Chapter 4.
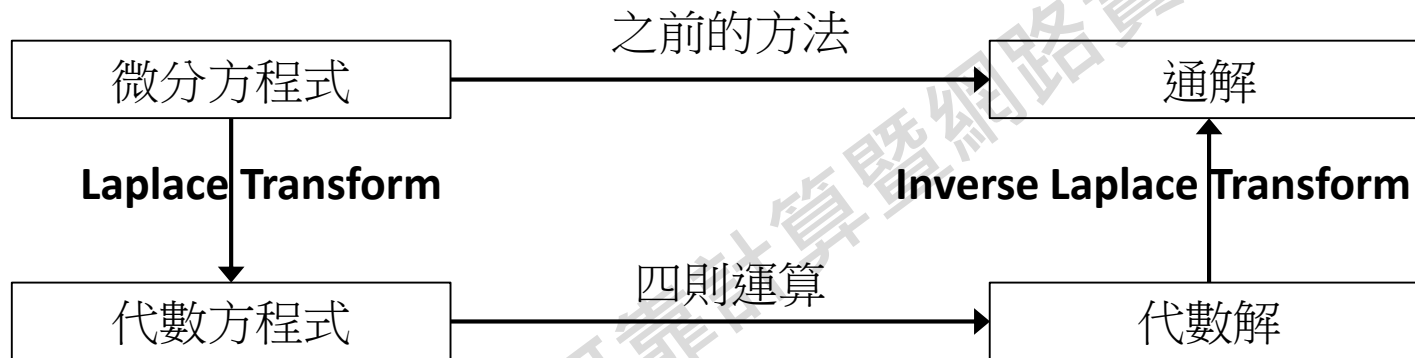# Laplace Transform

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Laplace Transform

- 想法：

微分方程式		通解

之前的方法

Laplace Transform　　　　　　　**Inverse Laplace Transform**

四則運算

代數方程式		代數解

Laplace Transform

From 微分方程式 to 代數方程式必須具備:

$$y^{''} + 3y^{'} + 5y = e^t$$

$$y^{''} + 3y^{'} + 5y = \cos t$$

$$y^{''} + 3y^{'} + 5y = \sin t$$

......

才可以進行 Laplace Transform

1. 基本函數

2. 一階、二階$(y^{'}, y^{''})$

Laplace Transform

- 定義：

$$L\{f(t)\} = f(t)$$函數的 Laplace Transform

$$= \int_0^\infty f(t)e^{-st}\,dt$$

$$= F(s)$$

$$f(t) = L^{-1}\{F(s)\} \Rightarrow \text{Inverse Laplace Transform}$$

$$= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)e^{st}\,ds \quad \text{複變(Complex Analysis)}$$

*轉換與逆轉換要同時學習

Laplace Transform

例：$y' + 2y = e^t$, $y(0) = 1$

$method\,(I)$

$y_h = Ce^{-2t}$

$y_p = \dfrac{1}{D+2}e^t = \dfrac{1}{3}e^t$

$y(t) = Ce^{-2t} + \dfrac{1}{3}e^t$, $y(0) = 1, C = \dfrac{2}{3}$

$y(t) = \dfrac{2}{3}e^{-2t} + \dfrac{1}{3}e^t$

Laplace Transform

例：$method(II)$

$$L\{y'\} + 2L\{y\} = L\{e^t\} \Rightarrow sY(s) - y(0) + 2Y(s) = \frac{1}{s-1}$$

$$(s+2)Y(s) = \frac{s}{s-1}$$

$$Y(s) = \frac{s}{(s-1)(s+2)} = \frac{a}{s-1} + \frac{b}{s+2}$$

$$a = \frac{1}{3}, b = \frac{2}{3}$$

$$Y(s) = \frac{1}{3(s-1)} + \frac{2}{3(s+2)} \Rightarrow \text{Inverse Laplace Transform}$$

