106-Calculus-CSIE

Midterm II

Solution

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December 8th, Fall 2017

Name: ____________________________

Student ID $\sharp$: ____________________________

Instructions:

1. This exam consists of 7 Problems with total of **110** points.
2. The maximum of the midterm is **100** points.
3. Put away books, notes, calculators, cell phones, and other electronic devices. No discussion during the exam.
4. It might be a good idea to finish the simpler questions first. Good luck!
5. Include all detail reasoning if possible.

1	2	3	4
5	6	7	

1. Consider the hyperbolic functions,

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \ \cosh(x) = \frac{e^x + e^{-x}}{2}, \ \tanh(x) = \frac{\sinh x}{\cosh x},$$

where $(-\infty, \infty)$ is the domain for all three functions.

(a) **(10 points)** The hyperbolic tangent function has the inverse function

$$f(x) = \tanh^{-1}(x), \ x \in (-1, 1).$$

Compute the derivative $f'(x)$. (*Hint: Instead of solving the inverse function* $f(x)$ *explicitly, try implicit differentiation.*)

Answer :

Method 1: Use implicit differentiation,

$$y = \tanh^{-1}(x) \Rightarrow \tanh y = x \overset{\frac{d}{dx}}{\Rightarrow} \operatorname{sech}^2 y \cdot \frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y} = \cosh^2 y \qquad (\mathbf{5points})$$

From the identities,

$$\begin{cases} x = \tanh y = \frac{\sinh y}{\cosh y} & \Rightarrow \ \sinh y = x \cdot \cosh y \\ \cosh^2 y - \sinh^2 y = 1 & \Rightarrow \ (1 - x^2)\cosh^2 y = 1 \end{cases}$$

$$\Rightarrow \ \therefore \cosh^2 y = \frac{1}{1 - x^2} \qquad (\mathbf{3points})$$

$$\Rightarrow \frac{d}{dx} \tanh^{-1}(x) = \frac{1}{1 - x^2} \qquad (\mathbf{2points})$$

Method 2: Solve the inverse function of $y = \tanh(x)$ and then take derivative,

$$y = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} < 1$$

$$\Rightarrow e^{2x} y + y = e^{2x} - 1 \qquad (\mathbf{2points})$$

$$\Rightarrow e^{2x}(y - 1) = -y - 1$$

$$\Rightarrow e^{2x} = \frac{1 + y}{1 - y}$$

$$\Rightarrow x = \frac{1}{2} \ln(\frac{1 + y}{1 - y})$$

$$\Rightarrow \ \tanh^{-1} x = \frac{1}{2} \ln(\frac{1 + x}{1 - x}), -1 < x < 1. \qquad (\mathbf{3points})$$

$$\therefore \frac{d}{dx} \tanh^{-1} x = \frac{1}{2}(\frac{1}{1 + x} - \frac{1}{1 - x} \cdot (-1)) = \frac{1}{1 - x^2} \qquad (\mathbf{5points})$$

(*b*) **(10 points)** Include your full reasoning to evaluate the limit .

$$\lim_{x\to 0} \frac{\tanh^{-1}(x) - x}{x^3}.$$

Answer : Check all the conditions before applying L'Hopital's rule.

(1) This is an indeterminate form as

$$\begin{cases} f(x) = \tanh^{-1} x - x \\ g(x) = x^3 \end{cases} \Rightarrow \lim_{x\to 0} f(x) = 0, \ \lim_{x\to 0} g(x) = 0 \qquad \textbf{(2points)}$$

(2) $f(x)$ and $g(x)$ are differentiable and $g'(x) = 3x^2 \neq 0$ if $x \neq 0$. **(3 points)**

(3) The limit after taking derivative is

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{\frac{1}{1-x^2} - 1}{3x^2} = \lim_{x\to 0} \frac{1}{3(1-x^2)} = \frac{1}{3} \qquad \textbf{(3points)}$$

By L'Hopital's rule, we can conclude

$$\lim_{x\to 0} \frac{\tanh^{-1}(x) - x}{x^3} = \frac{1}{3} \qquad \textbf{(2points)}$$

2.(10 points) In a murder investigation, the temperature of the corpse was 32.5°C at 1:30 PM and 30.3°C an hour later. Normal body temperature is 37.0°C and the temperature of the surroundings was 20.0°C. When did the murder take place?

