Chapter 2.
First-Order Ordinary Differential Equations

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Exact Examples

例: $(e^x y + 6x + 5y)dx + (e^x + 5x)dy = 0$

$$\underset{\text{M}}{(e^x y + 6x + 5y)}dx + \underset{\text{N}}{(e^x + 5x)}dy = 0$$

$$\frac{\partial \text{M}}{\partial y} = e^x + 5 \quad \Leftrightarrow \quad \frac{\partial N}{\partial x} = e^x + 5$$

(相等代表為正合)

$$\text{M} = \frac{\partial u}{\partial x} \qquad\qquad \text{N} = \frac{\partial u}{\partial y}$$

$$\partial u = (e^x y + 6x + 5y)\partial x \quad \Leftrightarrow \quad \partial u = (e^x + 5x)\partial y$$

Exact Examples

$$u = \int \left(e^x y + 6x + 5y \right) \partial x + f\left(y \right) \Leftrightarrow \mathrm{u} = \int \left(\mathrm{e}^{\mathrm{x}} + 5x \right) \partial y + g\left(x \right)$$

$$= \mathrm{e}^{\mathrm{x}} y + 3x^2 + 5xy + f\left(y \right) \qquad\qquad = \mathrm{e}^{\mathrm{x}} y + 5xy + g\left(x \right)$$

因此 $g\left(x \right) = 3x^2, f\left(y \right) = 0$

$$\mathrm{u}\left(\mathrm{x,y} \right) = \mathrm{e}^{\mathrm{x}} y + 5xy + 3x^2 = C$$

Exact Examples

例: $(\cos y + 8x)dx + (-x\sin y + 3y^2)dy = 0$

$$\underset{\mathrm{M}}{(\cos y + 8x)}dx + \underset{\mathrm{N}}{(-x\sin y + 3y^2)}dy = 0$$

$$\frac{\partial M}{\partial y} = -\sin y \quad \Leftrightarrow \quad \frac{\partial N}{\partial x} = -\sin y$$

(相等代表為正合)

$$M = \frac{\partial u}{\partial x} \qquad \mathrm{N} = \frac{\partial u}{\partial y}$$

$$\partial u = (\cos y + 8x)\partial x \qquad \partial u = (-x\sin y + 3y^2)\partial y$$

$$u = \int(\cos y + 8x)\partial x \quad \Leftrightarrow \quad \mathrm{u} = \int(-x\sin y + 3y^2)\partial y$$

Exact Examples

$$u = \int (\cos y + 8x)\partial x \qquad \Leftrightarrow \mathrm{u} = \int (-x\sin y + 3y^2)\partial y$$

$$= x\cos y + 4x^2 + f(y) \text{ —(1)} \qquad = x\cos y + y^3 + g(x) \text{—(2)}$$

比較(1)(2)

得 $f(y) = y^3, g(x) = 4x^2$

$\therefore u(x, y) = x\cos y + y^3 + 4x^2 = C$

Non-Exact

- 若不為正合情況

$$u(x,y)=C\ldots\ldots\ldots(\mathrm{A})$$

$$\downarrow$$

$$\mathrm{M}(x,y)dx+N(x,y)dy=0\ldots\ldots(B)$$

若有乘法消去項,怎麼辦？

想法：

還它

Non-Exact

方法：

解(B)式,應先將消去項,歸還回,使得(B)成為正合。

怎麼知道消去哪些項？

(1)假設消去 $I(x,y)$

$$I(x,y)M(x,y)dx + I(x,y)N(x,y)dy = 0$$

若 $\dfrac{\partial \mathrm{M}_1(x,y)}{\partial y} = \dfrac{\partial N_1(x,y)}{\partial x}$, 則$M_1dx + N_1dy = 0$為正合

Non-Exact

(2) $$\frac{\partial}{\partial y}\left(I(x,y)M(x,y)\right) = \frac{\partial}{\partial x}\left(I(x,y)N(x,y)\right)$$

$$M(x,y)\frac{\partial I(x,y)}{\partial y} + I(x,y)\frac{\partial M(x,y)}{\partial y} = \mathrm{N}(\mathrm{x,y})\frac{\partial I(x,y)}{\partial x} + I(x,y)\frac{\partial N(x,y)}{\partial x}$$

$$-N(x,y)\frac{\partial I(x,y)}{\partial x} + M(x,y)\frac{\partial I(x,y)}{\partial y} = I(x,y)\left[\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}\right]$$

