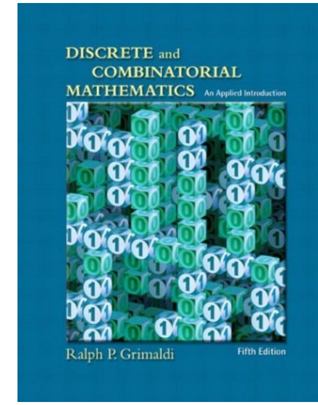
Discrete Mathematics
-- *Chapter 9: Generating Function*



Hung-Yu Kao (高宏宇)
Department of Computer Science and Information Engineering,
National Cheng Kung University

Outline

- Calculational Techniques
- Partitions of Integers
- The Exponential Generating Function
- The Summation Operator

Enumeration again

- Chapter 1: $c_1+c_2+c_3+c_4=25$, where $c_i >= 0$
- Chapter 8: $c_1+c_2+c_3+c_4=25$, where **10>**$c_i >= 0$
- In chapter 9, c_2 to be even and c_3 to be a multiple of 3

- the coefficient xy^2 in $(x+y)^3$
- the coefficient x^4 in $(x+x^2)(x^2+x^3+x^4)(1+x+2x^2)$

9.1 Introductory Examples

- **Ex 9.1** :
  - One mother buys 12 oranges for three children, Grace, Mary, and Frank.
  - Grace gets at least four, and Mary and Frank gets at least two, but Frank gets no more than five.
  - **Solution**
    - $c_1 + c_2 + c_3 = 12$, where $4 \le c_1, 2 \le c_2$, and $2 \le c_3 \le 5$
    - Generating function:
      $f(x) = (x^4 + x^5 + x^6 + x^7 + x^8)(x^2 + x^3 + x^4 + x^5 + x^6)(x^2 + x^3 + x^4 + x^5)$
      product $x^j x^j x^k \rightarrow$ every triple (i, j, k)
    - The coefficient of x^{12} in $f(x)$ yields the solution.

Table 9.1

G	M	F	G	M	F
4	3	5	6	2	4
4	4	4	6	3	3
4	5	3	6	4	2
4	6	2	7	2	3
5	2	5	7	3	2
5	3	4	8	2	2
5	4	3			
5	5	2			

Introductory Examples

- **Ex 9.2** :
  - There is an unlimited number of red, green, white, and black jelly beans.
  - In how many ways can we select 24 jelly beans so that we have an even number of white beans and at least six black ones?
  - **Solution**
    - red (green): $1+ x^1+ x^2+\ldots+ x^{23}+ x^{24}$
    - white: $1+ x^2+ x^4+\ldots+ x^{22}+ x^{24}$
    - black: $x^6+ x^7+\ldots+ x^{23}+ x^{24}$
    - Generating function:
      $f(x) = (1+ x^1+ x^2+\ldots+ x^{23}+ x^{24})^2(1+ x^2+ x^4+\ldots+ x^{22}+ x^{24})$
      $(x^6+ x^7+\ldots+ x^{23}+ x^{24})$
    - The coefficient of x^{24} in $f(x)$ is the answer.

Introductory Examples

- Ex 9.3 : How many nonnegative integer solutions are there for $c_1+c_2+c_3+c_4 = 25$?
  - **Solution**
  - Alternatively, in how many ways 25 pennies can be distributed among four children?
  - Generating function:
    $f(x) = (1+ x^1+ x^2+\ldots+ x^{24}+ x^{25})^4$ (polynomial)
  - The coefficient of x^{25} is the solution.

Note:

- $g(x) = (1+ x^1+ x^2+\ldots+ x^{24}+ x^{25} + x^{26}+\ldots)^4$ (**power series**) can also generate the answer

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

- Easier to compute with a power series than with a polynomial

9.2 Definition and Examples: Calculational Techniques

- Definition 9.1:

Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers. The function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{i=0}^{\infty} a_i x^i$$

is called the *generating function* for the given sequence.

- **Ex 9.4** : $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$

so, $(1+x)^n$ is the generating function for the sequence

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \cdots, \binom{n}{n}, 0, 0, 0, \cdots$$

Definition and Examples: Calculational Techniques

- **Ex 9.5** :

a) $(1 - x^{n+1})/(1 - x)$ is the generating function for the sequence <u>1, 1, 1,…, 1, 0, 0, 0,…</u>, where the first $n+1$ terms are 1.

