

# Chapter 1. Introduction to differential equations

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## Linear Differential Equations

- Linear D.E. 線性微分方程式，不具有下列任何一項

1. 因變數的自乘項
2. 因變數導數的自乘項
3. 因變數及其導數的互乘項

則稱為線性微分方程式；反之，若一微分方程式，具有上述1,2,3中任何一項，即稱為非線性D.E.。

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# Linear Differential Equations

例:

(1)  $y'(x) + 5y''(x) = e^x \Rightarrow$  線性2階1次O.D.E

(2)  $\frac{\partial u(x, y)}{\partial x} + u(x, y) \frac{\partial u(x, y)}{\partial y} + 3\mu(x, y) = 0$

$\Rightarrow$  非線性(Disobey#3), 1階1次P.D.E.

(3)  $\frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial y} = 0 \quad (u = u(x, y))$

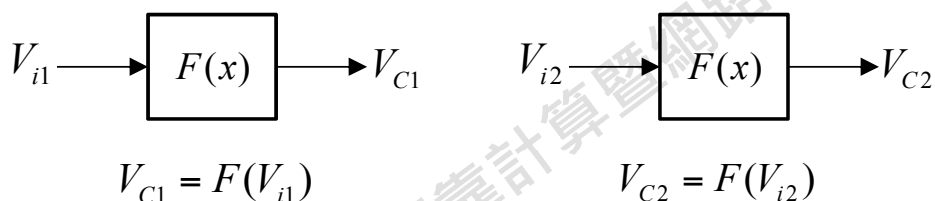
$\Rightarrow$  線性(x是自變數), 2階1次P.D.E.

(4)  $y'''(x) + 4y''(x) + 2y(x) = x^2 \Rightarrow$  線性3階1次O.D.E.

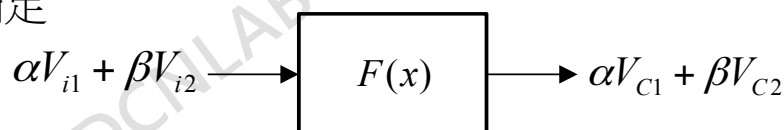
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# Linear Differential Equations

- 一線性微分方程式會滿足重疊定理(Superposition)



若滿足



$$\alpha V_{C1} + \beta V_{C2} = F(\alpha V_{i1} + \beta V_{i2}) \quad \alpha, \beta \text{ 為任意純量}$$

則為線性。

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# Differential Equations

- 目的：分析微分方程式的解  
⇒ 了解微分方程式由何而來

例：

$$(1) \ y(x) = C \quad C \in \text{常數 (Constant)}$$

如何消去C?

⇒ 應用微分，才可以消去C

$$\frac{dy(x)}{dx} = 0, \text{ 1階1次O.D.E.}$$

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# Differential Equations

$$(2) \ y(x) = C_1x + C_2 \quad C_1, C_2 \in \text{Constant}$$

$$\Rightarrow \frac{dy(x)}{dx} = C_1$$

$$\frac{d^2y(x)}{dx^2} = 0 \quad \Rightarrow \text{線性, 2 階 1 次 O.D.E.}$$

$$(3) \ y(x) = Cx + C^2 \quad C \in \text{constant}$$

$$\Rightarrow \frac{dy(x)}{dx} = C$$

一個常數，只能微分一次

( $\because$  一次不定積分，只會產生一個常數)

$$\Rightarrow y(x) = \frac{dy(x)}{dx}x + \left(\frac{dy(x)}{dx}\right)^2 \quad \Rightarrow \text{非線性, 1 階 2 次 O.D.E.}$$

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# Differential Equations

$$(4) \ y(x) = C_1 e^{2x} + C_2 e^x \quad - (a) , \ C_1, C_2 \in \text{Constant}$$