Answer :
$T(t)$ = Temperature of body at time t in hour.

Set t=0 $\Leftrightarrow$ 1:30 pm. $\therefore \begin{cases} T(0) = 32.5°\text{C} \\ T(1) = 30.3°\text{C} \end{cases}$

The surrounding temperature is $T_s = 20.0°$C.

Newton's law of cooling:

$$\begin{aligned} \frac{dT}{dt} &= k(T - T_s) \\ \Rightarrow \frac{d}{dt}(T - T_s) &= d(T - T_s) \\ \Rightarrow T - T_s &= Ce^{kt} \\ \Rightarrow T(t) &= T_s + Ce^{kt} \end{aligned}$$

The given conditions imply that

$$\begin{cases} 32.5 = T(0) = 20 + Ce^{0*k} \\ 30.3 = T(1) = 20 + Ce^{1*k} \end{cases}$$

$$\therefore C = 12.5, k = \ln \frac{10.3}{12.5} \qquad \textbf{(5points)}$$

$$T(t) = 20 + 12.5e^{kt} = 20 + 12.5(\frac{103}{125})^t$$

Solve

$$37 = T(t) = 20 + 12.5(\frac{103}{125})^{2t} \Rightarrow \ t = \frac{\ln \frac{17}{12.5}}{\ln \frac{103}{125}}.$$

The murder took place at t hours before. **(5 points)**

3. Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} x^3 \ln x, & x > 0 \\ 0, & x = 0 \end{cases}.$$

(a) **(10 points)** Show that $\lim_{x\to 0^+} f(x) = 0$. Remember to write your full reasoning.

Answer : We check those conditions for applying L'Hopital's rule.

(1) $\lim_{x\to 0^+} x^3 \ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-3}} (\frac{-\infty}{+\infty})$ [indeterminant form] **(2 points)**

(2) $\begin{cases} f(x) = \ln x \\ g(x) = x^{-3} \end{cases}$ are differentiable with $\begin{cases} \lim_{x\to 0^+} f(x) = -\infty \\ \lim_{x\to 0^+} g(x) = +\infty \end{cases}$ and $g'(x) = -3x^{-4} \neq 0$ if $x \neq 0$. **(2 points)**

(3) Taking derivatives, we have

$$\lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = \lim_{x\to 0^+} \frac{1/x}{-3x^{-4}} = \lim_{x\to 0^+} \frac{x^3}{-3} = 0 \quad (\textbf{3points})$$

By L'Hopital's rule, $\lim_{x\to 0^+} f(x) = 0$. **(2 points)**

(b) **(10 points)** Part (a) says that $f(x)$ is continuous on $[0, \infty)$. Does $f(x)$ have any absolute maximum and absolute minimum on $[0, 2]$? If so, find them. Remember to write your full reasoning.

Answer : Since $f(x)$ is continuous on $[0, 2]$, by EVT, f(x) has an absolute maximum and absolute minimum on $[0, 2]$. **(2 points)**

Use **Closed Interval Method**, we first find critical points and end points, and then compare their values to determine the extremal values.

Critical point:

$$\begin{aligned}
&\Leftrightarrow f'(x) = 0 \\
&\Leftrightarrow 3x^2 \ln x + x^3 \frac{1}{x} = 0 \\
&\Leftrightarrow x^2(3 \ln x + 1) = 0 \\
&\Leftrightarrow x = e^{-1/3} (< e^{1/2} < 2) \\
&\Rightarrow f(e^{-1/3}) = e^{-1}(-\frac{1}{3})
\end{aligned}$$

End point : $f(0) = 0,\ f(2) = 8 \ln 2.$ **(2 points)**

Absolute minimum : $f(e^{-1/3}) = -\frac{e^{-1}}{3} = -\frac{1}{3e}$ **(2 points)**

Absolute maximum : $f(2) = 8 \ln 2$ **(2 points)**

4. Consider the function $f(x) = x^2$ on $[1, 2]$.

(a) **(10 points)** Write down the upper sum $U(f, \mathcal{P})$ and the lower sum $L(f, \mathcal{P})$ if the partition $\mathcal{P}$ is given by

$$\mathcal{P} : x_0 = 1 < x_1 = 1 + \frac{1}{n} < \cdots < x_{n-1} = 1 + \frac{n-1}{n} < x_n = 2.$$

