Chapter 24
Single-Source Shortest Paths

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Shortest paths

- How to find the shortest route between two points on a map.
  - Input:
    - Directed graph $G = (V, E)$
    - Weight function $w : E \rightarrow \mathbf{R}$
  - ***Weight of path*** $p = \langle v_0, v_1, \ldots, v_k \rangle$

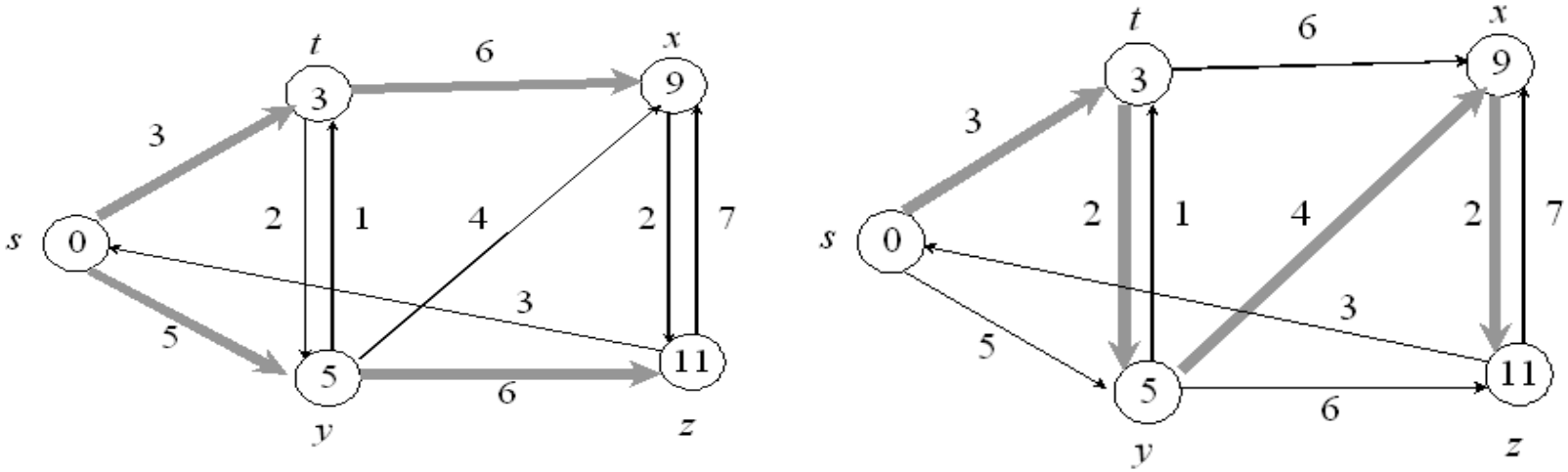
$$= \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

= sum of edge weights on path p.

- ***Shortest-path weight*** u to v :
  - $\delta(u,v) = \begin{cases} \min\{w(p) : u \overset{p}{\rightsquigarrow} v\}, & \text{if there exists a path } u \rightsquigarrow v \\ \infty, & \text{otherwise.} \end{cases}$
  - Shortest path u to v is any path p such that $w(p) = \delta(u, v)$.

- ***Example***: shortest paths from *s*

  [*d values appear inside vertices. Shaded edges show shortest paths.*]



  - This example shows that the shortest path might not be unique.
  - It also shows that when we look at shortest paths from one vertex to all other vertices, the shortest paths are organized as tree.

- Can think of weights as representing any measure that
  - accumulates linearly along a path.
  - we want to minimize.

- Example: time, cost, penalties, loss.
  - Generalization of breadth-first search to weighted graphs.

Variants

- ***Single-source***
  - Find shortest paths from a given source vertex $s \in V$ to every vertex $v \in V$.
- ***Single-destination***
  - Find shortest paths to a given destination vertex.
- ***Single-pair***
  - Find shortest path from u to v. No way known that's better in worst case than solving single-source.
- ***All-pairs***
  - Find shortest path from u to v for all u , $v \in V$.
  - We'll see algorithms for all-pairs in the next chapter.

Negative-weight edges

- OK, as long as no negative-weight cycles are reachable from the source.
  - If we have a negative-weight cycle, we can just keep going around it, and get $w(s, v) = -\infty$ for all v on the cycle.
  - But OK if the negative-weight cycle is not reachable from the source.
  - Some algorithms work only if there are no negative-weight edges in the graph.

    We'll be clear when they're allowed and not allowed.

Optimal substructure

- ***Lemma***

Any subpath of a shortest path is a shortest path.

***Proof*:** Cut - and - paste.

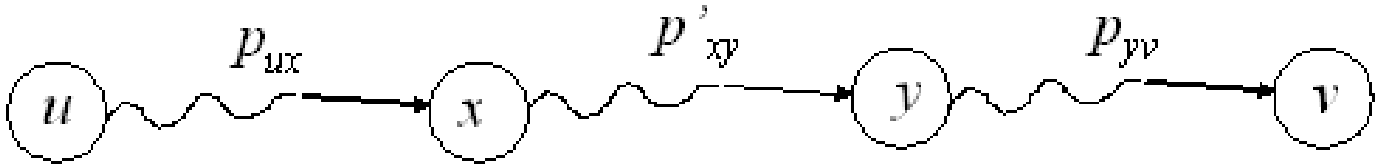
Suppose this path p is a shortest path from u to v.

Then $\delta(u, v) = w(p) = w(p_{ux}) + w(p_{xy}) + w(p_{yv})$.

Now suppose there exists a shorter path $x \overset{P'_{xy}}{\rightsquigarrow} y$.

Then $w(p'_{xy}) < w(p_{xy})$

Construct p'



Then $$w(p') = w(p_{ux}) + w(p'_{xy}) + w(p_{yv})$$
$$< w(p_{ux}) + w(p_{xy}) + w(p_{yv})$$
$$= w(p)$$

So p wasn't shortest path after all ! ■ (lemma)

Cycles

- Shortest paths can't contain cycles :
  - Already ruled out negative-weight cycles.
  - Positive-weight ⇒ we can get a shorter path by omitting the cycle.
  - Zero-weight: no reason to use them ⇒ assume that our solutions won't use them.

Output of single-source shortest-path algorithm

- For each vertex $v \in V$:
  - $d[v] = \delta(s, v)$.
    - Initially, $d[v] = \infty$.
    - Reduces as algorithms progress. But always maintain $d[v] \geq \delta(s, v)$.
    - Call $d[v]$ a ***shortest-path estimate***.
  - $\pi[v]$ = predecessor of v on a shortest path from s.
    - If no predecessor, $\pi[v]$ = NIL.
    - π induces a tree ------ ***shortest-path tree***.
    - We won't prove properties of π in lecture ------ see text.