$$\because (1-x^{n+1}) = (1-x)(1+x+x^2+\cdots+x^n).$$

b) 1/(1-x) is the generating function for the sequence <u>1, 1, 1, 1,…</u> $\because$ while $|x|<1, \frac{1}{1-x} = 1+x+x^2+x^3+\cdots$

c) $1/(1\text{-x})^2$ is the generating function for the sequence <u>1, 2, 3, 4,…</u>

$$\because \frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}(1-x)^{-1} = (-1)(1-x)^{-2}(-1)$$

$$= \frac{1}{(1-x)^2} = \frac{d}{dx}(1+x+x^2+x^3+\cdots) = 1+2x+3x^2+4x^3+\cdots$$

d) $x/(1\text{-x})^2$ is the generating function for the sequence <u>0,1,2,3,….</u>

$$\because \frac{x}{(1-x)^2} = 0+1x+2x^2+3x^3+4x^4+\cdots$$

Definition and Examples: Calculational Techniques

- **Ex 9.5** :

e) $(x+1)/(1-x)^3$ is the generating function for the sequence $1^2, 2^2, 3^2, 4^2, \ldots$

$$\because \frac{d}{dx}\frac{x}{(1-x)^2} = \frac{d}{dx}(0 + x + 2x^2 + 3x^3 + \cdots)$$

$$\frac{x+1}{(1-x)^3} = 1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots$$

$$\because \frac{d}{dx}\frac{x}{(1-x)^2}$$
$$= \frac{d}{dx} x(1-x)^{-2}$$
$$= (1-x)^{-2} + x(-2)(1-x)^{-3}(-1)$$
$$= \frac{(1-x)+2x}{(1-x)^3} = \frac{x+1}{(1-x)^3}$$

f) $x(x+1)/(1-x)^3$ is the generating function for the sequence $0^2, 1^2, 2^2, 3^2, 4^2, \ldots$

$$\because \frac{x(x+1)}{(1-x)^3} = 0 + 1x + 2^2 x^2 + 3^2 x^3 + \cdots$$

Definition and Examples: Calculational Techniques

- **Ex 9.5** :
  - g) Further extensions:

$$f_0(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$f_1(x) = x\frac{d}{dx}f_0(x) = \frac{x}{(1-x)^2}$$
$$= 0 + x + 2x^2 + 3x^3 + \cdots$$

$$f_2(x) = x\frac{d}{dx}f_1(x) = \frac{x^2+x}{(1-x)^3}$$
$$= 0^2 + 1^2x + 2^2x^2 + 3^2x^3 + \cdots$$

$$f_3(x) = x\frac{d}{dx}f_2(x) = \frac{x^3+4x^2+x}{(1-x)^4}$$
$$= 0^3 + 1^3x + 2^3x^2 + 3^3x^3 + \cdots$$

$$f_4(x) = x\frac{d}{dx}f_3(x) = \frac{x^4+11x^3+11x^2+x}{(1-x)^5}$$
$$= 0^4 + 1^4x + 2^4x^2 + 3^4x^3 + \cdots$$

Definition and Examples: Calculational Techniques

- **Ex 9.6** :
  a) $1/(1 - ax)$ is the generating function for the sequence $a^0, a^1, a^2, a^3, \ldots$
  b) $f(x) = 1/(1- x)$ is the generating function for the sequence 1, 1, 1, 1,… Then
     - $g(x) = f(x) - x^2$ is the generating function for the sequence 1, 1, 0, 1, 1, 1,…
     - $h(x) = f(x) + 2x^3$ is the generating function for the sequence 1, 1, 1, 3, 1, 1,…
  c) Can we find a generating function for the sequence 0, 2, 6, 12, 20, 30, 42,…?

Definition and Examples: Calculational Techniques

- **Ex 9.6** :

c) Observe $0, 2, 6, 12, 20, \ldots$

$$a_0 = 0 = 0^2 + 0, \qquad a_1 = 2 = 1^2 + 1,$$

$$a_2 = 6 = 2^2 + 2, \qquad a_3 = 12 = 3^2 + 3,$$

$$a_4 = 20 = 4^2 + 4, \cdots$$

$$\therefore a_n = n^2 + n$$

$$\frac{x(x+1)}{(1-x)^3} + \frac{x}{(1-x)^2} = \frac{x(x+1) + x(1-x)}{(1-x)^3} = \frac{2x}{(1-x)^3}$$

is the generating function.

Extension of Binomial Theorem

- Binomial theorem: $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$
  - When $n \in \mathbf{Z}^+$, we have $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$
  - If $n \in \mathbf{R}$, we define $\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$
  - If $n \in \mathbf{Z}^+$, we have $\binom{-n}{r} = \frac{(-n)(-n-1)(-n-2)\ldots(-n-r+1)}{r!}$

$$= \frac{(-1)^r(n)(n+1)\ldots(n+r-1)}{r!}$$

$$= \frac{(-1)^r(n+r-1)!}{(n-1)!\,r!} = (-1)^r\binom{n+r-1}{r}$$

Extension of Binomial Theorem

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

- **Ex 9.7** :

For $n \in Z^+$, the Maclaurin series expansion for $(1+x)^{-n}$ is given by

$$(1+x)^{-n}$$

$$= 1 + (-n)x + (-n)(-n-1)x^2/2! + (-n)(-n-1)(-n-2)x^3/3! + \dots$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(-n)(-n-1)(-n-2)\dots(-n-r+1)}{r!}x^r$$

$$= \sum_{r=0}^{\infty} (-1)^r \binom{n+r-1}{r} x^r.$$

$(1\text{-}x)^{-n}$?