$$y(t) = \frac{1}{3}e^t + \frac{2}{3}e^{-2t}$$

Laplace Transform

(1) $f(t) = e^{at}, a \in const$

$$F(s) = L\{f(t)\} = \int_0^\infty e^{at} e^{-st} dt$$

$$= \int_0^\infty e^{-(s-a)t} dt$$

$$= -\frac{1}{s-a} e^{-(s-a)t} \Big|_0^\infty$$

$$= -\frac{1}{s-a} e^{-(s-a)\infty} - (-\frac{1}{s-a}), s-a > 0 \because 要收斂$$

$$= \frac{1}{s-a}$$

$$f(t) = e^{at} \Rightarrow F(s) = \frac{1}{s-a}$$

Laplace Transform

例：$f(t) = e^{-2t}$

$$\Rightarrow F(s) = \frac{1}{s+2}$$

例：$G(s) = \dfrac{2}{s+3}$

$$\Rightarrow g(t) = L^{-1}\left\{ 2\frac{1}{s-(-3)} \right\}$$

$$= 2e^{-3t}$$

Laplace Transform

(2) $f(t) = \cos at = \dfrac{e^{iat} + e^{-iat}}{2}$

$F(s) = \displaystyle\int_0^\infty f(t)e^{-st}dt = \int_0^\infty \cos(at)e^{-st}dt$

$L\{\cos at\} = L\left\{\dfrac{e^{iat} + e^{-iat}}{2}\right\}$

$= \dfrac{1}{2}L\{e^{iat}\} + \dfrac{1}{2}L\{e^{-iat}\} = \dfrac{1}{2}\dfrac{1}{(s-ia)} + \dfrac{1}{2}\dfrac{1}{(s+ia)}$

$= \dfrac{1}{2}\dfrac{2s}{(s-ia)(s+ia)} = \dfrac{s}{(s-ia)(s+ia)}$

$= \dfrac{s}{s^2 + a^2}$

$f(t) = \cos at \Rightarrow F(s) = \dfrac{s}{s^2 + a^2}$

Laplace Transform

例：$\mathcal{L}\{\cos 3t\} = \dfrac{s}{s^2 + 3^2} = \dfrac{s}{s^2 + 9}$

完成 $\displaystyle\int_0^\infty \cos at \cdot e^{-st}\,dt$

令 $dv = e^{-st}\,dt, \quad v = \dfrac{-1}{s}e^{-st}$

$u = \cos at, \quad du = -a\sin at\,dt$

$\displaystyle\int_0^\infty \cos at\, e^{-st}\,dt$

$= \cos at \cdot \dfrac{-1}{s}e^{-st}\,\bigg|_0^\infty - \int_0^\infty \dfrac{-1}{s}e^{-st}(-a\sin at)\,dt, \quad s > 0$

$= [0 - \dfrac{-1}{s}] - \dfrac{a}{s}\displaystyle\int_0^\infty \sin at\, e^{-st}\,dt$

Laplace Transform

$$\diamondsuit u = \sin at, du = a\cos at dt$$

$$dv = e^{-st}dt, v = \frac{-1}{s}e^{-st}$$

$$= \frac{1}{s} - \frac{a}{s}[\sin at \cdot \frac{-1}{s}e^{-st}\Big|_0^\infty - \int_0^\infty (\frac{-1}{s}e^{-st})a\cos at dt], s > 0$$

$$= \frac{1}{s} - \frac{a}{s} \cdot \frac{a}{s}\int_0^\infty \cos at \cdot e^{-st}dt$$

$$\Rightarrow F(s) = \frac{1}{s} - \frac{a^2}{s^2}F(s)$$

$$\Rightarrow F(s) = \frac{s}{s^2 + a^2}$$

Laplace Transform

(3) $f(t) = \sin at$

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty \sin at \cdot e^{-st}\,dt$$

利用 $\sin at = \dfrac{e^{iat} - e^{-iat}}{2i}$

$$\Rightarrow \mathscr{L}\{\sin at\} = \frac{1}{2i}\mathscr{L}\{e^{iat}\} - \frac{1}{2i}\mathscr{L}\{e^{-iat}\}$$

$$= \frac{1}{2i}\frac{1}{s-ai} - \frac{1}{2i}\frac{1}{s+ai} = \frac{a}{s^2 + a^2}$$

Laplace Transform

例：$f(t) = \sin 2t$

$\Rightarrow F(s) = \dfrac{2}{s^2 + 4}$

例：$F(s) = \dfrac{s}{s^2 + 16}$

$\Rightarrow f(t) = \mathcal{L}^{-1}\{F(s)\} = \cos 4t$

例：$F(s) = \dfrac{3}{s^2 + 16}$

$\Rightarrow f(t) = \mathcal{L}^{-1}\{F(s)\} = \dfrac{3}{4} * \dfrac{4}{s^2 + 16} = \dfrac{3}{4}\sin 4t$

