Ch 4. Laplace Transform

◎. 想法:

$$\begin{tikzcd} (\mathrm{I})\ \boxed{\text{微分方程式}} \arrow[r, "\text{原先}"] \arrow[d, "\text{Laplace transform}"] & \boxed{\text{解}} \\ (\mathrm{II})\ \boxed{\text{代數方程式}} \arrow[r, "\text{四則運算}"] & \boxed{\text{代數解}} \arrow[u, "\text{Inverse Laplace transform}"'] \end{tikzcd}$$

from (I) to (II) 必須具備「基本函數」2. y', y'' ...

如. $y''+3y'+5y = e^t$, $\cos t$, $\sin t$ 這类不傾複雜的式子

才可進行 Laplace transform

⇒ 定义 $f(t)$.

$L\{f(t)\}$ = $f(t)$ 函數的 Laplace transform

$$= \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= F(s).$$

$f(t) = L^{-1}\{F(s)\}$ = Inverse Laplace transforms.

$$= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s) \cdot e^{st}\, ds$$

(└ 複变 (complex analysis))

* 轉換與逆轉換要同時平行學習喔 ^.<

ex. $y' + 2y = e^t$, $y(0) = 1$

sol ① (old method).

$$\left.\begin{aligned} y_h &= ce^{-2t} \\ y_p &= \frac{1}{D+2} e^t = \frac{1}{3} e^t \end{aligned}\right\} \Rightarrow y = ce^{-2t} + \frac{1}{3} e^t$$

又 $y(0) = c + \frac{1}{3} = 1 \Rightarrow c = \frac{2}{3}$

$\Rightarrow y = \frac{2}{3} e^{-2t} + \frac{1}{3} e^t$

sol ② (new method).

$$y' + 2\cdot y = e^{t}$$

$$L\{y'\} \quad L\{y\} \quad L\{e^t\}$$

$$sY(s) - y(0) \quad Y(s) \quad = \frac{s}{s-1}$$

$$\Rightarrow (s+2)Y(s) = 1 + \frac{1}{s-1} = \frac{s}{s-1}$$

$$\Rightarrow Y(s) = \frac{s}{(s-1)(s+2)} = \frac{a}{s-1} + \frac{b}{s+2}.$$

$$\left(\begin{aligned} &\text{又} \quad a = \frac{s}{s+2}\Big|_{s=1} = \frac{1}{3} \\ &\quad\ \ b = \frac{s}{s-1}\Big|_{s=-2} = \frac{2}{3} \end{aligned} \right)$$

$$\Rightarrow \frac{\frac{1}{3}}{s-1} + \frac{\frac{2}{3}}{s+2}$$

$\Rightarrow$ Inverse L.T $= y(t) = \frac{1}{3}e^{t} + \frac{2}{3}e^{-2t}$ #

(1) $f(t) = e^{at}$, $a \in$ const.

$$F(s) = L\{f(t)\} = \int_0^\infty e^{at}\cdot e^{-st}\,dt$$

$$= \int_0^\infty e^{-(s-a)t}\,dt = -\frac{1}{s-a}e^{-(s-a)t}\Big|_0^\infty$$

$$= -\frac{1}{s-a}e^{-(s-a)\infty} - \left(-\frac{1}{s-a}\right)$$

($s-a>0$ ∵ 要收斂).

$$= \frac{1}{s-a}$$

$$\Rightarrow f(t) = e^{at} \Rightarrow F(s) = \frac{1}{s-a}$$

ex. $f(t) = e^{-2t}$

$$\Rightarrow F(s) = \frac{1}{s+2}$$

ex. $G(s) = \frac{2}{s+3}$

$$\Rightarrow g(t) = L^{-1}\left\{\frac{2}{s+3}\right\} = 2\cdot L^{-1}\left\{\frac{1}{s-(-3)}\right\} = 2\cdot e^{-3t}$$

#

(2). $f(t) = \cos at$

$$F(s) = \int_0^\infty f(t) e^{-st} dt$$

$$= \int_0^\infty \cos at \cdot e^{-st} dt$$ 窒礙難行 ==

可利用 $\cos at = \frac{e^{iat} + e^{-iat}}{2}$ 的式子.

$\Rightarrow L\{\cos at\} = L\left\{\frac{e^{iat} + e^{-iat}}{2}\right\}$ (L有線性の特性)

$$= \frac{1}{2} L\{e^{iat}\} + \frac{1}{2} L\{e^{-iat}\}$$

$$= \frac{1}{2}\frac{1}{s - ia} + \frac{1}{2}\frac{1}{s + ia} = \frac{s}{s^2 + a^2}$$ #.