Initialization

- All the shortest-paths algorithms start with INIT-SINGLE-SOURCE.

  - INIT-SINGLE-SOURCE ($\boldsymbol{V}$, s)
    For each $v \in \boldsymbol{V}$
        do $d[v] \leftarrow \infty$
            $\pi[v] \leftarrow$ NIL
    $d[s] \leftarrow 0$

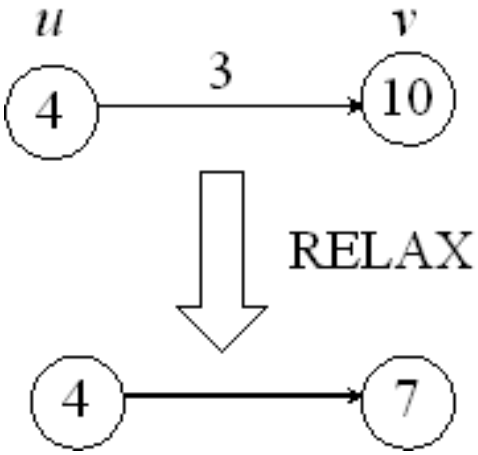
Relaxing an edge (u, v)

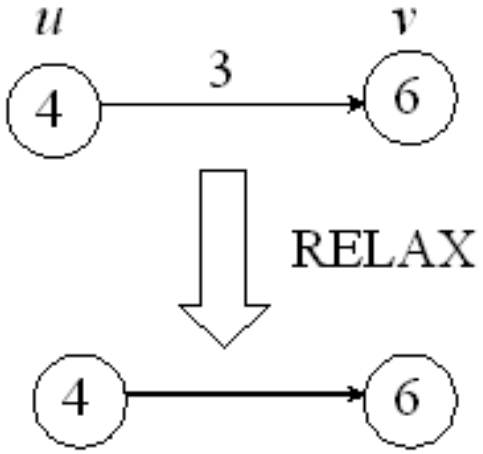
- Can we improve the shortest-path estimate for v by going through u and taking (u, v) ?
  - RELAX (u , v , w)

    If $d[v] > d[u] + w(u, v)$

    then $d[v] \leftarrow d[u] + w(u, v)$

    $\pi[v] \leftarrow u$





- For all the single-source shortest-paths algorithms we'll look at,
  - start by calling INIT-SINGLE-SOURCE,
  - then relax edges.

- The algorithms differ in the order and how many times they relax each edge.

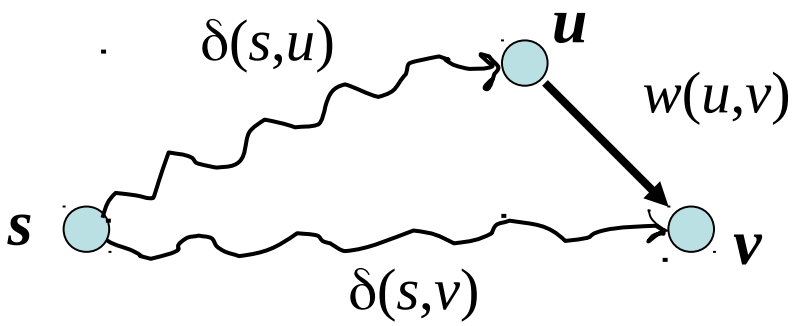
Shortest-paths properties

- Based on calling INIT-SINGLE-SOURCE once and then calling RELAX zero or more times.
- **Triangle inequality**
  - For all $(u, v) \in \boldsymbol{E}$, we have $\delta(s, v) \le \delta(s, u) + w(u, v)$.

    ***Proof*:**

    Weight of shortest path $s \rightsquigarrow v$ is $\le$ weight of any path $s \rightsquigarrow v$.

    Path $s \rightsquigarrow u \rightarrow v$ is a path $s \rightsquigarrow v$, and if we use a shortest path $s \rightsquigarrow u$, its weight is$\delta(s, u) + w(u, v)$. ■



Upper-bound property

- Always have $d[v] \geq \delta(s, v)$ for all v.

  Once $d[v] = \delta(s, v)$, it never changes.

***Proof*:** Initially true.

Suppose there exists a vertex such that $d[v] < \delta(s, v)$.

Without loss of generality, v is first vertex for which this happens.

Let u be the vertex that causes $d[v]$ to change.

Then $d[v] = d[u] + w(u, v)$.

So, $d[v] < \delta(s, v)$

$\leq \delta(s, u) + w(u, v)$ (triangle inequality)

$\leq d[u] + w(u, v)$ (v is first violation)

$\Rightarrow d[v] < d[u] + w(u, v)$. (Contradicts $d[v] = d[u] + w(u, v)$)

Once $d[v]$ reaches $\delta(s, v)$, it never goes lower. It never goes up, since relaxations only lower shortest-path estimates. ■

No-path property

- If $\delta(s, v) = \infty$, then $d[v] = \infty$ always.

***Proof*:**

$d[v] \geq \delta(s, v) = \infty \Rightarrow d[v] = \infty$. ■

Convergence property

- If $s \rightsquigarrow u \rightarrow v$ is a shortest path, $d[u] = \delta(s, u)$, and we call RELAX(u, v, w), then $d[v] = \delta(s, v)$ afterward.

  ***Proof*:**

  After relaxation:

  $d[v] \le d[u] + w(u, v)$ (RELAX code)

  $= \delta(s, u) + w(u, v)$

  $= \delta(s, v)$ (lemma ----- optimal substructure)

  Since $d[v] \ge \delta(s, v)$, must have $d[v] = \delta(s, v)$. ■

Path relaxation property

- Let $p = < v_0, v_1, ..., v_k >$ be a shortest path from $s = v_0$ to v_k . If we relax, *in order*, $(v_0, v_1), (v_1, v_2), ..., (v_{k-1}, v_k)$, even intermixed with other relaxations, then $d[v_k] = \delta(s, v_k)$.

***Proof*:**

Induction to show that $d[v_i] = \delta(s, v_i)$ after (v_{i-1}, v_i) is relaxed.

Basis:

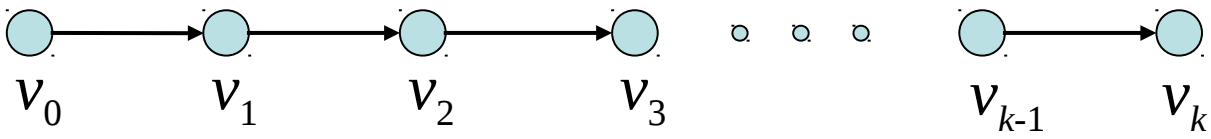
$i = 0$. Initially, $d[v_0] = 0 = \delta(s, v_0) = \delta(s, s)$.

