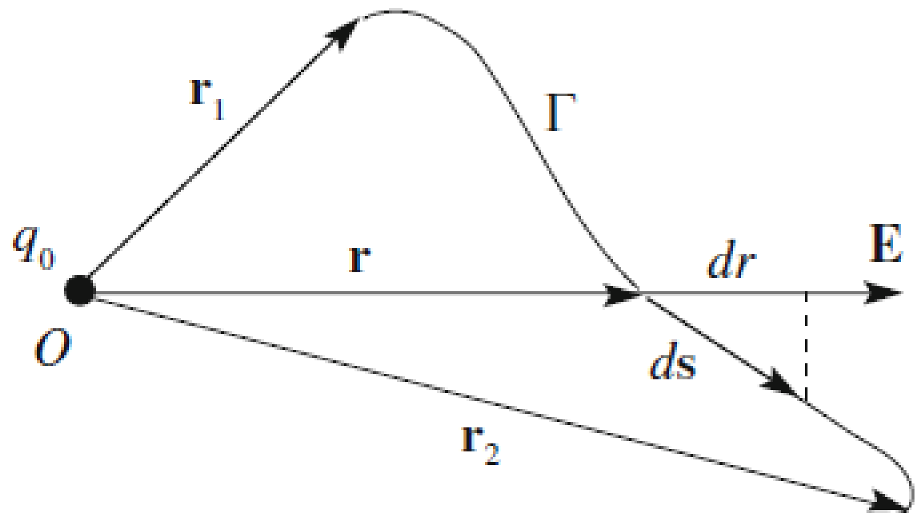
Electric Potential & Electric Potential Energy

Electric Potential Energy

$$\mathbf{F}(\mathbf{r}) = q\frac{q_0}{4\pi\varepsilon_0}\frac{\mathbf{u}_r}{r^2}$$

Potential energy difference = work against the field force

$$W = -\int_{1,\Gamma}^{2} \mathbf{F}\cdot d\mathbf{s} = -q\frac{q_0}{4\pi\varepsilon_0}\int_{1,\Gamma}^{2}\frac{1}{r^2}\mathbf{u}_r\cdot d\mathbf{s} = -q\frac{q_0}{4\pi\varepsilon_0}\int_{1}^{2}\frac{1}{r^2}dr$$



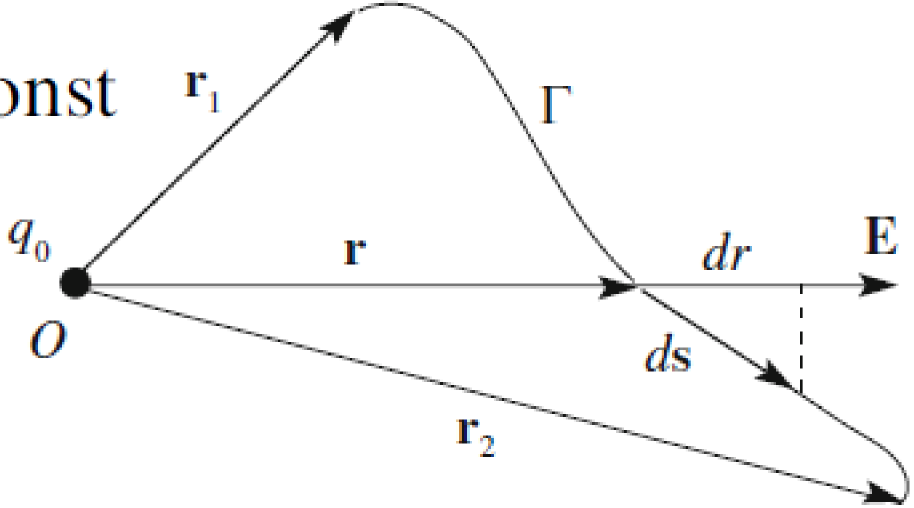
Electric Potential Energy

Potential Energy difference = work against the field force

$$W = U(\mathbf{r}_2) - U(\mathbf{r}_1)$$

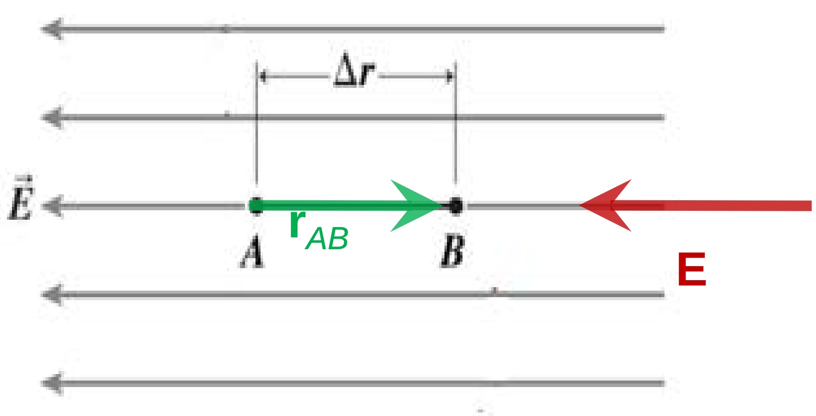
Electric Potential energy U can be defined up to a constant

$$U(\mathbf{r}) = q\frac{q_0}{4\pi\varepsilon_0}\frac{1}{r} + \text{const}$$



Electric Potential Difference

Electric potential difference ≡ electric potential energy difference per unit charge



Electric Potential Difference

Electric potential difference ≡ electric potential energy difference per unit charge

