Gauss's Law

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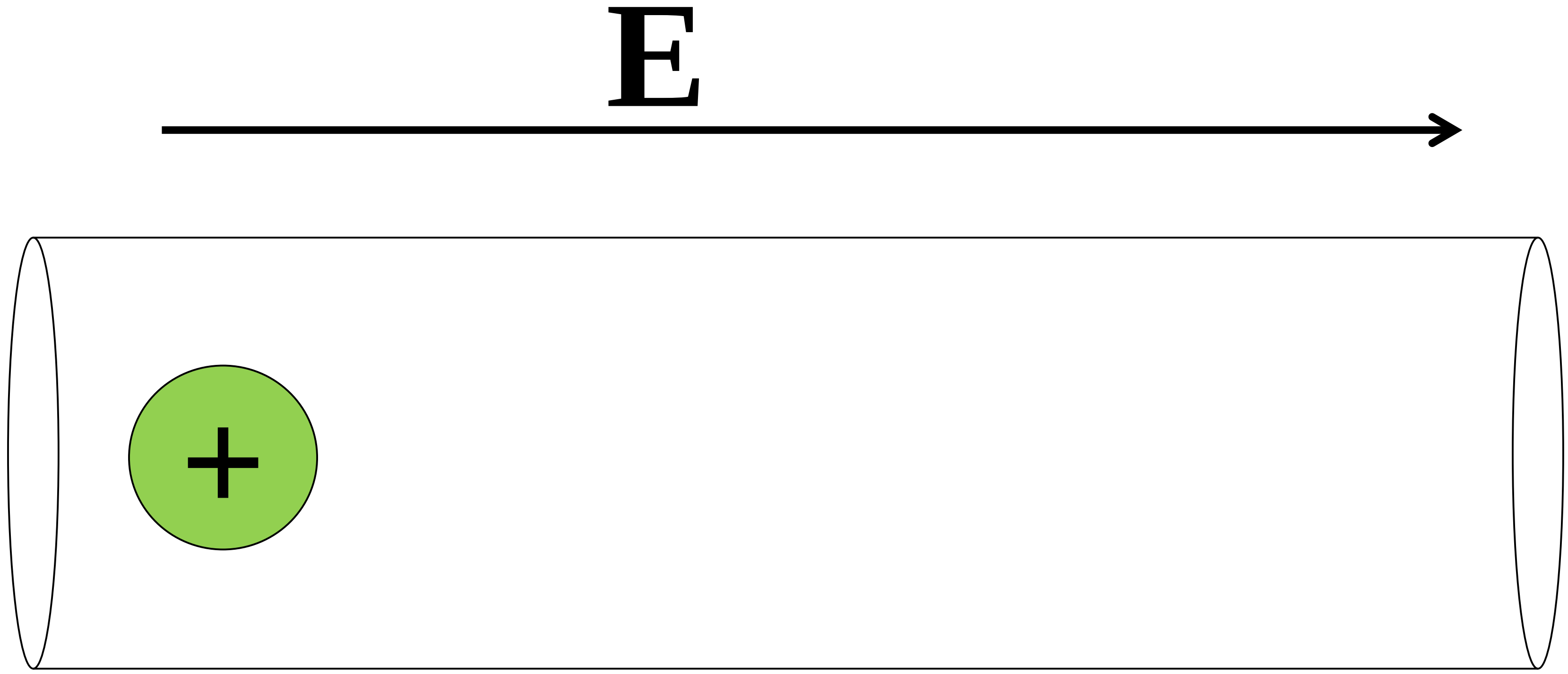
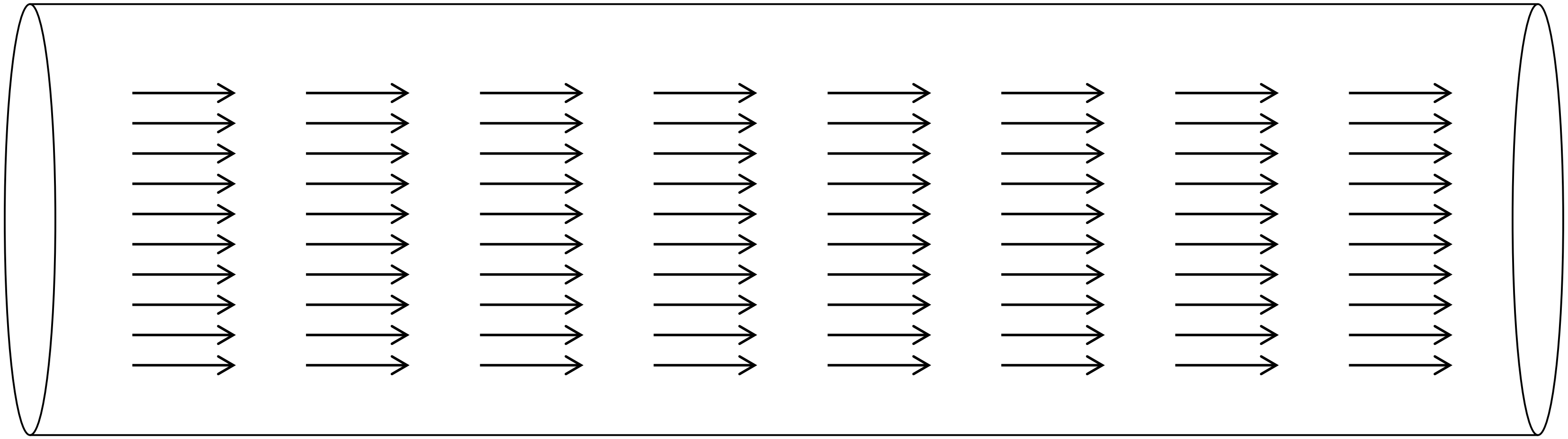
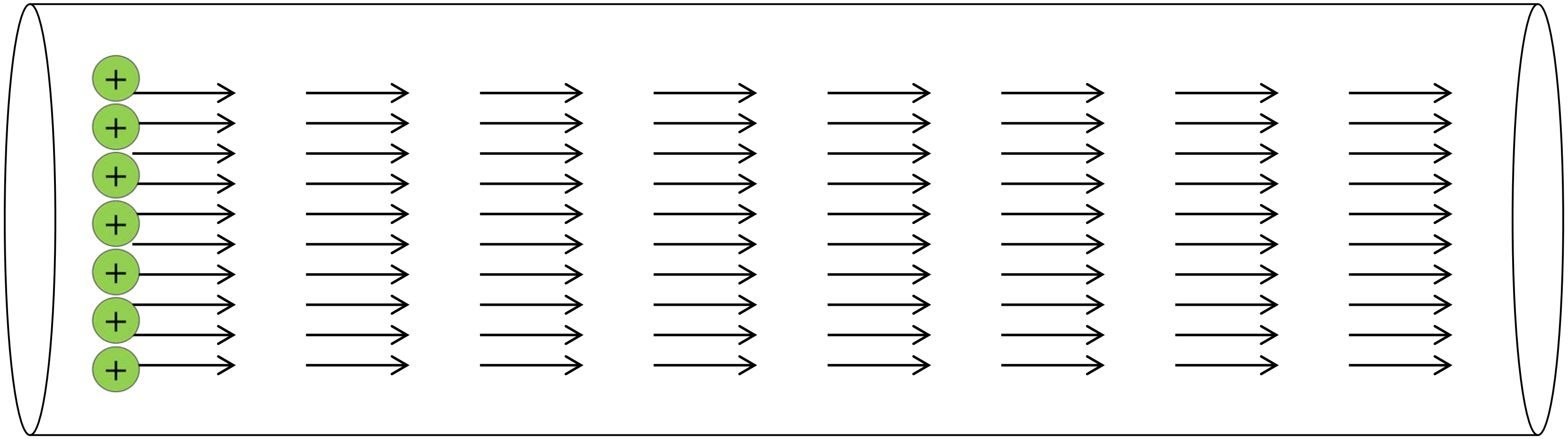

