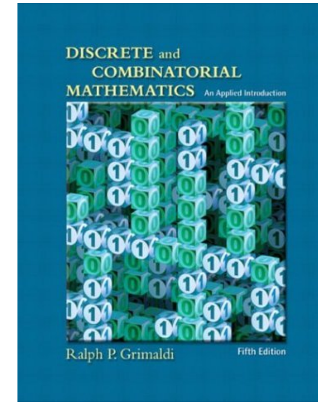
Discrete Mathematics
-- Chapter 8: The Principle of Inclusion and Exclusion



Hung-Yu Kao (高宏宇)

Department of Computer Science and Information Engineering,

National Cheng Kung University

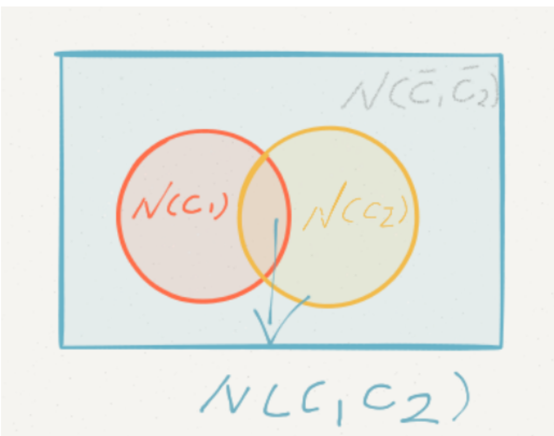
Outline

- The Principle of Inclusion and Exclusion
- Generalization of the Principle
- Derangements: Nothing Is in Its Right Place
- Rook Polynomials
- Arrangements with Forbidden Positions

8.1 The principle of Inclusion and Exclusion

- For a given finite set S ($|S| = N$) with conditions C_i
- $N(\overline{c_1}\,\overline{c_2}) = N - [N(c_1) + N(c_2)] + N(c_1 c_2)$ $\boxed{N(\overline{c_1} \text{ and } \overline{c_2})}$

$$= N(\overline{c_1}) - N(\overline{c_1} c_2)$$



$$N(\overline{c_1 c_2}) = N - N(c_1 c_2)$$

$$\neq N(\overline{c_1}\,\overline{c_2}) \quad \boxed{N(\overline{c_1} \text{ or } \overline{c_2})}$$

- $N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}) = N - [N(c_1) + N(c_2) + N(c_3)]$

$$+ [N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] - N(c_1c_2c_3)$$

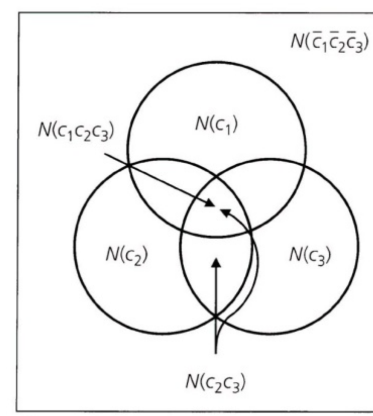


Figure 8.2

Four sets

- Ex 8.3 :

$$N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}) = N - [N(c_1) + N(c_2) + N(c_3) + N(c_4)]$$
$$+ [N(c_1c_2) + N(c_1c_3) + N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_3c_4)]$$
$$- [N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4) + N(c_2c_3c_4)] + N(c_1c_2c_3c_4)$$

- For each element $x \in S$, we have five cases:
  - (0) x satisfies none of the four conditions;
  - (1) x satisfies only one of the four conditions;
  - (2) x satisfies exactly two of the four conditions;
  - (3) x satisfies exactly three of the four conditions;
  - (4) x satisfies all the four conditions.

Four sets

$$N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}) = N - [N(c_1)+N(c_2)+N(c_3)+N(c_4)]$$
$$+[N(c_1c_2)+N(c_1c_3)+N(c_1c_4)+N(c_2c_3)+N(c_2c_4)+N(c_3c_4)]$$
$$-[N(c_1c_2c_3)+N(c_1c_2c_4)+N(c_1c_3c_4)+N(c_2c_3c_4)]+N(c_1c_2c_3c_4)$$

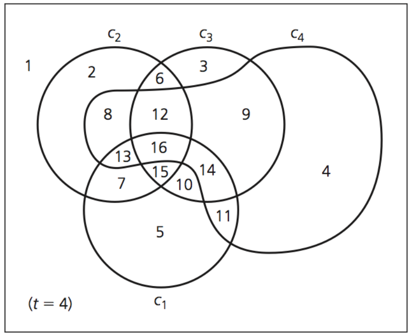


Figure 8.5

1. x satisfies no condition. x is counted once in $N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4})$ and once in N. [1=1]
2. x satisfies c_1. It is not counted on the left side. It is counted once in N and once in $N(c_1)$. [0 = 1-1 = 0]
3. x satisfies c_2 and c_4. It is not counted on the left side. It is counted once in N, $N(c_2)$, $N(c_4)$ and $N(c_2c_4)$.
   $$[0 = 1-(1+1)+1 = 1-\binom{2}{1}+\binom{2}{2} = 0]$$
4. x satisfies c_1, c_2 and c_4. It is not counted on the left side. It is counted once in N, $N(c_1)$, $N(c_2)$, $N(c_4)$, $N(c_1c_2)$, $N(c_1c_4)$, $N(c_2c_4)$ and $N(c_1c_2c_4)$.
   $$[0 = 1-(1+1+1)+(1+1+1)-1 = 1-\binom{3}{1}+\binom{3}{2}-\binom{3}{3} = 0]$$
5. x satisfies all conditions. It is not counted on the left side. It is counted once in all the subsets on the right side.
   $$[0 = 1-\binom{4}{1}+\binom{4}{2}-\binom{4}{3}+\binom{4}{4} = 0]$$

Four sets

$$
\begin{aligned}
N(c_1\overline{c}_2\overline{c}_3\overline{c}_4) &= \underline{N(\overline{c}_2\overline{c}_3\overline{c}_4) - N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4)} \\
&= \{N - [N(c_2) + N(c_3) + N(c_4)] + [N(c_2c_3) + N(c_2c_4) + N(c_3c_4)] \\
&\quad - N(c_2c_3c_4)\} - \{N - [N(c_1) + N(c_2) + N(c_3) + N(c_4)] \\
&\quad + [N(c_1c_2) + N(c_1c_3) + N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_3c_4)] \\
&\quad - [N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4) + N(c_2c_3c_4)] + N(c_1c_2c_3c_4)\}, \text{ or}
\end{aligned}
$$

$$
\begin{aligned}
N(\underline{c_1}\overline{c}_2\overline{c}_3\overline{c}_4) &= N(\underline{c_1}) - [N(\underline{c_1}c_2) + N(\underline{c_1}c_3) + N(\underline{c_1}c_4)] \\
&\quad + [N(\underline{c_1}c_2c_3) + N(\underline{c_1}c_2c_4) + N(\underline{c_1}c_3c_4)] - N(\underline{c_1}c_2c_3c_4).
\end{aligned}
$$