Gravitational Potential Energy: $\Delta U_g = mg\Delta y = m\Delta H$

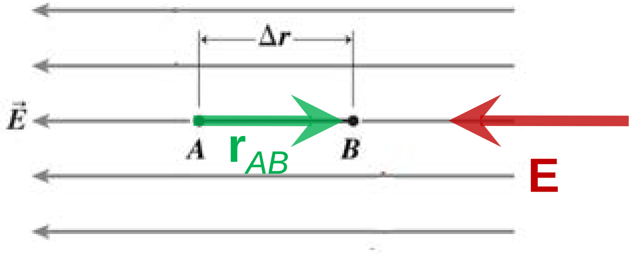
Electric Potential Energy: $\Delta U_e = q\Delta\phi$

$$\Delta\phi = \phi(\mathbf{r}_2) - \phi(\mathbf{r}_1) = -\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E}\cdot d\mathbf{r}$$

$[\,\phi\,]$ = J/C = Volt = V

For a uniform field:

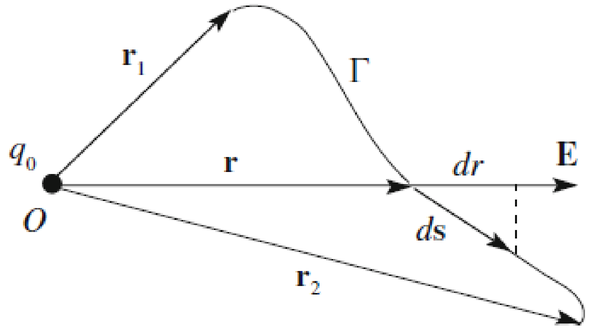
$$\Delta\phi = -\mathbf{E}\cdot(\mathbf{r}_B - \mathbf{r}_A)$$



Electric Potential Difference

The electric potential between r2 and r1

$$\Delta\phi = \phi(\mathbf{r}_2) - \phi(\mathbf{r}_1) = -\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E} \cdot d\mathbf{r}$$



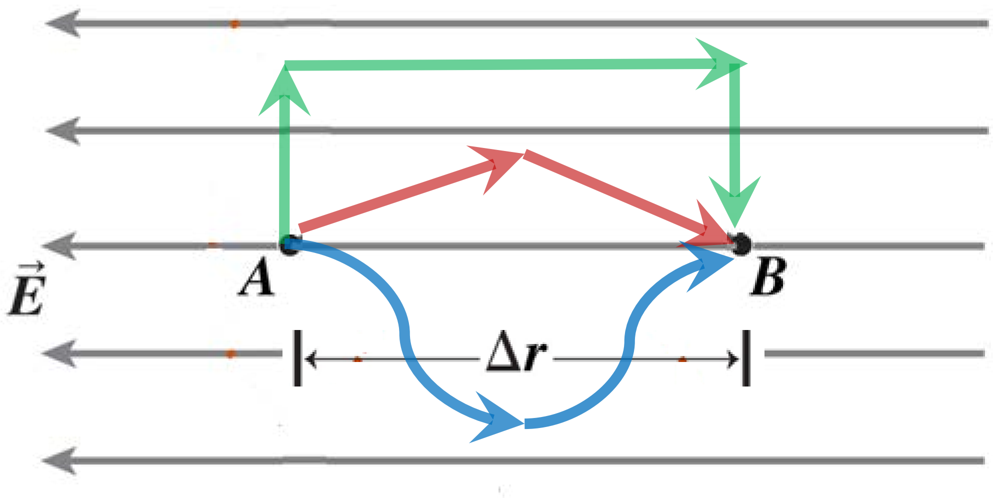
$$\phi(\mathbf{r}_2) - \phi(\mathbf{r}_1) = \frac{q_0}{4\pi\varepsilon_0}\frac{1}{r_2} - \frac{q_0}{4\pi\varepsilon_0}\frac{1}{r_1}$$

The potential at a point r generated by q0 at the origin

$$\phi(\mathbf{r}) = \frac{q_0}{4\pi\varepsilon_0}\frac{1}{r} + \text{const}$$

Electric Potential Difference

Potential difference ΔV_{AB} depends only on positions of *A* & *B*.



Calculating along any paths (1, 2, or 3) gives $\Delta V_{AB} = E\,\Delta r$.

The Volt & the Electronvolt

$[\phi] = J/C = \text{Volt} = V$

$$\Delta U = q\Delta\phi$$

E.g., for a 12V battery, 12J of work is done on every 1C charge that moves from its negative to its positive terminals.

Voltage = potential difference when no **B**(t) is present.

Electronvolt (eV) = energy gained by a particle carrying 1 elementary charge when it moves through a potential difference of 1 volt.

1 elementary charge = 1.6×10^{-19} C = e

1 eV = 1.6×10^{-19} J

Typical Potential Differences

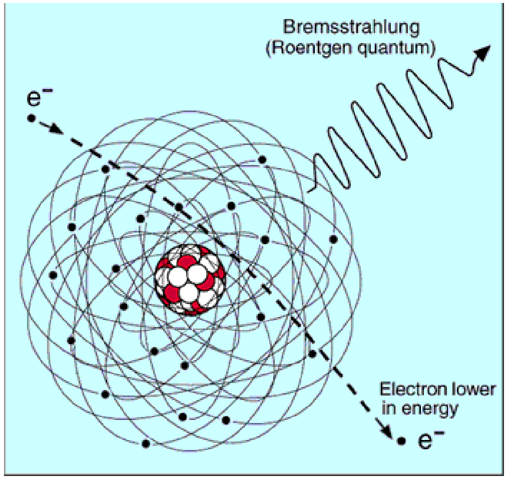
Between human arm & leq due to heart’s electrical activity	1 mV
Across biological cell membrane	80 mV
Between terminals of flashlight battery	1.5 V
Car battery	12 V
Electric outlet (depends on country)	100-240 V
Between lon-distance electric transmission line & ground	365 kV
Between base of thunderstorm cloud & ground	100 MV

Example: Cathode ray tube & X ray

In an X-ray tube, a uniform electric field of 300 kN/C extends over a distance of 10 cm, from an electron source to a target; the field points from the target towards the source.

Find the potential difference between source & target and the energy gained by an electron as it accelerates from source to target (where its abrupt deceleration produces X-rays).

Express the energy in both electronvolts & joules.



