Chapter 3.
Higher-Order Differential Equations

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Review

例: $y'' + 6y' + 5y = 0$

$\lambda^2 + 6\lambda + 5 = 0$

$\lambda = -1, -5$

$\Rightarrow y = C_1 e^{-x} + C_2 e^{-5x}$ ← 這個 y 真的是解嗎?

Review

$$y'' + ay' + by = 0 \qquad a,b \in const.$$

$$\lambda^2 + a\lambda + b = 0$$

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

$$\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0$$

$$\Rightarrow a = -(\lambda_1 + \lambda_2), \quad b = \lambda_1\lambda_2$$

$$y'' - (\lambda_1 + \lambda_2)y' + \lambda_1\lambda_2 y = 0$$

Differential Operator

- Def.

$$D \equiv \frac{d}{dx}(\text{微分運算子})$$

$$D^k \equiv \frac{d^k}{dx^k}$$

$$D^2 y - (\lambda_1 + \lambda_2)Dy + \lambda_1\lambda_2 y = 0$$

$$\left(D^2 - (\lambda_1 + \lambda_2)D + \lambda_1\lambda_2\right)y = 0$$

$$(D - \lambda_1)\underbrace{(D - \lambda_2)y}_{= z} = 0$$

$$z = k_1 e^{\lambda 1 x}$$

$$\begin{cases} (D - \lambda_1)z(x) = 0 \\ (D - \lambda_2)z(x) = 0 \end{cases}$$

Differential Operator

$$令(D-\lambda_2)y = z$$

$(D-\lambda_1)z = 0$ −>還原 $Dz - \lambda_1 z = 0$ $\quad \frac{dz}{dx} - \lambda_1 z = 0$

$$z' - \lambda_1 z = 0$$

$$z = k_1 e^{\lambda_1 x}$$

$\Rightarrow (D-\lambda_2)y = \overset{z}{\overset{\shortparallel}{(k_1 e^{\lambda_1 x})}}$ −>還原 $y' - \lambda_2 y = z(x)$

$$y' - \lambda_2 y = k_1 e^{\lambda_1 x}$$

$$I = e^{\int -\lambda_2 dx} = e^{-\lambda_2 x}$$

Differential Operator

$$\Rightarrow y = k_2 I^{-1} + I^{-1}\int I k_1 e^{\lambda_1 x} dx$$

$$= k_2 e^{\lambda_2 x} + e^{\lambda_2 x}\int k_1 e^{(\lambda_1 - \lambda_2)x} dx = k_2 e^{\lambda_2 x} + e^{\lambda_2 x}\frac{k_1}{\lambda_1 - \lambda_2} e^{(\lambda_1 - \lambda_2)x}$$

$$= k_2 e^{\lambda_2 x} + \frac{k_1}{\lambda_1 - \lambda_2} e^{\lambda_1 x}$$

ps. $\lambda_1 \neq \lambda_2$

C_2 $\qquad$ C_1

Differential Operator

Case(3):

$\lambda_1 = \lambda_2 = \alpha$ (重根)

$\therefore (\lambda - \alpha)(\lambda - \alpha) = 0$ (特性方程式)

$\lambda^2 - 2\alpha\lambda + \alpha^2 = 0$

$\Rightarrow y'' - 2\alpha y' + \alpha^2 y = 0$ (原D.E.)

$y = c_1 e^{\alpha x} + c_2 ?$

$D^2 y - 2\alpha D y + \alpha^2 y = 0$

$(D - \alpha)(D - \alpha) y = 0$

令$(D - \alpha) y = z$

$(D - \alpha) z = 0$

$z' - \alpha z = 0$

$z = C_1 e^{\alpha x}$

Differential Operator

$\Rightarrow (D - \alpha) y = C_1 e^{\alpha x}$

$y' - \alpha y = C_1 e^{\alpha x}$

$I = e^{\int -\alpha dx} = e^{-\alpha x}$

$\Rightarrow y = C_2 e^{\alpha x} + e^{\alpha x} \int e^{-\alpha x} C_1 e^{\alpha x} dx$

$= C_2 e^{\alpha x} + C_1 x e^{\alpha x}$

$\Rightarrow y = C_2 e^{\alpha x} + C_1 x e^{\alpha x}$

Differential Operator

例: $y'' - 2y' + y = 0$

$\lambda^2 - 2\lambda + 1 = 0$

$\lambda = 1,1$

$\Rightarrow y = C_1 e^x + C_2 x e^x$

例: $y'' + 4y' + 13y = 0$

$\lambda^2 + 4\lambda + 13 = 0$

$\lambda = -2 \pm 3i$

$\Rightarrow y = e^{-2x}\left(C_1 \cos 3x + C_2 \sin 3x\right)$

Differential Operator

例: $y'' + 6y' + 8y = 0$

$\lambda^2 + 6\lambda + 8 = 0$

$\lambda = -4, -2$

$\Rightarrow y = C_1 e^{-4x} + C_2 e^{-2x}$

例: $y'' + 10y' + 25y = 0$

$\lambda^2 + 10\lambda + 25 = 0$

$\lambda = -5, -5$

$\Rightarrow y = C_1 e^{-5x} + C_2 x e^{-5x}$

N-Order Constant Coefficients D.E

- 推廣N階常係數 O.D.E.

