National Cheng Kung University

Chapter 25
All-Pairs Shortest Paths

Sun-Yuan Hsieh

謝孫源　教授

成功大學資訊工程學系

Overview

- Given a directed graph $G=(V,E)$, weight function $w: E \to \Re, |V| = n$.
- Goal: create an $n \times n$ matrix of shortest-path distances $\delta(u, v)$.
- Could run BELLMAN-FORD once from each vertex:
  $O(V^2E)$ — which is $O(V^4)$ if the graph is **dense** ($E=\theta(V^2)$).
- If no negative-weight edges, could run Dijkstra's algorithm once from each vertex:
  $O(VE \lg V)$ with binary heap—$O(V^3 \lg V)$ if dense.
  $O(V^2 \lg V + VE)$ with Fibonacci heap — $O(V^3)$ if dense.
- We'll see how to do in $O(V^3)$ in all cases, with no fancy data structure.

Shortest paths and matrix multiplication

- Assume that G is given as adjacency matrix of weights: $W = (w_{ij})$, with vertices numbered 1 to n.

$$w_{ij} = \begin{cases} 0 & \text{if } i = j, \\ \text{weight of } (i, j) & \text{if } i \neq j, (i, j) \in E, \\ \infty & \text{if } i \neq j, (i, j) \notin E. \end{cases}$$

- Output is matrix $D = (d_{ij})$, where $d_{ij} = \delta(i,j)$. Won't worry about predecessor—see book.
- Will use dynamic programmingat first.
- ***Optimal substructure***: Recall: subpaths of shortest paths are shortest paths.
- ***Recursive solution***: Let $l_{ij}^{(m)}$ = weight of shortest path $i \rightsquigarrow j$ that contains ≦m edges.

- $m = 0$
  there is a shortest path $i \leadsto j$ with $\leqq m$ edges if and only if $i=j$

$$\Rightarrow \quad l_{ij}^{(0)} = \begin{cases} 0 & \text{if } i = j, \\ \infty & \text{if } i \neq j. \end{cases}$$

- $m \geqq 1$

$$\Rightarrow \quad l_{ij}^{(m)} = \min\left(l_{ij}^{(m-1)}, \min_{1 \le k \le n}\left\{l_{ik}^{(m-1)} + w_{kj}\right\}\right) \qquad (k \text{ is all possible predecessors of } j)$$

$$= \min_{1 \le k \le n}\left\{l_{ik}^{(m-1)} + w_{kj}\right\} \qquad \text{since } w_{jj} = 0 \text{ for all } j$$

- Observer that when m=1, must have $l_{ij}^{(1)} = w_{ij}$.
  Conceptually, when the path is restricted to at most 1 edge, the weight of the shortest path $i \leadsto j$ must be w_{ij}.
  And the math works out, too:

$$
\begin{aligned}
l_{ij}^{(1)} &= \min_{1 \le k \le n} \left\{ l_{ik}^{(0)} + w_{kj} \right\} \\
&= l_{ii}^{(0)} + w_{ij} \qquad (l_{ii}^{(0)} \text{ is the only non-}\infty \text{ among } l_{ik}^{(0)}) \\
&= w_{ij}.
\end{aligned}
$$

All simple shortest paths contain ≦n-1 edges

$$\Rightarrow \delta(i, j) = l_{ij}^{(n-1)} = l_{ij}^{(n)} = l_{ij}^{(n+1)} = \ldots$$

***Compute a solution bottom-up*:** Compute $L^{(1)}, L^{(2)}, \ldots, L^{(n-1)}$.

Start with $L^{(1)} = W$, since $l_{ij}^{(1)} = w_{ij}$.

Go from $L^{(m-1)}$ to $L^{(m)}$:

EXTEND(L, W, n)
create L', an $n \times n$ matrix

for $i \leftarrow 1$ **to** n
  do for $j \leftarrow 1$ **to** n
    do $l'_{ij} \leftarrow \infty$
      for $k \leftarrow 1$ **to** n
        do $l'_{ij} \leftarrow \min(l'_{ij}, l_{ik} + w_{kj})$
return L'

Compute each $L^{(m)}$:

SLOW-APSP(W, n)
$L^{(1)} \leftarrow W$
for $m \leftarrow 2$ **to** n-1
  do $L^{(m)} \leftarrow$ EXTEND($L^{(m-1)}$, W, n)
return $L^{(n-1)}$

***Time*:**

- EXTEND: $\Theta(n^3)$.
- SLOW-APSP: $\Theta(n^4)$.

