

# Chapter 3. Higher-Order Differential Equations

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1

## Linear Independent & Dependent

$W(y_1, y_2)$

補充說明

一組函數集合  $\{u_1(x), u_2(x), \dots, u_n(x)\}$

考慮  $C_1 u_1(x) + C_2 u_2(x) + \dots + C_n u_n(x) = 0$  -----(1)

其中  $C_1, C_2, \dots, C_n \in \text{const}$

(A) 若(1)成立  $\Leftrightarrow C_1 = C_2 = \dots = C_n = 0$

則  $u_1(x), u_2(x), \dots, u_n(x)$  為線性獨立(Linear Independent)

(B)  $\exists C_i \neq 0$  s.t. (1)成立

則  $u_1(x), u_2(x), \dots, u_n(x)$  為線性相依(Linear Dependent)

2

# Linear Independent & Dependent

例:  $u_1(x) = x^2, u_2(x) = x$

$$C_1x^2 + C_2x = 0 \Leftrightarrow C_1 = C_2 = 0$$

$\therefore u_1(x), u_2(x)$  線性獨立

例:  $u_1(x) = 3x, u_2(x) = -2x$

$$C_1(3x) + C_2(-2x) = 0$$

$$(3C_1 - 2C_2) = 0$$

$$\therefore 3C_1 = 2C_2 \quad \text{取 } C_1 = 1, C_2 = \frac{3}{2}$$

$u_1(x), u_2(x)$  線性相依

3

# Linear Independent & Dependent

Pf:

設  $C_i \neq 0$

$$C_1u_1 + C_2u_2 + \dots + C_nu_n = 0$$

$$u_i = \frac{-C_1}{C_i}u_1 + \frac{-C_2}{C_i}u_2 + \dots + \frac{-C_n}{C_i}u_n$$

$$= k_1u_1 + k_2u_2 + \dots + k_nu_n$$

上述例子只有兩個函數容易判斷  $C_1, C_2$

4

# Linear Independent & Dependent

Extend to n 個函數?

$$\Rightarrow \begin{cases} C_1 u_1 + C_2 u_2 + \dots + C_n u_n = 0 \\ C_1 u_1' + C_2 u_2' + \dots + C_n u_n' = 0 \\ \dots \\ C_1 u_1^{(n-1)} + C_2 u_2^{(n-1)} + \dots + C_n u_n^{(n-1)} = 0 \end{cases}$$

回顧:  $\begin{cases} ax + by = 0 \\ cx + dy = 0 \end{cases}$

# Linear Independent & Dependent

$$x = \frac{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = 0, y = \frac{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = 0$$

$$\Rightarrow (1)x = y = 0 \Leftrightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0. (2)x, y \text{ 具有非零解} \Leftrightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$$

# Linear Independent & Dependent

上述聯立方程式之行列式為

$$\begin{vmatrix} u_1 & u_2 & \dots & u_n \\ u_1' & u_2' & \dots & u_n' \\ \vdots & \vdots & \dots & \vdots \\ u_1^{(n-1)} & u_2^{(n-1)} & \dots & u_n^{(n-1)} \end{vmatrix} = \text{wronski}(u_1 \quad u_2 \quad \dots \quad u_n)$$

7

# Linear Independent & Dependent

$$(A) w(u_1 \quad u_2 \quad \dots \quad u_n) = 0$$

$\Leftrightarrow$  至少存在  $C_1 C_2 \dots C_n$  有非零解

$\Leftrightarrow u_1 \quad u_2 \quad \dots \quad u_n$  線性相依

$$(B) w(u_1 \quad u_2 \quad \dots \quad u_n) \neq 0$$

$$\Leftrightarrow C_1 = C_2 = \dots = C_n = 0$$

$\Leftrightarrow u_1 \quad u_2 \quad \dots \quad u_n$  線性獨立

8

# Linear Independent & Dependent

例:  $3x, -2x$

$$w = \begin{vmatrix} 3x & -2x \\ 3 & -2 \end{vmatrix} \quad L.D.$$

例:  $x, x^2$

$$w = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2 \neq 0, \forall x \quad L.I.$$

# Linear Independent & Dependent

例:  $e^x, e^{2x}, e^{3x}$

$$w = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x} \neq 0 \quad L.I.$$

# Euler-Cauchy Differential Equations

變係數微分方程式

Euler-Cauchy Differential Eqs.