The Principle of Inclusion and Exclusion

- Theorem 8.1:
  - $|S| = N$, conditions c_i, $1 \le i \le t$
  - $\overline{N} = N(\overline{c_1}\,\overline{c_2}\cdots\overline{c_t})$ denote the number of elements of S that satisfy none of the conditions.

$$\overline{N} = N - \sum_{1 \le i \le t} N(c_i) + \sum_{1 \le i < j \le t} N(c_i c_j) - \sum_{1 \le i < j < k \le t} N(c_i c_j c_k) + \cdots \quad (2)$$

$$+ (-1)^t N(c_1 c_2 c_3 \cdots c_t)$$

The other possibility is that x satisfies *exactly* r of the conditions where $1 \le r \le t$. In this case x contributes nothing to $\overline{N}$. But on the right-hand side of Eq. (2), x is counted

(1) One time in N.

(2) r times in $\sum_{1 \le i \le t} N(c_i)$. (Once for each of the r conditions.)

(3) $\binom{r}{2}$ times in $\sum_{1 \le i < j \le t} N(c_i c_j)$. (Once for each pair of conditions selected from the r conditions it satisfies.)

(4) $\binom{r}{3}$ times in $\sum_{1 \le i < j < k \le t} N(c_i c_j c_k)$. (Why?)

. . . .

($r+1$) $\binom{r}{r} = 1$ time in $\sum N(c_{i_1} c_{i_2} \cdots c_{i_r})$, where the summation is taken over all selections of size r from the t conditions.

Consequently, on the right-hand side of Eq. (2), x is counted

$$1 - r + \binom{r}{2} - \binom{r}{3} + \cdots + (-1)^r \binom{r}{r} = [1 + (-1)]^r = 0^r = 0 \text{ times},$$

The Principle of Inclusion and Exclusion

- Corollary 8.1: $N(c_1 \text{ or } c_2 \text{ or} \ldots \text{or } c_t) = N - \overline{N}.$
- Some notation for simplifying Theorem 8.1

$$S_0 = N,$$
$$S_1 = [N(c_1) + N(c_2) + \cdots + N(c_t)],$$
$$S_2 = [N(c_1c_2) + N(c_1c_3) + \cdots + N(c_1c_t) + N(c_2c_3) + \cdots + N(c_{t-1}c_t)],$$

and, in general,

$$S_k = \sum N(c_{i_1}c_{i_2}\cdots c_{i_k}),\ 1 \le k \le t,$$

where the summation is taken over all selections of size k from the collection of t conditions. Hence S_k has $\binom{t}{k}$ summands in it.

Using this notation we can rewrite the result in Eq. (2) as

$$\overline{N} = S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^t S_t.$$

The Principle of Inclusion and Exclusion

- **Ex 8.4** : Determine the number of positive integers n where $1 \le n \le 100$ and n is not divisible by 2, 3 or 5.
  - Condition c_1 if n is divisible by 2.
  - Condition c_2 if n is divisible by 3.
  - Condition c_3 if n is divisible by 5.
  - Then the answer to this problem is

Q: Find the number of positive integers n where 1<= n <= 3000, and n is not a perfect square, cube, or fifth power. (A: 2933)

$$N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}) = S_0 - S_1 + S_2 - S_3 \qquad \lfloor 100/(2*3) \rfloor = 16$$

$$= N - [N(c_1) + N(c_2) + N(c_3)] + [N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] - N(c_1c_2c_3)$$

$$= 100 - [50 + 33 + 20] + [16 + 10 + 6] - 3 = 26.$$

The Principle of Inclusion and Exclusion

- **Ex 8.5** : Determine the number of nonnegative integer solutions to the equation
  $x_1 + x_2 + x_3 + x_4 = 18$ and $x_i \leq 7$ for all i.
  - We say that a solution x_1, x_2, x_3, x_4 satisfies condition c_i if $x_i > 7$ (i.e., $x_i \geq 8$) .
  - Then the answer to this problem is

$$N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4) = S_0 - S_1 + S_2 - S_3 + S_4 =$$

$$\binom{21}{18} - \binom{4}{1}\binom{13}{10} + \binom{4}{2}\binom{5}{2} - 0 + 0 = 246.$$

$$\binom{4+18-1}{18} \qquad \binom{4+10-1}{10} \qquad \binom{4+2-1}{2}$$

The Principle of Inclusion and Exclusion

- **Ex 8.6** : For finite sets A, B, where $|A| = m \geq n = |B|$, and function $f: A \to B$, determine the number of onto functions f.
  - Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$.
  - Let c_i be the condition that b_i is not in the range of f.
    Then $N(c_1)$ is the number of functions in S that have b_i in their range.
  - Then the answer to this problem is $N(\overline{c_1}\,\overline{c_2}\ldots\ldots\overline{c_n})$.

$$N(\overline{c}_1\overline{c}_2\overline{c}_3\cdots\overline{c}_n) = S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^n S_n$$

$$N = S_0 = |S| = n^m$$
$$N(c_i) = (n-1)^m \Rightarrow S_1 = \binom{n}{1}(n-1)^m$$
$$N(c_i c_j) = (n-2)^m \Rightarrow S_2 = \binom{n}{2}(n-2)^m$$

$$= n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \binom{n}{3}(n-3)^m$$

$$+ \cdots + (-1)^n (n-n)^m = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^m$$

$$= \sum_{i=0}^{n} (-1)^i \binom{n}{n-i} (n-i)^m.$$

The Principle of Inclusion and Exclusion

- **Ex 8.7** : In how many ways can the 26 letters of the alphabet be permuted so that none of the patterns car, dog, pun, or byte occurs?

How about "spin", "game", "path", and "net"?

$$N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4) = 26! - [3(24!) + 23!] + [3(22!) + 3(21!)] - [20! + 3(19!)] + 17!$$

- **Ex 8.8** : Let $\phi(n)$ be the number of positive integers m, where $1 \leq m < n$ and $\gcd(m, n)=1$, i.e., m and n are relatively prime. [2008台大資工]
  - Consider $n = p_1^{e_1} p_2^{e_2} p_3^{e_3} p_4^{e_4}$
  - For $1 \leq i \leq 4$, let c_i denote that k is divisible by p_i.
    - $N = S_0 = n$; $N(c_i) = n/p_i$; $N(c_ic_j) = n/(p_ip_j)$; …
  - Then the answer to this problem is $N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4})$.