E
+

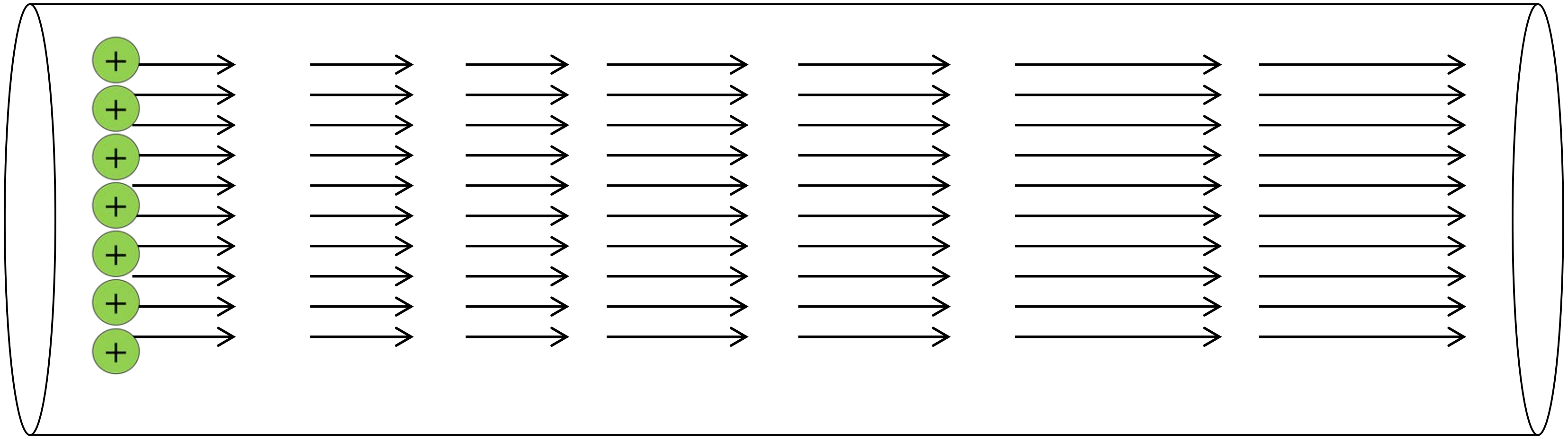
Flux



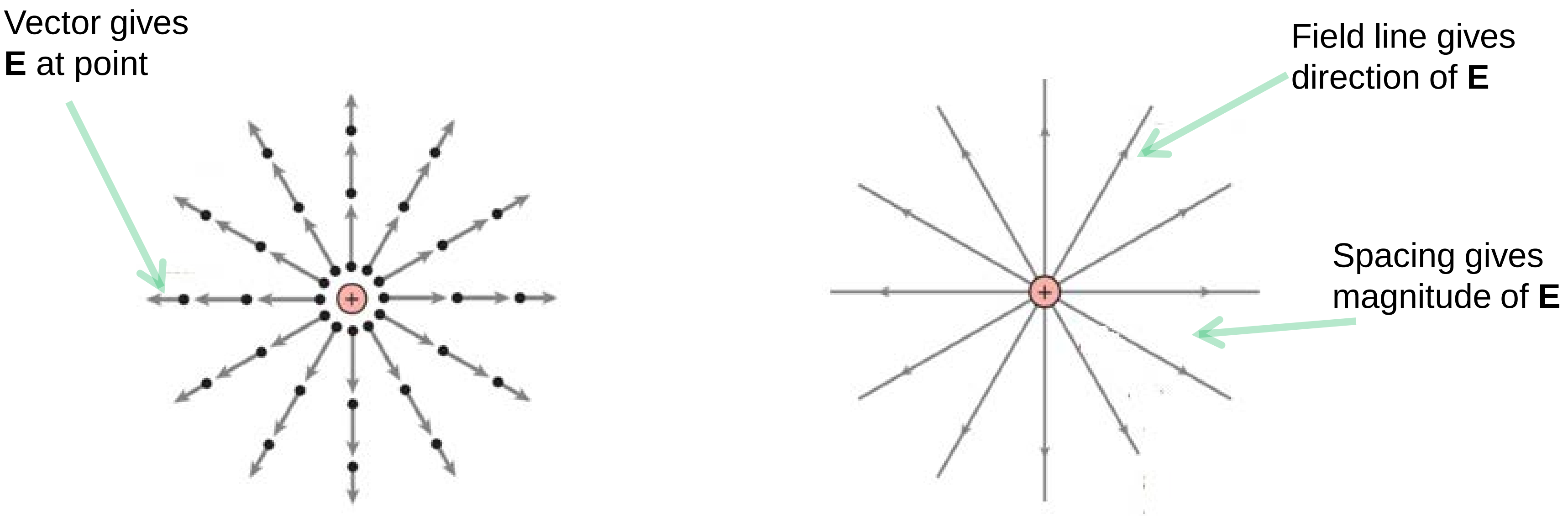
Flux



Flux



Electric Field Lines

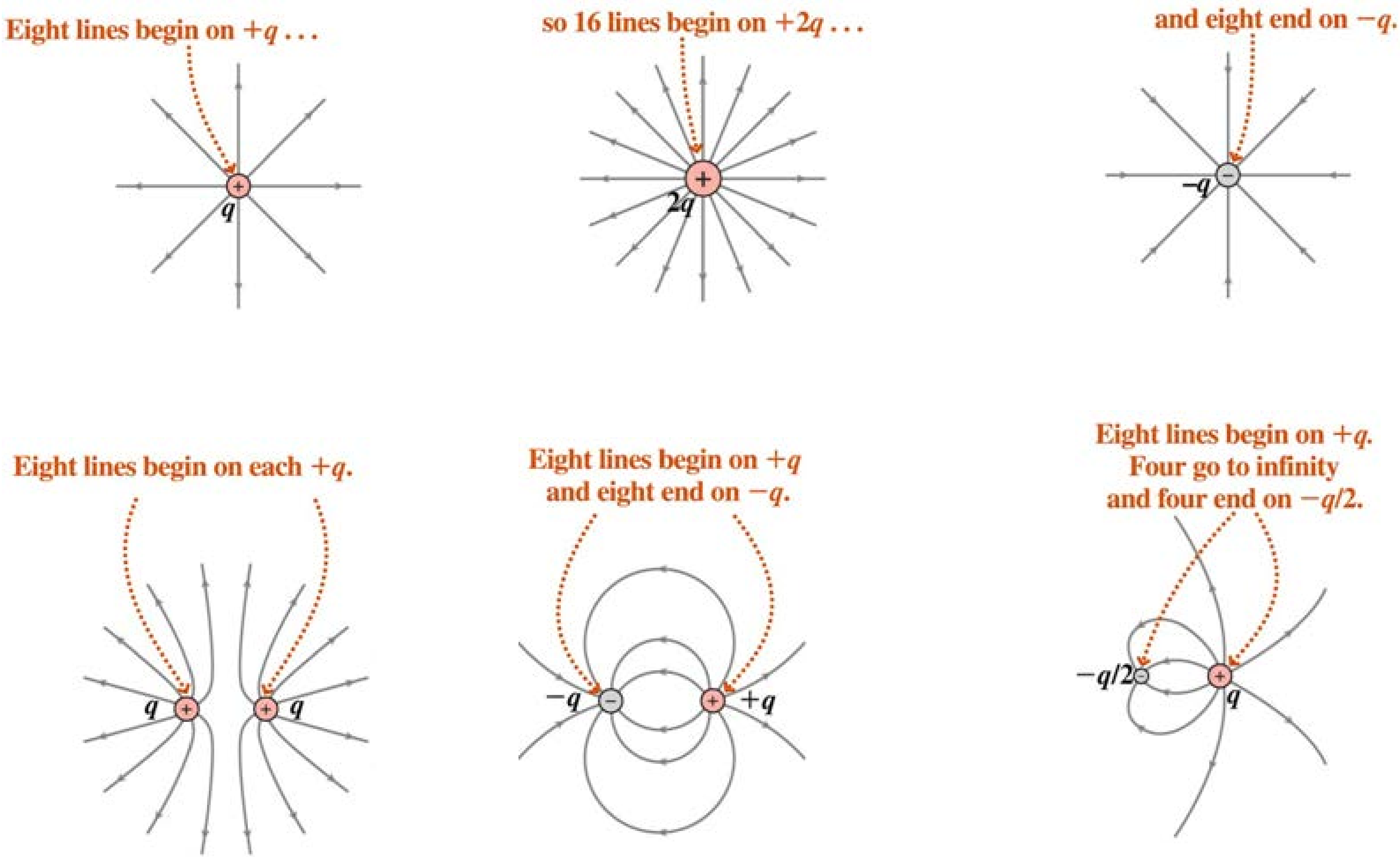


Electric field lines = Continuous lines whose tangent is everywhere // **E**.

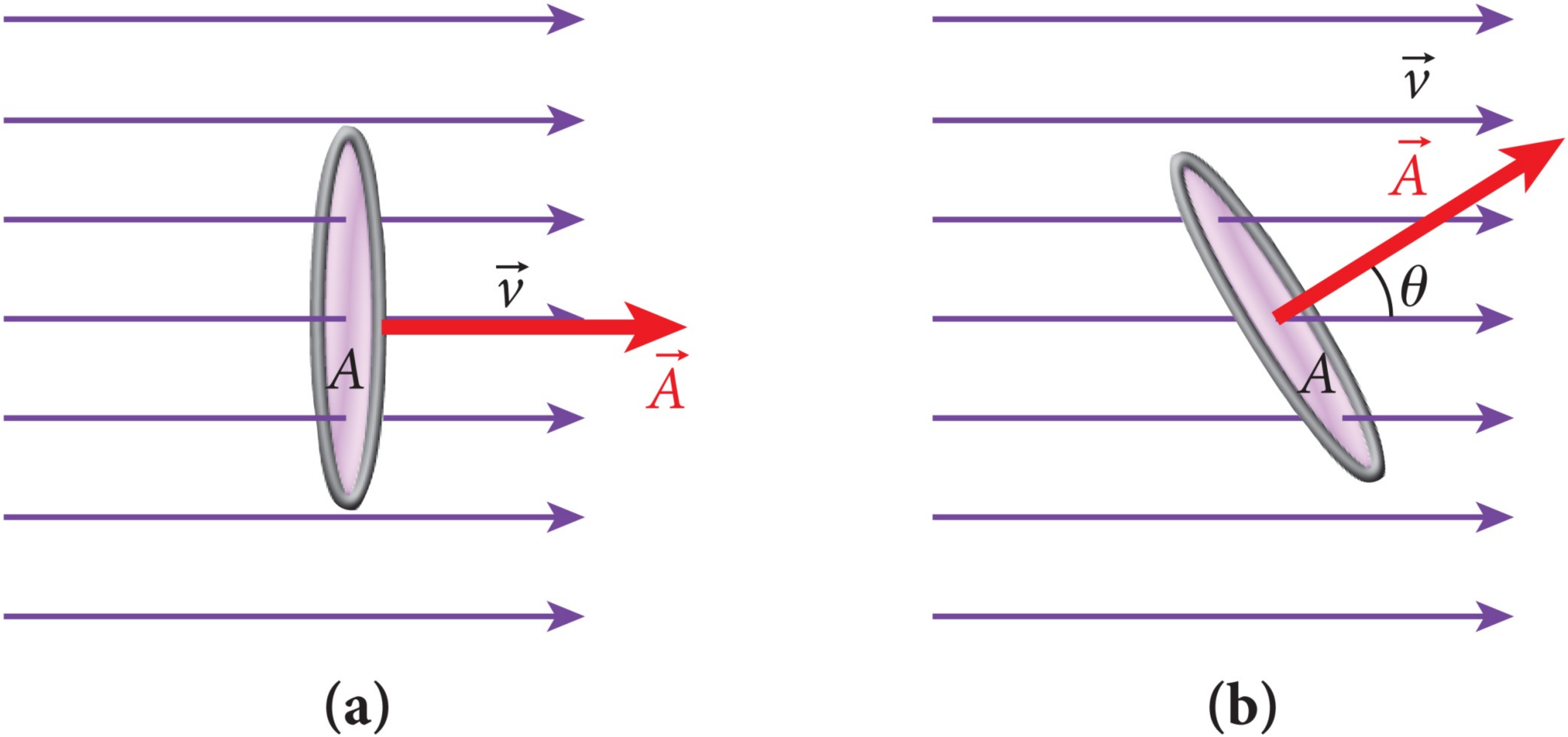
They begin at + charges & end at – charges or ∞.

