Engineering Mathematics

Final Term, 2010/01/11

1. （10%）根據定義，證明:（According to the definition, please prove）

（a） $\mathcal{L}\{\cos(\mathrm{at})\} = \dfrac{\mathrm{S}}{\mathrm{S}^2 + a^2}$

（b） Prove $\mathcal{L}\{f(t) * g(t)\} = F(s) \cdot G(s)$, 其中 $\mathrm{F(s)} = \mathcal{L}\{f(t)\}, G(s) = \mathcal{L}\{f(t)\}$

Ans：

(a)

$$f(t) = \cos at = \frac{e^{iat} + e^{-iat}}{2}$$

$$F(s) = \int_0^\infty f(t) e^{-st} dt = \int_0^\infty \cos(at) e^{-st} dt$$

$$\mathcal{L}\{\cos at\} = \mathcal{L}\left\{\frac{e^{iat} + e^{-iat}}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^{iat}\} + \frac{1}{2}\mathcal{L}\{e^{-iat}\}$$

$$= \frac{1}{2}\frac{1}{(s - ia)} + \frac{1}{2}\frac{1}{(s + ia)}$$

$$= \frac{1}{2}\frac{2s}{(s - ia)(s + ia)}$$

$$= \frac{s}{(s - ia)(s + ia)}$$

$$= \frac{s}{s^2 + a^2}$$

（b）

$$\mathcal{L}\{f(t) * g(t)\}$$

$$= \int_0^\infty [f(t) * g(t)] e^{-st} dt$$

$$= \int_0^\infty \left(\int_0^t f(\lambda) g(t - \lambda) d\lambda\right) e^{-st} dt$$

$$= \int_0^\infty \int_0^t f(\lambda) g(t - \lambda) e^{-st} d\lambda dt$$ λ 和 t 積分範圍相關，想對調

$$= \int_0^\infty \int_\lambda^\infty f(\lambda) g(t - \lambda) e^{-st} dt d\lambda$$ 要在積分區域相同條件 下

$$= \int_0^\infty f(\lambda) \int_\lambda^\infty g(t - \lambda) e^{-st} dt d\lambda$$ 令 $x = t - \lambda, dx = dt$

$$= \int_0^\infty f(\lambda) \int_\lambda^\infty g(x) e^{-s(\lambda + x)} dx d\lambda$$

$$= \int_0^\infty f(\lambda) e^{-s\lambda} \int_\lambda^\infty g(x) e^{-st} dx d\lambda$$

$$= G(s) \int_0^\infty f(\lambda) e^{-s\lambda} d\lambda$$

$$= G(s) F(s)$$

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2. （15%）

（a） $\mathcal{L}^{-1}\{\frac{s}{(s+2)^2+4}\}=?$

（b） $\mathcal{L}^{-1}\{e^{-3s}\cdot\frac{s+1}{(s+1)^2+1}\}=?$

（c） $\mathcal{L}^{-1}\{\frac{1}{(s^2+k^2)^2}\}=?$

Ans：

（a）

$$\mathcal{L}^{-1}\{\frac{s}{(s+2)^2+4}\}=\mathcal{L}^{-1}\{\frac{s+2-2}{(s+2)^2+4}\}$$

$$=\mathcal{L}^{-1}\{\frac{s+2}{(s+2)^2+4}\}+L^{-1}\{\frac{-2}{(s+2)^2+4}\}$$

$$\Rightarrow f(t)=e^{-2t}\cos 2t-e^{-2t}\sin 2t$$

（b）

$$\mathcal{L}^{-1}\{e^{-3s}\cdot\frac{s+1}{(s+1)^2+1}\}=e^{-(t-3)}\cos(t-3)H(t-3)$$

（c）

$$F(s)=G(s)=\frac{1}{s^2+k^2}$$

$$f(t)=g(t)=\frac{1}{k}\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}=\frac{1}{k}\sin kt$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\}=\frac{1}{k^2}\int_0^t \sin k\tau \sin k(t-\tau)d\tau$$

$$\sin A\sin B=(\frac{1}{2})[\cos(A-B)-\cos(A+B)]$$

$$\text{set } A=k\tau, B=k(t-\tau)$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\}=\frac{1}{2k^2}\int_0^t[\cos k(2\tau-t)-\cos kt]d\tau$$

$$=\frac{1}{2k^2}\left[\frac{1}{2k}\sin k(2\tau-t)-\tau\cos kt\right]\Bigg|_0^t$$

$$=\frac{\sin kt-kt\cos kt}{2k^3}$$

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3. (8%) $f(t) = t^2 + 3t + 2$

