Chapter 2.
First-Order Ordinary Differential Equations

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Review

- 上周 $Mdx + Ndy = 0$
  若發現 $\frac{\partial N}{\partial x} \neq \frac{\partial M}{\partial y}$ 則檢查 $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

  除以 $-N \rightarrow I(x)$
  除以 $M \rightarrow I(y)$
  除以 $-N + M \rightarrow I(x+y)$
  除以 $-Y \times N + X \times M \rightarrow I(x \times y)$

Integrating Factor Examples

- 今天

例: $(xy + y^2 + 1)dx + (xy + x^2 + 1)dy = 0$

$$\Rightarrow \quad x(ydx + xdy) + y(ydx + xdy) + dx + dy = 0$$

$$xd(xy) + yd(xy) + d(x + y) = 0$$

$$(x + y)d(xy) + d(x + y) = 0 \Rightarrow d(xy) + \frac{1}{x + y}d(x + y) = 0$$

$$\int d(xy) + \int \frac{1}{x + y}d(x + y) = \int 0$$

$$xy + \ln(x + y) = C$$

Integrating Factor Examples

例: $(y\cos x - \sin 2x)dx + dy = 0$

$$\frac{\partial M}{\partial y} = \cos x, \frac{\partial N}{\partial x} = 0$$

$$\frac{0 - \cos x}{-N} = \frac{-\cos x}{-1} = \cos x$$

$$\cos x dx = \frac{dI}{I}$$

$$I = e^{\sin x}$$

$$(y(\cos x)e^{\sin x} - (\sin 2x)e^{\sin x})dx + e^{\sin x}dy = 0$$

Integrating Factor Examples

$$\frac{\partial u}{\partial x} = y\cos x e^{\sin x} - \sin 2x e^{\sin x}$$

$$\partial u = (y\cos x e^{\sin x} - \sin 2x e^{\sin x})dx$$

$$\int \partial u = \int y\cos x e^{\sin x} dx - \int \sin 2x e^{\sin x} dx$$

$$u = \int y \underbrace{\cos x}_{d\sin x} e^{\sin x} dx - \int \underbrace{\sin 2x}_{2\sin x\cos x} e^{\sin x} dx$$

$= d\sin x$

$\uparrow$ (1) $\qquad\qquad$ $\uparrow$ (2)

Integrating Factor Examples

(1): $\int y\cos x e^{\sin x} dx$

$= y\int \cos x e^{\sin x} dx$

$= ye^{\sin x}$

(2): $\int \sin 2x e^{\sin x} dx$

$= \int 2\sin x\cos x e^{\sin x} dx$

$= 2\int te^{t} dt$ (令t=sinx)

$= 2(te^{t} - e^{t})$

$$u = ye^{\sin x} - 2\sin x e^{\sin x} + 2e^{\sin x} + f(y)$$

$$\frac{\partial u}{\partial y} = e^{\sin x} \qquad \partial u = e^{\sin x} dy$$

$$\int \partial u = \int e^{\sin x} dy$$

Integrating Factor Examples

$$u = \int e^{\sin x} dy + g(x)$$
$$= ye^{\sin x} + g(x)$$
$$f(y) = 0, g(x) = -2\sin x e^{\sin x} + 2e^{\sin x}$$
$$u = ye^{\sin x} - 2\sin x e^{\sin x} + 2e^{\sin x} = C$$

Integrating Factor Examples

例: $\dfrac{dy}{dx} = 3x^2 - 3x^2 y$ (分離變數法可解)

$$(3x^2 - 3x^2 y)dx - dy = 0$$
$$\frac{\partial M}{\partial y} = -3x^2, \frac{\partial N}{\partial x} = 0$$
$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3x^2$$

除以-N $\dfrac{3x^2}{-(-1)} = 3x^2$

Integrating Factor Examples

$$3x^2 dx = \frac{dI}{I}$$

$$I = e^{x^3}$$

$$e^{x^3}(3x^2 - 3x^2 y)dx - e^{x^3} dy = 0$$

$$\frac{\partial u}{\partial x} = e^{x^3}(3x^2 - 3x^2 y)$$

$$u = \int e^{x^3}(3x^2 - 3x^2 y)dx + f(y)$$

$$u = e^{x^3} - ye^{x^3} + f(y)$$

Integrating Factor Examples

$$\frac{\partial u}{\partial y} = -e^{x^3}$$

$$u = -ye^{x^3} + g(x)$$

$$g(x) = e^{x^3}, f(y) = 0$$

$$u = e^{x^3} - ye^{x^3} = C$$

Separation of Variables

- 2.2分離變數法(補充)