Their density is $\propto$ field strength or charge magnitude.

Field Lines

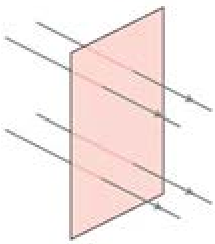


Eight lines begin on $+q$ . . .
q
so 16 lines begin on $+2q$ . . .
$2q$
and eight end on $-q$.
$-q$
Eight lines begin on each $+q$.
q
q
Eight lines begin on $+q$
and eight end on $-q$.
$-q$
$+q$
Eight lines begin on $+q$.
Four go to infinity
and four end on $-q/2$.
$-q/2$
q



$\vec{v}$
A
$\vec{A}$
(a)
$\vec{v}$
$\vec{A}$
θ
A
(b)

Electric Flux

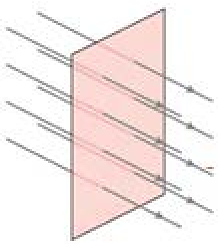


A flat surface is represented by a vector $\mathbf{A} = A\,\hat{\mathbf{a}}$

where A = area of surface and $\hat{\mathbf{a}}$ // normal of surface

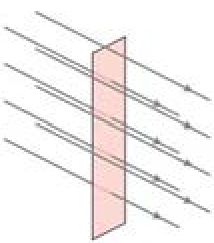
Electric flux through flat surface **A** :

$$\Phi = \mathbf{E} \cdot \mathbf{A} \qquad [\,\Phi\,] = \mathrm{N\,m^2/C}.$$



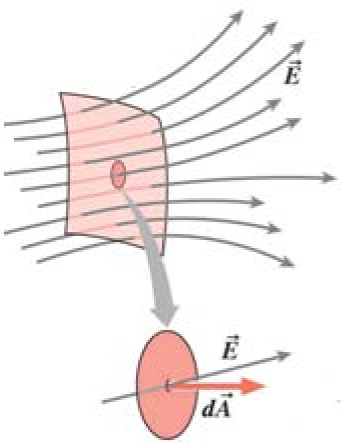
E↑, Φ↑

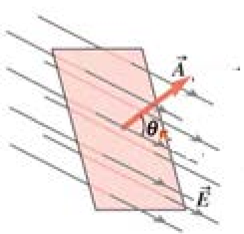
$$\Phi = \int_{surface} \mathbf{E} \cdot d\mathbf{A}$$



A↓, Φ↓

Open surface: can get from 1 side to the other w/o crossing surface. Direction of **A** ambiguous.

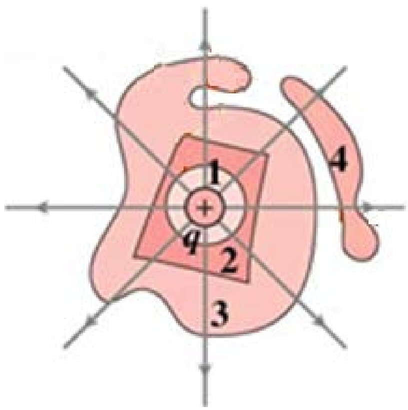




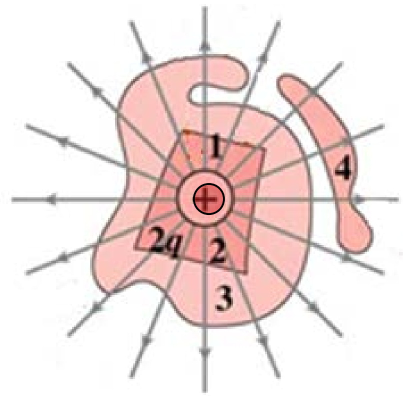
Closed surface: can't get from 1 side to the other w/o crossing surface. **A** defined to point outward.

Electric Flux & Field

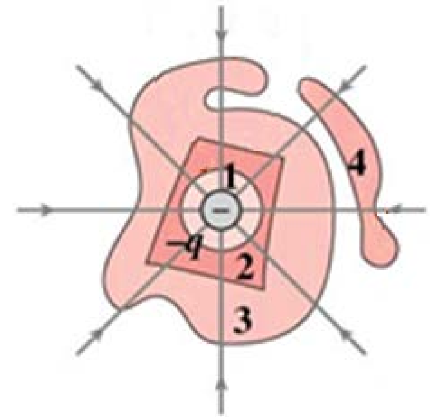
8 lines out of surfaces 1, 2, & 3. But 8–8 = 0 out of 4.



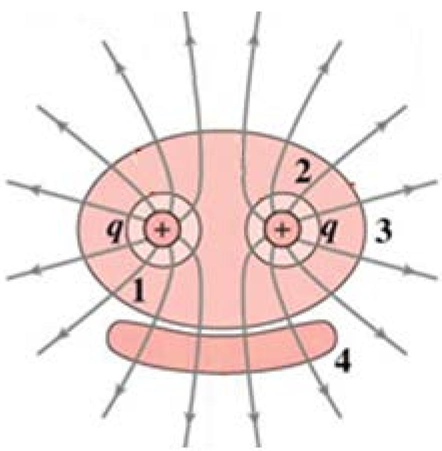
16 lines out of surfaces 1, 2, & 3. But 0 out of 4.



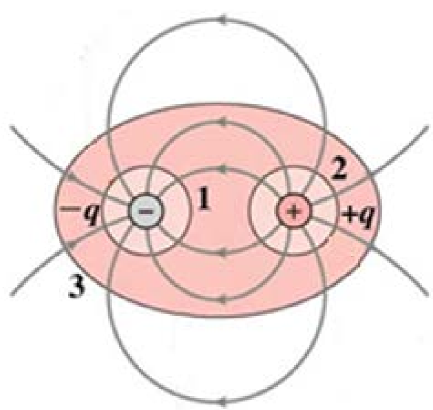
–8 lines out of surfaces 1, 2, & 3. But 0 out of 4.



8 lines out of surfaces 1 & 2.
16 lines out of surface 3.
0 out of 4.

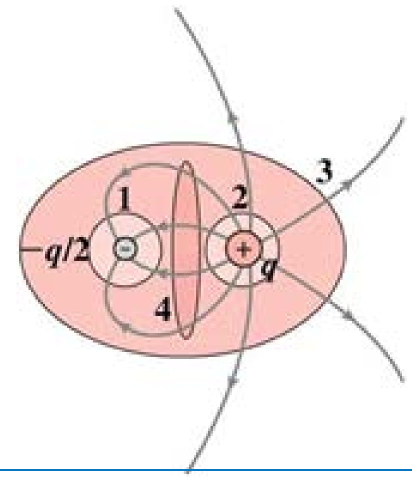


– 8 lines out of surface 1.
8 lines out of surface 2.
8–8 = 0 lines out of surface 3.



Count these.