***Observation*:** EXTEND is like matrix multiplication:

$L \rightarrow A$

$W \rightarrow B$

$L' \rightarrow C$

$\min \rightarrow +$

$+ \rightarrow \cdot$

$\infty \rightarrow 0$

create C, an $n \times n$ matrix

for $i \leftarrow 1$ **to** n

  do for $j \leftarrow 1$ **to** n

    do $c_{ij} \leftarrow 0$

     for $k \leftarrow 1$ **to** n

      do $c_{ij} \leftarrow a_{ik} \cdot \; b_{kj}$

So, we can view EXTEND as just like matrix multiplication!

Why do we care?

Because our goal is to compute $L^{(n-1)}$ as fast as we can. Don't need to Compute *all* the intermediate $L^{(1)}, L^{(2)}, L^{(3)}, \ldots, L^{(n-2)}$.

Suppose we had a matrix A and we wanted to compute A^{n-1} (like calling EXTEND n-1 times).

Could compute $A, A^2, A^4, A^8, \ldots$

If we knew $A^m = A^{n-1}$ for all $m \geqq n-1$, could just finish with A^r, where r is the smallest power of 2 that's $\geqq n-1$. ($r = 2^{\lceil \lg(n-1) \rceil}$)

FASTER-APSP(W, n)

$L^{(1)} \leftarrow W$

$m \leftarrow 1$

while $m < n$-1

do $L^{(2m)} \leftarrow$ EXTEND($L^{(m)}$, $L^{(m)}$, n)

$m \leftarrow 2m$

return $L^{(m)}$

OK to overshoot, since products don't change after $L^{(n-1)}$.

Time:$\Theta(n^3 \lg n)$

Floyd-Warshall algorithm

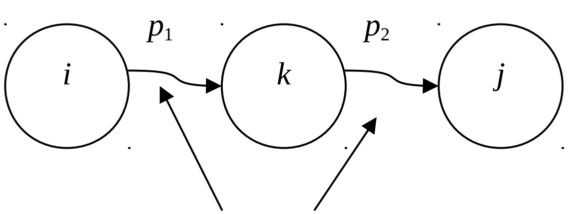
A different dynamic-programming approach.

For path $p=\langle v_1, v_2,\ldots, v_l\rangle$, an ***intermediate vertex*** is any vertex of p other than v_1 or v_l.

Let $d_{ij}^{(k)}$ = shortest-path weight of any path $i \leadsto j$ with all intermediate vertices in $\{1, 2,\ldots, k\}$.

Consider a shortest path $i \overset{p}{\leadsto} j$ with all intermediate vertices in $\{1,2,\ldots,k\}$:

- If k is not an intermediate vertex, then all intermediate vertices of p are in $\{1, 2,\ldots, k\text{-}1\}$.
- If k is an intermediate vertex:



all intermediate vertices in $\{1, 2,\ldots, k\text{-}1\}$

Recursive formulation

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0, \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1. \end{cases}$$

(Have $d_{ij}^{(0)} = w_{ij}$ because can't have intermediate vertices $\Rightarrow$ ≦1 edges.)

Want $D^{(n)} = (d_{ij}^{(n)})$, since all vertices numbered ≦n.

Compute bottom-up

Compute in increasing order of k:

Floyd-Warshall(W, n)

$D^{(0)} \leftarrow W$

for $k \leftarrow 1$ **to** n

  do for $i \leftarrow 1$ **to** n

    do for $j \leftarrow 1$ **to** n

      do $d_{ij}^{(k)} \leftarrow \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$

return $D^{(n)}$

Can drop superscripts. (See Exercise 25.2-4 in text.)

Time: $\Theta(n^3)$.

Transitive closure

Given $G(V,E)$, directed.

Compute $G^* = (V, E^*)$.

- $E^* = \{ (i, j) : \text{there is a path } i \rightsquigarrow j \text{ in } G\}$.

Could assign weight of 1 to each edge, then run FLOYD-WARSHALL.

- If $d_{ij} < n$, then there is a path $i \rightsquigarrow j$.
- Otherwise, $d_{ij} = \infty$ and there is no path.

Simpler way: Substitute other values and operators in FLOYD-WARSHALL.