Hence $(1+x)^{-n} = \binom{-n}{0} + \binom{-n}{1}x + \binom{-n}{2}x^2 + \dots \sum_{r=0}^{\infty} \binom{-n}{r} x^r.$

This generalizes the binomial theorem of Ch1 and shows us that $(1+x)^{-n}$ is the generalating function for the sequence

$$\binom{-n}{0}, \binom{-n}{1}, \binom{-n}{2}, \dots$$

Extension of Binomial Theorem

- **Ex 9.8** : Find the coefficient of x^5 in $(1-2x)^{-7}$.
  - **Solution**

$$(1-2x)^{-7} = \sum_{r=0}^{\infty} \binom{-7}{r}(-2x)^r$$

The coefficient of x^5 :

$$\binom{-7}{5}(-2)^5 = (-1)^5 \binom{7+5-1}{5}(-32) = (32)\binom{11}{5}$$

- **Ex 9.9** : Find the coefficient of all x^i in $(1+3x)^{-1/3}$

$$(1+3x)^{-1/3} = 1 + \sum_{r=1}^{\infty} \frac{(-1/3)(-4/3)(-7/3)\cdots((-3r+2)/3)}{r!}(3x)^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(-1)(-4)(-7)\cdots(-3r+2)}{r!}x^r,$$

$$1 + \sum_{r=1}^{\infty} \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r$$

Definition and Examples: Calculational Techniques

- **Ex 9.10** : Determine the coefficient of x^{15} in $f(x) = (x^2+x^3+x^4+\ldots)^4$.
  - **Solution**
    - $(x^2+x^3+x^4+\ldots) = x^2(1+x+x^2+\ldots) = x^2/(1-x)$
    - $f(x)=(x^2/(1-x))^4 = x^8/(1-x)^4$
    - Hence the solution is the coefficient of x^7 in $(1-x)^{-4}$: $C(-4, 7)(-1)^7 = (-1)^7C(4+7-1, 7)(-1)^7 = C(10, 7) = 120$.

Table 9.2

For all $m, n \in \mathbf{Z}^+$, $a \in \mathbf{R}$,

1) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$
2) $(1+ax)^n = \binom{n}{0} + \binom{n}{1}ax + \binom{n}{2}a^2x^2 + \cdots + \binom{n}{n}a^nx^n$
3) $(1+x^m)^n = \binom{n}{0} + \binom{n}{1}x^m + \binom{n}{2}x^{2m} + \cdots + \binom{n}{n}x^{nm}$
4) $(1-x^{n+1})/(1-x) = 1 + x + x^2 + \cdots + x^n$
5) $1/(1-x) = 1 + x + x^2 + x^3 + \cdots = \sum_{i=0}^{\infty} x^i$
6) $1/(1-ax) = 1 + (ax) + (ax)^2 + (ax)^3 + \cdots$

$$= \sum_{i=0}^{\infty}(ax)^i = \sum_{i=0}^{\infty} a^i x^i$$

$$= 1 + ax + a^2x^2 + a^3x^3 + \cdots$$

7) $1/(1+x)^n = \binom{-n}{0} + \binom{-n}{1}x + \binom{-n}{2}x^2 + \cdots$

$= \sum_{i=0}^{\infty} \binom{-n}{i} x^i$

$= 1 + (-1)\binom{n+1-1}{1}x + (-1)^2\binom{n+2-1}{2}x^2 + \cdots$

$= \sum_{i=0}^{\infty} (-1)^i \binom{n+i-1}{i} x^i$

8) $1/(1-x)^n = \binom{-n}{0} + \binom{-n}{1}(-x) + \binom{-n}{2}(-x)^2 + \cdots$

$= \sum_{i=0}^{\infty} \binom{-n}{i} (-x)^i$

$= 1 + (-1)\binom{n+1-1}{1}(-x) + (-1)^2\binom{n+2-1}{2}(-x)^2 + \cdots$

$= \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i$

check n=1

If $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{i=0}^{\infty} b_i x^i$, and $h(x) = f(x)g(x)$, then $h(x) = \sum_{i=0}^{\infty} c_i x^i$, where for all $k \geq 0$,

$$c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0 = \sum_{j=0}^{k} a_j b_{k-j}.$$

Definition and Examples: Calculational Techniques

- **Ex 9.11** : In how many ways can we select, with repetition allowed, r objects from n distinct objects?
  - **Solution**
  - For each object (with repetitions), $1+x+x^2+\ldots$ represents the possible choices for that object (namely none, one, two,…)
  - Consider all of the n distinct objects, the generating function is $f(x) = (1+x+x^2+\ldots)^n$

$$(1+x+x^2+\cdots)^n = \left(\frac{1}{1-x}\right)^n = \frac{1}{(1-x)^n} = \sum_{i=0}^{\infty}\binom{n+i-1}{i}x^i.$$

  - The answer is the coefficient of x^r in $f(x)$, $\binom{n+r-1}{r}$.

