System Of Equations

Find out the current and the resistance

EXAMPLE 8–20 Given the circuit of Figure 8–45, find the total resistance, R_T, and the total current, I.

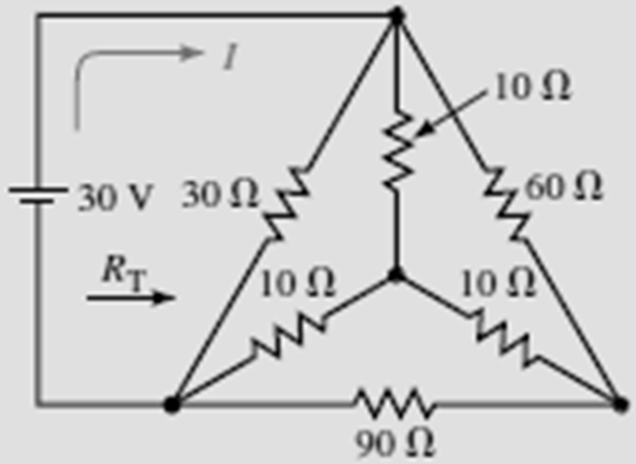
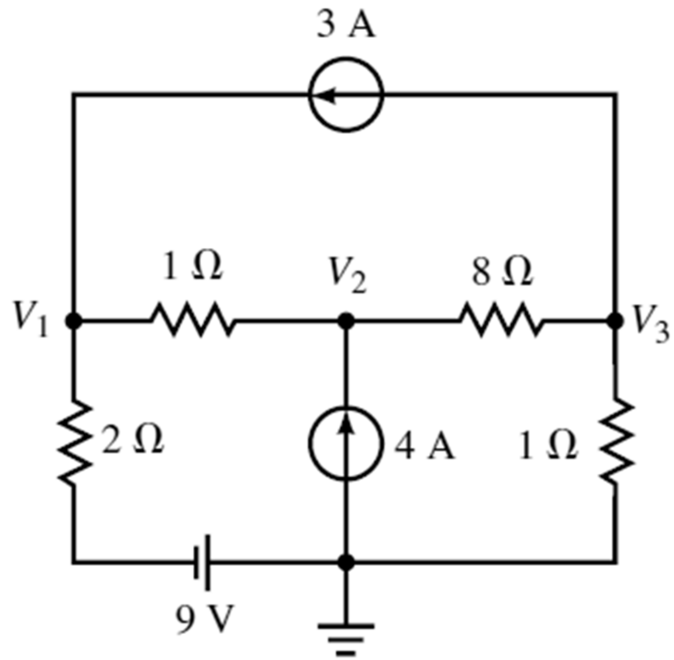


FIGURE 8–45

Find out V1, V2 and V3

FIGURE 8–38



Answers: $V_1 = 3.00$ V, $V_2 = 6.00$ V, $V_3 = -2.00$ V

Linear Systems of Equations

- Constant coefficients with several variables.
- Typically, the number of condition is the same as variables. (Condition shall still be examined)

$$E_1 : a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$
$$E_2 : a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$

$$E_n : a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n$$

Linear Systems of Equations

- Analytical solutions have been taught in high school and in the university.
- Huge scale equations are not easy to solve by hand.
- Try to solve these equations with computer

System Of Equations

- Direct Method
  - Gaussian Elimination with Backward Substitution
  - Pivoting Strategy
  - Matrix Factorization (LU decomposition)

- Iterative Methods

Example (I)

- How do you solve these equations

$$
\begin{array}{lllllll}
E_1 : x_1 & + x_2 & & + 3x_4 & = & 4 \\
E_2 : 2x_1 & + x_2 & - x_3 & + x_4 & = & 1 \\
E_3 : 3x_1 & - x_2 & - x_3 & + 2x_4 & = & -3 \\
E_4 : -x_1 & + 2x_2 & + 3x_3 & - x_4 & = & 4
\end{array}
$$

- Crammer's Rule
- 加減消去法，帶入消去法，Gauss Elimination

Example (I)

- $(E_2\text{-}2E_1) \text{->} E_2$, $E_3\text{-}3E_1 \text{->} E_3$, $E_4\text{+}E_1 \text{->} E_1$

$$
\begin{aligned}
E_1 &: x_1 && + x_2 && && + 3x_4 = && 4 \\
E_2 &: 2x_1 && + x_2 && - x_3 && + x_4 = && 1 \\
E_3 &: 3x_1 && - x_2 && - x_3 && + 2x_4 = && -3 \\
E_4 &: -x_1 && + 2x_2 && + 3x_3 && - x_4 = && 4
\end{aligned}
$$

Example (I)

- $(E_3-4E_2)\text{->}E_3$, $E_4+3E_2\text{->}E_4$

$$
\begin{array}{llllll}
E_1: x_1 & +x_2 & & +3x_4 = & 4 \\
E_2: & -x_2 & -x_3 & -5x_4 = & -7 \\
E_3: & -4x_2 & -x_3 & -7x_4 = & -15 \\
E_4: & 3x_2 & +3x_3 & +2x_4 = & 8
\end{array}
$$

Example (I)

