Homework 2

Section 2.5 56, 62
Section 2.6 51
Section 2.7 61
Section 2.8 50
Problem Plus 1, 2, 3, 8, 9, 14
Section 3.1 76
Section 3.4 46, 75, 99

Section 2.5 Continuity

EX.56

The equation $\sin x = x^2 - x$ is equivalent to the equation $\sin x - x^2 + x = 0$. $f(x) = \sin x - x^x + x$ is continuous on the interval [1,2], $f(1) = \sin 1 \approx 0.84$, and $f(2) = \sin 2 - 2 \approx -1.09$. Since $\sin 1 > 0 > \sin 2 - 2$, there is a number c in (1,2) such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\sin x - x^2 + x = 0$, or $\sin x = x^2 - x$, in the interval (1,2).

EX.62

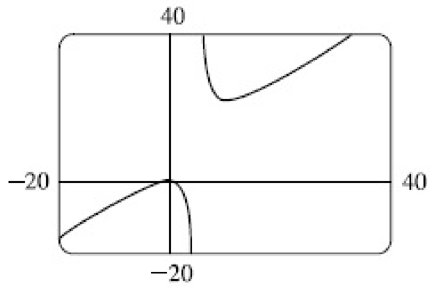
Let $f(x) = x^2 - 3 + \frac{1}{x}$. Then f is continuous on (0,2] since f is a rational function whose domain is $(0,\infty)$. By inspection, we see that $f(\frac{1}{4} = \frac{17}{16} > 0)$, $f(1) = -1 < 0$, and $f(2) = \frac{3}{2} > 0$. Appling the Intermediate Value Theorem on $[\frac{1}{4}$,1] and then on [1,2], we see there are numbers c and d in $(\frac{1}{4}$,1) and (1,2) such that $f(c) = f(d) = 0$. Thus, f has at least two x-intercepts in (0,2).

Section 2.6 Limits at Infinity; Horizontal Asymptotes

EX.51

$y = f(x) = \frac{x^3-x}{x^2-6x+5} = \frac{x(x^2-1)}{(x-1)(x-5)} = \frac{x(x+1)(x-1)}{(x-1)(x-5)} = \frac{x(x+1)}{x-5} = g(x)$ for $x \neq 1$.
The graph of g is the same as the graph of f with the exception of a hole in the graph of f at $x = 1$. By long division, $g(x) = \frac{x^2+x}{x-5} = x+6+\frac{30}{x-5}$. As $x \to \pm\infty$, $g(x) \to \pm\infty$, so there is no horizontal asymptote. The denominator of g is

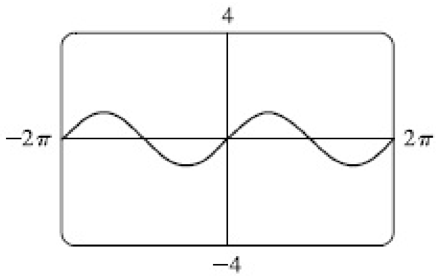
zero when $x = 5$. $\lim_{x \to 5^-} g(x) = -\infty$ and $\lim_{x \to 5^+} g(x) = \infty$ so $x = 5$ is a vertical asymptote.



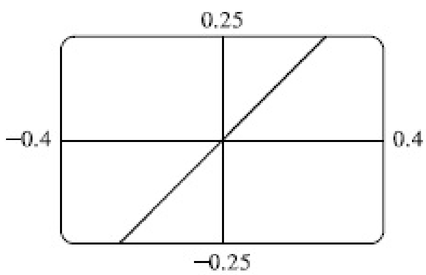
Section 2.7 Derivative and Rates of Change

EX.61

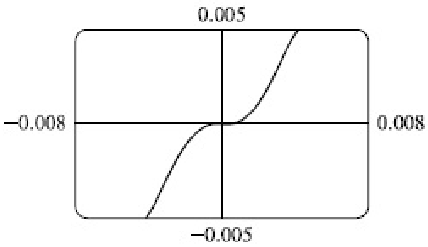
(a) The slope at the origin appears to be 1.



(b) The slope at the origin still appears to be 1.



(c) Yes, the slope at the origin now appears to be 0.



Section 2.8 The Derivative as a Function

EX.50

Where d has horizontal tangents, only c is 0, so $d' = c$. c has negative tangents for $x < 0$ and b is the only graph that is negative for $x < 0$, so $c' = b$. b has positive tangents on $\mathbb{R}$ (except at $x = 0$), and the only graph that is positive on the same domain is a, so $b' = a$. We conclude that $d = f, c = f', b = f''$, and $a = f'''$.