Definition and Examples: Calculational Techniques

- **Ex 9.12** : Counting the compositions of a positive integer n.
  - **Solution**
  - E.g., $n = 4$
    - One-summand: $(x^1+ x^2+x^3+x^4+\ldots) = [x/(1-x)]$, coefficient of $x^4 =1$
    - Two-summand: $(x^1+ x^2+x^3+x^4+\ldots)^2 = [x/(1-x)]^2$, coefficient of $x^4 =3$
    - Three-summand: $(x^1+ x^2+x^3+x^4+\ldots)^3 = [x/(1-x)]^3$, coefficient of $x^4 =3$
    - Four-summand: $(x^1+ x^2+x^3+x^4+\ldots)^4 = [x/(1-x)]^4$, coefficient of $x^4 =1$
    - The number of compositions of 4: coefficient of x^4 in $\sum_{i=1}^{4}[x/(1-x)]^i$

Total=1+3+3+1=8=2^3

How about n=5?

Definition and Examples: Calculational Techniques

- **Ex 9.12** : Counting the compositions of a positive integer n.
  - The number of ways to form an integer n is the coefficient of x^n in the following generating function.

$$\sum_{i=1}^{\infty}(x^1+x^2+x^3+\ldots)^i=\sum_{i=1}^{\infty}[x/(1-x)]^i$$

$f(x)=\sum_{i=1}^{\infty}[x/(1-x)]^i$. But if we set $y=x/(1-x)$, it then follows that

$$f(x)=\sum_{i=1}^{\infty}y^i=y\sum_{i=0}^{\infty}y^i=y\left(\frac{1}{1-y}\right)=\left(\frac{x}{1-x}\right)\left[\frac{1}{1-\left(\frac{x}{1-x}\right)}\right]=\left(\frac{x}{1-x}\right)\left[\frac{1}{\frac{1-x-x}{1-x}}\right]$$

$$=x/(1-2x)=x[1+(2x)+(2x)^2+(2x)^3+\cdots]$$

$$=2^0x+2^1x^2+2^2x^3+2^3x^4+\cdots.$$

So the number of compositions of a positive integer n is the coefficient of x^n in $f(x)$ — and this is $\underline{2^{n-1}}$ (as we found earlier in Examples 1.37, 3.11, and 4.12.)

Definition and Examples: Calculational Techniques

- **Ex 9.14** : In how many ways can a police captain distribute 24 rifle shells to four police officers, so that each officer gets at least three shells but not more than eight.
  - **Solution**

    $x1+x2+x3+x4=24,\ 8 \geqq x_i \geqq 3$
  - $f(x)= (x^3+x^4+ x^5+ x^6+x^7+x^8)^4$

    $= x^{12}(1+x+x^2+x^3+x^4+x^5)^4$

    $= x^{12}[(1-x^6)/(1-x)]^4$
  - The answer is the coefficient of x^{12} in $(1-x^6)^4(1-x)^{-4}$

$$=[1-\binom{4}{1}x^6+\binom{4}{2}x^{12}-\binom{4}{3}x^{18}+x^{24}][\binom{-4}{0}+\binom{-4}{1}(-x)+\binom{-4}{2}(-x)^2+\cdots]$$

$$[\binom{-4}{12}(-1)^{12}-\binom{4}{1}\binom{-4}{6}(-1)^6+\binom{4}{2}\binom{-4}{0}]=[\binom{15}{12}-\binom{4}{1}\binom{9}{6}+\binom{4}{2}]=125$$

Can you explain the last term by the principle of inclusion and exclusion?

Definition and Examples: Calculational Techniques

- **Ex 9.16** : Determine the coefficient of x^8 in $\dfrac{1}{(x-3)(x-2)^2}$.
  - **Solution**

Since $1/(x-a) = (-1/a)(1/(1-(x/a))) = (-1/a)[1 + (x/a) + (x/a)^2 + \cdots]$ for any $a \neq 0$, we could solve this problem by finding the coefficient of x^8 in $1/[(x-3)(x-2)^2]$ expressed as $(-1/3)[1 + (x/3) + (x/3)^2 + \cdots](1/4)\left[\binom{-2}{0} + \binom{-2}{1}(-x/2) + \binom{-2}{2}(-x/2)^2 + \cdots\right]$.

$$\frac{1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}.$$

$$1 = A(x-2)^2 + B(x-2)(x-3) + C(x-3),$$

$$\begin{aligned}
\frac{1}{(x-3)(x-2)^2} &= \frac{1}{x-3} - \frac{1}{x-2} - \frac{1}{(x-2)^2} \\
&= \left(\frac{-1}{3}\right)\frac{1}{1-(x/3)} + \left(\frac{1}{2}\right)\frac{1}{1-(x/2)} + \left(\frac{-1}{4}\right)\frac{1}{(1-(x/2))^2} \\
&= \left(\frac{-1}{3}\right)\sum_{i=0}^{\infty}\left(\frac{x}{3}\right)^i + \left(\frac{1}{2}\right)\sum_{i=0}^{\infty}\left(\frac{x}{2}\right)^i \\
&\quad + \left(\frac{-1}{4}\right)\left[\binom{-2}{0} + \binom{-2}{1}\left(\frac{-x}{2}\right) + \binom{-2}{2}\left(\frac{-x}{2}\right)^2 + \cdots\right].
\end{aligned}$$

The coefficient of x^8 is $(-1/3)(1/3)^8 + (1/2)(1/2)^8 + (-1/4)\binom{-2}{8}(-1/2)^8 = -[(1/3)^9 + 7(1/2)^{10}]$.