Inductive step:

Assume $d[v_{i-1}] = \delta(s, v_{i-1})$.

Relax (v_{i-1}, v_i).

By Convergence Property, $d[v_i] = \delta(s, v_i)$ afterward and $d[v_i]$ never changes.



- $d(v_1) = \delta(s,v_0) + w(v_0,v_1) = \delta(s,v_1)$

$d(v_2) = \delta(s,v_1) + w(v_1,v_2) = \delta(s,v_2)$

$d(v_3) = \delta(s,v_2) + w(v_2,v_3) = \delta(s,v_3)$

$\vdots$

$d(v_k) = \delta(s,v_{k-1}) + w(v_{k-1},v_k) = \delta(s,v_k)$

The Bellman-Ford algorithm

- Allows negative-weight edges.
- Computes $d[v]$ and $\pi[v]$ for all $v \in \boldsymbol{V}$.
- Returns TRUE if no negative-weight cycles reachable from s, FALSE otherwise.

- BELLMAN-FORD (***V***, ***E***, *w*, *s*)

  INIT-SINGLE-SOURCE (***V***, *s*)

  for *i* ← 1 to |***V***| - 1

  do for each edge (*u*, *v*) ∈ ***E***

  do RELAX (*u*, *v*, *w*)

  for each edge (*u* , *v*) ∈ ***E***

  do if *d* [*v*] > *d* [*u*] + *w*(*u*, *v*)

  then return FALSE

  return TRUE

  Core: The first **for** loop relaxes all edges |***V***| - 1 times.

  Time: Θ(***VE***).

- $i = 1$

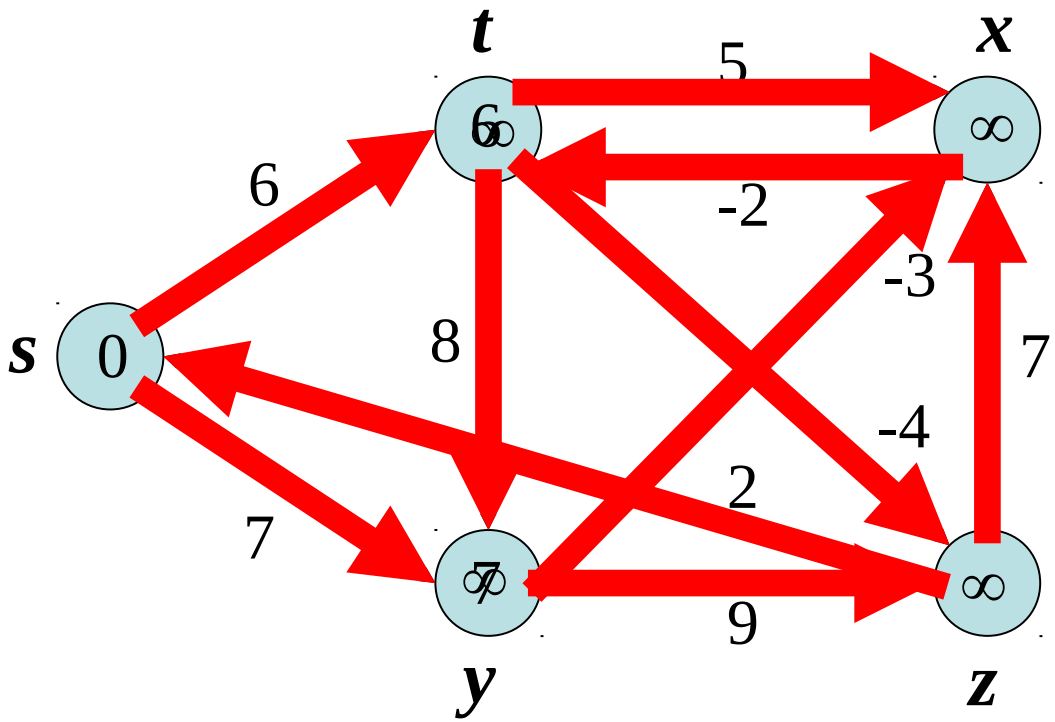
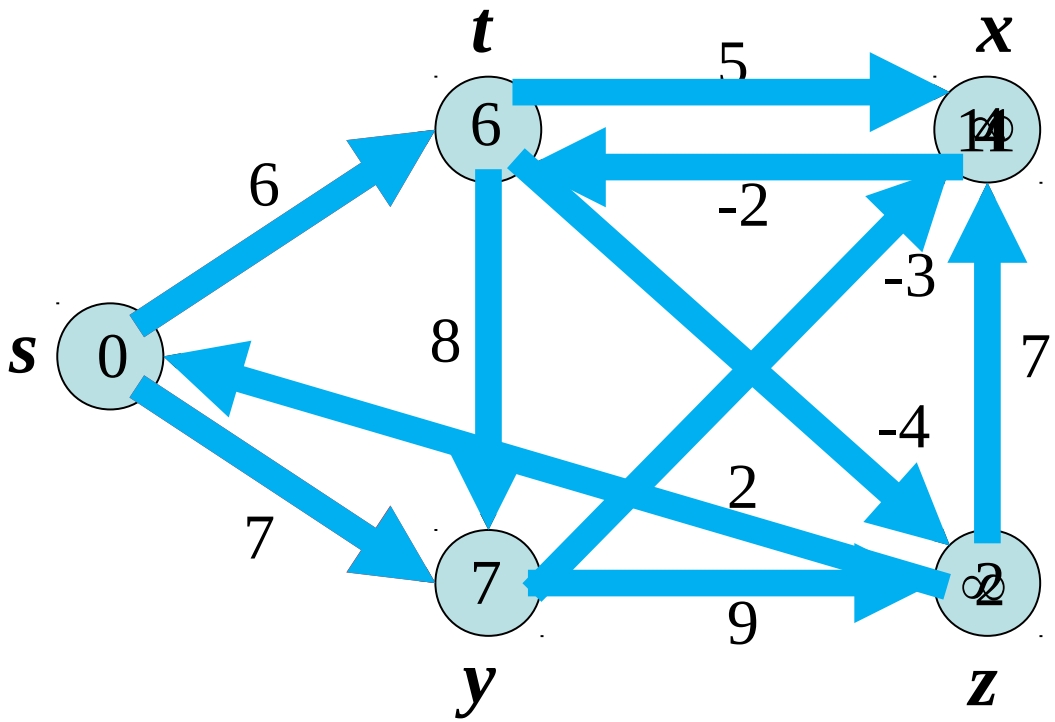


Figure 24.4 The execution of the Bellman-Ford algorithm. The source is vertex s. The d values are shown within the vertices, and shaded edges indicate predecessor values: if edge (u, v) is shaded, then $\pi[v] = u$. In this particular example, each pass relaxes the edges in the order (t, x), (t, y), (t, z), (x, t), (y, x), (y, z), (z, x), (z, s), (s, t), (s, y). **(a)** The situation just before the first pass over the edges. **(b)–(e)** The situation after each successive pass over the edges. The d and π values in part (e) are the final values. The Bellman-Ford algorithm returns TRUE in this example.

