Chapter 1.
Introduction to differential equations

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Linear Differential Equations

- Linear D.E. 線性微分方程式，不具有下列任何一項
  1. 因變數的自乘項
  2. 因變數導數的自乘項
  3. 因變數及其導數的互乘項

  則稱為線性微分方程式；反之，若一微分方程式，具有上述1,2,3中任何一項，即稱為非線性D.E.。

Linear Differential Equations

例:

(1) $y'(x)+5y''(x)=e^{x} \Rightarrow$ 線性2階1次O.D.E

(2) $\dfrac{\partial u(x,y)}{\partial x}+u(x,y)\dfrac{\partial u(x,y)}{\partial y}+3\mu(x,y)=0$

$\Rightarrow$ 非線性(Disobey#3)，1階1次P.D.E.

(3) $\dfrac{\partial^{2}u}{\partial x^{2}}+x\dfrac{\partial u}{\partial y}=0 \qquad (u=u(x,y))$

$\Rightarrow$ 線性(*x*是自變數)，2階1次P.D.E.

(4) $y'''(x)+4y''(x)+2y(x)=x^{2} \Rightarrow$ 線性3階1次O.D.E.

Linear Differential Equations

- 一線性微分方程式會滿足重疊定理(Superposition)

$$V_{i1} \longrightarrow \boxed{F(x)} \longrightarrow V_{C1} \qquad V_{i2} \longrightarrow \boxed{F(x)} \longrightarrow V_{C2}$$

$$V_{C1}=F(V_{i1}) \qquad V_{C2}=F(V_{i2})$$

若滿足

$$\alpha V_{i1}+\beta V_{i2} \longrightarrow \boxed{F(x)} \longrightarrow \alpha V_{C1}+\beta V_{C2}$$

$\alpha V_{C1}+\beta V_{C2}=F(\alpha V_{i1}+\beta V_{i2}) \qquad \alpha$、$\beta$為任意純量

則為線性。

Differential Equations

- 目的： 分析微分方程式的解

  $\Rightarrow$ 了解微分方程式由何而來

例:

(1) $y(x)=C$ $\quad$ $C\in$常數 (Constant)

如何消去C?

$\Rightarrow$ 應用微分，才可以消去C

$\dfrac{dy(x)}{dx}=0$, 1階1次O.D.E.

Differential Equations

(2) $y(x)=C_1x+C_2$ $\quad$ $C_1, C_2\in$Constant

$\Rightarrow$ $\dfrac{dy(x)}{dx}=C_1$

$\dfrac{d^2y(x)}{dx}=0$ $\quad\Rightarrow$ 線性, 2 階 1 次 O.D.E.

(3) $y(x)=Cx+C^2$ $\quad$ C $\in$ constant

$\Rightarrow\dfrac{dy(x)}{dx}=C$

一個常數，只能微分一次

($\because$ 一次不定積分 , 只會產生一個常數)

$\Rightarrow y(x)=\dfrac{dy(x)}{dx}x+\left(\dfrac{dy(x)}{dx}\right)^2$ $\quad\Rightarrow$ 非線性,1 階 2 次 O.D.E.

Differential Equations

(4) $y(x)=C_1e^{2x}+C_2e^{x}\quad -(a)\ ,\ C_1,C_2\in\text{Constant}$

$$\frac{dy(x)}{dx}=2C_1e^{2x}+C_2e^{x}\quad -(b)$$

$$\frac{d^2y(x)}{dx^2}=4C_1e^{2x}+C_2e^{x}\quad -(c)$$

$(a)\times 2+(b)\times -3+(c)\times 1$

$\Rightarrow\ y''(x)-3y'(x)+2y(x)=0$

Differential Equations

f 函數 —利用微分的技巧 及簡易運算, 消去 所有的未知常數→ 微分方程式 $D.E.$

函數: 包含了某些未知的任意常數

(Rule：有幾個未知任意常數，就可微分幾次)

