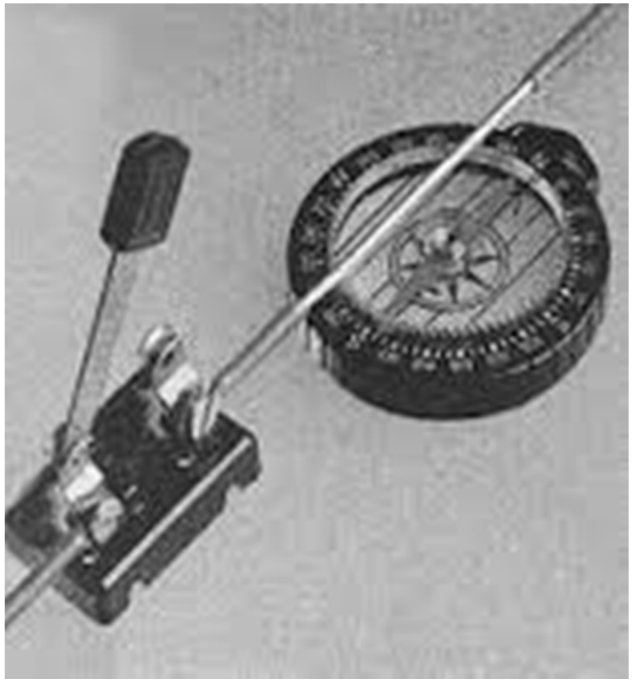
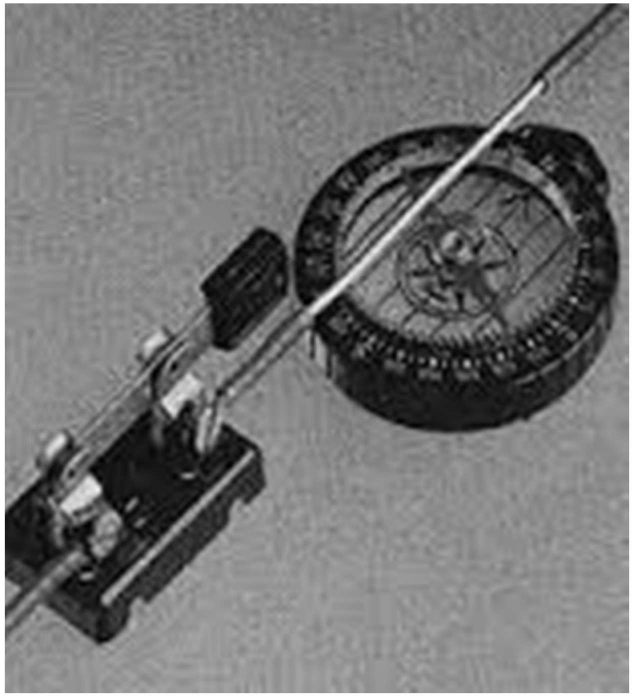
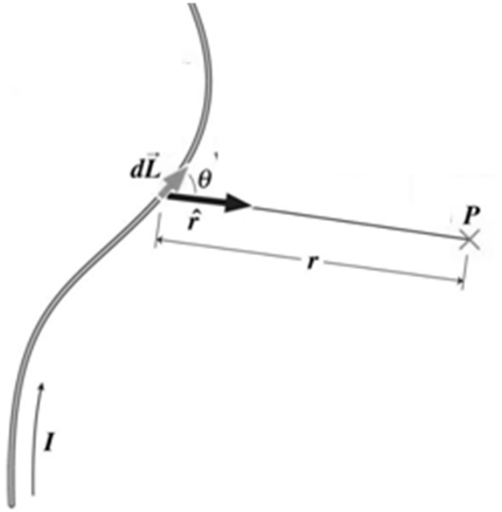
Magnetic Field



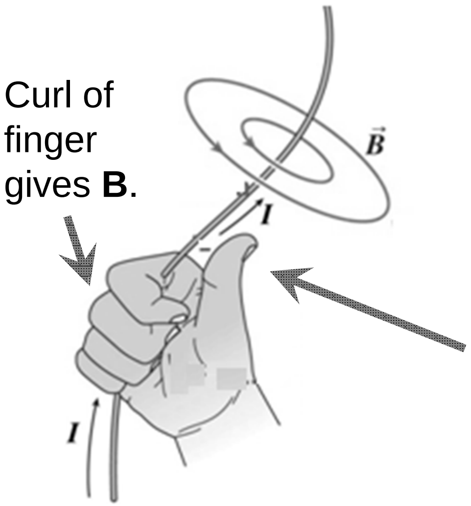


Origin of the Magnetic Field

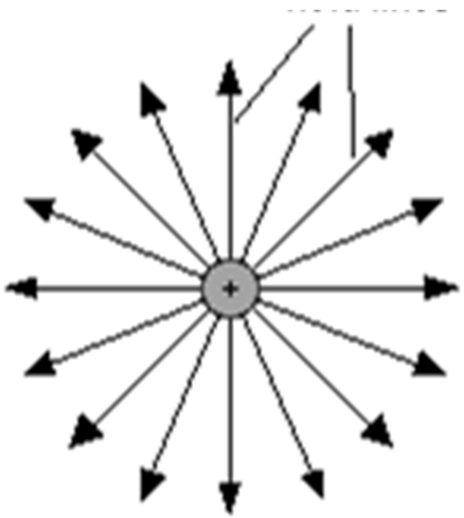


Biot-Savart Law:

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I\, d\mathbf{L} \times \hat{\mathbf{r}}}{r^2}$$



