

Chapter 3. Higher-Order Differential Equations

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Bernoulli Equation

- 非線性→線性

型式(1)

$$f'(y) \frac{dy}{dx} + p(x)f(y) = g(x)$$

$$\text{令 } z = f(y)$$

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

$$= \frac{df(y)}{dy} \cdot \frac{dy}{dx} = f'(y) \cdot \frac{dy}{dx}$$

$$\text{原式} : \frac{dz}{dx} + p(x)z = q(x)$$

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Bernoulli Equation

例: $x^2 \cos y \frac{dy}{dx} = 2x \sin y - 1$

令 $z = \sin y$ 則 $\frac{dz}{dx} = \cos y \cdot \frac{dy}{dx}$

$$\frac{dz}{dx} - \frac{2}{x} dz = \frac{-1}{x^2}$$

$$z = CI^{-1} + I^{-1} \int I r dx$$

對於型式(1)

$f(y)$ 同常等於 $y^2, y^3, \dots, \sin(y) \dots e^y$

Bernoulli Equation

型式(2)

Bernoulli

型式(3)

Riccati

$$\frac{dy}{dx} + p(x)y = q(x) + y^2 r(x)$$

若 y_1 為上式之一特解

則令 $y = y_1 + \frac{1}{z}$ 得Z的線性D.E.

Bernoulli Equation

- 例: $y' + (2x - 1)y = x^2 - x + 1 + y^2$

$$p = 2x - 1$$

$$q = x^2 - x + 1$$

$$r = 1$$

find $y_1 \Rightarrow$ *guess*? $1, x, x^2, \sin x, \cos x, \dots$

$$\therefore y_1 = x$$

令 $y = y_1 + \frac{1}{z}$ 代入原D.E.得 $z' + z = -1$

$$z = Ce^{-x} - 1$$

$$y = x + \frac{1}{ce^{-x} - 1}$$

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Review

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Review

$$M(x, y)dx + N(x, y)dy = 0$$

$$\text{正合} : \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\text{非正合} : \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\underline{I(x, y)M(x, y)dx + I(x, y)N(x, y)dy = 0}$$

$$M_1$$

$$N_1$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

$$M(x, y) \frac{\partial I(x, y)}{\partial y} + I(x, y) \frac{\partial M(x, y)}{\partial y} = N(x, y) \frac{\partial I(x, y)}{\partial x} + I(x, y) \frac{\partial N(x, y)}{\partial x}$$

$$M(x, y) \frac{\partial I(x, y)}{\partial y} - N(x, y) \frac{\partial I(x, y)}{\partial x} = +I(x, y) \frac{\partial N(x, y)}{\partial x} - I(x, y) \frac{\partial M(x, y)}{\partial y}$$

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Review

一階P.D.E.

$$P(x, y, z) \frac{\partial z}{\partial x} + Q(x, y, z) \frac{\partial z}{\partial y} = R(x, y, z)$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \text{ (輔助)}$$

$$u(x, y, z) = \alpha$$

$$v(x, y, z) = \beta$$

$$\varphi(u, v) = 0 \text{ or } C$$

$$v = f(u)$$

$$\frac{dx}{-N(x, y)} = \frac{dy}{M(x, y)} = \frac{dI}{I \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)}$$

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Review

(i)

希望 $I(x)$

$$\frac{dx}{-N(x,y)} = \frac{dI}{I \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)}$$

$$\frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)}{-N} dx = \frac{dI}{I} \quad \text{if } f(x) = \frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)}{-N}$$

$$I = e^{\int f(x) dx} = I(x)$$

(ii)

希望 $I(y)$

$$\frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)}{M} dy = \frac{dI}{I} \quad f(y) dy = \frac{dI}{I}$$

$$I = e^{\int f(y) dy}$$

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Review

(iii)

希望 $I(x+y)$

$$\frac{dx + dy}{-N + M} = \frac{dI}{I \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)}$$

$$\frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)}{-N + M} d(x+y) = \frac{dI}{I}$$

$$I = e^{\int f(x+y) d(x+y)}$$

(iv)