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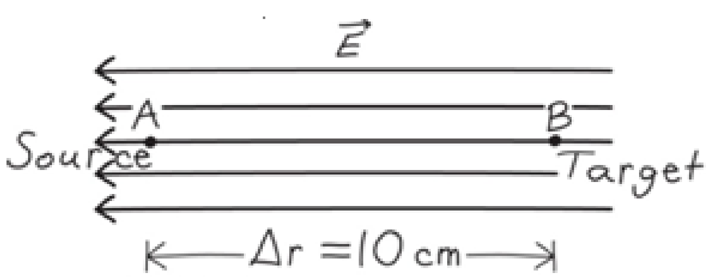


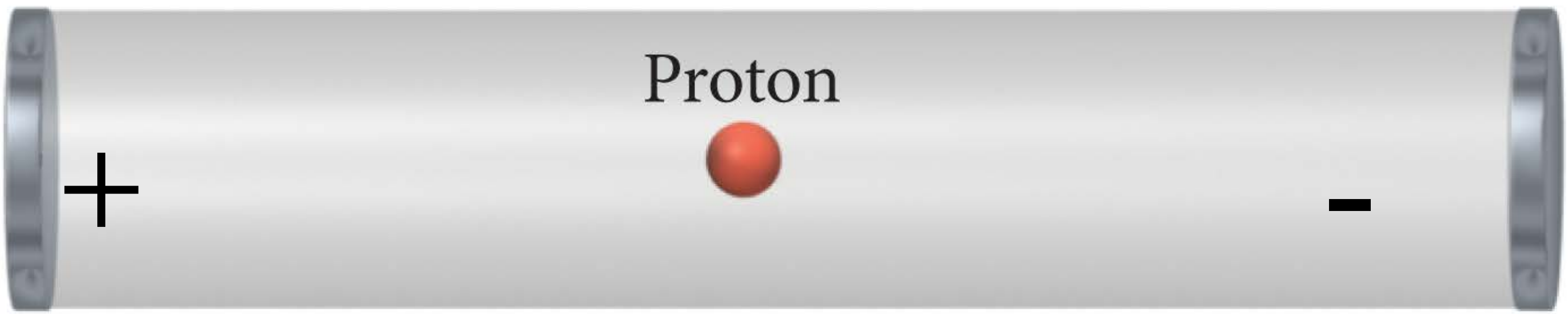
$$\Delta V_{AB} = -(-E)\,\Delta r = (300\ kN/C)(0.10\ m) = 30\ kV$$

$$\Delta U_{AB} = -e\,\Delta V_{AB} = -30\ keV$$

$$\Delta K_{AB} = -\Delta U_{AB} = 30\ keV = 4.8\times10^{-15}\ J = 4.8\ fJ$$

Example: Particle accelerator

- A proton enters the region between two parallel plates a distance 20cm apart. The uniform electric field is $3 \times 10^5 V/m$. If the initial speed of the proton is $5 \times 10^6\ m/s$, what is its final speed?



Defining the reference of $\phi(r)$

- Electric field of a point-charge field

$$\vec{E} = \frac{kq}{r^2}\hat{r} \qquad \Delta\phi_{\infty r} = \phi(r) - \phi(\infty) = \frac{kq}{r}$$

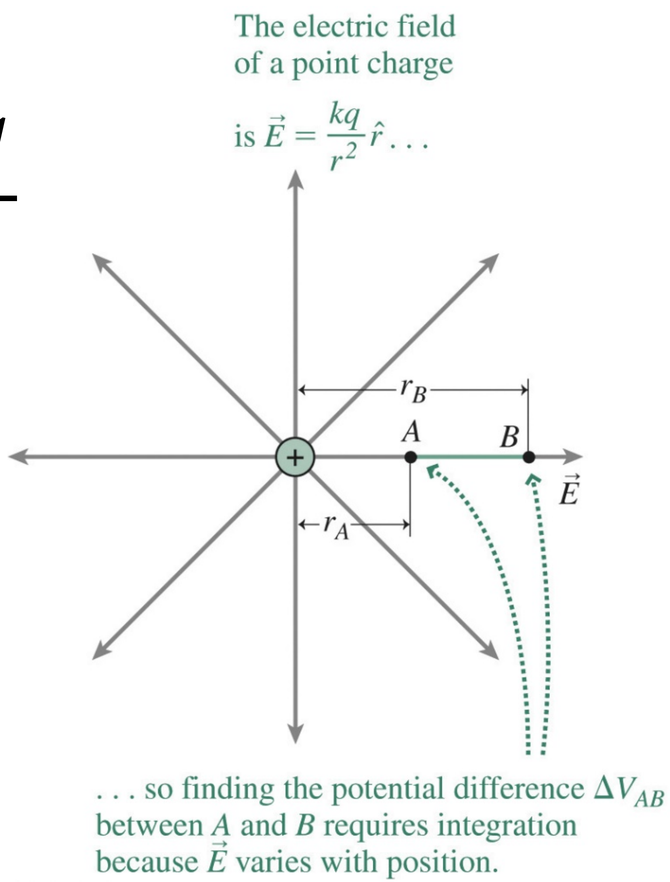
  - The potential of a point charge

$$\Delta\phi_{AB} = \phi_B - \phi_A = \frac{kq}{r_B} - \frac{kq}{r_A}$$

  - Taking the zero of potential at infinity gives

$$\Delta\phi_{\infty r} \equiv \phi(r) = \frac{kq}{r}$$

    for the potential difference between infinity and any point a distance *r* from the point charge.



