Chapter 4.
Laplace Transform

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Laplace Transform

- 想法：

Laplace Transform

From 微分方程式 to 代數方程式必須具備:

$y'' + 3y' + 5y = e^t$

$y'' + 3y' + 5y = \cos t$

$y'' + 3y' + 5y = \sin t$

……

才可以進行 Laplace Transform

1. 基本函數
2. 一階、二階(y', y'')

Laplace Transform

- 定義：

$L\{f(t)\}=f(t)$函數的 Laplace Transform

$= \int_0^\infty f(t)e^{-st}dt$

$= F(s)$

$f(t) = L^{-1}\{F(s)\} \Rightarrow$ Inverse Laplace Transform

$= \dfrac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)e^{st}ds$ 複變(Complex Analysis)

*轉換與逆轉換要同時學習

Laplace Transform

例：$y' + 2y = e^t, y(0) = 1$

$method(I)$

$y_h = Ce^{-2t}$

$y_p = \frac{1}{D+2}e^t = \frac{1}{3}e^t$

$y(t) = Ce^{-2t} + \frac{1}{3}e^t, y(0) = 1, C = \frac{2}{3}$

$y(t) = \frac{2}{3}e^{-2t} + \frac{1}{3}e^t$

Laplace Transform

例：$method(II)$

$L\{y'\} + 2L\{y\} = L\{e^t\} \Rightarrow sY(s) - y(0) + 2Y(s) = \frac{1}{s-1}$

$(s+2)Y(s) = \frac{s}{s-1}$

$Y(s) = \boxed{\frac{s}{(s-1)(s+2)}} = \frac{a}{s-1} + \frac{b}{s+2}$

$a = \frac{1}{3}, b = \frac{2}{3}$

$Y(s) = \frac{1}{3(s-1)} + \frac{2}{3(s+2)} \Rightarrow$ Inverse Laplace Transform

$y(t) = \frac{1}{3}e^t + \frac{2}{3}e^{-2t}$

Laplace Transform

(1) $f(t)=e^{at}, a \in const$

$$F(s)=L\{f(t)\}=\int_0^{\infty} e^{at}e^{-st}dt$$

$$=\int_0^{\infty} e^{-(s-a)t}dt$$

$$=-\frac{1}{s-a}e^{-(s-a)t}\Big|_0^{\infty}$$

$$=-\frac{1}{s-a}e^{-(s-a)\infty}-(-\frac{1}{s-a}), s-a>0 \because \text{要收斂}$$

$$=\frac{1}{s-a}$$

$$f(t)=e^{at} \Rightarrow F(s)=\frac{1}{s-a}$$

s係數要為1

Laplace Transform

例：$f(t)=e^{-2t}$

$$\Rightarrow F(s)=\frac{1}{s+2}$$

例：$G(s)=\frac{2}{s+3}$

$$\Rightarrow g(t)=L^{-1}\left\{2\frac{1}{s-(-3)}\right\}$$

$$=2e^{-3t}$$

Laplace Transform

(2) $f(t) = \cos at = \dfrac{e^{iat} + e^{-iat}}{2}$

$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty \cos(at)e^{-st}dt$

$L\{\cos at\} = L\left\{\dfrac{e^{iat} + e^{-iat}}{2}\right\}$

$= \dfrac{1}{2}L\{e^{iat}\} + \dfrac{1}{2}L\{e^{-iat}\} = \dfrac{1}{2}\dfrac{1}{(s-ia)} + \dfrac{1}{2}\dfrac{1}{(s+ia)}$

$= \dfrac{1}{2}\dfrac{2s}{(s-ia)(s+ia)} = \dfrac{s}{(s-ia)(s+ia)}$

$= \dfrac{s}{s^2 + a^2}$

$f(t) = \cos at \Rightarrow F(s) = \dfrac{s}{s^2 + a^2}$

Laplace Transform

例：$\pounds\{\cos 3t\} = \dfrac{s}{s^2 + 3^2} = \dfrac{s}{s^2 + 9}$

完成$\int_0^\infty \cos at \cdot e^{-st}dt$

令$dv = e^{-st}dt, v = \dfrac{-1}{s}e^{-st}$

$u = \cos at, du = -a\sin atdt$

$\int_0^\infty \cos ate^{-st}dt$

$= \cos at \cdot \dfrac{-1}{s}e^{-st}\Big|_0^\infty - \int_0^\infty \dfrac{-1}{s}e^{-st}(-a\sin at)dt, s > 0$

