos x) = \sin 2y = 2\sin y\cos y = 2(\sqrt{1-x^2})(x) = 2x\sqrt{1-x^2}$$



Section 2.1 The Tangent and Velocity Problems

EX.5

(a) $y = y(t) = 10t - 4.9t^2$. At $t = 1.5$, $y = 10(1.5) - 4.9(1.5)^2 = 3.975$. The average velocity between times 1.5 and 1.5+h is

$v_{ave} = \frac{y(1.5+h)-y(1.5)}{(1.5+h)-1.5} = \frac{[10(1.5+h)-4.9(1.5+h)^2]-3.975}{h} = \frac{15+10h-11.025-14.7h-4.9h^2-3.975}{h} = \frac{-4.7h-4.9h^2}{h} = -4.7 - 4.9h$, if $h \neq 0$.

(i)$[1.5, 2] : h = 0.5, v_{ave} = -7.15m/s$

(ii)$[1.5, 1.6] : h = 0.1, v_{ave} = -5.19m/s$

(iii)$[1.5, 1.55] : h = 0.05, v_{ave} = -4.945m/s$

(iv)$[1.5, 1.51] : h = 0.01, v_{ave} = -4.749m/s$

(b) The instantaneous velocity when t=1.5 (h approaches 0) is -4.7 m/s.

EX.8

(a)

(i) $s = s(t) = 2\sin \pi t + 3\cos \pi t$. On the interval [1,2], $v_{ave} = \frac{s(2)-s(1)}{2-1} = \frac{3-(-3)}{1} = 6cm/s$.

(ii) On the interval [1, 1.1], $v_{ave} = \frac{s(1.1)-s(1)}{1.1-1} \approx \frac{-3.471-(-3)}{0.1} = -4.71cm/s$.

(iii) On the interval [1,1.01], $v_{ave} = \frac{s(1.01)-s(1)}{1.01-1} \approx \frac{-3.0613-(-3)}{0.01} = -6.13cm/s$.

(iv) On the interval [1,1.001], $v_{ave} = \frac{s(1.001)-s(1)}{1.001-1} \approx \frac{-3.00627-(-3)}{0.001} = -6.27cm/s$.

(b) The instantaneous velocity of the particle when t = 1 appears to be about -6.3 cm/s.

Section 2.2 The Limit of a Function

EX.6

(a) h(x) approaches 4 as x approaches -3 from the left, so $\lim_{x\to\ -3^-} h(x) = 4$.

(b) h(x) approaches 4 as x approaches -3 from the right, so $\lim_{x\to\ -3^+} h(x) = 4$.

(c) $\lim_{x\to -3} h(x) = 4$ because the limits in part (a) and part (b) are equal.

(d) h(-3) is not defined, so it doesn't exist.

(e) h(x) approaches 1 as x approaches 0 from the left, so $\lim_{x\to\ 0^-} h(x) = 1$.

(f) h(x) approaches -1 as x approaches 0 from the right, so $\lim_{x\to\ 0^+} h(x) = -1$.

(g) $\lim_{x\to\ 0}$ h(x) does not exist because the limits in part (e) and part (f) are not equal.

(h) h(0) $=$ 1 since the point (0,1) is on the graph of h.

(i) Since $\lim_{x\to\ 2^-} h(x) = 2$ and $\lim_{x\to\ 2^+} h(x) = 2$, we have $\lim_{x\to\ 2} h(x) = 2$.

(j) h(2) is not defined, so it doesn't exist.

(k) h(x) approaches 3 as x approaches 5 from the right, so $\lim_{x\to\ 5^+} h(x) = 3$

(l) h(x) does not approach any one number as x approaches 5 from the left , so $\lim_{x\to\ 5^-} h(x)$ does not exist.

EX.9

(a) $\lim\limits_{x\to -7} f(x) = -\infty$

(b) $\lim\limits_{x\to -3} f(x) = \infty$

(c) $\lim\limits_{x\to 0} f(x) = \infty$

(d) $\lim\limits_{x\to 6^-} f(x) = -\infty$

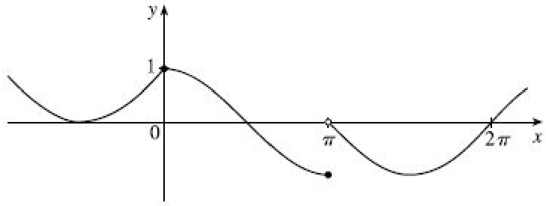
(e) $\lim\limits_{x\to 6^+} f(x) = \infty$

(f) the equations of the vertical asymptotes are $x = -7, x = -3, x = 0$, and $x = 6$

EX.12

From the graph of

$$f(x) = \begin{cases} 1+\sin(x) & \text{if } x \le 0 \\ \cos(x) & \text{if } 0 \le x \le \pi \\ \sin(x) & \text{if } \pi < x \end{cases}$$



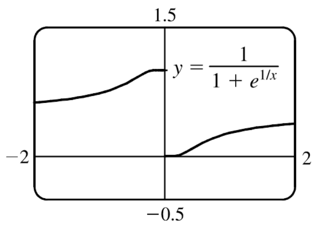
we see that $\lim_{x\to a} f(x)$ exists for all a except $a = \pi$. Notice that the right and left limits are different at $a = \pi$.

