# Chapter 4.
# Laplace Transform

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# Laplace Transform

例:

$$3e^{2t} \xrightarrow{\mathscr{L}} \frac{3}{s-2}$$

$$2\cos 3t \xrightarrow{\mathscr{L}} \frac{2s}{s^2+9}$$

$$3\sin 2t \xrightarrow{\mathscr{L}} \frac{6}{s^2+4}$$

$$t \xrightarrow{\mathscr{L}} \frac{1}{s^2}$$

$$H(t) \xrightarrow{\mathscr{L}} \frac{1}{s}$$

$$\delta(t) \xrightarrow{\mathscr{L}} 1$$

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# Laplace Transform

- Properties：

1. Linear
2. 第一移位

$$f(t) \to F(s)$$

$$e^{at} f(t) \to F(s-a)$$

$EX$：

$$e^{2t}\cos 3t \xrightarrow{\mathscr{L}} \left.\frac{s}{s^2+9}\right|_{s=s-2} = \frac{s-2}{(s-2)^2+9}$$

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# Laplace Transform

3. 第二移位

$$f(t) \to F(s)$$

$$H(t-a)f(t-a) \to e^{-as}F(s)$$

$EX:$

$$F(s) = \frac{2}{s^2+4}e^{-2s}, f(t) = H(t-2)\sin 2(t-2)$$

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# Laplace Transform

4. $f(t) \to F(s)$

$$tf(t) \to \frac{-dF(s)}{ds}$$

$$t^2 f(t) \to \frac{-d}{ds}\left(\frac{-dF(s)}{ds}\right)$$

5. $f(t) \to F(s)$

$$\frac{1}{t} f(t) \to \int_s^{\infty} F(s)ds$$

$$\frac{1}{t^2} f(t) \to \int_s^{\infty}\int_s^{\infty} F(s)dsds$$

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# Laplace Transform

EX: $\frac{1}{t}\sin t \xrightarrow{\mathscr{L}} ?$

$$\sin t \xrightarrow{\mathscr{L}} \frac{1}{s^2+1}$$

$$\frac{1}{t}\sin t \xrightarrow{\mathscr{L}} \int_s^\infty \frac{1}{s^2+1}\,ds = \tan^{-1} s\Big|_s^\infty = \frac{\pi}{2} - \tan^{-1} s$$

微積分內瑕積分

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# Laplace Transform

$$\int_0^\infty \frac{\sin t}{t} dt$$

$$= \lim_{t\to\infty} \int_0^t \frac{\sin t}{t} dt$$

$$\lim_{t\to 0} \int_0^\infty \frac{\sin t}{t} e^{-st} dt$$

$$= \lim_{s\to 0} \mathscr{L}\{\frac{\sin t}{t}\} = \lim_{s\to 0}\left(\frac{\pi}{2} - \tan^{-1} s\right) = \frac{\pi}{2}$$

$$\int_0^\infty \lim_{s\to 0} \frac{\sin t}{t} e^{-st} dt$$

$$= \int_0^\infty \frac{\sin t}{t} dt = \text{原式}$$

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# Laplace Transform

結合性質4.和性質5.

$$\begin{tikzcd}[column sep=huge, row sep=large]
\mathrm{G(s)} \arrow[d, "\mathcal{L}^{-1}"'] & F(s) \arrow[l, "\int_s^{\infty} F(s)ds"'] \arrow[r, "\frac{-dF(s)}{ds}"] \arrow[d, "\mathcal{L}^{-1}"] & G(s) \arrow[d, "\mathcal{L}^{-1}"] \\
g(t) \arrow[r, "\times t"] & f(t) & g(t) \arrow[l, "\div t"']
\end{tikzcd}$$

$$\mathcal{L}^{-1}\left(\int_s^{\infty} \mathcal{L}\left\{g(t)\cdot t\right\} ds\right) = g(t)$$