$$\frac{dy(x)}{dx} = 2C_1 e^{2x} + C_2 e^x \quad - (b)$$

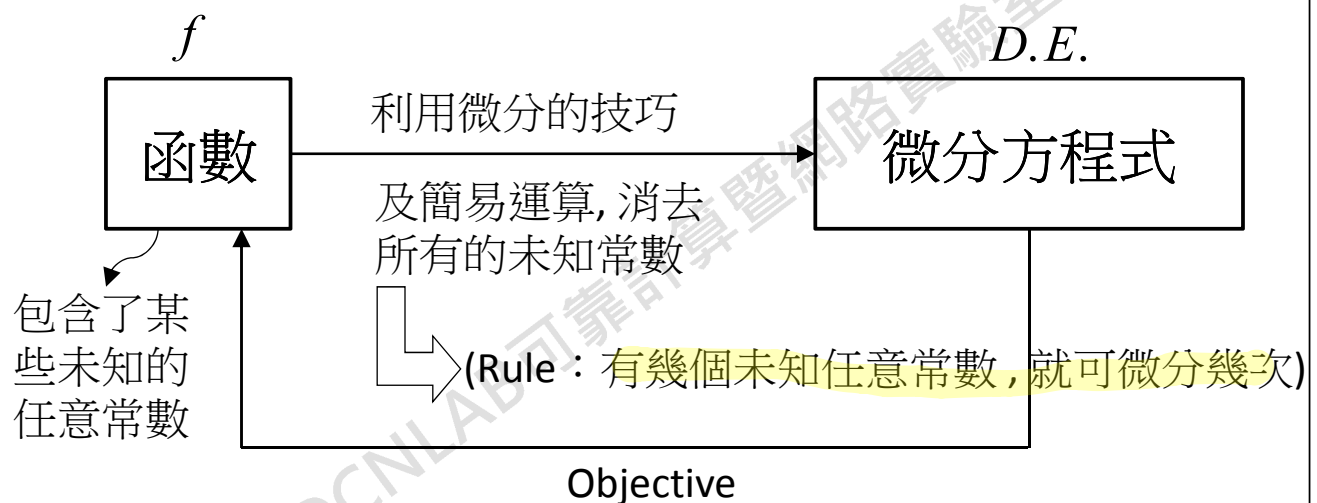
$$\frac{d^2 y(x)}{dx^2} = 4C_1 e^{2x} + C_2 e^x \quad - (c)$$

$$(a) \times 2 + (b) \times -3 + (c) \times 1$$

$$\Rightarrow y''(x) - 3y'(x) + 2y(x) = 0$$

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# Differential Equations



$\Rightarrow$  Apply for O.D.E. Only

$f$  稱為  $D.E.$  的通解(general solution) or 原函數

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# Differential Equations

例:

$$y(x) = C_1 \cos 3x + C_2 \sin 3x, \quad C_1, C_2 \in \text{Constant}$$
$$\Rightarrow D.E. = ?$$

Sol:

$$\frac{dy}{dx} = -3C_1 \sin 3x + 3C_2 \cos 3x$$

$$\frac{d^2 y}{dx^2} = -9C_1 \cos 3x - 9C_2 \sin 3x$$

$$\Rightarrow y''(x) + 9y(x) = 0$$

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## Chapter 2. First-Order Ordinary Differential Equations

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# First-Order Differential Equation

- 一般一階O.D.E., 可表成

1.  $M(x, y)dx + N(x, y)dy = 0$

2.  $y'(x) = f(x, y) \Rightarrow y'(x) = \frac{dy}{dx} = f(x, y)$

$$f(x, y)dx = dy \Rightarrow f(x, y)dx - dy = 0$$

$$\Rightarrow M(x, y)dx + N(x, y)dy = 0$$

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}$$

$$y' = f(x, y)$$

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# First-Order Differential Equation

- 定理 P14 §1.2.1

對  $y' = f(x, y)$  之D.E.有一個Initial condition(I.C.)

$$y(x_0) = y_0$$

若  $f(x, y)$ ,  $\frac{\partial f(x, y)}{\partial y}$  於  $(x_0, y_0)$  之鄰域為連續,

則存在  $\varepsilon > 0$ , 使得  $y(x)$  於  $(x_0 - \varepsilon, x_0 + \varepsilon)$

間有唯一解

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# First-Order Differential Equation

例: 下列各問題, 何者有唯一解

(1)  $y' = e^{xy^2}$  ,  $y(0) = 1$

(2)  $y' = \sqrt{y}$  ,  $y(0) = 0$

(3)  $y' = \sqrt{y}$  ,  $y(0) = 1$

(4)  $y' = -\sqrt{1-y^2}$  ,  $y(0) = 0$

(5)  $y' = -\sqrt{1-y^2}$  ,  $y(0) = 1$

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# First-Order Differential Equation

• Sol:

(1)  $y' = e^{xy^2}$  ,  $y(0) = 1$

$f(x, y) = e^{xy^2}$  (0,1)

$\frac{\partial f(x, y)}{\partial y} = 2xye^{xy^2}$  (0,1)

$\Rightarrow$  具唯一解

(2)  $y' = \sqrt{y}$  ,  $y(0) = 0$

$f(x, y) = \sqrt{y}$  (0,0)

$\frac{\partial f(x, y)}{\partial y} = \frac{1}{2\sqrt{y}}$  (0,0)

$\Rightarrow$  不具唯一解

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# First-Order Differential Equation

(3)  $y' = \sqrt{y}$  ,  $y(0) = 1$

$(0,1) \Rightarrow$  具唯一解

(4)  $y' = -\sqrt{1-y^2}$  ,  $y(0) = 0$

$f(x,y) = -\sqrt{1-y^2}$  (0,0)

$\frac{\partial f(x,y)}{\partial y} = \frac{y}{\sqrt{1-y^2}}$  (0,0)

$\Rightarrow$  具唯一解

(5)  $y' = -\sqrt{1-y^2}$  ,  $y(0) = 1$

$(0,1) \Rightarrow$  不具唯一解

$\Rightarrow$  (1) (3) (4) 具唯一解

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## Exact Equation

- 如何解  $M(x,y)dx + N(x,y)dy = 0 \cdots \cdots (B)$

令  $u(x,y) = C \cdots \cdots (A)$  ,  $C \in \text{const}$  .