EX.13

(a) $\lim_{x\to\ 0^-} f(x) = 1$

(b) $\lim_{x\to\ 0^+} f(x) = 0$

(c) $\lim_{x\to\ 0} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.



EX.34

$\lim\limits_{x\to 3^-} \dfrac{\sqrt{x}}{(x-3)^5} = -\infty$ since the numerator is positive and the denominator approaches 0 from the negative side as $x \to 3^-$

EX.37

$\lim_{x\to\ (\pi/2)^+}(\frac{1}{x}\sec x) = -\infty$ since $\frac{1}{x}$ is positive and $\sec x \to -\infty$ as $x \to (\pi/2)^+$.

EX.52

(a) For any positive integer n, if $x = \frac{1}{n\pi}$, then

$$f(x) = \tan\frac{1}{x} = \tan n\pi = 0$$

(Remember that the tangent function has period π)

(b) For any nonnegative number n , if $x=\frac{4}{(4n+1)\pi}$, then

$$f(x)=\tan\frac{1}{x}=\tan\frac{(4n+1)\pi}{4}=\tan\left(\frac{4n\pi}{4}+\frac{\pi}{4}\right)=\tan\left(n\pi+\frac{\pi}{4}\right)=\tan\frac{\pi}{4}=1$$

(c) From part (a), $f(x)=0$ infinitely often as $x\to 0$. From part (b), f(x)=1 infinitely often as $x\to 0$. Thus, $\lim\limits_{x\to 0}\tan\dfrac{1}{x}$ does not exist since $f(x)$ does not get close to a fixed number as $x\to 0$

Section 2.3 Calculating Limits Using the Limit Laws

EX.20

We use the difference of squares in the numerator and the difference of cubes in the denominator.

$$\lim_{t\to 1}\frac{t^4-1}{t^3-1}=\lim_{t\to 1}\frac{(t^2-1)(t^2+1)}{(t^2+t+1)(t-1)}$$
$$=\lim_{t\to 1}\frac{(t+1)(t-1)(t^2+1)}{(t^2+t+1)(t-1)}=\lim_{t\to 1}\frac{(t+1)(t^2+1)}{(t^2+t+1)}=\frac{2(2)}{3}=\frac{4}{3}$$

EX.27

$$\lim_{x\to 16}\frac{4-\sqrt{x}}{16x-x^2}=\lim_{x\to 16}\frac{(4-\sqrt{x})(4+\sqrt{x})}{(16x-x^2)(4+\sqrt{x})}=\lim_{x\to 16}\frac{16-x}{x(16-x)(4+\sqrt{x})}=\lim_{x\to 16}\frac{1}{x(4+\sqrt{x})}$$
$=\frac{1}{16(4+\sqrt{16})}$
$=\frac{1}{16(8)}=\frac{1}{128}$

EX.29

$$\lim_{t\to 0}\frac{1}{t\sqrt{(1+t)}}-\frac{1}{t}=\lim_{t\to 0}\frac{1-\sqrt{1+t}}{t\sqrt{(1+t)}}=\lim_{t\to 0}\frac{(1-\sqrt{1+t})(1+\sqrt{1+t})}{t\sqrt{(1+t)}(1+\sqrt{1+t})}$$
$$=\lim_{t\to 0}\frac{-t}{t\sqrt{(1+t)}(1+\sqrt{1+t})}=\lim_{t\to 0}\frac{-1}{\sqrt{(1+t)}(1+\sqrt{1+t})}=\frac{-1}{\sqrt{(1+0)}(1+\sqrt{1+0})}$$
$=-\frac{1}{2}$

EX.32.

$\lim\limits_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}=\lim\limits_{h\to 0}\frac{\frac{x^2-(x+h)^2}{(x+h)^2x^2}}{h}=\lim\limits_{h\to 0}\frac{x^2-(x^2+2xh+h^2)}{hx^2(x+h)^2}=\lim\limits_{h\to 0}\frac{-h(2x+h)}{hx^2(x+h)^2}=\lim\limits_{h\to 0}\frac{-(2x+h)}{x^2(x+h)^2}=$
$\frac{-2x}{x^2\cdot x^2}=\frac{-2}{x^3}$

EX.39.