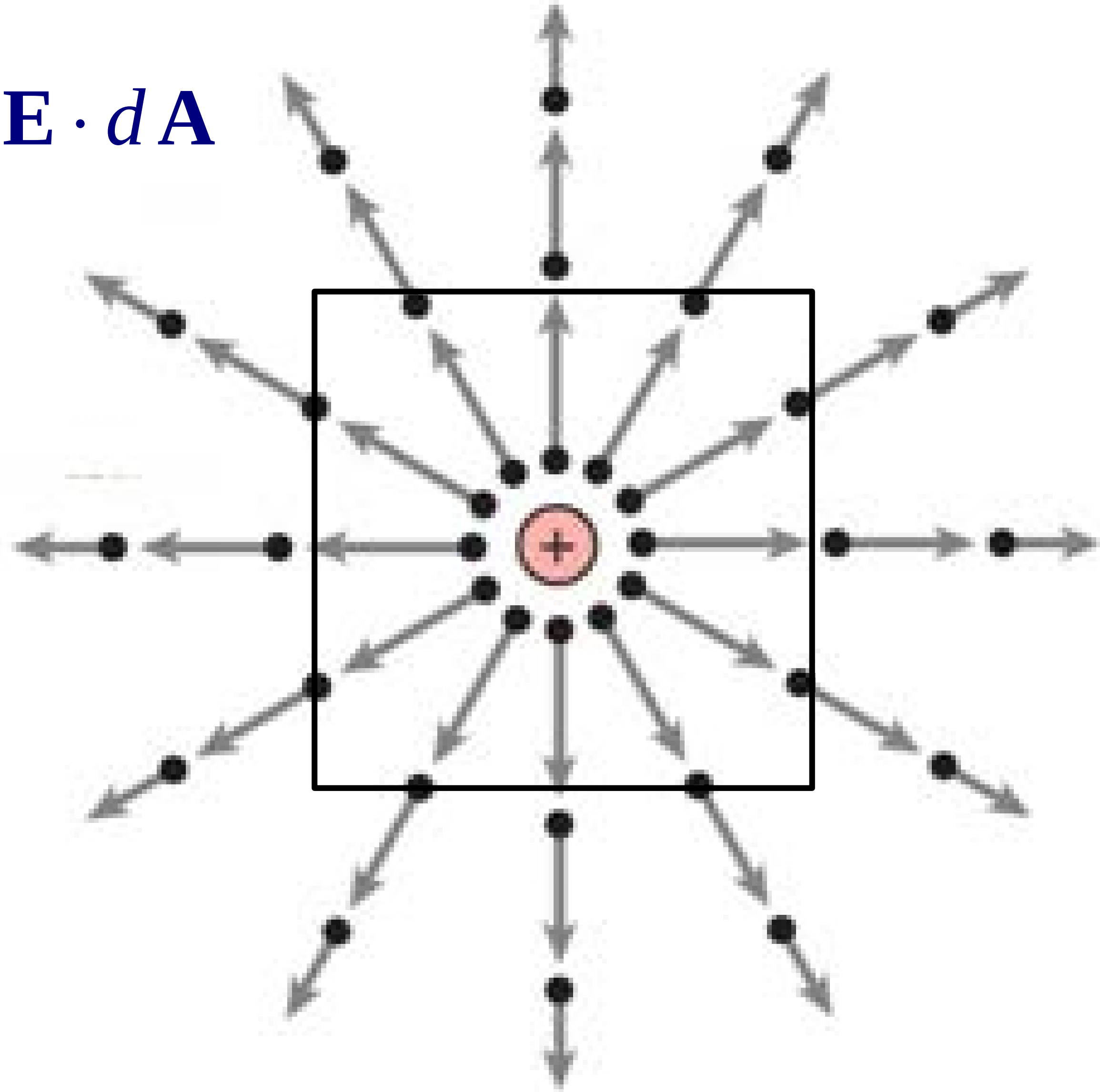
1: –4
2: 8
3: 4
4: 0



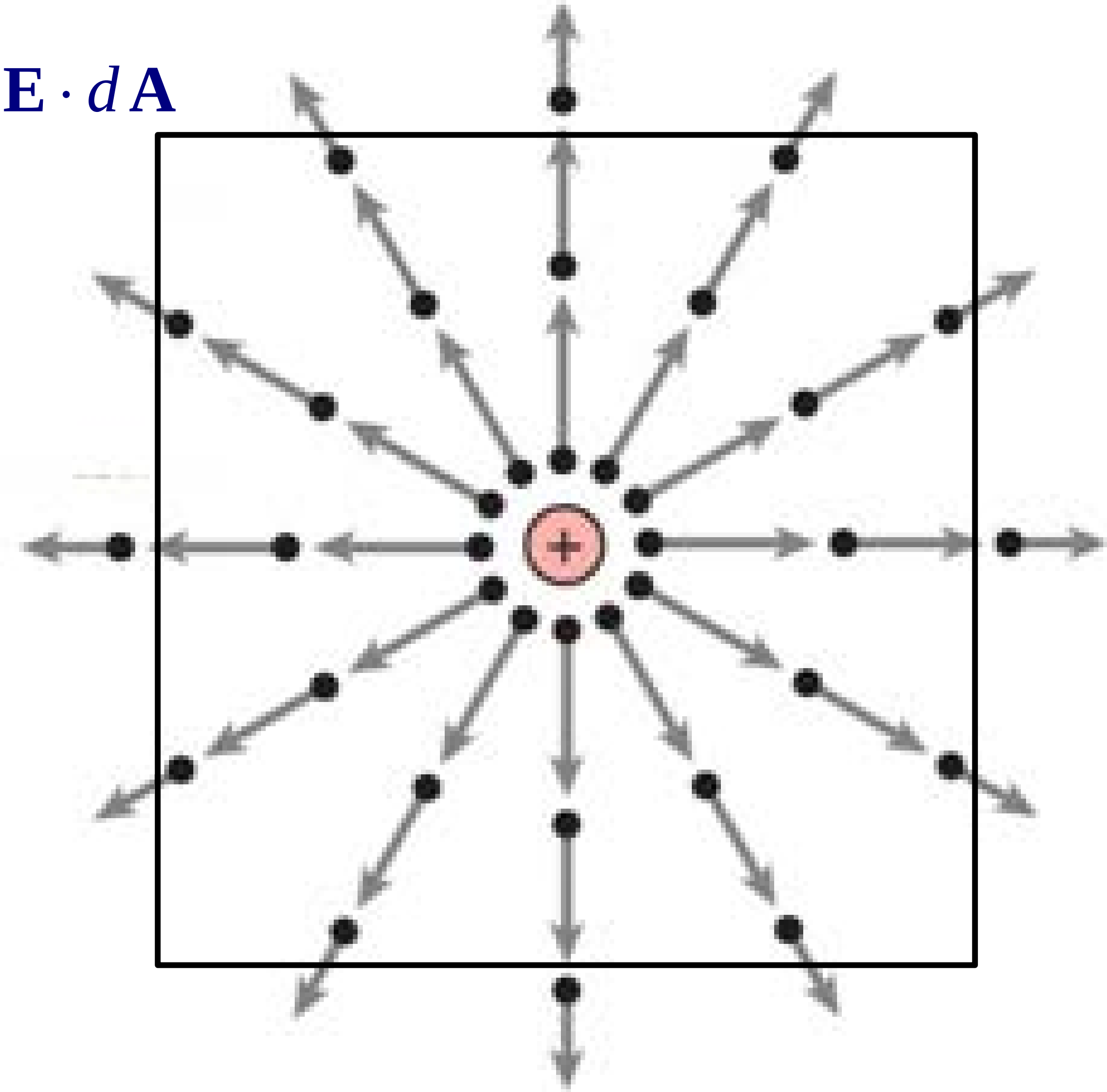
Number of field lines out of a closed surface ∝ net charge enclosed.

Analogy : Field Lines or Electric Flux is like water flow

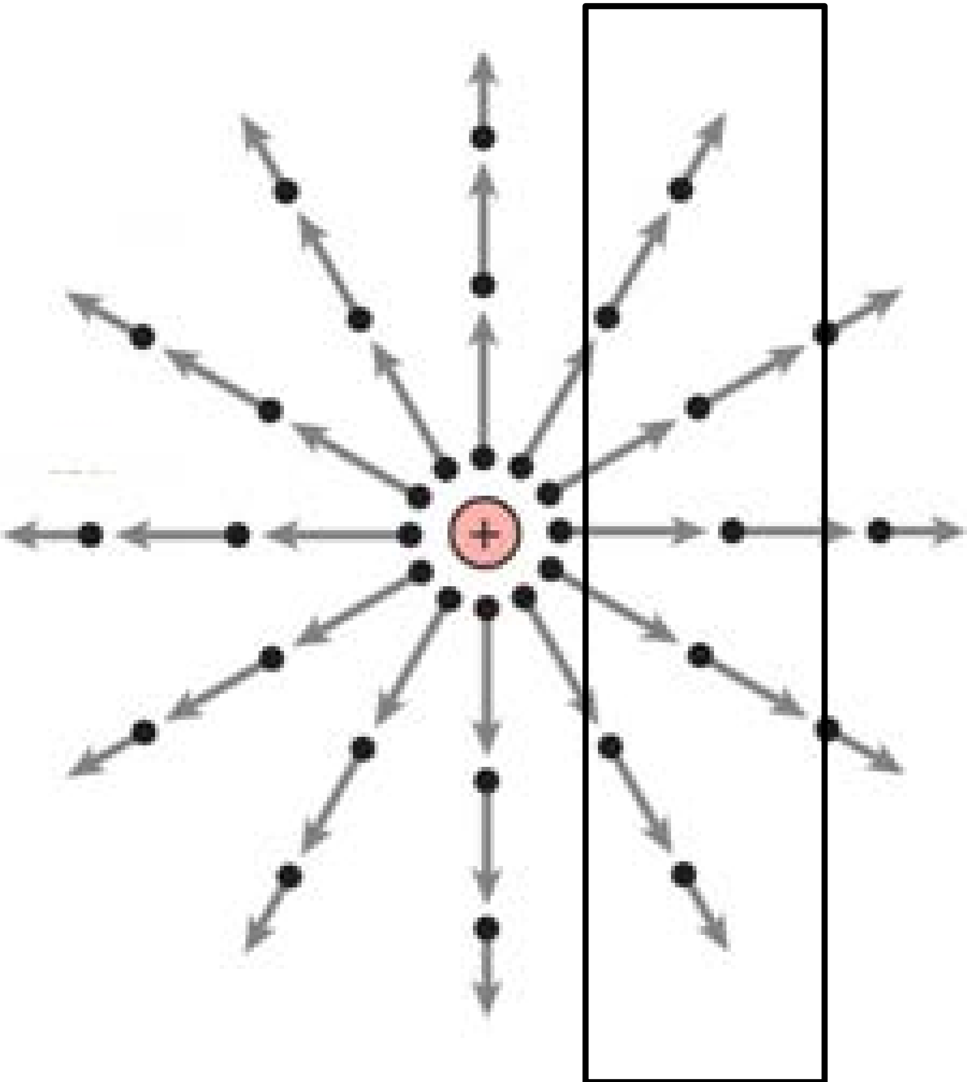
$$\Phi = \int_{surface} \mathbf{E} \cdot d\,\mathbf{A}$$



$$\Phi = \int_{surface} \mathbf{E} \cdot d\mathbf{A}$$



$$\Phi = \int_{surface} \mathbf{E} \cdot d\,\mathbf{A}$$



Gauss's Law

Gauss's law: The electric flux through any closed surface is proportional to the net charges enclosed.

$$\Phi = \oint \vec{E} \cdot d\vec{A} = \alpha Q_{enclosed}$$

α depends on units.