微分方程式 → 函數: Objective

$\Rightarrow$ Apply for O.D.E. Only

f 稱為 $D.E.$ 的通解(general solution) or 原函數

Differential Equations

例:

$$y(x) = C_1 \cos 3x + C_2 \sin 3x \qquad , \ C_1, C_2 \in \text{Constant}$$

$$\Rightarrow D.E. = ?$$

Sol:

$$\frac{dy}{dx} = -3C_1 \sin 3x + 3C_2 \cos 3x$$

$$\frac{d^2y}{dx^2} = -9C_1 \cos 3x - 9C_2 \sin 3x$$

$$\Rightarrow y''(x) + 9y(x) = 0$$

Chapter 2.
First-Order Ordinary Differential Equations

First-Order Differential Equation

- 一般一階O.D.E.，可表成
  1. $M(x,y)dx + N(x,y)dy = 0$
  2. $y'(x) = f(x,y) \Rightarrow y'(x) = \dfrac{dy}{dx} = f(x,y)$

$$f(x,y)dx = dy \Rightarrow f(x,y)dx - dy = 0$$

$$\Rightarrow M(x,y)dx + N(x,y)dy = 0$$

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$$

$$y' = f(x,y)$$

First-Order Differential Equation

- 定理 P14 §1.2.1

對 $y' = f(x,y)$ 之D.E.有一個Initial condition(I.C.)

$$y(x_0) = y_0$$

若 $f(x,y)$，$\dfrac{\partial f(x,y)}{\partial y}$ 於 (x_0, y_0) 之鄰域為連續，

則存在 $\varepsilon > 0$, 使得 $y(x)$ 於 $(x_0 - \varepsilon, x_0 + \varepsilon)$

間有唯一解

First-Order Differential Equation

例: 下列各問題, 何者有唯一解

(1) $y' = e^{xy^2} \quad , y(0) = 1$

(2) $y' = \sqrt{y} \quad , y(0) = 0$

(3) $y' = \sqrt{y} \quad , y(0) = 1$

(4) $y' = -\sqrt{1-y^2} \quad , y(0) = 0$

(5) $y' = -\sqrt{1-y^2} \quad , y(0) = 1$

First-Order Differential Equation

- Sol:

(1) $y' = e^{xy^2} \quad , y(0) = 1$

$$f(x,y) = e^{xy^2} \qquad (0,1)$$

$$\frac{\partial f(x,y)}{\partial y} = 2xye^{xy^2} \qquad (0,1) \qquad \Rightarrow 具唯一解$$

(2)

$$y' = \sqrt{y} \quad , y(0) = 0$$

$$f(x,y) = \sqrt{y} \qquad (0,0) \qquad \Rightarrow 不具唯一解$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{1}{2\sqrt{y}} \qquad (0,0)$$

First-Order Differential Equation

(3) $y' = \sqrt{y}$, $y(0) = 1$

$(0,1)$ $\Rightarrow$具唯一解

(4) $y' = -\sqrt{1-y^2}$, $y(0) = 0$

$f(x,y) = -\sqrt{1-y^2}$ $(0,0)$

$\dfrac{\partial f(x,y)}{\partial y} = \dfrac{y}{\sqrt{1-y^2}}$ $(0,0)$

$\Rightarrow$具唯一解

(5) $y' = -\sqrt{1-y^2}$, $y(0) = 1$

$(0,1)$ $\Rightarrow$ 不具唯一解

$\Rightarrow$ (1) (3) (4) 具唯一解

Exact Equation

- 如何解$M(x,y)dx + N(x,y)dy = 0 \cdots\cdots (B)$

令$u(x,y) = C \cdots\cdots (A)$, $C \in \text{const}$.