Answer:

$$\begin{aligned} U(f, \mathcal{P}) =& \sum_{i=1}^{n} f(1 + \frac{i}{n}) \cdot \frac{1}{n} = \sum_{i=1}^{n} \frac{(n+i)^2}{n^2} \frac{1}{n} \\ =& \frac{1}{n^3}[(n+1)^2 + (n+2^2) + \cdots + (2n)^2] \qquad \textbf{(5points)} \end{aligned}$$

$$\begin{aligned} L(f, \mathcal{P}) =& \sum_{i=1}^{n} f(1 + \frac{i-1}{n}) \cdot \frac{1}{n} = \sum_{i=1}^{n} \frac{(n+i-1)^2}{n^2} \frac{1}{n} \\ =& \frac{1}{n^3}[n^2 + (n+1)^2 + (n+2)^2 + \cdots + (2n-1)^2] \qquad \textbf{(5points)} \end{aligned}$$

(*b*) **(5 points)** Compute the definite integral $\int_1^2 f(x)dx$ from (*a*).

Answer:

Since

$$L(f,\mathcal{P}) \leq \int_1^2 f(x)dx \leq U(f,\mathcal{P})$$

and

$$\begin{cases} U(f,\mathcal{P}) = \frac{1}{n^3}\left(\frac{1}{6}(2n)(2n+1)(4n+1) - \frac{1}{6}(n)(n+1)(2n+1)\right) \\ L(f,\mathcal{P}) = \frac{1}{n^3}\left(\frac{1}{6}(2n-1)(2n)(4n-1) - \frac{1}{6}(n-1)(n)(2n-1)\right) \end{cases}$$

Take n$\to\infty$, we have

$$\lim_{n\to\infty} L(f,\mathcal{P}) = \frac{1}{6}\cdot 2\cdot 2\cdot 4 - \frac{1}{6}\cdot 1\cdot 1\cdot 2 = \frac{14}{6} = \lim_{n\to\infty} U(f,\mathcal{P})$$

and by squeeze theorem $\int_1^2 f(x)dx = \frac{7}{3}$. **(5 points)**

5. **(20 points)** Let $f(x) = \frac{x^2(x-2)}{(x+1)^2}$, where

$$f'(x) = \frac{x^3 + 3x^2 - 4x}{(x+1)^3}, \quad f''(x) = \frac{2(7x-2)}{(x+1)^4}.$$

Answer the following questions.

(a) The domain of $y = f(x)$ is ________________________.

(b) $y = f(x)$ has critical point(s) at ________________________.

(c) $f''(x) =$ ________________________.

(d) Determine the intervals of increasing and decreasing.
  $y = f(x)$ is increasing on interval(s) ________________________.
  $y = f(x)$ is decreasing on interval(s) ________________________.

(e) Determine the intervals of concave up and down.
  $y = f(x)$ is concave up on interval(s) ________________________.
  $y = f(x)$ is concave down on interval(s) ________________________.

(f) Find the (x, y)-coordinates of the following points if they exists:
  Local maximum point(s):________________________.
  Local minimum point(s):________________________.
  Inflection point(s):________________________.

(g) Find the asymptotes of the graph of $y = f(x)$ if exist.
  Vertical asymptotes(s):________________________.
  Horizontal asymptote(s):________________________ as $x \to$ ________________.
  Slant asymptote(s):________________________ as $x \to$ ________________.

(h) Sketch the graph of $y = f(x)$.

Answer:

(a) Since $(x+1)^2 \neq 0$, so the domain of $y = f(x)$ is $(-\infty, -1) \bigcup (-1, \infty)$ **(1 points)**

(b) $f'(x) = \dfrac{x^3 + 3x^2 - 4x}{(x+1)^3} = \dfrac{x(x+4)(x-1)}{(x+1)^3}$, so critical points are $x = -4, 0, 1$ **(2 points)**

(c) $f''(x) = \dfrac{2(7x-2)}{(x+1)^4}$

(d) Increasing intervals: $(-\infty, -4) \bigcup (-1, 0) \bigcup (1, \infty)$
  Decreasing intervals: $(-4, -1) \bigcup (0, 1)$
  (3 points)