For point charge enclosed by a sphere centered on it:

$$\Phi = k\frac{q}{r^2}\left(4\pi r^2\right) = \alpha\, q$$

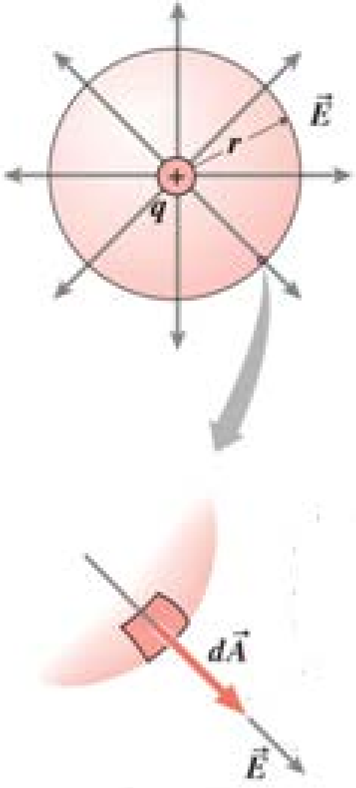
SI units

$$\rightarrow \qquad \alpha = 4\pi\, k = \frac{1}{\varepsilon_0}$$

$$\varepsilon_0 = \frac{1}{4\pi\, k} = 8.85\times 10^{-12} C^2 / N\cdot m^2$$ = vacuum permittivity

Field of point charge: $\mathbf{E} = k\frac{q}{r^2}\hat{\mathbf{r}} = \frac{q}{4\pi\,\varepsilon_0\, r^2}\hat{\mathbf{r}}$

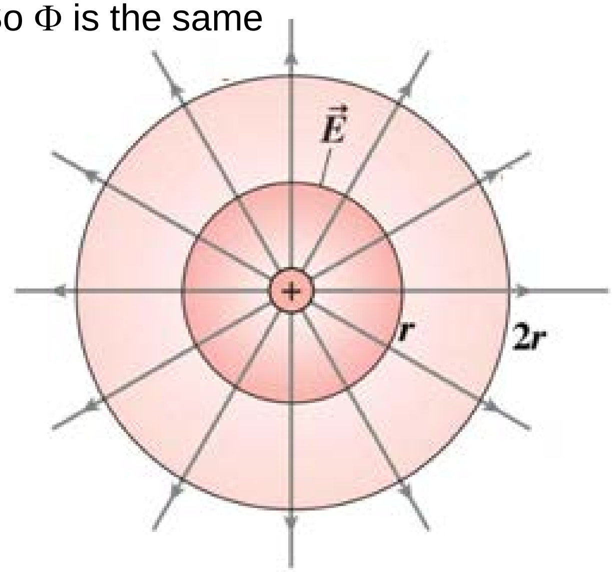
Gauss's law: $$\Phi = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\epsilon_0}$$



Gauss & Coulomb

Outer sphere has 4 times area.
But E is 4 times weaker.

So Φ is the same



For a point charge:

$$E \propto r^{-2}$$

$$A \propto r^{2}$$

$\rightarrow$ Φ indep of r.

Principle of superposition $\rightarrow$ argument holds for all charge distributions

Gauss' & Colomb's laws are both expression of the inverse square law.

For a given set of field lines going out of / into a point charge,

inverse square law $\rightarrow$ density of field lines $\propto E$ in 3-D.

Using Gauss's Law

Useful only for symmetric charge distributions.

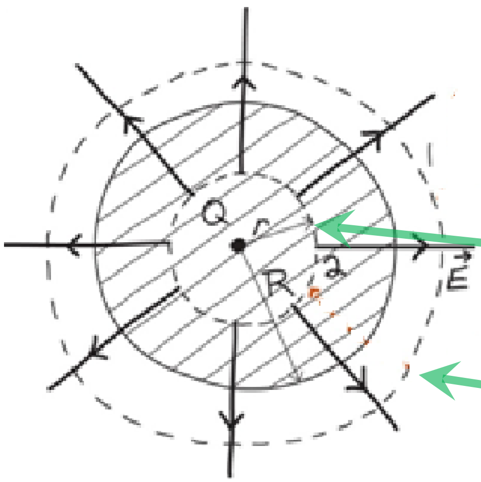
Spherical symmetry: $\rho(\mathbf{r}) = \rho(r)$ (point of symmetry at origin)

$\rightarrow \quad \mathbf{E}(\mathbf{r}) = E(r)\,\hat{\mathbf{r}}$

24.2: Uniformily Charged Sphere

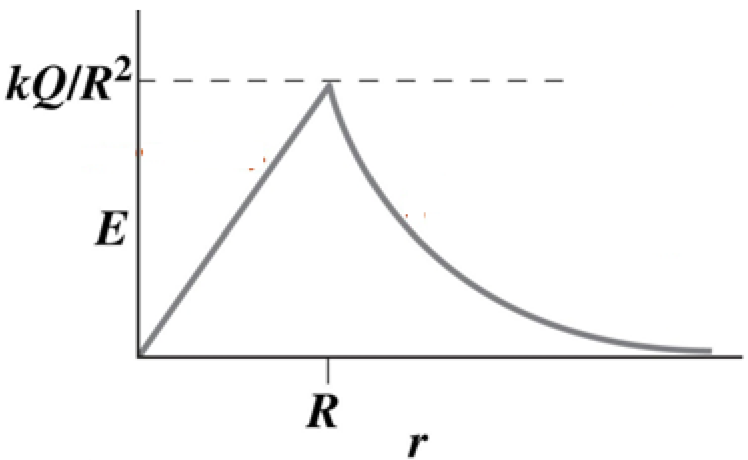
A charge *Q* is spreaded uniformily throughout a sphere of radius *R*.

Find the electric field at all points, first inside and then outside the sphere.



$$\mathbf{E}(\mathbf{r}) = E(r)\,\hat{\mathbf{r}} \qquad \rho(\mathbf{r}) = \frac{Q}{\frac{4\pi}{3}R^3}$$

$$\rightarrow \qquad \Phi = 4\pi r^2 E \quad = \begin{cases} \dfrac{Q}{\varepsilon_0}\left(\dfrac{r^3}{R^3}\right) & r \le R \\[2ex] \dfrac{Q}{\varepsilon_0} & r \ge R \end{cases}$$



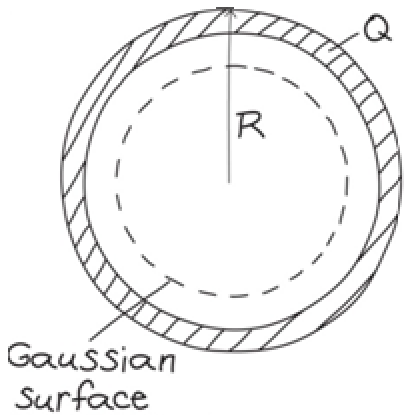
$$\therefore \qquad E = \begin{cases} \dfrac{Q\,r}{4\pi\,\varepsilon_0 R^3} & r \le R \\[2ex] \dfrac{Q}{4\pi\,\varepsilon_0\, r^2} & r \ge R \end{cases}$$

True for arbitrary spherical $\rho(r)$.

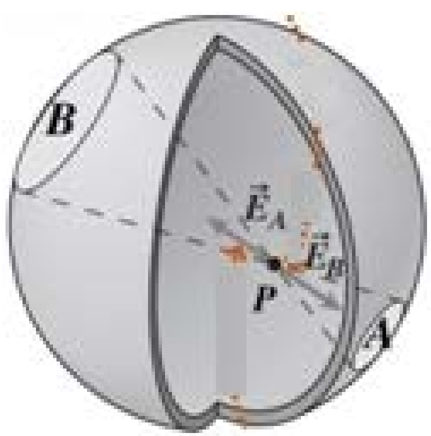
24.1: Hollow Spherical Shell

A thin, hollow spherical shell of radius *R* contains a total charge of *Q*. distributed uniformly over its surface.

Find the electric field both inside and outside the sphere.



$$\Phi = 4\pi r^2 E \quad = \begin{cases} 0 & r < R \\ \dfrac{Q}{\varepsilon_0} & r > R \end{cases}$$



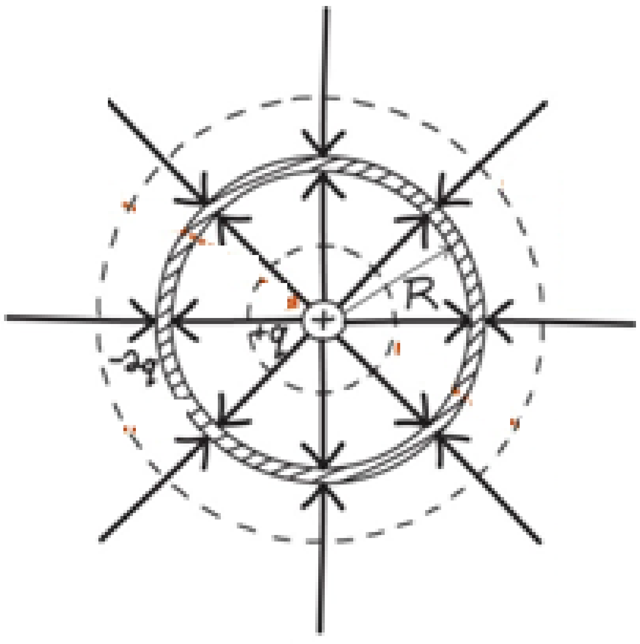
$$\therefore \qquad E = \begin{cases} 0 & r < R \\ \dfrac{Q}{4\pi\,\varepsilon_0\, r^2} & r > R \end{cases}$$

Contributions from A & B cancel.

Example: Point Charge Within a Shell

A positive point charge $+q$ is at the center of a spherical shell of radius R carrying charge $-2q$, distributed uniformly over its surface.

Find the field strength both inside and outside the shell.