Definition and Examples: Calculational Techniques

- **Ex 9.17** : How many four-element subsets of $S = \{1, 2, \ldots, 15\}$ contains no consecutive integers?

Solution

- E.g., one subset $\{1, 3, 7, 10\}$, $\mathbf{1} \le 1 < 3 < 7 < 10 \le \mathbf{15}$, difference 0, 2, 4, 3, 5, difference sum = 14.
- These suggest the integer solutions to $c_1 + c_2 + c_3 + c_4 + c_5 = 14$ where $0 \le c_1, c_5$ and $2 \le c_2, c_3, c_4$.
- The answer is the coefficient of x^{14} in $f(x) = (1+x+x^2+x^3+\ldots)(x^2+x^3+x^4+\ldots)^3(1+x+x^2+x^3+\ldots)$
  $= x^6(1-x)^{-5}$
- The coefficient of x^8 in $(1-x)^{-5}$.

$$\binom{-5}{8}(-1)^8 = \binom{5+8-1}{8} = \binom{12}{8} = 495$$

Convolution of Sequences

- **Ex 9.19** : Let
  - $f(\mathrm{x}) = \mathrm{x}/(1\text{-}\mathrm{x})^2 = 0+1\mathrm{x}+2\mathrm{x}^2+3\mathrm{x}^3+\ldots$, for the sequence $a_k = k$
  - $g(\mathrm{x}) = \mathrm{x}(\mathrm{x}+1)/(1\text{-}\mathrm{x})^3 = 0+1^2\mathrm{x}+2^2\mathrm{x}^2+3^2\mathrm{x}^3+\ldots$, for the sequence $b_k = k^2$
  - $h(\mathrm{x}) = f(\mathrm{x})g(\mathrm{x})$
    $= a_0b_0 + (a_0b_1+a_1b_0)\mathrm{x} + (a_0b_2+a_1b_1+a_2b_0)\mathrm{x}^2 + \ldots$, for the sequence $c_k = a_0b_k + a_1b_{k-1}+a_2b_{k-2}+ \ldots +a_{k-2}b_2+a_{k-1}b_1+a_kb_0$
  - $$c_k = \sum_{i=0}^{k} i(k-i)^2.$$

$c_0 = 0\times0^2$
$c_1 = 0\times1^2 + 1\times0^2 = 0$
$c_2 = 0\times2^2 + 1\times1^2 + 2\times0^2 = 1$
$c_3 = 6$

- The sequence $c_0, c_1, c_2, \ldots$ is the convolution of the sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$

Convolution of Sequences

- Ex 9.20 : Let
  - $f(x) = 1/(1-x) = 1+x+x^2+x^3+ \ldots$
  - $g(x) = 1/(1+x) = 1-x+x^2-x^3+ \ldots$
  - $h(x) = f(x)g(x)$
    $= 1/[(1-x)(1+x)] = 1/(1-x^2) = 1+x^2+x^4+x^6+ \ldots$
- The sequence 1, 0, 1, 0, … is the convolution of the sequences 1, 1, 1, 1, … and 1, -1, 1, -1, …

9.3 Partition of Integers

- $p(n)$: the number of partitioning a positive integer n

$$
\begin{aligned}
p(1) = 1: &\quad 1 \\
p(2) = 2: &\quad 2 = 1 + 1 \\
p(3) = 3: &\quad 3 = 2 + 1 = 1 + 1 + 1 \\
p(4) = 5: &\quad 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1 \\
p(5) = 7: &\quad 5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 \\
&\qquad = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1
\end{aligned}
$$

- The number of 1's is 0 or 1 or 2 or 3…. The power series is $1+x+x^2+x^3+x^4+\ldots$.
- The number of 2's can be kept tracked by the power series $1+x^2+x^4+x^6+x^8+\ldots$.
- For n, the number of 3's can be kept tracked by the power series $1+x^3+x^6+x^9+x^{12}+\ldots$.

