Engineering Mathematics Solution

Midterm Exam, Fall 2011/11/14

請詳細列出計算過程，如用到公式,請列出公式的通式。請記得在答案卷上簽名。

1. (10%)求通解(Find general solution)

$$y\cos(x^2)dx + \frac{2}{x}\sin(x^2)\,dy = 0$$

Ans:

同乘 x,

得到 $xy\cos(x^2)dx + 2\sin(x^2)\,dy = 0$ --- (1)

$$\frac{\partial M}{\partial y} = x\cos(x^2) \neq \frac{\partial N}{\partial x} = 4\,x\cos(x^2)$$

$$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3x\cos(x^2)$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{dI}{I} \quad \Rightarrow \quad \frac{3x\cos(x^2)}{xy\cos(x^2)} = \frac{3}{y}$$

$I = e^{\int \frac{3}{y}dy} = y^3$, 乘回式(1)

$$xy^4\cos(x^2)dx + 2y^3\sin(x^2)\,dy = 0$$

$$M = \frac{\partial u}{\partial x}$$

$$u = \int xy^4\cos(x^2)dx + f(y)$$

$$= \frac{1}{2}y^4\sin(x^2) + f(y)$$

$$N = \frac{\partial u}{\partial y}$$

$$u = \int 2y^3\sin(x^2)\,dy + g(x)$$

$$= \frac{1}{2}y^4\sin(x^2) + g(x)$$

$$\because f(y) = 0, g(x) = 0$$

$$\therefore u = \frac{1}{2}y^4\sin(x^2) = C,\ x \neq 0$$

2. (10%) 假設 $y_1(x), y_2(x)$為二階常微分方程式$y''(x) + p(x)y'(x) + q(x)y(x) = r(x)$的二個齊性解($y_h$),若其特解($y_p$)為$\phi_1 y_1+\phi_2 y_2$. 請說明如何求出$\phi_1$與$\phi_2$,並將此$\phi_1$與$\phi_2$表示出來。

Ans:

$y_h = C_1y_1(x) + C_2y_2(x)$ 且 $y_1'' + py_1' + qy_1 = 0, y_2'' + py_2' + qy_2 = 0$

$y_p = y_1\phi_1 + y_2\phi_2$

$y_p' = y_1'\phi_1 + y_1\phi_1' + y_2'\phi_2 + y_2\phi_2'$

$= (y_1'\phi_1 + y_2'\phi_2) + (y_1\phi_1' + y_2\phi_2')$ 令 $y_1\phi_1' + y_2\phi_2' = 0$

$y_p'' = y_1''\phi_1 + y_1'\phi_1' + y_2''\phi_2 + y_2'\phi_2'$

代入 $y_p'' + py_p' + qy_p = r$

$(Y_1''\phi_1 + y_1'\phi_1' + y_2''\phi_2 + y_2'\phi_2') + p(y_1'\phi_1 + y_2'\phi_2) + q(y_1\phi_1 + y_2\phi_2) = r$

$\phi_1(y_1'' + py_1' + qy_1) + \phi_2(y_2'' + py_2' + qy_2) + y_1'\phi_1' + y_2'\phi_2' = r$

⇨ $\begin{cases} y_1'\phi_1' + y_2'\phi_2' = r(x) \\ y_1\phi_1' + y_2\phi_2' = 0 \end{cases}$ 要與上式都滿足

⇨ $\phi_1' = \dfrac{\begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \dfrac{-ry_2}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$, $\phi_2' = \dfrac{\begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \dfrac{ry_1}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$

⇨ $\phi_1 = \int \dfrac{-ry_2}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} dx, \phi_2 = \int \dfrac{ry_1}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} dx$

$y_p = y_1\phi_1 + y_2\phi_2$

$= y_1 \int \dfrac{-ry_2}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} dx + y_2 \int \dfrac{ry_1}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} dx$

3. (10%) 求通解(Find general solution)

$(x^2 + 2xy - y^2)dx + (y^2 + 2xy - x^2)dy = 0$

Ans:

令 $u = \frac{y}{x}$, $y = ux$

$(x^2 + 2x^2u - x^2u^2)dx + (x^2u^2 + 2x^2u - x^2)(udx + xdu) = 0$

$(x^2u^3 + x^2u^2 + x^2u + x^2)dx + (x^3u^2 + 2x^3u - x^3)du = 0$

$-\frac{1}{x}dx = \frac{u^2+2u-1}{u^3+u^2+u+1}du$

$\frac{1}{x}dx = \left(\frac{1}{u+1} - \frac{2u}{u^2+1}\right)du$

$\int \frac{1}{x}dx = \int \left(\frac{1}{u+1} - \frac{2u}{u^2+1}\right)du$

$\ln x + \ln|C| = \ln|u+1| - \ln(u^2+1)$

$\ln|C|\, x(u^2+1) = \ln|u+1|$

$u + 1 = Cx(u^2+1)$

$\frac{y}{x} + 1 = Cx\left(\left(\frac{y}{x}\right)^2 + 1\right)$

⇨ $x+y=C(x^2+y^2)$

4. (10%)求通解(Find general solution)(Hint: 利用 Linear D.E 概念)

$[y(1-x\tan(x))+x^2\cos x]dx-xdy=0$

Ans:

$$\frac{dy}{dx}=y\left(\frac{1}{x}-\tan(x)\right)+\ x\cos x$$

$$\therefore \frac{dy}{dx}+y\left(\tan(x)-\frac{1}{x}\right)=\ x\cos x, \text{符合}\left(y'+py=r(x)\right)$$

$$I=\ e^{\int\left(\tan(x)-\frac{1}{x}\right)dx}=e^{\ln|\sec x|-\ln x}\ =e^{\ln\left|\frac{\sec x}{x}\right|}=\frac{\sec x}{x}=\frac{1}{x\cos x}$$

代入 $y=CI^{-1}+I^{-1}\int Ir(x)dx$

$$y=Cx\cos x+x\cos x\int\frac{1}{x\cos x}\cdot x\cos x dx=Cx\cos x+x^2\cos x$$

5. (10%)設一齊性微分方程式的特性方程式的根,分別為

$\lambda_{1\sim16}=(-1\pm5i),(-1\pm5i),(-1\pm5i),-3\pm2i,-2\pm3i,4,4,4,1,2,3$

則其通解為何?(Find general solution)

Ans:

$$\begin{aligned}y=&C_1e^x+C_2e^{2x}+C_3e^{3x}+C_4e^{4x}+C_5xe^{4x}+C_6x^2e^{4x}+\\&e^{-2x}(C_7\cos3x+C_8\sin3x)+e^{-3x}(C_9\cos2x+C_{10}\sin2x)+\\&e^{-x}(C_{11}\cos5x+C_{12}\sin5x)+xe^{-x}(C_{13}\cos5x+C_{14}\sin5x)+\\&x^2e^{-x}(C_{15}\cos5x+C_{16}\sin5x)\end{aligned}$$

6. (10%) 求通解(Find General Solution)

$$x^3y'''+4x^2y''-5xy'-15y=0$$

Ans:

令$x=e^t, \wp=\frac{d}{dt}$

$$\wp(\wp-1)(\wp-2)y+4\wp(\wp-1)y-5\wp y-15y=0$$

$$(\wp^3+\wp^2-7\wp-15)y=0$$

$$\lambda^3+\lambda^2-7\lambda-15=0$$

$$(\lambda-3)(\lambda^2+4\lambda+5)=0$$

$$\lambda=3,-2\pm i$$

$$y=C_1e^{3t}+e^{-2t}(C_2\cos t+C_3\sin t)$$

$$\ \ =C_1x^3+x^{-2}(C_2\cos(\ln x)+C_3\sin(\ln x))$$

7. (10%)求通解(Find General Solution)

$$y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$$

Ans:

$$y'' - 2y' - 3y = 0$$

⇨ $y_c = C_1e^{-x} + C_2e^{3x}$

$y_p = y_{p1} + y_{p2}$

$y_{p1} = A_1x + A_2, y_{p2} = B_1xe^{2x} + B_2e^{2x}$

$y_p = A_1x + A_2 + B_1xe^{2x} + B_2e^{2x}$

$$\begin{aligned} y_p'' - 2y_p' - 3y_p &= (4B_1e^{2x} + 4B_1xe^{2x} + 4B_2e^{2x}) - \\ &\quad 2(A_1 + B_1e^{2x} + 2B_1xe^{2x} + 2B_2e^{2x}) - \\ &\quad 3(A_1x + A_2 + B_1xe^{2x} + B_2e^{2x}) \\ &= -3A_1x - (2A_1 + 3A_2) + (2B_1 - 3B_2)e^{2x} - 3B_1xe^{2x} \\ &= 4x - 5 + 6xe^{2x} \end{aligned}$$

$$\begin{cases} -3A_1 = 4 \\ 2A_1 + 3A_2 = 5 \\ 2B_1 - 3B_2 = 0 \\ -3B_1 = 6 \end{cases}, A_1 = \frac{-4}{3}, A_2 = \frac{23}{9}, B_1 = -2, B_2 = \frac{-4}{3}$$

$$y_p = \frac{-4}{3}x + \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x}$$

$$y = y_h + y_p = C_1e^{-x} + C_2e^{3x} - \frac{4}{3}x + \frac{23}{9} - 2xe^{2x} + \frac{4}{3}e^{2x}$$

8. (10%) $6y'' + 54y = 8\cos^3 x - 12\sin^3 x - 6\cos x + 9\sin x$

求 $y = y_h + y_p$

Ans:

$$6y'' + 54y = 2\cos 3x + 3\sin 3x$$

$$y'' + 9y = \frac{1}{3}\cos 3x + \frac{1}{2}\sin 3x$$

$\lambda^2 + 9 = 0,\ \lambda = \pm 3i$

$y_h = C_1\cos 3x + C_2\sin 3x$

$y_p = y_{p1} + y_{p2}$

⇨ $y_{p1} = \frac{1}{3} \times \frac{1}{D^2 + 3^2}\cos 3x$

$$= \frac{1}{3}\lim_{\Delta\to\infty} \frac{1}{-(3+\Delta)^2 + 3^2}\cos(3+\Delta)\,x$$

$= \frac{1}{3}\lim_{\Delta\to\infty}\frac{1}{-2\times3\times\Delta-\Delta^2}\cos(3+\Delta)\,x$

$$\begin{cases} \text{Cos(t) 於 } t = ax \text{ 之 Taylor 展開} \\ \cos(t) = \cos(ax) - (t-ax)\sin(ax) - \frac{1}{2!}(t-ax)^2\cos(ax) + \frac{1}{3!}(t-ax)^3\sin(ax) + \cdots \end{cases}$$

令 $a = 3, t = (a+\Delta)x = (3+\Delta)x$

⇨ $y_{p1} = \frac{1}{3}\lim_{\Delta\to\infty}\frac{1}{-\Delta(2\times3+\Delta)}[\cos3x - \Delta x\sin3x - \frac{1}{2!}(\Delta x)^2\cos3x + \frac{1}{3!}(\Delta x)^3\sin3x + \ldots]$

(因為y_h已含 cos3x, 故可忽略)

$= \frac{1}{3}\lim_{\Delta\to0}\frac{1}{-\Delta(2\times3+\Delta)}[-x\sin3x - \frac{1}{2!}\Delta x^2\cos3x + \frac{1}{3!}\Delta^2x^3\sin3x + \ldots]$

$= \frac{1}{3}\times\frac{1}{-2\times3}(-x\sin3x)$

$= \frac{1}{18}x\sin3x$

同理可證

⇨ $y_{p2} = \frac{1}{2}\times\frac{1}{-2\times3}\cos3x = -\frac{1}{12}x\cos3x$

⇨ $y = y_h + y_p = C_1\cos3x + C_2\sin3x + \frac{1}{18}x\sin3x - \frac{1}{12}x\cos3x$