- Generally an augmented array is used to simplify the representation of Gauss Elimination

$$\begin{aligned} E_1 &: x_1 + x_2 + 3x_4 = 4 \\ E_2 &: 2x_1 + x_2 - x_3 + x_4 = 1 \\ E_3 &: 3x_1 - x_2 - x_3 + 2x_4 = -3 \\ E_4 &: -x_1 + 2x_2 + 3x_3 - x_4 = 4 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 4 \\ 2 & 1 & -1 & 1 & 1 \\ 3 & -1 & -1 & 1 & -3 \\ -1 & 2 & 3 & -1 & 4 \end{array}\right]$$

Example (I)

- $(E_2-2E_1) \rightarrow E_2$, $E_3-3E_1 \rightarrow E_3$, $E_4+E_1 \rightarrow E_1$

$$\left[\begin{array}{cccc:c} 1 & 1 & 0 & 3 & 4 \\ 2 & 1 & -1 & 1 & 1 \\ 3 & -1 & -1 & 1 & -3 \\ -1 & 2 & 3 & -1 & 4 \end{array}\right]$$

Example (I)

- $(E_2\text{-}2E_1)\text{->}E_2$, $E_3\text{-}3E_1\text{->} E_3$, $E_4\text{+}E_1 \text{->}E_1$

$$\left[\begin{array}{cccc:c} 1 & 1 & 0 & 3 & 4 \\ 2 & 1 & -1 & 1 & 1 \\ 3 & -1 & -1 & 1 & -3 \\ -1 & 2 & 3 & -1 & 4 \end{array}\right]$$

$$\left[\begin{array}{cccc:c} 1 & 1 & 0 & 3 & 4 \\ 0 & -1 & -1 & -5 & -7 \\ 0 & -4 & -1 & -7 & -15 \\ 0 & 3 & 3 & 2 & 8 \end{array}\right]$$

Example (I)

- E_3-4E_2->E_3, E_4+3E_2-> E_4

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 4 \\ 0 & -1 & -1 & -5 & -7 \\ 0 & -4 & -1 & -7 & -15 \\ 0 & 3 & 3 & 2 & 8 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 4 \\ 0 & -1 & -1 & -5 & -7 \\ 0 & 0 & 3 & 13 & 13 \\ 0 & 0 & 0 & -13 & -13 \end{array}\right]$$

Example (I)

- Row Echelon Form

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 4 \\ 0 & -1 & -1 & -5 & -7 \\ 0 & 0 & 3 & 13 & 13 \\ 0 & 0 & 0 & -13 & -13 \end{array}\right]$$

$$\begin{aligned} x_1 \quad + x_2 \qquad\qquad + 3x_4 &= 4 \\ - x_2 \quad - x_3 \quad - 5x_4 &= -7 \\ 3x_3 \quad + 13x_4 &= 13 \\ -13x_4 &= -13 \end{aligned}$$

Example (I)

- Backward Substitution

$$\begin{aligned}
x_1 \quad + x_2 \qquad\qquad + 3x_4 &= 4 \\
- x_2 \quad - x_3 \quad - 5x_4 &= -7 \\
3x_3 \quad + 13x_4 &= 13 \\
-13x_4 &= -13
\end{aligned}$$

$$x_4 = \frac{-13}{-13} = 1$$

$$x_3 = \frac{13 - (13x_4)}{3} = 0$$

$$x_2 = \frac{-7 - (-x_3 - 5x_4)}{(-1)} = 2$$

$$x_1 = \frac{4 - (x_2 + 0x_3 + 3x_4)}{1} = -1$$

Gauss Elimination with backward substitution

- Converting the problem to the augmented matrix

$$E_1 : a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$

$$E_2 : a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$

$$E_n : a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n$$

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} & b_1 \\ a_{21} & a_{22} & ... & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & ... & a_{nn} & b_1 \end{bmatrix}$$

Gauss Elimination

- For i = 1:n-1

  For j = i+1 : n

  Apply $(E_j-(a_{ji}/a_{ii})E_i) \rightarrow E_j$

  End

  End

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_1 \end{matrix}$$

- Row Echelon Form

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} & a_{1,n+1} \\ 0 & a_{22} & \cdots & a_{2n} & a_{2,n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} & a_{n,n+1} \end{matrix}$$

Backward substitution

- Check a_{nn} first.

$$x_i = \frac{a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}}$$

$$\begin{matrix} a_{11} & a_{12} & \dots & a_{1n} & a_{1,n+1} \\ 0 & a_{22} & \dots & a_{2n} & a_{2,n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_{nn} & a_{n,n+1} \end{matrix}$$

$$x_n = \frac{a_{n,n+1}}{a_{n,n}}$$

$$x_{n-1} = \frac{a_{n-1,n+1} - a_{n-1,n} x_n}{a_{n-1,n-1}}$$

$$x_{n-2} = \frac{a_{n-2,n+1} - (a_{n-2,n} x_n + a_{n-2,n-1} x_{n-1})}{a_{n-2,n-2}}$$

$$\vdots$$

$$x_1 = \frac{a_{1,n+1} - \sum_{j=2}^{n} a_{1j} x_j)}{a_{11}}$$

Example (II)

- Consider the linear system

$$
\begin{aligned}
E_1 &: x_1 - x_2 + 2x_3 - x_4 = 4 \\
E_2 &: 2x_1 - 2x_2 + 3x_3 - 3x_4 = -7 \\
E_3 &: x_1 + x_2 + x_3 = -15 \\
E_4 &: x_1 - x_2 + 4x_3 + 3x_4 = 8
\end{aligned}
$$

Example (II)

- After eliminating x_1,the pivot element of row 2 is 0.