Integrating Factor

- 目的：

解$I(x,y)$所形成一階P.D.E.；$I(x,y)$稱為積分因子(Integrating Factor)。

考慮一階P.D.E.

$$P(x,y,z)\frac{\partial z}{\partial x}+Q(x,y,z)\frac{\partial z}{\partial y}=R(x,y,z)\ldots\ldots(*)$$

$\Rightarrow$由下列等式決定出兩個獨立解

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$ 稱輔助方程組、lagrange方程組

Integrating Factor

其中，(*)的通解可以是

$\varphi(u,v)=0$ 隱函數表示法

$v = f(u)$ 顯函數表示法

比較前式

$$-N\frac{\partial I}{\partial x}+M\frac{\partial I}{\partial y}=I\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$$

$$\boxed{\frac{dx}{-N}=\frac{dy}{M}=\frac{dI}{I\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}} \Rightarrow$$

可以利用此式來求解 I，而非要求 (*) 的通解

如何求 $\Longrightarrow$ TRY !!

Integrating Factor

- 招數：

(1)猜 I 是X 的函數，看看有沒有解。i.e. I (x)

$$\frac{dx}{-N} = \frac{dI}{I\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)} \Rightarrow \frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx}{-N} = \frac{dI}{I}$$

預達到希望(猜對)，則需 $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{-N} = f(x)$

Integrating Factor

$$f\left(x\right)dx = \frac{dI}{I}$$

$$(\text{兩邊積分}) \Rightarrow \int f\left(x\right)dx = \ln I$$

$$\Rightarrow I = e^{\int f\left(x\right)dx}$$

(2) 猜 I 是 y 的函數。 i.e. I (y)

$$\frac{dy}{M} = \frac{dI}{I\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)} \quad \Rightarrow \quad \frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}{M}dy = \frac{dI}{I}$$

Integrating Factor

預達到希望(猜對)，則需 $\dfrac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}=g(\mathrm{y})$

$$g(y)\,dy=\frac{dI}{I}$$

(兩邊積分) $\Rightarrow \int g(y)\,dy=\ln I$

$$\Rightarrow I=e^{\int g(y)dy}$$

Integrating Factor

(3)猜I是的 $(x+y)$ 函數

$$\frac{d}{dx}(x+y)=1+\frac{dy}{dx}$$

同乘dx $\Rightarrow d(x+y)=dx+dy$

Hint: (和分比概念)

$$\frac{1}{2}=\frac{2}{4}=\frac{1+2}{2+4}=\frac{1\times 2}{2\times 2}=\frac{2\times 3}{4\times 3}=\frac{1\times 2+2\times 3}{2\times 2+4\times 3}$$

Integrating Factor

$$\frac{dx+dy}{-N+M}=\frac{dI}{I\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)} \Rightarrow \frac{d\left(x+y\right)}{-N+M}=\frac{dI}{I\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}$$

預達到希望(猜對)，則需 $\dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{-N+M}=f\left(x+y\right)$

$$f\left(x+y\right)d\left(x+y\right)=\frac{dI}{I}$$

(兩邊積分) $\Rightarrow \int f\left(x+y\right)d\left(x+y\right)=\ln I$

$$\Rightarrow I=e^{\int f\left(x+y\right)d\left(x+y\right)}$$

Integrating Factor

(4)猜 I 是 xy 的函數。

其中：

$$d(xy) = ?$$

$$\frac{d(xy)}{dx} = y\frac{dx}{dx} + x\frac{dy}{dx}$$

$$\frac{ydx + xdy}{y(-N) + xM} = \frac{dI}{\mathrm{I}\left(\frac{\partial \mathrm{N}}{\partial x} - \frac{\partial M}{\partial y}\right)}$$

Integrating Factor

$$\frac{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)d(xy)}{y(-N+xM)}=\frac{dI}{I}$$

