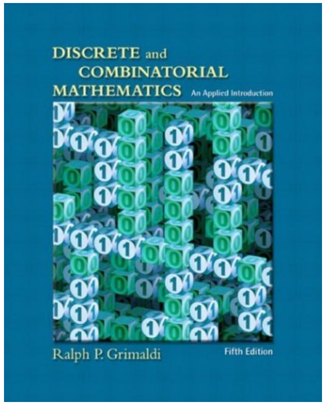
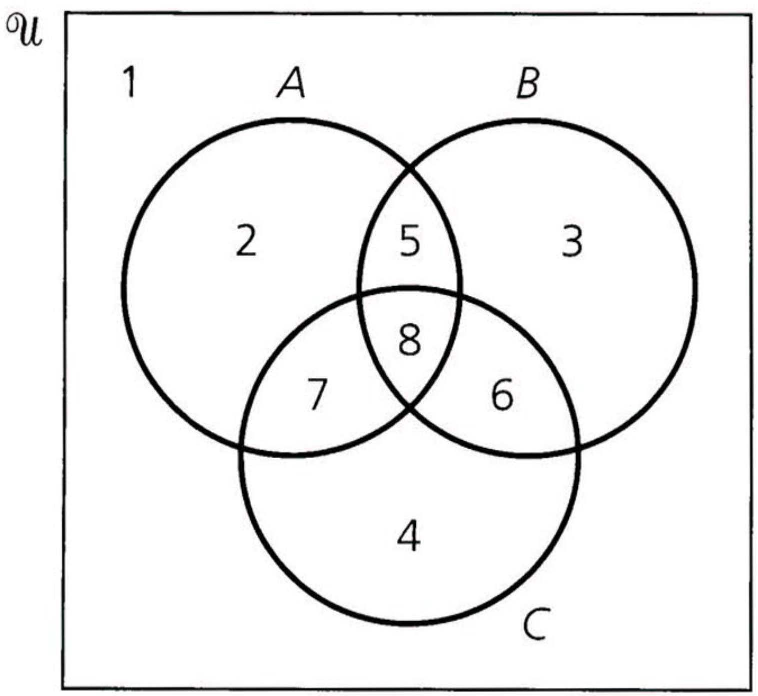
Discrete Mathematics
-- *Chapter 3: Set Theory*



Hung-Yu Kao (高宏宇)
Department of Computer Science and Information Engineering,
National Cheng Kung University

Outline

Why Set?



- Sets are the most simple, yet non-trivial structures in mathematics.
  - Many other mathematical objects and properties can be defined by them.
- For us, sets are useful to understand the principles of counting and probability theory.

3.1 Set and Subsets

- **Set**: should be a well-defined collection of objects.
- **Elements** (**members**): These objects are called elements or members of the set.
  - could be another set, $1 \neq \{1\} \neq \{\{1\}\}$
- Capital letters represent sets: A, B, C
  lowercase letters represent elements: x, y
  - E.g., $x \in A$, $y \notin B$
- A set can be designated by listing its elements within set braces "{","}".
  - E.g., $A = \{1, 2, 3, 4, 5\}$, $B = \{x \mid x \text{ is an integer, and } 1 \leq x \leq 5\}$
- **Cardinality** (**size**): $|A|$ denotes the number of elements in A.
  - for finite sets

Set and Subsets

- **Universe** (**Universe of discourse**): $\mathscr{U}$ denotes the range of all elements to form any set.
- Definition 3.1: If C and D are sets from a universe $\mathscr{U}$
  - **Subset**: $C \subseteq D$ $(D \supseteq C)$, if every element of C is an element of D.
  - **Proper subset**: $C \subset D$ $(D \supset C)$, if, in addition, D contains an element that is not in C.
  - $C \subseteq D \Leftrightarrow \forall x\,[x \in C \Rightarrow x \in D]$

$C \not\subseteq D$ (i.e., C is not a subset of D)

