QUIZ 6

NAME:______________________________ **NCKUid:**____________________

1.Compute the indefinite integral

$$\int \frac{x^2}{\sqrt{9-25x^2}}\,dx$$

Answer :
Let $x = \frac{3}{5}\sin\theta$, $(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2})$, so $dx = \frac{3}{5}\cos\theta\,d\theta$.
Then

$$\begin{aligned}\int \frac{x^2}{\sqrt{9-25x^2}}\,dx &= \int \frac{(\frac{3}{5})^2\sin^2\theta}{3\cos\theta}\left(\frac{3}{5}\cos\theta\,d\theta\right) = \frac{9}{125}\int \sin^2\theta\,d\theta \\ &= \frac{9}{125}\int \frac{1-\cos 2\theta}{2}\,d\theta = \frac{9}{250}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C\end{aligned}$$

We know

$$x = \frac{3}{5}\sin\theta \Rightarrow \theta = \sin^{-1}(\frac{5}{3}x)$$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{5}{3}x\cdot\frac{\sqrt{9-25x^2}}{3} = \frac{10x\sqrt{9-25x^2}}{9}$$

So,

$$\begin{aligned}\frac{9}{250}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C &= \frac{9}{250}\left(\sin^{-1}(\frac{5}{3}x) - \frac{1}{2}\cdot\frac{10x\sqrt{9-25x^2}}{9}\right) + C \\ &= \frac{9}{250}\sin^{-1}\frac{5}{3}x - \frac{x\sqrt{9-25x^2}}{50} + C\end{aligned}$$

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2.Evaluate the improper integral

$$\int_0^1 \ln x \, dx$$

Answer :
When $x \to 0$, $\ln x \to -\infty$

$$\begin{aligned}
\int_0^1 \ln x \, dx &= \lim_{t \to 0^+} \int_t^1 \ln x \, dx \left(\text{Let } u = \ln x, dv = dx, \text{then } du = \frac{1}{x} dx, v = x \right) \\
&= \lim_{t \to 0^+} \left([x \ln x]_t^1 - \int_t^1 x \cdot \frac{1}{x} dx \right) \\
&= \lim_{t \to 0^+} \left(0 - t \ln t - x|_t^1 \right) \\
&= \lim_{t \to 0^+} \left(-t \ln t - (1 - t) \right) \\
&= \lim_{t \to 0^+} \left(-t \ln t - 1 \right)
\end{aligned}$$

Consider

$$\lim_{t \to 0^+} t \ln t = \lim_{t \to 0^+} \frac{\ln t}{\frac{1}{t}}$$

(1) This is an indeterminate form as

$$f(t) = \ln t \Rightarrow \lim_{t \to 0^+} \ln t = -\infty$$

$$g(t) = \frac{1}{t} \Rightarrow \lim_{t \to 0^+} \frac{1}{t} = \infty$$

(2) f(t) and g(t) are differentiable and g'(t)$\neq$0
(3) The limit after taking derivative is

$$\lim_{t \to 0^+} \frac{f'(t)}{g'(t)} = \lim_{t \to 0^+} \frac{\frac{1}{t}}{-\frac{1}{t^2}} = \lim_{t \to 0^+} -t = 0$$

So,

$$\int_0^1 \ln x \, dx = \lim_{t \to 0^+} \left(-t \ln t - 1 \right) = -1$$