Problem Plus

EX.1

Let $t = \sqrt[6]{x}$, so $x = t^6$. Then $t \to 1$ as $x \to 1$, so

$$\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\sqrt{x}-1} = \lim_{t\to 1} \frac{t^2-1}{t^3-1} = \lim_{t\to 1} \frac{(t-1)(t+1)}{(t-1)(t^2+t+1)} = \lim_{t\to 1} \frac{t+1}{t^2+t+1}$$
$$= \frac{1+1}{1+1+1} = \frac{2}{3}$$

EX.2

First rationalize the numerator:

$$\lim_{x\to 0} \frac{\sqrt{ax+b}-2}{x} \cdot \frac{\sqrt{ax+b}+2}{\sqrt{ax+b}+2} = \lim_{x\to 0} \frac{ax+b-4}{x(\sqrt{ax+b}+2)}$$

Now since the denominator approaches 0 as $x \to 0$, the limit will exist only if the numerator also approaches 0 as $x \to 0$. So we require that $a(0)+b-4 = 0 \Rightarrow b = 4$. So the equation becomes $\lim_{x\to 0} \dfrac{a}{\sqrt{ax+4}+2} = 1 \Rightarrow$ $\dfrac{a}{\sqrt{ax+4}+2} = 1 \Rightarrow a = 4$. Therefore, $a = b = 4$.

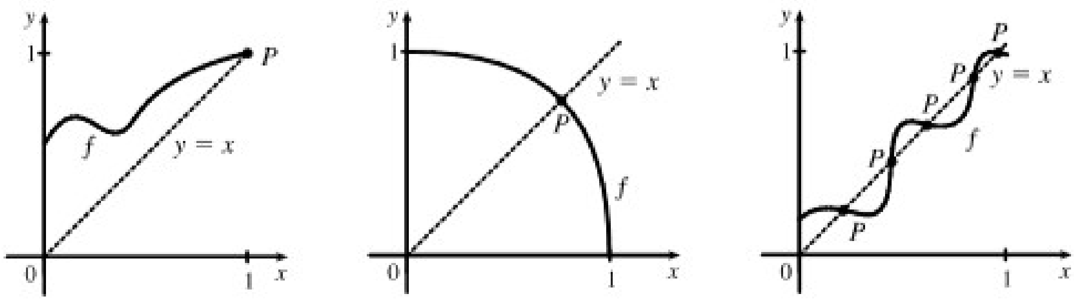
EX.3

For $-\frac{1}{2} < x < \frac{1}{2}$, we have $2x-1<0$ and $2x+1>0$, so $|2x-1| = -(2x-1)$ and $|2x+1| = 2x+1$

Therefore, $\lim\limits_{x\to 0} \dfrac{|2x-1|-|2x+1|}{x} = \lim\limits_{x\to 0} \dfrac{-(2x-1)-(2x+1)}{x} = \lim\limits_{x\to 0} \dfrac{-4x}{x}$
$= \lim\limits_{x\to 0}(-4) = -4$

EX.8

(a) Here are a few possibilities:



(b) The "obstacle" is the line $x = y$(see diagram). Any intersection of the graph of f with the line $y = x$ constitutes a fixed point, and if the graph of the function does not cross the line somewhere in (0,1), then it must either start at (0,0)(in which case 0 is a fixed point) or finish at (1,1)(in which case 1 is a fixed point).
(c) Consider the function $F(x) = f(x) - x$, where f is any continuous function with domain [0,1] and range in [0,1]. We shall prove that f has a fixed point. Now if $f(0) = 0$ then we are done: f has a fixed point(the number 0), which is what we are trying to prove. So assume $f(0) \neq 0$. For the same reason we can assume that $f(1) \neq 1$. Then $F(0) = f(0) > 0$ and $F(1) = f(1) - 1 < 0$. So by the Intermediate Value Theorem, there exists some number c in the interval (0,1) such that $F(c) = f(c) - c = 0$. So $f(c) = c$, and therefore f has a fixed point.

EX.9

$$\begin{cases} \lim\limits_{x\to a}[f(x)+g(x)] = 2 \\ \lim\limits_{x\to a}[f(x)-g(x)] = 1 \end{cases} \Rightarrow \begin{cases} \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x) = 2 \cdots \textbf{(1)} \\ \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x) = 1 \cdots \textbf{(2)} \end{cases}$$

Adding equations (1) and (2) gives us $2\lim_{x\to a} f(x) = 3 \Rightarrow \lim_{x\to a} f(x) = \frac{2}{3}$. From equation(1), $\lim_{x\to a} g(x) = \frac{1}{2}$.
Thus, $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4}$.

EX.14

We are given that $|f(x)| \leq x^2$ for all x. In particular, $|f(0)| \leq 0$, but $|a| \geq 0$ for all a. The only conclusion is that $f(0) = 0$. Now $|\frac{f(x)-f(0)}{x-0}| = |\frac{f(x)}{x}| = \frac{|f(x)|}{|x|} \leq \frac{x^2}{|x|} = \frac{|x^2|}{|x|} = |x| \Rightarrow -|x| \leq \frac{f(x)-f(0)}{x-0} \leq |x|$. But $\lim_{x\to\infty}(-|x|) = 0 = \lim_{x\to 0}|x|$, so by the Squeeze Theorem, $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = 0$. So by the definition of a derivative, f is differentiable at 0, and furthermore, $f'(0) = 0$.

