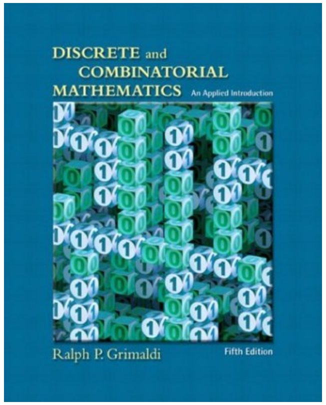
Discrete Mathematics

Part 3, GRAPH THEORY AND APPLICATIONS (outline)



-- Chapter 11: An Introduction to Graph Theory
-- Chapter 12: Trees
-- Chapter 13: Optimization and Matching

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Subgraph, complements and graph isomorphism

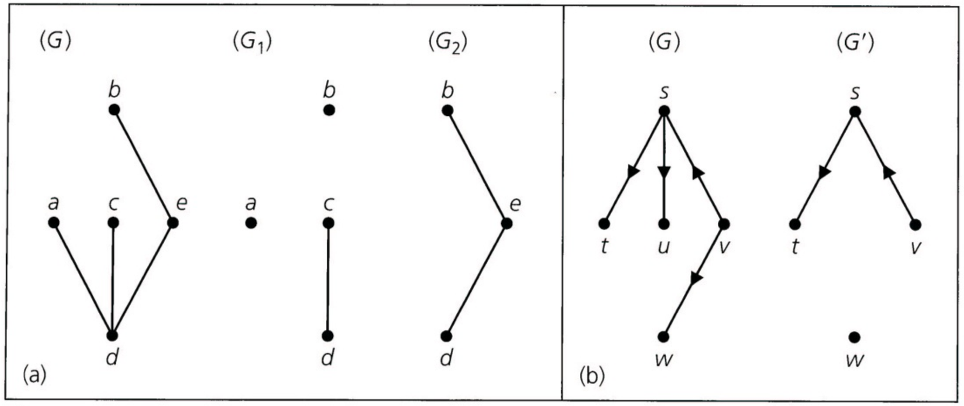


Figure 11.14

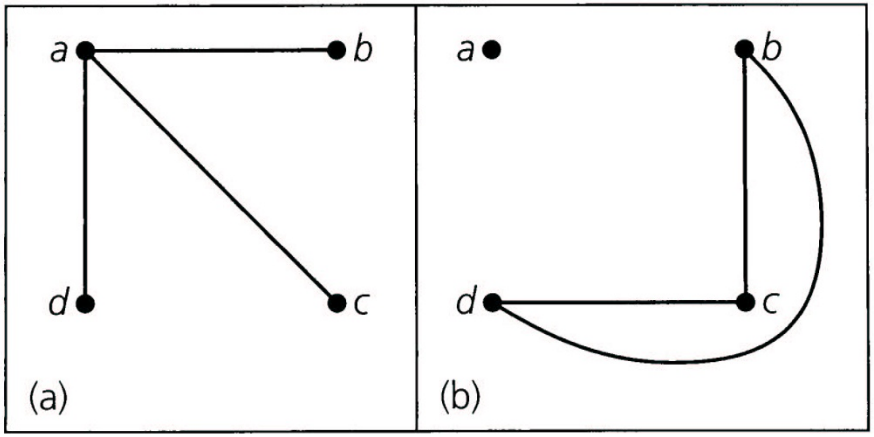


Figure 11.19

Subgraph, complements and graph isomorphism

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two undirected graphs. A function $f: V_1 \to V_2$ is called a *graph isomorphism* if (a) f is one-to-one and onto, and (b) for all $a, b \in V_1$, $\{a, b\} \in E_1$ if and only if $\{f(a), f(b)\} \in E_2$. When such a function exists, G_1 and G_2 are called *isomorphic graphs*.

