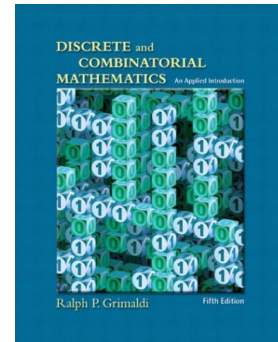
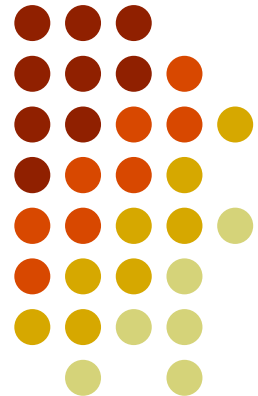


Discrete Mathematics

-- Chapter 1: Fundamental Principles of Counting



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Outline



- Sum & Product
- Permutations
- Combinations: The Binomial Theorem (二項式定理)
- Combinations with Repetition
- The Catalan Numbers
- Summary



1.1 The Rules of Sum and Product

- **The Rule of Sum**

- If a first task can be performed in m ways, while a second task can be performed in n ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any one of $m+n$ ways.

- **The Rule of Product**

- If a procedure can be broken down into first and second stages, and if there are m possible outcomes for the first stage and if, for each of these outcomes, there are n possible outcomes for the second stage, then the total procedure can be carried out, in the designated order, in mn ways.
- **Ex 1.6:** If a license plate consists of two letters followed by four digits, how many different plates are there? $26 \times 26 \times 10 \times 10 \times 10 \times 10$

Rule combination



1.2 Permutations

- Permutation: counting linear arrangements of (**distinct**) objects
- If there are n distinct objects and r is an integer, with $1 \leq r \leq n$, then **by the rule of product**,
 - The number of permutations of size r for the n objects is

$$\begin{aligned} P(n, r) &= n(n-1)(n-2) \cdots (n-r+1) \\ &= n(n-1)(n-2) \cdots (n-r+1) \times \frac{(n-r)(n-r-1) \cdots (3)(2)(1)}{(n-r)(n-r-1) \cdots (3)(2)(1)} \\ &= \frac{n!}{(n-r)!} \quad \longleftarrow n \text{ factorial} \end{aligned}$$

- **Ex 1.9:** Given 10 students, *three* are to be chosen and seated in a row. How many such linear arrangements are possible?



Permutations with Repeated Objects

- If there are n objects with n_1 **indistinguishable** objects of a first type, n_2 indistinguishable objects of a second type, ..., and n_r indistinguishable objects of an r th type, where $n_1 + n_2 + \dots + n_r = n$.

→ the number of (linear) arrangements of the given n objects

$$= \frac{n!}{n_1! n_2! \dots n_r!}$$

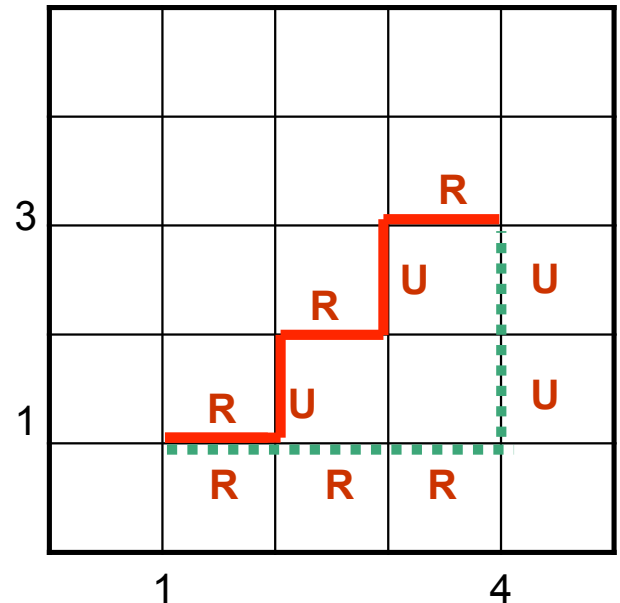
- **Ex 1.13:** Arranging all of the letters in MASSASAUGA, we find there are $\frac{10!}{4! 3! 1! 1! 1!}$ possible arrangements, $\frac{7!}{3! 1! 1! 1!}$

arrangements while all four A's are together.