$-1 \leq \cos(\frac{2}{x}) \leq 1 \Rightarrow -x^4 \leq x^4\cos(\frac{2}{x}) \leq x^4$. Since $\lim\limits_{x\to 0}(-x^4) = 0$ and $\lim\limits_{x\to 0}(x^4) = 0$, we have $\lim\limits_{x\to 0}[x^4\cos(\frac{2}{x})] = 0$ by the Squeeze Theorem.

EX.43

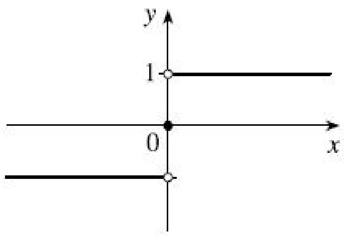
$\mid 2x^3 - x^2 \mid = \mid x^2(2x-1) \mid = \mid x^2 \mid * \mid 2x-1 \mid = x^2 \mid 2x-1 \mid$

$$\mid 2x-1 \mid = \begin{cases} 2x-1 & \text{if } 2x-1 \geq 0 \\ -(2x-1) & \text{if } 2x-1 < 0 \end{cases} = \begin{cases} 2x-1 & \text{if } x \geq 0.5 \\ -(2x-1) & \text{if } x < 0.5 \end{cases}$$

So $\mid 2x^3 - x^2 \mid = x^2[-(2x-1)]$ for $x < 0.5$.

Thus, $= \lim\limits_{x\to 0.5^-} \dfrac{2x-1}{\mid 2x^3 - x^2} \mid = \lim\limits_{x\to 0.5^-} \dfrac{2x-1}{x^2 \mid -(2x-1) \mid} = \lim\limits_{x\to 0.5^-} \dfrac{-1}{x^2}$
$= \frac{-1}{0.5^2} = \frac{-1}{0.25} = -4$

EX.47



(i) Since sgn $x = 1$ for $x > 0$, $\lim\limits_{x\to 0^+} \text{sgn}x = \lim\limits_{x\to 0^+} 1 = 1$

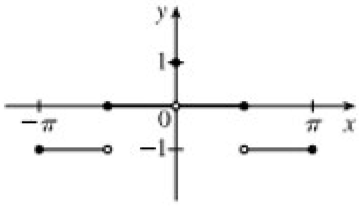
(ii)Since sgn $x = -1$ for $x < 0$, $\lim\limits_{x\to 0^-} \text{sgn}x = \lim\limits_{x\to 0^-} -1 = -1$

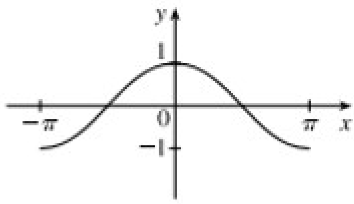
(iii)Since $\lim\limits_{x\to 0^+} \text{sgn}x \neq \lim\limits_{x\to 0^-} \text{sgn}x$, $\lim\limits_{x\to 0} \text{sgn}x$ does not exist.

(iv)Since $\mid \text{sgn}x \mid = 1$ for $x \neq 0$, $\lim\limits_{x\to 0} \mid \text{sgn}x \mid = \lim\limits_{x\to 0} 1 = 1$.

EX.54.

(a) See the graph of $y = \cos x$. Since $-1 \le \cos x < 0$ on $[-\pi, -\frac{\pi}{2})$, we have $y = f(x) = [\cos x] = -1$ on $[-\pi, -\frac{\pi}{2})$. Since $0 \le \cos x < 1$ on $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$, we have $f(x) = 0$ on $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$. Since $-1 \le \cos x < 0$ on $(\frac{\pi}{2}, \pi]$, we have $f(x) = -1$ on $(\frac{\pi}{2}, \pi]$. Note that $f(0) = 1$.





(b)(i) $\lim\limits_{\mathrm{x} \to 0^-} \mathrm{f(x)} = 0$ and $\lim\limits_{\mathrm{x} \to 0^+} \mathrm{f(x)} = 0$, so $\lim\limits_{\mathrm{x} \to 0} \mathrm{f(x)} = 0$.

(ii)As $\mathrm{x} \to (\dfrac{\pi}{2})^-$, $\mathrm{f(x)} \to 0$, so $\lim\limits_{\mathrm{x} \to (\frac{\pi}{2})^-} \mathrm{f(x)} = 0$.

(iii)As $\mathrm{x} \to (\dfrac{\pi}{2})^+$, $\mathrm{f(x)} \to -1$, so $\lim\limits_{\mathrm{x} \to (\frac{\pi}{2})^+} \mathrm{f(x)} = -1$.