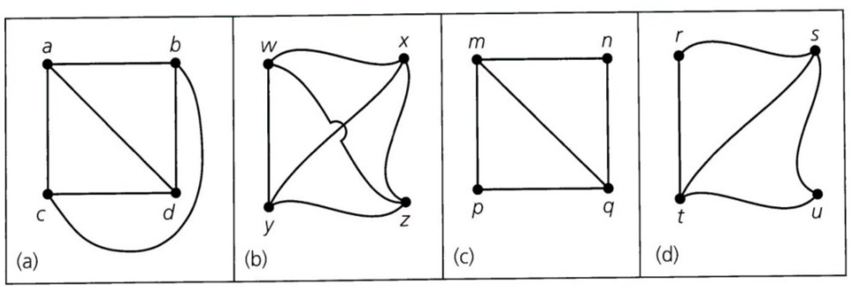


Figure 11.24

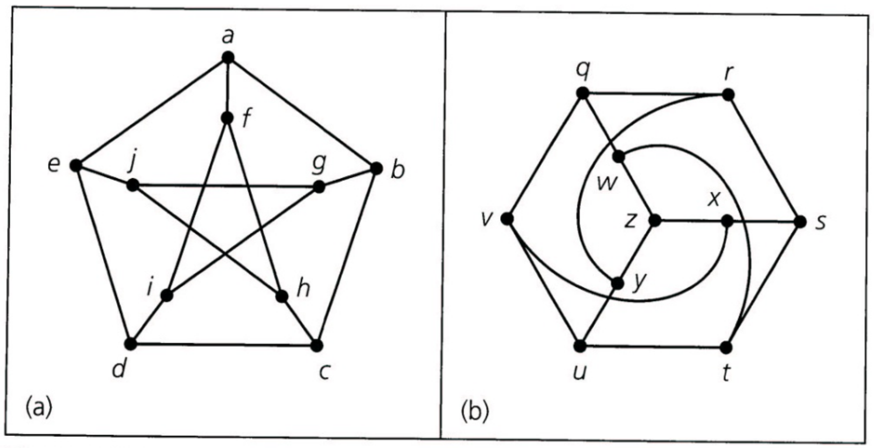


Figure 11.25

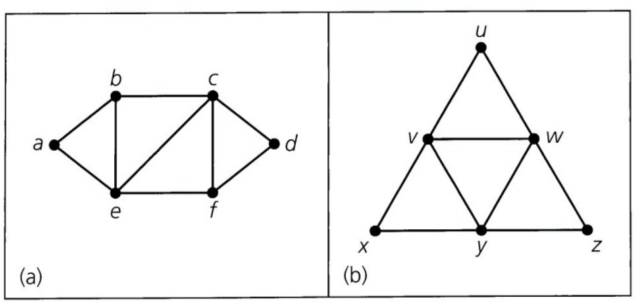


Figure 11.26

Euler Trails and Circuits

If $G = (V, E)$ is an undirected graph or multigraph, then $\sum_{v \in V} \deg(v) = 2|E|$.