Permutations with Repeated Objects

- **Ex 1.14:** Determine the number of (staircase) paths in the xy-plane from $(1, 1)$ to $(4, 3)$, where each such path is made up of individual steps going one unit to the right or one unit upward.
- As for xyz-space, from $(1, 1, 1)$ to $(4, 3, 2)$?





Permutations with Repeated Objects

(Exercise in textbook P45)

29. a) In how many ways can a particle move in the xy -plane from the origin to the point $(7, 4)$ if the moves that are allowed are of the form:

(R): $(x, y) \rightarrow (x + 1, y)$; (U): $(x, y) \rightarrow (x, y + 1)$?

b) How many of the paths in part (a) do not use the path from $(2, 2)$ to $(3, 2)$ to $(4, 2)$ to $(4, 3)$ shown in Fig. 1.12?

c) Answer parts (a) and (b) if a third type of move

(D): $(x, y) \rightarrow (x + 1, y + 1)$

is also allowed.

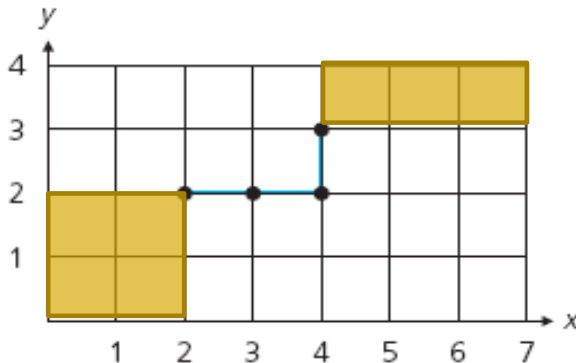


Figure 1.12

$$a) \frac{11!}{7! * 4!}$$

$$b) \frac{11!}{7! * 4!} - \frac{4!}{2! * 2!} * \frac{4!}{3! * 1!}$$

$$1D = 1R + 1U$$



Combinatorial Proofs

- Prove that $\frac{(2k)!}{2^k}$ is an integer.
 - Consider $2k$ symbols $x_1, x_1, x_2, x_2, \dots, x_k, x_k$.
 - The number of ways they can be arranged is

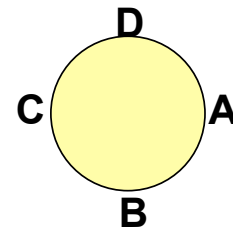
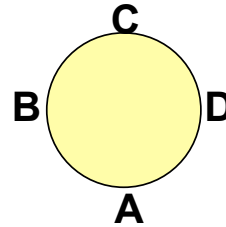
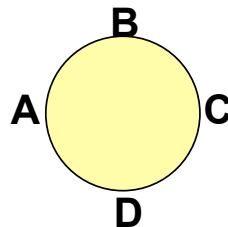
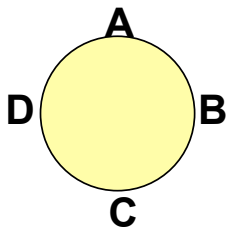
$$\frac{(2k)!}{2^k} = \frac{(2k)!}{2!2!\cdots 2!}$$

- It must be an integer.
- Prove that $\frac{(mk)!}{(m!)^k}$ is also an integer.



Arrangement around a Circle

- Consider n distinct objects
- Two arrangements are considered the same when one can be obtained from the other by rotation.
- How many different circular arrangements?
 - Thinking distinct linear arrangements for 4 objects, e.g., ABCD, BCDA, CDAB, and ...
 - So, the number of circular arrangements is? **$4!/4 = 3!$**





Arrangement around a Circle

- **Ex 1.17**: arrange six people (3 males, 3 females) around the table so that the sexes alternate.