Origin of the Electric Field

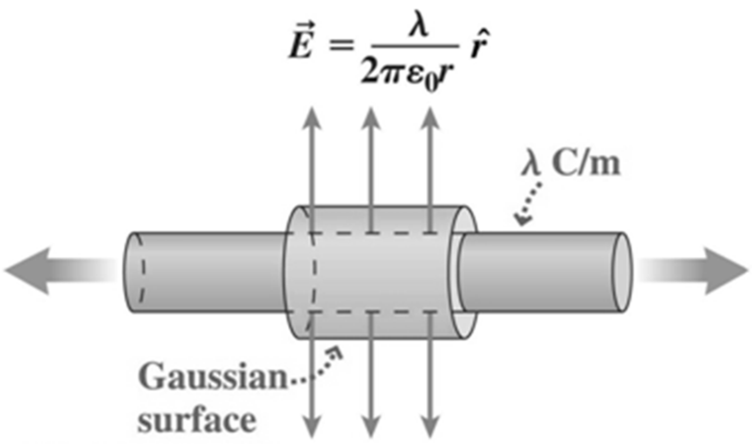


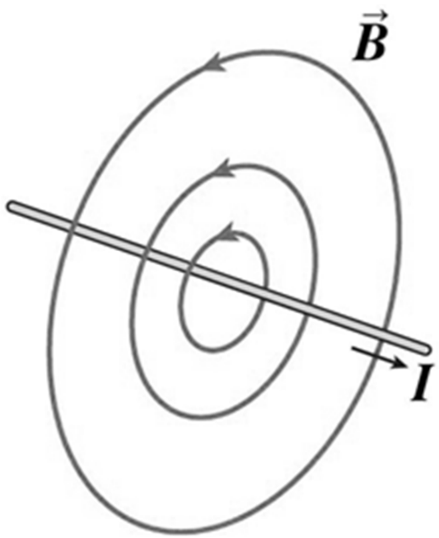
Comparison with Electric Field

Source of an electric field: ΔQ

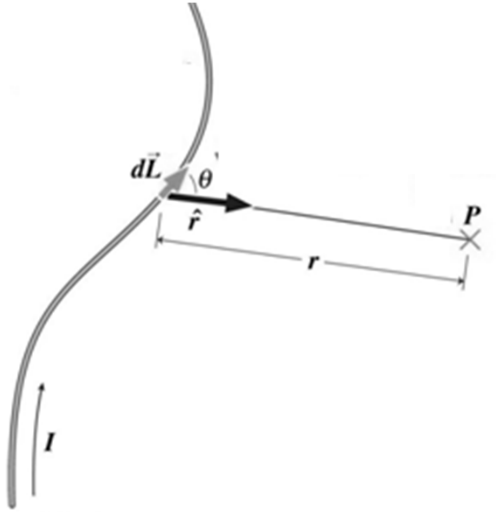
Field from the source:

$$\Delta E = \frac{k\Delta Q}{r^2}\,\hat{r} = \frac{1}{4\pi\epsilon_0}\frac{\Delta Q}{r^2}\,\hat{r}$$


$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$
λ C/m
Gaussian
surface


$\vec{B}$
I

Origin of the Magnetic Field



Comparison with Electric Field

Source of a magnetic field

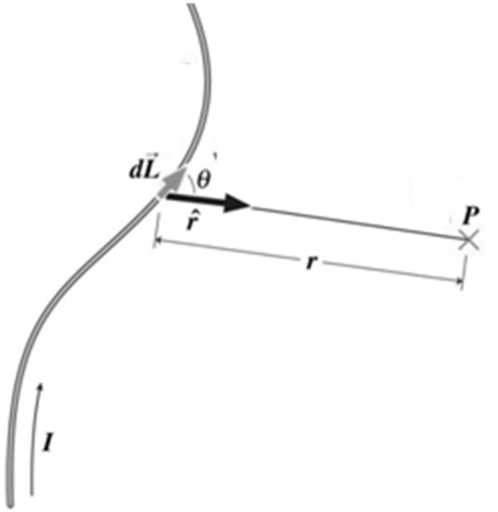
$$I\Delta L$$

Field from the source:

$$\Delta B = \frac{\mathrm{k_m}(I\Delta L)}{r^2}\left(\hat{L} \times \hat{r}\right)$$

$$= \frac{\mu_0}{4\pi}\frac{I\Delta L}{r^2}\left(\hat{L} \times \hat{r}\right)$$

Origin of the Magnetic Field



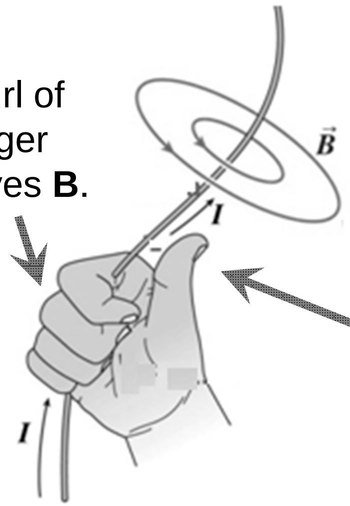
Biot-Savart Law: $$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I \, d\mathbf{L} \times \hat{\mathbf{r}}}{r^2}$$

$$\frac{\mu_0}{4\pi} = 10^{-7} \; N / A^2 \quad = 10^{-7} \; T \cdot m / A$$

permeability constant

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{I \, d\mathbf{L} \times \hat{\mathbf{r}}}{r^2} \quad = \frac{\mu_0}{4\pi} \int d^3 r \; \frac{\mathbf{J} \times \hat{\mathbf{r}}}{r^2}$$

Curl of finger gives **B**.



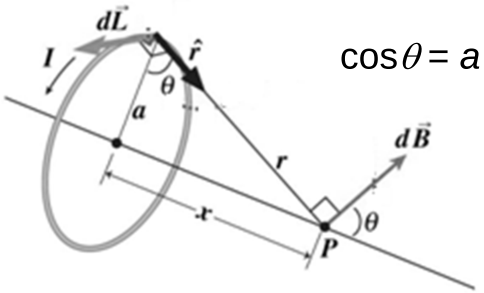
Thumb // *I*

C.f. $$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \int d^3 r \; \frac{\rho}{r^2} \, \hat{\mathbf{r}}$$

Current Loop

Find the magnetic field at an arbitrary point *P* on the axis of a circular loop of radius *a* carrying current *I*.

d**L** $\perp$ **r**



$\cos\theta = a\,/\,r$

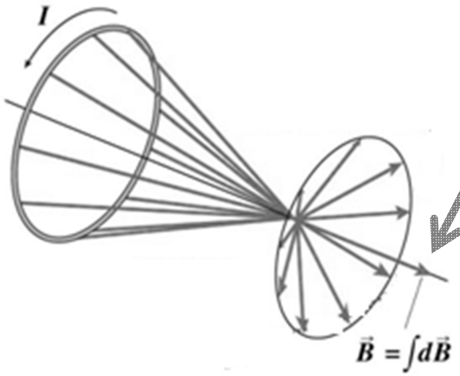
$$\mathbf{B} = \frac{\mu_0}{4\pi}\int \frac{I\, d\mathbf{L}\times\hat{\mathbf{r}}}{r^2}$$

By symmetry, only $B_x \neq 0$.

$$\left(d\mathbf{L}\times\hat{\mathbf{r}}\right)_x = \cos\theta\, dL \;= \frac{a}{r}\, dL$$

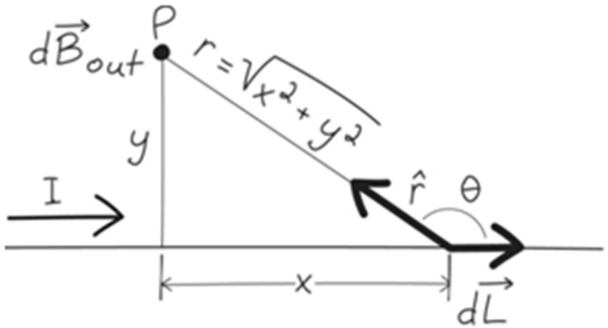
$$B_x = \frac{\mu_0}{4\pi}\int \frac{I\,a}{r^3}\, dL \;= \frac{\mu_0}{4\pi}\,\frac{I\,a}{r^3}\left(2\,\pi\,a\right) = \frac{\mu_0}{2}\,\frac{I\,a^2}{r^3}$$

$$= \frac{\mu_0}{2}\,\frac{I\,a^2}{\left(x^2+a^2\right)^{3/2}}$$



Straight Line

Find the magnetic field produced by an infinitely long straight wire carrying current *I*.