(a) $\mathcal{L}\{f(t)\}$

(b) $\mathcal{L}\{f(t-1)\}$

(c) $\mathcal{L}\{f(t)H(t-1)\}$

(d) $\mathcal{L}\{f(t-1)H(t-1)\}$

Ans:

(a)

$$\mathcal{L}\{f(t)\} = \frac{2!}{s^3} + 3\frac{1}{s^2} + 2\frac{1}{s}$$

(b)

$$\begin{aligned}
&\mathcal{L}\{f(t-1)\} \\
&= \mathcal{L}\{(t-1)^2 + 3(t-1) + 2\} \\
&= \mathcal{L}\{t^2 + t\} \\
&= \frac{2!}{s^3} + \frac{1}{s^2}
\end{aligned}$$

(c)

$$\begin{aligned}
&\mathcal{L}\{f(t)H(t-1)\} \\
&= \mathcal{L}\{(t^2 + 3t + 2)H(t-1)\} \\
&= \mathcal{L}\{((t-1)^2 + A(t-1) + B)H(t-1)\} \qquad A = 5, B = 6 \\
&= \frac{2}{s^3}e^{-s} + 5\frac{1}{s^2}e^{-s} + 6\frac{1}{s}e^{-s}
\end{aligned}$$

(d)

$$\mathcal{L}\{f(t-1)H(t-1)\} = [\frac{2!}{s^3} + 3\frac{1}{s^2} + 2\frac{1}{s}]e^{-s}$$

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4. （10%）

（a） $\int_0^{\infty} t(\sin t)(\cos t)dt = ?$

（b） $\mathcal{L}\{e^{2t}\int_0^t \mathrm{t}\cdot e^{3t}\cdot \sin t\, dt\} = ?$

Ans：

（a）

$$\int_0^{\infty} t(\sin t)(\cos t)dt$$

$$= \lim_{s\to 0}\int_0^{\infty} t(\sin t)(\cos t) * e^{-st}dt$$

$$= \lim_{s\to 0}\int_0^{\infty} \frac{1}{2} t\sin(2t) * e^{-st}dt$$

$$= \frac{1}{2}\lim_{s\to 0}\mathcal{L}\{t\sin(2t)\}$$

$$= \lim_{s\to 0}\frac{2s}{(s^2+4)^2} = 0$$

（b）

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$$

$$\mathcal{L}\{t\sin t\} = \frac{-d}{ds}\frac{1}{s^2+1} = \frac{2s}{(s^2+1)^2}$$

$$\mathcal{L}\{e^{3t}t\sin t\} = \frac{2s}{(s^2+1)^2}\Big|_{s=s-3} = \frac{2(s-3)}{((s-3)^2+1)^2} = \frac{2s-6}{(s^2-6s+10)^2}$$

$$\mathcal{L}\left\{\int_0^t e^{3t}(t)(\sin t)dt\right\} = \frac{1}{s}\frac{2s-6}{(s^2-6s+10)^2}$$

$$\mathcal{L}\left\{e^{2t}\int_0^t e^{3t}(t)(\sin t)dt\right\} = \frac{2s-6}{s(s^2-6s+10)^2}\Big|_{s=s-2}$$

$$= \frac{2(s-2)-6}{(s-2)((s-2)^2-6(s-2)+10)^2}$$

$$= \frac{2s-10}{(s-2)(s^2-10s+26)^2}$$

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5. (15%) 求 $f(t) = ?$

(a) $F(s) = \dfrac{s}{(s+1)(s-2)^2}$

(b) $F(s) = \dfrac{s}{(s-1)(s^2+4s+13)}$

(c) $F(s) = \dfrac{s^2+6s+9}{(s-1)(s-2)(s+4)}$

Ans：

(a)

$$F(s) = \frac{s}{(s+1)(s-2)^2} = \frac{k_1}{s+1} + \frac{k_2}{s-2} + \frac{k_3}{(s-2)^2}$$

$$\Rightarrow k_1 = \frac{-1}{9}, k_2 = \frac{1}{9}, k_3 = \frac{2}{3}$$

$$\Rightarrow f(t) = \frac{-1}{9}e^{-t} + \frac{1}{9}e^{-2t} + \frac{2}{3}te^{-2t}$$