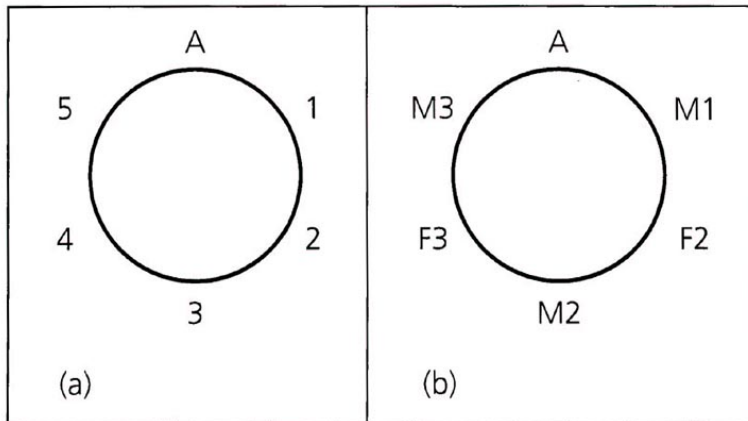


Figure 1.3

No constraint $6!/6 = 5! = 120$

Sexes alternate
 $3 \times 2 \times 2 \times 1 \times 1 = 12$



1.3 Combinations: The Binomial Theorem

- If there are n distinct objects and r is an integer, with $1 \leq r \leq n$
- The number of combinations (*selections without reference to order*) of size r for the n objects is

$$C(n, r) = \boxed{\binom{n}{r}} = \frac{P(n, r)}{r!} = \frac{n(n-1) \cdots (n-r+1)}{r(r-1) \cdots 1} = \frac{n!}{r!(n-r)!}$$

- are sometimes read “ n choose r ”.
- $C(n, 0) = C(n, n) = 1$, for all $n \geq 0$.
- $C(n, r) = C(n, n-r)$, for all $n \geq 0$.



Combinations

- Ex 1.19
 - To win the grand prize for PowerBall one must match five numbers selected from 1 to 49 **inclusive** and then must also match the powerball, an integer from 1 to 42 inclusive.

- Consequently, by the rule of product, they can select the six numbers in $\binom{49}{5} \binom{42}{1} = 80,089,128$ ways.

How about the case in Taiwan?

$$\binom{38}{6} \binom{8}{1} = 22,085,448$$





Combinations

- Ex 1.20

- A student taking a history examination is directed to answer any seven of 10 essay questions. There is no concern about order here.

- She can answer the examination in $\binom{10}{7} = 120$ ways

- If the student must answer three questions from the first five and four questions from the last five.

$$\binom{5}{3} \binom{5}{4} = 5 \times 10 = 50$$

- If the student must answer **at least** three questions from the first five.

$$\binom{5}{3} \binom{5}{4} + \binom{5}{4} \binom{5}{3} + \binom{5}{5} \binom{5}{2} = 50 + 50 + 10 = 110$$

$$\text{also, } \sum_{i=3}^5 \binom{5}{i} \binom{5}{7-i}$$

● 窮舉

● 反例



Combinations

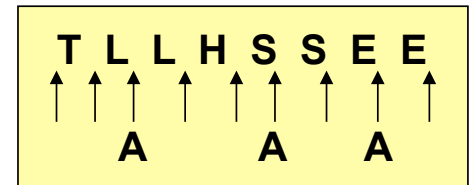
- Ex 1.23

- The number of arrangements of the letters in TALLAHASSEE is?

$$\frac{11!}{3! 2! 2! 2! 1! 1!} = 831,600 \quad \text{Permutations with Repeated Objects}$$

- How many of these arrangements have no adjacent A's?

$$\left(\frac{8!}{2! 2! 2! 1! 1!} \right) \binom{9}{3} = 5040 \times 84 = 423,360$$





Overcounting

• Ex 1.25

- How many ways to draw five cards from a standard deck of 52 cards with no clubs?

$$\binom{39}{5}$$

- How many ways to draw five cards with at least one club?

$$\binom{13}{1} \binom{51}{4} = 3,248,700$$



What are the repetition cases ?