- $i = 2$



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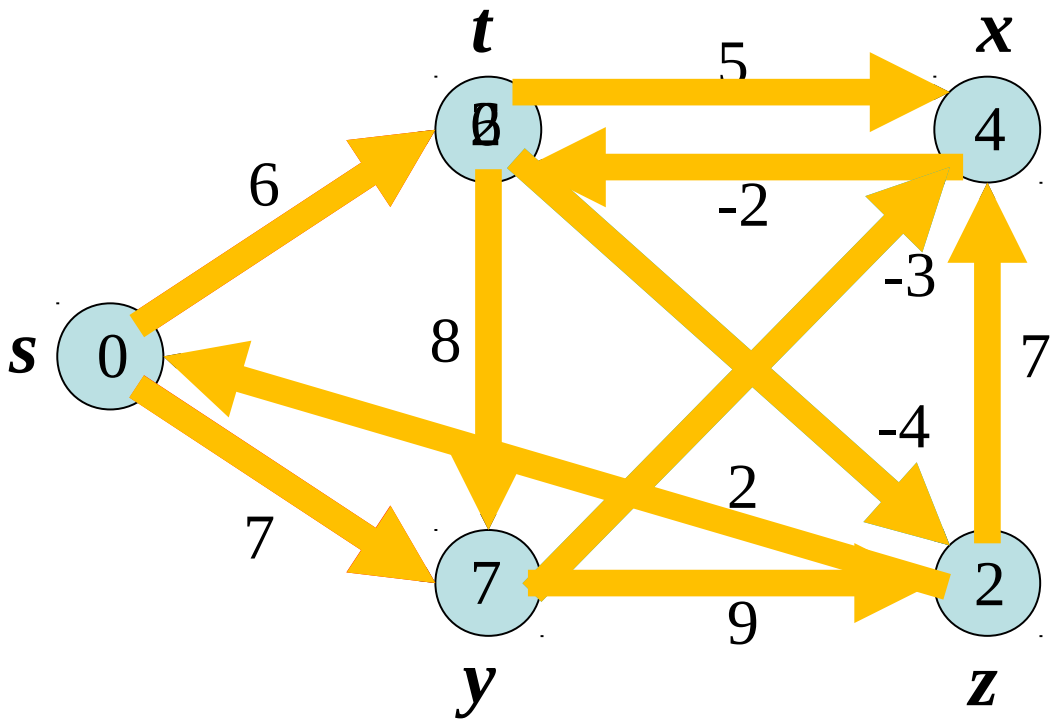
- $i = 3$



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- $i = 4$

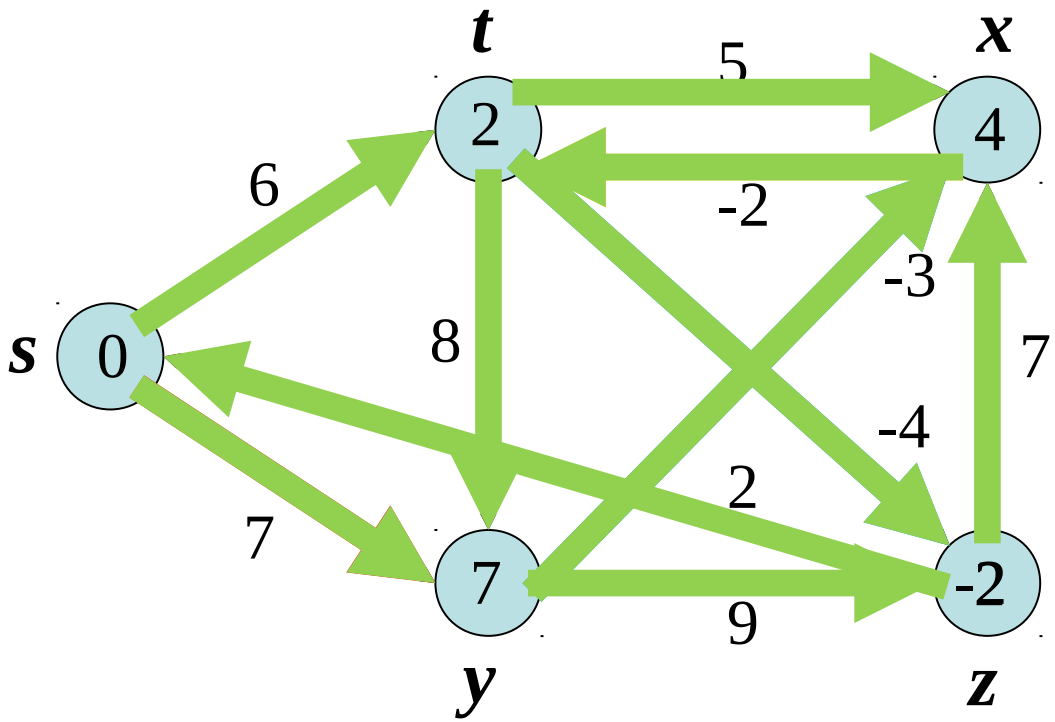
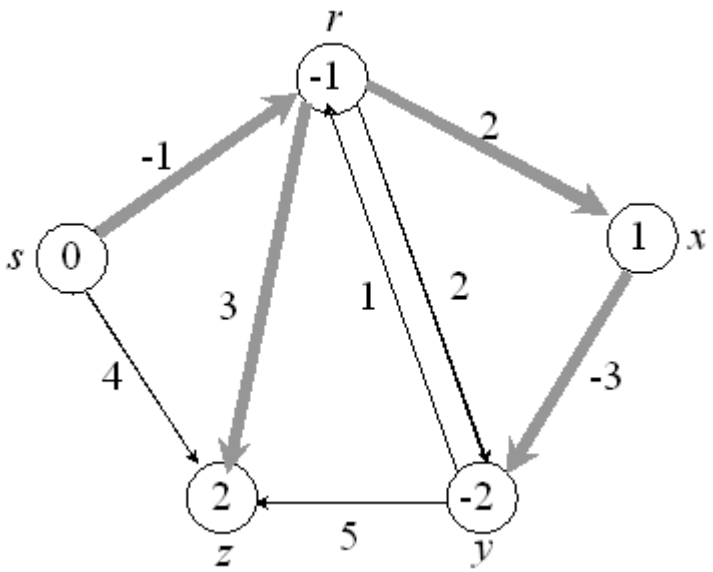