$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{I \, d\mathbf{L} \times \hat{\mathbf{r}}}{r^2}$$

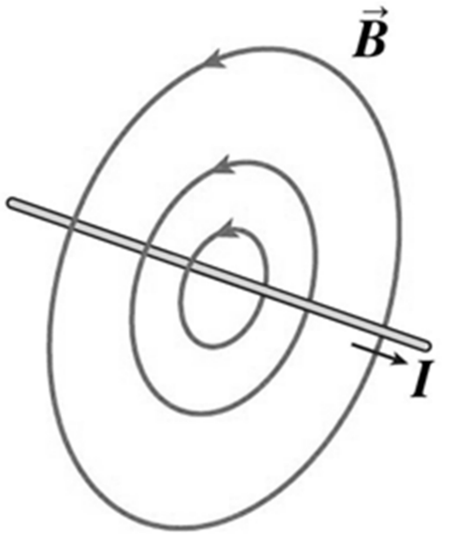
$$d\mathbf{L} \times \hat{\mathbf{r}} = \sin\theta \, dL \, \hat{\mathbf{z}} \quad = \frac{y}{r} dL \, \hat{\mathbf{z}}$$

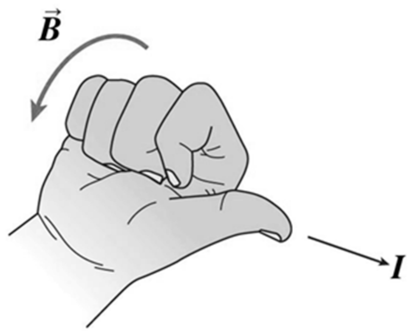
$$B_z = \frac{\mu_0}{4\pi} \int \frac{I \, y}{r^3} dL \quad = \frac{\mu_0}{4\pi} I \, y \int_{-\infty}^{\infty} \frac{1}{\left(x^2 + y^2\right)^{3/2}} dx$$

$$= \frac{\mu_0 \, I}{2\pi \, y}$$

$$\int \frac{1}{\left(x^2 + a^2\right)^{3/2}} dx = \frac{x}{a^2 \sqrt{x^2 + a^2}}$$

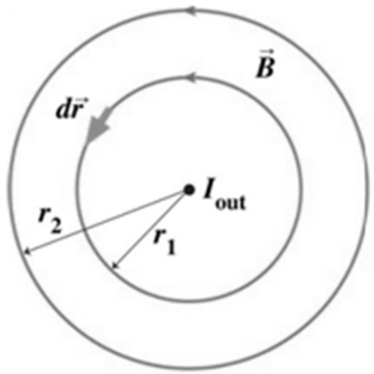
$$\lim_{x \to \pm\infty} \frac{x}{a^2 \sqrt{x^2 + a^2}} = \lim_{x \to \pm\infty} \frac{x}{a^2 |x|} \quad = \pm \frac{1}{a^2}$$





$$\boxed{\mathbf{B} = \frac{\mu_0 \, I}{2\pi \, d} \hat{\boldsymbol{\varphi}}}$$

Ampere’s law



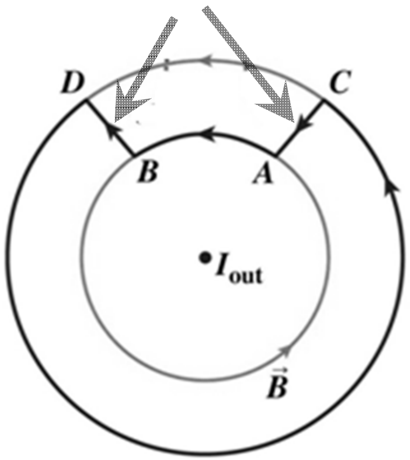
Field around long wire carrying current *I* :

$$\mathbf{B} = \frac{\mu_0\, I}{2\,\pi\, r}\,\hat{\boldsymbol{\varphi}}$$ from Biot-Savart law

$$\rightarrow \boxed{\oint \vec{B}\cdot d\vec{r}} = \int_0^{2\pi} \frac{\mu_0\, I}{2\,\pi\, r}\, r\, d\varphi \quad = \mu_0\, I$$

True for arbitrary closed paths & steady currents:

B·d**r** = 0

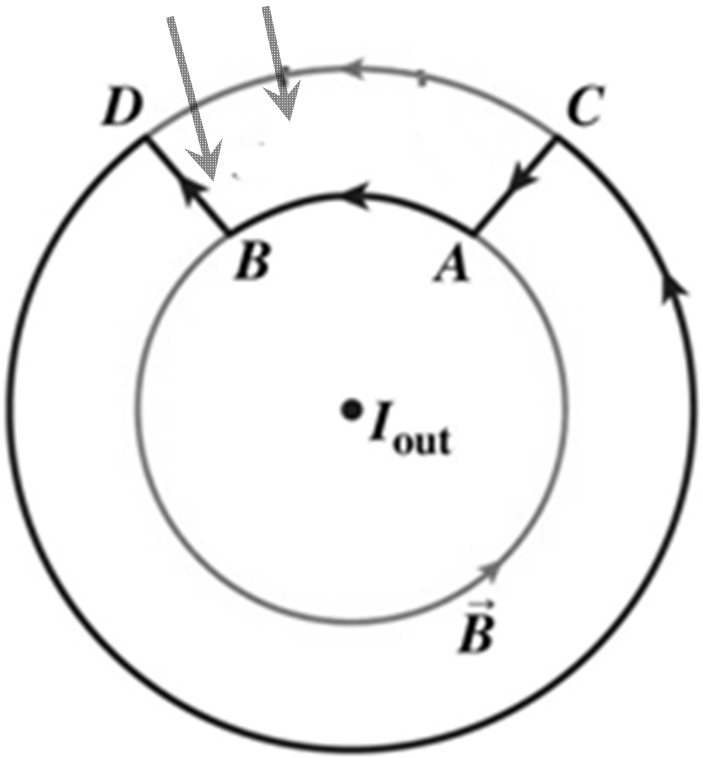


$$\boxed{\oint \vec{B}\cdot d\vec{r}} = \mu_0\, I_{enclosed}$$ Ampere’s law

net field from ALL sources

Ampere's law

B·d**r** = 0



Using Ampere's Law

STRATEGY Ampère's Law :

Base on symmetry, choose the amperian loop such that **B** is either // or $\perp$ to it.