希望 $I(xy)$

$$\frac{\cancel{y}dx + \cancel{x}dy}{-yN + xM} d(x,y) = \frac{dI}{I}$$

$$I = e^{\int f(x,y) dxy}$$

$$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$$

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Review

(v)

希望 $I(x^a y^b)$

$$d(x^a y^b) = ?$$

$$\frac{d(x^a y^b)}{dx} = ax^{a-1} y^b + bx^a y^{b-1} \frac{dy}{dx}$$

$$d(x^a y^b) = ax^{a-1} y^b dx + bx^a y^{b-1} dy = x^{a-1} y^{b-1} (aydx + bxdy)$$

$$\frac{aydx + bxdy}{-ayN + bxM} = \frac{dI}{I \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)}$$

$$\frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) d(x^a y^b)}{(-ayN + bxM) x^{a-1} y^{b-1}} = \frac{dI}{I}$$

Review

$$\text{if } f(x^a y^b) = \frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)}{(-ayN + bxM) x^{a-1} y^{b-1}}$$

$$I = e^{\int f(x^a y^b) d(x^a y^b)}$$

$$\frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)}{(-ayN + bxM)} \cdot \frac{1}{x^a y^b} d(x^a y^b) = \frac{dI}{I}$$

$$\frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)}{(-ayN + bxM)} = 1 \Rightarrow I = x^a y^b$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{(-ayN + bxM)}{xy} = \frac{-aN}{x} + \frac{bM}{y}$$

Review

• 例: $(4xy + 6y^2)dx + (2x^2 + 6xy)dy = 0$

$$\frac{\partial M}{\partial y} = 4x + 12y, \frac{\partial N}{\partial x} = 4x + 6y, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -6y$$

若 $I = x^a y^b$, 則 $\frac{-aN}{x} + \frac{bM}{y} = -6y$

$$\frac{-a(2x^2 + 6xy)}{x} + \frac{b(4xy + 6y^2)}{y} = -a(2x + 6y) + b(4x + 6y)$$

$$= (-2a + 4b)x + (-6a + 6b)y$$

$$\begin{cases} -2a + 4b = 0 \\ -6a + 6b = -6 \end{cases} \Rightarrow \begin{cases} a = 2 \\ b = 1 \end{cases}$$

Review

$\because I = x^2 y$

$$x^2 y (4xy + 6y^2)dx + x^2 y (2x^2 + 6xy)dy = 0$$

$$\frac{\partial u}{\partial x} = 4x^3 y^2 + 6x^2 y^3, \frac{\partial u}{\partial y} = 2x^4 y + 6x^3 y^2$$

$$u = x^4 y^2 + 2x^3 y^3 + f(y), u = x^4 y^2 + 2x^3 y^3 + g(x)$$

$$f(y) = 0, g(x) = 0$$

$$\therefore u = x^4 y^2 + 2x^3 y^3 = 0$$

Review

- § 2.2 分離變數

例4 : $\cos x(e^{2y} - y) \frac{dy}{dx} = e^y \sin 2x$

$$(e^y \sin 2x) dx - (\cos x)(e^{2y} - y) dy = 0$$

$$\frac{\partial M}{\partial y} = e^y \sin 2x, \frac{\partial N}{\partial x} = \sin x(e^{2y} - y)$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \sin x(e^{2y} - y) - e^y \sin 2x$$

$$I = ?$$

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Review

$$e^y + ye^{-y} + e^{-y} + 2 \cos x = C$$

$$\frac{\partial u}{\partial x} = -2 \sin x, \frac{\partial u}{\partial y} = e^y + e^{-y} + y(-e^{-y}) + (-e^{-y}) = e^y - ye^{-y}$$

$$-2 \sin x dx + e^{-y}(e^{2y} - y) dy = 0$$

$$I = -e^y \cos x$$

$$\frac{e^{2y} - y}{e^y} dy = \frac{\sin 2x}{\cos x} dx$$

$$\int (e^y - ye^{-y}) dy = 2 \int \sin x dx + C$$

$$e^y + ye^{-y} + e^{-y} = -2 \cos x + C$$

$$u = C$$

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Review

- § 2.3 線性微分方程式

例1 : $\frac{dy}{dx} - 3y = 6$

$$y = CI^{-1} + I^{-1} \int I r dx$$

$$I = e^{\int -3 dx} = e^{-3x}$$

$$y = Ce^{3x} + e^{3x} \int e^{-3x} 6 dx$$

$$= Ce^{3x} - 2$$

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Review

例2 : $x \frac{dy}{dx} - 4y = x^6 e^x \Rightarrow \frac{dy}{dx} - 4 \frac{y}{x} = x^5 e^x$