Figure 24.4 The execution of the Bellman-Ford algorithm. The source is vertex s. The d values are shown within the vertices, and shaded edges indicate predecessor values: if edge (u, v) is shaded, then $\pi[v] = u$. In this particular example, each pass relaxes the edges in the order (t, x), (t, y), (t, z), (x, t), (y, x), (y, z), (z, x), (z, s), (s, t), (s, y). **(a)** The situation just before the first pass over the edges. **(b)–(e)** The situation after each successive pass over the edges. The d and π values in part (e) are the final values. The Bellman-Ford algorithm returns TRUE in this example.

- ***Example***:



- Values you get on each pass and how quickly it converges depends on order of relaxation.
- But guaranteed to converge after $|V| - 1$ passes, assuming no negative-weight cycles.

- ***Proof***: Use path-relaxation property.

  Let v be reachable from s, and let $p = < v_0, v_1, ..., v_k >$ be a shortest path from s to v, where $v_0 = s$ and $v_k = v$.

  Since p is acyclic, it has $\leq |\boldsymbol{V}| - 1$edges, so $k \leq |\boldsymbol{V}| - 1$.

  Each iteration of the **for** loop relaxes all edges:

  - First iteration relaxes (v_0, v_1).
  - Second iteration relaxes (v_1, v_2).
  - kth iteration relaxes (v_{k-1}, v_k).

  By the path-relaxation property, $d[v] = d[v_k] = \delta(s, v_k) = \delta(s, v)$.

■

- How about the TRUE/FALSE return value?
  - Suppose there is no negative-weight cycle reachable from s.

    At termination, for all $(u, v) \in \boldsymbol{E}$,

    $d[v] = \delta(s, v)$

    $\leq \delta(s, u) + w(u, v)$ (triangle inequality)

    $= d[u] + w(u, v)$

    So BELLMAN-FORD returns TRUE.

Now suppose there exists negative-weight cycle

$c = < v_0, v_1, ..., v_k >$ where $v_0 = v_k$, reachable from s.

Then $\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$

Suppose (for contradiction) that BELLMAN-FORD returns TRUE.

Then $d[v_i] \leq d[v_{i-1}] + w(v_{i-1}, v_i)$ for $i = 1,2,\ldots,k$.

sum around c:

$$\sum_{i=1}^{k} d[v_i] \leq \sum_{i=1}^{k} (d[v_{i-1}] + w(v_{i-1}, v_i))$$

$$= \sum_{i=1}^{k} d[v_{i-1}] + \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

Each vertex appears once in each summation $\sum_{i=1}^{k} d[v_i]$ and $\sum_{i=1}^{k} d[v_{i-1}]$.

$$\Rightarrow 0 \sum_{i=1}^{k} w(v_{i-1}, v_i).$$

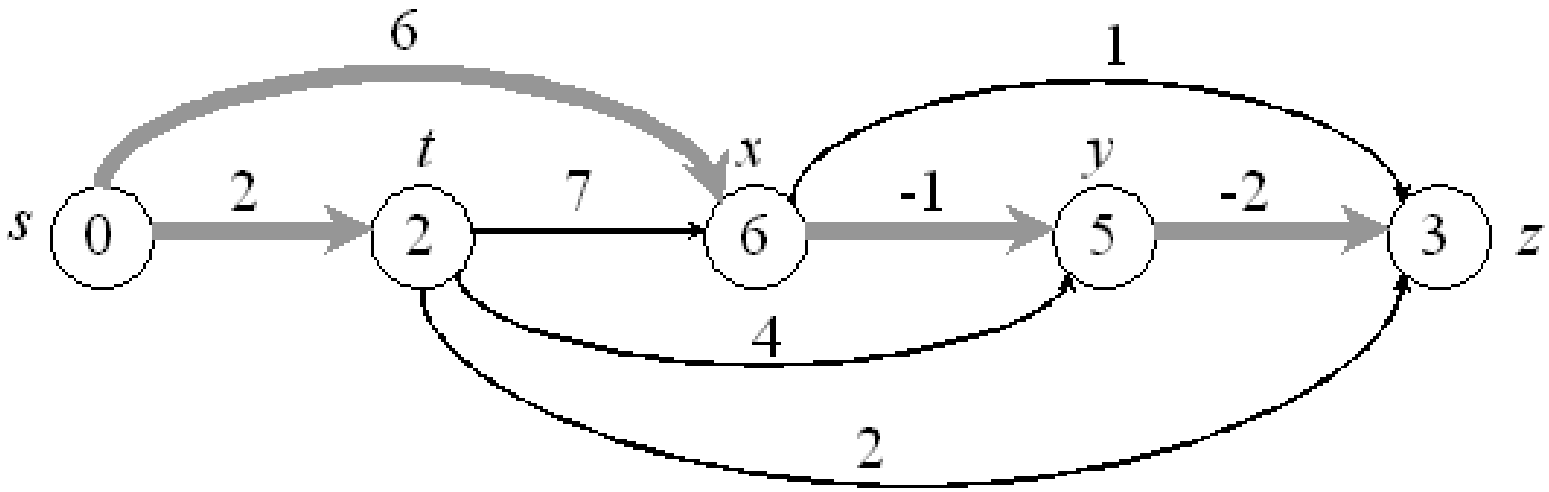
This contradicts C being a negative-weight cycle! ■

Single-source shortest paths in a directed acyclic graph

- Since a dag, we're guaranteed no negative-weight cycles.

DAG-SHORTEST-PATHS (***V***, ***E***, *w*, *s*)
topologically sort the vertices
INIT-SINGLE-SOURCE (***V***, *s*)
for each vertex *u*, taken in topologically sorted order
    do for each vertex $v \in Adj\ [u]$
        do RELAX (*u*, *v*, *w*)

- ***Example***:



- ***Time***:$\Theta(V + E)$.