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# Laplace Transform

EX: $F(s)=\ln\frac{s+1}{s+2},\ f(t)=?$

$$\frac{dF(s)}{ds}=\frac{d}{ds}\ln\frac{s+1}{s+2}=\frac{s+2}{s+1}\left(\frac{d}{ds}\cdot\frac{s+1}{s+2}\right)$$

$$=\frac{s+2}{s+1}\cdot\frac{(s+2)-(s+1)}{(s+2)^2}=\frac{1}{(s+2)(s+1)}=\frac{a}{s+1}+\frac{b}{s+2}$$

$$=\frac{1}{s+1}+\frac{-1}{s+2}$$

$$=\mathscr{L}(e^{-t}-e^{-2t})$$

$$\Rightarrow \mathscr{L}^{-1}\left(\frac{-dF(s)}{ds}\right)=e^{-2t}-e^{-t}$$

$$f(t)=\frac{1}{t}\left(e^{-2t}-e^{-t}\right)$$

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# Laplace Transform

$$\int_0^\infty \frac{1}{t}\left(e^{-2t} - e^{-t}\right)dt$$

$$= \int_0^\infty \lim_{s\to 0} \frac{1}{t}\left(e^{-2t} - e^{-t}\right)e^{-st}dt = \lim_{s\to 0}\int_0^\infty \frac{1}{t}\left(e^{-2t} - e^{-t}\right)e^{-st}dt$$

$$= \lim_{s\to 0}\int_0^\infty \frac{1}{t}\left(e^{-2t} - e^{-t}\right)e^{-st}dt$$

$$= \lim_{s\to 0} \mathscr{L}\{\frac{1}{t}\left(e^{-2t} - e^{-t}\right)\} = \lim_{s\to 0} \ln\frac{s+1}{s+2} = -\ln 2$$

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# Laplace Transform

EX: $F(s)=\dfrac{2s}{(s^2+4)^2},\ f(t)=?$

越微分越複雜

$$\int_0^\infty F(s)ds=\int_s^\infty \frac{2s}{(s^2+4)^2}ds$$

令$t=s^2+4, dt=2sds, ds=\dfrac{1}{2s}dt$

$$=\int_{s^2+4}^\infty \frac{2s}{t^2}\frac{1}{2s}dt$$

$$=\int_{s^2+4}^\infty \frac{2s}{t^2}\frac{1}{2s}dt$$

$$=\int_{s^2+4}^\infty \frac{1}{t^2}dt$$

$$=-\frac{1}{t}\bigg|_{s^2+4}^{\infty}$$

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# Laplace Transform

$$= -\frac{1}{t}\Big|_{s^2+4}^{\infty}$$

$$= 0 - \left(-\frac{1}{s^2+4}\right)$$

$$= \frac{1}{s^2+4}$$

$$\Rightarrow g(t) = \frac{1}{2}\sin 2t$$

$$\Rightarrow f(t) = \frac{1}{2}t\sin 2t$$

$$\int_0^{\infty} \frac{1}{2}t\sin 2t dt = 0$$

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# Laplace Transform

6.

$$f(t) \xrightarrow{\mathscr{L}} F(s)$$

$$f'(t) \xrightarrow{\mathscr{L}} sF(s) - f(0)$$

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# Laplace Transform

EX:

$$y' + 2y = e^t, y(0) = 0$$

$$\mathscr{L}\{y' + 2y\} = \mathscr{L}\{e^t\}$$

$$\mathscr{L}\{y'\} + \mathscr{L}\{2y\} = \mathscr{L}\{e^t\}$$

$$SY(s) - y(0) + 2Y(s) = \frac{1}{s-1}$$

$$(s+2)Y(s) = \frac{1}{s-1}$$

$$Y(s) = \frac{1}{(s-1)(s+2)} = \frac{\frac{1}{3}}{s-1} + \frac{-\frac{1}{3}}{s+2}$$

$$y(t) = \frac{-1}{3}e^{-2t} + \frac{1}{3}e^t$$

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# Laplace Transform

Pf:

$$\mathscr{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}dt$$

$$= \int_0^\infty e^{-st}df(t)$$

$$= e^{-st}f(t)\Big|_0^\infty - \int_0^\infty f(t)(-se^{-st})dt$$

$$= (0 - f(0)) + s\int_0^\infty f(t)e^{-st}dt$$

$$= sF(s) - f(0)$$

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# Laplace Transform

## 性質6.1

$$\mathscr{L}\left\{f''(t)\right\} = L\left\{(f'(t))'\right\} \text{ 令 } g(t) = f'(t)$$