$$I = e^{\int \frac{-4}{x} dx} = e^{-4 \ln x} = e^{\ln x^{-4}} = x^{-4}$$

$$y = CI^{-1} + I^{-1} \int I r dx = Cx^4 + x^4 \int x^{-4} x^5 e^x dx$$

$$= Cx^4 + x^5 e^x - x^4 e^x$$

例3 : $(x^2 - 9) \frac{dy}{dx} + xy = 0 \Rightarrow \frac{dy}{dx} + \frac{1}{x^2 - 9} xy = 0$

$$I = e^{\int \frac{x}{x^2 - 9} dx} = e^{\int \frac{1}{2x^2 - 9} d(x^2 - 9)} = e^{\frac{1}{2} \ln |x^2 - 9|} = (x^2 - 9)^{\frac{1}{2}}$$

$$y = CI^{-1} = \frac{C}{\sqrt{x^2 - 9}}$$

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Review

- § 2.4 正合方程式

例1： $2xydx + (x^2 - 1)dy = 0$

$$\frac{\partial u}{\partial x} = 2xy, \frac{\partial u}{\partial y} = x^2 - 1$$

$$u = x^2 y + f(y), u = x^2 y - y + g(x)$$

$$f(y) = -y, g(x) = 0$$

$$u = x^2 y - y$$

Review

例4： $xydx + (2x^2 + 3y^2 - 20)dy = 0$

$$\frac{\partial M}{\partial y} = x, \frac{\partial N}{\partial x} = 4x$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3x$$

$$\frac{3x dy}{M} = \frac{dI}{I} \Rightarrow \frac{3 dy}{y} = \frac{dI}{I}$$

$$I = e^{\int \frac{3}{y} dy} = e^{3 \ln y} = y^3$$

Review

$$xy(y^3)dx + (2x^2 + 3y^2 - 20)(y^3)dy = 0$$

$$xy^4 dx + (2x^2 y^3 + 3y^5 - 20y^3)dy$$

$$\frac{\partial u}{\partial x} = xy^4, \frac{\partial u}{\partial y} = 2x^2 y^3 + 3y^5 - 20y^3$$

$$u = \frac{1}{2}x^2 y^4 + f(y), u = \frac{1}{2}x^2 y^4 + \frac{1}{2}y^6 - 5y^4 + g(x)$$

$$f(y) = \frac{1}{2}y^6 - 5y^4, g(x) = 0$$

$$u = \frac{1}{2}x^2 y^4 + \frac{1}{2}y^6 - 5y^4 = C$$

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Review

• § 2.5 取代法

例1 : $(x^2 + y^2)dx + (x^2 - xy)dy = 0$

$$\frac{\partial M}{\partial y} = 2y, \frac{\partial N}{\partial x} = 2x - y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - 3y$$

$$\Leftrightarrow \frac{y}{x} = u, y = ux \Rightarrow dy = udx + xdu$$

$$(x^2 + u^2 x^2)dx + (x^2 - ux^2)(udx + xdu) = 0$$

$$(x^2 + u^2 x^2 + ux^2 - u^2 x^2)dx + (x^3 - ux^3)du = 0$$

$$x^2(1+u)dx + x^3(1-u)du = 0$$

$$(1+u)dx + x(1-u)du = 0$$

→ 分離變數法

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Review

$$\frac{\partial M}{\partial u} = 1, \frac{\partial N}{\partial x} = 1 - u$$

$$\frac{1-u}{1+u} du + \frac{1}{x} dx = 0$$

$$\frac{1}{x} dx = -\left(\frac{1-u}{1+u}\right) du$$

$$\ln x = -\left(-1 + \frac{2}{1+u}\right) du$$

$$\ln x = -(-u + 2 \ln|1+u|) + C$$

Review

$$代 u = \frac{y}{x}$$

$$\ln x = u - 2 \ln \left| 1 + \frac{y}{x} \right| + C$$

$$\ln x + 2 \ln \left| 1 + \frac{y}{x} \right| = \frac{y}{x} + \ln C'$$

$$\frac{x(1 + \frac{y}{x})^2}{C'} = e^{\frac{y}{x}}$$

$$(x+y)^2 = C' x e^{\frac{y}{x}}$$

Review

- P.D.E

例： $x \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} - 3z = 0$, Initial Condition $z(x,1) = x^2 e^{x^2}$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{3z}$$