Outside & Inside a Wire

A long, straight wire of radius *R* carries a current *I* distributed uniformly over its cross section.

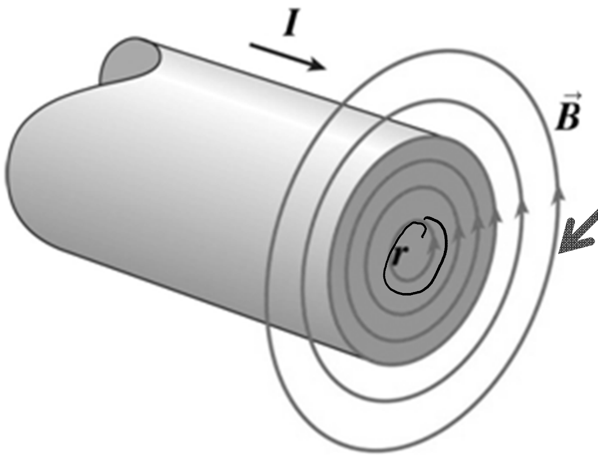
Find the magnetic field

(a) outside and

(b) inside the wire.

By symmetry, **B** is azimuthal.

Amperian loop is a circle.



$$\boxed{\oint \vec{B} \cdot d\vec{r}} = 2\,\pi\,r\,B$$

(a) $I_{enclosed} = I$ $\qquad B = \dfrac{\mu_0\, I}{2\,\pi\,r}$

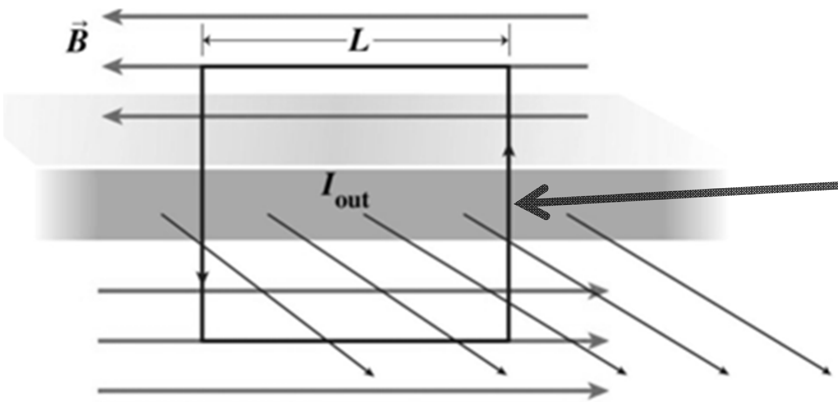
(b) $I_{enclosed} = I\,\dfrac{\pi\,r^2}{\pi\,R^2}$ $\qquad B = \dfrac{\mu_0\, I\, r}{2\,\pi\,R^2}$

Current Sheet

An infinite flat sheet carries current out of this page.

The current is distributed uniformly along the sheet, with current per unit width given by J_S .

Find the magnetic field of this sheet.



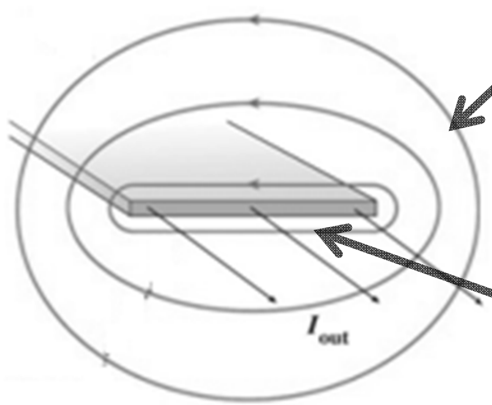
By symmetry, **B** is // to sheet & $\perp$ *I*.

Amperian loop is a rectangle.

