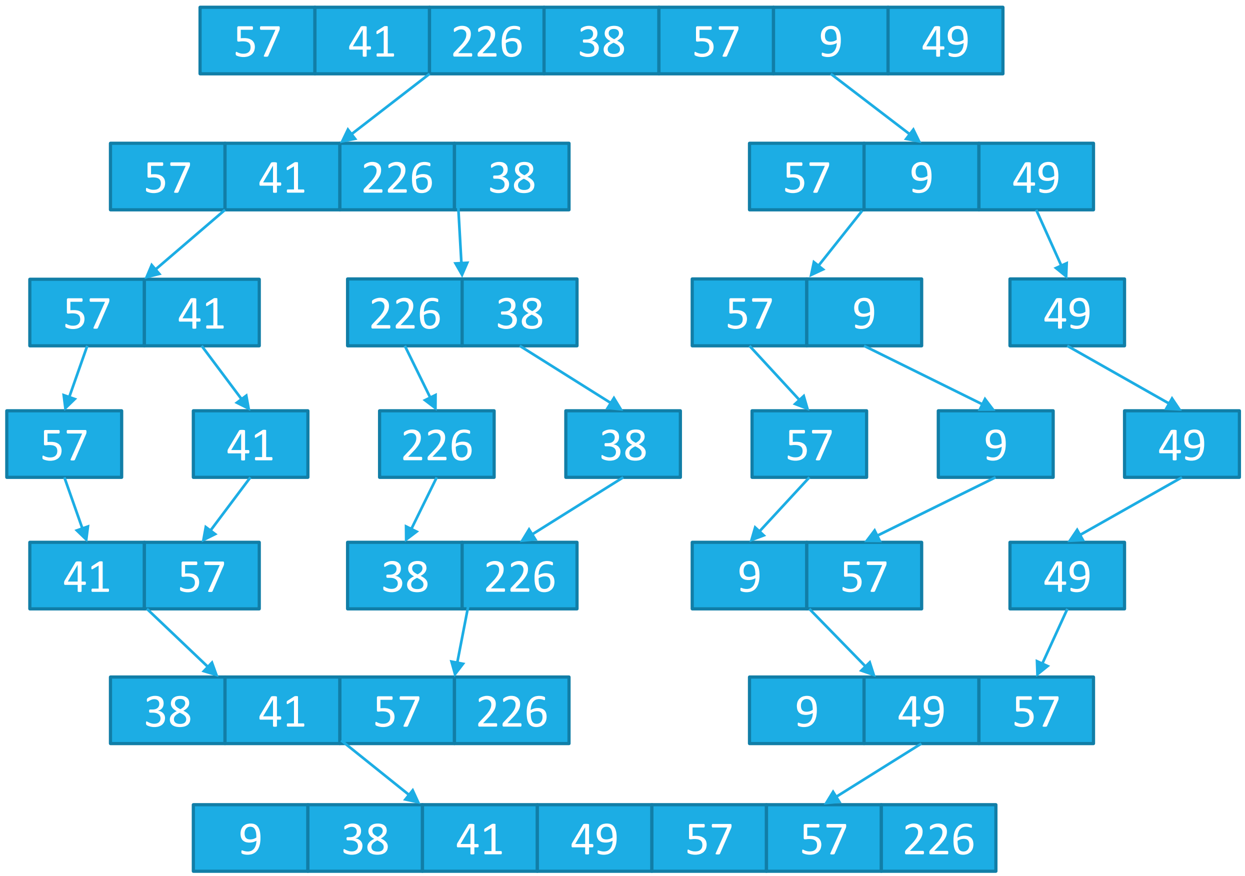
Algorithm 2017 Spring
Homework 1 Solutions

指導教授：謝孫源 教授

助教：許景添 陳琮皓 林玉陞 何岱璇

1. (10pts)Illustrate the operation of merge sort on the array A= 〈57,41,226,38,57,9,49〉.



2. Answer “true” or “false” first, then explain the reason.

a)(5pts) $2^{n+2} = O(2^n)$

True.

$2^{n+2} = 4 * 2^n = O(2^n)$

b)(5pts) $2^{2n} = O(2^n)$

False.

$2^{2n} = 4^n \neq O(2^n)$**. We have no constant c asymptotically may be less than or equal to** $c * 2^n$**.**

Question 3(10pts)

Solution:

先算O:

- $\text{Log}(n!) = \log(n) + \log(n-1) + \cdots + \log(1)$
- $\leq \log(n) + \log(n) + \cdots + \log(n) = n\log n$
- 所以$\log(n!) = O(n\log n)$

再算Ω:

- $\text{Log}(n!) = \log(n) + \log(n-1) + \cdots + \log(\frac{n}{2}) + \cdots + \log(1)$
- $\geq \log(\frac{n}{2}) + \log(\frac{n}{2}) + \cdots + \log(\frac{n}{2}) = (\frac{n}{2})\log(\frac{n}{2})$
- 所以$\log(n!) = \Omega(n\log n)$

有上述可得$\log(n!) = \theta(n\log n)$

Question 4(10pts)

1. $\theta(x^2)$

2. $\Rightarrow \dfrac{n(n+1)}{2} \Rightarrow \dfrac{n^2}{2} + \dfrac{1}{2} \Rightarrow \theta(n^2)$