Partition of Integers

- Determine $p(10)$
- The coefficient of x^{10} in $f(x)$ $=(1+x+x^2+x^3+\ldots)(1+x^2+x^4+x^6+\ldots)(1+x^3+x^6+x^9+\ldots)\ldots(1+x^{10}+x^{20}+\ldots)$

$$f(x) = \frac{1}{(1-x)}\frac{1}{(1-x^2)}\frac{1}{(1-x^3)}\cdots\frac{1}{(1-x^{10})} = \prod_{i=1}^{10}\frac{1}{(1-x^i)}$$

- By the coefficient of x^n in $P(x) = \prod_{i=1}^{\infty}\frac{1}{(1-x^i)}$, we get the sequence $p(0), p(1), p(2), p(3), \ldots$

Partition of Integers

- **Ex 9.21** : Find the generating function for the number of ways an advertising agent can purchase n minutes of air time if the time slots come in blocks of 30, 60, or 120 seconds.

Solution

- Let 30 seconds represent one time unit.
- Find integer solutions to $a+2b+4c = 2n$
- Generating function:
$f(x) = (1+x+x^2+x^3+x^4+\ldots)(1+x^2+x^4+x^6+x^8+\ldots)(1+x^4+x^8+x^{12}+\ldots)$
$$= \frac{1}{(1-x)}\frac{1}{(1-x^2)}\frac{1}{(1-x^4)}.$$
- Answer: the coefficient of x^{2n} is the number of partitions of $2n$ into 1's, 2's, and 4's.

Partition of Integers

- **Ex 9.22** : Find the generating function for $p_d(n)$, the number of partitions of a positive integer n into **distinct** summands.
  - One time of occurrence per summand
  - $P_d(x) = (1+x)(1+x^2)(1+x^3)\ldots..$

6 = 1+5
6 = 1+2+3
6 = 2+4

- **Ex 9.23** : Find the generating function for $p_o(n)$, the number of partitions of a positive integer n into **odd** summands.
  - $P_o(x) = (1+x+x^2+x^3+\ldots)(1+x^3+x^6+\ldots)(1+x^5+x^{10}+\ldots)\ldots$
  - $= 1/(1-x) \times 1/(1-x^3) \times 1/(1-x^5) \times 1/(1-x^7) \times \ldots$
  - $P_d(x) = P_o(x)$?

6 = 1+1+1+3
6 = 1+5
6 = 3+3

Now because

$$1+x = \frac{1-x^2}{1-x}, \qquad 1+x^2 = \frac{1-x^4}{1-x^2}, \qquad 1+x^3 = \frac{1-x^6}{1-x^3}, \qquad \ldots,$$

we have

$$\begin{aligned} P_d(x) &= (1+x)(1+x^2)(1+x^3)(1+x^4)\cdots \\ &= \frac{1-x^2}{1-x}\,\frac{1-x^4}{1-x^2}\,\frac{1-x^6}{1-x^3}\,\frac{1-x^8}{1-x^4}\cdots = \frac{1}{1-x}\,\frac{1}{1-x^3}\cdots = P_o(x). \end{aligned}$$

Partition of Integers

- **Ex 9.24** : Find the generating function for the number of partitions of a positive integer n into odd summands and occurring an odd number of times.

Solution

$$f(x) = (1+x+x^3+x^5+\ldots)(1+x^3+x^9+x^{15}+\ldots)(1+x^5+x^{15}+x^{25}+\ldots)\ldots$$

$$= \prod_{k=0}^{\infty}\left(1+\sum_{i=0}^{\infty} x^{(2k+1)(2i+1)}\right).$$

6 = 1+1+1+3
6 = 1+5
6 = 3+3

Partition of Integers

- Ferrers graph uses rows of dots to represent a partition of an integer
- In fig. 9.2, two Ferrers graphs are transposed each other for the partitions of 14.
  - (a) 14 = 4+3+3+2+1+1
  - (b) 14 = 6+4+3+1

*The number of partitions of an integer **n** into **m summands** is equal to the number of partitions of n into summands where **m is the largest summand**.*

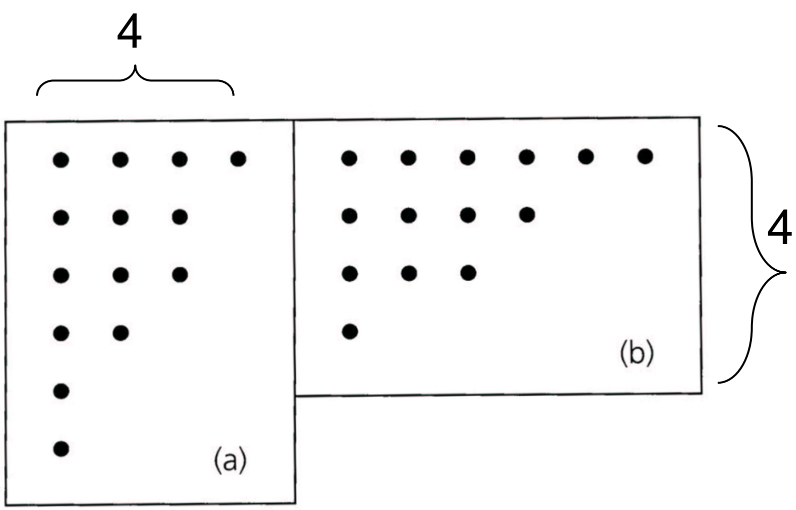


Figure 9.2

http://mathworld.wolfram.com/FerrersDiagram.html

9.4 The Exponential Generating Function

Now for all $0 \le r \le n$,

$$C(n,r) = \frac{n!}{r!(n-r)!} = \left(\frac{1}{r!}\right) P(n,r),$$

where $P(n,r)$ denotes the number of permutations of n objects taken r at a time. So