(b)

$$F(s) = \frac{s}{(s-1)(s^2+4s+13)} = \frac{k_1}{(s-1)} + \frac{k_2 s + k_3}{(s^2+4s+13)}$$

$$\Rightarrow k_1 = \frac{1}{18}, k_2 = -\frac{1}{18}, k_3 = \frac{13}{18}$$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\left\{\frac{\frac{1}{18}}{(s-1)} + \frac{-\frac{1}{18}s + \frac{13}{18}}{(s^2+4s+13)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{\frac{1}{18}}{(s-1)} + \frac{-\frac{1}{18}(s+2) + \frac{15}{18}}{(s+2)^2+9}\right\}$$

$$= \frac{1}{18}e^{t} - \frac{1}{18}\cos(3t)e^{-2t} + \frac{5}{18}\sin(3t)e^{-2t}$$

(c)

$$F(s) = \frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4}$$

$$\Rightarrow s^2+6s+9$$

$$= A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2)$$

$$\Rightarrow s = 1, 2, -4$$

$$A = -16/5, B = 25/6, C = 1/30$$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$$

$$= -\frac{16}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{25}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{30}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$$

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6. （7%）$y''+2ty'-4y=6, y(0)=0, y'(0)=0,$ 求 $y=?$

Ans：

$$\Rightarrow S^2Y(s)+2\left(-\frac{d(SY(s))}{ds}\right)-4Y(s)=\frac{6}{s}$$

$$S^2Y(s)+2Y(s)-2SY'(s)-4Y(s)=\frac{6}{s}$$

$$-2SY'(s)+(s^2-6)Y(s)=\frac{6}{s}$$

$$y'+py=r$$

$$y=CI^{-1}+I^{-1}\int Irdx$$

$$I=e^{\int pdx}$$

$$Y'(s)+\frac{s^2-6}{-2s}Y(s)=\frac{6}{s(-2s)}$$

$$I=e^{\int\frac{s^2-6}{-2s}ds}=e^{\int\left(\frac{-s}{2}+\frac{3}{s}\right)ds}=e^{\frac{-1}{4}s^2+3\ln s}=e^{\frac{-1}{4}s^2}\cdot e^3$$

$$Y(s)=Ce^{\frac{1}{4}s^2}\cdot e^{-3}+e^{\frac{1}{4}s^2}\cdot e^{-3}\int e^{\frac{-1}{4}s^2}\cdot e^3\cdot\frac{6}{s(-2s)}ds$$

$$\left(\text{令 } u=\frac{-s^2}{4}, du=\frac{-1}{2}sds\right)$$

$$=Ce^{\frac{1}{4}s^2}\cdot S^{-3}+e^{\frac{1}{4}s^2}\cdot S^{-3}\int 6e^u du$$

$$=Ce^{\frac{1}{4}s^2}\cdot S^{-3}+6S^{-3}$$

如何解C，利用初值定理

$$y(0)=\lim_{s\to\infty}SY(s)=\lim_{s\to\infty}\left(Ce^{\frac{s^2}{4}}s^{-2}+\frac{6}{s^2}\right)=0$$

$$\therefore C=0$$

$$\therefore Y(s)=6s^{-3}$$

$$y(t)=3t^2$$

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7. (15%)Solve the following problems

(a) $y'' + y = f(t), f(t) = \begin{cases} 0, 0 \le t < \pi \\ 1, \pi \le t < 2\pi, y(0) = 0, y'(0) = 1 \\ 0, t \ge 2\pi \end{cases}$

(b) $y'' + 4y' + 3y = 3\delta(t-2) + H(t-1), y(0) = 0, y'(0) = 0$

(c) $f(t) = 3t^5 + \int_0^t f(t-\tau)e^{-\tau}d\tau$

Ans：

(a)

$$\left(S^2Y(s) - Y(0) - Y'(0)\right) + Y = \frac{1}{s}e^{-\pi s} - \frac{1}{s}e^{-2\pi s}$$

$$\left(S^2 + 1\right)Y(s) = 1 + \frac{1}{s}\left[e^{-\pi s} - e^{-2\pi s}\right]$$

$$Y(s) = \frac{1}{\left(S^2+1\right)} + \frac{1}{S\left(S^2+1\right)}\left(e^{-\pi s} - e^{-2\pi s}\right)$$

$$= \frac{1}{\left(S^2+1\right)} + \left(\frac{1}{s} + \frac{-s}{\left(S^2+1\right)}\right)\left(e^{-\pi s} - e^{-2\pi s}\right)$$

$$y(t) = \sin t + [1 - \cos(t-\pi)]H(t-\pi) - [1 - \cos(t-2\pi)]H(t-2\pi)$$

(b)

$$S^2Y(s) - SY(0) - Y'(0) + 4\left(SY(0) - Y(0)\right) + 3Y(s) = 3e^{-2s} + \frac{1}{s}e^{-s}$$