尤拉科西 or 等維

$$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \dots + a_{n-1} x^1 y' + a_n y = r(x)$$

If  $r(x) = 0$  homogeneous case

$$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \dots + a_{n-1} x^1 y' + a_n y = 0$$

想法: 常係數時積  $y = e^{\lambda x}$  求  $\lambda$

想辦法變成常系數

想法: 令  $x = e^t$

11

# Euler-Cauchy Differential Equations

例:  $x^2 y'' - 2xy' + 2y = 0$

令  $x = e^t$

$$\frac{dx}{dt} = e^t = x$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} \text{ (chain rule)}$$

$$= \frac{dy}{dx} x \text{ --- (1)}$$

12

# Euler-Cauchy Differential Equations

$$\frac{d^2 y}{dt^2} = \frac{d}{dt} \left( \frac{dy}{dt} \right) = \left( \frac{d}{dx} \times \left( \frac{dy}{dt} \right) \right) \times \frac{dx}{dt} \text{ (chain rule)}$$

$$= \frac{d}{dx} \times \left( x \frac{dy}{dx} \right) \times x = \left[ x \frac{d^2 y}{dx^2} + \frac{dy}{dx} \times 1 \right] x$$

$$= x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} \text{ --- (2)}$$

$$xy' = x \frac{dy}{dx} = \frac{dy}{dt} \text{ --- (1)}$$

$$x^2 y'' = x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} - x \frac{dy}{dx} = \frac{d^2 y}{dt^2} - \frac{dy}{dt}$$

13

# Euler-Cauchy Differential Equations

$$\text{原式 } x^2 y'' - 2xy' + 2y = 0$$

$$\frac{d^2 y}{dt^2} - \frac{dy}{dt} - 2 \frac{dy}{dt} + 2y = 0$$

$$\frac{d^2 y}{dt^2} - 3 \frac{dy}{dt} + 2y = 0 \quad \text{常係數}$$

$$(1) x \frac{dy}{dx} = \frac{dy}{dt} \Rightarrow x \overset{D_x}{D} y = \overset{D_t}{D} y$$

$$(2) x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} - \frac{dy}{dt} \Rightarrow x^2 D^2 y = \overset{D_t}{D}^2 y - \overset{D_t}{D} y = (\overset{D_t}{D}^2 - \overset{D_t}{D}) y$$

$$(3) x^3 \frac{d^3 y}{dx^3} = \frac{d^3 y}{dt^3} - 3 \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} \Rightarrow x^3 D^3 y = \overset{D_t}{D}^3 y - 3 \overset{D_t}{D}^2 y + 2 \overset{D_t}{D} y$$

$$= (\overset{D_t}{D}^3 - 3 \overset{D_t}{D}^2 + 2 \overset{D_t}{D}) y = \overset{D_t}{D} (\overset{D_t}{D} - 1) (\overset{D_t}{D} - 2) y$$

14

# Euler-Cauchy Differential Equations

$$\begin{aligned}\text{其中 } \frac{d^3 y}{dt^3} &= \frac{d}{dt} \left( \frac{d^2 y}{dt^2} \right) \\ &= \frac{d}{dx} \left( \frac{d^2 y}{dt^2} \right) \frac{dx}{dt} \\ &= \frac{d}{dx} \left( x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} \right) x \\ &= \left[ x^2 \frac{d^3 y}{dx^3} + 2x \frac{d^2 y}{dx^2} + x \frac{d^2 y}{dx^2} + \frac{dy}{dx} \right] x \\ &= x^3 \frac{d^3 y}{dx^3} + 3x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx}\end{aligned}$$

15

# Euler-Cauchy Differential Equations

$$\begin{aligned}x^3 \frac{d^3 y}{dx^3} &= \frac{d^3 y}{dt^3} - 3x \frac{d^2 y}{dx^2} - x \frac{dy}{dx} \\ &= \frac{d^3 y}{dt^3} - 3 \left( \frac{d^2 y}{dt^2} - \frac{dy}{dt} \right) - \frac{dy}{dt} \\ &= \frac{d^3 y}{dt^3} - 3 \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt}\end{aligned}$$