(e) Concave up interval: $(\frac{2}{7}, \infty)$
  Concave down interval: $(-\infty, -1) \bigcup (-1, \frac{2}{7})$
  (3 points)

(f) Since critical points are $(-4, -\frac{32}{3}), (0, 0)$ and $(1, -\frac{1}{4})$. By (d), we have:
  Local maximum points: $(0, 0), (-4, -\frac{32}{3})$ **(1 points)**
  Local minimum point: $(1, -\frac{1}{4})$ **(1 points)**

On the other hand, $f''(x) = 0 \;\Rightarrow\; x = \frac{2}{7}$ and concavity changes from down to up. So the infection point is: $(\frac{2}{7}, -\frac{16}{189})$**(1 points)**

(g) $\displaystyle\lim_{x\to -1^+} \frac{x^2(x-2)}{(x+1)^2} = \lim_{x\to -1^-} \frac{x^2(x-2)}{(x+1)^2} = \infty$, so the vertical asymptote is: $x = -1$.**(1 points)**

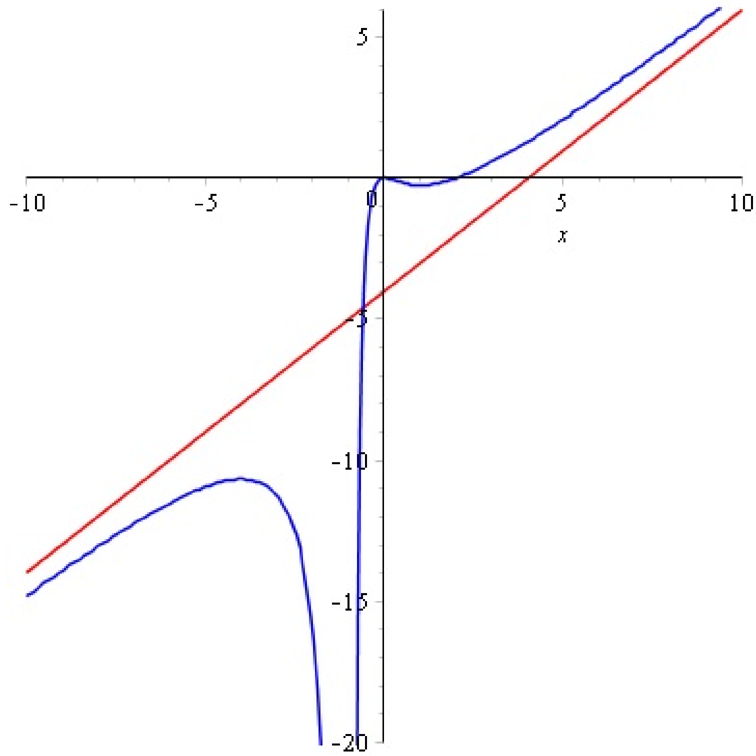
$\displaystyle\lim_{x\to\pm\infty} \frac{x^2(x-2)}{(x+1)^2} = \infty$, so there is no horizontal asymptote**(1 points)**.

Suppose the slant asymptote is $y = mx + b$, then

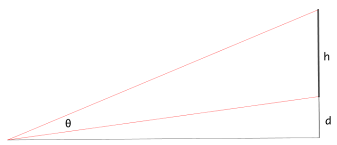
$\displaystyle\lim_{x\to\pm\infty} [(x+4) + \frac{7x-4}{(x+1)^2} - (mx+b)] = 0 \;\Rightarrow\; y = x - 4$

So, the slant asymptote is $y = x - 4$.**(2 points)**

(h) Graph **(4 points)**

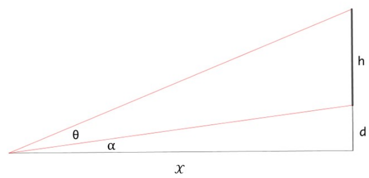


6. (10 points) A painting in an art gallery has height h and is hung so that its lower edge is a distance d above the eye of an observer (as in the figure). How far from the wall should the observer stand to get the best view? (In other words, where should the observer stand so as to maximize the angle θ subtended at his eye by the painting?)
You might use the formula,



$$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}.$$

Answer:



$\tan\alpha = \frac{d}{x}$ **(1 point)**

$\tan(\theta+\alpha) = \frac{h+d}{x} = \frac{\tan\theta+\tan\alpha}{1-\tan\theta\cdot\tan\alpha} = \frac{\tan\theta+\frac{d}{x}}{1-\tan\theta\cdot\frac{d}{x}}$

$\Rightarrow \frac{h+d}{x} - \tan\theta\cdot\frac{d(h+d)}{x^2} = \tan\theta + \frac{d}{x}$

$\Rightarrow \frac{h}{x} = \tan\theta\cdot(\frac{x^2+d(h+d)}{x^2})$**(1 point)**

$\Rightarrow \tan\theta = \frac{x\cdot h}{x^2+dh+d^2}$, $x\in(0,\infty)$

$\Rightarrow \theta = \theta(x) = \tan^{-1}(\frac{xh}{x^2+dh+d^2})$ **(2 points)**

$\frac{d\theta}{dx} = \frac{1}{1+(\frac{xh}{x^2+dh+d^2})^2}\cdot\frac{h\cdot(x^2+dh+d^2)-xh\cdot(2x)}{(x^2+dh+d^2)^2} = \frac{h(dh+d^2-x^2)}{(x^2+dh+d^2)^2+(xh)^2}$ **(2 points)**

Hence $\theta'(x)$ is increasing on $(-\infty,\sqrt{dh+d^2})$ and decreasing on $(\sqrt{dh+d^2},\infty)$. **(2 points)**

Since $\theta(x)$ is increasing before $x=\sqrt{dh+d^2}$ and decreasing after.

$\Rightarrow$ When $x = \sqrt{dh + d^2}$, where

$$\tan\theta = \frac{\sqrt{dh + d^2} \cdot h}{(\sqrt{d(d+h)})^2 + dh + d^2} = \frac{h \cdot \sqrt{dh + d^2}}{2(dh + d^2)} = \frac{h}{2\sqrt{dh + d^2}},$$

the angle

$$\theta = \tan^{-1}(\frac{h}{2\sqrt{dh + d^2}})$$

is the absolute maximum. **(2 points)**

7. Let $f : [0, \infty) \to \mathbb{R}$ be continuous and $f(0) = 0$. Suppose that $f'(x)$ exists for all $x \in (0, \infty)$ and $f'(x)$ is increasing on $(0, \infty)$.

(a) **(10 points)** Show that for a fixed $x \neq 0$, there is a number $c \in (0, x)$ such that $f'(c) = f(x)/x$.

(b) **(5 points)** Show that the function $g(x) = \frac{f(x)}{x}$ is increasing on $(0, \infty)$.

Answer: (a) Since $f(x)$ is continuous on $[0, x]$ and is differentiable over $(0, x)$, by **Mean Value Theorem** there is $c \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(c).$$

As $f(0) = 0$, we have $f'(c) = f(x)/x$. **(10 points)**

(b) We use the first derivative test. Consider the derivative of $g(x)$ for a fixed $x \in (0, \infty)$, then

$$g'(x) = \frac{f'(x) \cdot x - f(x)}{x^2} = \frac{f'(x) - \frac{f(x)}{x}}{x} = \frac{f'(x) - f'(c)}{x} \qquad \textbf{(2points)}$$

where the last equality is from part (a). By hypothesis, $f'(x)$ is an increasing function and hence $f'(x) \geq f'(c)$ if $c \in (0, x)$. It follows that

$$g'(x) = \frac{f'(x) - f'(c)}{x} \geq 0,$$

and $g(x)$ is increasing on $(0, \infty)$. **(3 points)**

WRONG ANSWER: $g(x) = \frac{f(x)}{x} = f'(c)$ from (a). Since $f'(x)$ is an increasing function, then $g(x)$ is increasing.

There are two possible mistakes:

1 Notice that $g(x) = f'(c)$ for some $0 < c < x$. It does NOT mean $g(x) = f'(x)$. Hence we don't know if $g(x)$ is increasing from that of $f'(x)$.

2. On the other hand, in the MVT the choice of c depends on x and is random. It is not clear that for $0 < x_1 < x_2$, the choices of c_1, c_2 with $g(x_1) = f'(c_1)$ and $g(x_2) = f'(c_2)$ will satisfy $c_1 \leq c_2$. Hence we can not say directly that

$$g(x_1) = f'(c_1) \leq f(c_2) = g(x_2).$$