$\Delta u(x,y) = u(x+\Delta x, y+\Delta y) - u(x,y)$

$= u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y) + u(x, y+\Delta y) - u(x,y)$

$= \frac{\partial u(x,y)}{\partial x} (x+\Delta x - x) + \frac{\partial u(x,y)}{\partial y} (y+\Delta y - y) = 0$

### ★ Mean-Value-Theorem 均值定理

$f(x)$  ,  $a \leq x \leq b$  , 一定存在一個  $C$  ,  $a \leq C \leq b$

St.  $f'(C) = \frac{f(b) - f(a)}{b - a} \Rightarrow f'(C)(b - a) = f(b) - f(a)$

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## Exact Equation

$$\Rightarrow \Delta u(x, y) = \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y = 0$$

$$\Rightarrow du(x, y) = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

$$= M(x, y)dx + N(x, y)dy = 0$$

$$\Rightarrow M(x, y) = \frac{\partial u}{\partial x} \quad N(x, y) = \frac{\partial u}{\partial y}$$

$$\partial u = M(x, y)\partial x$$

$$\partial u = N(x, y)\partial y$$

$$\int \partial u = \int M(x, y)\partial x + f(y) \quad \int \partial u = \int N(x, y)\partial y + g(x)$$

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## Exact Equation

$$\Rightarrow u = \begin{cases} \int \int M(x, y)\partial x + f(y) \cdots \cdots \cdots (1) \\ \int N(x, y)\partial y + g(x) \cdots \cdots \cdots (2) \end{cases}$$

經過比較(1)及(2)式，決定  $f(x)$  和  $g(x)$

$$\Rightarrow u(x, y) = C$$

$$\int \ln x = x \ln x - x$$

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## Exact Equation

- Q: 如何知道 (A)  $\rightarrow$  (B) 只有微分而已?

$$M(x, y) = \frac{\partial u}{\partial x}$$

$$N(x, y) = \frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{M(x, y)}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} = \frac{N(x, y)}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

假設  $u(x, y)$  具有連續二階偏導數

$$\Rightarrow M(x, y)dx + N(x, y)dy = 0$$

若  $\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$

則稱此微分方程式為“正合”(Exact)

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## Exact Equation

例:

$$u(x, y) = x^2 y^3 = C$$

$$du(x, y) = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

$$\Rightarrow 2xy^3 dx + x^2 3y^2 dy = 0$$

Check:  $\frac{\partial M(x, y)}{\partial y} = 2x3y^2 = \frac{\partial N(x, y)}{\partial x}$

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# Exact Equation

例:

$$u(x, y) = xy^2 + 3x + 5y = C$$

$$du(x, y) = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

$$\Rightarrow (y^2 + 3)dx + (2xy + 5)dy = 0$$

$$\frac{\partial M(x, y)}{\partial y} = 2y = \frac{\partial N(x, y)}{\partial x} \Rightarrow \text{正合}$$

$$\Rightarrow \frac{\partial u(x, y)}{\partial x} = y^2 + 3$$

$$\int \partial u(x, y) = \int (y^2 + 3)dx + f(y)$$

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# Exact Equation

$$\Rightarrow \frac{\partial u(x, y)}{\partial y} = 2xy + 5$$

$$\int \partial u(x, y) = \int (2xy + 5)dy + g(x)$$

$$\Rightarrow u = \begin{cases} xy^2 + 3x + f(y) \\ xy^2 + 5y + g(x) \end{cases}$$

$$\Rightarrow f(y) = 5y, \quad g(x) = 3x$$

$$\Rightarrow u(x, y) = xy^2 + 3x + 5y = C$$

三、反三角函數之微分

$$1. \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \quad |x| < 1$$

$$2. \frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}, \quad |x| < 1$$

$$3. \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}, \quad x \in \mathbb{R}$$

$$4. \frac{d}{dx} \cot^{-1} x = \frac{-1}{1+x^2}, \quad x \in \mathbb{R}$$

$$5. \frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1$$

$$6. \frac{d}{dx} \csc^{-1} x = \frac{-1}{|x|\sqrt{x^2-1}}, \quad |x| > 1$$