Engineering Mathematics　　Homework 6-Solution

1. $Solve:\ y''+a^2y=\sin ax$

Sol:

$\lambda^2+a^2=0 \quad \lambda=\pm ai$

$y_h=C_1\cos ax+C_2\sin ax$

$(D^2+a^2)y_p=\sin ax$

$$y_p=\frac{1}{D^2+a^2}\sin ax=\lim_{\Delta\to 0}\frac{1}{-(a+\Delta)^2+a^2}\sin(a+\Delta)x$$

$$=\lim_{\Delta\to 0}\frac{1}{-2a\Delta-\Delta^2}\sin(a+\Delta)x$$

$\sin t$於$t=ax$ 之$Taylor$展開

$$\sin t=\sin ax+(t-ax)\cos ax-\frac{1}{2!}(t-ax)^2\sin ax-\frac{1}{3!}(t-ax)^3\cos ax+...$$

令$t=(a+\Delta)x$

$$\sin(a+\Delta)x=\sin ax+(ax+\Delta\cdot x-ax)\cos ax-\frac{1}{2!}(ax+\Delta\cdot x-ax)^2\sin ax$$

$$-\frac{1}{3!}(ax+\Delta\cdot x-ax)^3\cos ax+...$$

$$=\sin ax+(\Delta\cdot x)\cos ax-\frac{1}{2!}(\Delta\cdot x)^2\sin ax-\frac{1}{3!}(\Delta\cdot x)^3\cos ax+...$$

$$y_p=\lim_{\Delta\to 0}\frac{1}{-2a\Delta-\Delta^2}\sin(a+\Delta)x$$

$$=\lim_{\Delta\to 0}\frac{1}{-2a\Delta-\Delta^2}[\sin ax+(\Delta\cdot x)\cos ax-\frac{1}{2!}(\Delta\cdot x)^2\sin ax-\frac{1}{3!}(\Delta\cdot x)^3\cos ax+...]$$

$(Note:\ y_h$ 已含$\sin ax)$

$$=\lim_{\Delta\to 0}\frac{1}{-2a-\Delta}[x\cos ax-\frac{1}{2!}\Delta\cdot x^2\sin ax-\frac{1}{3!}\Delta^2\cdot x^3\cos ax+...]=\frac{1}{-2a}x\cos ax$$

$$y=y_h+y_p=C_1\cos ax+C_2\sin ax-\frac{1}{2a}x\cos ax$$

2. *Find* y_p: $y''-8y'+25y=5x^3e^{-x}-7e^{-x}$

Sol:

$$r_1(x)=5x^3e^{-x}$$

$$y_{p1}''-8y_{p1}'+25y_{p1}=5x^3e^{-x}\Rightarrow(D^2-8D+25)y_{p1}=5x^3e^{-x}$$

$$y_{p1}=\frac{1}{D^2-8D+25}5x^3e^{-x}$$

$$=5e^{-x}\frac{1}{(D-1)^2-8(D-1)+25}x^3$$

$$=5e^{-x}\frac{1}{D^2-10D+34}x^3$$

$$=5e^{-x}\frac{1}{34(1+\frac{D^2-10D}{34})}x^3$$

$$=5e^{-x}\frac{1}{34}[1-\frac{D^2-10D}{34}+(\frac{D^2-10D}{34})^2-(\frac{D^2-10D}{34})^3+...]x^3$$

$$=\frac{5}{34}e^{-x}(x^3-\frac{6x-30x^2}{34}+\frac{-120+600x}{34^2}+\frac{6000}{34^3})$$

$$r_2(x)=-7e^{-x}$$

$$y_{p2}''-8y_{p2}'+25y_{p2}=-7e^{-x}$$

$$y_{p2}=\frac{-7}{34}e^{-x}$$

$$y_p=y_{p1}+y_{p2}=(\frac{5}{34}x^3+\frac{75}{578}x^2+\frac{495}{9826}x-\frac{33191}{167042})e^{-x}$$