3. $\log(n!) = \theta(n\log n)$

$n\log(n!) = \theta(n^2 \log n)$

原式 $\Rightarrow \theta(n^3)$

5. Give tight asymptotic bounds for the following recurrences

a)(3pts) $T(n) = 8T\left(\frac{n}{3}\right) + n^2$

by master method

a=8, b=3, $f(n) = n^2, n^{\log_a b} = n^{\log_3 8}$

Case3 ⇨ $T(n) = \Theta(n^2)$

b)(3pts) $T(n) = 4T\left(\frac{n}{4}\right) + n\lg n$

by master method

a=4, b=4, $f(n) = n\lg n, n^{\log_a b} = n$

Case2 with k=1 ⇨ $T(n) = \Theta(n\lg^2 n)$

5. Give tight asymptotic bounds for the following recurrences

c)(4pts) $T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\lg n}$

$$= 4T\left(\frac{n}{4}\right) + \frac{n}{\lg\frac{n}{2}} + \frac{n}{\lg n}$$

$$= 8T\left(\frac{n}{8}\right) + \frac{n}{\lg\frac{n}{4}} + \frac{n}{\lg\frac{n}{2}} + \frac{n}{\lg n}$$

$$= \cdots$$

$$= nT(1) + \frac{n}{1} + \frac{n}{2} + \cdots + \frac{n}{\lg n}$$

$$= n + n\left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{\lg n}\right)$$

$$\doteqdot n + n\lg\lg n$$

➡ $T(n) = O(n\lg\lg n)$

6. Answer "true" or "false" first, then explain the reason or give a counterexample.

a)(3pts) $\mathrm{f}(n) = O\left(g(n)\right)$ implies $\mathrm{g}(n) = \Omega\left(f(n)\right)$

True. 根據定義

b)(3pts) $\mathrm{f}(n) = \omega\left(f(n)\right)$

False. 根據定義

c)(4pts) If $\mathrm{f}(n) = O\left(g(n)\right)$ then $2^{\mathrm{f}(n)} = O\left(2^{\mathrm{g}(n)}\right)$

False.

$\mathrm{f}(n) = 2n = O(n), \mathrm{g}(n) = n \Rightarrow 2^{2n} \neq O(2^n)$

7. Prove that $x^{\log_b y} = y^{\log_b x}$

- **法一：**

同取*log* => $\log x^{\log_b y} = \log y^{\log_b x}$

$$=> log_b\, y\, log\, x = log_b\, x\, log\, y$$

$$=> \frac{log\, y}{log\, b} log\, x = \frac{log\, x}{log\, b} y$$

$$=> log\, y\, log\, x = log\, x\, log\, y$$ --- 得證

- **法二：**

$x^{\frac{log\, y}{log\, b}} = x^{\frac{\log_x y}{\log_x b}} = y^{\frac{1}{log_x\, b}} = y^{log_b\, x}$ --- 得證

8. Assume that $a_k > 0$.
Show that $p(n) = \sum_{i=0}^{k} a_i n^i$ is in $\theta(n^k)$.

法一：要證明$p(n) \in \theta(n^k)$:

$$\lim_{n\to\infty} \frac{p(n)}{n^k} = \lim_{n\to\infty} \frac{a_0 n^0 + a_1 n^1 + a_2 n^2 + \cdots + a_{k-1} n^{k-1} + a_k n^k}{n^k}$$

$$= \lim_{n\to\infty} \left(\frac{a_0}{n^k} + \frac{a_1}{n^{k-1}} + \frac{a_2}{n^{k-2}} + \cdots + \frac{a_{k-1}}{n} + a_k\right)$$

$= a_k = const$ --- 得證

法二：要證明$p(n) \in \theta(n^k)$:

$$a_0 n^0 + a_1 n^1 + \cdots + a_k n^k \leq a_k n^k + \ldots + a_k n^k = (k+1) a_k n^k$$

$$\Rightarrow p(n) \in O(n^k)$$

$$a_0 n^0 + a_1 n^1 + \cdots + a_k n^k \geq a_k n^k \Rightarrow p(n) \in \Omega(n^k)$$

9. (10pts) Partition the following functions by their asymptotic order.

(That is, f and g are in the same partition if, and only if, $f \in \theta(g)$.)
Then list them from the lowest asymptotic order to highest asymptotic order.

$$n,\ 2^n,\ n\, log\, n,\ n^3,\ n^2,\ 7n^5 - n^3 + n,\ n^2 + log\, n,\ e^n,$$
$$\sqrt{n},\ 2^{n-1},\ \log\log n,\ log\, n,\ log^2 n,\ n!,\ n^{1+\varepsilon}(0 < \varepsilon < 1)$$

Ans. 相同時間複雜度必須標記

$$\log\log n < log\, n < log^2 n < \sqrt{n} < n < n\, log\, n <$$
$$n^{1+\varepsilon}(0 < \varepsilon < 1) < n^2 + log\, n = n^2 < n^3 < 7n^5 - n^3 + n <$$
$$2^{n-1} = 2^n < e^n < n!$$

Powers always grow faster than logarithms, for n sufficiently large.

10.(10pts)

Define a function $g(x) = max\left\{p \middle| log_2^{(p)}(x) \geq 1\right\}$.

Compute a good upper bound for $g(10^{100})$.

good upper bound→要符合$log_2^{(p)}(x) \geq 1$

$g(x) = max\left\{p \middle| log_2^{(p)}(x) \geq 1\right\}$, 找$g(10^{100})$的上界

Step1. : p=1 => $log_2^1 10^{100} = 100\log_2 10 \cong 100 * \frac{1}{0.301} \cong 332.226$

Step2. : p=2 => $log_2^2 10^{100} = log_2(log_2 10^{100}) \cong log332.226 \cong 8.XX$

Step3. : p=3 => $log_2^3 10^{100} \cong 3.XX$

Step4. : p=4 => $log_2^4 10^{100} \cong$ 1.XX

Step5. : p=5 => $log_2^5 10^{100} \cong$ 0.XX

又$log_2^{(p)}(x) \geq 1$ => p=4