$\Leftrightarrow \neg \forall x\,[x \in C \Rightarrow x \in D]$

$\Leftrightarrow \exists x \neg\,[x \in C \Rightarrow x \in D]$

$\Leftrightarrow \exists x \neg\,[\neg(x \in C) \vee x \in D]$

$\Leftrightarrow \exists x\,[\neg\neg(x \in C) \wedge \neg(x \in D)]$

$\Leftrightarrow \exists x\,[x \in C \wedge x \notin D]$

Set and Subsets

- Definition 3.2: The sets C and D are equal for a given universe $\mathscr{U}$,
$$C = D \Leftrightarrow (C \subseteq D) \wedge (D \subseteq C)$$
- Let A, B, $C \subseteq \mathscr{U}$,
  a) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$
  b) If $A \subset B$ and $B \subseteq C$, then $A \subset C$
  c) If $A \subseteq B$ and $B \subset C$, then $A \subset C$
  d) If $A \subset B$ and $B \subset C$, then $A \subset C$
- Let $\mathscr{U} = \{1, 2, 3, 4, 5\}$ with $A = \{1, 2, 3\}$, $B = \{3, 4\}$, and $C = \{1, 2, 3, 4\}$. Then the following subset relations hold:
  - a) $A \subseteq C$ b) $A \subset C$ c) $B \subset C$
    d) $A \subseteq A$ e) $B \nsubseteq A$ f) $A \not\subset A$ ← *A is not a proper subset of A*

Set and Subsets

- **Null set** (**empty set**), $\varnothing$ or { }: is the set containing no elements.
  - $|\varnothing|=0$ but $\{0\} \neq \varnothing$
  - $\varnothing \neq \{\varnothing\}$
- **Power set**, $P(A)$: is the collection (set) of all subsets of the set A from universe $\mathscr{U}$.
- Example: $A = \{1, 2, 3\}$
  - $P(A) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, A\}$
- For any finite set A with $|A|=n$
  - A has 2^n subsets and $|P(A)|= 2^n$
  - There are $\binom{n}{k}$ subsets of size k, $0 \leq k \leq n$
  - Counting the subsets of A (binomial theorem)

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Set and Subsets

- Theorem 3.2

For any universe $\mathscr{U}$, let $A \subseteq \mathscr{U}$. Then $\emptyset \subseteq A$, and if $A \neq \emptyset$, then $\emptyset \subset A$.

Proof: If the first result is not true, then $\emptyset \nsubseteq A$, so there is an element x from the universe with $x \in \emptyset$ but $x \notin A$. But $x \in \emptyset$ is impossible. So we reject the assumption $\emptyset \nsubseteq A$ and find that $\emptyset \subseteq A$. In addition, if $A \neq \emptyset$, then there is an element $a \in A$ (and $a \notin \emptyset$), so $\emptyset \subset A$.

$\phi \subseteq \{\phi\}$? T $\phi \subset \{\phi\}$? T

$\phi \subseteq \phi$? T $\phi \subset \phi$? F

Set and Subsets

- **Ex 3.11**

2^6

Table 3.1

	Composition of 7		Determining Subset of {1, 2, 3, 4, 5, 6}
(i)	$1+1+1+1+1+1+1$	(i)	$\emptyset$
(ii)	$1+2+1+1+1+1$	(ii)	$\{2\}$
(iii)	$1+1+3+1+1$	(iii)	$\{3, 4\}$
(iv)	$2+3+2$	(iv)	$\{1, 3, 4, 6\}$
(v)	$4+3$	(v)	$\{1, 2, 3, 5, 6\}$
(vi)	7	(vi)	$\{1, 2, 3, 4, 5, 6\}$

Set and Subsets

- **Ex 3.12**

A={x, 1, 2, 3, 4}, r=3

{x, 1, 2}	{1, 2, 3}
{x, 1, 3}	{1, 2, 4}
{x, 1, 4}	{1, 3, 4}
{x, 2, 3}	{2, 3, 4}
{x, 2, 4}	
{x, 3, 4}	

C(4, 2) + C(4, 3) = C(5,3)

Let $A = \{x, a_1, a_2, \cdots, a_n\}$ and consider all subsets of A that contain r elements.