Potential from multiple charges

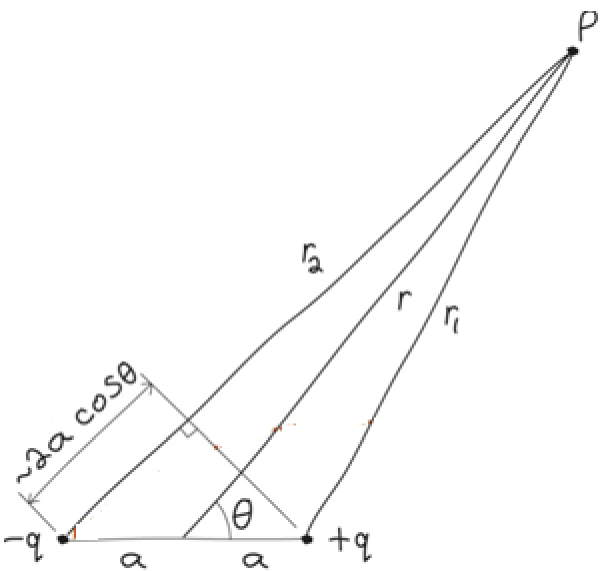
- The potential at a point r_0 due to n charges $q_1 \; in \; r_1, q_2 \; in \; r_2, \dots, q_n \; in \; r_n$

$$\phi(\mathbf{r_0}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_{i0}} + \mathrm{const}$$

Dipole Potential

An electric dipole consists of point charges $\pm q$ a distance $2a$ apart.

Find the potential at an arbitrary point P, and approximate for the case where the distance to P is large compared with the charge separation.



$$V(P) = k\frac{q}{r_1} + k\frac{(-q)}{r_2} \quad = kq\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \quad = kq\,\frac{r_2 - r_1}{r_2\, r_1}$$

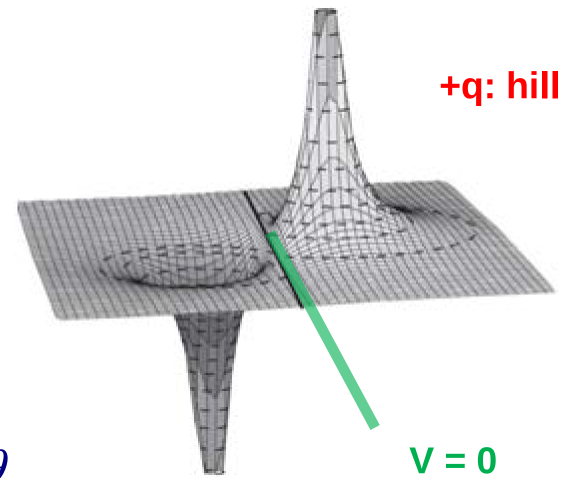
$$r_1^2 = r^2 + a^2 - 2r\,a\,\cos\theta$$

$$r_2^2 = r^2 + a^2 + 2r\,a\,\cos\theta$$

$$r_2^2 - r_1^2 = 4\,r\,a\,\cos\theta$$

$$= (r_2 - r_1)(r_1 + r_2)$$

$r >> a \;\rightarrow\quad r_2 - r_1 \approx 2\,a\,\cos\theta$



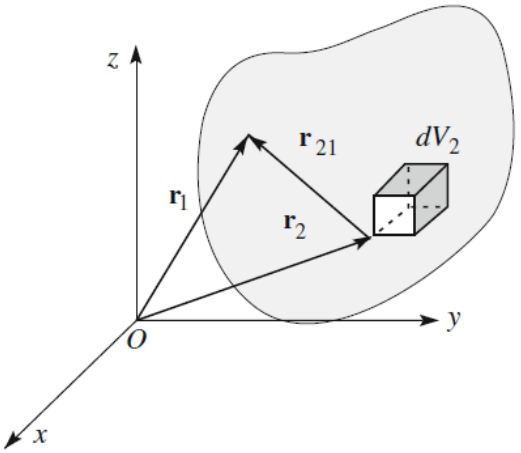
$$V(P) \approx k\,q\,\frac{r_2 - r_1}{r^2} \quad = k\,\frac{2\,qa\cos\theta}{r^2} \quad = k\,\frac{p\cos\theta}{r^2}$$

$p = 2qa$ = dipole moment

Potential Difference of a Charge Distribution

- The potential at a point $r_1 = (x_1, y_1, z_1)$ due to a continuous charge distribution with density $\rho(x_2, y_2, z_2)$

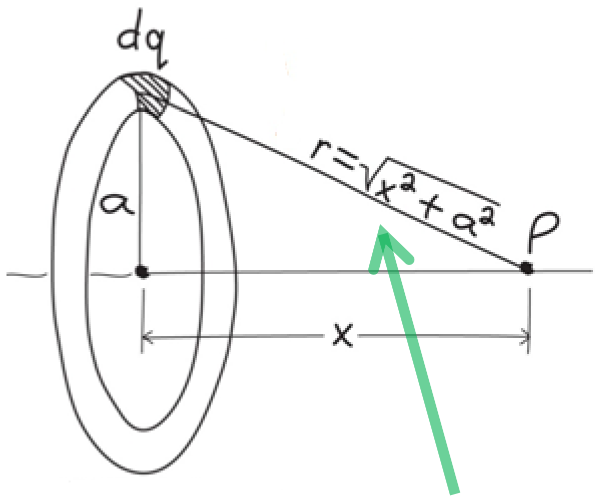
$$\phi(\mathbf{r_0}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(x_2, y_2, z_2)}{r_{21}} dV_2 + \text{const}$$



Charged Ring

A total charge Q is distributed uniformly around a thin ring of radius a.

Find the potential on the ring's axis.



