HW4

HW4 總分 30 分

題目為設計一組題目，這一個題目無法被一般數學歸納法證明，並使用強數學歸納法嚴謹的證明設計正確。

1. 證明設計的這組題目無法被一般數學歸納法證明(5 分)
2. 強數學歸納法 (20 分)
   - Basis step <10 分>
   - n=n0 時，命題成立 <5 分>
   - 假設 n0<=n<=K 命題成立時，n=k+1 命題亦成立 <5 分>
3. 其他細節 (5 分)

→ Please design a proposition and prove it by strong mathematics induction. (Cannot be proved by general mathematics induction).

→ Proposition: if $n \in N$, then $12 \mid (n^4 - n^2)$.

→ General mathematics induction:

When $n=1$, $12 \mid (n^4 - n^2) = 0$ ⇒ true.

Assume that $12 \mid (k^4 - k^2)$ is true, such that $(n^4 - n^2) = 12a$ for some $a \in \mathbb{N}$

When $n = k+1$,

$$(k+1)^4 - (k+1)^2 = k^4 + 4k^3 + 6k^2 + 4k + 1 - (k^2 + 2k + 1) = k^4 + 4k^3 + 5k^2 + 2k$$

$$= (k^4 \ k^2) + 4k^3 + 6k^2 + 2k$$

$$= 12a + 2(2k^3 + 3k^2 + k)$$

⇒ leads to nothing

→ Strong mathematical induction.

Base case:

$n=1 \Rightarrow 12 \mid (1^4 - 1^2) = 0 = 12 * 0$

$n=2 \Rightarrow 12 \mid (2^4 - 2^2) = 12 = 12 * 1$

$n=3 \Rightarrow 12 \mid (3^4 - 3^2) = 72 = 12 * 6$

$n=4 \Rightarrow 12 \mid (4^4 - 4^2) = 240 = 12 * 20$

$n=5 \Rightarrow 12 \mid (5^4 - 5^2) = 600 = 12 * 50$

$n=6 \Rightarrow 12 \mid (6^4 - 6^2) = 1260 = 12 * 105$

↓ 背

Induction step:

let $k \geq 6 \in \mathbb{N}$ and assume that $12 \mid (m^4 - m^2)$ for $1 \leq m \leq k$

when $n = (k+1)$ $\rightarrow$ prove $12 \mid [(k+1)^4 - (k+1)^2]$ is true.

Define $m = k-5$, according to the proposition,

$m^4 - m^2 = 12a$ for some value of a.

Therefore, I only have to prove $12 \mid (m+6)^4 - (m+6)^2$ is true.

$$(m+6)^4 - (m+6)^2$$
$$= m^4 + 24m^3 + 180m^2 + 864m + 1296 - (m^2 + 12m + 36).$$
$$= m^4 + 24m^3 + 179m^2 + 852m + 1260$$
$$= (m^4 - m^2) + 24m^3 + 180m^2 + 852m + 1260$$
$$= 12a + 12(2m^3 + 15m^2 + 71m + 105)$$

↳ is clearly divisible by 12.

Proposition : if $n \in N$, then $12 \mid (n^4 - n^2)$

If we use weak induction.....

(Base) when $n=1$ $\quad 12 \mid 1^4 - 1^2 = 12 \mid 0 = 0$, true

(Induction) Assume $12 \mid (k^4 - k^2)$ is true , which means $(n^4 - n^2) = 12a$ for some $a \in N$

For $k+1$, $(k+1)^4 - (k+1)^2 = k^4 + 4k^3 + 6k^2 + 4k + 1 - k^2 - 2k - 1 = (k^4 - k^2) + 4k^3 + 6k^2 + 2k = \underline{12a + 4k^3 + 6k^2 + 2k}$

↳無法證明是12的倍數

If we use strong induction.....

(Base) $n=1 : 12 \mid (1^4 - 1^2) = 12 \mid (1-1) = 0$

$n=2 : 12 \mid (2^4 - 2^2) = 12 \mid 16 - 4 = 12$

$n=3 : 12 \mid (3^4 - 3^2) = 12 \mid 81 - 9 = 72 = 6 \times 12$

$n=4 : 12 \mid (4^4 - 4^2) = 12 \mid 256 - 16 = 240 = 20 \times 12$

$n=5 : 12 \mid (5^4 - 5^2) = 12 \mid 625 - 25 = 600 = 50 \times 12$

$n=6 : 12 \mid (6^4 - 6^2) = 12 \mid 1296 - 36 = 1260 = 105 \times 12$

(Induction)

let $k \geq 6$ and $k \in N$, and we assume that $12 \mid (m^4 - m^2)$ for $1 \leq m \leq k$

We define $r = k - 5$, from our assumption we say $(r^4 - r^2) = 12a$ for some $a \in N$

Now we need to prove that $12 \mid (k+1)^4 - (k+1)^2$ is true ($k+1$ 的情況)

and $k+1 = r+6$

$(k+1) \Rightarrow (r+6)^4 - (r+6)^2 = r^4 + 24r^3 + 216r^2 + 864r + 1296 - (r^2 + 12r + 36) = (r^4 - r^2) + 24r^3 + 216r^2 + 852r + 1260$

$= \underline{12a + 12(2r^3 + 18r^2 + 71r + 105)}$

So we prove the proposition.