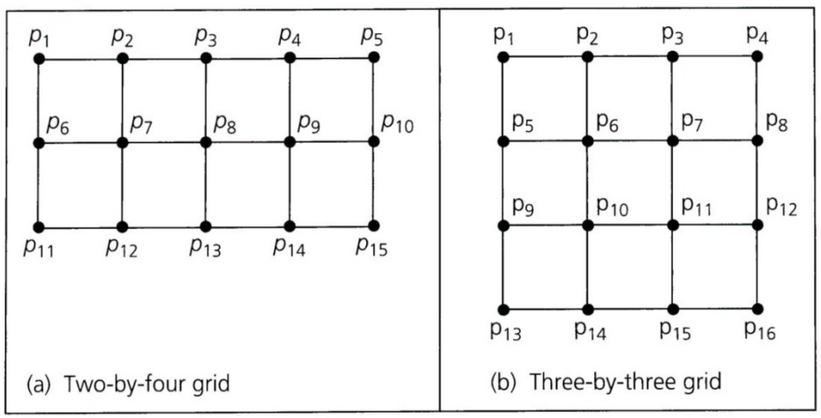


Figure 11.34

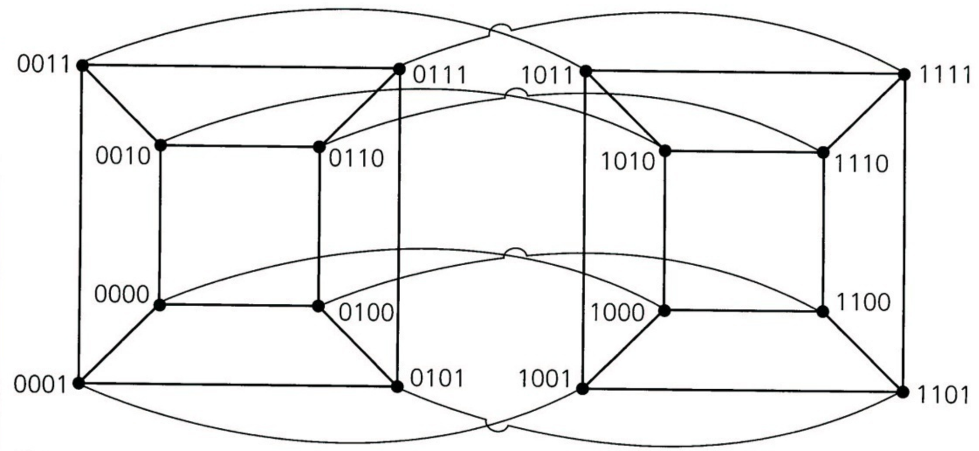


Figure 11.36

Hypercube

Euler Trails and Circuits

Let $G = (V, E)$ be an undirected graph or multigraph with no isolated vertices. Then G is said to have an *Euler circuit* if there is a circuit in G that traverses every edge of the graph exactly once. If there is an open trail from a to b in G and this trail traverses each edge in G exactly once, the trail is called an *Euler trail*.

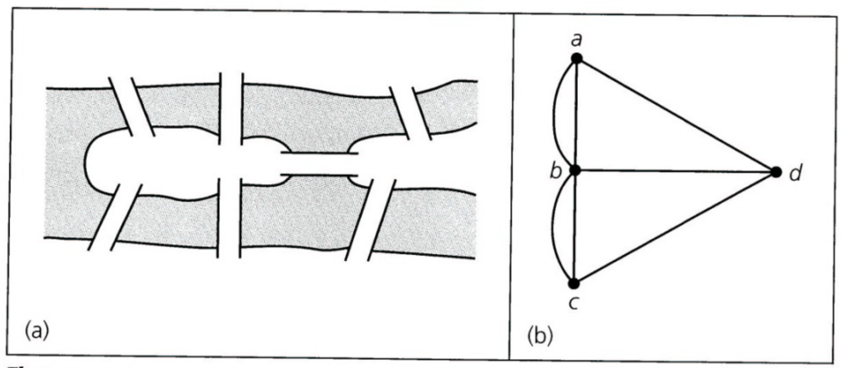
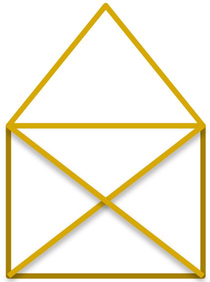


Figure 11.37



Let $G = (V, E)$ be an undirected graph or multigraph with no isolated vertices. Then G has an Euler circuit if and only if G is connected and every vertex in G has even degree.

If G is an undirected graph or multigraph with no isolated vertices, then we can construct an Euler trail in G if and only if G is connected and has exactly two vertices of odd degree.

Planar Graphs

A graph (or multigraph) G is called *planar* if G can be drawn in the plane with its edges intersecting only at vertices of G. Such a drawing of G is called an *embedding* of G in the plane.