Same r for all dq

$$V(x)=\int k\,\frac{dq}{r} \quad = \frac{k}{\sqrt{x^2+a^2}}\int dq$$

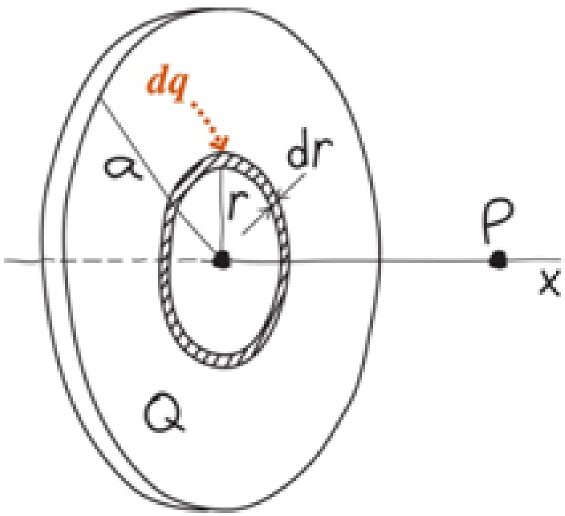
$$= \frac{k}{\sqrt{x^2+a^2}}Q$$

$$\approx \frac{kQ}{|x|} \quad if\ |x| \gg a$$

Charged Disk

A charged disk of radius *a* carries a charge *Q* distributed uniformly over its surface.

Find the potential at a point *P* on the disk axis, a distance *x* from the disk.



$$\mathrm{d}V(x) = \frac{k}{\sqrt{x^2 + r^2}} dQ$$

$$dQ = \sigma dA = \sigma(2\pi r) dr$$

$$\int_0^a \mathrm{d}V(x) = \int_0^a \frac{k\sigma(2\pi r)}{\sqrt{x^2 + r^2}} dr$$

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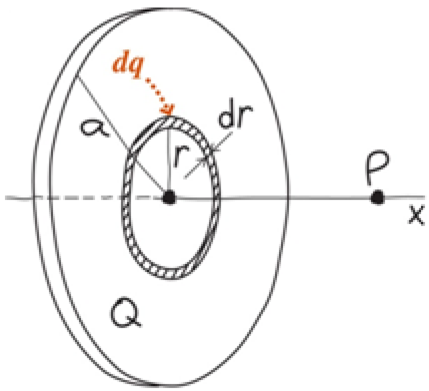
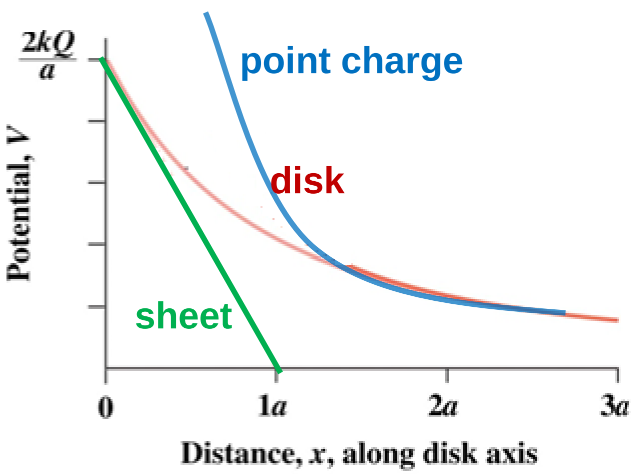


$$V(x) = \int dV \quad = \int \frac{k}{\sqrt{x^2 + r^2}}\, dq$$

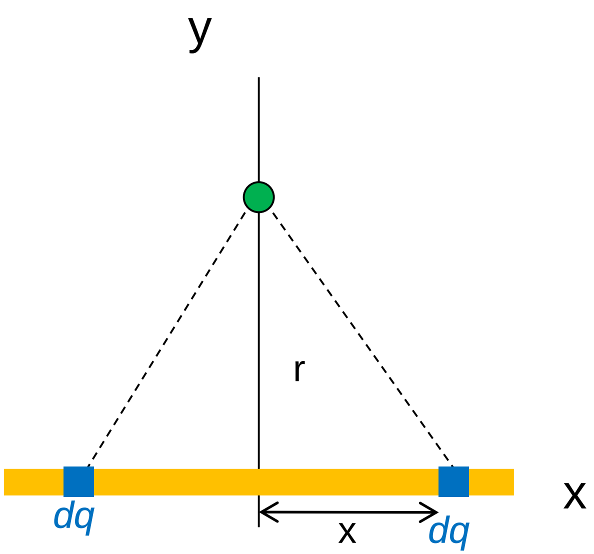
$$= \int_0^a \frac{k}{\sqrt{x^2 + r^2}} \left(\frac{Q}{\pi a^2} \right) 2\pi\, r\, dr$$

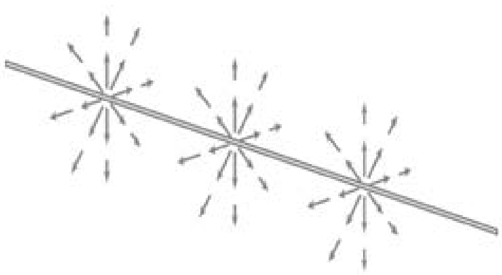
$$= \frac{2k\,Q}{a^2} \left(\sqrt{x^2 + a^2} - |x| \right)$$

$$\approx \begin{cases} \dfrac{2k\,Q}{a^2}\left(a - |x|\right) = \dfrac{2kQ}{a} - \dfrac{\sigma}{2\varepsilon_0}|x| & x \ll a \\ \dfrac{k\,Q}{|x|} & x \gg a \end{cases}$$



Linear distributed charges





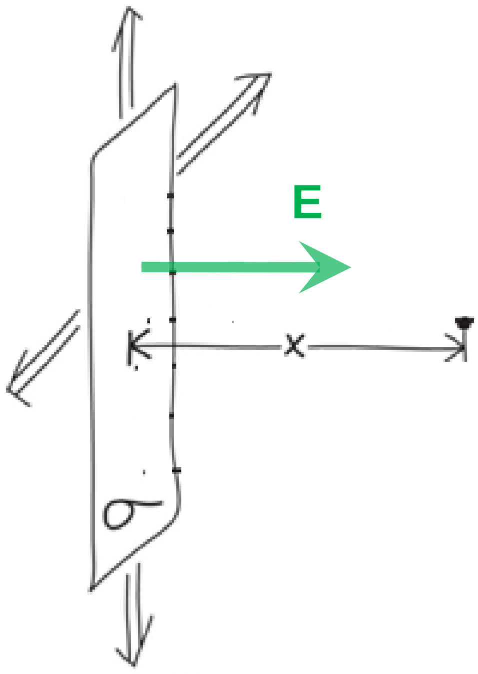
$$\phi(r') - \phi(r'_0)$$

$$= -\int_{r'_0}^{r'} \mathbf{E}.d\mathbf{s} = -\frac{\lambda}{2\pi\varepsilon_0}\int_{r'_0}^{r'} \frac{dr'}{r'}$$

$$= -\frac{\lambda}{2\pi\varepsilon_0}\ln r' + \frac{\lambda}{2\pi\varepsilon_0}\ln r'_0.$$

