QUIZ 3

NAME:__________________________ **NCKU id:**__________________

1. Consider $f(x) = \tan^{-1} x$ over the interval $[a, b]$ with $1 < a \leq b < \infty$.

(a) Write down the precise statement of the Mean Value Theorem and the result when apply to $f(x)$.

(b) Use (a) to prove that

$$\tan^{-1} b - \tan^{-1} a \leq \frac{b-a}{2}.$$

Answer :

(a) Suppose that $f(x)$ is a function continuous on $[a, b]$ and differentiable on (a, b). Then there is a number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

(b) Case (i) : For $a = b$, $\tan^{-1} b - \tan^{-1} a = 0 = \frac{b-a}{2}$.

Case (ii) : For $a < b$, the inverse tangent function is continuous on $[a, b]$ and differentiable on (a, b) with $f'(x) = \frac{1}{1+x^2}$. By the Mean Value Theorem, there is a number $c \in (a, b)$ such that

$$\frac{1}{1+c^2} = \frac{\tan^{-1} b - \tan^{-1} a}{b - a}.$$

Since $1 < a < b < \infty$, we have

$$\frac{1}{1+c^2} < \frac{1}{2}$$

$$\Rightarrow \frac{\tan^{-1} b - \tan^{-1} a}{b - a} < \frac{1}{2}$$

$$\Rightarrow \tan^{-1} b - \tan^{-1} a < \frac{b-a}{2}.$$

Date: November 17th.

2. Two cars start moving from the same point. One travels south at 30 km/h and the other travels west at 72 km/h. At what rate is the distance between the cars increasing two hours later?

Answer : Let $x(t)$ be the position of first car at time t and $y(t)$ be the position of second one. We are given that $\frac{dx}{dt} = 30$ km/h and $\frac{dy}{dt} = 72$ km/h. The distance $z(t)$ between these two cars is subject to the equation,

$$
\begin{aligned}
& z^2 = x^2 + y^2 \\
\Rightarrow\ & 2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\
\Rightarrow\ & \frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)
\end{aligned}
$$

After 2 hours, $x = 30 \times 2 = 60$, $y = 72 \times 2 = 144$, and

$$z = \sqrt{60^2 + 144^2} = 156,$$

so

$$\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = \frac{60 \times 30 + 144 \times 72}{156} = 78 \text{ km/h}.$$