Chapter 3.
Higher-Order Differential Equations

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Advanced R(x)

例: $y'' + 3y' + 2y = \cos x + x = r_1(x) + r_2(x)$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\lambda = -1, -2$$

$$y''_{p1} + 3y'_{p1} + 2y_{p1} = r_1(x) = \cos x$$

$$y''_{p2} + 3y'_{p2} + 2y_{p2} = r_2(x) = x$$

$$(y_{p1} + y_{p2})'' + 3(y_{p1} + y_{p2})' + 2(y_{p1} + y_{p2}) = r_1(x) + r_2(x)$$

Advanced R(x)

$$y_p = y_{p1} + y_{p2}$$

$$y_{p1}'' + 3y_{p1}' + 2y_{p1} = \cos x$$

$$(D^2 + 3D + 2)y_{p1}(x) = \cos x$$

$$y_{p1}(x) = \frac{1}{D^2 + 3D + 2}\cos x \quad (a = 1)$$

Advanced R(x)

- 用特性3. $L(D^2)\cos ax = L(-a^2)\cos ax$

$$y_{p1}(x) = \frac{1}{-1 + 3D + 2}\cos x = \frac{1}{3D + 1}\cos x$$

$$= \frac{1 - 3D}{(1 - 3D)(1 + 3D)}\cos x$$

$$= \frac{1 - 3D}{1 - 9D^2}\cos x$$

$$= \frac{1 - 3D}{1 - 9(-1)}\cos x = \frac{1}{10}(1 - 3D)\cos x$$

$$= \frac{1}{10}\cos x + \frac{3}{10}\sin x$$

Advanced R(x)

$$y''_{p2} + 3y'_{p2} + 2y_{p2} = x$$

$$(D^2 + 3D + 2)y_{p2}(x) = x$$

$$y_{p2}(x) = \frac{1}{D^2 + 3D + 2}x = \frac{1}{2\left(1 + \frac{D^2 + 3D}{2}\right)}x$$

$$= \frac{1}{2}\left(1 - \frac{D^2 + 3D}{2} + \left(\frac{D^2 + 3D}{2}\right)^2 - \cdots\right)x$$

$$= \frac{1}{2}\left(x - \frac{3}{2}\right) = \frac{1}{2}x - \frac{3}{4}$$

$$y_p = y_{p1} + y_{p2} = \frac{1}{10}\cos x + \frac{3}{10}\sin x + \frac{1}{2}x - \frac{3}{4}$$

$$y = y_h + y_p = C_1 e^{-x} + C_2 e^{-2x} + \frac{1}{10}\cos x + \frac{3}{10}\sin x + \frac{1}{2}x - \frac{3}{4}$$

Variation of Variable

- Method 4: Variation of Variable

  概念：

$$y' + p(x)y = r(x)$$

$$I = e^{\int p(x)dx}$$

$$y = CI^{-1} + I^{-1}\int Ir(x)dx$$

$$= Ce^{-\int p(x)dx} + e^{-\int p(x)dx}\int e^{\int p(x)dx} r(x)dx$$

$$= y_h + y_p$$

$$\Rightarrow y_p = y_h\phi$$

Variation of Variable

例: $y' + 2y = e^x$

$y_h = Ce^{-2x}$

$y_p = e^{-2x}\varphi(x)$

$y_p' = e^{-2x}\varphi'(x) - 2e^{-2x}\varphi(x)$

$e^{-2x}\varphi'(x) - 2e^{-2x}\varphi(x) + 2e^{-2x}\varphi(x) = e^x$

$\varphi'(x) = e^{3x}$

$\varphi(x) = \frac{1}{3}e^{3x} + k \quad (k\text{可略})$

$y = y_h + y_p = Ce^{-2x} + \frac{1}{3}e^x$

Variation of Variable

考慮二階常微分方程式

$y''(x) + p(x)y'(x) + q(x)y(x) = r(x)$

設 $y_1(x), y_2(x)$分別為此方程式的齊性解

$\Rightarrow y_h = C_1y_1(x) + C_2y_2(x)$且$y_1'' + py_1' + qy_1 = 0, y_2'' + py_2' + qy_2 = 0$

$y_p = y_1\varphi_1 + y_2\varphi_2$

$y_p' = y_1'\varphi_1 + y_1\varphi_1' + y_2'\varphi_2 + y_2\varphi_2'$

$= (y_1'\varphi_1 + y_2'\varphi_2) + (y_1\varphi_1' + y_2\varphi_2') \quad$ 令$y_1\varphi_1' + y_2\varphi_2' = 0$

$y_p'' = y_1''\varphi_1 + y_1'\varphi_1' + y_2''\varphi_2 + y_2'\varphi_2'$

Variation of Variable

代入 $y_p'' + py_p' + qy_p = r$

$$\left(y_1''\varphi_1 + y_1'\varphi_1' + y_2''\varphi_2 + y_2'\varphi_2'\right) + p\left(y_1'\varphi_1 + y_2'\varphi_2\right) + q\left(y_1\varphi_1 + y_2\varphi_2\right) = r$$

$$\varphi_1\underbrace{\left(y_1'' + py_1' + qy_1\right)}_{為0} + \varphi_2\underbrace{\left(y_2'' + py_2' + qy_2\right)}_{為0} + y_1'\varphi_1' + y_2'\varphi_2' = r$$

$$\Rightarrow \boxed{\begin{cases} y_1'\varphi_1' + y_2'\varphi_2' = r(x) \\ y_1\varphi_1' + y_2\varphi_2' = 0 \end{cases}}$$

要與上式都滿足

Kramer's Law

$$\left(\begin{cases} ax + by = c \\ dx + ey = f \end{cases} \Rightarrow x = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}, y = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} \right)$$