Charged Sheet

An isolated, infinite charged sheet carries a uniform surface charge density σ. Find an expression for the potential difference from the sheet to a point a perpendicular distance *x* from the sheet.



$$\Delta V_{0x} = -E\left(x-0\right) \quad = -\frac{\sigma}{2\varepsilon_0}x$$

Science Museum

The Hall of Electricity at the Boston Museum of Science contains a large Van de Graaff generator, a device that builds up charge on a metal sphere.

The sphere has radius R = 2.30 m and develops a charge Q = 640 μC.

Considering this to be a single isolate sphere, find

(a) the potential at its surface,

(b) the work needed to bring a proton from infinity to the sphere's surface,

(c) the potential difference between the sphere's surface & a point $2R$ from its center.

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(b) the work needed to bring a proton from infinity to the sphere's surface,

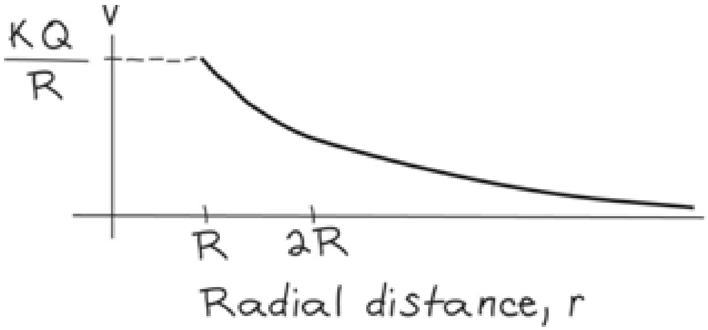
(c) the potential difference between the sphere's surface & a point 2R from its center.

(a) $$V(R) = k\frac{Q}{R} = \left(9.0\times10^{9}\ Vm/C\right)\frac{\left(640\times10^{-6}\ C\right)}{2.30\ m} = 2.50\ MV$$

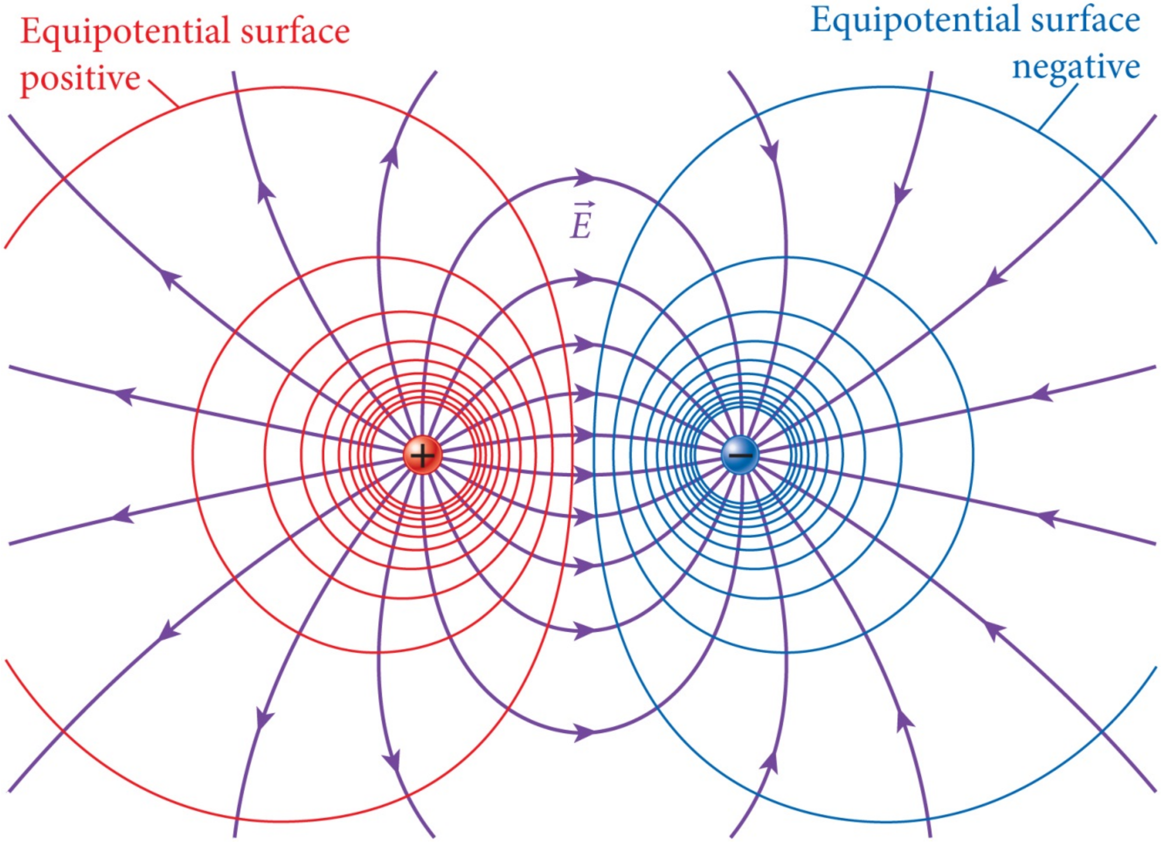
(b) $$W = eV(R) = 2.50\ MeV = \left(1.6\times10^{-19}\ C\right)\left(2.50\ MV\right) = 4.0\times10^{-13}\ J$$