- Use unweighted adjacency matrix
- min → $\vee$ (OR)
- + → $\wedge$ (AND)
- $$t_{ij}^{(k)} = \begin{cases} 1 & \text{if there is path } i \rightsquigarrow j \text{ with all intermediate vertices} \\ & \text{in } \{1, 2, \ldots, k\}, \\ 0 & \text{otherwise.} \end{cases}$$
- $$t_{ij}^{(0)} = \begin{cases} 0 & \text{if } i \neq j \text{ and } (i, j) \notin E, \\ 1 & \text{if } i = j \text{ or } (i, j) \in E. \end{cases}$$
- $$t_{ij}^{(k)} = t_{ij}^{(k-1)} \vee (t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)}).$$

TRANSITIVE-CLOSURE(E, n)

for $i \leftarrow 1$ **to** n

  do for $j \leftarrow 1$ **to** n

    do if $i=j$ or $(i, j) \in E[G]$

      then $t_{ij}^{(0)} \leftarrow 1$

      else $t_{ij}^{(0)} \leftarrow 0$

for $k \leftarrow 1$ **to** n

  do for $i \leftarrow 1$ **to** n

    do for $j \leftarrow 1$ **to** n

      do $t_{ij}^{(k)} \leftarrow t_{ij}^{(k-1)} \vee (t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)})$

return $T^{(n)}$

Time: $\Theta(n^3)$, but simpler operations than FLOYD-WARSHALL.

- ***Idea*:** If the graph is sparse, it pays to run Dijkstra's algorithm once from each vertex.
- If we use a Fibonacci heap for the priority queue, the running time is down to $O(V^2 \lg V + VE)$, which is better than FLOYD-WARSHALL's $\theta(V^3)$ time if $E=o(V^2)$.
- But Dijkstra's algorithm requires that all edge weights be nonnegative.
- Donald Johnson figured out how to make an equivalent graph that *does* have all edge weights≧0.

Reweighting

Compute a new weight function $\hat{w}$ such that

1. For all $u, v \in V$, p is a shortest path $u \leadsto v$ using w if and only if p is a shortest path $u \leadsto v$ using $\hat{w}$.
2. For all $(u, v) \in E$, $\hat{w}(u,v) \geq 0$.

- Property(1) says that it suffices to find shortest paths with $\hat{w}$.
- Property(2) says we can do so by running Dijkstra's algorithm from each vertex.

- How to come up with $\hat{w}$?
- Lemma shows it's easy to get property(1):

Lemma (Rewighting doesn't change shortest paths)

- Given a directed, weighted graph $G=(V,E)$, w: $E \rightarrow \mathbf{R}$. Let h be any function such that h:$V \rightarrow \boldsymbol{R}$. For all $(u, v) \in E$, define $\hat{w}(u,v) = w(u,v) + h(u) - h(v)$. Let $p=<v_0, v_1,\ldots,v_k>$ be any path $v_0 \leadsto v_k$.
- Then, p is a shortest path $v_0 \leadsto v_k$ with w if and only if p is a shortest path $v_0 \leadsto v_k$ with $\hat{w}$. Also, G has a negative-weight cycle with weight w iff G has a negative-weight cycle with weight $\hat{w}$.

Proof

- First, we'll show that $\hat{w}(p) = w(p) + h(v_0) - h(v_k)$:

$$\begin{aligned}
\hat{w}(p) &= \sum_{i=1}^{k} \hat{w}(v_{i-1}, v_i) \\
&= \sum_{i=1}^{k} (w(v_{i-1}, v_i) + h(v_{i-1}) - h(v_i)) \\
&= \sum_{i=1}^{k} w(v_{i-1}, v_i) + h(v_0) - h(v_k) \qquad \text{(sum telescopes)} \\
&= w(p) + h(v_0) - h(v_k).
\end{aligned}$$

- Therefore, any path $v_0 \xrightarrow{p} v_k$ has $\hat{w}(p) = w(p) + h(v_0) - h(v_k)$. Since $h(v_0)$ and $h(v_k)$ don't depend on the path from v_0 to v_k, if one path $v_0 \rightarrow v_k$ is shorter than another with w, it's also shorter with $\hat{w}$.
- Now show there exists a negative-weight cycle with w if and only if there exists a negative-weight cycle with $\hat{w}$:

- Let cycle $C=\langle v_0, v_1,\ldots, v_k\rangle$, where $v_0=v_k$.
- Then $\hat{w}(C) = w(C) + h(v_0) - h(v_k)$

$$= w(C) \qquad\qquad \text{(since } v_0 = v_k\text{)}.$$

Therefore, C has a negative-weight cycle with w if and only if it has a negative-weight cycle with $\hat{w}$. ■ (lemma)

So, now to get property(2), we just need to come up with a function $h: V \rightarrow \boldsymbol{R}$ such that when we compute $\hat{w}(u,v) = w(u,v) + h(u) - h(v)$, it's ≥ 0.