$$
\begin{aligned}
\phi(n) &= S_0 - S_1 + S_2 - S_3 + S_4 \\
&= n - \left[\frac{n}{p_1} + \cdots + \frac{n}{p_4}\right] + \left[\frac{n}{p_1 p_2} + \frac{n}{p_1 p_3} + \cdots + \frac{n}{p_3 p_4}\right] \\
&\quad - \left[\frac{n}{p_1 p_2 p_3} + \cdots + \frac{n}{p_2 p_3 p_4}\right] + \frac{n}{p_1 p_2 p_3 p_4} \\
&= n\left[1 - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_4}\right) + \left(\frac{1}{p_1 p_2} + \frac{1}{p_1 p_3} + \cdots + \frac{1}{p_3 p_4}\right)\right. \\
&\quad \left. - \left(\frac{1}{p_1 p_2 p_3} + \cdots + \frac{1}{p_2 p_3 p_4}\right) + \frac{1}{p_1 p_2 p_3 p_4}\right] \\
&= \frac{n}{p_1 p_2 p_3 p_4}\left[p_1 p_2 p_3 p_4 - (p_2 p_3 p_4 + p_1 p_3 p_4 + p_1 p_2 p_4 + p_1 p_2 p_3)\right. \\
&\quad + (p_3 p_4 + p_2 p_4 + p_2 p_3 + p_1 p_4 + p_1 p_3 + p_1 p_2) \\
&\quad \left. - (p_4 + p_3 + p_2 + p_1) + 1\right] \\
&= \frac{n}{p_1 p_2 p_3 p_4}\left[(p_1 - 1)(p_2 - 1)(p_3 - 1)(p_4 - 1)\right] \\
&= n\left[\frac{p_1 - 1}{p_1} \cdot \frac{p_2 - 1}{p_2} \cdot \frac{p_3 - 1}{p_3} \cdot \frac{p_4 - 1}{p_4}\right] = n \prod_{i=1}^{4}\left(1 - \frac{1}{p_i}\right).
\end{aligned}
$$

$$\boxed{(p_1 - 1)p_1^{e_1 - 1}(p_2 - 1)p_2^{e_2 - 1}(p_3 - 1)p_3^{e_3 - 1}(p_4 - 1)p_4^{e_4 - 1}}$$

The Principle of Inclusion and Exclusion

- **Ex 8.9** : Six married couples are to be seated at a circular table. In how many ways can they arrange themselves so that no wife sits next to her husband?
  - For $1 \le i \le 6$, let c_i denote the condition where a seating arrangement has couple i seated next to each other. $N(c_i) = 2(11-1)!$
  - Then the answer to this problem is $N(\overline{c_1}\,\overline{c_2}\ldots\overline{c_6})$.

$N(c_1c_2c_3) = 2^3(8!)$, $S_3 = \binom{6}{3}2^3(8!)$ $\quad$ $N(c_1c_2c_3c_4) = 2^4(7!)$, $S_4 = \binom{6}{4}2^4(7!)$

$N(c_1c_2c_3c_4c_5) = 2^5(6!)$, $S_5 = \binom{6}{5}2^5(6!)$ $\quad$ $N(c_1c_2c_3c_4c_5c_6) = 2^6(5!)$, $S_6 = \binom{6}{6}2^6(5!)$.

The Principle of Inclusion and Exclusion

- **Ex 8.10** : In a certain area of the countryside are five villages. An engineer is to devise a system of two-way roads so that after the system is completed, no village will be isolated. In how many ways can he do this?
  - Let c_i denote the condition that a system of these roads isolates village a, b, c, d, and e, respectively.

$$N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4\overline{c}_5) = \underbrace{2^{10}}_{C(5,2)} - \binom{5}{1}\underbrace{2^6}_{C(4,2)} + \binom{5}{2}2^3 - \binom{5}{3}2^1 + \binom{5}{4}2^0 - \binom{5}{5}2^0 = 768.$$

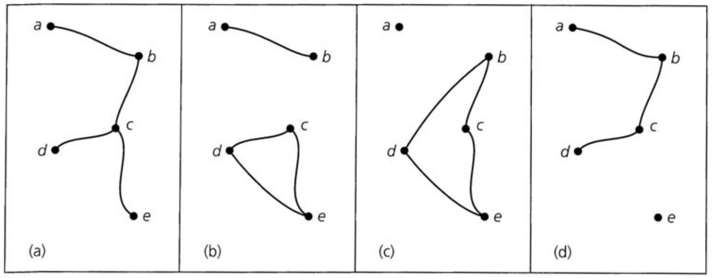


Figure 8.3

example

Let A be a set containing n elements. Consider the binary operation on A,
(a) How many of them are neither reflexive nor symmetric?
(b) How many of them are symmetric but not reflexive?
(c) How many of them are reflexive but not antisymmetric? (90 NCTU)

$$N = 2^{n^2}, N(a_1) = 2^{n^2-n}, N(a_2) = 2^{\frac{n^2+n}{2}}, N(a_3) = ?$$

$$N(a_1a_2) = 2^{\frac{n^2-n}{2}}, N(a_1a_3) = 3^{\binom{n}{2}}$$

8.2 Generalizations of the Principle

- E_m denotes the number of elements in S that satisfy exactly m of the t conditions.
- $E_1 = N(c_1\overline{c_2}\,\overline{c_3}\cdots\overline{c_t}) + N(\overline{c_1}c_2\overline{c_3}\cdots\overline{c_t}) + \cdots + N(\overline{c_1}\,\overline{c_2}\cdots\overline{c_{t-1}}c_t)$.

  $E_2 = N(c_1c_2\overline{c_3}\cdots\overline{c_t}) + N(c_1\overline{c_2}c_3\cdots\overline{c_t}) + \cdots + N(\overline{c_1}\,\overline{c_2}\cdots\overline{c_{t-2}}c_{t-1}c_t)$.

- $E_1 = \mathbf{2}+\mathbf{3}+\mathbf{4} = N(c_1)+N(c_2)+N(c_3) - 2[N(c_1c_2)+ N(c_1c_3)+ N(c_2c_3)] + 3N(c_1c_2c_3)$

  $= S_1 - 2S_2 + 3S_3 = S_1 - \binom{2}{1}S_2 + \binom{3}{2}S_3$
- $E_2 = \mathbf{5}+\mathbf{6}+\mathbf{7} = N(c_1c_2)+N(c_1c_3)+N(c_2c_3) - 3N(c_1c_2c_3)$

  $= S_2 - 3S_3 = S_2 - \binom{3}{1}S_3$
- $E_3 = \mathbf{8} = N(c_1c_2c_3) = S_3$

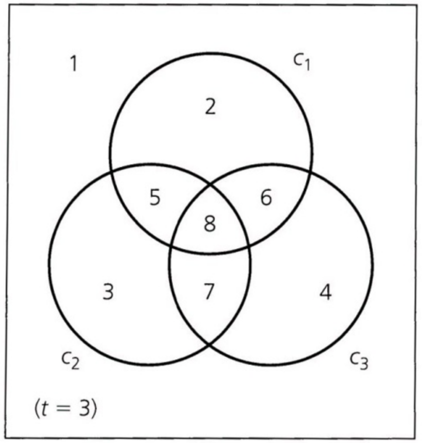


Figure 8.4

E_1= **2+3+4+5**=$N(c_1)+N(c_2)+N(c_3)+N(c_4)$
-2[$N(c_1c_2)+ N(c_1c_3)+ N(c_2c_3)+N(c_1c_4)+ N(c_2c_4)+ N(c_3c_4)$]
+ 3[$N(c_1c_2c_3)+N(c_1c_2c_4)+N(c_2c_3c_4)+N(c_1c_3c_4)$]
-4$N(c_1c_2c_3c_4)$