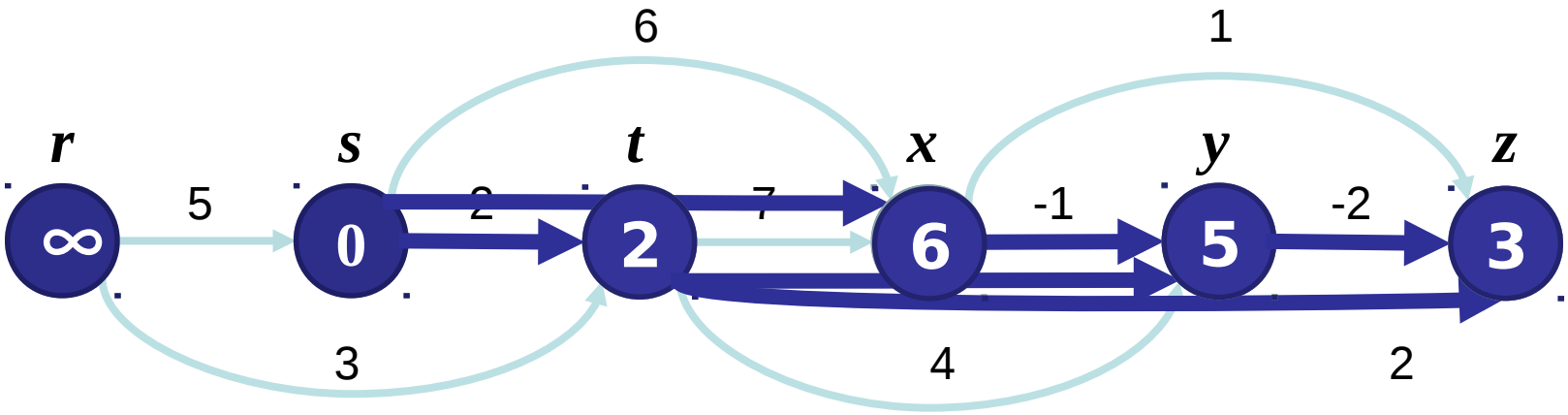


Figure 24.5 The execution of the algorithm for shortest paths in a directed acyclic graph. The vertices are topologically sorted from left to right. The source vertex is *s*. The *d* values are shown within the vertices, and shaded edges indicate the π values.

- ***Correctness***: Because we process vertices in topologically sorted order, edges of any path must be relaxed in order of appearance in the path.

  ⇒ Edges on any shortest path are relaxed in order.

  ⇒ By path-relaxation property, correct. ■

Dijkstra's algorithm

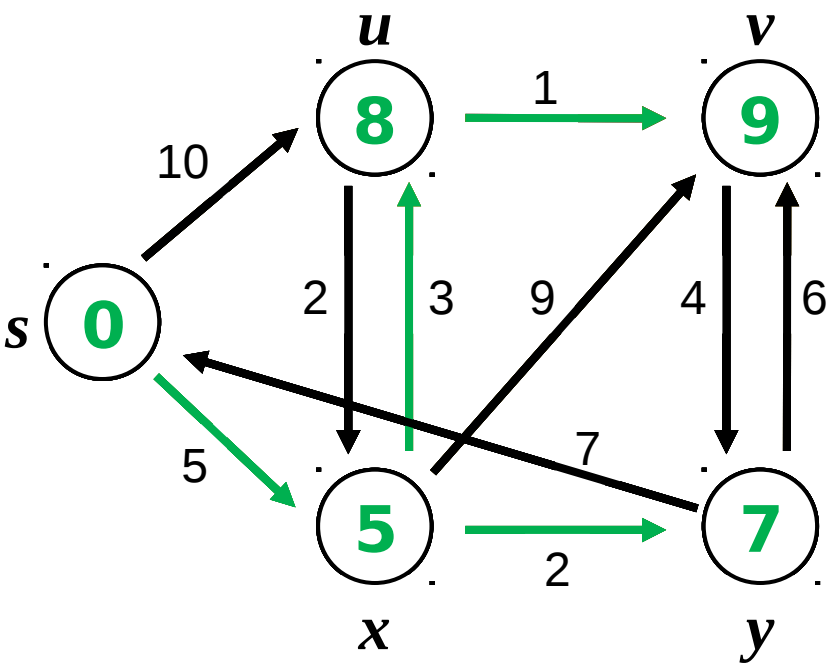
- No negative-weight *edges*.

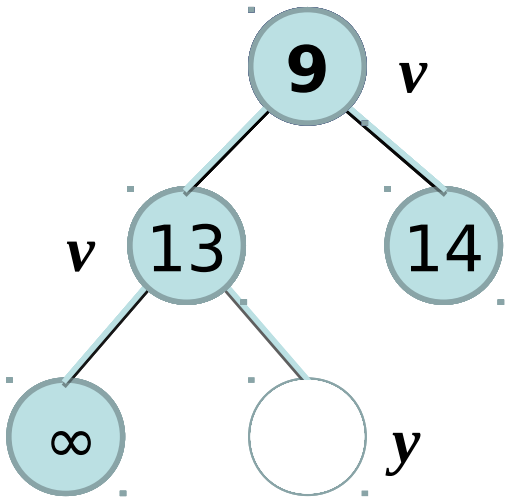
- Essentially a weighted version of breadth-first search.
  - Instead of a **FIFO** queue, uses a priority queue.
  - Keys are shortest-path weights ($d[v]$).

- Have two sets of vertices:
  - S = vertices whose final shortest-path weights are determined.
  - Q = priority queue = $\boldsymbol{V} - S$.

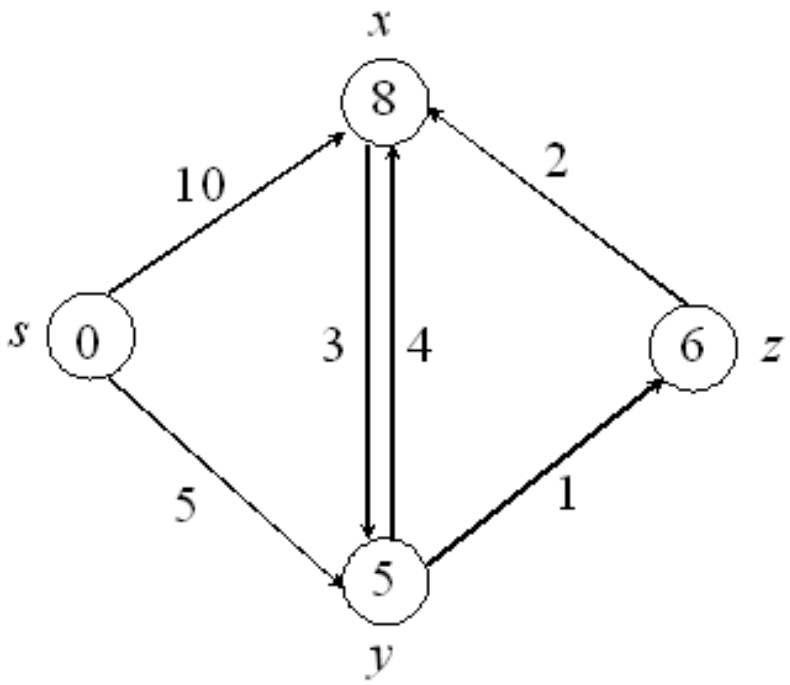
```
DIJKSTRA (V, E, w, s)
INIT-SINGLE-SOURCE (V, s)
S ← Ø
Q ← V          //  i.e., insert all vertices into Q
While Q ≠ Ø
    do u ← EXTRACT-MIN(Q)
        S ← S ∪ {u}
        for each vertex v ∈ Adj [u]
            do RELAX (u, v, w)
```