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -8 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & 2 & 4 & 12 \end{bmatrix}$$

Example (II)

- interchange Row 2 and Row 3, then carry on Gaussian Elimination

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 2 & 4 & 12 \end{bmatrix}$$

Example (II)

- $E_4 + 2E_3$ -> E_4

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix}$$

Gaussian Elimination with Pivoting and Backward Substitution

- To solve the nxn linear system

$$E_1 : a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$E_2 : a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$E_n : a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$$

- INPUT: number of unknows, the augmented amtrix
- OUPUT: $x_1, \ldots x_n$ or other message

Gaussian Elimination with Pivoting and Backward Substitution

- For i = 1:n-1 (row index)

  {Find the column index of the 1st non-zero term:p}
  {Interchange i-th row to p-th row}

  {if first n column elements are all 0, no unique sol}

  – For j = i+1:n
    $m_{ji} = a_{ji}/a_{ii}$;
    $E_j = E_j - m_{ji} * E_i$
  – end
- End

Gaussian Elimination with Pivoting and Backward Substitution

- If a_{nn}=0, {no unique solution}
- $x_n = a_{n,n+1}/a_{n,n}$
- For i = 1:n-1

  for j =0:i-1

  $x_{n-i}=x_{n-i}+x_{n-j}a_{n-i,n-j}$

  end

  $x_{n-i} = [a_{n-i,n+1}-x_{n-i}]/a_{n-i,n-i}$;
- End
- Output x_1-x_n;

Computation Complexity

- Arithmetic only takes places at calculating the ratio to the pivoting element, subtraction each row , and backward substitution

- And addition/subtraction and multiplication/division are separated counted as they takes different time to perform calculation

Computation Complexity (Row Echelon Form)

- At the step $E_j - m_{ji} E_i$
  - M_{ji}: n-i division
  - $m_{ji} E_i$ (n-i)(n-i+1) multiplications
  - $E_j - m_{ji} E_i$: (n-i)(n-i+1) subtractions
- Because I = 1:n
  - Multiplication/Division : $\frac{2n^3 + 3n^2 - 5n}{6}$
  - Addition / Subtraction: $\frac{n^3 - n}{3}$

Computation Complexity (Backward Substitution)

- Backward substitution

$$x_i = \frac{a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}}$$

- $a_{ij}x_j$: (n-i) multiplication
- Sigma: n-i-1 addition
- And i = 1:n-1
  - Multiplication/Division : $\dfrac{n^2 + n}{2}$
  - Addition / Subtraction: $\dfrac{n^2 - n}{2}$

Total Number of computation

- Multiplication/Division :

$$\frac{2n^3+3n^2-5n}{6}+\frac{n^2+n}{2}=\frac{n^3}{3}+n^2-\frac{n}{3}$$

- Addition / Subtraction

$$\frac{n^3-n}{3}+\frac{n^2-n}{2}=\frac{n^3}{3}+\frac{n^2}{2}-\frac{5n}{6}$$

Gaussian Elimination with Backward Substitution

- A very basic approach to find the roots of a linear system of equations

- The computation complexity is proportional to n^3

A case where the round-off error becomes severe

- Explore the root of the equation using 4 digit arithmetic

$$
\begin{aligned}
E_1 &: \quad 0.003000x_1 + 59.14x_2 = 59.17 \\
E_2 &: \quad 5.291x_1 - 6.130x_2 = 46.78
\end{aligned}
$$

$$m_{11} = \frac{5.291}{0.003000} = 1763.66...$$

A case where the round-off error becomes severe

- E_2-1764*E_1 -> E_2

$$E_1: \quad 0.003000x_1 \quad +59.14x_2 \quad = 59.17$$

$$E_2: \quad -104300x_2 \quad = -104400$$

$$\Rightarrow x_2 \approx \frac{104400}{104300} = 1.001 \qquad x_1 \approx \frac{59.17-(59.14)*1.001)}{0.003000} = -10.00$$

- correct answer should be $x_1 = 10, x_2 = 1$

Reconsider Example (III)

- What if the two equations are interchange prior to the elimination performs?

$$
\begin{aligned}
E_1: &\quad 5.291x_1 &&-6.130x_2 &&= 46.78 \\
E_2: &\quad 0.003x_1 &&+59.14x_2 &&= 59.17
\end{aligned}
$$

$$m_{21} = \frac{0.003000}{5.291} = 0.0005670...$$

Reconsider Example (III)

- E_2-0.0005670*E_1->E_2

$$\begin{aligned} E_1: &\quad 5.291x_1 \quad - 6.130x_2 \quad = 46.78 \\ E_2: &\quad \qquad\qquad + 59.14x_2 \quad = 59.14 \end{aligned}$$

$$\Rightarrow x_2 \approx \frac{59.14}{59.14} = 1 \qquad\qquad x_1 \approx \frac{59.17 - (59.14) * 1.00)}{0.003000} = 10.00$$

- correct answer should be $x_1 = 10, x_2 = 1$

Maximal Column Pivoting

- Before performing Gaussian elimination, the maximum pivot element of each column will be interchanged to the upper row.

- Maximum column pivoting can "SOMEWHAT" avoid round-off error in Gaussian elimination.