There are $\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$ subsets.

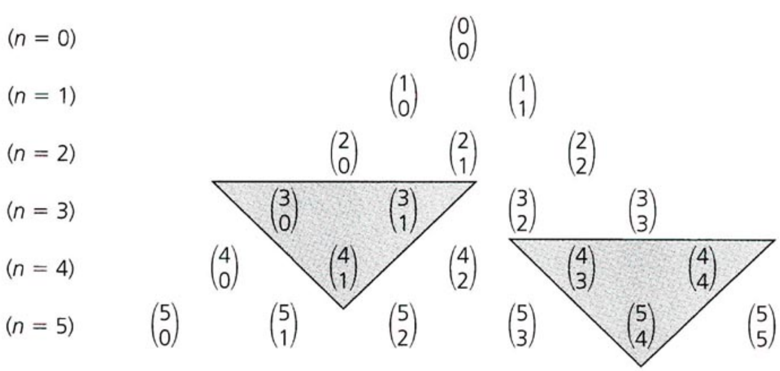


Figure 3.3

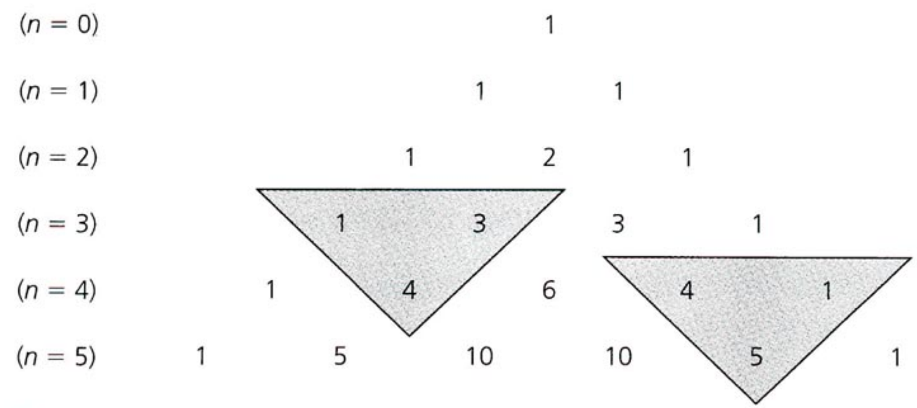


Figure 3.4

Prove C(2n, 2) = 2C(n, 2)+n (95NTHU)

Prove nC(n-1 r) = (r+1)C(n, r+1)

Set and Subsets

- **Ex 3.13** the number of nonnegative integer solutions of the inequality $x_1 + x_2 + \cdots + x_6 < 10$

$\forall k, 0 \le k \le 9$, the number of solution to $x_1 + x_2 + \cdots + x_6 = k$ is

$$\binom{6+k-1}{k} = \binom{5+k}{k}$$

in chapter 1, $\binom{7+9-1}{9} = \binom{15}{9}$

$$\binom{5}{0} + \binom{6}{1} + \binom{7}{2} + \binom{8}{3} + \cdots + \binom{14}{9}$$

$$= \left[\binom{6}{0} + \binom{6}{1}\right] + \binom{7}{2} + \binom{8}{3} + \cdots + \binom{14}{9}, \quad \text{since } \binom{5}{0} = 1 = \binom{6}{0}$$

$$= \left[\binom{7}{1} + \binom{7}{2}\right] + \binom{8}{3} + \cdots + \binom{14}{9}, \quad \text{since } \binom{6}{0} + \binom{6}{1} = \binom{7}{1}$$

$$= \left[\binom{8}{2} + \binom{8}{3}\right] + \binom{9}{4} + \cdots + \binom{14}{9}, \quad \text{since } \binom{7}{1} + \binom{7}{2} = \binom{8}{2}$$

$$= \left[\binom{9}{3} + \binom{9}{4}\right] + \cdots + \binom{14}{9} = \cdots = \binom{14}{8} + \binom{14}{9} = \binom{15}{9} = 5005.$$

Set and Subsets

?