$$\boxed{\oint \vec{B} \cdot d\vec{r}} = 2\,B\,L$$

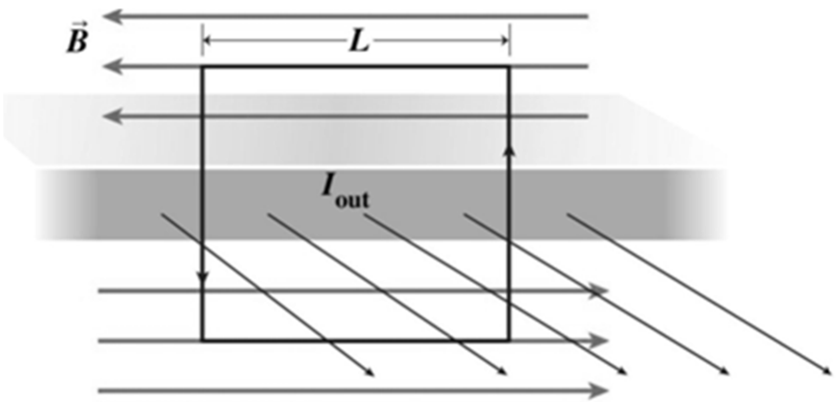
$$I_{enclosed} = J_S\,L$$

$$B = \frac{1}{2}\mu_0 J_S$$

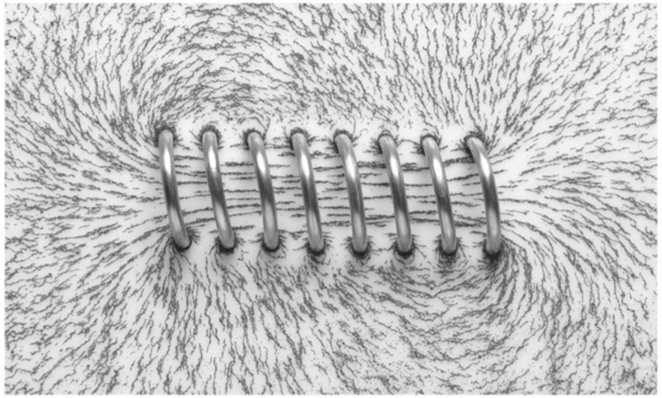


Far out, **B** lines nearly circular

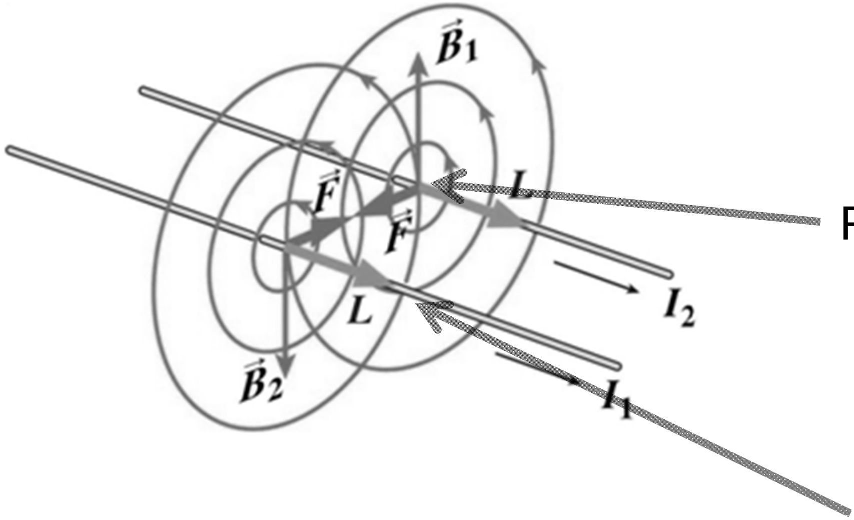
Close in, **B** lines ~ infinite sheet



$\vec{B}$
L
I_{out}



The Magnetic Force Between Conductors



$$\mathbf{F} = I\ \mathbf{L} \times \mathbf{B}$$

Field of I_1 at I_2 : $\mathbf{B}_1 = \dfrac{\mu_0\ I_1}{2\ \pi\ d}\ \hat{\mathbf{b}}_1$

Force on length L of I_2 :

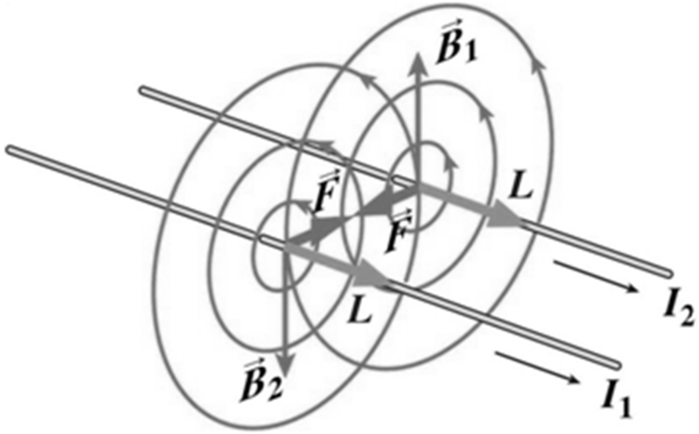
$$\mathbf{F}_2 = I_2\ \mathbf{L} \times \mathbf{B}_1 = \frac{\mu_0\ I_1\ I_2}{2\ \pi\ d}\ L\ \hat{\mathbf{d}}$$

Force per unit length on I_2 : $\mathbf{f}_2 = \dfrac{\mathbf{F}_2}{L} = \dfrac{\mu_0\ I_1\ I_2}{2\ \pi\ d}\ \hat{\mathbf{d}}$ points toward I_1

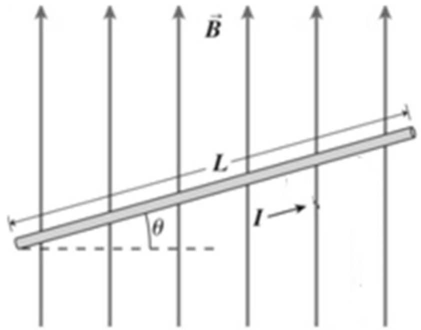
Hum from electric equipments are vibrations of transformers in response to AC.

Definition: 1 A is the current in two long, parallel wire 1 m apart & exerting 2×10^{-7} N per meter of length.
1C is the charge passing in 1s through a wire carrying 1A.

The Magnetic Force Between Conductors



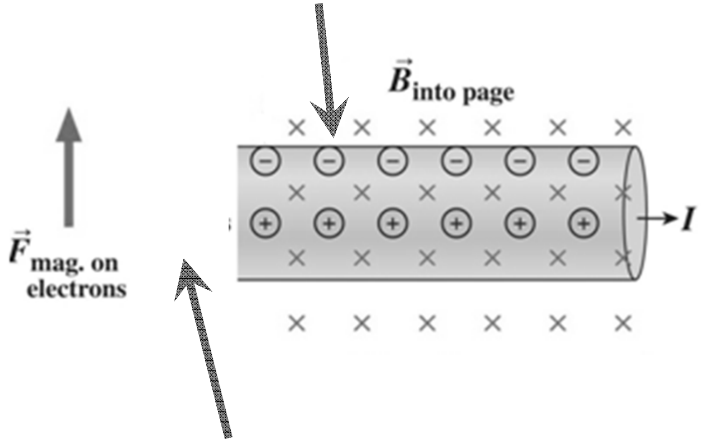
The Magnetic Force on a moving charge



F out of paper

Force on carrier in wire: $\mathbf{f} = q\,\mathbf{v}_d \times \mathbf{B}$

e moving left deflected upward …

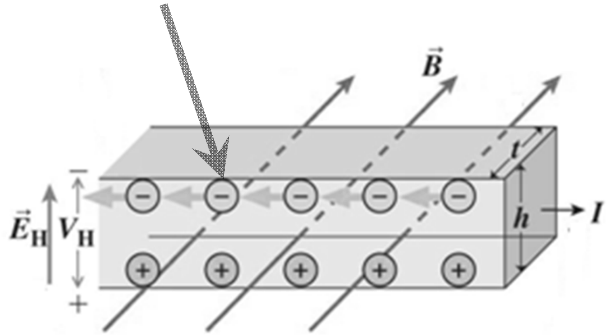


… resulting charge separation leads to upward force on whole wire

$\mathbf{F}_{mag}$ on + charge is also upward

The Hall Effect

e moves to left &
deflected upward

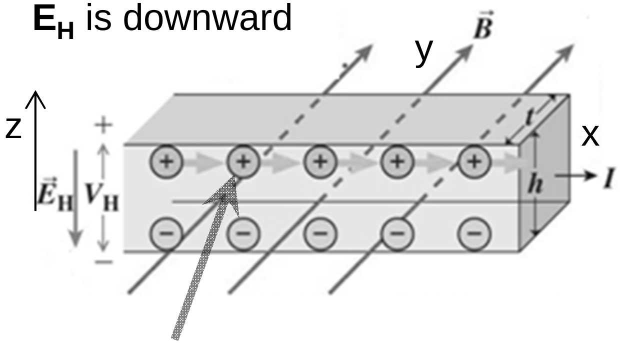


$\mathbf{E_H}$ is upward

Direction of $\mathbf{F_B}$ depends on **I**,
not on sign of charged carriers.