16



# Euler-Cauchy Differential Equations

例:  $x^2 y'' + 4xy' + 2y = 0$

$$\text{令 } x = e^t \quad \wp \equiv \frac{d}{dt}$$

$$\wp(\wp - 1)y + 4\wp y + 2y = 0$$

$$(\wp^2 + 3\wp + 2)y = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\lambda = -1, -2$$

$$y = C_1 e^{-t} + C_2 e^{-2t}$$

$$= C_1 x^{-1} + C_2 x^{-2}$$

17

# Euler-Cauchy Differential Equations

例:  $x^3 y''' + 4x^2 y'' - 5xy' - 15y = 0$

$$\text{令 } x = e^t, \wp \equiv \frac{d}{dt}$$

$$\wp(\wp - 1)(\wp - 2)y + 4\wp(\wp - 1)y - 5\wp y - 15y = 0$$

$$(\wp^3 + \wp^2 - 7\wp - 15)y = 0$$

$$\lambda^3 + \lambda^2 - 7\lambda - 15 = 0$$

$$(\lambda - 3)(\lambda^2 + 4\lambda + 5) = 0$$

$$\lambda = 3, -2 \pm i$$

$$y(t) = C_1 e^{3t} + e^{-2t} (C_2 \cos t + C_3 \sin t)$$

$$y(x) = C_1 x^3 + x^{-2} (C_2 \cos(\ln x) + C_3 \sin(\ln x))$$

18

# Euler-Cauchy Differential Equations

例:  $x^2 y'' - xy' - 3y = 4x$

$$y = y_h + y_p$$

$$y_h : x^2 y_h'' - xy_h' - 3y_h = 0$$

$$\text{令 } x = e^t, \rho = \frac{d}{dt}$$

$$\rho(\rho - 1)y - \rho y - 3y = 0$$

$$(\rho^2 - 2\rho - 3)y = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$\lambda = 3, -1$$

19

# Euler-Cauchy Differential Equations

$$y_h = C_1 e^{3t} + C_2 e^{-t}$$

$$= C_1 x^3 + C_2 x^{-1}$$

$$y_p : x^2 y_p'' - xy_p' - 3y_p = 4x$$

[法1] *Undetermined Coefficient*

$$y_p'' - \frac{1}{x} y_p' - \frac{3}{x^2} y_p = \frac{4}{x}$$

換成t的函數  $y_p = ae^t$

最後記得將  $e^t$  換回  $x$

20

# Euler-Cauchy Differential Equations

[法2] *Order Reduction*

$$(\wp^2 - 2\wp - 3)y_p = 4e^t$$

$$(\wp - 3)(\wp + 1)y_p = 4e^t, z(t) = (\wp + 1)y_p$$

$$z'(t) - 3z(t) = 4e^t$$

$$z_0(t) = I_1^{-1} \int I_1 r dt$$

$$I_1 = e^{-3t}, r = 4e^t$$

$$(\wp + 1)y_p = z_p = I_1^{-1} \int I_1 r dt$$

21

# Euler-Cauchy Differential Equations

$$y_p' + y_p = I_1^{-1} \int I_1 r dt$$

$$y_p = CI_2^{-1} + I_2^{-1} \int I_2 r' dt$$

$$I_2 = e^t$$

$$r' = I_1^{-1} \int I_1 r dt$$

$$y_p = I_2^{-1} \int I_2 I_1^{-1} \int I_1 r dt dt$$

$$= e^{-t} \int e^t e^{3t} \int e^{-3t} 4e^t dt dt$$

$$= e^{-t} \int -2e^{2t} dt$$

$$= e^{-t} (-1)e^{2t} = -e^t = -x$$

22

# Euler-Cauchy Differential Equations

[法3] *Differential Operator*

$$(\rho^2 - 2\rho - 3) = 4e^t$$

$$\begin{aligned} y_p &= \frac{1}{\rho^2 - 2\rho - 3} \times 4e^t \\ &= \frac{1}{1 - 2 - 3} 4e^t = -e^t = -x \end{aligned}$$