(c) $$\Delta V_{R,2R} = V(2R) - V(R) = k\frac{Q}{2R} - k\frac{Q}{R}$$

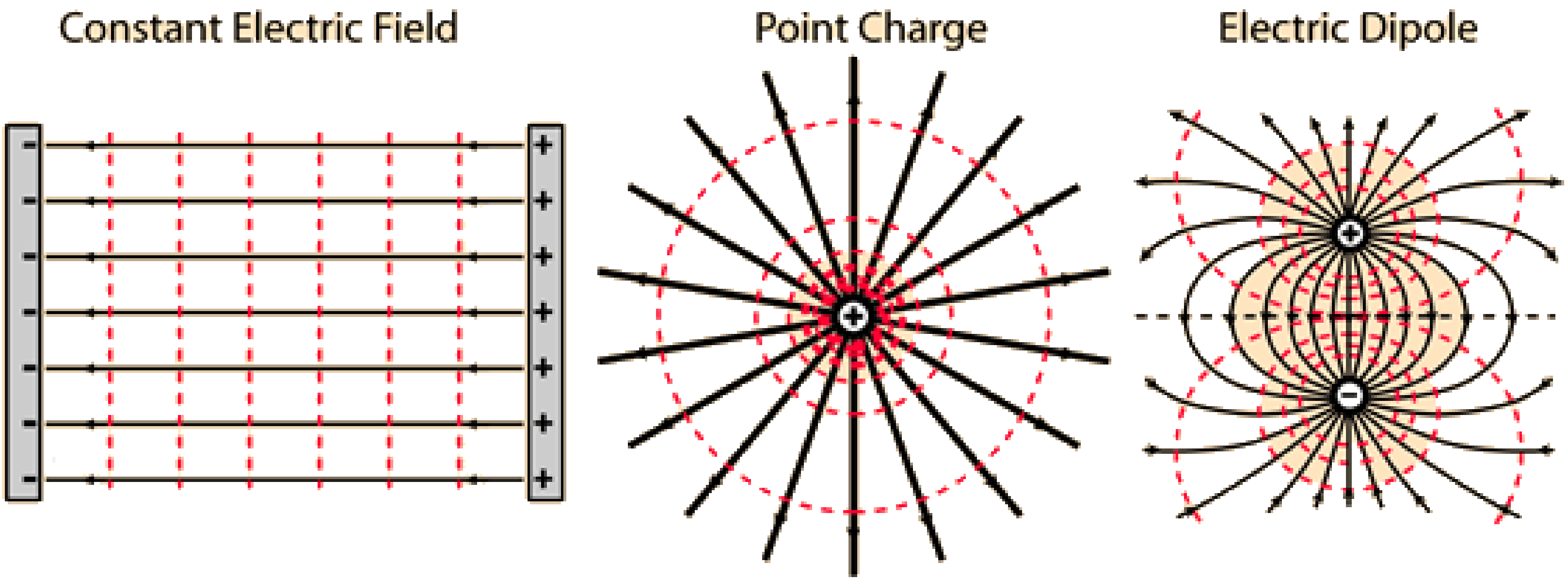
$$= -k\frac{Q}{2R} = -1.25\ MV$$



Equipotential



Equipotential and Field



Dashed lines are equipotential lines while solid lines are electric field lines.
Click on one of the diagrams for further detail.