$$= \mathscr{L}\left\{g'(t)\right\}$$

$$= s\mathscr{L}\left\{g(t)\right\} - g(0)$$

$$= s(sF(s) - f(0)) - f'(0)$$

$$= s^2F(s) - sf(0) - f'(0)$$

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# Laplace Transform

## 性質6.2

$$\mathscr{L}\left\{ f^{(n)}(t) \right\}$$

$$= s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{'}(0) - \cdots - f^{(n-1)}(0)$$

*用數學歸納法可證

$$y'' + 3y' + 2y = e^t, y(0) = 0, y'(0) = 0$$

$$\mathscr{L}\left\{ y^{''} \right\} = s^2 Y(s) - sy(0) - y^{'}(0) \qquad \mathscr{L}\left\{ y^{'} \right\} = sY(s) - y(0)$$

$$s^2 Y(s) + 3sY(s) + 2Y(s) = \frac{1}{s-1}$$

$$Y(s) = \frac{1}{(s-1)(s+1)(s+2)} = \frac{\frac{1}{6}}{s-1} + \frac{\frac{-1}{2}}{s+1} + \frac{\frac{1}{3}}{s+2}$$

$$y(t) = \frac{1}{6} e^t - \frac{1}{2} e^{-t} + \frac{1}{3} e^{-2t}$$

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# Laplace Transform

性質7

$$f(t) \xrightarrow{\mathscr{L}} F(s)$$

$$\int_0^t f(t)dt \xrightarrow{\mathscr{L}} \frac{F(s)}{s}$$

$$1 \xrightarrow{\mathscr{L}} \frac{1}{s}$$

$$t \xrightarrow{\mathscr{L}} \frac{1}{s^2}$$

$$\int_0^t 1dt \xrightarrow{\mathscr{L}} \frac{1}{s^2}$$

$$\mathscr{L}\left\{\int_0^t \cdots \int_0^t f(t)dt \cdots dt\right\} = \frac{1}{s^k} F(s)$$

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# Laplace Transform

pf：

$$\mathscr{L}\left\{\int_0^t f(t)\right\}$$

$$=\int_0^\infty \int_0^t f(t)dte^{-st}dt$$

$$=\int_0^\infty \int_0^t f(\lambda)d\lambda e^{-st}dt \qquad u=\int_0^t f(\lambda)d\lambda \quad du=f(t)dt \quad dv=e^{-st}dt \quad v=\frac{-1}{s}e^{-st}$$

$$=uv\Big|_0^\infty-\int_0^\infty vdu$$

$$=(\int_0^t f(\lambda)d\lambda)(\frac{-1}{s}e^{-st})\Big|_0^\infty-\int_0^\infty \frac{-1}{s}e^{-st}f(t)dt$$

$$=(0-0)+\frac{1}{s}\int_0^\infty f(t)e^{-st}dt$$

$$=\frac{1}{s}F(s)$$

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EX： $\mathscr{L}\left\{e^{2t}\int_0^t e^{3t}(t)(\sin t)dt\right\}$

$$\mathscr{L}\{\sin t\} = \frac{1}{s^2+1}$$

$$\mathscr{L}\{t\sin t\} = \frac{-d}{ds}\frac{1}{s^2+1} = \frac{2s}{(s^2+1)^2}$$

$$\mathscr{L}\left\{e^{3t}t\sin t\right\} = \frac{2s}{(s^2+1)^2}\Big|_{s=s-3} = \frac{2(s-3)}{((s-3)^2+1)^2} = \frac{2s-6}{(s^2-6s+10)^2}$$

$$\mathscr{L}\left\{\int_0^t e^{3t}\times(t)\times(\sin t)dt\right\} = \frac{1}{s}\frac{2s-6}{(s^2-6s+10)^2}$$

$$\mathscr{L}\left\{e^{2t}\int_0^t e^{3t}\times(t)\times(\sin t)dt\right\} = \frac{2s-6}{s(s^2-6s+10)^2}\Big|_{s=s-2}$$

