Discrete Mathematics (2008 Spring) Final

(total:110 points, max:100 points)

1. (30 points) For each of the following statements, determine and explain (required) whether it is correct or not.
   (1). The number of derangements of 1,2,3,4,5 is 44.
   (2). The number of integer solutions for $c_1 + c_2 + c_3 + c_4 + c_5 = 30$, $1 \le c_i$ for all i, with c_2 even and c_3 odd is equal to the coefficient of x^{25} in $(x+x^2+x^3+\ldots)^3(x^2+x^4+\ldots)(x+x^3+x^5+\ldots)$.
   (3). $\sum_{k=0}^{n}(-1)^k\binom{n}{n-k}(n-k)^n = 0.$
   (4). The coefficient of x^{50} in $(x^6+x^7+x^8+\ldots)^7$ is $\binom{13}{8}$.
   (5). $1/(1-x)$ is the exponential generating function for the sequence $0!, 1!, 2!, 3!, \ldots$.
   (6). $(p \vee q) \rightarrow [q \rightarrow (\phi \subset \{\phi\})]$ is a tautology.

2. (10 points) (a) Find the coefficient of x^3y^2z in the expansion of $[(x/2) + 4y^2 - 3z]^5$. (b) What is the sum of all coefficients in the complete expansion?

3. (10 points) (1) Determine the sequence generated by the generating function $f(x) = \dfrac{1}{(3-2x)}$. (2) What is the generating function for the number of partitions of $n \in N$ into summands that cannot exceed 12 and cannot occur more than five times?

4. (10 points, 4, 6) Determine how many integer solutions there are to $x_1 + x_2 + x_3 + x_4 = 18$, if (1) $1 \le x_i$ for all i, (2) $0 \le x_1 \le 4$, $0 \le x_2 \le 6$, $2 \le x_3 \le 7$, $3 \le x_4 \le 7$.

5. (10 points) (1) How many ways can one distributes eight distinct prizes among four students with exactly two students getting nothing? (2) How many ways have at least two students getting nothing?

6. (10 points) For A={1, 2, 3, 4, 5} and B={u, v, w, x, y}, determine the number of one-to-one functions f : A→B where $f(1) \neq$ v, w, $f(2) \neq$ u, w, $f(3) \neq$ x, y and $f(4) \neq$ v, x, y.

7. (10 points, 2,2,3,3) Let A={a, b, c, d}, f: A→A and g:AxA→A. (1) How many one-to-one correspondence functions in f? (2) How many commutative functions in g? (3) How many closed binary operations on A have c as the identity? (4) How many functions in part (3) are commutative?

8. (10 points) For $n \ge 1$, let D_n be the n x n determinant in the right figure. Find and solve a recurrence relation for the value D_n.

$$\begin{vmatrix} 5 & 3 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 2 & 5 & 3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 3 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & & & & & \ddots & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 & 5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 2 & 5 \end{vmatrix}$$

9. (10 points) Solve the recurrence relation $a_{n+2} - 2a_{n+1} + a_n = 2^n$. $n \ge 0$, $a_0 = 1$, $a_1 = 2$ by the method of *generating functions*.