預達到希望(猜對)，則需 $\dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{y(-N)+xM}=f(xy)$

$$f(xy)\,d(xy)=\frac{dI}{I}$$

(兩邊積分) $\Rightarrow \int f(xy)\,d(xy)=\ln I$

$$\Rightarrow I=e^{\int f(xy)\,d(xy)}$$

Integrating Factor

- Summary：

(1)(2)(3)(4)分子都是 $\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}$

若發現 $\frac{\partial N}{\partial x}\neq\frac{\partial M}{\partial y}$

則算出 $(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y})$

除以 –N → I(x)　　除以 –N+M → I(x+y)

除以 M → I(y)　　除以 –y*N+x*M → I(x*y)

Integrating Factor Examples

例: $\underbrace{(x^2+y^2+x)}_{M}dx+\underbrace{xy}_{N}dy=0$

$$\frac{\partial M}{\partial y}=2y \quad \longleftarrow \quad 不相等 \quad \longrightarrow \quad \frac{\partial N}{\partial x}=y$$

$$\frac{dx}{x}=\frac{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=-y}{-N}dx=\frac{dI}{I}$$

$$I=e^{\int\frac{1}{x}dx}=e^{\ln x}=x$$

Integrating Factor Examples

原式=> $(x^3 + xy^2 + x^2)dx + x^2 ydy = 0$

$$\frac{\partial M}{\partial y} = 2xy \qquad = \qquad \frac{\partial N}{\partial x} = 2xy$$

正合

$$M = \frac{\partial u}{\partial x} \qquad N = \frac{\partial u}{\partial y}$$

$$\partial u = (x^3 + xy^2 + x^2)\partial x$$

$$u = \frac{1}{4}x^4 + \frac{1}{2}x^2 y^2 + \frac{1}{3}x^3 + f(y)$$

Integrating Factor Examples

$$\partial u = (x^2 y)\partial y$$

$$u = \frac{1}{2}x^2 y^2 + g(x)$$

$$\therefore\ f(y) = 0$$

$$g(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3$$

$$\mu(x, y) = \frac{1}{2}x^2 y^2 + \frac{1}{4}x^4 + \frac{1}{3}x^3 = C$$

Integrating Factor Examples

Sol2(用微積分解) 不好解，但可以用來當驗算題: $(x^2+y^2+x)dx+xydy=0$

$$y^2dx+xydy+(x^2+x)dx=0$$

$$y(ydx+xdy)+(x^2+x)dx=0$$

$$yd(xy)+(x^2+x)dx=0$$

同乘x => $xyd(xy)+(x^3+x^2)dx=0$

=> $\frac{1}{2}x^2y^2+\frac{1}{4}x^4+\frac{1}{3}x^3=C$

Integrating Factor Examples

例: $2\sin(y^2)dx + xy\cos(y^2)dy = 0$

$$\frac{\partial M}{\partial y} = 2\cos(y^2)2y = 4y\cos(y^2) \qquad \frac{\partial N}{\partial x} = y\cos(y^2)$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -3y\cos(y^2)$$

$$\Rightarrow \frac{-3y\cos(y^2)}{-N}dx = \frac{-3y\cos(y^2)}{-xy\cos(y^2)}dx = \frac{3}{x}dx = \frac{dI}{I}$$

$$\Rightarrow I = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$$

Integrating Factor Examples

得 $2x^3 \sin(y^2)dx + x^4 y \cos(y^2)dy = 0$

$$M = \frac{\partial u}{\partial x} \qquad N = \frac{\partial u}{\partial y}$$

$$u = \int 2x^3 \sin(y^2)dx + f(y)$$

$$= \frac{1}{2}x^4 \sin(y^2) + f(y)$$

$$u = \int x^4 y \cos(y^2)dy + g(x)$$

$$t = y^2, dt = 2ydy, dy = \frac{dt}{2y}$$

$$= \frac{1}{2}x^4 \sin(y^2) + g(x)$$

Integrating Factor Examples

$$\because g(x)=0, f(y)=0 \qquad \therefore u=\frac{1}{2}x^4\sin(y^2)=C$$

其中在 u中沒有 x 的式

$$\int y\cos(y^2)dy = \int y\cos(t)\frac{dt}{2y} = \int\frac{1}{2}\cos(t)dt$$

$$=\frac{1}{2}\sin(t)$$

$$=\frac{1}{2}\sin(y^2)$$

Integrating Factor Examples

例: $xy\cos(x^2)dx + 2\sin(x^2)dy = 0$

$$\frac{\partial M}{\partial y} = x\cos(x^2) \neq \frac{\partial N}{\partial x} = 2\cos(x^2)2x = 4x\cos(x^2)$$

$$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3x\cos(x^2)$$

$$\frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = 3x\cos(x^2)}{xy\cos(x^2)} dy = \frac{3}{y}dy \quad I = e^{\int \frac{3}{y}dy} = y^3$$