- aka. Partial Pivoting

Gaussian Elimination

- For i = 1:n-1 (row index)

  {Find the column index of the 1st non-zero term:p}
  {Interchange i-th row to p-th row}

  {if first n column elements are all 0, no unique sol}

  – For j = i+1:n
    $m_{ji} = a_{ji}/a_{ii}$;
    $E_j = E_j - m_{ji} * E_i$
  – end
- End

Gaussian Elimination with Max Column Pivoting

- For i = 1:n-1 (row index)

  {from i-th row to n-th row,
  find the max abs(coefficient) of i-th COLUMN, P}
  {Interchange i-th row to p-th row}
  {if all i-th column elements are 0, no unique sol}

  – For j = i+1:n
  $m_{ji} = a_{ji}/a_{ii}$;
  $E_j = E_j - m_{ji} * E_i$
  – end
- End

Example (III)

- What if original E_1 is multiplied by 10000

$$
\begin{aligned}
E_1: &\quad 30.00x_1 && +591400x_2 && = 59.1700 \\
E_2: &\quad 5.291x_1 && -6.130x_2 && = 46.78
\end{aligned}
$$

- The equations satisfy maximum column pivoting criteria.

$$\Rightarrow x_2 \approx \frac{104400}{104300} = 1.001 \qquad x_1 \approx \frac{591700 - (591400) * 1.001)}{30.00} = -10.00$$

Scaled Column Pivoting

- Finding out the maximum coefficient of each row.

- Scale pivoting column with the maximum coefficient

- Use maximum column pivoting strategy according to scaled pivot column elements

Scaled Column Pivoting

- $a_{11}/\max(E_1) = 30/591400 = 0.5073\text{x}10^{-4}$
- $a_{21}/\max(E_2) = 5.291/6.130 = 0.8631$

- E_1 and E_2 shall be interchanged

$$\begin{aligned} E_1: &\quad 30.00x_1 &\quad + 591400x_2 &\quad = 59.1700 \\ E_2: &\quad 5.291x_1 &\quad - 6.130x_2 &\quad = 46.78 \end{aligned}$$

Gauss Elimination with Maximal Column Pivoting

- For $i = 1{:}n-1$ (row index)

  {from i-th row to n-th row,
  find the max abs(coefficient) of i-th COLUMN, P}
  {Interchange i-th row to p-th row}

  {if first n column elements are all 0, no unique sol}

  – For $j = i+1{:}n$
    $m_{ji} = a_{ji}/a_{ii}$;
    $E_j = E_j - m_{ji} * E_i$
  – end
- End

Gauss Elimination with Scaled Column

- Find max row elements of each row, s(i)
- For I = 1:n-1 (row index)

  {from i-th row to n-th row,
  find the max abs(coefficient/s(i)) of i-th COLUMN, P}
  {Interchange i-th row to p-th row}
  {if all i-th column elements are 0, no unique sol}

  – For j = i+1:n
  $m_{ji} = a_{ji}/a_{ii}$;
  $E_j = E_j - m_{ji} * E_i$
  – end
- End

Computation Complexity

- For each row, there should be n-1 comparisons
  - Total n(n-1) comparisons
- To determine the interchange of 1^{st} column
  - n divisions and n-1 comparisons
- To determine interchange of k-th column
  - n-k+1 divisions and n-k comparisons

Computation Complexity

- Divisions

$$\sum_{k=1}^{n-1}(n-k+1)=\frac{n(n+1)}{2}-1$$

- Comparisons

$$n(n-1)+\sum_{k=1}^{n-1}(n-k)=\frac{3}{2}n(n-1)$$

- Still way smaller than Gauss Elimination$O(n^3)$

Iterative Methods

Similar to finding a root for nonlinear equations

Iterative Methods

- These methods are mostly similar to those used to solve non-linear equation of single variable.

- What can be used to solve a system of non-linear equations can also be used to solve linear equations.

Iterative Methods

- Pick up an initial guess
- Use a rational iteration process to approach the true answer
- Rational “Iteration Relationship” and “Stop Criteria” will be needed

Iterative Methods

- Fixed Point Method
  - Linear and non-linear system
- Newton's Method
  - Non-linear system
- Conjugate Gradient
  - Symmetric and positive definite system
- Steepest Descent
  - Linear and non-linear

When to stop an iteration?

- For the case of a single variable (e.g. $x - cosx = 0$)
  - Criterion the absolute error: $|x_n - x_{n+1}| < \epsilon$
  - Criterion the relative error: $\frac{|x_n - x_{n+1}|}{|x_{n+1}|} < \epsilon = 10^{-10}$
  - | x | denotes the distance/length for a 1D vector
- For the case of multiple variables, such as a vector $[x_1, x_2, x_3, \ldots]$
  - Distance/Length shall be defined for a vector: Vector Norm

Definition of the Vector Norm

- p-norm of a vector $\vec{v}$ is defined as

$$|\,\vec{v}\,|_p = \left(\sum_{i=1}^{N} |v_i|^p \right)^{\frac{1}{p}}$$

The norm of a vector represents the length of the vector in a specific geometry representation.