例: $(1+x)dy - ydx = 0$

$\Rightarrow \quad -ydx + (1+x)dy = 0$

$$\frac{\partial M}{\partial y} = -1, \frac{\partial N}{\partial x} = 1$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2$$

除以M $\Rightarrow \quad \dfrac{2}{-y}dy = \dfrac{dI}{I}$

Separation of Variables

$$I = y^{-2}$$

$$-y^{-1}dx + (1+x)y^{-2}dy = 0$$

$$\frac{\partial u}{\partial x} = -y^{-1} \qquad \frac{\partial u}{\partial y} = (1+x)y^{-2}$$

$$u = \int -y^{-1}dx + f(y) \qquad u = \int (1+x)y^{-2}dy + g(x)$$

$$= -xy^{-1} + f(y) \qquad = -(1+x)y^{-1} + g(x)$$

$$f(y) = -y^{-1}, g(x) = 0$$

$$u = -(1+x)y^{-1} = C$$

$y = C(1+x) \quad \Leftarrow$ 課本答案

Separation of Variables

$$(1+x)dy - ydx = 0$$

除以 $(1+x)y$

$$\frac{dy}{y} - \frac{dx}{1+x} = 0 \qquad \int\frac{dy}{y} - \int\frac{dx}{1+x} = \int 0$$

$$\ln y = \ln(1+x) + C$$

$$y = (1+x)e^{c}$$

$$= (1+x)C$$

Separation of Variables Example

例: $\frac{dy}{dx} = y^2 - 4$

$$\frac{dy}{y^2-4} = dx \qquad \left(\frac{\frac{1}{4}}{y-2} - \frac{\frac{1}{4}}{y+2}\right)dy = dx$$

$$\frac{1}{4}\ln(y-2) - \frac{1}{4}\ln(y+2) = x + C$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4x + C$$

$$\frac{y-2}{y+2} = \frac{e^{4x+C}}{1} \qquad \frac{2y}{4} = \frac{e^{4x+C}+1}{1-e^{4x+C}} \qquad y = 2\frac{1+e^{4x+C}}{1-e^{4x+C}}, y = \pm 2$$

First-Order O.D.E.

- 2.3複習:一階O.D.E.(線性 OR 非線性)

一階線性O.D.E.(常微分)

$$y'(x) + p(x)y(x) = r(x)$$

(1) $r(x) = 0$ Homogeneous 齊性

(2) $r(x) \neq 0$ Non- Homogeneous 非齊性

First-Order O.D.E. (homogeneous)

Case (1):

$r(x) = 0 \Rightarrow y'(x) + p(x)y(x) = 0$ $\quad y'(x) = \dfrac{dy}{dx} = \ldots$

$$\frac{dy(x)}{dx} = -p(x)y(x) \qquad \frac{dy(x)}{y(x)} = -p(x)dx$$

$$\int \frac{dy(x)}{y(x)} = \int -p(x)dx \qquad \ln y(x) = -\int p(x)dx + k$$

$$y(x) = e^{-\int p(x)dx} e^{k}$$

$$= Ce^{-\int p(x)dx} \quad (\text{令 } C = e^{k})$$

First-Order O.D.E.(Non-homogeneous)

Case (2):

$$r(x) \neq 0 \Rightarrow y'(x) + p(x)y(x) = r(x)$$

$$\frac{dy}{dx} + p(x)y(x) - r(x) = 0$$

$$\left(p(x)y(x) - r(x)\right)dx + dy = 0$$

$$\frac{\partial M}{\partial y} = p(x), \qquad \frac{\partial N}{\partial x} = 0$$

$$\frac{0 - p(x)}{-N} = p(x) \qquad p(x)dx = \frac{dI}{I}$$

First-Order O.D.E.(Non-homogeneous)

$$I = e^{\int p(x)dx}$$

$$e^{\int p(x)dx}\left(p(x)y(x) - r(x)\right)dx + e^{\int p(x)dx}dy = 0$$

$$\frac{\partial u}{\partial x} = e^{\int p(x)dx}\left(p(x)y(x) - r(x)\right)$$

$$\partial u = e^{\int p(x)dx}\left(p(x)y(x) - r(x)\right)\partial x \quad y(x)\int e^{\int p(x)dx}dx$$