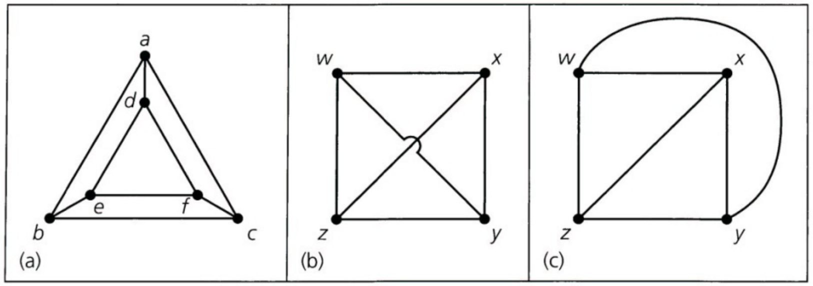


Figure 11.47

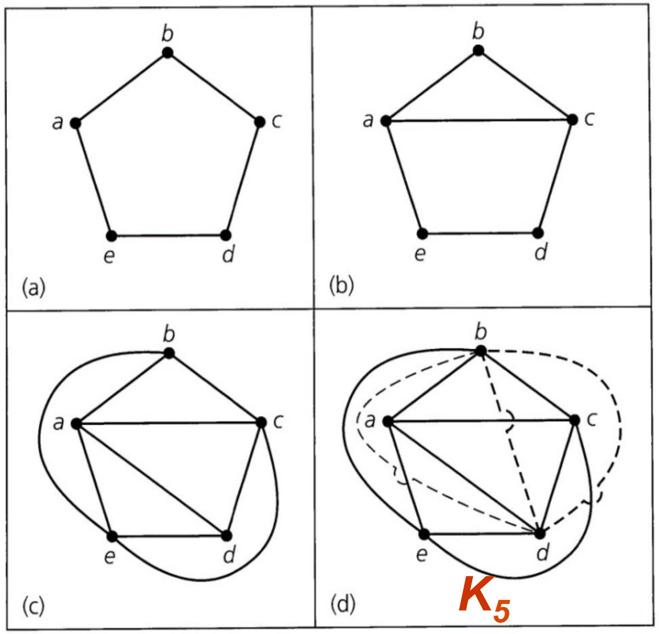


Figure 11.48

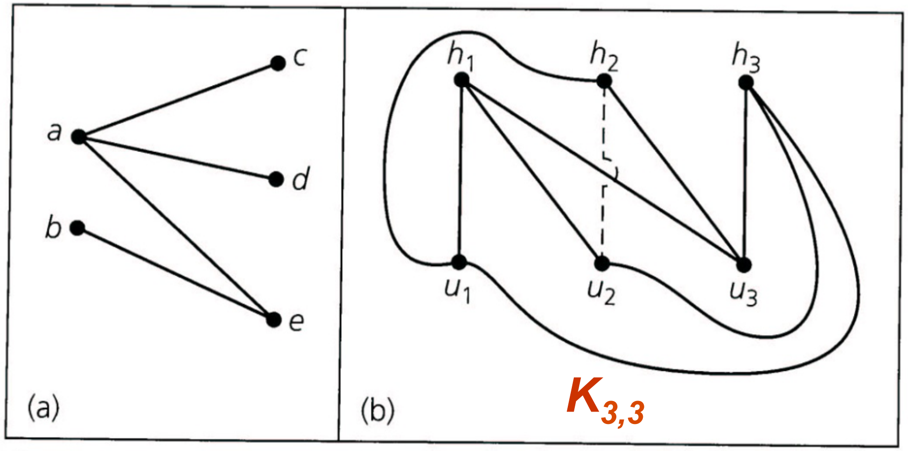


Figure 11.50

Kuratowski's Theorem. A graph is nonplanar if and only if it contains a subgraph that is homeomorphic to either K_5 or $K_{3,3}$.

Planar Graphs

Let $G = (V, E)$ be a connected planar graph or multigraph with $|V| = v$ and $|E| = e$. Let r be the number of regions in the plane determined by a planar embedding (or, depiction) of G; one of these regions has infinite area and is called *the infinite region*. Then $v - e + r = 2$.

Proof: The proof is by induction on e. If $e = 0$ or 1, then G is isomorphic to one of the graphs in Fig. 11.56. The graph in part (a) has $v = 1, e = 0$, and $r = 1$; so, $v - e + r = 1 - 0 + 1 = 2$. For graph (b), $v = 1$, $e = 1$, and $r = 2$. Graph (c) has $v = 2$, $e = 1$, and $r = 1$. In both cases, $v - e + r = 2$.