Vector Norm

- For a vector x

$$\mathbf{x} = [x_1, x_2, \ldots, x_n]$$

- The L2 norm (Euclidean norm) is defined

$$\|x\|_2 = (|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2)^{\frac{1}{2}}$$

- The L∞ norm is defined

- $\|x\|_\infty = \lim_{k\to\infty}\left(|x_1|^k + |x_2|^k + \cdots + |x_n|^k\right)^{\frac{1}{k}}$

$$= \max(|x_i|)$$

Example

- $\vec{v} = [1, -2, 3, 0, 4]$
  - $|\vec{v}|_2 = \sqrt{1 + 4 + 9 + 0 + 16} = \sqrt{30}$
  - $|\vec{v}|_\infty = \max(|v_i|) = 4$

- $\vec{u} = [2, -9, 3, 6, 5]$
  - $|\vec{u}|_2 = \sqrt{2 + 81 + 9 + 36 + 25} = \sqrt{153}$
  - $|\vec{u}|_\infty = \max(|v_i|) = 9$

- $|\vec{u} - \vec{v}|_\infty = |[1, -7, 0, 6, 1]|_\infty = 7$

Norm and stop criteria

- L2 norm is a common measurement of the error or residual length for the stop criteria in interative method
- And in the following discussion only L2 Norm will be used

- Stop criteria:

$$\|\mathbf{Ax} - \mathbf{b}\| < \varepsilon \qquad\qquad \left\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\right\| < \varepsilon$$

Jacobi Method

Example

- Solve the following equation using Jacobi Method

$$3x + 2y - z = 1$$
$$x - 3y + 2z = 5$$
$$2x + y - 3z = -4$$

Jacobi Method

- For a system of linear equations

$$\mathbf{Ax=b}$$

- The simples fixed point counterpart is the following form

$$\mathbf{x} = \mathbf{Tx} + \mathbf{c}$$

$$\mathbf{x}^{(k+1)} = \mathbf{Tx}^{(k)} + \mathbf{c}$$

Example

- Solve the following equation using Jacobi Method

$$
\begin{aligned}
E_1 &: 10x_1 - x_2 + 2x_3 = 6 \\
E_2 &: -x_1 + 11x_2 - x_3 + 3x_4 = 25 \\
E_3 &: 2x_1 - x_2 + 10x_3 - x_4 = -11 \\
E_4 &: 3x_2 - x_3 + 8x_4 = 15
\end{aligned}
$$

Example

- Fixed point form

$$
\begin{aligned}
x_1 &= \qquad\quad \tfrac{1}{10}x_2 - \tfrac{1}{5}x_3 \qquad\qquad + 3/5 \\
x_2 &= 1/11x_1 \qquad\quad + 1/11x_3 - 3/11x_4 + 25/11 \\
x_3 &= -1/5x_1 + 1/10x_2 \qquad\quad + 1/10x_4 - 11/10 \\
x_4 &= \qquad\quad -3/8x_2 + 1/8x_3 \qquad\qquad + 15/8
\end{aligned}
$$

- Start with [0,0,0,0]

Example

- 1^{st} Iteration

$$
\begin{aligned}
0.6000 &= & & 1/10(0) & -1/5(0) & & +3/5 \\
2.2727 &= 1/11(0) & & & +1/11(0) & -3/11(0) & 25/11 \\
-1.1000 &= -1/5(0) & +1/10(0) & & & +1/10(0) & -11/10 \\
1.8750 &= & & -3/8(0) & +1/8(0) & & 15/8
\end{aligned}
$$

- 2^{nd} Iteration

$$
\begin{aligned}
1.0473 &= & & 1/10(2.2727) & -1/5(-1.1) & & +3/5 \\
1.7159 &= 1/11(0.6) & & & +1/11(-1.1) & -3/11(1.875) & 25/11 \\
-0.8052 &= -1/5(0.6) & +1/10(2.2727) & & & +1/10(1.875) & -11/10 \\
0.8852 &= & & -3/8(2.2727) & +1/8(-1.1) & & 15/8
\end{aligned}
$$

Example

k	0	1	2	3	4	5	6	7	8	9	10
$x_1^{(k)}$	0.000	0.6000	1.0473	0.9326	1.0152	0.9890	1.0032	0.9981	1.0006	0.9997	1.0001
$x_2^{(k)}$	0.0000	2.2727	1.7159	2.053	1.9537	2.0114	1.9922	2.0023	1.9987	2.0004	1.9998
$x_3^{(k)}$	0.0000	−1.1000	−0.8052	−1.0493	−0.9681	−1.0103	−0.9945	−1.0020	−0.9990	−1.0004	−0.9998
$x_4^{(k)}$	0.0000	1.8750	0.8852	1.1309	0.9739	1.0214	0.9944	1.0036	0.9989	1.0006	0.9998

Gauss Seidel Method

Jacobi Method

- Iteration of Jacobi Method

$$\mathbf{x}^{(k+1)} = \mathbf{T}\mathbf{x}^{(k)} + \mathbf{c}$$

- The explicit form

$$x_1^{(k+1)} = T_{12}x_2^{(k)} + T_{13}x_3^{(k)} + \ldots + c_1$$
$$x_2^{(k+1)} = T_{21}x_1^{(k)} + T_{23}x_3^{(k)} + \ldots + c_2$$
$$\vdots$$
$$x_n^{(k+1)} = T_{n1}x_1^{(k)} + T_{n2}x_2^{(k)} + \ldots + c_n$$

Gauss Seidel Method

- Conjecture: The k+1-th result shall be more accurate than the result of k-th iteration

$$x_1^{(k+1)} = T_{12}x_2^{(k)} + T_{13}x_3^{(k)} + \ldots + c_1$$

$$x_2^{(k+1)} = T_{21}x_1^{(k+1)} + T_{23}x_3^{(k)} + \ldots + c_2$$

$$x_3^{(k+1)} = T_{31}x_1^{(k+1)} + T_{32}x_2^{(k+1)} + T_{34}x_4^{(k)} + \ldots + c_3$$

$$\vdots$$

$$x_n^{(k+1)} = T_{n1}x_1^{(k+1)} + T_{n2}x_2^{(k+1)} + \ldots + T_{n,n-1}x_{n-1}^{(k+1)} + c_n$$