(兩邊同積分)

$$= y(x)\int 1 de^{\int p(x)dx}$$

$$u = \int e^{\int p(x)dx}\left(p(x)y(x) - r(x)\right)dx + f(y)$$

First-Order O.D.E.(Non-homogeneous)

令 $t = \int p(x)dx$

$$dt = p(x)dx \quad , \quad dx = \frac{dt}{p(x)}$$

$$u = \int e^t y(x)dt - \int e^{\int p(x)dx} r(x)dx + f(y)$$

$$= y(x)e^t - \int e^{\int p(x)dx} r(x)dx + f(y)$$

$$= y(x)e^{\int p(x)dx} - \int e^{\int p(x)dx} r(x)dx + f(y)$$

First-Order O.D.E.(Non-homogeneous)

$$\frac{\partial u}{\partial y} = e^{\int p(x)dx}$$

$$\partial u = e^{\int p(x)dx}\partial y + g(x)$$

(兩邊同積分)

$$u = \int e^{\int p(x)dx} dy + g(x)$$

$$= ye^{\int p(x)dx} + g(x)$$

$$g(x) = -\int e^{\int p(x)dx} r(x)dx \quad , \quad f(y) = 0$$

$$u = ye^{\int p(x)dx} - \int e^{\int p(x)dx} r(x)dx = C \ \#$$

$$I = e^{\int p(x)dx}$$

First-Order O.D.E.(Non-homogeneous)

$$y = C\underbrace{e^{-\int p(x)dx}}_{I^{-1}} + \underbrace{e^{-\int p(x)dx}}_{I^{-1}} \int \underbrace{e^{\int p(x)dx}}_{I} r(x)\,dx$$

$$= y_h + y_p$$

若能看出線性，務必以公式解（不要用正合）

y_h ：$r(x) = 0$ homogeneous solution

y_p ：特解(*particular solution*),

互補解(*complementary solution*)

記法：$y' + p(x)y(x) = r(x)$之解

$$\boxed{y = CI^{-1} + I^{-1}\int I\gamma(x)d(x)}$$

First-Order O.D.E.(Non-homogeneous)

例: $y' + 2xy = 3x$

$$y = Ce^{-x^2} + e^{-x^2}\int e^{x^2} 3x\,dx$$

$$= \underbrace{Ce^{-x^2}}_{y_h} + \underbrace{\frac{3}{2}}_{y_p} \quad \#$$

(1) $y_h' + p(x)y_p = 0$

(2) Theorem:

若 $y_p(x)$滿足非齊性方程式 $\Leftrightarrow y_p' + p(x)y_p(x) = r(x)$

特解

First-Order O.D.E.(Non-homogeneous)

Proof:

$\because y = y_h + y_p$ 代入原方程式

$y' + Py = r$

$(y_h + y_p)' + P(y_h + y_p) = r$

$y_h' + y_p' + Py_h + Py_p = r$

$\because y_h' + Py_h = 0$

$\therefore y_p' + Py_p = r$

Non-homogeneous Example

例: $y' + 2xy = x$

$I = e^{\int 2x dx} = e^{x^2}$

$y = Ce^{-x^2} + e^{-x^2}\int e^{x^2} x dx$

$= Ce^{-x^2} + \frac{1}{2}$

Chapter 3. Higher-Order Differential Equations

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First-Order O.D.E.

$$y' + ay = r(x) \quad a \in const.$$

Case (1):

$r(x) = 0$ Homogeneous

$$y' + ay = 0$$

$$y = Ce^{-\int a dx}$$

$$= Ce^{-ax}$$

$$= y_h(x)$$

常係數 $\Rightarrow$ $y_h(x)$ 的部分必為指數函數 $e^{\lambda x}$ $\lambda \in const.$

First-Order O.D.E.

例: $y' + 2y = 0$

$y = Ce^{-2x}$

apply $\quad y = e^{\lambda x}$代入 $y' + 2y = 0$

$\lambda e^{\lambda x} + 2e^{\lambda x} = 0$

$(\lambda + 2)e^{\lambda x} = 0$, $\lambda + 2 = 0 \Rightarrow$ 特性方程式

$\lambda = -2$

$y = Ce^{-2x}$

Higher-Order O.D.E.