23

# Euler-Cauchy Differential Equations

[法4] *Variation of Variable* 換成  $y'' - \frac{1}{x}y' - \frac{3}{x^2}y = \frac{4}{x}$

$$y_1 = e^{3t}, y_2 = e^{-t}$$

$$w(y_1, y_2) = \begin{vmatrix} e^{3t} & e^{-t} \\ 3e^{3t} & -e^{-t} \end{vmatrix} = -4e^{2t}$$

$$y_p = e^{3t} \int \frac{-4e^t \times e^{-t}}{-4e^{2t}} dt + e^{-t} \int \frac{e^{3t} \times 4e^t}{-4e^{2t}} dt$$

$$= e^{3t} \int e^{-2t} dt + e^{-t} \int -e^{2t} dt$$

$$= e^{3t} \left( -\frac{1}{2} e^{-2t} \right) + e^{-t} \left( -\frac{1}{2} e^{2t} \right) = -\frac{1}{2} e^t - \frac{1}{2} e^t = -e^t = -x$$

24

# Euler-Cauchy Differential Equations

\*注意 若是用  $y_h = C_1 x^3 + C_2 x^{-1}$

$$w(y_1, y_2) = \begin{vmatrix} x^3 & x^{-1} \\ 3x^2 & -x^{-2} \end{vmatrix} = -4x$$

$$\begin{aligned} y_p &= y_1 \int \frac{-y_2 r}{w} dx + y_2 \int \frac{y_1 r}{w} dx \\ &= x^3 \int \frac{-x^{-1}(r(x))}{-4x} dx + x^{-1} \int \frac{x^3(r(x))}{-4x} dx \\ &= -x \end{aligned}$$

25

# Euler-Cauchy Differential Equations

例:  $(2x-3)^2 y'' - 6(2x-3) y' + 12y = 0$

令  $u = 2x - 3$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \text{ (chain rule)}$$

$$= \frac{dy}{du} \times 2$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( 2 \frac{dy}{du} \right)$$

$$= \frac{d}{du} [\dots] \frac{du}{dx}$$

$$= \frac{d}{du} \left[ 2 \frac{dy}{du} \right] \times 2 = 4 \frac{d^2 y}{du^2}$$

26

# Euler-Cauchy Differential Equations

$$\text{原式} = u^2 \times 4 \frac{d^2 y}{du^2} - 6u \times 2 \frac{dy}{du} + 12y = 0$$

$$4u^2 \frac{d^2 y}{du^2} - 12u \frac{dy}{du} + 12y = 0$$

$$\text{令 } u = e^t, \wp = \frac{d}{dt}$$

$$[4\wp(\wp - 1) - 12\wp + 12]y = 0$$

$$[4\wp^2 - 16\wp + 12]y = 0$$

$$[\wp^2 - 4\wp + 3]y = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 1)(\lambda - 3) = 0$$

$$\lambda = 1, 3$$

27

# Euler-Cauchy Differential Equations

$$y(t) = C_1 e^t + C_2 e^{3t}$$

$$\Rightarrow y(u) = C_1 u + C_2 u^3$$

$$y(x) = C_1 (2x - 3) + C_2 (2x - 3)^3$$

$$\text{例: } (3x + 4)^2 y'' - 6(3x + 4)y' + 18y = 9 \ln(3x + 4)$$

$$\text{令 } u = 3x + 4$$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = 3 \frac{dy}{dx}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{du} \frac{dy}{dx} \frac{du}{dx}$$

28

# Euler-Cauchy Differential Equations

$$= \frac{d}{du} \times 3 \times \frac{dy}{du} \times 3 = 9 \frac{d^2 y}{du^2}$$

$$\text{原式: } u^2 \times 9 \frac{d^2 y}{du^2} - 6u \times 3 \frac{du}{dx} + 18y = 9 \ln(u)$$

$$u^2 \frac{d^2 y}{du^2} - 2u \frac{du}{dx} + 2y = \ln(u)$$

$$\text{令 } u = e^t, \wp = \frac{d}{dt}$$

$$(\wp(\wp - 1) - 2\wp + 2)y = \ln(u) = t$$

$$(\wp^2 - 3\wp + 2)y = t$$

29

# Euler-Cauchy Differential Equations

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 1)(\lambda - 2) = 0$$

$$\lambda = 1, 2$$

$$y_h = C_1 e^t + C_2 e^{2t}$$

$$= C_1 u + C_2 u^2$$

$$= C_1 (3x + 4) + C_2 (3x + 4)^2$$

30

# Euler-Cauchy Differential Equations

$$\begin{aligned}y_p &= \frac{1}{D^2 - 3D + 2} \cdot t \\&= \frac{1}{2} \cdot \frac{1}{1 + \frac{D^2 - 3D}{2}} \cdot t \\&= \frac{1}{2} \left[ 1 - \frac{D^2 - 3D}{2} + \left( \frac{D^2 - 3D}{2} \right)^2 - \dots \right] t \\&= \frac{1}{2} t - \frac{1}{4} \cdot 0 + \frac{3}{4}\end{aligned}$$