Carriers of both signs are deflected upwards

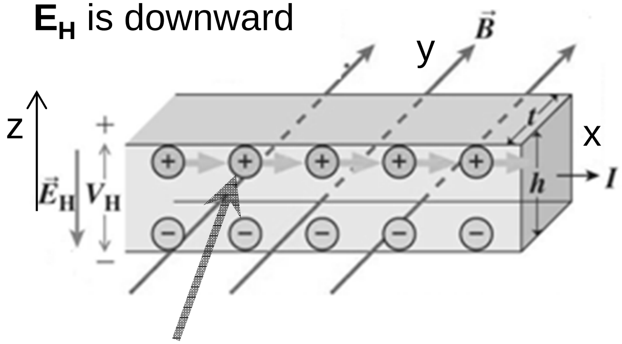
$\mathbf{E_H}$ is downward



p moves to right &
deflected upward

Direction of **E** due to built-up charges
depends on signs of charged carriers:
Hall effect.

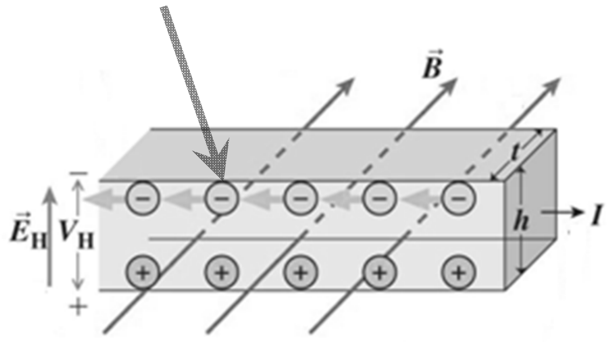
E_H is downward



p moves to right &
deflected upward

The Hall Effect

e moves to left &
deflected upward

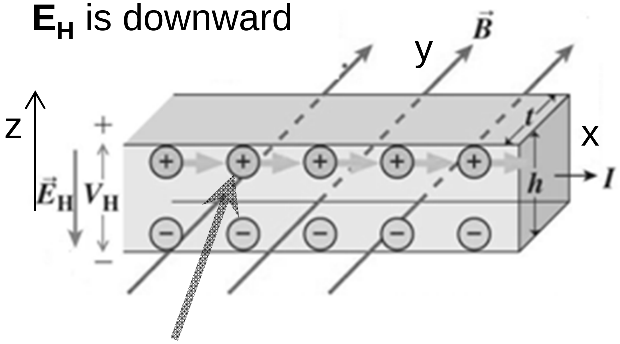


E_H is upward

Steady state, $F_z = 0$: $q\,E_H + q\,v_d\,B = 0$

$$\rightarrow \quad E_H = -\,v_d\,B$$

E_H is downward



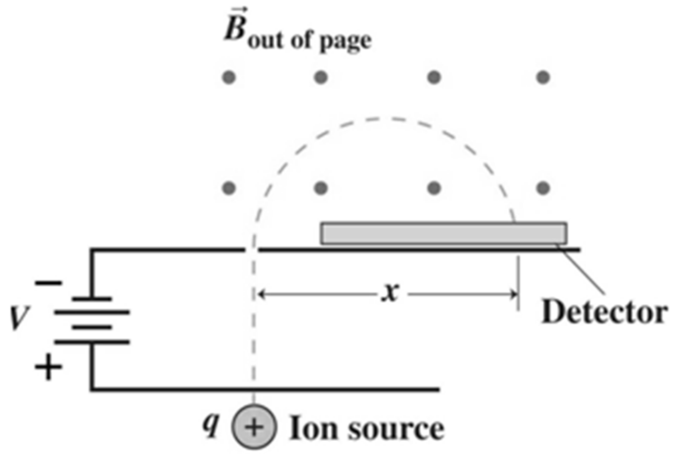
p moves to right &
deflected upward

Hall potential: $V_H = v_d\,B\,h \;=\; \dfrac{I}{n\,q\,A}\,B\,h \;=\; \dfrac{I}{n\,q\,t}\,B$

Hall coefficient: $R_H = \dfrac{1}{n\,q}$

A mass spectrometer

Two isotopes of an element with masses $m_1 \; and \; m_2$ are accelerated by a potential difference V. They then enter a uniform field B normal to the magnetic field lines. What is the ratio of the radii of their paths



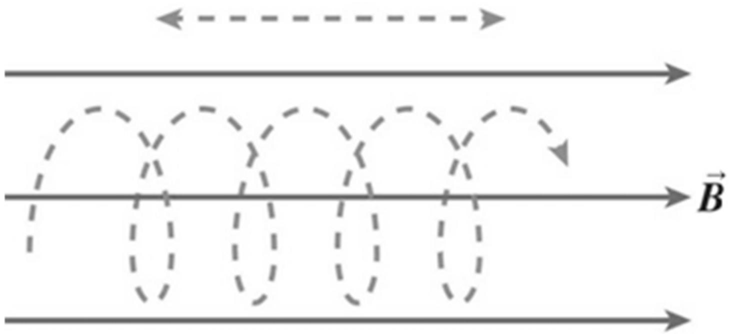
The Cyclotron

Period of particle in circular orbit in uniform **B**:

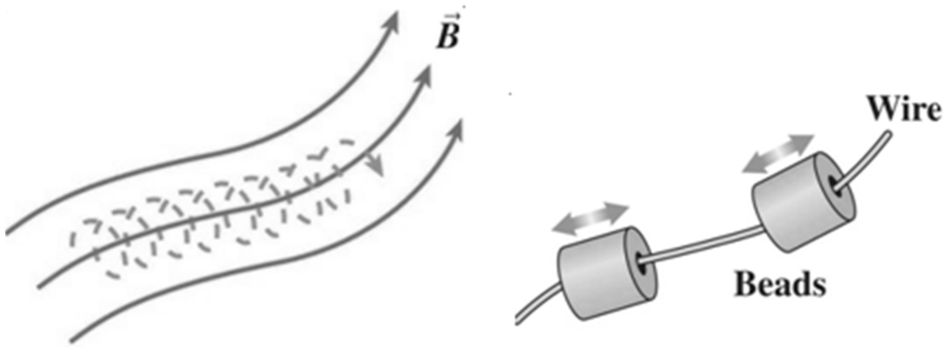
$$T = \frac{2\pi r}{v} = \frac{2\pi}{v}\frac{mv}{qB} = \frac{2\pi m}{qB}$$

$$\omega = \frac{2\pi}{T} = \frac{qB}{m}$$ Cyclotron frequency

Motion // **B** not affected by it

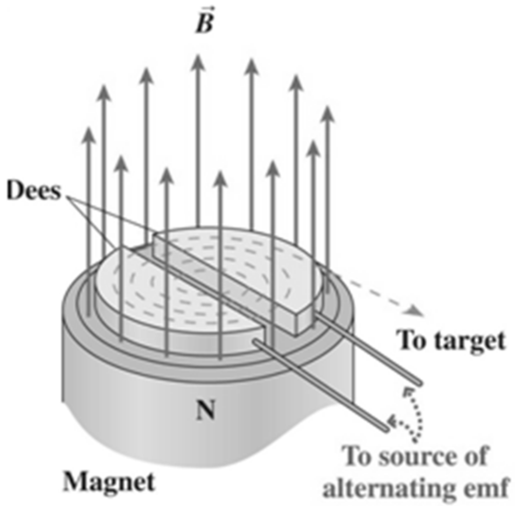


Trajectory in 3-D



Charged particles frozen to **B** field lines.