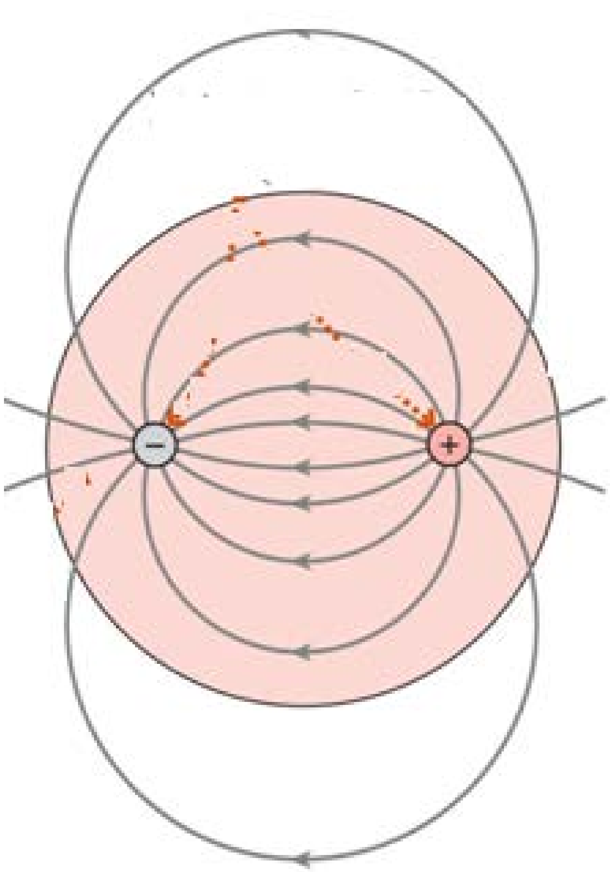
$$\Phi = 4\pi r^2 E = \begin{cases} \dfrac{1}{\varepsilon_0}(+q) & r < R \\ \dfrac{1}{\varepsilon_0}(+q-2q) & r > R \end{cases}$$

$$\therefore \quad E = \begin{cases} \dfrac{q}{4\pi\,\varepsilon_0\,r^2} & r < R \\ -\dfrac{q}{4\pi\,\varepsilon_0\,r^2} & r > R \end{cases}$$

Tip: Symmetry Matters

Spherical charge distribution inside a spherical shell is zero
$\rightarrow$ E = 0 inside shell

E $\neq$ 0 if either shell or distribution is not spherical.



Q = q–q = 0

But **E** $\neq$ 0 on or inside surface

Line Symmetry

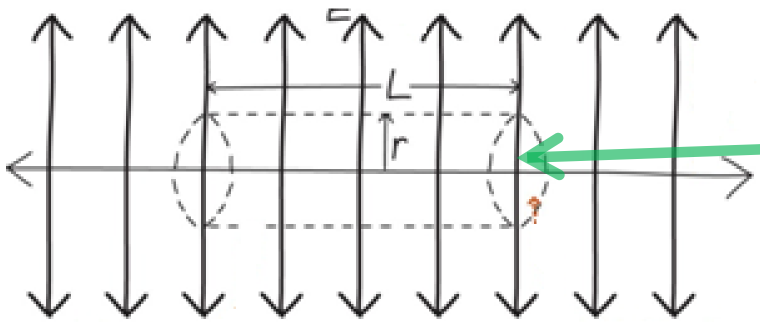
Line symmetry: $\rho(\mathbf{r}) = \rho(r_\perp)$ $r_\perp$ = perpendicular distance to the symm. axis.

Distribution is indpendent of $r_{//}$ $\rightarrow$ it must extend to infinity along symm. axis.

$$\rightarrow \quad \mathbf{E}(\mathbf{r}) = E(r_\perp)\,\hat{\mathbf{r}}_\perp$$

Infinite Line of Charge

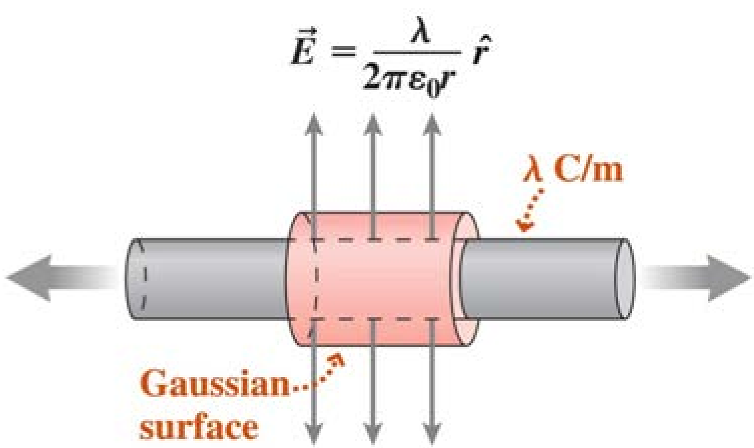
Use Gauss' law to find the electric field of an infinite line charge carrying charge density λ in C/m.



$$\mathbf{E}(\mathbf{r}) = E(r)\,\hat{\mathbf{r}}_{\perp} \qquad \text{(radial field)}$$

No flux thru ends

$$\rightarrow \quad \Phi = 2\pi\, r_{\perp}\, L\, E \quad = \frac{\lambda\, L}{\varepsilon_0}$$



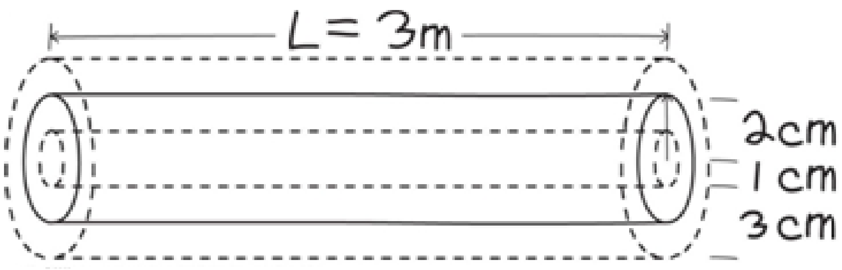
$$\therefore \quad E = \frac{\lambda}{2\pi\, \varepsilon_0\, r_{\perp}} \qquad \text{c.f. Eg. 20.7}$$

True outside arbitrary radial $\rho(r_{\perp})$.

Example: A Hollow Pipe

A thin-walled pipe 3.0 m long & 2.0 cm in radius carries a net charge q = 5.7 μC distributed uniformly over its surface.

Fine the electric field both 1.0 cm & 3.0 cm from the pipe axis, far from either end.



$$\Phi = 2\pi\, r_\perp\, L\, E \;=\begin{cases} 0 & r < 2.0\,cm \\ \dfrac{1}{\varepsilon_0}(5.7\,\mu C) & r > 2.0\,cm \end{cases}$$

$$\therefore \quad E = \begin{cases} 0 & r < 2.0\,cm \\ \dfrac{1}{2\pi\,\varepsilon_0\,L\,r_\perp}(5.7\,\mu C) & r > 2.0\,cm \end{cases}$$

$E = 0$ at r = 1.0 cm

$$E = 2\times\left(9\times10^{9}\ N\ m^{2}\ C^{-2}\right)\frac{1}{(3.0\,m)(0.03\,m)}\left(5.7\times10^{-6}\,C\right) \;= 1.1\,M\ N/C$$

at r = 3.0 cm

Plane Symmetry

Plane symmetry: $\rho(\mathbf{r}) = \rho(r_\perp)$ $r_\perp$ = perpendicular distance to the symm. plane.

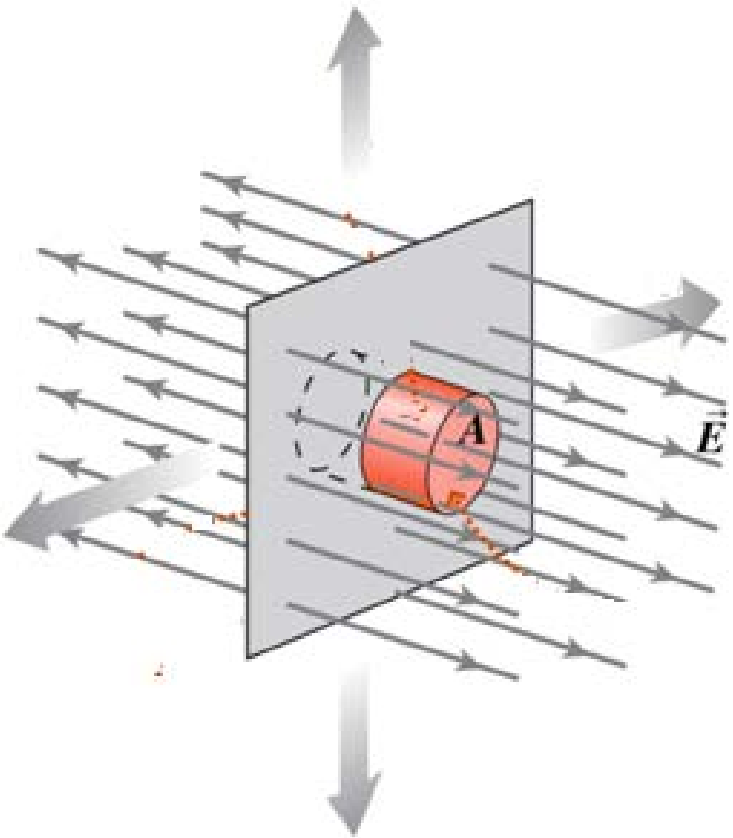
Distribution is indpendent of $r_{//}$ $\rightarrow$ it must extend to infinity in symm. plane.

$\rightarrow$ $\mathbf{E}(\mathbf{r}) = E(r_\perp)\,\hat{\mathbf{r}}_\perp$

A Sheet of Charge

An infinite sheet of charge carries uniform surface charge density σ in C/m^2.

Find the resulting electric field.