Def:

$$y^{(n)} \equiv \frac{d^n y}{dx^n}$$

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y = 0$$

$$a_1, a_2, \cdots, a_n \in const.$$

N-Order Constant Coefficients D.E

Case(1) 相異實根

$$\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n \in \Re$$

$$\Rightarrow y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \cdots + C_n e^{\lambda_n x}$$

Case(2) 相等實根

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = \alpha \in \Re$$

$$\Rightarrow y = C_1 e^{\alpha x} + C_2 x e^{\alpha x} + \cdots + C_n x^{n-1} e^{\alpha x}$$

N-Order Constant Coefficients D.E

Case(3) 共軛複數根

$$\lambda_1, \lambda_2, \cdots, \lambda_{2k} \quad n = 2k$$

$$\alpha_j \pm \beta_j i \quad j = 1, 2, \cdots, k$$

$$\Rightarrow y = e^{\alpha_1 x}\left(C_1 \cos \beta_1 x + C_2 \sin \beta_1 x\right)$$

$$+ e^{\alpha_2 x}\left(C_3 \cos \beta_2 x + C_4 \sin \beta_2 x\right)$$

$$\cdots$$

$$+ e^{\alpha_k x}\left(C_{2k-1} \cos \beta_k x + C_{2k} \sin \beta_k x\right)$$

N-Order Constant Coefficients D.E

Case(4)共軛複數根重根

$(\alpha \pm \beta i)^k$ $\quad k$個重根

$$\Rightarrow y = e^{\alpha x}\left(C_1 \cos \beta x + C_2 \sin \beta x\right)$$

$$+ x e^{\alpha x}\left(C_3 \cos \beta x + C_4 \sin \beta x\right)$$

$$\cdots$$

$$+ x^{k-1} e^{\alpha x}\left(C_{2k-1} \cos \beta x + C_{2k} \sin \beta x\right)$$

P

N-Order Constant Coefficients D.E

例: 設一微分方程式的特性方程式的根分別為：

$$x_{1\sim16}=1,\ 2,\ 3,\ 4,\ 4,\ 4,\ -2\pm 3i,\ -3\pm 2i,\ (-1\pm 5i)^3$$

$$Y=?$$

Sol:

$$\begin{aligned}y&=C_1e^{x}+C_2e^{2x}+C_3e^{3x}+C_4e^{4x}+C_5xe^{4x}\\&+C_6x^2e^{4x}+e^{-2x}\left(C_7\cos 3x+C_8\sin 3x\right)\\&+e^{-3x}\left(C_9\cos 2x+C_{10}\sin 2x\right)\\&+e^{-x}\left(C_{11}\cos 5x+C_{12}\sin 5x\right)\\&+xe^{-x}\left(C_{13}\cos 5x+C_{14}\sin 5x\right)\\&+x^2e^{-x}\left(C_{15}\cos 5x+C_{16}\sin 5x\right)\end{aligned}$$

Determine y_p

- 如何決定 y_p

例: $y'+2y=e^{3x}$

$$y_h\Rightarrow y'+2y=0$$

$$\lambda+2=0$$

$$\lambda=-2$$

$$y_h=Ce^{-2x}$$

Undetermined Coefficient

- Method 1 : Undetermined Coefficient (未定係數法)

$y_p' + 2y_p = e^{3x}$

猜 $y_p = ke^{3x}$ [依照 r (x) 函數的型式決定 y_p]