N^+, N^0

a) $\mathbf{Z}$ = the set of *integers* = $\{0, 1, -1, 2, -2, 3, -3, \ldots\}$

b) $\mathbf{N}$ = the set of *nonnegative integers* or *natural numbers* = $\{0, 1, 2, 3, \ldots\}$

c) $\mathbf{Z}^+$ = the set of *positive integers* = $\{1, 2, 3, \ldots\} = \{x \in \mathbf{Z} \mid x > 0\}$

d) $\mathbf{Q}$ = the set of *rational numbers* = $\{a/b \mid a, b \in \mathbf{Z}, b \neq 0\}$

e) $\mathbf{Q}^+$ = the set of *positive rational numbers* = $\{r \in \mathbf{Q} \mid r > 0\}$

f) $\mathbf{Q}^*$ = the set of *nonzero rational numbers*

g) $\mathbf{R}$ = the set of *real numbers*

h) $\mathbf{R}^+$ = the set of *positive real numbers*

i) $\mathbf{R}^*$ = the set of *nonzero real numbers*

j) $\mathbf{C}$ = the set of *complex numbers* = $\{x + yi \mid x, y \in \mathbf{R}, i^2 = -1\}$

k) $\mathbf{C}^*$ = the set of *nonzero complex numbers*

l) For each $n \in \mathbf{Z}^+$, $\mathbf{Z}_n = \{0, 1, 2, \ldots, n-1\}$

m) For real numbers a, b with $a < b$, $[a, b] = \{x \in \mathbf{R} \mid a \leq x \leq b\}$, $(a, b) = \{x \in \mathbf{R} \mid a < x < b\}$, $[a, b) = \{x \in \mathbf{R} \mid a \leq x < b\}$, $(a, b] = \{x \in \mathbf{R} \mid a < x \leq b\}$. The first set is called a *closed interval*, the second set an *open interval*, and the other two sets *half-open intervals*.

○ $Z^+ \subseteq Q^+$

○ $R^+ \cap C = R^+$

✕ $R^+ \subseteq Q$

✕ $Q^* \cap Z = Z$

○ $Z^+ \cup R^+ = R^+$

3.2 Set Operations and the Laws of Set Theory

- Definition 3.5: For A and $B \subseteq \mathscr{U}$

a) $A \cup B$ (the union of A and B) $= \{x \mid x \in A \vee x \in B\}$

b) $A \cap B$ (the intersection of A and B) $= \{x \mid x \in A \wedge x \in B\}$

c) $A \Delta B$ (the symmetric difference of A and B)

$$= \{x \mid (x \in A \vee x \in B) \wedge x \notin A \cap B\} = \{x \mid x \in A \cup B \wedge x \notin A \cap B\}$$

- Definition 3.6: The sets S, $T \subseteq \mathscr{U}$, are called disjoint (mutually disjoint), when $S \cap T = \phi$.

Set Operations and the Laws of Set Theory

- Theorem 3.3: If S, $T \subseteq \mathscr{U}$ are disjoint if and only if $S \cup T = S \Delta T$
  - Proof: (1) $\underline{x \in S \cup T} \Rightarrow x \in S \cup x \in T$

    But S and T disjoint, i.e., $x \notin S \cap T, \Rightarrow \underline{x \in S \Delta T}$

    $\therefore S \cup T \subseteq S \Delta T$

    (2) $\underline{y \in S \Delta T} \Rightarrow y \in S \cup y \in T$

    $\Rightarrow \underline{y \in S \cup T}$

    $\therefore S \Delta T \subseteq S \cup T$

    $\because S \cup T \subseteq S \Delta T$ and $S \Delta T \subseteq S \cup T$

    $\therefore S \cup T = S \Delta T$

Prove the converse by the method of proof by contradiction

Set Operations and the Laws of Set Theory

- Definition 3.7: For a set $A \subseteq \mathscr{U}$, the **complement** of A, denoted $\mathscr{U} - A$ or $\overline{A}$, is given by $\{x \mid x \in \mathscr{U} \wedge x \notin A\}$
- Definition 3.8: For $A, B \subseteq \mathscr{U}$, the (relative) complement of A in B, denoted $B - A$ or, is given by $\{x \mid x \in B \wedge x \notin A\}$
- **Ex 3.18** : For $\mathscr{U} = \mathbf{R}$, $A = [1, 2]$, $B = [1, 3)$
  - a) $A = \{x \mid 1 \le x \le 2\} \subseteq \{x \mid 1 \le x < 3\} = B$