Generalizations of the Principle

$$E_1 = S_1 - 2S_2 + 3S_3 - 4S_4$$
$$= S_1 - \binom{2}{1}S_2 + \binom{3}{2}S_3 - \binom{4}{3}S_4$$
$$E_2 = S_2 - 3S_3 + 6S_4$$
$$= S_2 - \binom{3}{1}S_3 + \binom{4}{2}S_4$$
$$E_3 = S_3 - 4S_4 = S_3 - \binom{4}{1}S_4$$
$$E_4 = S_4$$

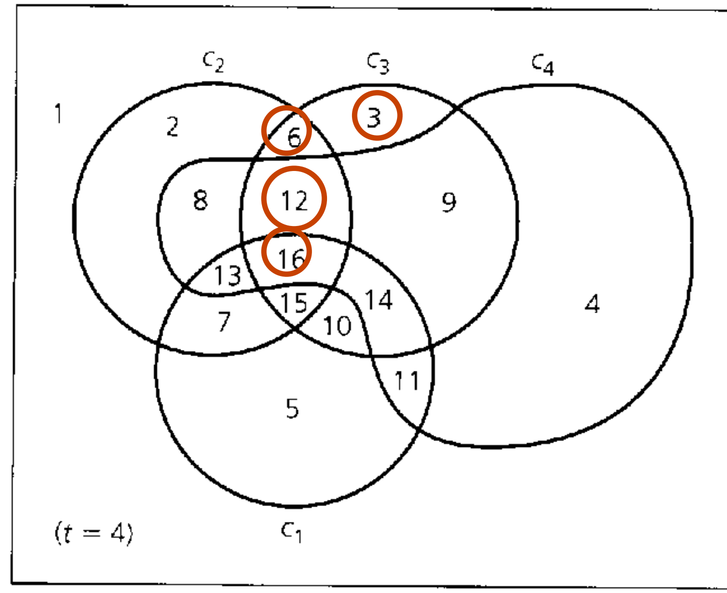


Figure 8.5

Table 8.1

S_2	S_3	S_4
$N(c_1c_2)$: 7, 13, 15, 16 $N(c_1c_3)$: 10, 14, 15, 16 $N(c_1c_4)$: 11, 13, 14, 16 $N(c_2c_3)$: 6, 12, 15, 16 $N(c_2c_4)$: 8, 12, 13, 16 $N(c_3c_4)$: 9, 12, 14, 16	$N(c_1c_2c_3)$: 15, 16 $N(c_1c_2c_4)$: 13, 16 $N(c_1c_3c_4)$: 14, 16 $N(c_2c_3c_4)$: 12, 16	$N(c_1c_2c_3c_4)$: 16

Theorem 8.2

$$E_m = S_m - \binom{m+1}{1}S_{m+1} + \binom{m+2}{2}S_{m+2} - \cdots + (-1)^{t-m}\binom{t}{t-m}S_t. \quad (1)$$

(If $m = 0$, we obtain Theorem 8.1.)

Proof: Arguing as in Theorem 8.1, let $x \in S$ and consider the following three cases.

a) When x satisfies fewer than m conditions, it contributes a count of 0 to each of the terms E_m, S_m, $S_{m+1}, \ldots, S_t$, so it is not counted on either side of the equation.

b) When x satisfies exactly m of the conditions, it is counted once in E_m and once in S_m, but not in $S_{m+1}, \ldots, S_t$. Consequently, it is included once in the count for either side of the equation.

c) Suppose x satisfies r of the conditions, where $m < r \leq t$. Then x contributes nothing to E_m. Yet it is counted $\binom{r}{m}$ times in S_m, $\binom{r}{m+1}$ times in $S_{m+1}, \ldots$, and $\binom{r}{r}$ times in S_r, but 0 times for any term beyond S_r. So on the right-hand side of the equation, x is counted $\binom{r}{m} - \binom{m+1}{1}\binom{r}{m+1} + \binom{m+2}{2}\binom{r}{m+2} - \cdots + (-1)^{r-m}\binom{r}{r-m}\binom{r}{r}$ times.

For $0 \le k \le r - m$,

$$\binom{m+k}{k}\binom{r}{m+k} = \frac{(m+k)!}{k!\,m!} \cdot \frac{r!}{(m+k)!(r-m-k)!}$$

$$= \frac{r!}{m!} \cdot \frac{1}{k!(r-m-k)!} = \frac{r!}{m!(r-m)!} \cdot \frac{(r-m)!}{k!(r-m-k)!}$$

$$= \binom{r}{m}\binom{r-m}{k}.$$

Consequently, on the right-hand side of Eq. (1), x is counted

$$\binom{r}{m}\binom{r-m}{0} - \binom{r}{m}\binom{r-m}{1} + \binom{r}{m}\binom{r-m}{2} - \cdots + (-1)^{r-m}\binom{r}{m}\binom{r-m}{r-m}$$

$$= \binom{r}{m}\left[\binom{r-m}{0} - \binom{r-m}{1} + \binom{r-m}{2} - \cdots + (-1)^{r-m}\binom{r-m}{r-m}\right]$$

$$= \binom{r}{m}[1-1]^{r-m} = \binom{r}{m} \cdot 0 = 0 \text{ times},$$

and the formula is verified.

