Homework 6

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Ch7 Problem Plus

EX.7

Recall that $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$. So

$$\begin{aligned}
f(x) &= \int_0^\pi \cos t \cos(x-t)\,dt = \frac{1}{2}\int_0^\pi [\cos(t+x-t) + \cos(t-x+t)]\,dt \\
&= \frac{1}{2}\int_0^\pi [\cos x + \cos(2t-x)]\,dt = \frac{1}{2}[t\cos x + \frac{1}{2}\sin(2t-x)]_0^\pi \\
&= \frac{\pi}{2}\cos x + \frac{1}{4}\sin(2\pi - x) - \frac{1}{4}\sin(-x) \\
&= \frac{\pi}{2}\cos x + \frac{1}{4}\sin(-x) - \frac{1}{4}\sin(-x) = \frac{\pi}{2}\cos x
\end{aligned}$$

The minimum of $\cos x$ on this domain is -1, so the minimum value of $f(x)$ is $f(\pi) = -\frac{\pi}{2}$.

EX.9

In accordance with the hint, we let $I_k = \int_0^1 (1-x^2)^k\,dx$, and we find an expression for I_{k+1} in terms of I_k. We integrate I_{k+1} by parts with $u = (1-x^2)^{k+1} \Rightarrow du = (k+1)(1-x^2)^k(-2x)$, $dv = dx \Rightarrow v = x$, and then split the remaining integral into identifiable quantities:

$$\begin{aligned}
I_{k+1} &= x(1-x^2)^{k+1}|_0^1 + 2(k+1)\int_0^1 x^2(1-x^2)^k\,dx \\
&= (2k+2)\int_0^1 (1-x^2)^k[1-(1-x^2)]\,dx = (2k+2)(I_k - I_{k+1})
\end{aligned}$$

So $I_{k+1}[1+(2k+2)] = (2k+2)I_k \Rightarrow I_{k+1} = \dfrac{2k+2}{2k+3}I_k$. Now to complete the proof, we use induction: $I_0 = 1 = \dfrac{2^0(0!)^2}{1!}$, so the formula holds for $n = 0$.

Now suppose it holds for $n = k$. Then

$$\begin{aligned}
I_{k+1} &= \frac{2k+2}{2k+3} I_k = \frac{2k+2}{2k+3}\left[\frac{2^{2k}(k!)^2}{(2k+1)!}\right] = \frac{2(k+1)2^{2k}(k!)^2}{(2k+3)(2k+1)!} \\
&= \frac{2(k+1)}{2k+2} \cdot \frac{2(k+1)2^{2k}(k!)^2}{(2k+3)(2k+1)!} = \frac{[2(k+1)]^2 2^{2k}(k!)^2}{(2k+3)(2k+2)(2k+1)!} \\
&= \frac{2^{2(k+1)}[(k+1)!]^2}{[2(k+1)+1]!}
\end{aligned}$$

So by induction, the formula holds for all integers $n \geq 0$.

Section 8.1 Arc Length

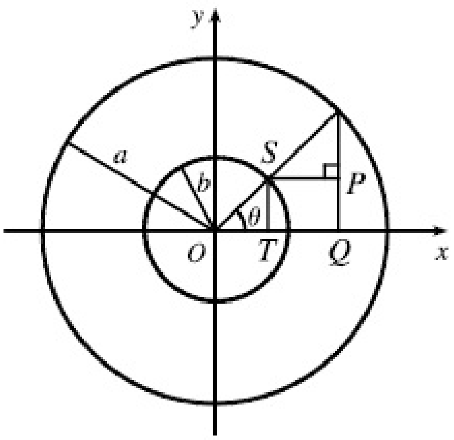
EX.42

Let $y = a - b\cosh cx$, where $a = 211.49, b = 20.96$, and $c = 0.03291765$. Then $y' = -bc\sinh cx \Rightarrow 1 + (y')^2 = 1 + b^2c^2\sinh^2(cx)$.
So $L = \int_{-91.2}^{91.2} \sqrt{1 + b^2c^2\sinh^2(cx)}\,dx \approx 451.137 \approx 451$, to the nearest meter.