    b) $A \cup B = ? \ = \{x \mid 1 \le x < 3\} = B$

    c) $A \cap B = ? \ = \{x \mid 1 \le x \le 2\} = A$

    d) $\overline{B} = (-\infty, 1) \cup [3, +\infty) \subseteq (-\infty, 1) \cup (2, +\infty) = \overline{A}$
- Theorem 3.4: The following statements are equivalent:

$$\text{(a) } A \subseteq B \quad \text{(b) } A \cup B = B \quad \text{(c) } A \cap B = A \quad \text{(d) } \overline{B} \subseteq \overline{A}$$

Sets and Logic

Logic and set theory go very well together.

The previous definitions can be made very succinct:

- $x \notin A$ if and only if $\neg(x \in A)$
- $A \subseteq B$ if and only if $(x \in A \rightarrow x \in B)$ is True
- $x \in (A \cap B)$ if and only if $(x \in A \wedge x \in B)$
- $x \in (A \cup B)$ if and only if $(x \in A \vee x \in B)$
- $x \in A-B$ if and only if $(x \in A \wedge x \notin B)$
- $x \in A \Delta B$ if and only if $(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)$
- $x \in \overline{A}$ if and only if $\neg(x \in A)$
- $X \in P(A)$ if and only if $X \subseteq A$

The Laws of Set Theory

1) Law of *Double Complement* : $\overline{\overline{A}} = A$

2) *DeMogran's* Laws: $\begin{cases} \overline{A \cup B} = \overline{A} \cap \overline{B} \\ \overline{A \cap B} = \overline{A} \cup \overline{B} \end{cases}$ 3) *Commutative* Laws: $\begin{cases} A \cup B = B \cup A \\ A \cap B = B \cap A \end{cases}$

4) *Associative* Laws: $\begin{cases} A \cup (B \cup C) = (A \cup B) \cup C \\ A \cap (B \cap C) = (A \cap B) \cap C \end{cases}$

5) *Distributive* Laws: $\begin{cases} A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \end{cases}$

6) *Idempotent* Laws: $\begin{cases} A \cup A = A \\ A \cap A = A \end{cases}$ 7) *Identity* Laws: $\begin{cases} A \cup \phi = A \\ A \cap U = A \end{cases}$

8) *Inverse* Laws: $\begin{cases} A \cup \overline{A} = U \\ A \cap \overline{A} = \phi \end{cases}$ 9) Domination Laws: $\begin{cases} A \cup U = U \\ A \cap \phi = \phi \end{cases}$

10) *Absorption* Laws: $\begin{cases} A \cup (A \cap B) = A \\ A \cap (A \cup B) = A \end{cases}$

The Laws of Set Theory

- **Ex 3.20**
  - Simplify the expression $\overline{\overline{(A\cup B)\cap C}\cup\overline{B}}$

	Reasons
$\overline{\overline{(A\cup B)\cap C}\cup\overline{B}}$	
$=\overline{\overline{((A\cup B)\cap C)}}\cap\overline{\overline{B}}$	DeMorgan's Law
$=((A\cup B)\cap C)\cap B$	Law of Double Complement
$=(A\cup B)\cap(C\cap B)$	Associative Law of Intersection
$=(A\cup B)\cap(B\cap C)$	Commutative Law of Intersection
$=[(A\cup B)\cap B]\cap C$	Associative Law of Intersection
$=B\cap C$	Absorption Law

How about expressing $\overline{A-B}$ in terms of $\cup$ and $\bar{\ }$? $\overline{A\cap\overline{B}}=\overline{A}\cup B$

Set Operations and the Laws of Set Theory

- Definition 3.9: The **dual** of s, s^d can be replaced mutually.
  - (1) $\cup$ and $\cap$ (2) ϕ and $\mathscr{U}$
- Theorem 3.5: **The Principle of Duality**, let s denote **a theorem** dealing with the equality of two set expressions. Then s^d is also a theorem.