Section 3.1 Derivatives of Polynomials and Exponential Functions

EX.76

$y = x^4 + ax^3 + bx^2 + cx + d \Rightarrow y(0) = d$. Since the tangent line $y = 2x + 1$ is equal to 1 at $x = 0$, we must have $d = 1$. $y' = 4x^3+3ax^2+2bx+c \Rightarrow y'(0) = c$. Since the slope of tangent line $y = 2x + 1$ at $x = 0$ is 2, we must have $c = 2$. Now $y(1) = 1 + a + b + c + d = a + b + 4$ and the tangent line $y = 2 - 3x$ at $x = 1$ has y-coordinate -1, so $a + b + 4 = -1$ or $a + b = -5$ **(1)**. Also, $y'(1) = 4 + 3a + 2b + c = 3a + 2b + 6$ and the slope of the tangent line $y = 2 - 3x$ at $x = 1$ is -3, so $3a + 2b + 6 = -3$ or $3a + 2b = -9$ **(2)**. Adding -2 times **(1)** to **(2)** gives us $a = 1$ and hence, $b = -6$. The curve has equation $y = x^4 + x^3 - 6x^2 + 2x + 1$.

EX.81

f is clearly differentiable for $x < 2$ and for $x > 2$. For $x < 2$, $f'(x) = 2x$, so $f'_-(2) = 4$. For $x > 2$, $f'(x) = m$, so $f'_+(2) = m$. For f to be differentiable at $x = 2$, we need $4 = f'_-(2) = f'_+(2) = m$. So $f(x) = 4x + b$. We must also have continuity at $x = 2$, so $4 = f(2) = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(4x + b) = 8 + b$. Hence, $b = -4$.

Section 3.4 The Chain Rule

EX.46

$y = [x + (x + \sin^2 x)^3]^4$
$\Rightarrow y' = 4[x + (x + \sin^2 x)^3]^3 \cdot [1 + 3(x + sin^2 x)^2 \cdot (1 + 2\sin x \cos x)]$

EX.75

$$
\begin{aligned}
y &= e^{2x}(A\cos 3x + B\sin 3x) \Rightarrow \\
y' &= e^{2x}(-3A\sin 3x + 3B\cos 3x) + (A\cos 3x + B\sin 3x)\cdot 2e^{2x} \\
&= e^{2x}(-3A\sin 3x + 3B\cos 3x + 2A\cos 3x + 2B\sin 3x) \\
&= e^{2x}[(2A+3B)\cos 3x + (2B-3A)\sin 3x) \Rightarrow \\
y'' &= e^{2x}[-3(2A+3B)\sin 3x + 3(2B-3A)\cos 3x] + [(2A+3B)\cos 3x) \\
&\quad + (2B-3A)\sin 3x]\cdot 2e^{2x}] \\
&= e^{2x}[-3(2A+3B) + 2(2B-3A)]\sin 3x + [3(2B-3A) + 2(2A+3B)]\cos 3x \\
&= e^{2x}[(-12A-5B)\sin 3x + (-5A+12B)\cos 3x]
\end{aligned}
$$

Substitute the expressions for y, y', and y'' in $y'' - 4y' + 13y$ to get

$$
\begin{aligned}
y'' - 4y' + 13y &= e^{2x}[(-12A-5B)\sin 3x + (-5A+12B)\cos 3x] \\
&\quad - 4e^{2x}[(2A+3B)\cos 3x + (2B-3A)\sin 3x] \\
&\quad + 13e^{2x}(A\cos 3x + B\sin 3x) \\
&= e^{2x}[(-12A - 5B - 8B + 12A + 13B)\sin 3x \\
&\quad + (-5A + 12B - 8A - 12B + 13A)\cos 3x] \\
&= e^{2x}[(0)\sin 3x + (0)\cos 3x] = 0
\end{aligned}
$$

Thus, the function y satisfies the differential equation $y'' - 4y' + 13y = 0$.

EX.99

The Chain Rule says that $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$, so

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}(\frac{dy}{du})(\frac{du}{dx}) = [\frac{d}{dx}(\frac{dy}{du})]\frac{du}{dx} + \frac{dy}{du}\frac{d}{dx}(\frac{du}{dx}) \\
&= [\frac{d}{du}(\frac{dy}{du})\frac{du}{dx}]\frac{du}{dx} + \frac{dy}{du}\frac{d^2u}{dx^2} = \frac{d^2y}{du^2}(\frac{du}{dx})^2 + \frac{dy}{du}\frac{d^2u}{dx^2}
\end{aligned}
$$