C2 – C5 D6 C9 H3
C5 – C9 H3 C2 D6
C9 – D6 C2 H3 C5

● 反例

$$\binom{52}{5} - \binom{39}{5} = 2,023,203$$

● 窮舉

$$\binom{13}{1} \binom{39}{4} + \binom{13}{2} \binom{39}{3} + \binom{13}{3} \binom{39}{2} + \binom{13}{4} \binom{39}{1} + \binom{13}{5} \binom{39}{0} = \sum_{i=1}^5 \binom{13}{i} \binom{39}{5-i} = 2,023,203$$



Theorem 1.1: The Binomial Theorem

- $(x + y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \dots + \binom{n}{n}x^ny^0 = \sum_{k=0}^n \binom{n}{k}x^ky^{n-k}$
- There are $C(n, k)$ ways to choose k x 's and $n-k$ y 's.
- $C(n, k)$ is often referred to as a binomial coefficient.

- In case $(x+y)^2$

$$\begin{array}{r}
 x \quad + \quad y \\
 x \quad + \quad y \\
 \hline
 xy \quad + \quad y^2 \\
 x^2 + xy \\
 \hline
 x^2y^0 + 2x^1y^1 + x^0y^2
 \end{array}$$



Theorem 1.1: The Binomial Theorem

- the coefficient of x^2y^2 in the expansion of $(x+y)^4$ is $\binom{4}{2} = 6$

$x + y$

$x + y$

$x + y$

$x + y$

Table 1.5

Factors Selected for x		Factors Selected for y	
(1)	1, 2	(1)	3, 4
(2)	1, 3	(2)	2, 4
(3)	1, 4	(3)	2, 3
(4)	2, 3	(4)	1, 4
(5)	2, 4	(5)	1, 3
(6)	3, 4	(6)	1, 2



The Binomial Theorem

- $(x + y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \dots + \binom{n}{n}x^ny^0 = \sum_{k=0}^n \binom{n}{k}x^ky^{n-k}$
- **Ex 1.26**
 - a) What is the coefficient of x^5y^2 in the expansion of $(x + y)^7$?
 - b) What is the coefficient of a^5b^2 in the expansion of $(2a - 3b)^7$?

a) the coefficient of x^5y^2 in $(x + y)^7$ is $\binom{7}{5}$

b) Set $x = 2a, y = -3b$

$$\binom{7}{5}x^5y^2 = \binom{7}{5}(2a)^5(-3b)^2 = \binom{7}{5}(2)^5(-3)^2a^5b^2 = 6048a^5b^2$$



Corollaries of The Binomial Theorem

- Corollary 1.1:
 - a) $\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = ?$ **2^n**
 - b) $\binom{n}{0} - \binom{n}{1} + \dots + (-1)^n \binom{n}{n} = ?$ **0**

- Proof

- Part (a) set $x=y=1$
- Part (b) set $x=-1$ and $y=1$

$$(x + y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \dots + \binom{n}{n}x^ny^0 = \sum_{k=0}^n \binom{n}{k}x^ky^{n-k}$$

how about $\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \dots + 2^k\binom{n}{k} + \dots + 2^n\binom{n}{n} = ?$

Theorem 1.2 : The Multinomial Theorem



- The coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$ in the expansion of

$$(x_1 + x_2 + \cdots + x_t)^n \text{ is } \frac{n!}{n_1! n_2! \cdots n_t!} = \binom{n}{n_1, n_2, \dots, n_t}$$

where $0 \leq n_i \leq n$, and $n_1 + n_2 + \cdots + n_t = n$.

Proof

- The number of ways we can select x_1 from n_1 of the n factors, x_2 from n_2 of the $n - n_1$ remaining factors, ...

$$\binom{n}{n_1} \binom{n - n_1}{n_2} \cdots \binom{n - n_1 - n_2 - \cdots - n_{t-1}}{n_t} = \frac{n!}{n_1! n_2! \cdots n_t!} = \binom{n}{n_1, n_2, \dots, n_t}$$

Multinomial coefficient
 $t=2 \rightarrow$ binomial coefficient



The Multinomial Theorem

- Ex 1.27

- What is the coefficient of x^5y^2 in the expansion of $(x + y + z)^7$?

$$\binom{7}{5 \ 2 \ 0} = \frac{7!}{5! \ 2! \ 0!} = 21$$

- What is the coefficient of $a^2b^3c^2d^5$ in the expansion of $(a + 2b - 3c + 2d + 5)^{16}$?