Corollary 8.2

- Let L_m denotes the number of elements in S that satisfy at least m of the t conditions.

$$L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} - \cdots + (-1)^{t-m} \binom{t-1}{m-1} S_t .$$

When $m = 1$, the result in Corollary 8.2 becomes

$$\begin{aligned} L_1 &= S_1 - \binom{1}{0} S_2 + \binom{2}{0} S_3 - \cdots + (-1)^{t-1} \binom{t-1}{0} S_t \\ &= S_1 - S_2 + S_3 - \cdots + (-1)^{t-1} S_t . \end{aligned}$$

Comparing this with the result in Theorem 8.1, we find that

$$L_1 = N - \overline{N} = |S| - \overline{N}.$$

Corollary 8.2

- **Ex 8.11** : In 8.10, a certain area of the countryside are five villages. An engineer is to devise a system of two-way roads so that after the system is completed, no village will be isolated. In how many ways can he do this?
  - Let c_i denote the condition that a system of these roads isolates village a, b, c, d, and e, respectively.

$$N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4\overline{c}_5) = 2^{10} - \binom{5}{1}2^6 + \binom{5}{2}2^3 - \binom{5}{3}2^1 + \binom{5}{4}2^0 - \binom{5}{5}2^0 = 768.$$

C(5,2) C(4,2)

E2 = S2-(3,1)S3+(4,2)S4-(5,3)S5=40
L2 = S2-(2,1)S3+(3,1)S4-(4,1)S5=51

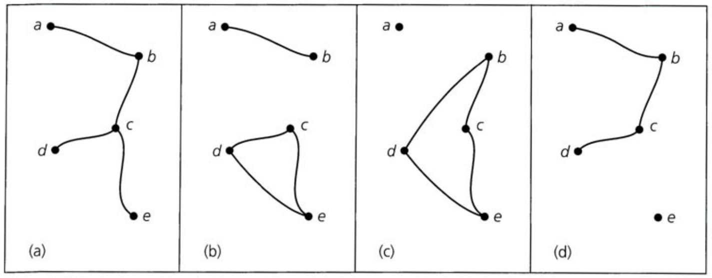


Figure 8.3

8.3 Derangements: Nothing Is in Its Right Place

- $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!},\ e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots$$
- e^{-1}=0.36788, 1-1+(1/2!)-(1/3!)+…-(1/7!) ≈ 0.36786

- **Derangement** means that all numbers are in the wrong positions.
- **Ex 8.12** : Determine the number of derangements of 1, 2,…,10. Let c_i be the condition that integer i is in the ith place.

$$d_{10} = N(\overline{c}_1\overline{c}_2\overline{c}_3 \cdots \overline{c}_{10}) = 10! - \binom{10}{1}9! + \binom{10}{2}8! - \binom{10}{3}7! + \cdots + \binom{10}{10}0!$$
$$= 10!\left[1 - \binom{10}{1}(9!/10!) + \binom{10}{2}(8!/10!) - \binom{10}{3}(7!/10!) + \cdots + \binom{10}{10}(0!/10!)\right]$$
$$= 10![1 - 1 + (1/2!) - (1/3!) + \cdots + (1/10!)] \doteq (10!)(e^{-1}).$$

Derangements: Nothing Is in Its Right Place

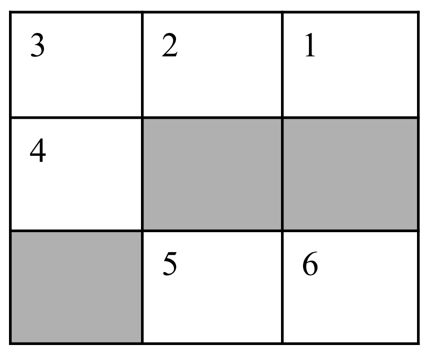
- The general formula:

$$d_n = n!e^{-1} = n![1-1+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-....\frac{1}{n!}] \qquad P=d_n/n!$$

- **Ex 8.14** : Peggy has seven books and hires seven reviewers. She wants two reviewers per book. In how many ways can she make the distributions?
  - The first time: 7! ways
  - The second time: d_7=7!* e^{-1}Ways (different position)
  - Totally, we have $7!\times d_7$ ways

8.4 Rook Polynomials

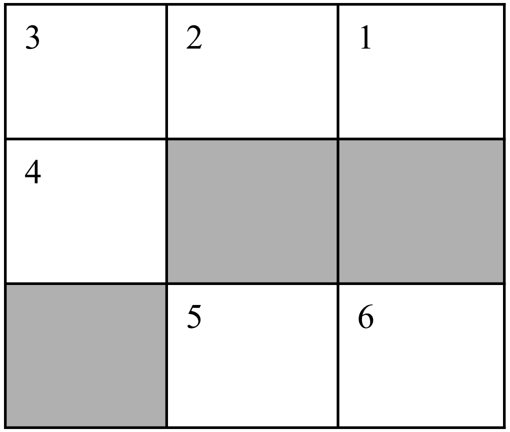
- In Fig. 8.6, we want to determine *the number of ways* in which *k rooks* can be placed on the unshaded squares of this chessboard so that no two of them can take each other—that is, *no two of them are in the same row or column of the chessboard C*. This number is denoted as $r_k(C)$.



Rook Polynomials

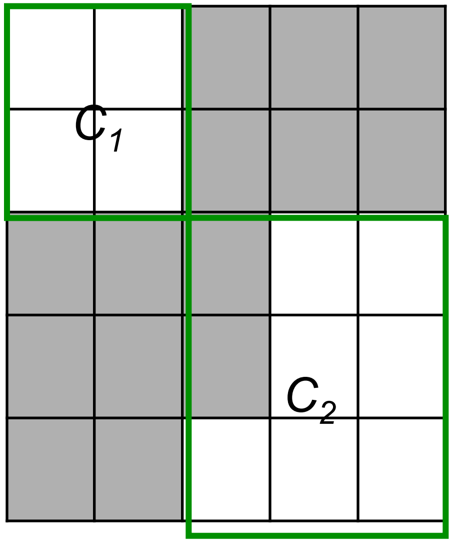
- In Fig. 8.6, we have r_0=1, r_1= 6, r_2= 8, r_3= 2 and r_k= 0 for $k \geq 4$.
- Rook polynomial: $r(C, x) = 1+6x+8x^2+2x^3$. For each $k \geq 0$, the coefficient of x^k is the number of ways we can place k nontaking rooks on chessboard C.

(1,4)(1,5)(2,4)(2,6)(3,5)(3,6)(4,5)(4,6)
(1,4,5)(2,4,6)



Disjoint Subboards

- Break up a larger board into smaller subboards.
- In Fig. 8.7, the chessboard contains two disjoint subboards that have no squares in the same column or row of C.
- $r(C, x) = r(C_1, x) \cdot r(C_2, x)$



$$r(C_1, x) = 1 + 4x + 2x^2$$

$$r(C_2, x) = 1 + 7x + 10x^2 + 2x^3$$

$$r(C, x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5$$

$$= r(C_1, x) \cdot r(C_2, x)$$

Disjoint Subboards

$$r(C_1, x) = 1 + 4x + 2x^2$$

$$r(C_2, x) = 1 + 7x + 10x^2 + 2x^3$$

- r_3 for C

  a) All three rooks are on subboard C_2 (and none is on C_1): $(2)(1) = 2$ ways.

  b) Two rooks are on subboard C_2 and one is on C_1: $(10)(4) = 40$ ways.

  c) One rook is on subboard C_2 and two are on C_1: $(7)(2) = 14$ ways.

- In general, if C is a chessboard made up of pairwise disjoint subboards $C_1, C_2, \ldots, C_n$, then $r(C, x) = r(C_1, x) \cdot r(C_2, x) \cdots r(C_n, x)$.