$\Rightarrow f(t) = \cos at \Rightarrow F(s) = \frac{s}{s^2 + a^2}$

ex. $L\{\cos 3t\}$.

$$= \frac{s}{s^2 + 9}$$

回到窒礙難行的部分.

$$F(s) = \int_0^\infty \underbrace{(\cos at)}_{u} \underbrace{(e^{-st} dt)}_{dv}$$ $\left(\begin{array}{l} dv = e^{-st} dt, \quad v = -\frac{1}{s} e^{-st} \\ u = \sin at, \quad du = a \cos at\, dt \end{array}\right.$

又 $\int u\,dv = uv - \int v\,du$.

$$= (\cos at) \cdot \left(-\frac{1}{s} e^{-st}\right)\Big|_0^\infty - \int_0^\infty -\frac{1}{s} e^{-st} \cdot (-a \sin at)\, dt$$

(s > 0 ∵要收斂)

$$\Rightarrow 0 - \left(-\frac{1}{s}\right) - \frac{a}{s}\int_0^\infty \sin at\, e^{-st} dt$$

$$= \frac{1}{s} - \frac{a}{s}\underbrace{\int_0^\infty \underbrace{(\sin at)}_{u} \underbrace{(e^{-st} dt)}_{dv}}$$ $\left(\begin{array}{l} dv = e^{-st} dt, \quad v = -\frac{1}{s} e^{-st} \\ u = \sin at, \quad du = a \cos at\, dt \end{array}\right.$

$$\sin at \cdot \left(-\frac{1}{s} e^{-st}\right)\Big|_0^\infty - \int \left(-\frac{1}{s} e^{-st}\right) a \cos at\, dt$$

$$= (0-0) + \frac{a}{s}\int e^{-st}\cos at\,dt$$

$$\Rightarrow F(s) = \frac{1}{s} - \frac{a}{s}\cdot\frac{a}{s}\int e^{-st}\cos at\,dt$$

$$= \frac{1}{s} - \frac{a^2}{s^2}F(s)$$

$$\Rightarrow F(s) = \frac{s}{s^2+a^2} \quad \#$$

實驗證明 2 种做法是一样的~

(3) $f(t) = \sin at$

$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty \sin at \cdot e^{-st}dt$ 自行練習.

利用 $\sin at = \frac{e^{iat}-e^{-iat}}{2i}$

$$\Rightarrow L\{\sin at\} = \frac{1}{2i}L\{e^{iat}\} - \frac{1}{2i}L\{e^{-iat}\}$$

$$= \frac{1}{2i}\cdot\frac{1}{s-ia} - \frac{1}{2i}\cdot\frac{1}{s+ia} = \frac{a}{s^2+a^2} \quad \#.$$

ex. $f(t) = \sin 2t$

$$\Rightarrow F(s) = \frac{2}{s^2+4}$$

ex. $F(s) = \frac{s}{s^2+16}$

$$\Rightarrow f(t) = L^{-1}\{F(s)\} = \cos 4t$$

ex. $F(s) = \frac{3}{s^2+16}$

$$\Rightarrow f(t) = L^{-1}\{F(s)\} = \frac{4}{s^2+16}\cdot\frac{3}{4} = \frac{3}{4}\sin 4t$$

ex. $L^{-1}\left\{\frac{3}{s+6} + \frac{5s}{s^2+25} + \frac{2}{s^2+36}\right\}$.

$$= 3e^{-6t} + 5\cos 5t + \frac{1}{3}\sin 6t$$

(4). $f(t) = t$

$$F(s) = \int_0^\infty \underbrace{t}_{u}\,\underbrace{e^{-st}\,dt}_{dv}$$

$$= t\cdot\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty - \int -\frac{1}{s}e^{-st}\,dt$$

$$= \lim_{t\to\infty}\frac{t}{-se^{st}} - 0 + \frac{1}{s}\int_0^\infty e^{-st}\,dt$$

$$\boxed{\lim_{t\to\infty}\frac{-1}{s^2e^{st}} = 0} \qquad (s>0) \Rightarrow \frac{1}{s}$$

$\Rightarrow L\{t\} = \frac{1}{s^2}$

$\Rightarrow f(t) = t \quad \Rightarrow \quad F(s) = \frac{1}{s^2}$

ex. $L\{3t\}$

$= \frac{3}{s^2}$.

ex. $G(s) = \frac{5}{s^2}$

$\Rightarrow g(t) = 5t$

ex. $G(s) = \frac{5s+1}{s^2}$

$\Rightarrow g(t) = 5 + t$

(5) $f(t) = H(t) = \begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$ 單位步階函數 = unit-step function.