$$\begin{aligned}(1+x)^n &= C(n,0) + C(n,1)x + C(n,2)x^2 + C(n,3)x^3 + \cdots + C(n,n)x^n \\ &= P(n,0) + P(n,1)x + P(n,2)\frac{x^2}{2!} + P(n,3)\frac{x^3}{3!} + \cdots + P(n,n)\frac{x^n}{n!}.\end{aligned}$$

For a sequence $a_0, a_1, a_2, a_3, \ldots$ of real numbers,

$$f(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!},$$

is called the *exponential generating function* for the given sequence.

The Exponential Generating Function

- **Ex 9.25** : Examining the Maclaurin series expansion for e^x, we find

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

so e^x is the exponential generating function for the sequence 1, 1, 1,….

What 's the sequence for the exponential generating function 1/(1-x) ?

Ans: 0!, 1!, 2!, 3!, ….

The Exponential Generating Function

- **Ex 9.26** : In how many ways can four of the letters in ENGINE be arranged?

Solution

Table 9.4

E E N N	4!/(2! 2!)	E G N N	4!/2!
E E G N	4!/2!	E I N N	4!/2!
E E I N	4!/2!	G I N N	4!/2!
E E G I	4!/2!	E I G N	4!

- Using exponential generating function: $f(x) = [1+x+(x^2/2!)]^2[1+x]^2$
  - E, N: $[1+x+(x^2/2!)]$
  - G, I: $[1+x]$
- The answer is the coefficient of $x^4/4!$.

In the complete expansion of $f(x)$, the term involving x^4 [and, consequently, $x^4/4!$] is

$$\left(\frac{x^4}{2!\,2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+x^4\right)$$

$$=\left[\left(\frac{4!}{2!\,2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+4!\right]\left(\frac{x^4}{4!}\right).$$

The Exponential Generating Function

- **Ex 9.27** : Consider the Maclaurin series expansion of e^x and e^{-x}.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots.$$

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.$$

The Exponential Generating Function

- **Ex 9.28** : A ship carries 48 flags, 12 each of the colors red, white, blue and black. Twelve flags are placed on a vertical pole to communicate signal to other ships.
  - How many of these signals use an even number of blue flags and an odd number of black flags?

$$f(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^2 \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$f(x) = (e^x)^2 \left(\frac{e^x + e^{-x}}{2}\right) \left(\frac{e^x - e^{-x}}{2}\right) = \left(\frac{1}{4}\right)(e^{2x})(e^{2x} - e^{-2x}) = \frac{1}{4}(e^{4x} - 1)$$

$$= \frac{1}{4}\left(\sum_{i=0}^{\infty} \frac{(4x)^i}{i!} - 1\right) = \left(\frac{1}{4}\right) \sum_{i=1}^{\infty} \frac{(4x)^i}{i!},$$

the coefficient of $x^{12}/12!$ in $f(x)$ yields $(1/4)(4^{12}) = 4^{11}$ signals made up of 12 flags with an even number of blue flags and an odd number of black flags.

The Exponential Generating Function

- how many of these use at least three white flags or no white flag at all?

$$g(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \underline{\left(1 + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)} \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^2$$

$$= e^x \left(e^x - x - \frac{x^2}{2!}\right) (e^x)^2 = e^{3x} \left(e^x - x - \frac{x^2}{2!}\right) = e^{4x} - xe^{3x} - \left(\frac{1}{2}\right) x^2 e^{3x}$$

$$= \sum_{i=0}^{\infty} \frac{(4x)^i}{i!} - x \sum_{i=0}^{\infty} \frac{(3x)^i}{i!} - \left(\frac{x^2}{2}\right) \left(\sum_{i=0}^{\infty} \frac{(3x)^i}{i!}\right).$$

The Exponential Generating Function

i) $\sum_{i=0}^{\infty} \frac{(4x)^i}{i!}$ — Here we have the term $\frac{(4x)^{12}}{12!} = 4^{12}\left(\frac{x^{12}}{12!}\right)$, so the coefficient of $x^{12}/12!$ is 4^{12};

ii) $x\left(\sum_{i=0}^{\infty} \frac{(3x)^i}{i!}\right)$ — Now we see that in order to get $x^{12}/12!$ we need to consider the term $x[(3x)^{11}/11!] = 3^{11}(x^{12}/11!) = (12)(3^{11})(x^{12}/12!)$, and here the coefficient of $x^{12}/12!$ is $(12)(3^{11})$; and

iii) $(x^2/2)\left(\sum_{i=0}^{\infty} \frac{(3x)^i}{i!}\right)$ — For this last summand we observe that $(x^2/2)[(3x)^{10}/10!] = (1/2)(3^{10})(x^{12}/10!) = (1/2)(12)(11)(3^{10})(x^{12}/12!)$, where this time the coefficient of $x^{12}/12!$ is $(1/2)(12)(11)(3^{10})$.