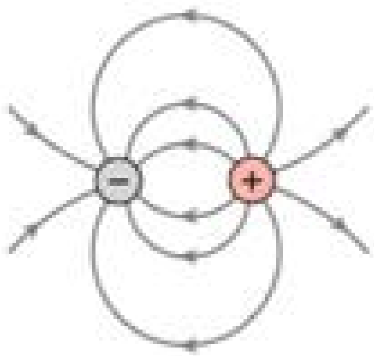
$$\mathbf{E}(\mathbf{r}) = E(r)\,\hat{\mathbf{r}}_\perp$$

$$\rightarrow \qquad \Phi = 2\,A\,E \quad = \frac{\sigma\,A}{\varepsilon_0}$$

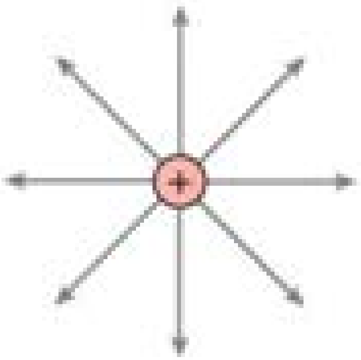
$$\therefore \qquad E = \frac{\sigma}{2\,\varepsilon_0}$$

$E > 0$ if it points away from sheet.

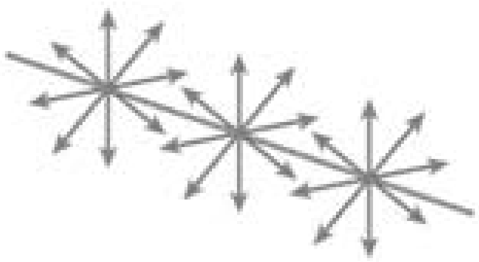
Example: Fields of Arbitrary Charge Distributions



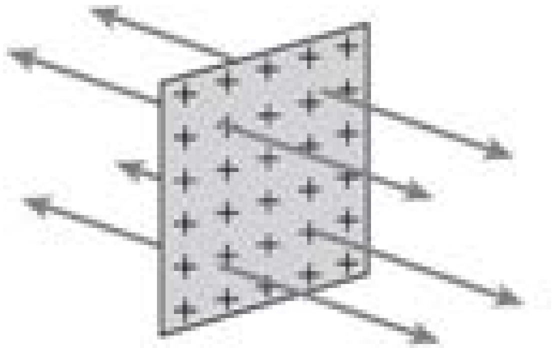
Dipole : $E \propto r^{-3}$



Point charge : $E \propto r^{-2}$



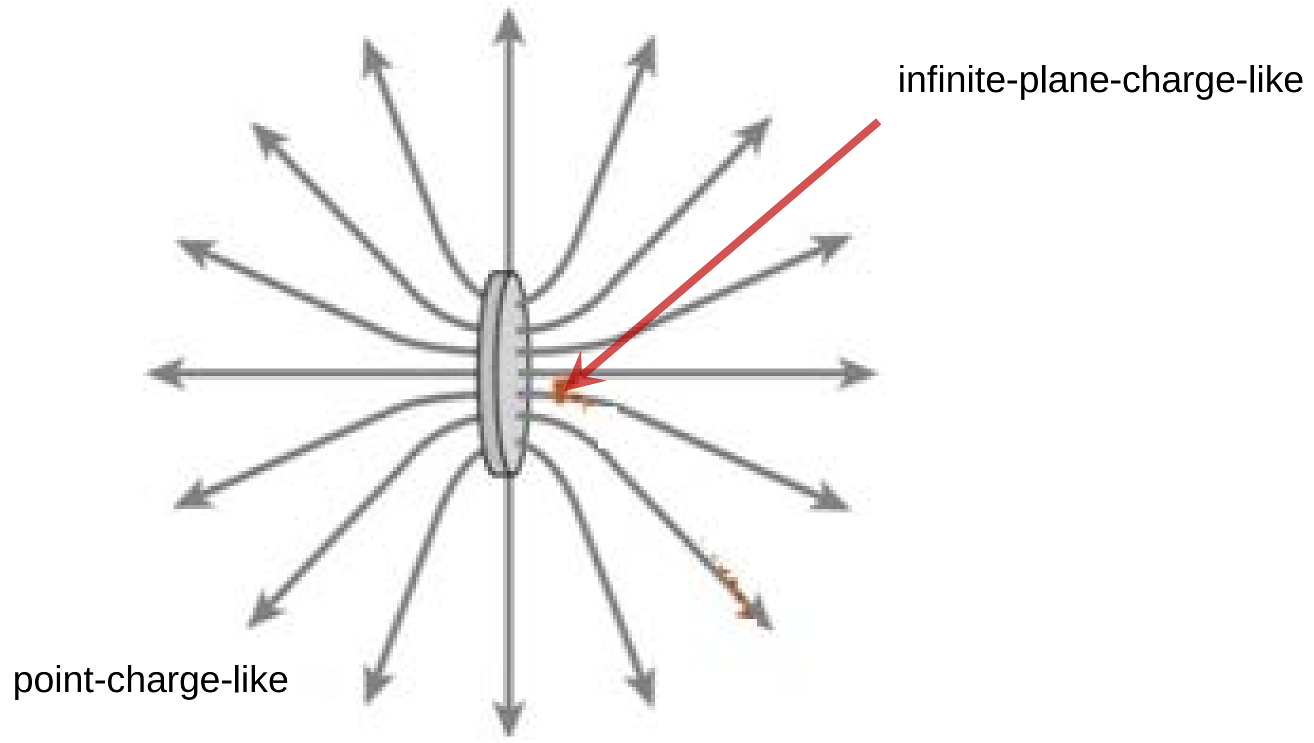
Line charge : $E \propto r^{-1}$



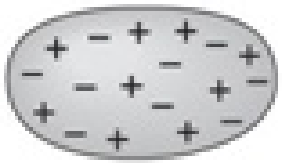
Surface charge : $E \propto const$

Concept Example: Charged Disk

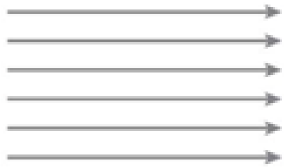
Sketch some electric field lines for a uniformly charged disk, starting at the disk and extending out to several disk diameters.



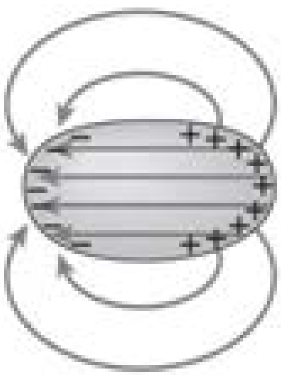
Gauss's Law & Conductors



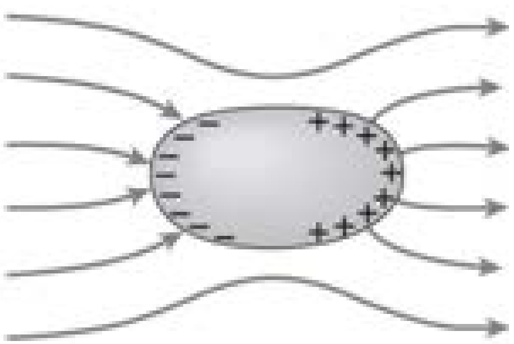
Neutral conductor



Uniform field



Induced polarization
cancels field inside



Net field

Electrostatic Equilibrium

Conductor = material with free charges

E.g., free electrons in metals.

External **E** $\rightarrow$ Polarization

$\rightarrow$ Internal **E**

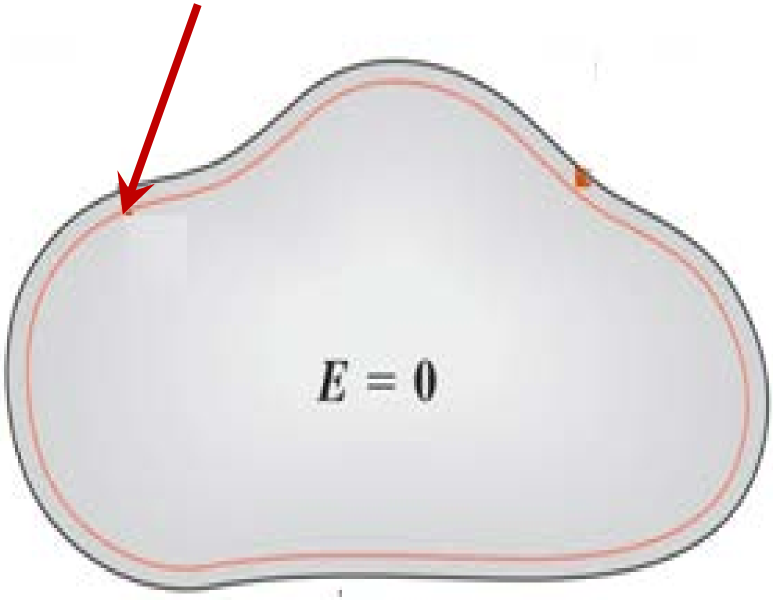
Total **E** = 0 : Electrostatic equilibrium

(All charges stationary)

Microscopic view: replace
above with averaged values.

Charged Conductors

$\Phi = 0$ thru this surface



Excess charges in conductor tend to keep away from each other

$\rightarrow$ they stay at the surface.