Set $v = a, w = 2b, x = -3c, y = 2d, z = 5$

the coefficient of $v^2w^3x^2y^5z^4$ in $(v + w + x + y + z)^{16}$ is $\binom{16}{2,3,2,5,4}$

$$\begin{aligned} \binom{16}{2,3,2,5,4} (a)^2 (2b)^3 (-3c)^2 (2d)^5 (5)^4 &= \binom{16}{2,3,2,5,4} (1)^2 (2)^3 (-3)^2 (2)^5 (5)^4 a^2 b^3 c^2 d^5 \\ &= 435,891,456,000,00 a^2 b^3 c^2 d^5 \end{aligned}$$



1.4 Combination with Repetition

- Ex 1.28
 - How many different purchases are possible for **seven** students each having one of the following, a cheeseburger, a hot dog, a taco, or a fish sandwich?

Possible way	Another way
c,c,h,h,t,t,f	xx xx xx x
c,c,c,c,h,t,f	xxxx x x x
c,c,c,c,c,c,f	xxxxxx x
h,t,t,f,f,f,f	x xx xxxx

$$7 \text{ x's} + 3 \text{ |'s} \quad \binom{10}{7} = \frac{10!}{7!3!} = \frac{(4+7-1)!}{7!(4-1)!}$$

- The number of combinations of n objects taken r at a time, with repetition, is (foods) (students)

$$H_{r_7}^n = C(n+r-1, r) = \frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r}$$



1.4 Combination with Repetition

- Ex 1.31**
 - In how many ways can we distribute **seven bananas** and **six oranges** among **four children** so that each child receives at least one banana?
 - Remaining bananas: $7-4=3$
 - 3 bananas was distributed 4 children: $(n=4, r=3)$**
 $C(4+3-1, 3) = C(6, 3) = 20$
 - 6 oranges was distributed 4 children: $(n=4, r=6)$
 $C(4+6-1, 6) = C(9, 6) = 84$
 - Thus, $20 \times 84 = 1680$

Distribute 3 bananas to 4 children	
c_1, c_2, c_3	b b b
c_1, c_3, c_3	b bb
c_3, c_4, c_4	 b bb
c_4, c_4, c_4	 bbb

3 b's + 3 |'s



1.4 Combination with Repetition

- **Ex 1.33**

- Determine all integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 7$, where $x_i \geq 0$ for all $1 \leq i \leq 4$.

- $n=4, r=7 \rightarrow C(4+7-1, 7)$

3' "+", 7' "1" linear permutation

- Equivalence: $C(n + r - 1, r)$

- The number of integer solutions of the equation $x_1 + x_2 + \dots + x_n = r$, where $x_i \geq 0$ for all $1 \leq i \leq n$.
- The number of selections, with repetition, of size r from a collection of size n .
- The number of ways r identical objects can be distributed among n distinct containers.

=the number of ways r distinct objects be distributed among n identical containers ?

- Difference

- r distinct objects can be distributed among n distinct containers in n^r ways.



1.4 Combination with Repetition

the number of ways r objects be distributed among n containers

	<i>r distinct</i>	<i>r identical</i>
<i>n distinct</i>	n^r	$C(n+r-1, r)$
<i>n identical</i>	$n^r/n!$ X <i>See Chapter 5</i>	<i>See Chapter 9</i>