Consequently, the number of 12 flag signals with at least three white flags, or none at all, is

$$4^{12} - 12(3^{11}) - (1/2)(12)(11)(3^{10}) = 10{,}754{,}218.$$

The Exponential Generating Function

- **Ex 9.29** : A company hires 11 new employees, and they will be assigned to four **different** subdivisions. Each subdivision has at least one new employee. In how many ways can these assignments be made?
  - **Solution**

$$f(x) = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)^4 = (e^x - 1)^4 = e^{4x} - 4e^{3x} + 6e^{2x} - 4e^x + 1.$$

The answer then is the coefficient of $x^{11}/11!$ in $f(x)$:

$$4^{11} - 4(3^{11}) + 6(2^{11}) - 4(1^{11}) = \sum_{i=0}^{4} (-1)^i \binom{4}{i} (4-i)^{11}.$$

9.5 The Summation Operator

- Let $f(x) = a_0+a_1x+a_2x^2+a_3x^3+\ldots.$ Then $f(x)/(1-x)$ generate the sequence of a_0, a_0+a_1, $a_0+a_1+a_2$, $a_0+a_1+a_2+a_3,\ldots$ So we refer to 1/(1-x) as the <u>summation operator</u>.

$$\frac{f(x)}{1-x} = f(x)\cdot\frac{1}{1-x} = [a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots][1 + x + x^2 + x^3 + \cdots]$$

$$= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + (a_0 + a_1 + a_2 + a_3)x^3 + \cdots,$$

The Summation Operator

Summation operator

- Ex 9.30 :
  - $1/(1-x)$ is the generating function for the sequence 1, 1, 1, 1, 1,…
  - $[1/(1-x)]\times [1/(1-x)]$ is the generating function for the sequence 1, 1+1, 1+1+1,… $\Rightarrow$ 1, 2, 3,…
  - $x+x^2$ is the generating function for the sequence 0, 1, 1, 0, 0, 0,…
  - $(x+x^2) \times [1/(1-x)]$ is the generating function for the sequence 0, 1, 2, 2, 2, 2, …
  - $(x+x^2)/(1-x)^2$ is the generating function for the sequence 0, 1, 3, 5, 7, 9, 11, …
  - $(x+x^2)/(1-x)^3$ is the generating function for the sequence 0, 1, 4, 9, 16, 25, 36, …

The Summation Operator

- **Ex 9.31** : Find a formula to express $0^2+1^2+2^2+\ldots+n^2$ as a function of n.

  Solution

$$f_0(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$f_1(x) = x\frac{d}{dx}f_0(x) = \frac{x}{(1-x)^2}$$

$$= 0 + x + 2x^2 + 3x^3 + \cdots$$

$$f_2(x) = x\frac{d}{dx}f_1(x) = \frac{x^2+x}{(1-x)^3}$$

$$= 0^2 + 1^2x + 2^2x^2 + 3^2x^3 + \cdots$$

so $x(1+x)/(1-x)^3$ generates $0^2, 1^2, 2^2, 3^2, \ldots$. As a consequence of our earlier observations about the summation operator, we find that

$$\frac{x(1+x)}{(1-x)^3}\frac{1}{(1-x)} = \frac{x(1+x)}{(1-x)^4}$$

is the generating function for $0^2, 0^2+1^2, 0^2+1^2+2^2, 0^2+1^2+2^2+3^2, \ldots$.

The Summation Operator

Hence the coefficient of x^n in $[x(1+x)]/(1-x)^4$ is $\sum_{i=0}^{n} i^2$. But the coefficient of x^n in $[x(1+x)]/(1-x)^4$ can also be calculated as follows:

$$\frac{x(1+x)}{(1-x)^4} = (x+x^2)(1-x)^{-4} = (x+x^2)\left[\binom{-4}{0} + \binom{-4}{1}(-x) + \binom{-4}{2}(-x)^2 + \cdots\right],$$

so the coefficient of x^n is

$$\begin{aligned}
&\binom{-4}{n-1}(-1)^{n-1} + \binom{-4}{n-2}(-1)^{n-2} \\
&\quad = (-1)^{n-1}\binom{4+(n-1)-1}{n-1}(-1)^{n-1} + (-1)^{n-2}\binom{4+(n-2)-1}{n-2}(-1)^{n-2} \\
&\quad = \binom{n+2}{n-1} + \binom{n+1}{n-2} = \frac{(n+2)!}{3!(n-1)!} + \frac{(n+1)!}{3!(n-2)!} \\
&\quad = \frac{1}{6}[(n+2)(n+1)(n) + (n+1)(n)(n-1)] \\
&\quad = \frac{1}{6}(n)(n+1)[(n+2)+(n-1)] = \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$