$$\left(S^2 + 4S + 3\right)Y(s) = 3e^{-2s} + \frac{1}{s}e^{-s}$$

$$Y(s) = \frac{3e^{-2s}}{\left(S^2+4S+3\right)} + \frac{e^{-s}}{s\left(S^2+4S+3\right)}$$

$$= \left(\frac{\frac{3}{2}}{s+1} + \frac{\frac{-3}{2}}{s+3}\right)e^{-2s} + \left(\frac{\frac{1}{3}}{s} + \frac{\frac{-1}{2}}{s+1} + \frac{\frac{1}{6}}{s+3}\right)e^{-s}$$

$$y(t) = \left[\frac{3}{2}e^{-(t-2)} - \frac{3}{2}e^{-3(t-2)}\right]H(t-2) + \left[\frac{1}{3} - \frac{1}{2}e^{-(t-1)} + \frac{1}{6}e^{-3(t-1)}\right]H(t-1)$$

(c)

$$F(s) = F(s)\frac{1}{s+1} + 3\frac{5!}{s^{5+1}}$$

$$\Rightarrow \frac{s}{s+1}F(s) = 3\frac{5!}{s^{5+1}}$$

$$F(s) = \frac{3\cdot 5!\cdot(s+1)}{s\cdot s^6} = \frac{3\cdot 5!}{s^6} + \frac{3\cdot 5!\cdot\frac{6!}{6!}}{s\cdot s^7}$$

$$f(t) = 3t^5 + \frac{1}{2}t^6$$

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8. (20%)

$$y' + y = 2x^2 + 3x + 1$$

(a) 求 x=2 的 Taylor 級數解(Find the Taylor series solution at x=2)

(b) 試著直接解方程式，驗證(a)的結果(Please verify the result in (a) by direct solving the differential equation)

Ans：

(a)

$$y(x) = \sum_{n=0}^{\infty} a_n (x-2)^n, |x-2| < \infty$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n (x-2)^{n-1}$$

$$\sum_{n=1}^{\infty} n a_n (x-2)^{n-1} + \sum_{n=0}^{\infty} a_n (x-2)^n = 2x^2 + 3x + 1$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} (x-2)^n + \sum_{n=0}^{\infty} a_n (x-2)^n = 2x^2 + 3x + 1$$

$$\sum_{n=0}^{\infty} [(n+1) a_{n+1} + a_n](x-2)^n = 2x^2 + 3x + 1 = m(x-2)^2 + n(x-2) + k$$

$$= 2(x-2)^2 + 11(x-2) + 15$$

$$n = 0, a_1 + a_0 = 15$$

$$n = 1, 2a_2 + a_1 = 11$$

$$n = 2, 3a_3 + a_2 = 2$$

$$n \ge 3, a_{n+1} = \frac{-1}{n+1} a_n$$

$$a_2 = 2 - 3a_3$$

$$a_1 = 11 - 2a_2 = 11 - 2(2 - 3a_3) = 7 + 6a_3$$

$$a_0 = 15 - a_1 = 1 - (7 + 6a_3) = 8 - 6a_3$$

$$a_4 = \frac{-1}{4} a_3$$

$$a_5 = \frac{-1}{5*4} a_3$$

$$a_n = \frac{6(-1)^{n-3}}{n!} a_3$$

$$\therefore y(x) = a_0 + a_1(x-2) + a_2(x-2)^2 + \cdots$$

$$= 8 - 6a_3 + (7 + 6a)(x-2) + (2 - 3a_3)(x-2)^2 + \cdots + \frac{6(-1)^{n-3}}{n!} a_3 (x-2)^n + \cdots$$

$$= 8 + 7(x-2) + 2(x-2)^2 - 6a_3[1 - (x-2) + \frac{1}{2}(x-2)^2 + \cdots + \frac{(-1)^n}{n!}(x-2)^n + \cdots]$$

$$= 8 + 7(x-2) + 2(x-2)^2 - 6a_3 e^{-(x-2)}$$

(b)

$$y' + y = 2x^2 + 3x + 1 = 2(x-2)^2 + 11(x-2) + 15$$

令$t = x - 2$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}$$

$$\Rightarrow y'(t) + y(t) = 2t^2 + 11t + 15$$

$$y(t) = y_h + y_p = Ce^{-t} + y_p$$

$$y_p = \frac{1}{D+1}(2t^2 + 11t + 15)$$

$$= (2t^2 + 11t + 15) - (4t + 11) + 4$$

$$= 2t^2 + 7\mathrm{t} + 8$$

$$y(t) = Ce^{-t} + 2t^2 + 7\mathrm{t} + 8$$

$$y(x) = Ce^{-t} + 2(x-2)^2 + 7(x-2) + 8$$