- **Ex 3.19** : find a dual for statement A$\subseteq$ B (Th. 3.4)
  - A$\cup$B=B ➔A$\cap$B=B (duality)

    But A$\cap$B=B $\Leftrightarrow$ B$\subseteq$A (the dual of A$\subseteq$ B)
  - or A$\cap$B=A ➔ A$\cup$B=A $\Leftrightarrow$ B$\subseteq$A

Set Operations and the Laws of Set Theory

- Theorem 3.6: **Generalized DeMorgan's Laws**, let I be an index set, then

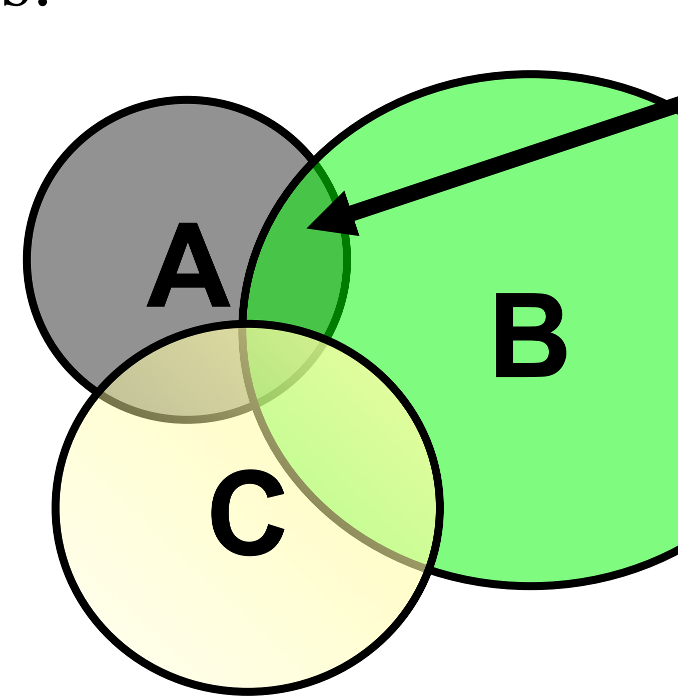
$$(1)\ \overline{\bigcup_{i\in I} A_i} = \bigcap_{i\in I} \overline{A_i} \quad (2)\ \overline{\bigcap_{i\in I} A_i} = \bigcup_{i\in I} \overline{A_i}$$

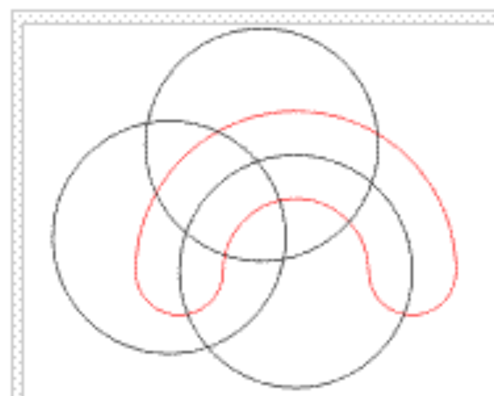
  - Proof:

$$(1)\ x \in \overline{\bigcup_{i\in I} A_i} \Leftrightarrow x \notin \bigcup_{i\in I} A_i \Leftrightarrow x \notin A_i \text{ for all } i \in I \Leftrightarrow x \in \overline{A_i} \text{ for all } i \in I \Leftrightarrow x \in \bigcap_{i\in I} \overline{A_i}$$

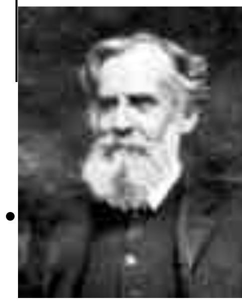
3.3 Counting and Venn Diagrams

- Venn diagrams are used to depict the various unions, subsets, complements, intersections etc. of sets:





Venn's construction for 4 sets



John Venn
(1834~1923)

Venn Diagrams

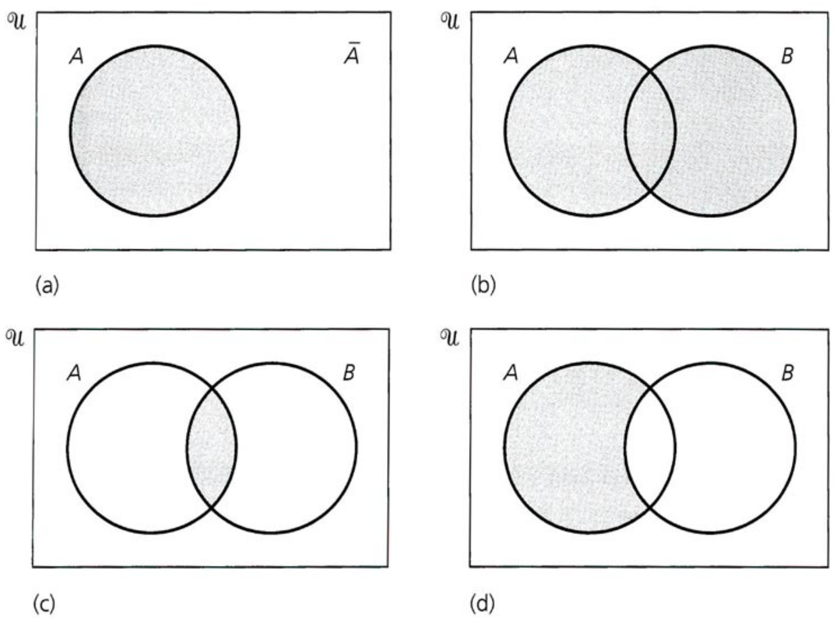


Figure 3.6

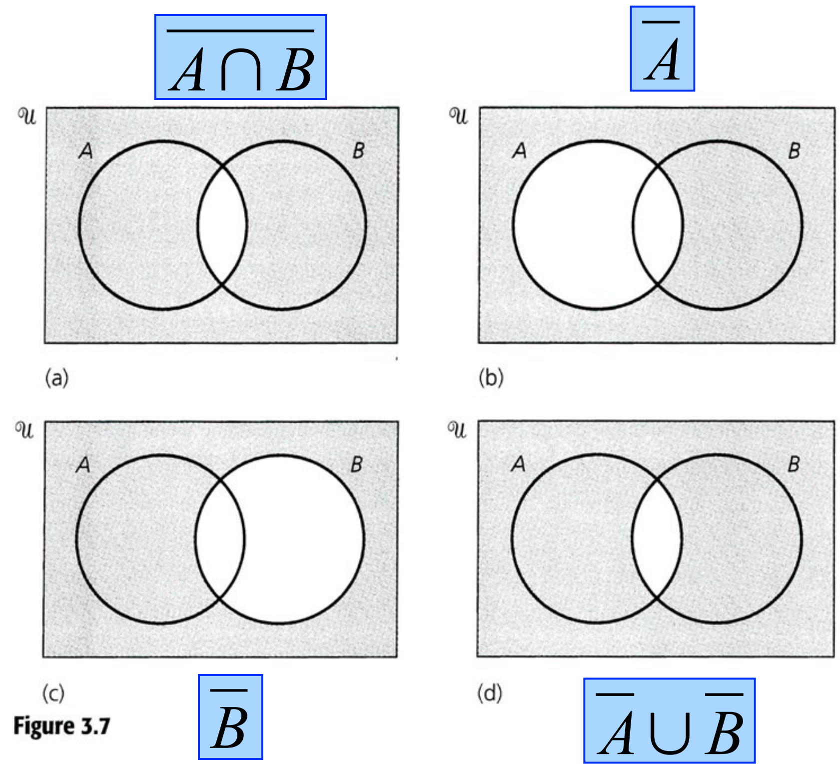
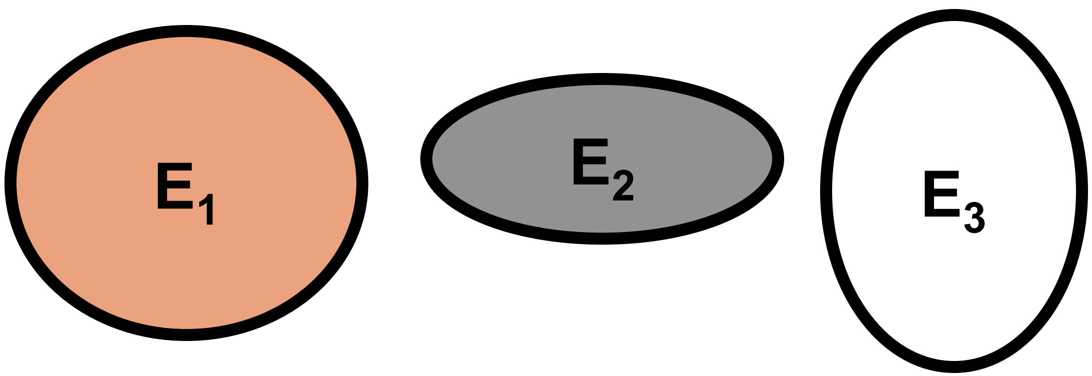