$= [0 - \dfrac{-1}{s}] - \dfrac{a}{s}\int_0^\infty \sin ate^{-st}dt$

Laplace Transform

令$u = \sin at, du = a\cos at dt$

$dv = e^{-st}dt, v = \frac{-1}{s}e^{-st}$

$= \frac{1}{s} - \frac{a}{s}[\sin at \cdot \frac{-1}{s}e^{-st}\Big|_0^\infty - \int_0^\infty (\frac{-1}{s}e^{-st})a\cos at dt], s > 0$

$= \frac{1}{s} - \frac{a}{s} \cdot \frac{a}{s}\int_0^\infty \cos at \cdot e^{-st}dt$

$\Rightarrow F(s) = \frac{1}{s} - \frac{a^2}{s^2}F(s)$

$\Rightarrow F(s) = \frac{s}{s^2 + a^2}$

Laplace Transform

(3) $f(t) = \sin at$

$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty \sin at \cdot e^{-st}dt$

利用$\sin at = \frac{e^{iat} - e^{-iat}}{2i}$

$\Rightarrow \mathscr{L}\{\sin at\} = \frac{1}{2i}\mathscr{L}\{e^{iat}\} - \frac{1}{2i}\mathscr{L}\{e^{-iat}\}$

$= \frac{1}{2i}\frac{1}{s - ai} - \frac{1}{2i}\frac{1}{s + ai} = \frac{a}{s^2 + a^2}$

Laplace Transform

例： $f(t) = \sin 2t$

$$\Rightarrow F(s) = \frac{2}{s^2 + 4}$$

例： $F(s) = \dfrac{s}{s^2 + 16}$

$$\Rightarrow f(t) = \mathscr{L}^{-1}\{F(s)\} = \cos 4t$$

例： $F(s) = \dfrac{3}{s^2 + 16}$

$$\Rightarrow f(t) = \mathscr{L}^{-1}\{F(s)\} = \frac{3}{4} * \frac{4}{s^2 + 16} = \frac{3}{4}\sin 4t$$

例： $\mathscr{L}^{-1}\{\dfrac{3}{s+6} + \dfrac{5s}{s^2 + 25} + \dfrac{2}{s^2 + 36}\}$

$$\Rightarrow 3e^{-6t} + 5\cos 5t + \frac{1}{3}\sin 6t$$

Laplace Transform

(4) $f(t) = t$

$$F(s) = \int_0^\infty \underbrace{t}_{=u}\,\underbrace{e^{-st}dt}_{=dV}$$

$$= t\left(\frac{-1}{s}e^{-st}\right)\Bigg|_0^\infty - \int_0^\infty \frac{-1}{s}e^{-st}dt$$

$$= \lim_{t\to\infty} \frac{-t}{se^{st}} - 0 + \frac{1}{s}\left(\frac{-1}{s}e^{-st}\right)\Bigg|_0^\infty$$

$$= \frac{1}{s^2}$$

$$\Rightarrow \mathscr{L}\{t\} = \frac{1}{s^2}$$

$$\Rightarrow f(t) = t \;\Rightarrow F(s) = \frac{1}{s^2}$$

Laplace Transform

例： $\mathscr{L}\{3t\}$

$$\Rightarrow F(s) = \frac{3}{s^2}$$

例： $G(s) = \dfrac{5}{s^2}$

$$\Rightarrow g(t) = 5t$$

例： $G(s) = \dfrac{5s+1}{s^2}$

$$\Rightarrow g(t) = \underbrace{5}_{5H(t)} + t \quad , t > 0$$

Laplace Transform

(5) $f(t) = H(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$

單位步階函數=unit-step function

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{H(t)\} = \mathscr{L}\{U(t)\}$$

$$= \int_0^\infty f(t)e^{-st}dt = \int_0^\infty 1 \cdot e^{-st}dt = \frac{1}{s}$$