- Like Prim's algorithm, but computing $d[v]$, and using shortest-path weights as keys.
- Dijkstra's algorithm can be viewed as greedy, since it always chooses the "lightest" ("closest"?) vertex in $\boldsymbol{V} - S$ to add to S.





- ***Example:***



Order of adding to *S*: *s, y, z, x.*

- ***Correctness***:
  - **Loop invariant** : At the start of each iteration of the **while** loop, $d[v] = \delta(s, v)$ for all $v \in S$.
  - **Initialization**: Initially, $S = \emptyset$, so trivially true.
  - **Termination**: At end, $Q = \emptyset \Rightarrow S = V \Rightarrow d[v] = \delta(s, v)$ for all $v \in V$.

- ***Maintenance***:

  Need to show that $d[u] = \delta(s, u)$ when u is added to S in each iteration.

  Suppose there exists u such that $d[u] \neq \delta(s, u)$. Without loss of generality, let u be the first vertex for which $d[u] \neq \delta(s, u)$ when u is added to S.

  Observations:

  - $u \neq s$, since $d[s] = \delta(s, s) = 0$.
  - Therefore, $s \in S$, so $S \neq \emptyset$.
  - There must be some path $s \rightsquigarrow u$,

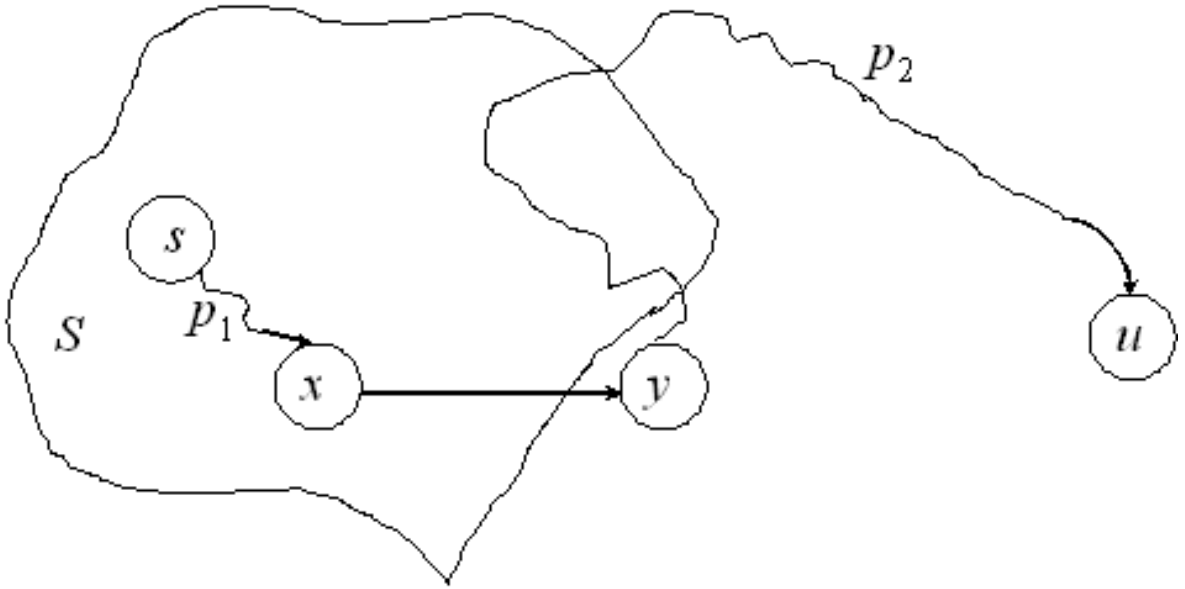
    since otherwise $d[u] = \delta(s, u) = \infty$ by no-path property.

So, there's a path $s \rightsquigarrow u$.

This means there's a shortest path $s \overset{p}{\rightsquigarrow} u$.

Just before u is added to S, path p connects a vertex in S (i.e., s) to a vertex in $\boldsymbol{V} - S$ (i.e., u).

Let y be first vertex along p that's in $\boldsymbol{V} - S$, and let $x \in S$ be y's predecessor.



Decompose p into $s \overset{p_1}{\rightsquigarrow} x \rightarrow y \overset{p_2}{\rightsquigarrow} u$.

(Could have $x = s$ or $y = u$, so that p_1 or p_2 may have no edges.)

- ***Claim***

  $d[y] = \delta(s, y)$ when u is added to S.

Proof

$x \in S$ and u is the first vertex such that $d[u] \neq \delta(s, u)$ when u is added to $S \Rightarrow d[x] = \delta(s, x)$ when x is added to S.

Relaxed (x, y) at that time, so by the convergence property,

$d[y] = \delta(s, y)$. ■ (claim)

Now can get a contradiction to $d[u] \neq \delta(s, u)$:

y is on shortest path $s \rightsquigarrow u$, and all edge weights are nonnegative

$\Rightarrow \delta(s, y) \leq \delta(s, u)$

$\Rightarrow d[y] = \delta(s, y)$

$\leq \delta(s, u)$

$\leq d[u]$ (upper-bound property).

Also, both y and u were in Q when we chose u, so $d[u] \leq d[y]$

$\Rightarrow d[u] = d[y]$.

Therefore, $d[y] = \delta(s, y) = \delta(s, u) = d[u]$.

Contradicts assumption that $d[u] \neq \delta(s, u)$.

Hence, Dijkstra's algorithm is correct. ■

- ***Analysis***: Like Prim's algorithm, depends on implementation of priority queue.
  - If binary heap, each operation takes $O(\lg \boldsymbol{V})$ time
    $\Rightarrow O(\boldsymbol{E} \lg \boldsymbol{V})$.
  - If a Fibonacci heap:
    - Each EXTRACT-MIN takes $O(1)$ amortized time.
    - There are $O(\boldsymbol{V})$ other operations, taking $O(\lg \boldsymbol{V})$ amortized time each.
    - Therefore, time is $O(\boldsymbol{V} \lg \boldsymbol{V} + \boldsymbol{E})$.