$$= \frac{2(s-2)-6}{(s-2)((s-2)^2-6(s-2)+10)^2}$$

$$= \frac{2s-10}{(s-2)(s^2-10s+26)^2}$$

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# Laplace Transform

性質8 Convolution theorem $\mathscr{L}\{f \otimes g\} = F(s)G(s)$

$$f(t) \otimes g(t) \triangleq \int_0^t f(\lambda)g(t-\lambda)d\lambda$$

EX： $f(t) = e^{2t}, g(t) = t$

$$f(t) \otimes g(t) = \int_0^t g(\lambda) f(t-\lambda)d\lambda \qquad 令x = t-\lambda, dx = -d\lambda$$

$$= \int_t^0 f(t-x)g(t)(-dx)$$

$$= \int_0^t f(t-x)g(x)dx \qquad 令x = \lambda$$

$$= \int_0^t f(t-\lambda)g(\lambda)d\lambda = g(t) \otimes f(t)$$

$$e^{2t} \otimes t = \int_0^t e^{2\lambda}(t-\lambda)d\lambda = \int_0^t \lambda e^{2(t-\lambda)}d\lambda$$

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# Laplace Transform

EX： $\int_0^t \cos\lambda e^{2(t-\lambda)} d\lambda$

$$= \cos t \otimes e^{2t}$$

$$f(t) \xrightarrow{\mathscr{L}} F(s)$$

$$g(t) \xrightarrow{\mathscr{L}} G(s)$$

$$f(t) \otimes g(t) \xrightarrow{\mathscr{L}} F(s)G(s)$$

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# Laplace Transform

EX： $\mathscr{L}\left\{\int_0^t e^{2\lambda}(t-\lambda)d\lambda\right\}$

$$=\frac{1}{s-2}\frac{1}{s}$$

EX： $f(t)=e^{2t}, g(t)=e^{3t}$

$$f(t)\otimes g(t)=\int_0^t e^{2\lambda}e^{3(t-\lambda)}d\lambda$$

$$=e^{3t}\int_0^t e^{-\lambda}d\lambda$$

$$=e^{3t}(-e^{-\lambda}\Big|_0^t)$$

$$=e^{3t}(-e^{-t}-(-1))$$

$$=-e^{2t}+e^{3t}$$

$$\mathscr{L}\left\{e^{3t}-e^{2t}\right\}$$

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# Laplace Transform

$$= \frac{1}{s-3} - \frac{1}{s-2}$$

$$= \frac{1}{(s-3)(s-2)}$$

$$= \frac{1}{s-3} \bullet \frac{1}{s-2}$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s-3} \bullet \frac{1}{s-2}\right\} = e^{3t} \otimes e^{2t}$$

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# Laplace Transform

pf：

$$L\{f(t)\otimes g(t)\}$$

$$=\int_0^{\infty}[f(t)\otimes g(t)]e^{-st}dt$$

$$=\int_0^{\infty}\left(\int_0^{t} f(\lambda)g(t-\lambda)d\lambda\right)e^{-st}dt$$

$$=\int_0^{\infty}\int_0^{t} f(\lambda)g(t-\lambda)e^{-st}d\lambda dt$$

$\lambda$ 和 $t$ 積分範圍相關,想對調

$$=\int_0^{\infty}\int_{\lambda}^{\infty} f(\lambda)g(t-\lambda)e^{-st}dtd\lambda$$

要在積分區域相同條件下

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# Laplace Transform

$$= \int_0^\infty f(\lambda) \int_0^\infty g(t-\lambda) e^{-st} dt d\lambda \qquad \text{令} x = t - \lambda, dx = dt$$

$$= \int_0^\infty f(\lambda) \int_0^\infty g(x) e^{-s(\lambda+x)} dx d\lambda$$

$$= \int_0^\infty f(\lambda) e^{-s\lambda} \int_0^\infty g(x) e^{-sx} dx d\lambda$$

$$= G(s) \int_0^\infty f(\lambda) e^{-s\lambda} d\lambda$$

$$= G(s) F(s)$$

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