Q: Show that every planar graph has a node of degree at most 5.
(Ex11.4-21, 2008nthu)

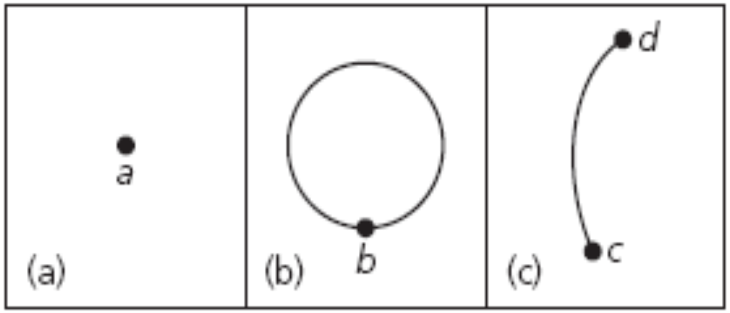


Figure 11.56

Sol:
$\deg(v) \geq 6$ then $2e = \Sigma\deg(v) \geq 6v$,
so $e \geq 3v$, contradiction

Loop-free planar graph ➔ $e \leq 3v-6$

11.5 Hamilton Paths and Cycles

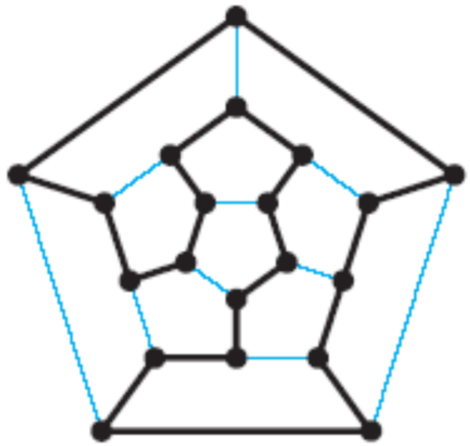


Figure 11.77

Let $G = (V, E)$ be a loop-free graph with $|V| = n \geq 2$. If $\deg(x) + \deg(y) \geq n - 1$ for all $x, y \in V$, $x \neq y$, then G has a Hamilton path.

If $G = (V, E)$ is a loop-free undirected graph with $|V| = n \geq 3$, and if $\deg(v) \geq n/2$ for all $v \in V$, then G has a Hamilton cycle.

THEOREM 11.9 Let $G = (V, E)$ be a loop-free undirected graph with $|V| = n \geq 3$. If $\deg(x) + \deg(y) \geq n$ for all nonadjacent $x, y \in V$, then G contains a Hamilton cycle.

12. Trees

Let $G = (V, E)$ be a loop-free undirected graph. The graph G is called a *tree*† if G is connected and contains no cycles.

If G is a directed graph, then G is called a *directed tree* if the undirected graph associated with G is a tree. When G is a directed tree, G is called a *rooted tree* if there is a unique vertex r, called the *root*, in G with the in degree of $r = id(r) = 0$, and for all other vertices v, the in degree of $v = id(v) = 1$.