Application: The Cyclotron

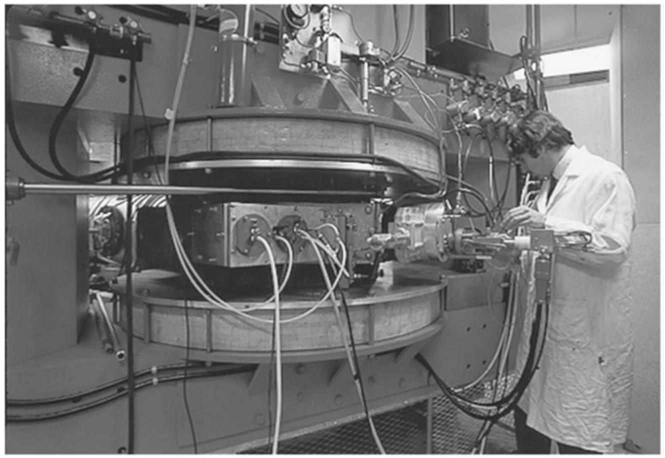


Whole device in vacuum chamber.

Small V across the dee's, which alternates polarities at the cyclotron frequency.

Particle injected at center of gap & spirals out.

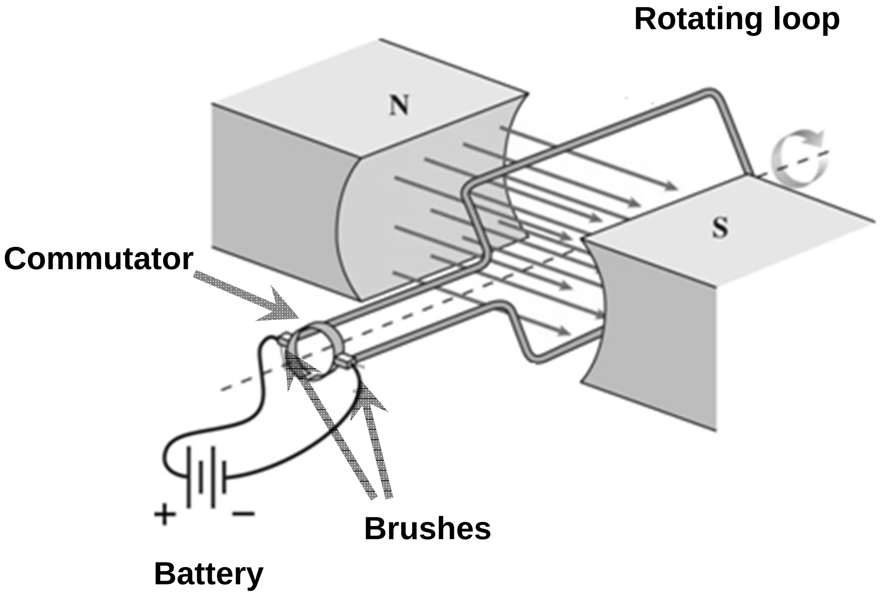
E ~ MeVs.

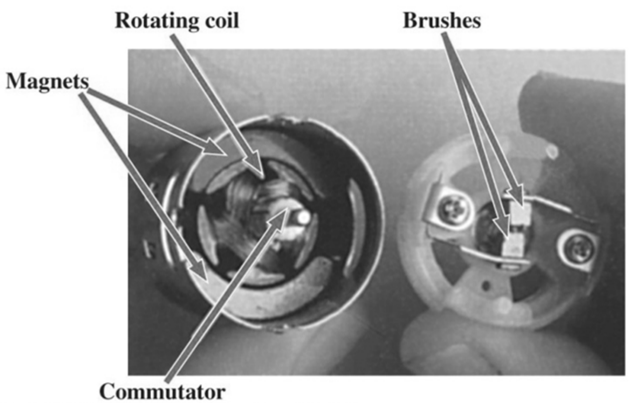


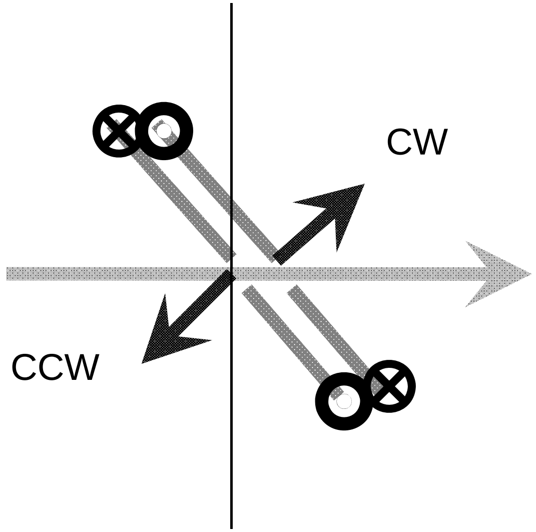
Applications:
Manufacture of radioactive isotopes.
e.g., PET (Positron Emission Tomography).