Difference constraints

- Given a set of inequalities of the form $x_j - x_i \leq b_k$.
  - x's are variables, $1 \leq i, j \leq n$,
  - b's are constants, $1 \leq k \leq m$.

Want to find a set of values for the x's that satisfy all m inequalities, or determine that no such values exist.

Call such a set of values a ***feasible solution***.

- ***Example*:**

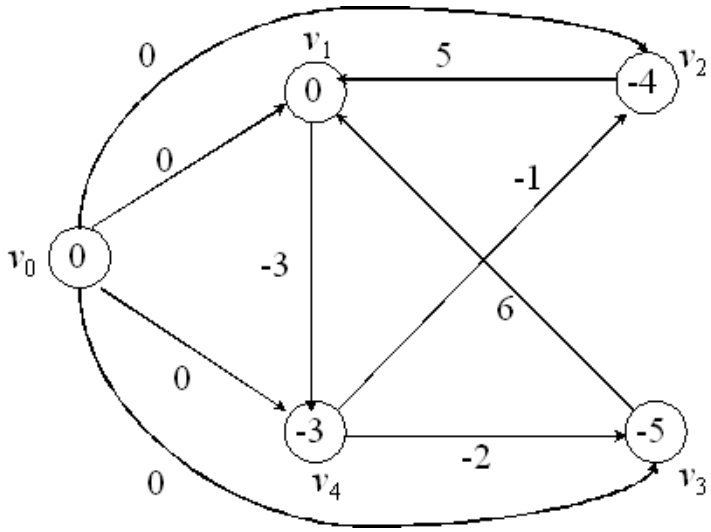
$x_1 - x_2 \leq 5$

$x_1 - x_3 \leq 6$

$x_2 - x_4 \leq -1$

$x_3 - x_4 \leq -2$

$x_4 - x_1 \leq -3$



Solution: $x = (0, -4, -5, -3)$

Also: $x = (5, 1, 0, 2) = [\text{above solution}] + 5$

Lemma

If x is a feasible solution, then so is $x + d$ for any constant d.

Proof

x is a feasible solution

$\Rightarrow \quad x_j - x_i \; \square \; b_k$ for all i, j, k.

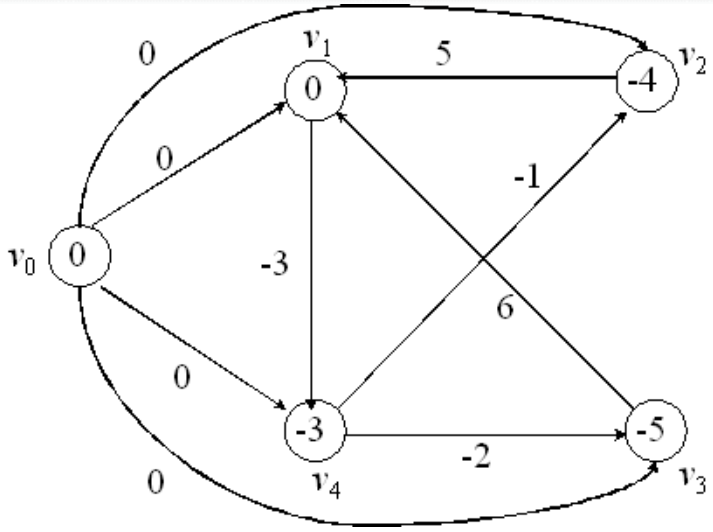
$\Rightarrow \quad (x_j + d) - (x_i + d) \; \square \; b_k$.

■ (lemma)

- **Constraint graph**

  $G = (V, E)$, weighted, directed.

  - $V = < v_0, v_1, v_2, ..., v_n >$: one vertex per variable + v_0
  - $E = \{(v_i, v_j) : x_j - x_i \leq b_k$ is a constraint$\}$
    $\cup \{(v_0, v_1), (v_0, v_2), ..., (v_0, v_n)\}$
  - $w(v_0, v_j) = 0$ for all j
  - $w(v_i, v_j) = b_k$ if $x_j - x_i \leq b_k$



- ***Theorem***

Given a system of difference constraints, let $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ be the corresponding Constraint graph.

1. If G has no negative-weight cycles, then $x = (\delta(v_0, v_1), \delta(v_0, v_2), ..., \delta(v_0, v_n))$ is a feasible solution.

2. If G has a negative-weight cycle, then there is no feasible solution.

- ***Proof***

1. Show no negative-weight cycles ⇒ feasible solution.

   Need to show that $x_j - x_i \leq b_k$ for all constraints. Use

   $x_j = \delta(v_0, v_j)$

   $x_i = \delta(v_0, v_i)$

   $b_k = w(v_i, v_j)$

   By the triangle inequality,

   $\delta(v_0, v_j) \leq \delta(v_0, v_i) + w(v_i, v_j)$

   $x_j \leq x_i + b_k$

   $x_j - x_i \leq b_k$

   Therefore, feasible.

2. Show negative-weight cycles ⇒ no feasible solution.

Without loss of generality, let a negative-weight cycle be $C = \langle v_1, v_2, \ldots, v_k \rangle$ where $v_1 = v_k$ (v_0 can't be on C, since v_0 has no entering edges.) C corresponds to the constraints

$$x_2 - x_1 \leq w(v_1, v_2)$$
$$x_3 - x_2 \leq w(v_2, v_3)$$
$$\vdots$$
$$x_{k-1} - x_{k-2} \leq w(v_{k-2}, v_{k-1})$$
$$x_k - x_{k-1} \leq w(v_{k-1}, v_k)$$

(The last two inequalities above are incorrect in the first three printings of the book. They were corrected in the fourth printing.)

If x is a solution satisfying these inequalities, it must satisfy their sum.

So add them up.

Each x_i is added once and subtracted once. ($v_1 = v_k$ ⇔ $x_1 = x_k$)

We get $0 \le w(C)$.

But $w(C) < 0$, since C is a negative-weight cycle.

Contradiction ⇒ no such feasible solution x exists.

■ (theorem)

- How to find a feasible solution

1. Form constraint graph.
   - $n + 1$ vertices.
   - $m + n$ edges.
   - $\Theta(m + n)$ time.
2. Run BELLMAN-FORD from v_0 .
   - $O((n + 1)(m + n)) = O(n^2 + nm)$ time.
3. If BELLMAN-FORD returns FALSE ⇒ no feasible solution.

   If BELLMAN-FORD returns TURE

   ⇒ set $x_i = \delta(v_0, v_i)$ for all i.