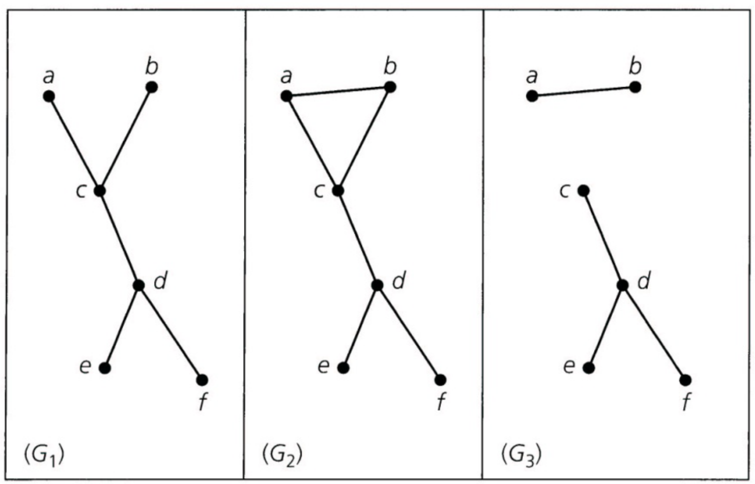


Figure 12.1

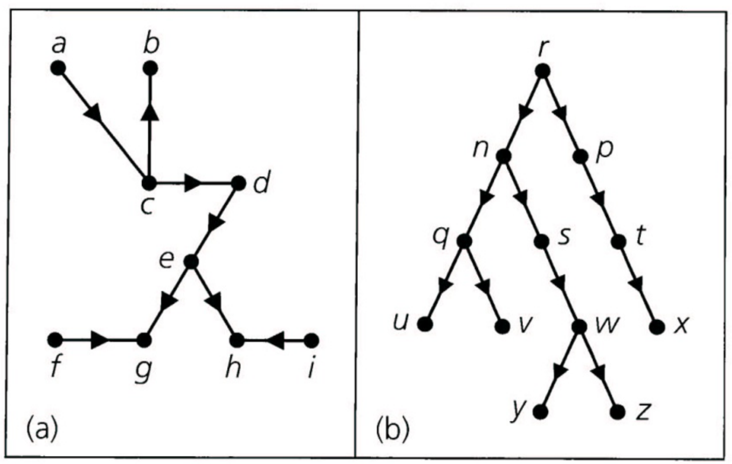


Figure 12.10

Others in Algorithm

- DFS, BFS
- MergeSort
- Huffman code

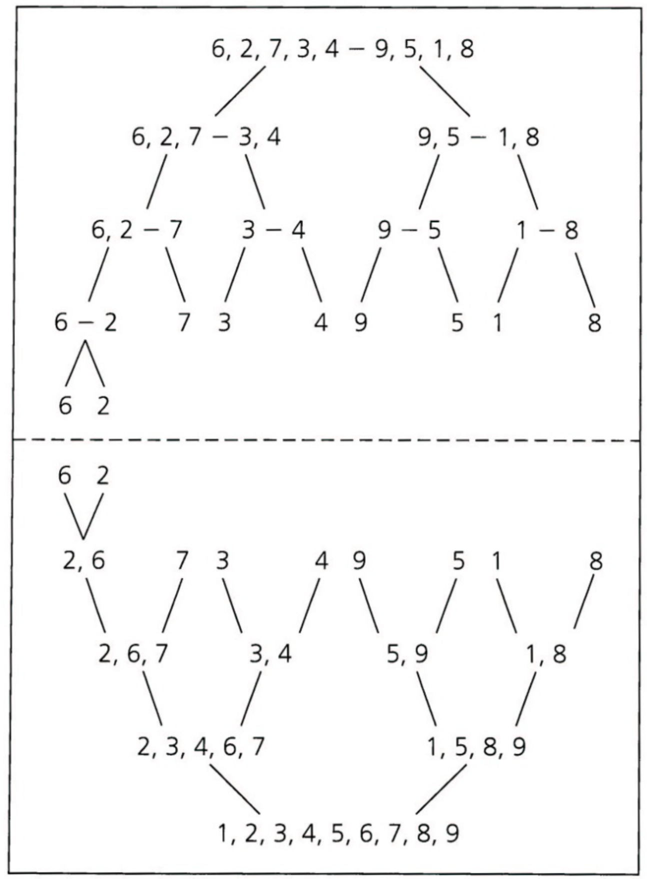


Figure 12.33

$$a_n=2a_{n/2}+n$$

13. Optimization

- Single-Source Shortest Paths *(algorithm ch24)*
  - The Bellman-Ford algorithm
  - Dijkstra's shortest-path algorithm
- All-Pairs Shortest Paths *(algorithm ch25)*
  - The Floyd-Warshall algorithm

- Minimum Spanning Trees *(algorithm ch23)*
  - The algorithm of Kruskal and Prim (Greedy)

- Transport Networks (Maximum Flow) *(algorithm ch26)*
  - The Edmonds-Karp algorithm
  - The Ford-Fulkerson method

14. Ring

- A long story!
- …
- …