Laplace Transform

(a) High-pass filter

$$g(t)=\begin{cases}1 & t>3\\ 0 & t<3\end{cases}$$

$$g(t)=H(t-3)$$

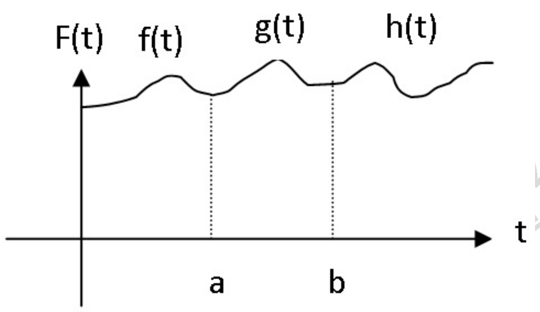
(b) Band-pass filter

$$g(t)=\begin{cases}1 & -3< \ t<3\\ 0 & t>3\text{或}t<-3\end{cases}$$

$$g(t)=H(t+3)-H(t-3)$$

Laplace Transform

例：



$$F(t)=\begin{cases}0,t<0\\ f(t),0<t<a\\ g(t),a<t<b\\ h(t),b<t\end{cases}$$

$$F(t)=f(t)\left[H(t)-H(t-a)\right]+g(t)\left[H(t-a)-H(t-b)\right]+h(t)[H(t-b)]$$
$$=f(t)+\left[g(t)-f(t)\right]H(t-a)+[h(t)-g(t)]H(t-b)$$

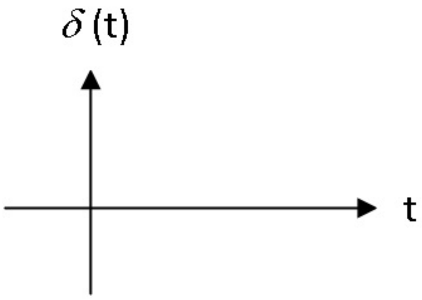
Laplace Transform

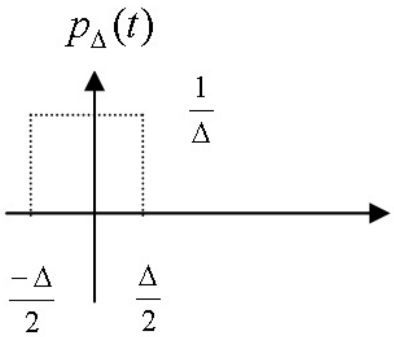
$\delta'(t) = 0$

(6) $\delta(t)\begin{cases}1, t=0\\0, t\neq 0\end{cases}$，其中1為面積值

$\delta(t) = unit-impulse\ function$ 單位脈衝函數

自然界沒有這函數，可用近似函數來取代





Laplace Transform

$$\lim_{\Delta\to 0} P_\Delta(t) = \begin{cases}1, t=0\\0, t\neq 0\end{cases}$$

$\mathscr{L}\{\delta(t)\}$

$$= \int_0^\infty \delta(t)e^{-st}dt$$

在 t 的積分範圍內，只有在 $t=0$ 時 δ(t)才有值

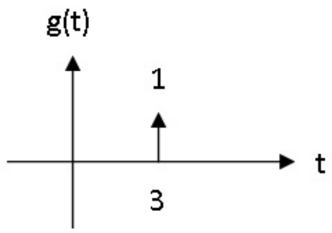
$$= \int_0^\infty \delta(t)e^{\overset{-s\cdot 0}{-0}}dt$$

$= 1$

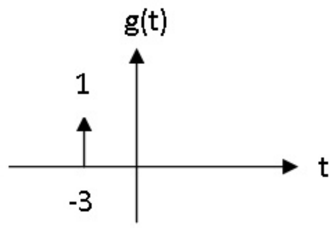
$\boxed{\delta(t) = H'(t)}$

Laplace Transform

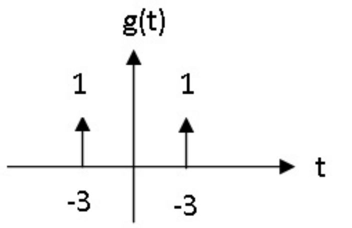
例：



$g(t)=\delta(t-3)$



$g(t)=\delta(t+3)$



$g(t)=\delta(t+3)+\delta(t-3)$

Laplace Transform

(7) $f(t^n) \quad \Rightarrow \quad F(s)=\dfrac{n!}{s^{n+1}}$

例: $\mathscr{L}\left\{e^t+3\sin 2t+5\cos 3t+3t^2+5\delta(t)\right\}$

$$=\frac{1}{s-1}+\frac{2}{s^2+4}\cdot 3+\frac{s}{s^2+9}\cdot 5+\frac{2!}{s^3}\cdot 3+5$$

例: $\mathscr{L}^{-1}\left\{7+\dfrac{2}{s^5}+\dfrac{2}{s^2+1}+\dfrac{s}{s^2+49}+\dfrac{3}{s+5}\right\}$

$$=7\delta(t)+\frac{2}{4!}t^4+2\sin t+\cos 7t+3e^{-5t}$$