Variation of Variable

$$\Rightarrow \boxed{\varphi_1' = \frac{\begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}} = \frac{-ry_2}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}, \boxed{\varphi_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}} = \frac{ry_2}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

$$\Rightarrow \varphi_1 = \int \frac{-ry_2}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} dx, \varphi_2 = \int \frac{ry_1}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} dx$$

$$y_p = y_1\varphi_1 + y_2\varphi_2$$

$$= y_1 \int \frac{-ry_2}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} dx + y_2 \int \frac{ry_1}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} dx$$

Variation of Variable

- 定義Wranski $(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W(y_1, y_2)$

$$y_p = y_1 \int \frac{-ry_2}{w(y_1, y_2)} dx + y_2 \int \frac{ry_1}{w(y_1, y_2)} dx$$

例: $y'' + 3y' + 2y = x$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\lambda = -1, -2$$

Variation of Variable

$$\Rightarrow y_1 = e^{-2x}, y_2 = e^{-x}$$

$$w(y_1, y_2) = -e^{-3x} + 2e^{-3x} = e^{-3x}$$

$$y_p = y_1 \int \frac{-ry_2}{W} dx + y_2 \int \frac{ry_1}{W} dx$$

$$= e^{-2x} \int \frac{-re^{-x}}{e^{-3x}} dx + e^{-x} \int \frac{re^{-2x}}{e^{-3x}} dx$$

$$= -e^{-2x} \left(\frac{1}{2} xe^{2x} - \frac{1}{4} e^{2x} \right) + e^{-x} \left(xe^{x} - e^{x} \right)$$

$$= \frac{1}{2} x - \frac{3}{4}$$

Other Method Verification

- 用Method2 (Order Reduction):

$$(D+1)(D+2)y_p = x$$

$$Z'(x) + Z(x) = x$$

$$Z(x) = CI_1^{-1} + I_1^{-1}\int I_1 r dx$$

$$Z_p(x) = I_1^{-1}\int I_1 r dx$$

$$(D+2)y_p = I_1^{-1}\int I_1 r dx$$

Other Method Verification

$$y_p = I_2^{-1}\int I_2 Z_p(x) dx$$

$$= I_2^{-1}\int I_2 I_1^{-1}\int I_1 r dx dx$$

$$I_1 = e^x, I_2 = e^{2x}$$

$$y_p = e^{-2x}\int e^{2x} e^{-x}\int e^x x dx dx$$

$$= e^{-2x}\int e^x\left(xe^x - e^x\right)dx$$

$$= e^{-2x}\left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} - \frac{1}{2}e^{2x}\right)$$

$$= \frac{1}{2}x - \frac{3}{4}$$

Example Practice

例: $y'' + 8y' + 16y = 3e^{-4x}$

$\lambda = -4, -4$

$y_h = C_1 e^{-4x} + C_2 x e^{-4x}$

Method1 (Undetermined Coefficient):

$y_p = kx^2 e^{-4x}$

$y_p' = kx^2(-4e^{-4x}) + 2kxe^{-4x}$

$= k\left(-4x^2 e^{-4x} + 2xe^{-4x}\right)$

Example Practice

$y_p'' = k\left(16x^2 - 8x - 8x + 2\right)e^{-4x}$

$y_p'' + 8y_p' + 16y_p$

$= k\left(16x^2 - 8x - 8x + 2\right)e^{-4x} + 8k\left(-4x^2 e^{-4x} + 2xe^{-4x}\right) + 16kx^2 e^{-4x}$

$= 2ke^{-4x} = 3e^{-4x}$

$\Rightarrow k = \dfrac{3}{2}$

$y_p = \dfrac{3}{2} x^2 e^{-4x}$

Example Practice

- Method2 (Order Reduction):

$$(D+4)(D+4)y_p = 3e^{-4x}$$

$$I_1 = e^{4x}, I_2 = e^{4x}$$

$$y_p = I_2^{-1}\int I_2 I_1^{-1}\int I_1 r dx dx$$

$$y_p = e^{-4x}\int e^{4x}e^{-4x}\int e^{4x}3e^{-4x}dxdx$$

$$= e^{-4x}\int 3x dx$$

$$= \frac{3}{2}x^2e^{-4x}$$

Example Practice

- Method3 (Differential Operator):

$$y_p = \frac{1}{(D+4)^2}3e^{-4x}$$

$$= 3e^{-4x}\frac{1}{(D+4-4)^2}$$

$$= 3e^{-4x}\frac{1}{D^2}$$

$$= 3e^{-4x}\iint 1 dx dx$$

$$= \frac{3}{2}x^2e^{-4x}$$

Example Practice

- Method4 (Variation of Variable):

$$y_1 = e^{-4x}, y_2 = xe^{-4x}$$

$$W(y_1, y_2) = \begin{vmatrix} e^{-4x} & xe^{-4x} \\ -4e^{-4x} & e^{-4x} - 4xe^{-4x} \end{vmatrix} = e^{-8x}$$

$$y_p = y_1 \int \frac{-ry_2}{W} dx + y_2 \int \frac{ry_1}{W} dx$$

$$= e^{-4x} \int \frac{-3e^{-4x}}{e^{-8x}} xe^{-4x} dx + xe^{-4x} \int \frac{3e^{-4x}}{e^{-8x}} e^{-4x} dx$$

$$= e^{-4x} \int -3x dx + xe^{-4x} \int 3 dx$$

Example Practice

$$= \frac{-3}{2} x^2 e^{-4x} + 3x^2 e^{-4x}$$

$$= \frac{3}{2} x^2 e^{-4x}$$

*分析：未定係數法及微分運算子法，受限於r(x)的型式