Recursive Formula

- Fig. 8.8 (a), For a given designated square (*), we either (b) place one rook here, or (c) do not use this square.
- $r_k(C) = r_{k\text{-}1}(C_s)+r_k(C_e)$
  - C_s: denote the remaining smaller subboard (Fig. 8.8(b))
  - C_e: C with the one designed square eliminated (Fig. 8.8(c))
- $r_k(C)x^k = r_{k\text{-}1}(C_s)\, x^k + r_k(C_e)x^k$ for $1 \le k \le$ n.

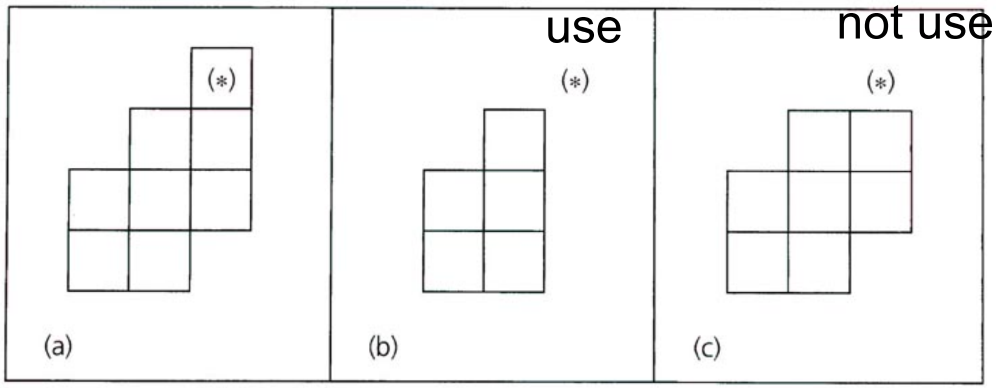


Figure 8.8

Recursive Formula

$$\sum_{k=1}^{n} r_k(C)x^k = \sum_{k=1}^{n} r_{k-1}(C_s)x^k + \sum_{k=1}^{n} r_k(C_e)x^k$$

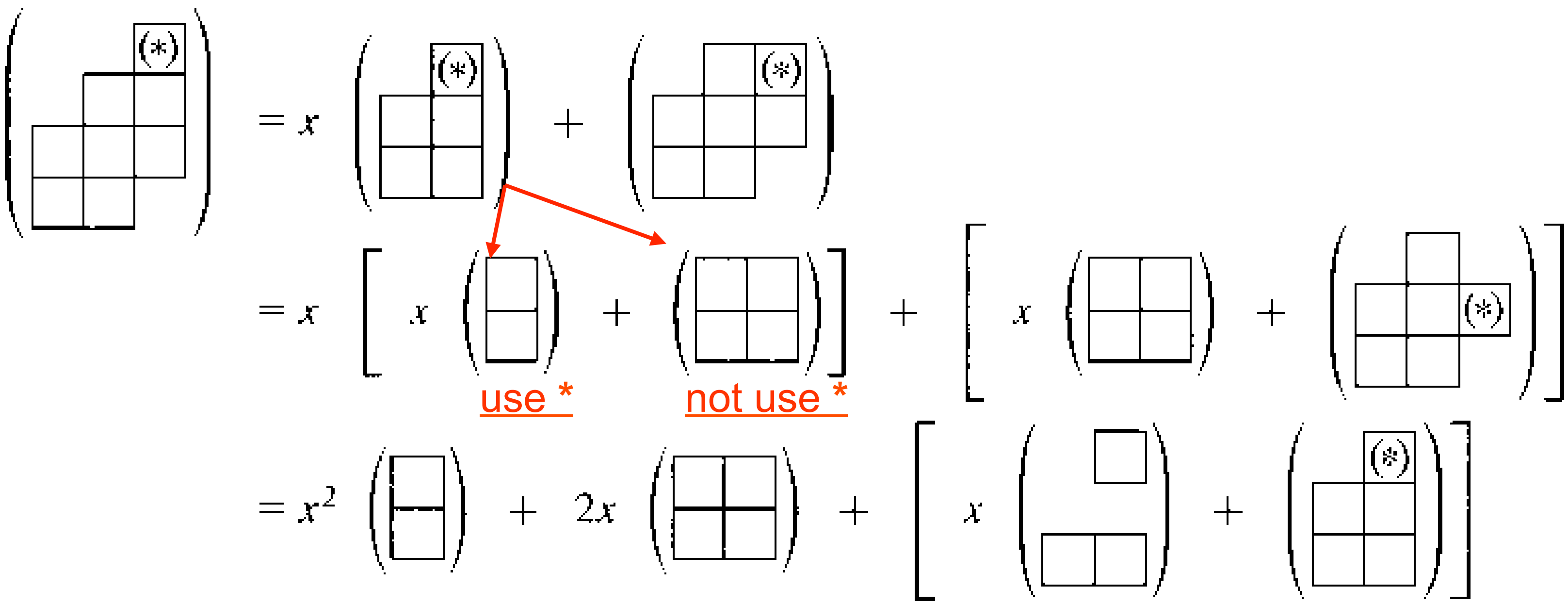
$$= x\sum_{k=1}^{n} r_{k-1}(C_s)x^{k-1} + \sum_{k=1}^{n} r_k(C_e)x^k$$

$$= x \cdot r(C_s, x) + \sum_{k=1}^{n} r_k(C_e)x^k$$

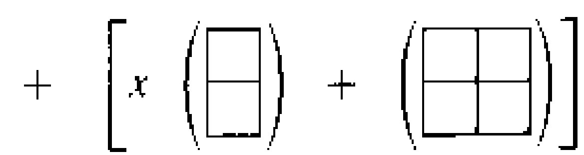
$$1 + \sum_{k=1}^{n} r_k(C)x^k = x \cdot r(C_s, x) + \sum_{k=1}^{n} r_k(C_e)x^k + 1$$

$$r(C, x) = x \cdot r(C_s, x) + r(C_e, x)$$

Apply the Recursive Formula



$$= x^2(1+2x) + 2x(1+4x+2x^2) + x(1+3x+x^2)$$



$$= 3x + 12x^2 + 7x^3 + \underline{x(1+2x) + (1+4x+2x^2)} = 1 + 8x + 16x^2 + 7x^3.$$

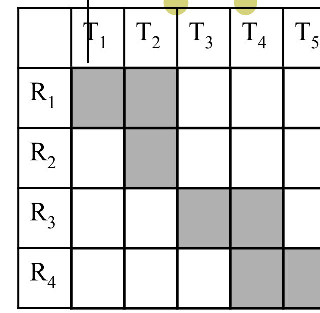
8.5 Arrangements with Forbidden Positions