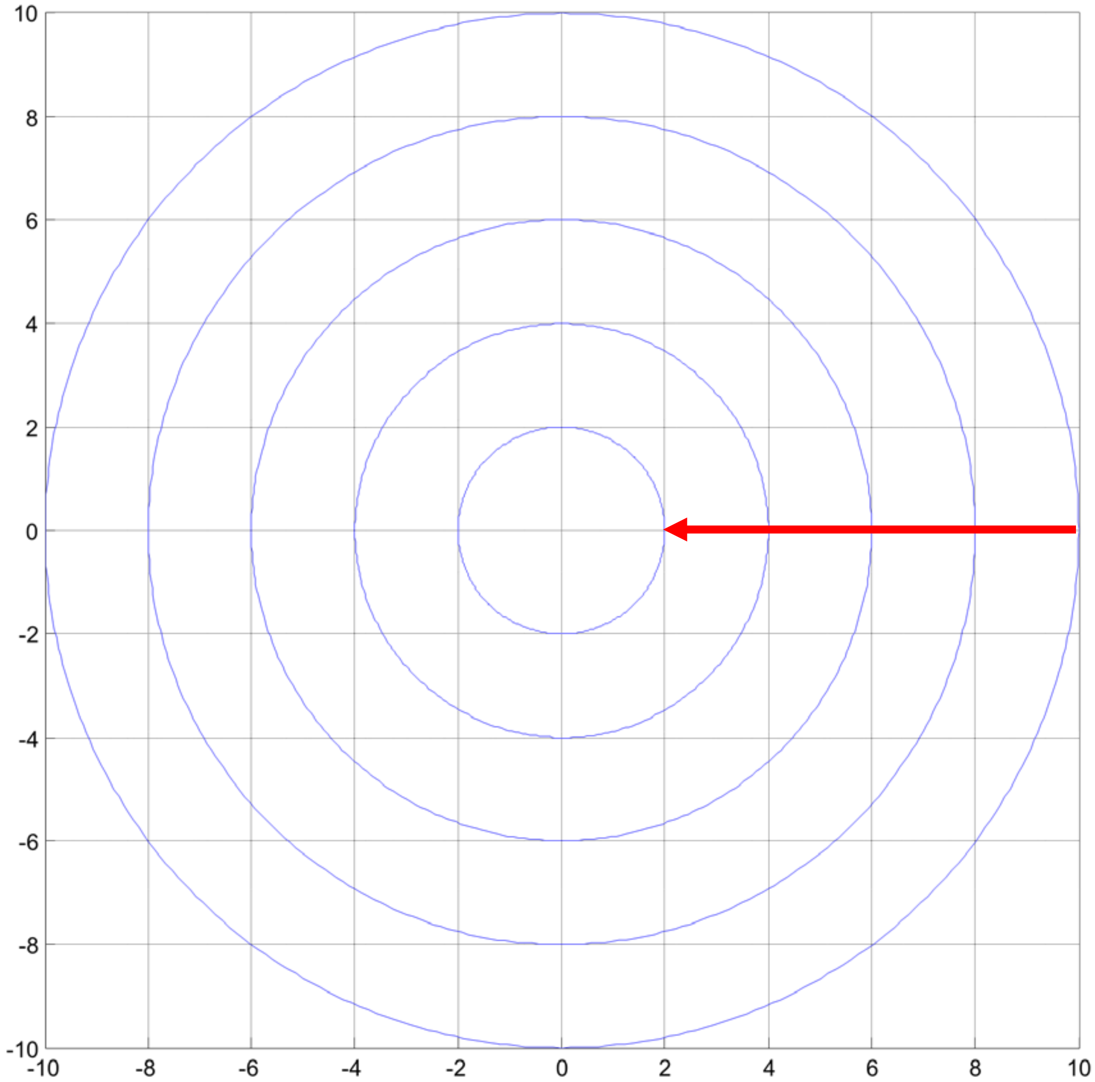
Gradient Theorem

aka fundamental theorem of line integral

$$\phi(\mathbf{r}_2) - \phi(\mathbf{r}_1) = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \nabla\phi \cdot d\mathbf{r}$$

$$\nabla\phi = \frac{\partial\phi(x,y,z)}{\partial x}\hat{\imath} + \frac{\partial\phi(x,y,z)}{\partial x}\hat{\jmath} + \frac{\partial\phi(x,y,z)}{\partial x}\hat{k}$$

$$= \left(\frac{\partial\phi(x,y,z)}{\partial x}, \frac{\partial\phi(x,y,z)}{\partial y}, \frac{\partial\phi(x,y,z)}{\partial z}\right)$$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6
8
10

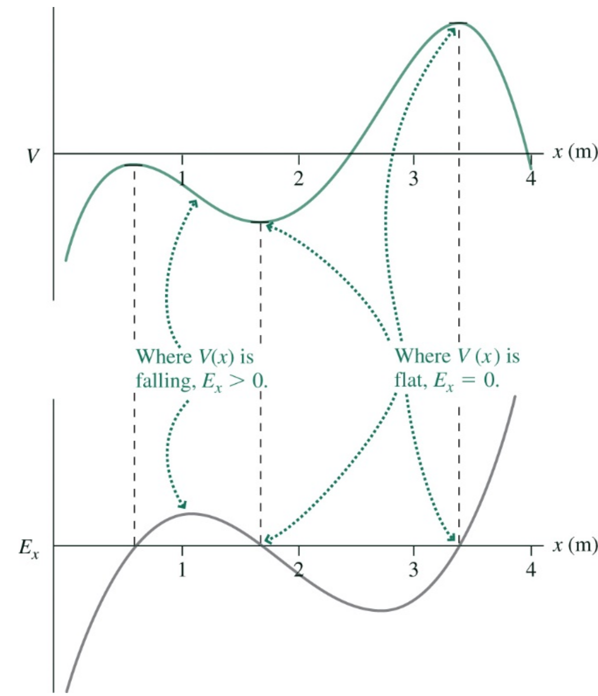
Potential Difference and the Electric Field

- Potential difference involves an integral over the electric field.
- So the field involves derivatives of the potential.
  - Specifically, the component of the electric field in a given direction is the negative of the rate of change (the derivative) of potential in that direction.

  - Then, given potential *V* (a scalar quantity) as a function of position, the electric field (a vector quantity) follows from

$$\vec{E} = -(\frac{\partial}{\partial x}V(\vec{r})\hat{x} + \frac{\partial}{\partial x}V(\vec{r})\hat{y} + \frac{\partial}{\partial x}V(\vec{r})\hat{k})$$

  - The derivatives here are **partial derivatives**, expressing the variation with respect to one variable alone.
    - This approach may be used to find the field from the potential.
      - Potential is often easier to calculate, since it's a scalar rather than a vector.



Force & Field, Potential Energy & Electric Potential

Quantity	Symbol / Equation	Units
Force	**F**	N
Electric field	**E** = **F** / *q*	N/C or V/m
Potential energy difference	$\Delta U = -\int_{r_0}^{r} \mathbf{F} \cdot d\mathbf{r}$ $\mathbf{F} = -\nabla U$	J
Electric potential difference	$\Delta \phi = \frac{\Delta U}{q}$ $\Delta \phi = -\int_{r_0}^{r} \mathbf{E} \cdot d\mathbf{r}$ $\mathbf{E} = -\nabla \phi$	J/C or V

Consider 2 widely separated, charged conducting spheres.

Their potentials are

$$V_1 = k\frac{Q_1}{R_1} \qquad V_2 = k\frac{Q_2}{R_2}$$

Same V

If we connect them with a thin wire,

there'll be charge transfer until $V_1 = V_2$, i.e., $\frac{Q_1'}{R_1} = \frac{Q_2'}{R_2}$

In terms of the surface charge densities $\sigma_j = \frac{Q_j'}{4\pi R_j^2}$

we have $\sigma_1 R_1 = \sigma_2 R_2 \quad \rightarrow \quad E_{1\perp} R_1 = E_{2\perp} R_2$

$\therefore$ Smaller sphere has higher field at surface.

An Irregular Conductor

Sketch some equipotentials & electric field lines for an isolated egg-shaped conductor.

Ans.

Surface is equipotential $\rightarrow$ $|E_\perp|$ is larger where curvature of surface is large.

$\therefore$ More field lines emerging from sharply curved regions.

From afar, conductor is like a point charge.