Integrating Factor Examples

得 $xy^4\cos(x^2)dx + 2y^3\sin(x^2)dy = 0$

$$M = \frac{\partial u}{\partial x} \qquad N = \frac{\partial u}{\partial y}$$

$$u = \int xy^4\cos(x^2)dx + f(y) \qquad u = \int 2y^3\sin(x^2)dy + g(x)$$

$$= \frac{1}{2}y^4\sin(x^2) + f(y) \qquad = \frac{1}{2}y^4\sin(x^2) + g(x)$$

$$\because f(y) = 0, g(x) = 0 \qquad \therefore u = \frac{1}{2}y^4\sin(x^2) = C$$

Integrating Factor Examples

例: $(xy + y^2 + 1)dx + (xy + x^2 + 1)dy = 0$

$$\frac{\partial M}{\partial y} = x + 2y \qquad \frac{\partial N}{\partial x} = y + 2x$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = x - y \qquad \frac{x-y}{-N+M}d(x+y)$$

$$= \frac{x-y}{y^2 - x^2}d(x+y) = \frac{-1}{x+y}d(x+y)$$

$$I = e^{\int \frac{-1}{x+y}d(x+y)} = e^{-\ln(x+y)} = e^{\ln\frac{1}{(x+y)}} = \frac{1}{x+y}$$

$$\frac{1}{x+y}(xy + y^2 + 1)dx + \frac{1}{x+y}(xy + x^2 + 1)dy = 0$$

Integrating Factor Examples

$$M = \frac{\partial u}{\partial x} \qquad N = \frac{\partial u}{\partial y}$$

$$u = \int \frac{xy + y^2 + 1}{x+y} dx + f(y) \qquad u = \int \frac{xy + x^2 + 1}{x+y} dx + g(x)$$

$$= \int (y + \frac{1}{x+y}) dx + f(y) \qquad = \int (x + \frac{1}{x+y}) dx + g(x)$$

$$= yx + \ln(x+y) + f(y) \qquad = xy + \ln(x+y) + g(x)$$

$$\because g(x) = 0, f(y) = 0 \quad \therefore u(x,y) = xy + \ln(x+y) = C$$

Integrating Factor Non-Unique

Note: 若取的不同，則積分因子可以不唯一，但答案一定相同。

如上題 $(xy+y^2+1)dx+(xy+x^2+1)dy=0$

$$\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=x-y$$

$$\frac{x-y}{-yN+xM}d(xy)=\frac{x-y}{-y(xy+x^2+1)+x(xy+y^2+1)}d(xy)$$

$$=d(xy)$$

$$I=e^{\int 1d(xy)}=e^{xy} \Leftarrow \text{ 積分因子不唯一}$$

Integrating Factor Non-Unique

得 $e^{xy}(xy+y^2+1)dx+e^{xy}(xy+x^2+1)dy=0$

$M=\dfrac{\partial\mu}{\partial x}\quad u=\int e^{xy}(xy+y^2+1)dx+f(y)$

其中 $\int e^{xy}xydx=?......(1)$

$$y\int e^{xy}xdx=y(x\frac{1}{y}e^{xy}-\int\frac{1}{y}e^{xy}dx)=xe^{xy}-\int e^{xy}dx$$

$\int e^{xy}y^2dx......(2)$

$$=\frac{1}{y}e^{xy}y^2=ye^{xy}$$

$\int e^{xy}dx......(3)$

Integrating Factor Non-Unique

因此 $u = (1) + (2) + (3) + f(y)$

$$= xe^{xy} + ye^{xy} + f(y)$$

同理 $N = \dfrac{\partial \mu}{\partial y}$

$$u = xe^{xy} + ye^{xy} + g(x)$$

$\because f(y) = 0, g(x) = 0 \quad \therefore u = xe^{xy} + ye^{xy} = C$