- **Ex 8.15** : In making seating arrangements, the shaded square of the figure means relative R_i will not sit at table T_j.
  - Determine the number of ways that we can seat these four relatives at five different tables.
  - Let |S| be the total number of ways we can place the four relatives. (|S| = 5!)
  - Let c_i be the condition that R_i is seated in a forbidden position but at different tables.
    - $N(c_1) = 4! + 4!$ ($R_1 \rightarrow T_1$ or $R_1 \rightarrow T_2$)
    - $N(c_2) = 4!$ ($R_2 \rightarrow T_2$)
    - $N(c_3) = ?$ $4! + 4!$
    - $N(c_4) = ?$ $4! + 4!$
  - $S_1 = 7(4!)$

number of shaded squares

	T_1	T_2	T_3	T_4	T_5
R_1	■	■			
R_2		■			
R_3			■	■	
R_4				■	■

Arrangements with Forbidden Positions



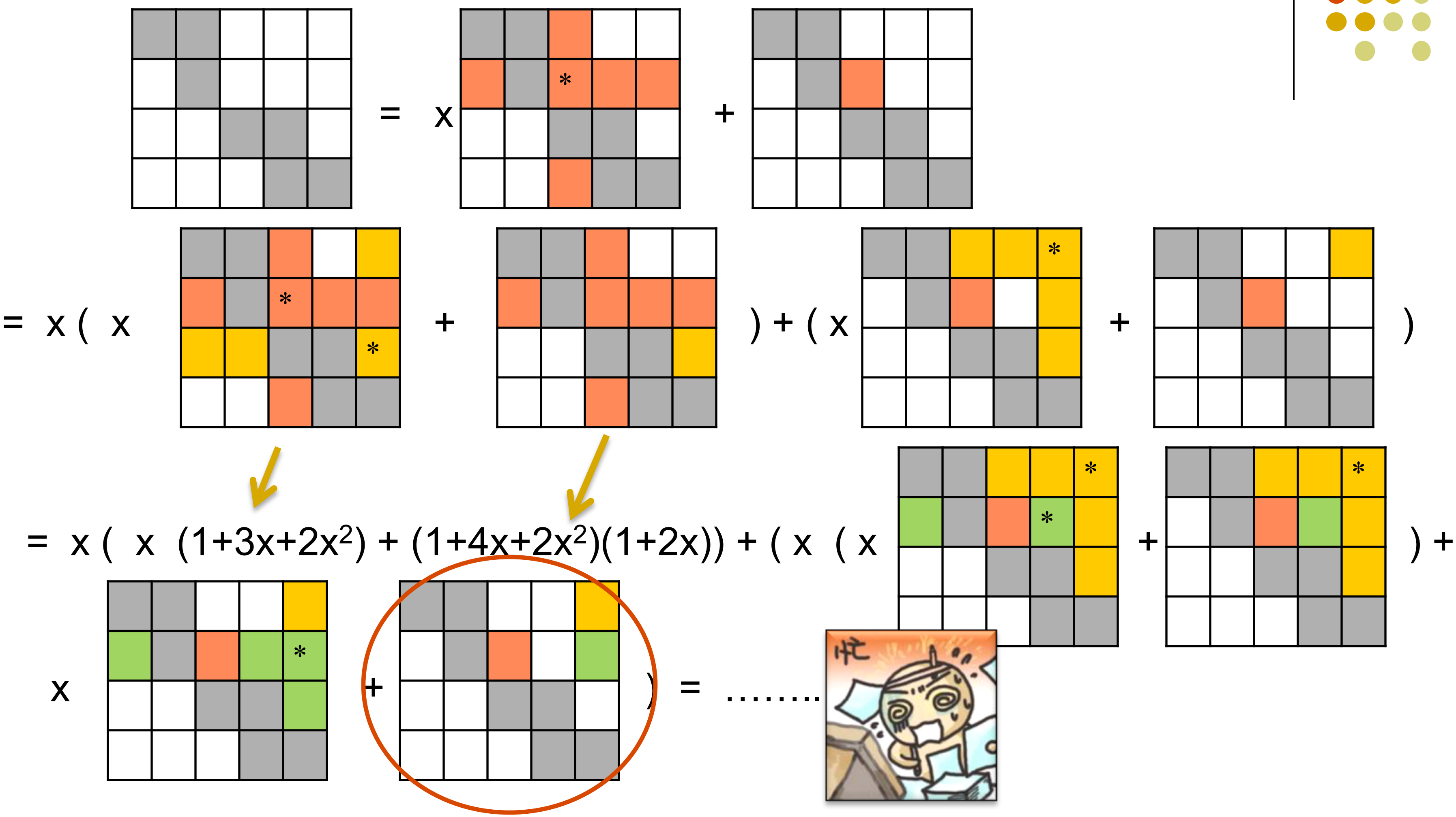
- $S_2 = 16(3!)$
  - $N(c_1c_2) = 3!$ ($R_1 \rightarrow T_1$ and $R_2 \rightarrow T_2$)
  - $N(c_1c_3) = \mathbf{4}(3!)$
  - $N(c_1c_4) = ?$, $N(c_2c_3) = ?$, $N(c_2c_4) = ?$, $N(c_3c_4) = ?$
- Observation: 16 is the number of ways two non-taking rooks can be placed on the shaded chessboard.
- Let r_i be the number of ways in which it is possible to place i non-taking rooks on the shaded chessboard.
  - For all $0 \le i \le 4$, $S_i = r_i(5 - i)!$
- Decompose C into the disjoint subboards in the upper left and lower right corners.
  - $r(C, x) = (1 + 3x + x^2)(1 + 4x + 3x^2) = 1 + 7x + 16x^2 + 13x^3 + 3x^4$

$$\therefore N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}) = S_0 - S_1 + S_2 - S_3 + S_4 = 1(5!) - 7(4!) + 16(3!) - 13(2!) + 3(1!)$$

$$= \sum_{i=0}^{4} (-1)^i r_i (5-i)! = 25$$

If you want to count the rook number in the white chessboard…
(skip the principle of inclusion and exclusion)