$$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \frac{y}{x} = \alpha = u$$

$$\frac{dx}{x} = \frac{dz}{3z} \Rightarrow \frac{z}{x^3} = \beta = v$$

$$\text{通解 } \Phi(u, v) = 0, \frac{z}{x^3} = f\left(\frac{y}{x}\right)$$

$z(x,1)$ 帶入通解把 f 求出

$$\because z(x,1) = x^2 e^{x^2}$$

$$z = x^3 f\left(\frac{y}{x}\right)$$

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Review

$$z(x,1) = x^3 f\left(\frac{1}{x}\right) = x^2 e^{x^2}$$

$$f\left(\frac{1}{x}\right) = \frac{1}{x} e^{x^2}$$

$$\text{令 } \mu = \frac{1}{x} \text{ 則 } x = \frac{1}{\mu}$$

$$f(\mu) = \mu e^{\frac{1}{\mu^2}}$$

$$\therefore z = x^3 f\left(\frac{y}{x}\right)$$

$$= x^3 \frac{y}{x} e^{\frac{1}{\left(\frac{y}{x}\right)^2}}$$

$$= x^2 y e^{\frac{x^2}{y^2}}$$

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Review

- Chap3

常O.D.E

法一U.C 3-3, 3-4

法二R.O. 補充

法三D.D. 補充

法四VV 3-5

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Review

3-3:

例9: $y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$

$$y = y_h + y_p$$

$$\lambda^2 - 6\lambda + 9 = 0 \Rightarrow \lambda = 3, 3$$

$$y_h = C_1 e^{3x} + C_2 x e^{3x}$$

法四: $y_p = y_{p1}, y_{p2}$

$$w = \begin{vmatrix} e^{3x} & x e^{3x} \\ 3e^{3x} & e^{3x} + 3x e^{3x} \end{vmatrix} = e^{6x} + 3x e^{6x} - 3x e^{6x} = e^{6x}$$

$$y_{p1} = e^{3x} \int \frac{-(6x^2 + 2)x e^{3x}}{e^{6x}} dx + x e^{3x} \int \frac{(6x^2 + 2)e^{3x}}{e^{6x}} dx$$

$$= e^{3x} \int -(6x^2 + 2)x e^{-3x} dx + x e^{3x} \int (6x^2 + 2)e^{-3x} dx$$

= 換方法

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Review

法三:

$$\begin{aligned}y_{p1} &= \frac{1}{D^2 - 6D + 9}(6x^2 + 2) \\&= \frac{1}{9(1 + \frac{D^2 - 6D}{9})}(6x^2 + 2) \\&= \frac{1}{9}[1 - \frac{D^2 - 6D}{9} + (\frac{D^2 - 6D}{9})^2 - (\frac{D^2 - 6D}{9})^3 + \dots](6x^2 + 2) \\&= \frac{1}{9}(6x^2 + 2 - \frac{12 - 72x}{9} + \frac{36 \cdot 12}{81}) \\&= \frac{1}{9}(6x^2 + 8x + 6) = \frac{2}{3}x^2 + \frac{8}{9}x + \frac{2}{3}\end{aligned}$$

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Review

$$\begin{aligned}y_{p2} &= \frac{1}{D^2 - 6D + 9}(-12e^{3x}) \\&= e^{3x} \frac{1}{(D + 3)^2 - 6(D + 3) + 9}(-12) \\&= (-12)e^{3x} \frac{1}{D^2} \times 1 \\&= e^{3x}(-6x^2) \\&= -6x^2 e^{3x}\end{aligned}$$

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