31

# Euler-Cauchy Differential Equations

$$\begin{aligned}&= \frac{1}{2} t + \frac{3}{4} \\&= \frac{1}{2} \ln(3x + 4) + \frac{3}{4}\end{aligned}$$

$$y = y_h + y_p$$

$$= C_1(3x + 4) + C_2(3x + 4)^2 + \frac{1}{2} \ln(3x + 4) + \frac{3}{4}$$

32



# Bernoulli Equation

- Bernoulli equation [非線性]

$$y'(x) + p(x)y = r(x)y^n$$

$$n = 1, y'(x) + (p - r)y = 0$$

$$n \neq 0, n \neq 1, y^{-n}y' + py^{1-n} = r \text{-----} (*)$$

$$\text{令 } z = y^{1-n}$$

$$\frac{dz}{dx} = (1-n)y^{1-n-1} \frac{dy}{dx}$$

$$= (1-n)y^{-n} \frac{dy}{dx}$$

$$y^{-n} \frac{dy}{dx} = \frac{1}{1-n} \frac{dz}{dx}$$

33

# Bernoulli Equation

$$\text{原式} (*) = \frac{1}{1-n} \frac{dz(x)}{dx} + p(x)z(x) = r(x)$$

$$\frac{dz(x)}{dx} + (1-n)p(x)z(x) = (1-n)r(x)$$

$$\frac{dz}{dx} + p'z = r'$$

$$z = CI^{-1} + I^{-1} \int Ir' dx$$

$$I = e^{\int p' dx}$$

$$z = Ce^{-\int (1-n)p dx} + e^{-\int (1-n)p dx} \int e^{\int (1-n)p dx} (1-n)r(x) dx = y^{1-n}$$

34

# Bernoulli Equation

例:  $x \frac{dy}{dx} + y = x^2 y^2$

$$\frac{dy}{dx} + \frac{1}{x} y = xy^2 = xy^n$$

$$\Rightarrow n = 2$$

$$\text{令 } z = y^{1-n} = y^{-1}$$

$$\frac{dz}{dx} - \frac{1}{x} z(x) = -x \quad I = e^{\int \frac{-1}{x} dx} = e^{-\ln x} = \frac{1}{x}$$

$$z(x) = C_1 (x^{-1})^{-1} + (x^{-1})^{-1} \int x^{-1} (-x) dx$$

$$= Cx - x^2 = y^{-1}, \quad y = \frac{1}{Cx - x^2}$$

35

# Bernoulli Equation

- 非線性  $\rightarrow$  線性  
型式(1)

$$f'(y) \frac{dy}{dx} + p(x) f(y) = g(x)$$

$$\text{令 } z = f(y)$$

$$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$$

$$= \frac{df(y)}{dy} \times \frac{dy}{dx} = f'(y) \times \frac{dy}{dx}$$

$$\text{原式} : \frac{dz}{dx} + p(x)z = q(x)$$

36

# Bernoulli Equation

例:  $x^2 \cos y \frac{dy}{dx} = 2x \sin y - 1$

令  $z = \sin y$  則  $\frac{dz}{dx} = \cos y \cdot \frac{dy}{dx}$  Chain Rule

$$\frac{dz}{dx} - \frac{2}{x} dz = \frac{-1}{x^2}$$

$$z = CI^{-1} + I^{-1} \int I r dx$$

對於型式(1)

$f(y)$ 同常等於  $y^2, y^3, \dots, \sin(y) \dots e^y$

37

# Bernoulli Equation

型式(2)

Bernoulli

型式(3)

Riccati

$$\frac{dy}{dx} + p(x)y = q(x) + y^2 r(x)$$

若  $y_1$  為上式之一特解

則令  $y = y_1 + \frac{1}{z}$  得Z的線性D.E.

38