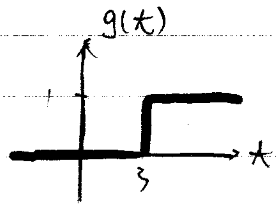
$L\{f(t)\} = L\{H(t)\} = L\{U(t)\}$.

$$= \int_0^\infty f(t)\cdot e^{-st}\,dt = \int_0^\infty 1\cdot e^{-st}\,dt = \frac{1}{s}$$

(a). high-pass filter.

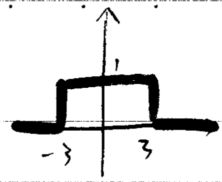
$g(t) = \begin{cases} 1, & t>3 \\ 0, & t<3 \end{cases}$

$g(t) = H(t-3)$



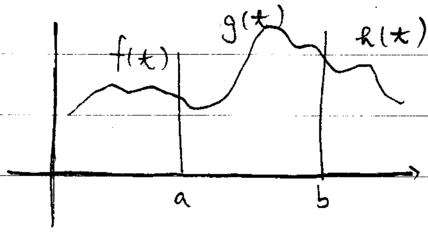
(b) band-pass filter

$$g(t) = \begin{cases} 1 & , -3 < t < 3 \\ 0 & , t > 3 \text{ 或 } t < -3 \end{cases}$$



$$g(t) = H(t-3) - H(t+3)$$

ex.

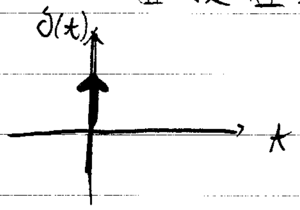


$$F(t) = \begin{cases} 0 & t < 0 \\ f(t) & 0 < t < a \\ g(t) & a < t < b \\ h(t) & b < t \end{cases}$$

$$= f(t)\cdot[H(t) - H(t-a)]$$
$$+ g(t)\cdot[H(t-a) - H(t-b)]$$
$$+ h(t)\cdot[H(t-b)]$$
$$= f(t)\cdot H(t) + [g(t) - f(t)]\cdot H(t-a) + [h(t) - g(t)]\cdot H(t-b) \ \#$$

(6). $\delta(t) = \begin{cases} 1 & , t = 0 \\ 0 & , t \neq 0 \end{cases}$ · 單位脈衝函數 unit-impulse function.

面積值.



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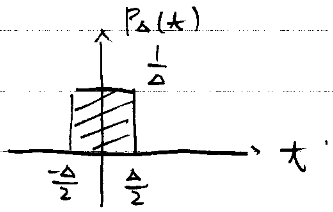


$$\Delta \to 0 \Rightarrow \delta(t) = \lim_{\Delta \to 0} \ , \ P_\Delta(t) = \begin{cases} 1 & , t = 0 \\ 0 & , t \neq 0 \end{cases}$$

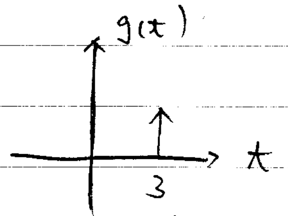
$$L\{\delta(t)\} = \int_0^\infty \delta(t) e^{-st}\, dt$$

No.
Date.

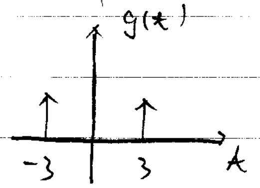
在t的積分範圍內，只有t=0時.
δ(t)才有值.

$$\Rightarrow \int_0^\infty \delta(t)\, e^{-s\cdot 0}\, dt = 1$$

ex.



$g(t) = \delta(t-3)$.



$g(t) = \delta(t+3) + \delta(t-3)$

(7). $f(t^n) \Rightarrow F(s) = \frac{n!}{s^{n+1}}$, $n \geq 0$ 擇日再證 ==

ex. $L\{e^t + 3\sin 2t + 5\cos 3t + 3t^2 + 5\delta(t)\}$.

$$= \frac{1}{s-1} + 3\cdot\frac{2}{s^2+4} + 5\cdot\frac{3}{s^2+9} + 3\cdot\frac{2!}{s^3} + 5 \quad \#$$

ex. $L^{-1}\{7 + \frac{2}{s^5} + \frac{2}{s^2+1} + \frac{s}{s^2+49} + \frac{3}{s+5}\}$

$$= 7\delta(t) + \frac{2}{4!}t^4 + 2\sin t + \cos 7t + 3e^{-5t} \quad \#$$