Do what we did for difference constraints:

- $G'=(V', E')$
- $V'=V\cup\{s\}$, where s is a new vertex.
- $E'=E\cup\{(s, v): v \in V\}$.
- $w(s, v) = 0$ for all $v \in V$.

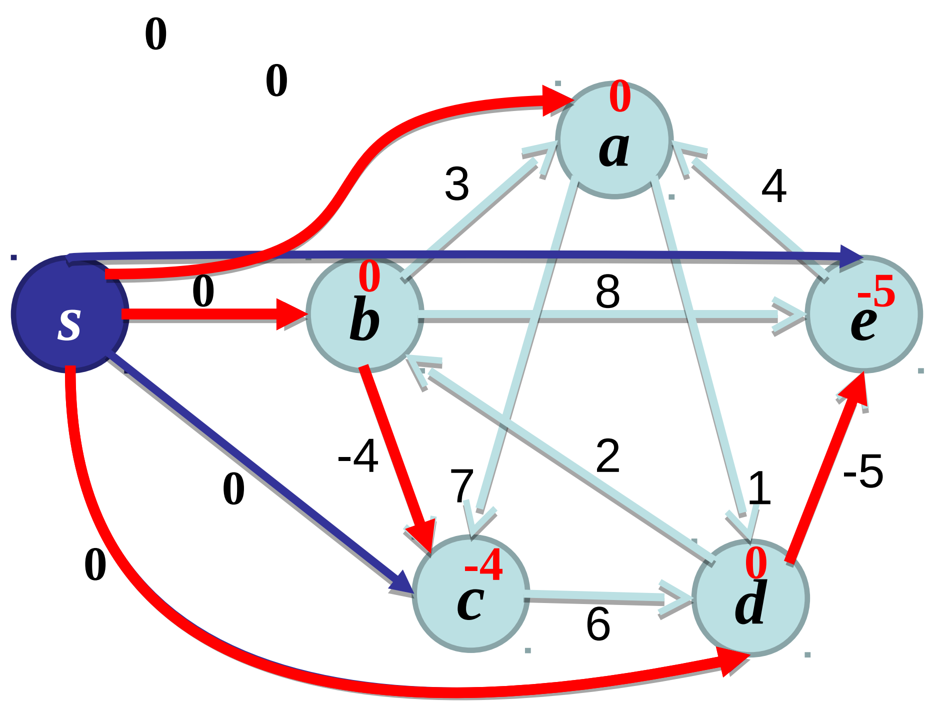


Figure 25.6 Johnson's all-pairs shortest-paths algorithm

(a) The graph *G'* with the original weight function *w*. The new vertex ***s*** is black.

- Since no edges enter s, G' has the same set of cycles as G. In particular, G' has a negative-weight cycle if and only if G does.

Define $h(v) = \delta(s, v)$ for all $v \in V$.

Claim

$\hat{w}(u,v) = w(u,v) + h(u) - h(v) \geq 0.$

Proof By the triangle inequality,

$$\delta(s, v) \leqq \delta(s, u) + w(u, v)$$

$$h(v) \leqq h(u) + w(u, v) .$$

Therefore, $w(u, v) + h(u) - h(v) \geqq 0$. ■(claim)

Johnson's algorithm

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from *G'*

run BELLMAN-FORD on *G'* to compute $\delta(s, v)$ for all $v \in V$

if BELLMAN-FORD returns FALSE

    then *G* has a negative-weight cycle

    else

        compute $\hat{w}(u,v) = w(u,v) + \delta(s,u) - \delta(s,v)$ for all $(u, v) \in E$

        for each vertex $u \in V$

            do run Dijkstra's algorithm from *u* using weight function $\hat{w}$

                    to compute $\hat{\delta}(u,v)$ for all $v \in V$

        for each vertex $v \in V$

            do ⊳Compute entry d_{uv} in matrix D

$$d_{uv} = \hat{\delta}(u,v) + \delta(s,v) - \delta(s,u)$$

because if p is a path $u \to v$,
then $\hat{w}(p) = w(p) + h(u) - h(v)$

Time:

- $\Theta(V+E)$ to compute G'.
- $O(VE)$ to run BELLMAN-FORD.
- $\Theta(E)$ to compute $\hat{w}$.
- $O(V^2 \lg V + VE)$ to run Dijkstra's algorithm $|V|$ times (using Fibonacci heap).
- $\Theta(V^2)$ to compute D matrix.

Total: $O(V^2 \lg V + VE)$.