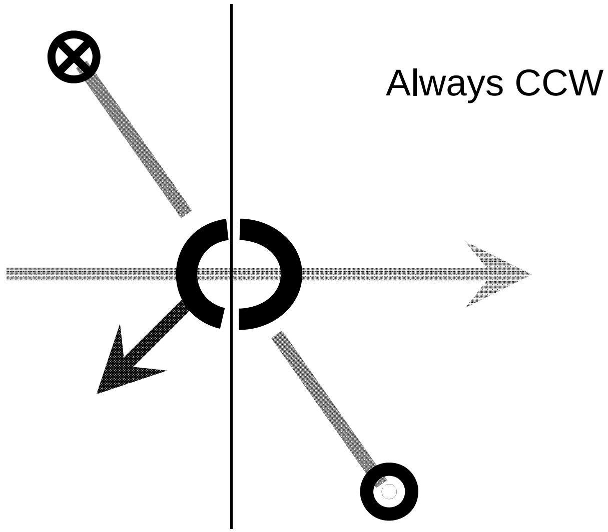
Higher energies:
Relativity effects → Synchrotron (**B** also varies)

Application: Electric Motors

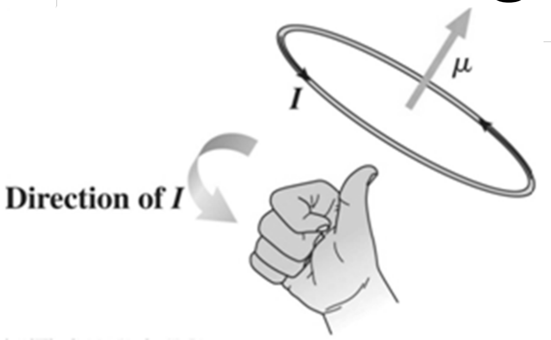






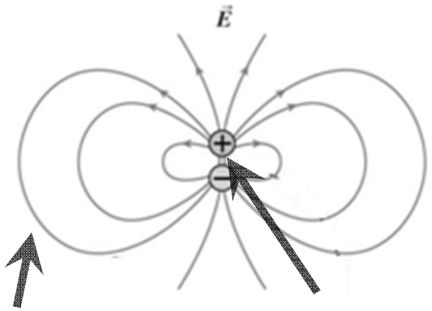


26.6. Magnetic Dipoles



Field on axis of current loop of radius *a* :

$$B_x = \frac{\mu_0}{2}\frac{I\,a^2}{\left(x^2+a^2\right)^{3/2}} \xrightarrow[x\ ?\ a]{} \frac{\mu_0}{2}\frac{I\,a^2}{x^3} = \frac{\mu_0}{2}\frac{I\,A}{\pi\,x^3}$$

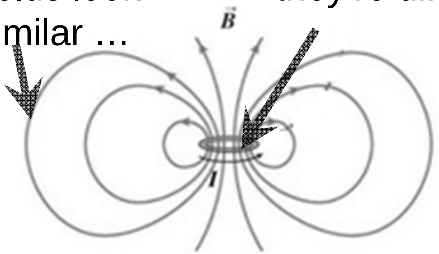


C.f. electric dipole: $E = 2k\,\frac{p}{x^3}$

Setting $k \leftrightarrow \frac{\mu_0}{4\pi}$ $\rightarrow$ $p \leftrightarrow \mu = I\,A$ magnetic dipole

Far away, fields look similar …

… but close in, they're different.



N-turn current loop: $\boldsymbol{\mu} = N\,I\,\mathbf{A}$

Detailed calculations show:

- $\mu = I\,A$ valid for arbitrary loop.
- Vector behavior of μ similar to that of p for $r \gg a$.

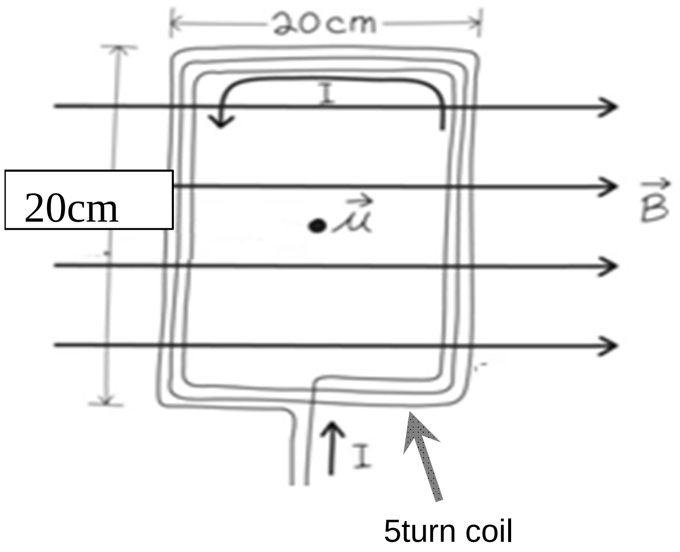
Multi-turn loops = electromagnets

Very strong field requires superconducting wires, e.g., MRI scanners.

Orbiting e in atom $\rightarrow$ μ.

Example

A square loop has sides of length 20cm. It has 5 turns and carries a current of 2A. Find: (a) The magnetic moment; (b) the torque on the loop if the B field is at 37^0 to the direction of the normal to the loop. (c) The work needed to rotate the loop from its position of minimum energy to the given orientation



Symmetric Counterparts of Electrostatics & Magnetostatics

Physical Quantity	*Electrostatics*	*Magnetosatics (steady current)*
Source:	ΔQ	$\Delta(I * \vec{L}) = I\Delta\vec{L}$
Field:	$\Delta \mathrm{E} = \mathrm{K_E}\left(\frac{\Delta Q}{r^2}\right)\cdot\hat{r}$ $\vec{E} = \int \mathrm{K_E}\left(\frac{\mathrm{d}Q}{r^2}\right)\cdot\hat{r}$	$\Delta B = K_B\left(\frac{I\Delta(\vec{L})}{r^2}\right)\times\hat{r}$ $\vec{B} = \int K_B\left(\frac{I\mathrm{d}\vec{\mathrm{L}}}{r^2}\right)\times\hat{r}$
Force:	$\vec{F} = \Delta Q\cdot\vec{E}$	$\vec{F} = \Delta(I * \vec{L})\times\vec{B}$
Potential:	$\Delta V = -\int_{r_A}^{r_B}\vec{E}\cdot d\vec{r}$	"Vector Potential" -- Complicated
Dipole:	$\vec{p} = Qd$	$\vec{\mu} = NI\vec{\mathrm{A}}$
Source moving at a constant velocity	$I = \frac{\Delta Q}{\Delta t} = nAq\overrightarrow{v_d}$	Not Applicable

$$K_E = \frac{1}{4\pi\epsilon_0} = 9\times 10^9\ \frac{N\cdot m^2}{C^2} \qquad K_B = \frac{\mu_0}{4\pi} = 10^{-7}\,\frac{T\cdot A}{m}$$

Symmetric Counterparts of Electrostatics & Magnetostatics

Dipole in a homogeneous field		
Torque:	$\vec{\tau} = \vec{p} \times \vec{E}$	$\vec{\tau} = \vec{\mu} \times \vec{B}$
Potential Energy:	$U = -\vec{p} \cdot \vec{E}$	$U = -\vec{\mu} \cdot \vec{B}$
Gauss' s Law in Vacuum	$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$	$\oint \vec{B} \cdot d\vec{A} = 0$
Ampère' s Law in Vacuum	Not Applicable	$\oint \vec{B} \cdot d\vec{r} = \mu_0 \int \vec{I} \cdot d\vec{A} = \mu_0 I_{enc}$