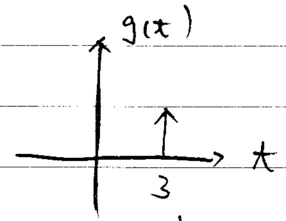
No.
Date.

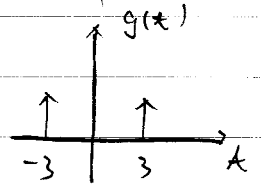
在 t 的積分範圍內，只有 $t=0$ 時，
$\delta(t)$ 才有值.

$$\Rightarrow \int_0^\infty \delta(t)\, e^{-s\cdot 0}\, dt = 1$$

ex.



$$g(t) = \delta(t-3).$$



$$g(t) = \delta(t+3) + \delta(t-3)$$

(7). $f(t^n) \Rightarrow F(s) = \dfrac{n!}{s^{n+1}}$, $n \geq 0$ 擇日再證 = =

ex. $L\{e^t + 3\sin 2t + 5\cos 3t + 3t^2 + 5\delta(t)\}$.

$$= \frac{1}{s-1} + 3\cdot\frac{2}{s^2+4} + 5\cdot\frac{3}{s^2+9} + 3\cdot\frac{2!}{s^3} + 5 \quad \#$$

ex. $L^{-1}\left\{7 + \dfrac{2}{s^5} + \dfrac{2}{s^2+1} + \dfrac{s}{s^2+49} + \dfrac{3}{s+5}\right\}$

$$= 7\delta(t) + \frac{2}{4!}t^4 + 2\sin t + \cos 7t + 3e^{-5t} \quad \#$$

Review:

1. $f(t) = e^{at} \xrightarrow{L} \dfrac{1}{s-a}$

2. $f(t) = \cos at \xrightarrow{L} \dfrac{s}{s^2+a^2}$

3. $f(t) = \sin at \xrightarrow{L} \dfrac{a}{s^2+a^2}$

4. $f(t)=t \xrightarrow{\mathcal{L}} \frac{1}{s^2}$

5. $f(t)=H(t) \xrightarrow{\mathcal{L}} \frac{1}{s}$

6. $f(t)=\delta(t) \xrightarrow{\mathcal{L}} 1$

7. $f(t)=t^n \xrightarrow{\mathcal{L}} \frac{n!}{s^{n+1}}$

現在 $f(t)=t\cdot e^{2t} \xrightarrow{\mathcal{L}} \int_0^\infty t\cdot e^{2t}\cdot e^{-st}\,dt$ …… 不好做.

⇒ 需要 Laplace Transform 之基本性質.

設 $L\{f(t)\}=F(s)$, $L\{g(t)\}=G(s)$ 已知.

1. $L\{k_1f(t)+k_2g(t)\}=k_1\cdot L\{f(t)\}+k_2\cdot L\{g(t)\}=k_1F(s)+k_2G(s)$

線性轉換的性質, $k_1, k_2 \in$ const.

pf: $L\{k_1f(t)+k_2g(t)\}=\int_0^\infty(k_1f(t)+k_2g(t))e^{-st}\,dt$

$=k_1\int_0^\infty f(t)e^{-st}\,dt+k_2\int_0^\infty f(t)e^{-st}\,dt$

$=k_1F(s)+k_2G(s)$ #.

2. First shifting thm (第一移位定理).

$f(t) \xrightarrow{\mathcal{L}} F(s)=L\{f(t)\}$.

$e^{at}f(t) \xrightarrow{\mathcal{L}} F(s-a)$

pf: $L\{e^{at}f(t)\}=\int_0^\infty e^{at}\cdot f(t)\cdot e^{-st}\,dt=\int_0^\infty f(t)\cdot e^{-(s-a)t}\,dt$

$=\int_0^\infty f(t)e^{-s't}\,dt = F(s')=F(s-a)$ #

ex: $f(t)=t\cdot e^{2t}=e^{2t}\cdot g(t) \xrightarrow{\mathcal{L}} G(s-2)=\frac{1}{(s-2)^2}$

ex. $H(t) \xrightarrow{\mathcal{L}} \frac{1}{s}$

ex. $e^{at} \xrightarrow{\mathcal{L}} \frac{1}{s-a}$ $\quad e^{at}\cdot H(t) \xrightarrow{\mathcal{L}} H(s-a)=\frac{1}{s-a}$

ex. $L\{e^{-2t}\cdot t\} = F(s+2) = \frac{1}{(s+2)^2}$

ex. $L\{e^{t}\cdot \cos 2t\}$

$\cos 2t \to f(t)$.