例：$\mathcal{L}^{-1}\{\dfrac{3}{s+6} + \dfrac{5s}{s^2 + 25} + \dfrac{2}{s^2 + 36}\}$

$\Rightarrow 3e^{-6t} + 5\cos 5t + \dfrac{1}{3}\sin 6t$

Laplace Transform

(4) $f(t) = t$

$$F(s) = \int_0^\infty t e^{-st} dt$$

$$= t(\frac{-1}{s} e^{-st})\Big|_0^\infty - \int_0^\infty \frac{-1}{s} e^{-st} dt$$

$$= \lim_{t \to \infty} \frac{-t}{s e^{st}} - 0 + \frac{1}{s}(\frac{-1}{s} e^{-st})\Big|_0^\infty$$

$$= \frac{1}{s^2}$$

$$\Rightarrow \mathscr{L}\{t\} = \frac{1}{s^2}$$

$$\Rightarrow f(t) = t \Rightarrow F(s) = \frac{1}{s^2}$$

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Laplace Transform

例： $\mathscr{L}\{3t\}$

$\Rightarrow F(s) = \dfrac{3}{s^2}$

例： $G(s) = \dfrac{5}{s^2}$

$\Rightarrow g(t) = 5t$

例： $G(s) = \dfrac{5s+1}{s^2}$

$\Rightarrow g(t) = 5 + t$

Laplace Transform

(5) $f(t) = H(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$

單位步階函數=unit-step function

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{H(t)\} = \mathscr{L}\{U(t)\}$$

$$= \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty 1 \cdot e^{-st}\,dt = \frac{1}{s}$$

Laplace Transform

(a) High-pass filter

$$g(t) = \begin{cases} 1 & t > 3 \\ 0 & t < 3 \end{cases}$$

$$g(t) = H(t-3)$$

(b) Band-pass filter

$$g(t) = \begin{cases} 1 & -3 < t < 3 \\ 0 & t > 3 \text{或} t < -3 \end{cases}$$
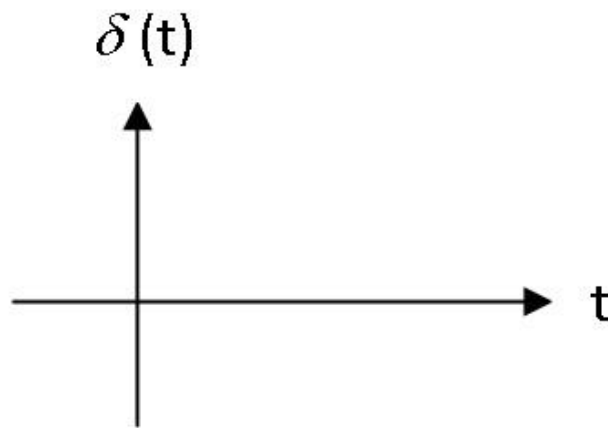
$$g(t) = H(t+3) - H(t-3)$$

Laplace Transform

例：
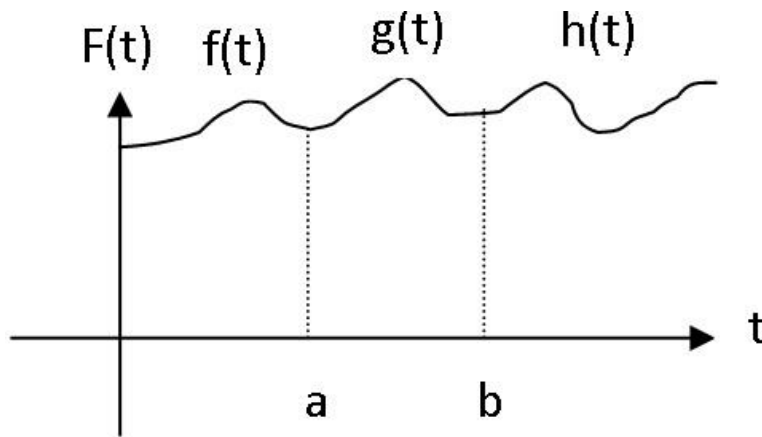
$$F(t) = \begin{cases} 0, t < 0 \\ f(t), 0 < t < a \\ g(t), a < t < b \\ h(t), b < t \end{cases}$$

$$F(t) = f(t)\left[H(t) - H(t-a)\right] + g(t)\left[H(t-a) - H(t-b)\right] + h(t)[H(t-b)]$$