$$\sum_{i=1}^n S(r, i)$$


Partitions of integers




$r=3$, distinct, $n=3$, distinct $\rightarrow 3 \times 3 \times 3 = 27$

ABC			BC	A		BC		A
AB	C		B	AC		B	C	A
AB		C	B	A	C	B		AC
AC	B		C	AB		C	B	A
A	BC			ABC			BC	A
A	B	C		AB	C		B	AC
AC		B	C	A	B	C		AB
A	C	B		AC	B		C	AB
A		BC		A	BC			ABC
n_1	n_2	n_3	n_1	n_2	n_3	n_1	n_2	n_3


r distinct, n identical

 $= 3 \times 2 = 6$

 $= 3 \times 2 \times 1 = 6$

Ans: 5 ($= S(3,1) + S(3,2) + S(3,3)$)
 $\{\{ABC\}, \{AB,C\}, \{AC,B\}, \{BC,A\}, \{A,B,C\}\}$

r identical, n distinct

 $= C(3,2) = 3$

Ans: $C(3+3-1, 3) = 10$

r identical, n identical

Ans: 3





1.4 Combination with Repetition

- Ex 1.35
 - How many nonnegative integer solutions to the inequality $x_1 + x_2 + \dots + x_6 < 10$?
 - Transform the problem to $x_1 + x_2 + \dots + x_6 + x_7 = 10$, $x_i \geq 0$ for all $1 \leq i \leq 6$, but $x_7 > 0$.
 - $y_1 + y_2 + \dots + y_6 + y_7 = 9$, where $y_i = x_i$ for all $1 \leq i \leq 6$, and $y_7 = x_7 - 1$.
- $C(7+9-1, 9) = 5005$.



1.4 Combination with Repetition

- Ex 1.36
 - In the binomial expansion for $(x + y)^n$, each term is of the form $C(n, k)x^k y^{n-k}$
 - The total number of terms in the expansion is the number of nonnegative integer solutions of $n_1 + n_2 = n$.
 - $C(2 + n - 1, n) = C(n+1, n) = n+1$.
- How many terms are there in the expansion of $(w+x+y+z)^{10}$?
 $C(4+10-1, 10)$



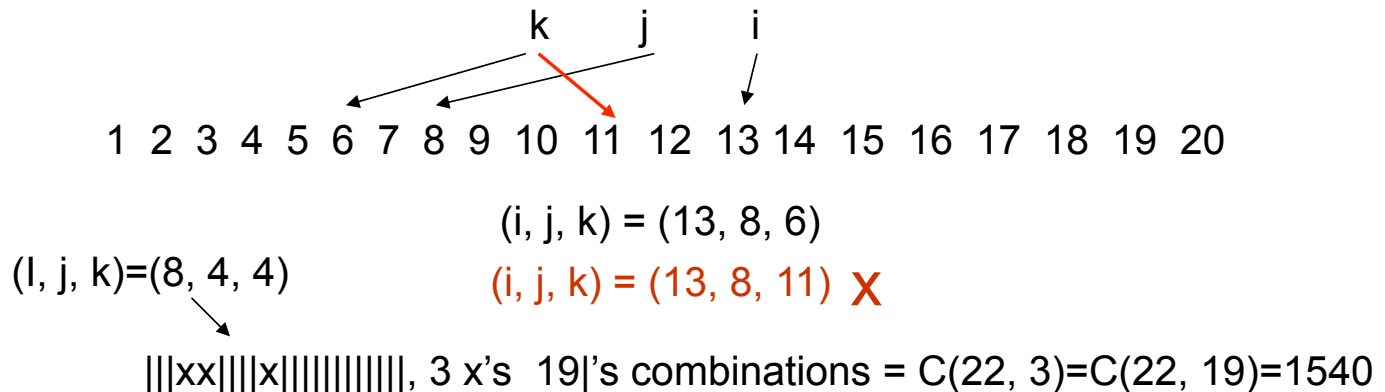
1.4 Combination with Repetition

- Ex 1.39

Consider the following program segment, where i , j , and k are integer variables.

```
for i := 1 to 20 do
  for j := 1 to i do
    for k := 1 to j do
      print (i * j + k)
```

How many times is the **print** statement executed in this program segment?