9. (10%) 求通解(Find general solution) $y'' + 3y' + 2y = \cos x + x$

Ans:

$\lambda^2 + 3\lambda + 2 = 0,\ \lambda = -1, -2$

$y_h = C_1e^{-x} + C_2e^{-2x}$

$y_p = y_{p1} + y_{p2}$

$y''_{p1} + 3y'_{p1} + 2y_{p1} = \cos x$

$(D^2 + 3D + 2)y_{p1} = \cos x$

$y_{p1} = \frac{\cos x}{D^2+3D+2} \quad (a = 1)$

$L(D^2)\cos ax = L(-a^2)\cos ax$

$y_{p1} = \frac{\cos x}{-1+3D+2} = \frac{\cos x}{3D+1}$

$= \frac{1-3D}{(1-3D)(1+3D)}\cos x$

$= \frac{1-3D}{1-9D^2}\cos x$

$= \frac{1}{10}\cos x - (-\frac{3}{10}\sin x)$

$= \frac{1}{10}\cos x + \frac{3}{10}\sin x$

$y_{p2} = \frac{x}{D^2+3D+2}$

$= \frac{x}{2\left(1+\frac{D^2+3D}{2}\right)}$

$= \frac{1}{2}\left(1 - \frac{D^2+3D}{2} + \left(\frac{D^2+3D}{2}\right)^2 - \cdots\right)x$

$= \frac{1}{2}\left(x - \frac{3}{2}\right) = \frac{1}{2}x - \frac{3}{4}$

$y_p = y_{p1} + y_{p2} = \frac{1}{10}\cos x + \frac{3}{10}\sin x + \frac{1}{2}x - \frac{3}{4}$

⇨ $y = y_h + y_p = C_1e^{-x} + C_2e^{-2x} + \frac{1}{10}\cos x + \frac{3}{10}\sin x + \frac{1}{2}x - \frac{3}{4}$

10. (10%) $(3x+4)^2 y'' - 6(3x+4)y' + 18y = 9\ln(3x+4)$

求 $y = y_h + y_p$

Ans:

令 $u = 3x + 4$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3\frac{dy}{du}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\frac{dy}{dx}\frac{du}{dx}$

$= \frac{d}{du} \times 3 \times \frac{dy}{du} \times 3 = 9\frac{d^2y}{du^2}$

原式$u^2 \times 9\frac{d^2y}{du^2} - 6u \times 3\frac{dy}{du} + 18y = 9\ln(u)$

$u^2\frac{d^2y}{du^2} - 2u\frac{dy}{du} + 2y = \ln(u)$

令 $u = e^t, \wp = \frac{d}{dt}$

$(\wp(\wp - 1) - 2\wp + 2)y = \ln(u) = t$

$(\wp^2 - 3\wp + 2)y = t$

$\lambda^2 - 3\lambda + 2 = 0,\ \lambda = 1,2$

$$
\begin{aligned}
y_h &= C_1 e^t + C_2 e^{2t} \\
&= C_1 u + C_2 u^2 \\
&= C_1(3x+4) + C_2(3x+4)^2
\end{aligned}
$$

$$
\begin{aligned}
y_p &= \frac{t}{\wp^2 - 3\wp + 2} \\
&= \frac{1}{2}\frac{t}{1+\frac{\wp^2-3\wp}{2}} \\
&= \frac{1}{2}\left[1 - \frac{\wp^2-3\wp}{2} + \left(\frac{\wp^2-3\wp}{2}\right)^2 - \cdots\right] t \\
&= \frac{1}{2}t + \frac{3}{4} + \frac{1}{2}\left(\frac{0}{4}\right) \\
&= \frac{1}{2}t + \frac{3}{4} \\
&= \frac{1}{2}\ln(3x+4) + \frac{3}{4}
\end{aligned}
$$

$$
y = y_h + y_p = C_1(3x+4) + C_2(3x+4)^2 + \frac{1}{2}\ln(3x+4) + \frac{3}{4}
$$