Figure 3.7

Remember the Sum Rule

If we have sets of mutually disjoint **events** $E_1, E_2, \ldots, E_m$, where E_j can occur in n_j ways (with $E_j \cap E_k = \varnothing$ for all $j \neq k$), then there are $n_1+n_2+\ldots+n_m$ possible events.

Think union of sets:
Let $|E_j| = n_j$ for all $1 \leq j \leq m$,
then $E_1 \cup E_2, \ldots \cup E_m$ has $n_1+n_2+\ldots+n_m$ elements.



Inclusion and Exclusion（排容原理）

- The Principle of Inclusion and Exclusion generalizes the Sum Rule to the cases where the events are not disjoint.

- We can use it when solving counting problems…

3.3 Counting and Venn Diagrams

$$(1)\ |A \cup B| = |A| + |B| - |A \cap B|$$

$$(2)\ |\overline{A} \cap \overline{B}| = |\overline{A \cup B}|$$

$$= |U| - |A \cup B|$$

$$= |U| - |A| - |B| + |A \cap B|$$

$$(3)\ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$(4)\ |\overline{A} \cap \overline{B} \cap \overline{C}| = |\overline{A \cup B \cup C}|$$

$$= |U| - |A \cup B \cup C|$$

$$= |U| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|$$

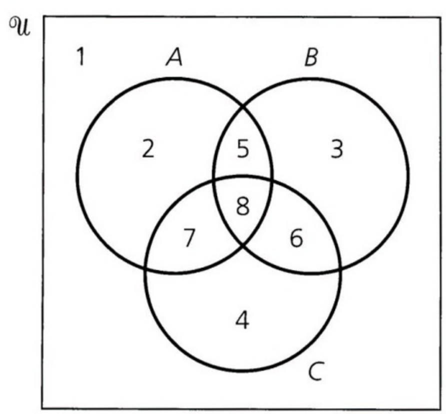


Figure 3.8

How to Derive the 3 Set Case

Using the various Laws for Sets:

$|A \cup B \cup C| = |A \cup (B \cup C)|$

$= |A| + |B \cup C| - |A \cap (B \cup C)|$

$= |A| + |B| + |C| - |B \cap C| - |A \cap (B \cup C)|$

$= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)|$

$= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap A \cap C|$

$= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$

Examples

Throw two dice, how many ways of throwing:
a total of 8, value 6+not value 6, or two identical values?

Counting the possibilities:

a total of eight:	5
value 6+not value 6:	10
two identical values:	6
total 8 and 6+not 6:	2
total 8 and two id-vs:	1
6+not 6 and two id-vs:	0
6+not 6, two id-vs, total 8:	0

Total:
5+10+6–2–1–0+0 = 18