(iv)Since the answers in parts (ii) and (iii) are not equal, $\lim\limits_{\mathrm{x} \to \frac{\pi}{2}} \mathrm{f(x)}$ does not exist.

(c) $\lim\limits_{\mathrm{x} \to \mathrm{a}} \mathrm{f(x)}$ exists for all a in the open interval $(-\pi, \pi)$ except $\mathrm{a} = -\dfrac{\pi}{2}$ and $\mathrm{a} = \dfrac{\pi}{2}$.

EX.65.

Since the denominator approaches 0 as $x \to -2$, the limit will exist only if the numerator also approaches 0 as $x \to -2$. In order for this to happen, we need
$\lim\limits_{x \to -2} (3x^2 + ax + a + 3) = 0 \Leftrightarrow 3(-2)^2 + a(-2) + a + 3 = 0 \Leftrightarrow 12 - 2a + a + 3 = 0 \Leftrightarrow a = 15$. With a=15, the limit becomes

$$\lim_{x \to -2} \frac{3x^2 + 15x + 18}{x^2 + x - 2} = \lim_{x \to -2} \frac{3(x+2)(x+3)}{(x-1)(x+2)} = \lim_{x \to -2} \frac{3(x+3)}{x-1} = \frac{3(-2+3)}{-2-1} = -1.$$

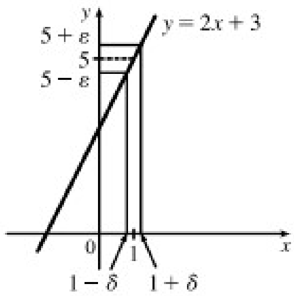
Section 2.4 The Precise Definition of a Limit

EX.3.

The leftmost question mark is the solution of $\sqrt{x} = 1.6$ and the rightmost, $\sqrt{x} = 2.4$. So the values are $1.6^2 = 2.56$ and $2.4^2 = 5.76$. On the left side, we need $| x - 4 |<| 2.56 - 4 |= 1.44$. On the right side, we need $| x - 4 |<| 5.76 - 4 |= 1.76$. To satisfy both conditions, we need the more restrictive condition to hold —namely, $| x - 4 |< 1.44$. Thus, we can choose $\delta = 1.44$, or any smaller positive number.

EX.17.

Given $\epsilon > 0$, we need $\delta > 0$ such that if $0 <| x-1 |< \delta$, then $| (2x+3)-5 |< \epsilon$. But $| (2x + 3) - 5 |< \epsilon \Leftrightarrow| 2x - 2 |< \epsilon \Leftrightarrow 2 | x - 1 |< \epsilon \Leftrightarrow| x - 1 |< \frac{\epsilon}{2}$. So if we choose $\delta = \frac{\epsilon}{2}$, then $0 <| x - 1 |< \delta \Rightarrow| (2x + 3) - 5 |< \epsilon$.Thus, $\lim\limits_{\infty\to 1}(2x + 3) = 5$ by the definition of a limit.



EX.25.

Given $\epsilon > 0$, we need $\delta > 0$ such that if $0 <| x - 0 |< \delta$, then $| x^2 - 0 |< \epsilon \Leftrightarrow x^2 < \epsilon \Leftrightarrow| x |< \sqrt{\epsilon}$. Take $\delta = \sqrt{\epsilon}$. Then $0 <| x - 0 |< \delta \Rightarrow| x^2 - 0 |< \epsilon$. Thus, $\lim\limits_{x\to 0}(x^2) = 0$ by the definition of a limit.

EX.29.

Given $\epsilon > 0$, we need $\delta > 0$ such that if $0 <| x - 2 |< \delta$, then $| (x^2 - 4x + 5) - 1 |< \epsilon \Leftrightarrow| x^2 - 4x + 4 |< \epsilon \Leftrightarrow| (x - 2)^2 |< \epsilon$. So take $\delta = \sqrt{\epsilon}$. Then $0 <| x - 2 |< \delta \Leftrightarrow| x - 2 |< \sqrt{\epsilon} \Leftrightarrow| (x - 2)^2 |< \epsilon$. Thus, $\lim\limits_{x\to 2}(x^2 - 4x + 5) = 1$ by the definition of a limit.

EX.43.

Given $M < 0$ we need $\delta > 0$ so that $\ln x < M$ wherever $0 < x < \delta$; that is, $x = e^{\ln x} < e^M$ whenever $0 < x < \delta$. This suggests that we take $\delta = e^M$. If $0 < x < e^M$, then $\ln x < \ln e^M = M$. By the definition of a limit, $\lim_{x \to 0^+} \ln x = -\infty$.