$= F(s-1) = \frac{(s-1)}{(s-1)^2+4} = \frac{s-1}{s^2-2s+5}$ #.

ex. $L\{e^{3t}\cdot \sin 5t\}$.

$\sin 5t \to f(t)$.

$= F(s-3) = \frac{5}{(s-3)^2+5^2} = \frac{5}{s^2-6s+34}$ #.

ex. $L^{-1}\{\frac{2}{s^2+2s+5}\}$

$\frac{2}{s^2+2s+5} \to \frac{2}{(s+1)^2+4}$

$= f(t) = \sin 2t\cdot e^{-t}$ #.

ex. $L^{-1}\{\frac{s}{(s+2)^2+4}\}$.

$\Rightarrow F(s) = \frac{s+2}{(s+2)^2+4} - \frac{2}{(s+2)^2+4}$.

$\Rightarrow f(s) = e^{-2t}\cdot\cos 2t - e^{-2t}\cdot\sin 2t$

3. second shift thm (第二移位定理).

$f(t) \xrightarrow{L} F(s)$

$f(t-a)\cdot H(t-a) \xrightarrow{L} F(s)\cdot e^{-as}$

pf: $L\{f(t-a)H(t-a)\} = \int_0^\infty f(t-a)H(t-a)e^{-st}dt$

$\left(\text{又 } H(t-a) = \begin{cases} 1 & (t-a)>0 \Rightarrow t>a \\ 0 & (t-a)<0 \Rightarrow t<a \end{cases}\right)$

$= \int_a^\infty f(t-a)e^{-st}dt$

令 $x = t - a \Rightarrow dx = dt \quad (t \to a \sim \infty \ , \ x \to 0 \sim \infty)$

$\Rightarrow L\{f(t-a)\cdot H(t-a)\} = \int_0^\infty f(x)\cdot 1 \cdot e^{-(x+a)}dx$

$= e^{-as}\int_0^\infty f(x)\cdot e^{-sx}dx = e^{-as}\cdot F(s)$ 得.

☆ 於積分过程中範圍会变!!

ex. $f(t) = e^{2t} \xrightarrow{L} \frac{1}{s-2}$

$L\{e^{2(t-3)}H(t-3)\} = e^{-3s}\cdot F(s) = e^{-3s}\cdot \frac{1}{s-2}$

ex. $L\{\underbrace{\cos(t-2)}_{\frac{s}{s^2+1}}\cdot \underbrace{H(t-2)}_{e^{-2s}}\} = e^{-2s}\cdot \frac{s}{s^2+1}$

ex. $G(s) = \underbrace{e^{-3s}}_{\because H(t-3)}\cdot \frac{s+1}{(s+1)^2+1} \to e^{-t}\cdot \cos t$

$\Rightarrow g(s) = e^{-(t-3)}\cdot \cos(t-3)\, H(t-3)$

ex. $G(s) = \underbrace{\frac{1}{s+3}}_{e^{-3t}}\cdot e^{-s} \to \because H(t-1)$

$\Rightarrow g(t) = e^{-3(t-1)}\cdot H(t-1)$

* 此処看些題目:

ex. $f(t) = t$.

a. $L\{f(t)\} = \frac{1}{s^2}$

b. $L\{f(t-2)\} = L\{t-2\} = L\{t\} - L\{2\} = \frac{1}{s^2} - \frac{2}{s}$

c. $L\{f(t)\cdot H(t-2)\} = L\{t\cdot H(t-2)\}$

$= L\{[(t-2)+2]\cdot H(t-2)\}$

$= L\{(t-2)\cdot H(t-2)\} + L\{2H(t-2)\}$

$= e^{-2s}\cdot \frac{1}{s^2} + 2\cdot \frac{1}{s}\cdot e^{-2s}$

ex. $f(t) = t^2+3t+2.$

a. $L\{f(t)\} = \frac{2!}{s^3} + 3\cdot\frac{1}{s^2} + 2\cdot\frac{1}{s}$

b. $L\{f(t-1)\} = L\{t^2+t\} = \frac{2!}{s^3} + \frac{1}{s^2}$

c. $L\{f(t)\cdot H(t-1)\}$ …. 等等再做.

d. $L\{f(t-1)\cdot H(t-1)\} = \left(\frac{2!}{s^3} + 3\cdot\frac{1}{s^2} + 2\cdot\frac{1}{s}\right)\cdot e^{-s}$

c. $L\{[(t-1)^2 + A(t-1)+B]\cdot H(t-1)\}$ $A=5,\ B=6.$

$= \frac{2}{s^3}e^{-s} + 5\cdot\frac{1}{s^2}\cdot e^{-s} + 6\cdot\frac{1}{s}\cdot e^{-s}$