Example

- Solve the following equation using Gauss Seidel Method

$$
\begin{array}{lllllll}
E_1 : 10x_1 & -x_2 & +2x_3 & & = & 6 \\
E_2 : -x_1 & +11x_2 & -x_3 & +3x_4 & = & 25 \\
E_3 : \ 2x_1 & -x_2 & +10x_3 & -x_4 & = & -11 \\
E_4 : & 3x_2 & -x_3 & +8x_4 & = & 15
\end{array}
$$

Example

- Fixed point form

$$
\begin{aligned}
x_1 &= \qquad\qquad 1/10x_2 \quad -1/5x_3 \qquad\qquad +3/5 \\
x_2 &= 1/11x_1 \qquad\qquad +1/11x_3 \quad -3/11x_4 \quad 25/11 \\
x_3 &= -1/5x_1 \quad +1/10x_2 \qquad\qquad +1/10x_4 \quad -11/10 \\
x_4 &= \qquad\qquad -3/8x_2 \quad +1/8x_3 \qquad\qquad 15/8
\end{aligned}
$$

- Start with [0,0,0,0]

Example

- 1st Iteration

$$\begin{array}{rlllll}
0.6 = & & 1/10(0) & -1/5(0) & & +3/5 \\
2.3272 = & 1/11(0.6) & & +1/11(0) & -3/11(0) & 25/11 \\
-0.9873 = & -1/5(0.6) & +1/10(2.3272) & & +1/10(0) & -11/10 \\
0.8789 = & & -3/8(2.3272) & +1/8(-0.9873) & & 15/8
\end{array}$$

- 2nd Iteration

$$\begin{array}{rlllll}
1.030 = & & 1/10(2.372) & -1/5(-0.9873) & & +3/5 \\
2.037 = & 1/11(1.030) & & +1/11(-0.9873) & -3/11(0.8789) & 25/11 \\
-1.014 = & -1/5(1.030) & +1/10(2.037) & & +1/10(0.8789) & -11/10 \\
0.9844 = & & -3/8(2.037) & +1/8(-1.014) & & 15/8
\end{array}$$

Example

k	0	1	2	3	4	5
$x_1^{(k)}$	0.0000	0.6000	1.030	1.0065	1.0009	1.0001
$x_2^{(k)}$	0.0000	2.3272	2.037	2.0036	2.0003	2.0000
$x_3^{(k)}$	0.0000	−0.9873	−1.014	−1.0025	−1.0003	−1.0000
$x_4^{(k)}$	0.0000	0.8789	0.9844	0.9983	0.9999	1.0000

Geometric Illustration of the 2D Fixed point method

- Consider a two variable linear system

$$
\begin{aligned}
E_1 &: \quad 5x_1 \quad - x_2 \quad = 6 \\
E_2 &: \quad x_1 \quad - 3x_2 \quad = 4
\end{aligned}
$$

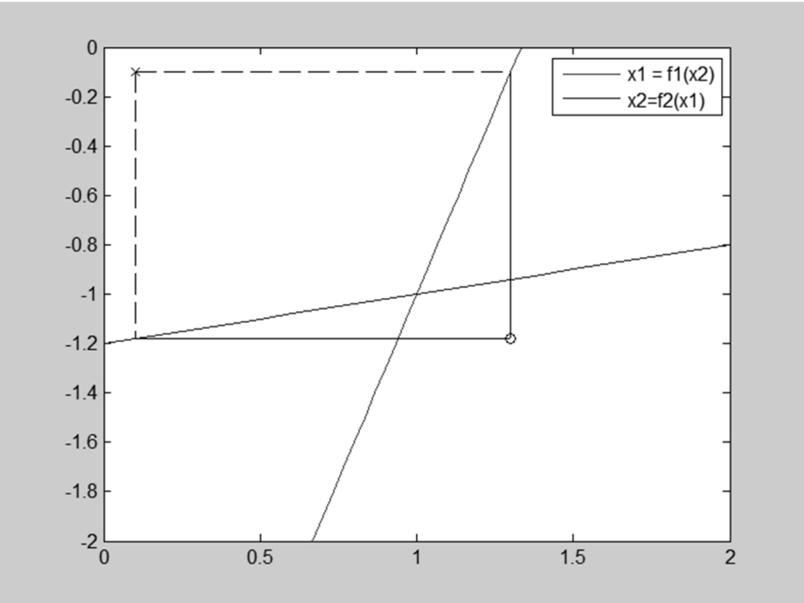
- The fixed point counterpart is

$$
\begin{aligned}
f_1(x_2) = x_1 &= \qquad\qquad 1/5x_2 \quad + 6/5 \\
f_2(x_1) = x_2 &= \quad 1/3x_1 \qquad\qquad - 4/3
\end{aligned}
$$