代入 $\left(ke^{3x}\right)' + 2\left(ke^{3x}\right) = e^{3x}$

代入要能消，
猜與r(x)相關

$3ke^{3x} + 2ke^{3x} = e^{3x}$

$5ke^{3x} = e^{3x}$

$\Rightarrow k = \frac{1}{5} \quad \therefore y_p = \frac{1}{5}e^{3x}$

Undetermined Coefficient

例: $y'' + 3y' + 2y = e^x$

$y_h \Rightarrow y_h'' + 3y_h' + 2y_h = 0$

$\lambda^2 + 3\lambda + 2 = 0$

$\Rightarrow y_h = C_1e^{-x} + C_2e^{-2x}$

$y_p \Rightarrow$ 猜 $y_p = ke^x$

$\Rightarrow y_p = ke^x$

Undetermined Coefficient

$$y_p'' + 3y_p' + 2y_p = e^x$$

$$y_p' = ke^x$$

$$y_p'' = ke^x$$

$$\Rightarrow ke^x + 3ke^x + 2ke^x = e^x$$

$$6ke^x = e^x \qquad k = \frac{1}{6}$$

$$\therefore y = y_h + y_p = C_1 e^{-x} + C_2 e^{-2x} + \frac{1}{6} e^x$$

Undetermined Coefficient

*考慮二階O.D.E. $\xleftrightarrow[\text{推廣}]{}$ n階 O.D.E.

$y'' + ay' + by = r(x)$ $\qquad$ $y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = r(x)$

I. 依照前述 $e^{\lambda x}$，D 的結果，y_h 可以先決定
II. $r(x)$ 決定 y_p

Undetermined Coefficient

r(x)的函數型式

(1) $e^{\alpha x} <--> y_p = ke^{\alpha x}$ <=> $y_p = k_1 e^{\alpha x} + k_2 e^{\beta x}$

(2) $\cos\beta x \,\&\, \sin\beta x <--> y_p = k_1 \cos\beta x + k_2 \sin\beta x$

(3) $x^k <--> y_p = k_0 x^k + k_1 x^{k-1} + \ldots + k_k$

(4) r $<--> y_p = k$ 猜特解 y_p，將每一項分開猜

(5) $e^{\alpha x}\cos\beta x \,\&\, e^{\alpha x}\sin\beta x <-->$

$y_p = e^{\alpha x}(k_1\cos\beta x + k_2 \sin\beta x)$

Undetermined Coefficient

(6) $e^{\alpha x} x^k <--> y_p = e^{\alpha x}(k_0 x^k + k_1 x^{k-1} + \ldots + k_k)$

(7) $(\cos\beta x)x^k \,\&\, (\sin\beta x)x^k <-->$

$y_p = \cos\beta x(A_0 x^k + A_1 x^{k-1} + \ldots + A_k) +$

$\sin\beta x(B_0 x^k + B_1 x^{k-1} + \ldots + B_k)$

(8) $\cos^2 x = \frac{\cos 2x + 1}{2}$

$y_p = \underline{k_1 \cos 2x + k_2 \sin 2x} + \underline{k}$

Undetermined Coefficient

- 觀察

$$y'+ay=e^{-ax} \qquad y=CI^{-1}+I^{-1}\int Irdx$$

$$y_h=Ce^{-ax} \qquad =Ce^{-ax}+xe^{-ax}$$

猜 $y_p=ke^{-ax}$ => 失效 Non-homogeneous

$$\Rightarrow y_p=kxe^{-ax} \quad k=1$$

=>當我們依r(x)的函數型式決定y_p後，將$y_p(x)$與$y_h(x)$比較是否有相同項。若有，必須將相同的部分乘上x的最低冪次，使其不再相同為止，之後再將修正後 y_p 代入，決定未定的係數。

Undetermined Coefficient

例: $y''+3y'+2y=e^{-2x}$

$$y=y_h+y_p$$

$$\Rightarrow y_h=C_1e^{-x}+C_2e^{-2x}$$

$$\left\{\begin{aligned} y_p&=ke^{-2x}\rightarrow y_p=kxe^{-2x} && \cdots *2\\ y_p'&=ke^{-2x}-2kxe^{-2x} && \cdots *3\\ y_p''&=-2ke^{-2x}-2k\ e^{-2x}+4kxe^{-2x} && \cdots *1 \end{aligned}\right.$$

Undetermined Coefficient

代入原O.D.E

$$\Rightarrow (2-6+4)xe^{-2x} + (3k-4k)e^{-2x} = e^{-2x}$$

$$k = -1$$

$$\Rightarrow y = C_1 e^{-x} + C_2 e^{-2x} - xe^{-2x}$$

Undetermined Coefficient

例: $y'' + 4y' + 4y = 3e^{-2x}$

$$y_h = C_1 e^{-2x} + C_2 xe^{-2x}$$

$$y_p = ke^{-2x} \rightarrow y_p = kx^2 e^{-2x}$$

(練習)

$$\Rightarrow k = \frac{3}{2}$$

$$y = C_1 e^{-2x} + C_2 xe^{-2x} + \frac{3}{2} x^2 e^{-2x}$$

Undetermined Coefficient

例: $y'' + 4y = \cos 2x$

$\lambda^2 + 4 = 0$

$\lambda = \pm 2i$

$y_h = C_1 \cos 2x + C_2 \sin 2x$

$y_p = (k_1 \cos 2x + k_2 \sin 2x)x \quad y_p' = \ldots \quad y_p'' = \ldots$

代入原O.D.E求出 k_1、k_2 … Exercise

Note:

1. 未定係數法，道理簡單卻費時
2. 有些函數，不知道如何猜 y_p $(ex: \csc x)$

Order Reduction Method

- Method 2: Order Reduction Method(降階法)

例: $y'' + 3y' + 2y = e^x$

$(D^2 + 3D + 2)y = e^x$

$(D+1)(D+2)y = e^x$

令$(D+2)y = Z_p$

$(D+1)Z_p = e^x$

$Z_p' + Z_p = e^x, I = e^x$

$\Rightarrow Z_p = I^{-1} \int I e^x dx$

Order Reduction Method

$$(D+2)y_p = I^{-1}\int Ie^x dx$$

n階ODE就有n個 $\underline{I^{-1}\int I}$

$$y_p' + 2y_p = I^{-1}\int Ie^x dx, \quad I_{new} = e^{2x}$$

$$\Rightarrow y_p = \cancel{CI_{new}^{-1}} + I_{new}^{-1}\int I_{new}(I^{-1}\int Ie^x dx)dx$$

$$= e^{-2x}\int e^{2x}(e^{-x}\int e^x e^x dx)dx$$

$$= \frac{1}{6}e^x$$

Order Reduction Method

例: $y'' + 4y' + 4y = e^{-2x}$

$$y'' + 4y' + 4y = e^{-2x}$$

$$y_h = C_1 e^{-2x} + C_2 x e^{-2x}$$

$$y_p = kx^2 e^{-2x}$$

$$(D+2)(D+2)y = e^{-2x}$$

$$I_1 = e^{2x}$$

$$I_2 = e^{2x}$$

Order Reduction Method

$$y_p = I_2^{-1}\int I_2(I_1^{-1}\int I_1 e^{-2x}dx)dx$$
$$= e^{-2x}\int e^{2x}(e^{-2x}\int e^{2x}e^{-2x}dx)dx$$
$$= \frac{1}{2}e^{-2x}x^2$$
$$\Rightarrow y = C_1e^{-2x} + C_2xe^{-2x} + \frac{1}{2}e^{-2x}x^2$$

Note:

降階的順序是否會影響 y_p?

Ans: NO!

Order Reduction Method

例: 設某個微分方程式的特性方程式的根為

$\lambda_1, \lambda_2, \ldots, \lambda_n$

$(D-\lambda_1)(D-\lambda_2)\ldots(D-\lambda_n)y_p = r(x)$

$y_p(x) = ?$

Sol:

$$y_p(x) = e^{\lambda_n x}\int e^{-\lambda_n x}(\ldots e^{\lambda_2 x}\int e^{-\lambda_2 x}(e^{\lambda x}\int e^{-\lambda x}r(x)dx)dx\ldots)dx$$

Order Reduction Method

例: $y''' + 6y'' + 11y' + 6y = e^x$

$$y_h = C_1 e^{-x} + C_2 e^{-2x} + C_3 e^{-3x}$$

$$y_p = e^{-3x} \int e^{3x} (e^{-2x} \int e^{2x} (e^{-x} \int e^x e^x dx) dx) dx$$

$$= \frac{1}{24} e^x$$

$$\Rightarrow y = C_1 e^{-x} + C_2 e^{-2x} + C_3 e^{-3x} + \frac{1}{24} e^x$$

Exercise : $y'' - y' + \frac{1}{4} y = 3 + e^{\frac{x}{2}}$

$$\lambda^2 - \lambda + \frac{1}{4} = 0$$

$$\left(\lambda - \frac{1}{2}\right)^2 = 0 \qquad \lambda = \frac{1}{2}$$

$$y_h = c_1 e^{\frac{x}{2}} + c_2 x e^{\frac{x}{2}}$$

$$y_p = y_{p_1} + y_{p_2}$$

$$y_{p_1} = t \qquad y_{p_2} = kx^2 e^{\frac{x}{2}}$$

$$y'_{p_2} = e^{\frac{x}{2}} \left(2kx + \frac{1}{2} kx^2\right)$$

$$y''_{p_2} = e^{\frac{x}{2}} \left(\frac{1}{4} kx^2 + 2kx + 2k\right)$$

$$y'' - y' + \frac{1}{4} y = \frac{t}{4} + 2ke^{-2x} = 3 + e^{\frac{x}{2}} \qquad t = 12, k = \frac{1}{2}$$

$$y = c_1 e^{\frac{x}{2}} + c_2 x e^{\frac{x}{2}} + \frac{1}{2} x^2 e^{\frac{x}{2}} + 12$$