$$= f(t) + \left[g(t) - f(t)\right]H(t-a) + [h(t) - g(t)]H(t-b)$$

Laplace Transform

(6) $\delta(t)\begin{cases}1, t = 0 \\ 0, t \neq 0\end{cases}$,其中1為面積值

$\delta(t) = unit-impulse\ function$ 單位脈衝函數

自然界沒有這函數，可用近似函數來取代
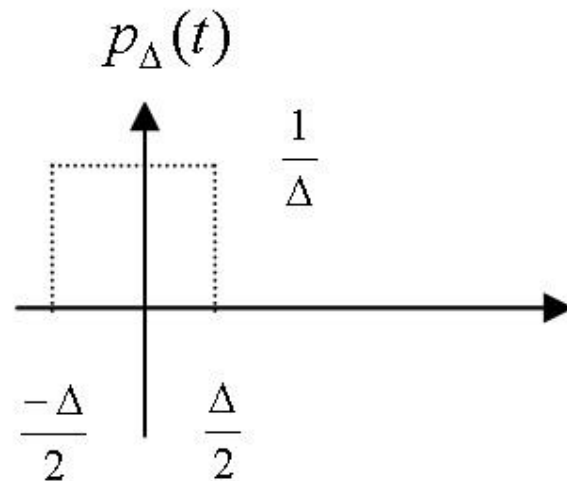
Laplace Transform

$$\lim_{\Delta \to 0} P_\Delta(t) = \begin{cases} 1, t = 0 \\ 0, t \neq 0 \end{cases}$$

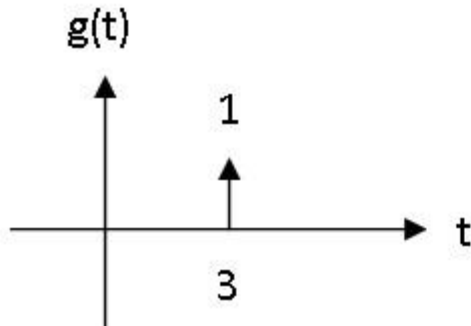$$\mathscr{L}\{\delta(t)\}$$

$$= \int_0^\infty \delta(t)e^{-st}dt$$

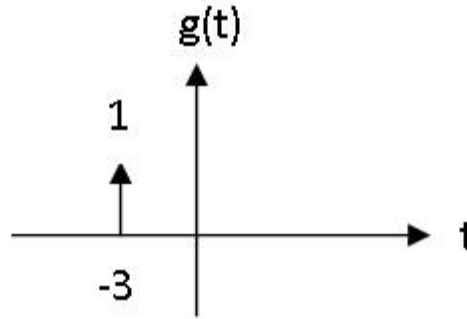在 t 的積分範圍內，只有在 $t = 0$ 時 $\delta(t)$ 才有值

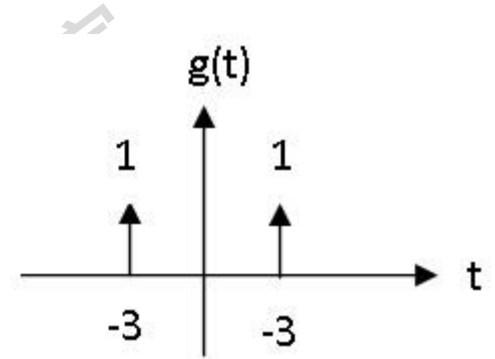$$= \int_0^\infty \delta(t)e^{-0}dt$$

$$= 1$$

Laplace Transform

例：

$$g(t) = \delta(t-3)$$

$$g(t) = \delta(t+3)$$

$$g(t) = \delta(t+3) + \delta(t-3)$$

Laplace Transform

(7) $f(t^n) \implies F(s) = \dfrac{n!}{s^{n+1}}$

例: $\mathcal{L}\{e^t + 3\sin 2t + 5\cos 3t + 3t^2 + 5\delta(t)\}$

$$= \frac{1}{s-1} + \frac{2}{s^2+4} \cdot 3 + \frac{s}{s^2+9} \cdot 5 + \frac{2!}{s^3} \cdot 3 + 5$$

例: $\mathcal{L}^{-1}\left\{7 + \dfrac{2}{s^5} + \dfrac{2}{s^2+1} + \dfrac{s}{s^2+49} + \dfrac{3}{s+5}\right\}$

$$= 7\delta(t) + \frac{2}{4!}t^4 + 2\sin t + \cos 7t + 3e^{-5t}$$