Geometric Illustration of the 2D Fixed point method

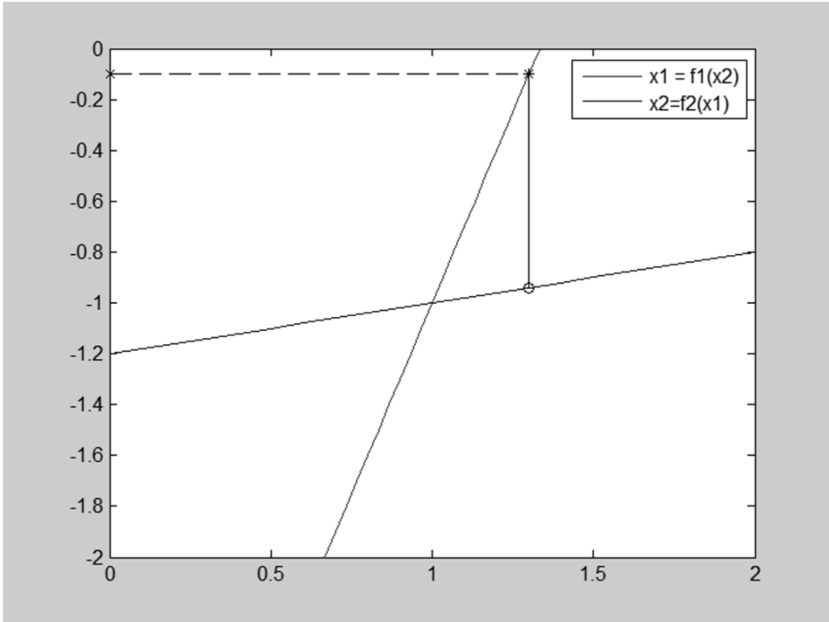
- Jacobi Iteration

$$x_1^{(k+1)} = 1/5x_2^{(k)} + 6/5$$
$$x_2^{(k+1)} = 1/3x_1^{(k)} - 4/3$$



- Gauss Seidel Iteration

$$x_1^{(k+1)} = 1/5x_2^{(k)} + 6/5$$
$$x_2^{(k+1)} = 1/3x_1^{(k+1)} - 4/3$$



Geometric Illustration of the 2D Fixed point method

- What if E1 and E2 interchanges?

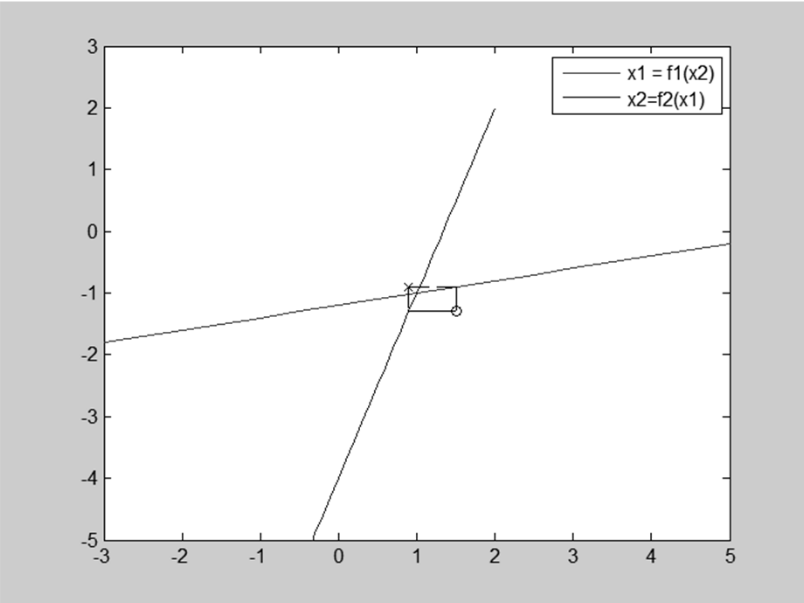
$$
\begin{aligned}
E_1 &: \quad x_1 - 3x_2 = 4 \\
E_2 &: \quad 5x_1 - x_2 = 6
\end{aligned}
\longrightarrow
\begin{aligned}
x_1 &= \qquad 3x_2 + 4 \\
x_2 &= 5x_1 \qquad - 6
\end{aligned}
$$