4. $f(t) \xrightarrow{L} F(s)$

$t f(t) \xrightarrow{L} -\frac{dF(s)}{ds}$

ex. $1 \xrightarrow{L} \frac{1}{s}$

$t \xrightarrow{L} \frac{1}{s^2}$

$\Rightarrow t\cdot 1 \xrightarrow{L} -\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2}$

$$\left(\begin{array}{ll} \text{* note} & t \xrightarrow{L} \frac{1}{s^2} \\ & t^2 \xrightarrow{L} -\left(\frac{d}{ds}\left(\frac{1}{s^2}\right)\right) = \frac{2}{s^3} \\ & \vdots \\ & t^n \xrightarrow{L} \frac{n!}{s^{n+1}} \end{array}\right)$$

pf: $L\{t f(t)\} = \int_0^\infty t f(t) e^{-st}\,dt$

$F(s) = \int_0^\infty f(t)\cdot e^{-st}\,dt$

$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt$$

$$= \int_0^\infty \frac{\partial}{\partial s} f(t)e^{-st}\,dt$$

$$= \int_0^\infty f(t)\cdot(-t)\cdot e^{-st}\,dt = -\int_0^\infty f(t)\cdot t e^{-st}\,dt$$

$$= -L\{tf(t)\}$$

ex. $L\{t\sin 2t\}$.

$$= -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = -\frac{0-2\cdot 2s}{(s^2+4)^2} = \frac{4s}{(s^2+4)^2}$$

ex. $L\{e^{2t}\cdot(t\cdot\sin 2t)\}$.

$$t\cdot\sin 2t \to \frac{4s}{(s^2+4)^2}$$

$$= F(s-2) = \frac{4(s-2)}{((s-2)^2+4)^2}$$

證：$L\{t^n\} = \dfrac{n!}{s^{n+1}}$

pf: $\because 1 \xrightarrow{L} \dfrac{1}{s}$

設 $n = k-1$ 時 $L\{t^{k-1}\} = \dfrac{(k-1)!}{s^k}$

$\Rightarrow n = k$ 時 $L\{t^k\} = L\{t\cdot t^{k-1}\}$

$$= -\frac{d}{ds}\cdot\frac{(k-1)!}{s^k}$$

$$= -(k-1)(-k)\cdot s^{-(k+1)} = \frac{k!}{s^{k+1}}$$ 故.

推廣 $t^n f(t) \xrightarrow{L} \underbrace{-\frac{d}{ds}\cdots\cdots\left(-\frac{d}{ds}F(s)\right)}_{n\text{次}}$

5. $\dfrac{1}{t}\cdot f(t) \xrightarrow{L} \displaystyle\int_s^\infty F(s)\,ds.$

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pf: $F(s) = \int_0^\infty f(t) \cdot e^{-st} dt$ —— (1).

$$\int_s^\infty (1)\, ds = \int_s^\infty \int_s^\infty f(t) e^{-st} dt\, ds$$
$$= \int_0^\infty \int_s^\infty f(t) \cdot e^{-st} ds\, dt$$

(積分次序可对調 ∵ s, t 在範圍内独立)

$$= \int_0^\infty f(t) \int_s^\infty e^{-st} ds \cdot dt$$
$$= \int_0^\infty f(t) \left[-\frac{1}{t} e^{-st} \Big|_s^\infty \right] dt$$
$$= \int_0^\infty f(t) \left[0 - \left(-\frac{1}{t} e^{-st} \right) \right] dt$$
$$= \int_0^\infty f(t) \cdot \frac{1}{t} e^{-st} dt = L\left\{ \frac{1}{t} f(t) \right\}. \text{ 卄.}$$

$$1 \xrightarrow{L} \frac{1}{s}$$
$$t \cdot 1 \xrightarrow{L} \frac{1}{s^2} = -\frac{d}{ds}\left(\frac{1}{s}\right)$$
$$\frac{1}{t} \cdot 1 \xrightarrow{L} \int_s^\infty \frac{1}{s} ds$$
$$\frac{1}{t^2} \xrightarrow{L} \int_s^\infty \int_s^\infty \frac{1}{s} ds\, ds$$
$$\Rightarrow L\left\{ \frac{1}{t} \sin t \right\} = \int_s^\infty \frac{s}{s^2+1} ds$$
$$= \tan^{-1} s \Big|_s^\infty = \frac{\pi}{2} - \tan^{-1} s \text{ 卄.}$$