More rigorously:

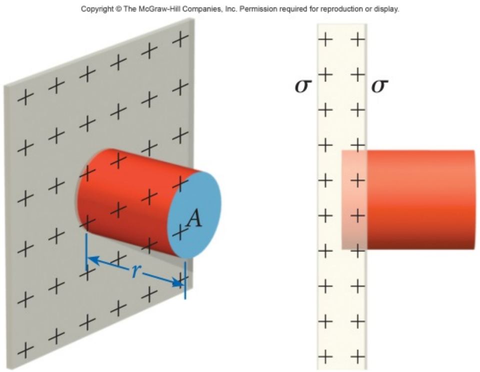
Gauss' law with **E** = 0 inside conductor

$\rightarrow$ $q_{enclosed} = 0$

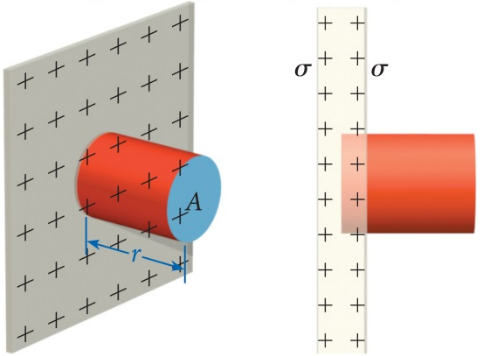
$\therefore$ For a conductor in electrostatic equilibrium, all charges are on the surface.

Example: Field of an Infinite Conducting Plane

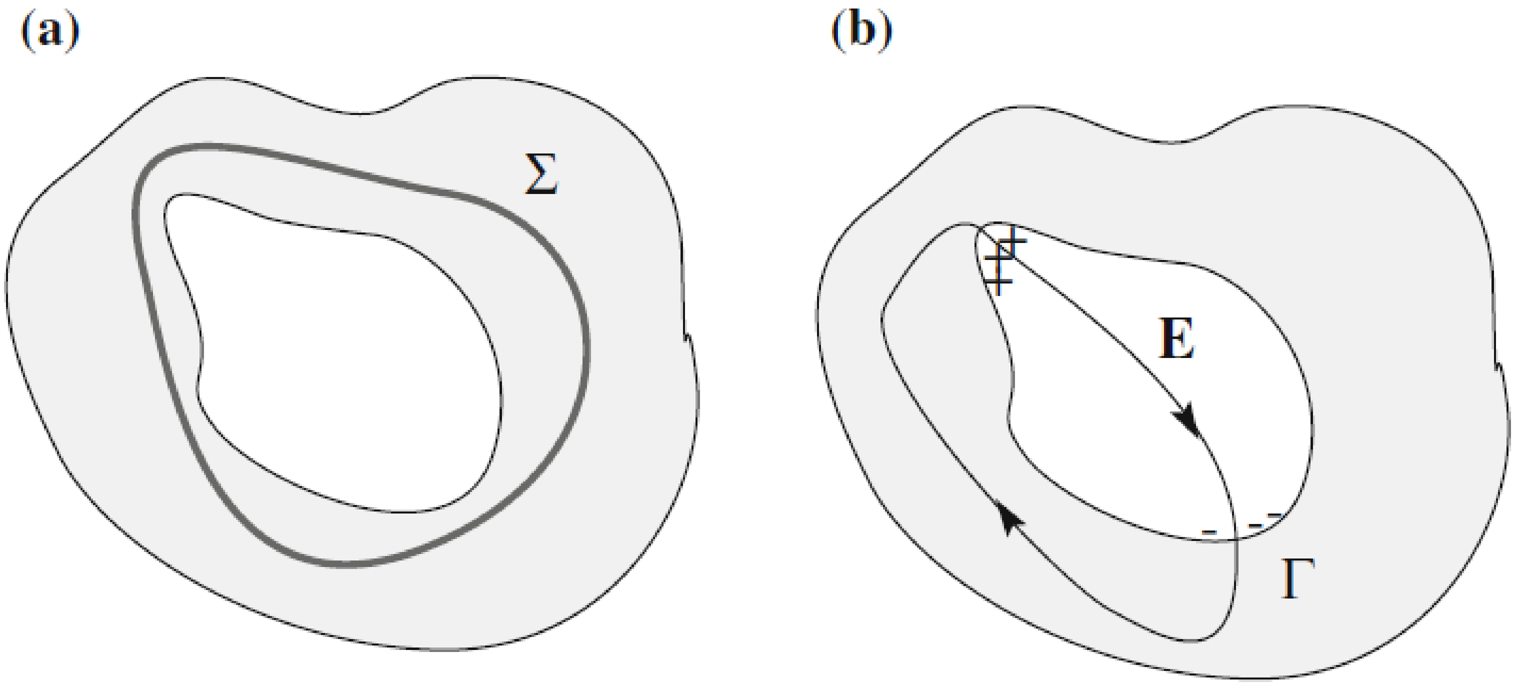
- Find the field due to an infinite conducting plate with a uniform surface charge $\sigma\ C/m^2$



Example: Field of an Infinite Conducting Plane



A Hollow Conductor



Example . A Hollow Conductor

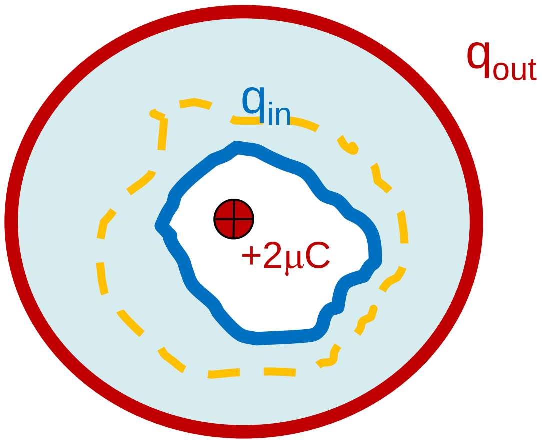
An irregularly shaped conductor has a hollow cavity.

The conductor itself carries a net charge of 1 μC,

and there's a 2 μC point charge inside the cavity.

Find the net charge on the cavity wall & on the outer

surface of the conductor, assuming electrostatic equilibrium.



E = 0 inside conductor

→ $\Phi = 0$ through dotted surface

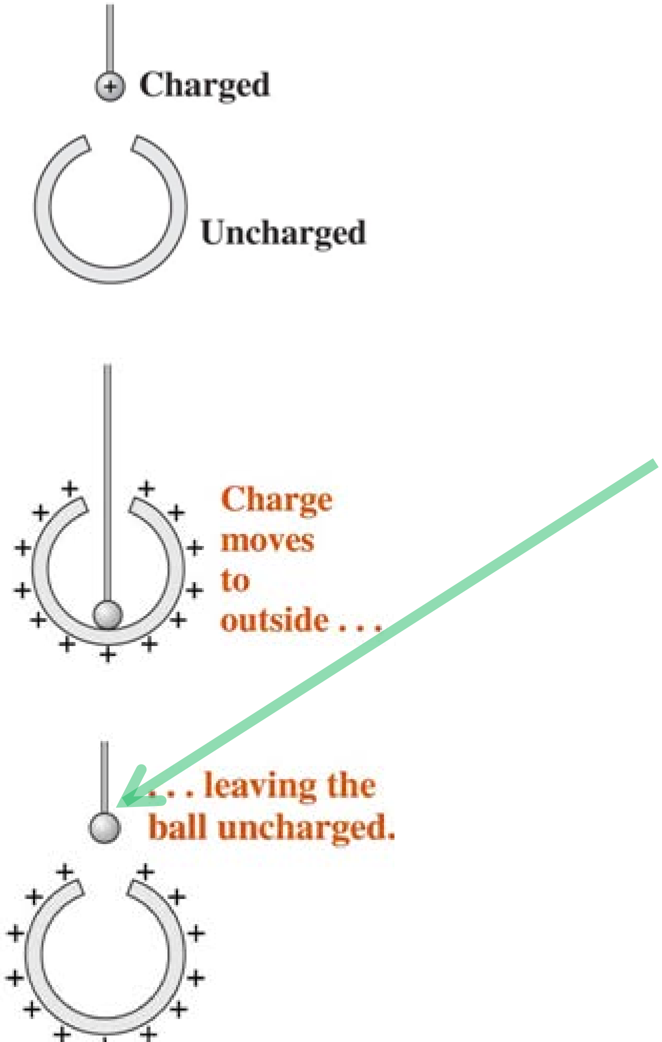
→ $q_{enclosed} = 0$

→ Net charge on the cavity wall $q_{in} = -2\ \mu C$

Net charge in conductor = 1 μC = $q_{out} + q_{in}$

→ charge on outer surface of the conductor $q_{out} = +3\ \mu C$

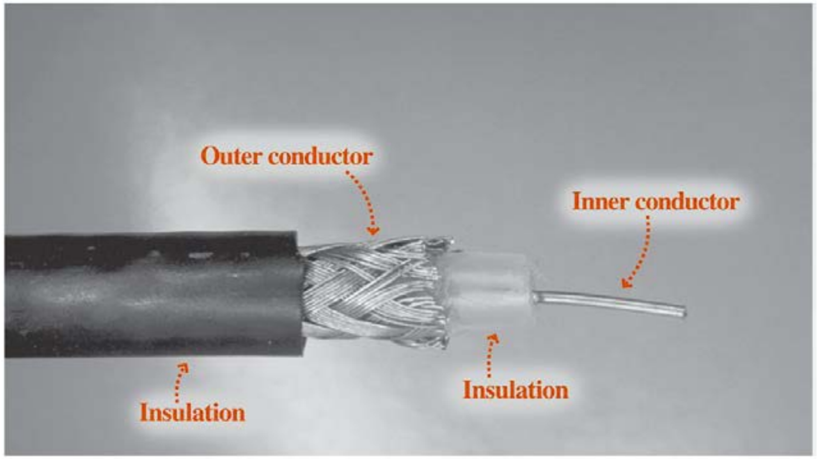
Experimental Tests of Gauss' Law



Measuring charge on ball is equivalent to testing the inverse square law.

The exponent 2 was found to be accurate to 10^{-16} .

Application: Shielding & Lightning Safety



Coaxial cable

Car hit by lightning,
driver inside unharmed.

Strictly speaking, Gauss law applies only to static **E**.

However, *e* in metal can respond so quickly that
high frequency EM field (radio, TV, MW) can also be blocked (skin effect).