- Linear algebra says, Two interchanged rows shall not affect the answer

Geometric Illustration of the 2D Fixed point method

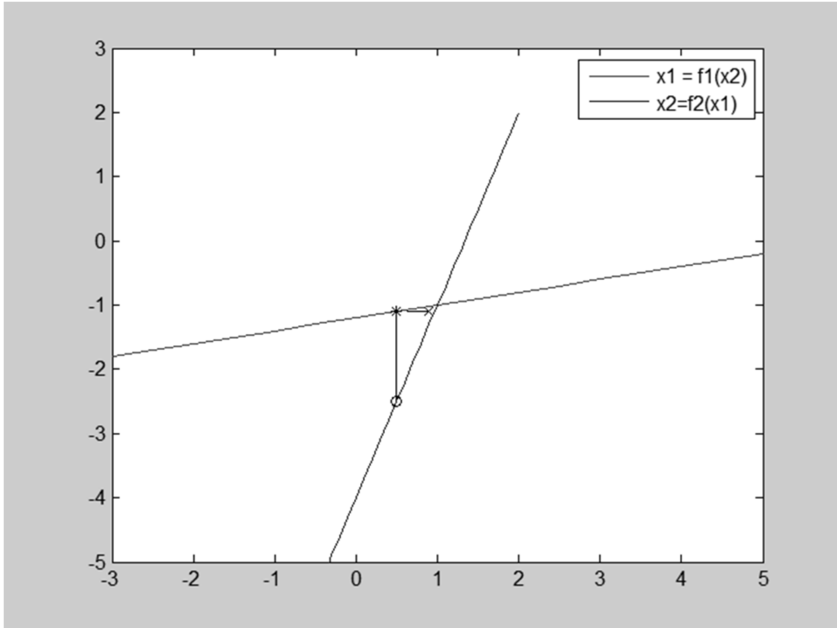
- Jacobi Iteration

$$x_1^{(k+1)} = 3x_2^{(k)} + 4$$
$$x_2^{(k+1)} = 5x_1^{(k)} - 6$$



- Gauss Seidel Iteration

$$x_1^{(k+1)} = 3x_2^{(k)} + 4$$
$$x_2^{(k+1)} = 5x_1^{(k+1)} - 6$$



Condition of Convergence

- Assume that p_0 is the correct fixed point of the following fixed point problem

$$\mathbf{x}^{(k+1)} = \mathbf{T}\mathbf{x}^{(k)} + \mathbf{c}$$

- The distance between an approximated point p and p_0 is

$$\left\| \mathbf{p}^{(k+1)} - \mathbf{p_0}^{(k+1)} \right\| = \left\| \mathbf{T}(\mathbf{p}^{(k)} - \mathbf{p_0}^{(k)}) \right\|$$

$$\left\| \mathbf{p}^{(k+1)} - \mathbf{p_0}^{(k+1)} \right\| \le \left\| \mathbf{T} \right\| \left\| \mathbf{p}^{(k)} - \mathbf{p_0}^{(k)} \right\|$$

Condition of Convergence

- A convergent fixed point algorithm shall generate a series of approximated point approaching to the true answer

$$\lim_{k\to\infty}\left\|\mathbf{p}^{(k+1)}-\mathbf{p}_0{}^{(k+1)}\right\|=\lim_{k\to\infty}\left\|\mathbf{T}(\mathbf{p}^{(k)}-\mathbf{p}_0{}^{(k)})\right\|=0$$

- L2-Norm of T shall be smaller than 1

Condition of Convergence

- For any x defined by

$$\mathbf{x}^{(k+1)} = \mathbf{T}\mathbf{x}^{(k)} + \mathbf{c}$$

- Converges to a unique solution if and only if the maximum eigen value of T is smaller than 1

Conditional Convergence

- For any eigen valu λ of T, 1- λ is an eigen value of (I-T). Because no eigen value is greater than 1, that means 1- λ cannot be zero leading that I-T is non-singular.
- So $(I-T)^{-1}$ exists
- Original equation is

$$(\mathbf{I}-\mathbf{T})\mathbf{x}=\mathbf{c} \qquad \mathbf{x}=(\mathbf{I}-\mathbf{T})^{-1}\mathbf{c}$$

Conditional Convergence

- Assume a matrix Sm is

$$\mathbf{S_m} = \mathbf{I} + \mathbf{T} + \mathbf{T^2} + \mathbf{...} + \mathbf{T^m}$$

- Multiply both side by (I+T)

$$(\mathbf{I} - \mathbf{T})\mathbf{S_m} = (\mathbf{I} - \mathbf{T})(\mathbf{I} + \mathbf{T} + \mathbf{T^2} + \mathbf{...} + \mathbf{T^m}) = \mathbf{I} - \mathbf{T^{m+1}}$$

- Because max eigen value of is smaller than 1

$$\lim_{m\to\infty} \mathbf{T^m} = 0$$

- therefore

$$\lim_{m\to\infty} \mathbf{S_m} = \mathbf{I} + \mathbf{T} + \mathbf{T^2} + \mathbf{...} + \mathbf{T^m} + \ldots = (\mathbf{I} - \mathbf{T})^{-1}$$

Conditional Convergence

- The expension of the iteration is

$$
\begin{aligned}
\mathbf{x}^{(k)} &= \mathbf{T}\mathbf{x}^{(k-1)} + \mathbf{c} = \mathbf{T}(\mathbf{T}\mathbf{x}^{(k-2)} + \mathbf{c}) + \mathbf{c} \\
&= \mathbf{T}^2\mathbf{x}^{(k-2)} + (\mathbf{T} + \mathbf{I})\mathbf{c} = \mathbf{T}^2(\mathbf{T}\mathbf{x}^{(k-3)} + \mathbf{c}) + (\mathbf{T} + \mathbf{I})\mathbf{c} \\
&= \mathbf{T}^3\mathbf{x}^{(k-3)} + (\mathbf{T}^2 + \mathbf{T} + \mathbf{I})\mathbf{c} \\
&\vdots \\
&= \mathbf{T}^k\mathbf{x}^{(0)} + (\mathbf{T}^{k-1} + \ldots + \mathbf{T}^2 + \mathbf{T} + \mathbf{I})\mathbf{c}
\end{aligned}
$$

- Taking the limit

$$
\begin{aligned}
\lim_{m\to\infty} \mathbf{x}^{(\mathbf{k})} &= \lim_{m\to\infty} \mathbf{T}^{\mathbf{k}}\mathbf{x}^{(\mathbf{0})} + \lim_{m\to\infty} (\mathbf{T}^{\mathbf{k-1}} + \mathbf{\ldots} + \mathbf{T} + \mathbf{I})\mathbf{c} \\
&= (\mathbf{I} - \mathbf{T})^{-1}\mathbf{c}
\end{aligned}
$$