$\Delta u(x,y) = u(x+\Delta x, y+\Delta y) - u(x,y)$

$= u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y) + u(x, y+\Delta y) - u(x,y)$

$= \dfrac{\partial u(x,y)}{\partial x}(x+\Delta x - x) + \dfrac{\partial u(x,y)}{\partial y}(y+\Delta y - y) = 0$

★ Mean-Value-Theorem 均值定理

$f(x)$, $a \le x \le b$, 一定存在一個 C , $a \le C \le b$

St. $f'(C) = \dfrac{f(b) - f(a)}{b - a} \Rightarrow f'(C)(b-a) = f(b) - f(a)$

Exact Equation

$$\Rightarrow \Delta u(x,y) = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y = 0$$

$$\Rightarrow du(x,y) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

$$= M(x,y)dx + N(x,y)dy = 0$$

$$\Rightarrow M(x,y) = \frac{\partial u}{\partial x} \qquad N(x,y) = \frac{\partial u}{\partial y}$$

$$\partial u = M(x,y)\partial x \qquad \partial u = N(x,y)\partial y$$

$$\int \partial u = \int M(x,y)\partial x + f(y) \qquad \int \partial u = \int N(x,y)\partial y + g(x)$$

Exact Equation

$$\Rightarrow \boxed{u = \begin{cases} \int M(x,y)\partial x + f(y) \\ \int N(x,y)\partial y + g(x) \end{cases}} \begin{matrix} \cdots\cdots\cdots(1) \\ \cdots\cdots\cdots(2) \end{matrix}$$

經過比較(1)及(2)式，決定 $f(x)$和 $g(x)$

$$\Rightarrow u(x,y) = C$$

$$\int \ln x = x\ln x - x$$

Exact Equation

- Q : 如何知道 (A) → (B) 只有微分而已 ?

$$M(x,y)=\frac{\partial u}{\partial x} \qquad N(x,y)=\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{M(x,y)}{\partial y}=\frac{\partial^2 u}{\partial x\partial y} = \frac{N(x,y)}{\partial x}=\frac{\partial^2 u}{\partial x\partial y}$$

假設 $u(x,y)$ 具有連續二階偏導數

$$\Rightarrow M(x,y)dx+N(x,y)dy=0$$

若 $\dfrac{\partial M(x,y)}{\partial y}=\dfrac{\partial N(x,y)}{\partial x}$

則稱此微分方程式為"正合"(Exact)

Exact Equation

例:

$$u(x,y)=x^2y^3=C$$

$$du(x,y)=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=0$$

$$\Rightarrow 2xy^3dx+x^2 3y^2dy=0$$

Check: $\dfrac{\partial M(x,y)}{\partial y}=2x3y^2=\dfrac{\partial N(x,y)}{\partial x}$

Exact Equation

例:

$$u(x,y) = xy^2 + 3x + 5y = C$$

$$du(x,y) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

$$\Rightarrow (y^2+3)dx + (2xy+5)dy = 0$$

$$\frac{\partial M(x,y)}{\partial y} = 2y = \frac{\partial N(x,y)}{\partial x} \Rightarrow \text{正合}$$

$$\Rightarrow \frac{\partial u(x,y)}{\partial x} = y^2 + 3$$

$$\int \partial u(x,y) = \int (y^2+3)dx + f(y)$$

Exact Equation

$$\Rightarrow \frac{\partial u(x,y)}{\partial y} = 2xy + 5$$

$$\int \partial u(x,y) = \int (2xy+5)dy + g(x)$$

$$\Rightarrow u = \begin{cases} xy^2 + 3x + f(y) \\ xy^2 + 5y + g(x) \end{cases}$$

$$\Rightarrow f(y) = 5y \quad , \quad g(x) = 3x$$

$$\Rightarrow u(x,y) = xy^2 + 3x + 5y = C$$

三、反三角函數之微分

1. $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$ ，$|x|<1$
2. $\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$ ，$|x|<1$
3. $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$ ，$x \in R$
4. $\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$ ，$x \in R$
5. $\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$ ，$|x|>1$
6. $\frac{d}{dx}\csc^{-1}x = \frac{-1}{|x|\sqrt{x^2-1}}$ ，$|x|>1$