1.4 Combination with Repetition

- Ex 40

- Summation formula

- $counter = C(n+2-1, 2) = C(n+1, 2)$

- Also, $counter = \sum_{i=1}^n i = 1 + 2 + 3 + \dots + n = \binom{n+1}{2} = \frac{n(n+1)}{2}$

```
counter := 0
for i := 1 to n do
  for j := 1 to i do
    counter := counter + 1
```



1.5 Catalan Number

- Count paths from $(0,0) \rightarrow (5,5)$ but never rise over the line $y=x$

- No constraint $C(10,5)$

- With constraint

- $C(10,5) - C(10,4)$ 扣除超出對角線

- Exchange R,U after the first “crossing” U

RUUURRRUUR

\rightarrow RUURUUURRU

將第一次超過對角線後的RU轉換

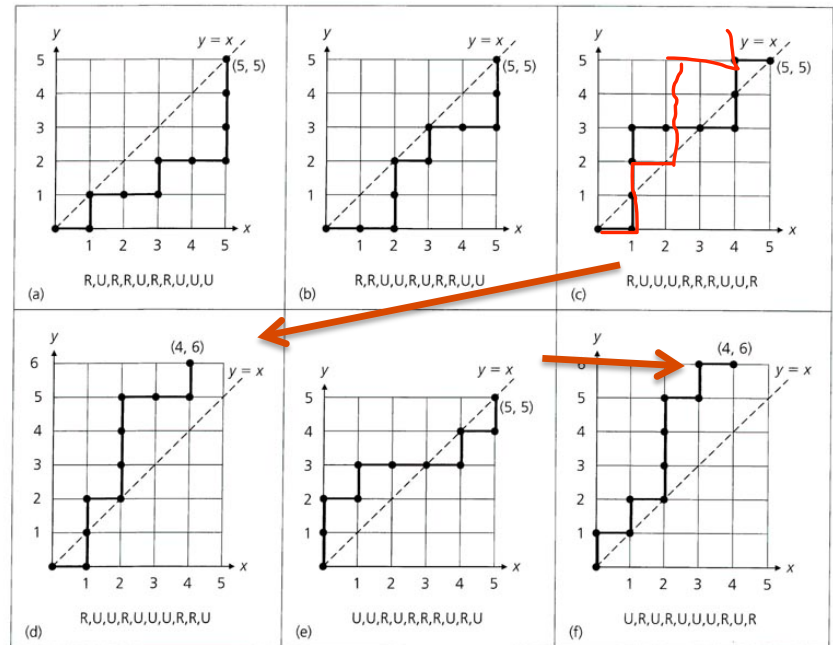


Figure 1.9



1.5 Catalan Number

- From $(0,0) \rightarrow (n,n)$

$$\# path = b_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1} \binom{2n}{n}$$

- which can determine the number of ways to parenthesize the product $x_1 x_2 x_3 x_4 \cdots x_n$.
- E.g., $n=4$, $\#parenthesis = b_3 = \frac{1}{4} * C(6,3) = 5$

$$(((x_1 x_2) x_3) x_4), ((x_1 (x_2 x_3) x_4)), ((x_1 x_2) (x_3 x_4)), \\ (x_1 ((x_2 x_3) x_4)), (x_1 (x_2 (x_3 x_4)))$$



1.6 Summary

- Fundamental techniques in counting:
 - **Top-down approach:** Divide the problems into subproblems suitable for discrete and combinatorial mathematics.

Order Is Relevant	Repetitions Are Allowed	Type of Result	Formula	Location in Text
Yes	Yes	Arrangement	n^r	Page 7
Yes	No	Permutation	$P(n, r) = \frac{n!}{(n-r)!}$	Page 7
No	No	Combination	$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$	Page 15
No	Yes	Combination with repetition	$C(n+r-1, r) = \binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}$	Page 27



Observation & Tricks

- The previous 4 cases are not answers by themselves. Instead they are tools that you have to learn to use.
- It is often not immediately clear which tool to use
- Experiences
 - try a small example and solve it by “brute force”.
 - Verify your answers in the special cases (e.g., $k=0$ or $k=n$).
 - More Practice, more experiences