The coefficient of x^4 in $r(C, x)$



Arrangements with Forbidden Positions

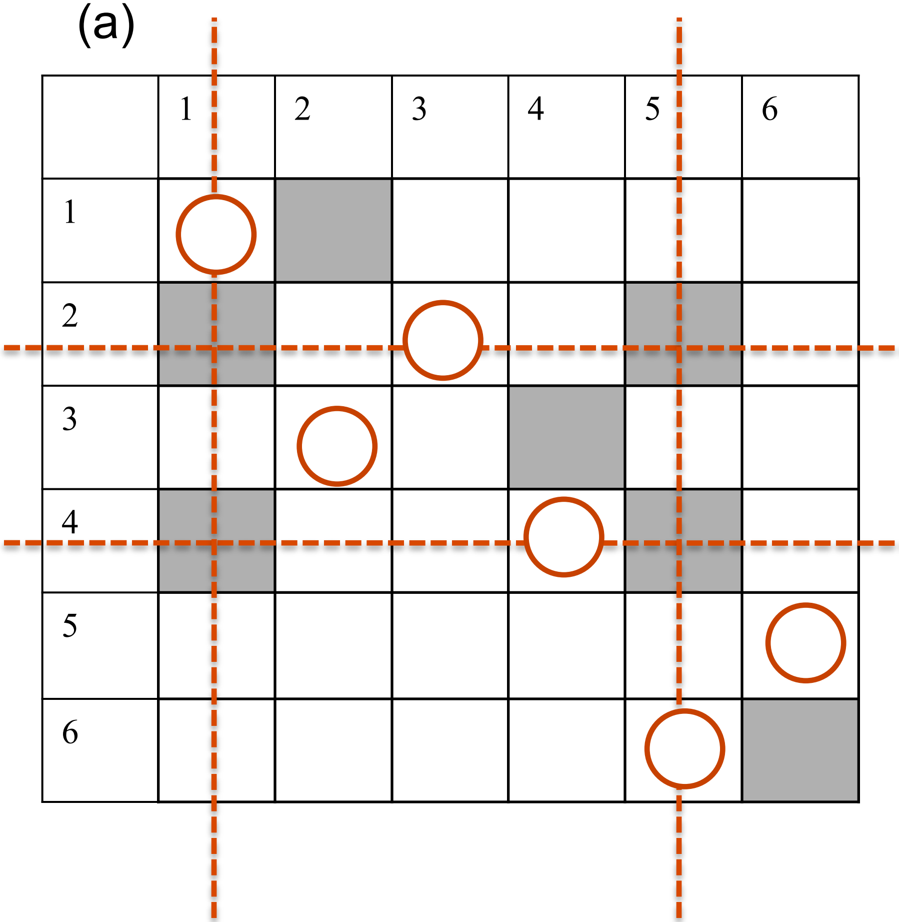
- **Ex 8.16** : We roll two dice six times, where one is red die and the other green die.
- We know the following pairs did not occur: (1, 2), (2, 1), (2, 5), (3, 4), (4, 1), (4, 5) and (6, 6).
- What is the probability that we obtain all six values both on red die and green die?
  - One of solutions is like (1, 1), (2, 3), (4, 4), (3, 2), (5, 6), (6, 5).
- In Fig. 8.10(b), chessboard *C* with seven shaded squares
  - $r(C, x) = (1+4x+2x^2)(1+x)^3 = 1+7x+17x^2+19x^3+10x^4+2x^5$

  - *c_i denotes that all six values occur on both the red and green dies, but i on the red die is paired with one of the forbidden numbers on the green die.*

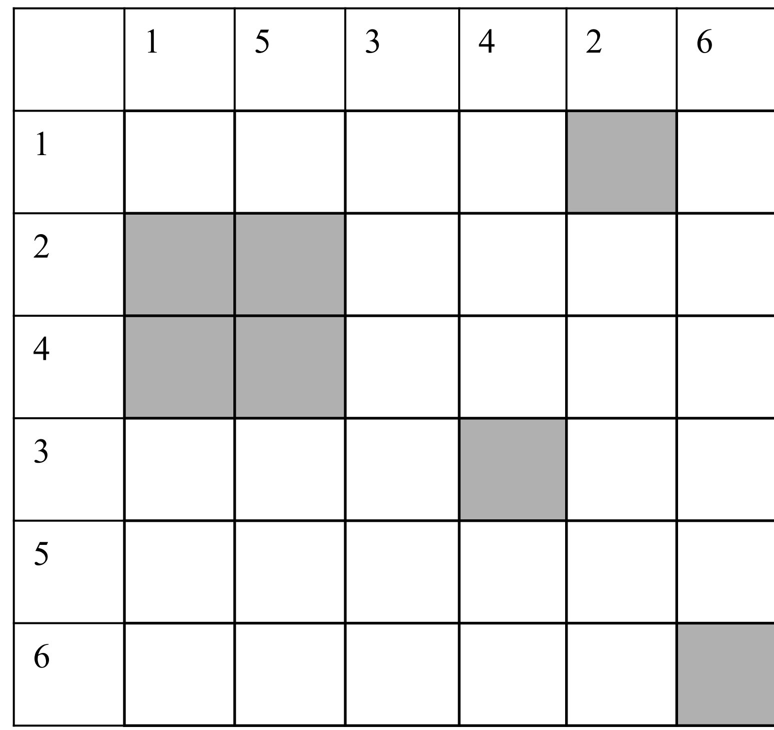
(1, 1), (2, 3), (4, 4), (3, 2), (5, 6), (6, 5)

Figure 8.10

(a)



(b)



$r(C, x) = (1+4x+2x^2)(1+x)^3 =$
$1+7x+17x^2+19x^3+10x^4+2x^5$

Arrangements with Forbidden Positions

$$\begin{aligned}(6!)N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4\overline{c}_5\overline{c}_6) &= (6!)\sum_{i=0}^{6}(-1)^i S_i = (6!)\sum_{i=0}^{6}(-1)^i r_i(6-i)! \\ &= 6![6! - 7(5!) + 17(4!) - 19(3!) + 10(2!) - 2(1!) + 0(0!)] \\ &= 6![192] = 138{,}240.\end{aligned}$$

Since the sample space consists of all sequences of six ordered pairs selected with repetition from the 29 unshaded squares of the chessboard, the probability of this event is $138{,}240/(29)^6 \doteq 0.00023$.

Arrangements with Forbidden Positions

- **Ex 8.17** : How many one-to-one functions f: A → B satisfy none of the following conditions:

$$c_1 : f(1) = u \text{ or } v$$
$$c_2 : f(2) = w$$
$$c_3 : f(3) = w \text{ or } x$$
$$c_4 : f(4) = x, y, \text{ or } z$$

B

A

	u	v	w	x	y	z
1						
2						
3						
4						

- $r(C, x) = (1+2x)(1+6x+9x^2+2x^3) = 1+8x+21x^2+20x^3+4x^4$
-

$$N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4) = S_0 - S_1 + S_2 - S_3 + S_4$$
$$= (6!/2!) - 8(5!/2!) + 21(4!/2!) - 20(3!/2!) + 4(2!/2!)$$

P(6,4) P(5,3)

$$= \sum_{i=0}^{4} (-1)^i r_i (6-i)!/2! = 76$$

Arrangements with Forbidden Positions

Finally, for $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$, suppose we want to count the number of one-to-one functions $h: A \to A$ where $h(i) \neq i$ for all $i \in A$. Here the rook polynomial would be

$$r(C, x) = (1 + x)^8 = \sum_{k=0}^{8} \binom{8}{k} x^k$$

and we find that the number of such one-to-one functions h is

$$\binom{8}{0}8! - \binom{8}{1}7! + \binom{8}{2}6! - \binom{8}{3}5! + \cdots + \binom{8}{8}0!$$

$$= 8!\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{8!}\right]$$

$= d_8$, the number of derangements of 1, 2, 3, . . . , 8.