例: $y' - 3y = 0$

$y = Ce^{3x}$ #

推廣 $y'' + ay' + by = 0$, $a,b \in const.$

猜 $e^{\lambda x}$ 是解

$\left(e^{\lambda x}\right)'' + a\left(e^{\lambda x}\right)' + b\left(e^{\lambda x}\right) = 0$

$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = 0 \qquad \because \forall x$, $e^{\lambda x} \neq 0$

$e^{\lambda x}\left(\lambda^2 + a\lambda + b\right) = 0 \qquad \therefore \lambda^2 + a\lambda + b = 0$

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Higher-Order O.D.E.

Case (1):

$\lambda_1 \neq \lambda_2 \in \Re$ 相異實根

$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

例: $y'' + 3y' + 2y = 0$

$\lambda^2 + 3\lambda + 2 = 0$

$\lambda = -1, -2$

$y(x) = C_1 e^{-x} + C_2 e^{-2x}$ #

例: $y'' + 6y' + 5y = 0$

$\lambda = -5, -1$

$y = C_1 e^{-x} + C_2 e^{-5x}$

Higher-Order O.D.E.

Case (2):

$\lambda_1 = \alpha + \beta\, i \quad , \quad \lambda_2 = \alpha - \beta\, i$ 共軛複數根

$y = k_1 e^{(\alpha+\beta\, i)x} + k_2 e^{(\alpha-\beta\, i)x}$ Euler's rule $\boxed{\begin{array}{l} e^{ix} = \cos x + i\sin x \\ e^{-ix} = \cos x - i\sin x \end{array}}$

$= e^{\alpha x}\left(k_1(\cos\beta x + i\sin\beta x) + k_2(\cos\beta x - i\sin\beta x)\right)$

(令$k_2 = ik_2'$)

$= e^{\alpha x}\left(k_1(\cos\beta x + i\sin\beta x) + ik_2'(\cos\beta x - i\sin\beta x)\right)$

$= e^{\alpha x}\left((k_1 + ik_2')\cos\beta x + (k_2' + ik_1)\sin\beta x\right)$

$= e^{\alpha x}\left(C_1\cos\beta x + C_2\sin\beta x\right)$

$\Rightarrow C_1 = k_1 + ik_2' \quad , \quad C_2 = k_2' + ik_1$

Higher-Order O.D.E. Examples

例: $y'' + 2y' + 10y = 0$

$\lambda = -1 \pm 3\,i$

$y = e^{-x}\left(C_1 \cos 3x + C_2 \sin 3x\right)$#

例: $y'' + 4y = 0$

$\lambda = \pm 2i$

$y = C_1 \cos 2x + C_2 \sin 2x$#

Higher-Order O.D.E.

- 補充 :

$y'' + ay' + by = 0$

$\Rightarrow \lambda^2 + a\lambda + b = 0$

$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$

$\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0$

$a = -(\lambda_1 + \lambda_2) \quad b = \lambda_1\lambda_2$

$y'' - (\lambda_1 + \lambda_2)y' + \lambda_1\lambda_2 y = 0$

Higher-Order O.D.E.

- 定義:

$\mathrm{D} \equiv \dfrac{d}{dx}$ ：微分運算子

$\mathrm{D}x^2 = \dfrac{d}{dx}x^2$ D只對右邊函式作運算 note: $\mathrm{D}x^2 \neq x^2 D$

$\mathrm{D}^k = \dfrac{d^k}{dx^k}$

Higher-Order O.D.E.

$$y'' - (\lambda_1 + \lambda_2)y' + \lambda_1\lambda_2 y = 0$$
$$\Rightarrow D^2 y - (\lambda_1 + \lambda_2)Dy + \lambda_1\lambda_2 y = 0$$
$$\left(D^2 - (\lambda_1 + \lambda_2)D + \lambda_1\lambda_2\right)y = 0$$
$$(D - \lambda_1)(D - \lambda_2)y = 0$$

$\circ$令$(D - \lambda_2)y = z(x)$

$\Rightarrow (D - \lambda_1)z(x) = 0$

$z'(x) - \lambda_1 z(x) = 0$

$z(x) = k_1 e^{-\lambda_1 x}$

$(D - \lambda_2)y = z(x) = k_1 e^{-\lambda_1 x}$

$y' - \lambda_2 y = k_1 e^{-\lambda_1 x}$

$y = CI^{-1} + I^{-1}\int Irdx$

$I = e^{-\int \lambda_2 dx} = e^{-\lambda_2 x}$