Section 10.1 Curves Defined by Parametric Equations

EX.41

It is apparent that $x = |OQ|$ and $y = |QP| = |ST|$. From the diagram, $x = |OQ| = a\cos\theta$ and $y = b\sin\theta$. To eliminate θ we rearrange: $\sin\theta = y/b \Rightarrow \sin^2\theta = (y/b)^2$ and $\cos\theta = x/a \Rightarrow \cos^2\theta = (x/a)^2$. Adding the two equations: $\sin^2\theta + \cos^2\theta = 1 = x^2/a^2 + y^2/b^2$. Thus, we have an ellipse.



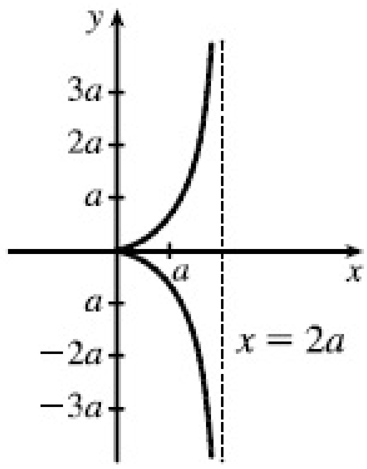
EX.44

(a) Let θ be the angle of inclination of segment OP. Then $|OB| = \dfrac{2a}{\cos\theta}$. Let $C = (2a, 0)$. Then by use of right triangle OAC we see that $|OA| = 2a\cos\theta$. Now

$$|OP| = |AB| = |OB| - |OA|$$
$$= 2a(\frac{1}{\cos\theta} - \cos\theta) = 2a\frac{1-\cos^2\theta}{\cos\theta} = 2a\frac{\sin^2\theta}{\cos\theta} = 2a\sin\theta\tan\theta$$

So P has coordinates $x = 2a\sin\theta\tan\theta\cdot\cos\theta = 2a\sin^2\theta$ and $y = 2a\sin\theta\tan\theta\cdot\sin\theta = 2a\sin^2\theta\tan\theta$.

(b) Graph



Section 10.2 Calculus with Parametric Curves

EX.34

By symmetry,

$$A = 4\int_0^a y\,dx = 4\int_{\pi/2}^0 a\sin^3\theta(-3a\cos^2\theta\sin\theta)\,d\theta$$
$$= 12a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta$$

Now

$$\int\sin^4\theta\cos^2\theta\,d\theta = \int\sin^2\theta(\frac{1}{4}\sin^2 2\theta)\,d\theta = \frac{1}{8}\int(1-\cos 2\theta)\sin^2 2\theta\,d\theta$$
$$= \frac{1}{8}\int\left[\frac{1}{2}(1-\cos 4\theta) - \sin^2 2\theta\cos 2\theta\right]\,d\theta$$
$$= \frac{1}{16}\theta - \frac{1}{64}\sin 4\theta - \frac{1}{48}\sin^3 2\theta + C$$

So $\displaystyle\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta = \left[\frac{1}{16}\theta - \frac{1}{64}\sin 4\theta - \frac{1}{48}\sin^3 2\theta\right]_0^{\pi/2} = \frac{\pi}{32}$.

Thus, $A = 12a^2\left(\frac{\pi}{32}\right) = \frac{3}{8}\pi a^2$.

EX.62

$x = 2t^2 + 1/t,\ y = 8\sqrt{t},\ 1 \le t \le 3.$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(4t - \frac{1}{t^2}\right)^2 + \left(\frac{4}{\sqrt{t}}\right)^2$$
$$= 16t^2 - \frac{8}{t} + \frac{1}{t^4} + \frac{16}{t} = 16t^2 + \frac{8}{t} + \frac{1}{t^4} = \left(4t + \frac{1}{t^2}\right)^2$$

$$S = \int_1^3 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_1^3 2\pi\left(8\sqrt{t}\right)\sqrt{\left(4t + \frac{1}{t^2}\right)^2}$$
$$= 16\pi\int_1^3 t^{1/2}(4t + t^{-2})\,dt = 16\pi\int_1^3(4t^{3/2} + t^{-3/2})\,dt$$
$$= 16\pi\left[\frac{8}{5}t^{5/2} - 2t^{-1/2}\right]_1^3 = \frac{32\pi}{15}\left(103\sqrt{3} + 3\right)$$