Engineering Mathematics Homework 13 Solution

1. Find the Taylor series solution at x=0 for the following equation

$$2x(1-x)y''+(1+x)y'-y=0$$

Sol:

$x=0, x=1$ singular points

$x\dfrac{(1+x)}{2x(1-x)}, x^2\dfrac{-1}{2x(1-x)}$ are differentiable

$\therefore x=0$ is a regular singular point

Exist a Frobenius series solution

$$y(x)=x^r\sum_{n=0}^{\infty}a_n x^n, |x-0|<1$$

$$y(x)=a_0x^r+a_1x^{r+1}+\ldots+a_nx^{r+n}+\ldots=\sum_{n=0}^{\infty}a_nx^{n+r}$$

$$y'(x)=ra_0x^{r-1}+(r+1)a_1x^r+\ldots+(r+n)a_nx^{r+n-1}+\ldots$$

$$=\sum_{n=0}^{\infty}(r+n)a_nx^{n+r-1}$$

$$y''(x)=\sum_{n=0}^{\infty}(r+n)(r+n-1)a_nx^{n+r-2}$$

Then, the original equation can be

$$2x(1-x)\sum_{n=0}^{\infty}(r+n)(r+n-1)a_nx^{n+r-2}+(1+x)\sum_{n=0}^{\infty}(r+n)a_nx^{n+r-1}$$

$$-\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

$$\Rightarrow 2\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{n+r-1} - 2\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{n+r} +$$

$$\sum_{n=0}^{\infty}(r+n)a_n x^{n+r-1} + \sum_{n=0}^{\infty}(r+n)a_n x^{n+r} - \sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\Rightarrow 2\sum_{n=-1}^{\infty}(r+n+1)(r+n)a_{n+1} x^{n+r} - 2\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{n+r} +$$

$$\sum_{n=-1}^{\infty}(r+n+1)a_{n+1} x^{n+r} + \sum_{n=0}^{\infty}(r+n)a_n x^{n+r} - \sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\Rightarrow 2r(r-1)a_0 x^{r-1} + ra_0 x^{r-1} + \sum_{n=0}^{\infty}\{[2(n+r+1)(n+r)+(n+r+1)]a_{n+1}$$

$$+[-2(n+r)(n+r-1)+(n+r)-1]a_n\}x^{n+r} = 0$$

$$\Rightarrow [2r(r-1)+r]a_0 x^{r-1} + \sum_{n=0}^{\infty}\{A(n,r)a_{n+1} + B(n,r)a_n\}x^{n+r} = 0$$

$1.[2r(r-1)+r]a_0 = 0$

$2.A(n,r)a_{n+1} + B(n,r)a_n = 0$

$\because a_0 \neq 0 \therefore 2r(r-1)+r = 0, r = 0 \ or \ \dfrac{1}{2}$

$Case(i) \ r = 0$

$$a_{n+1} = -\frac{B(n,r)}{A(n,r)}a_n = -\frac{B(n,0)}{A(n,0)}a_n$$

$$= -\frac{(-2n(n-1)+n-1)}{2(n+1)n+n+1}a_n$$

$$= \frac{(2n-1)(n-1)}{(n+1)(2n+1)}a_n$$

$$n = 0, a_1 = \frac{(-1)\times(-1)}{1\times 1}a_0 = a_0$$

$$n = 1, a_2 = \frac{1\times 0}{2\times 3}a_1 = 0$$

$$y_1(x) = a_0 + a_0 x = a_0(1+x)$$

$$Case(ii)\ r=\frac{1}{2}$$

$$a_{n+1}=-\frac{B(n,r)}{A(n,r)}a_n=-\frac{B(n,\frac{1}{2})}{A(n,\frac{1}{2})}a_n$$

$$=-\frac{(-2(n+\frac{1}{2})n+n+\frac{1}{2}-1)}{2(n+\frac{1}{2}+1)(n+\frac{1}{2})+n+\frac{1}{2}+1}a_n$$

$$=\frac{(2n-1)n}{(2n+3)(n+\frac{3}{2})}a_n$$

$$n=0, a_1=\frac{0}{3\times\frac{3}{2}}a_0=0$$

$$a_n=0$$

$$y_2(x)=a_0$$

$$y(x)=C_1x^{\frac{1}{2}}a_0+C_2(1+x)a_0$$

2. Find the Taylor series solution at x=2 for the following equation

$$y'+y=2x^2+3x+1$$

Sol:

$$y(x)=\sum_{n=0}^{\infty}a_n(x-2)^n, |x-2|<\infty$$

$$y'(x)=\sum_{n=1}^{\infty}na_n(x-2)^{n-1}$$

$$\sum_{n=1}^{\infty} na_n(x-2)^{n-1} + \sum_{n=0}^{\infty} a_n(x-2)^n = 2x^2+3x+1$$

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}(x-2)^n + \sum_{n=0}^{\infty} a_n(x-2)^n = 2x^2+3x+1$$

$$\sum_{n=0}^{\infty} [(n+1)a_{n+1}+a_n](x-2)^n = 2x^2+3x+1$$

$$= m(x-2)^2 + n(x-2) + k$$

$$= 2(x-2)^2 + 11(x-2) + 15$$

$$n=0, a_1 + a_0 = 15$$

$$n=1, 2a_2 + a_1 = 11$$

$$n=2, 3a_3 + a_2 = 2$$

$$n \geq 3, a_{n+1} = \frac{-1}{n+1} a_n$$

$$a_2 = 2 - 3a_3$$

$$a_1 = 11 - 2a_2 = 11 - 2(2-3a_3) = 7 + 6a_3$$

$$a_0 = 15 - a_1 = 1 - (7 + 6a_3) = 8 - 6a_3$$

$$a_4 = \frac{-1}{4} a_3$$

$$a_5 = \frac{-1}{5 \times 4} a_3$$

$$a_n = \frac{6(-1)^{n-3}}{n!} a_3$$

$$\therefore y(x) = a_0 + a_1(x-2) + a_2(x-2)^2 + \ldots$$

$$= 8 - 6a_3 + (7+6a)(x-2) + (2-3a_3)(x-2)^2 + \ldots$$

$$+ \frac{6(-1)^{n-3}}{n!} a_3 (x-2)^n + \ldots$$

$$= 8 + 7(x-2) + 2(x-2)^2 - 6a_3[1 - (x-2) + \frac{1}{2}(x-2)^2$$

$$+ \ldots + \frac{(-1)^n}{n!}(x-2)^n + \ldots]$$

$$= 8 + 7(x-2) + 2(x-2)^2 - 6a_3 e^{-(x-2)}$$

3. Find the Taylor series solution at x=0 for the following equation

$$x(x-1)y''+(3x-1)y'+y=0$$

Sol:

$$y(x)=\sum_{n=0}^{\infty}a_n x^{n+r}, |x|<1$$

$$y'(x)=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''(x)=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}-\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1}$$

$$+3\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}-\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\Rightarrow\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}-\sum_{n=-1}^{\infty}(n+r+1)(n+r)a_{n+1}x^{n+r}$$

$$+3\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}-\sum_{n=-1}^{\infty}(n+r+1)a_{n+1}x^{n+r}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\Rightarrow -r(r-1)a_0x^{r-1}-ra_0x^{r-1}+\sum_{n=0}^{\infty}\{\left[-(n+r+1)(n+r)-(n+r+1)\right]a_{n+1}$$

$$+\left[(n+r)(n+r-1)+3(n+r)+1\right]a_n\}x^{n+r}=0$$

$$1.(-r(r-1)-r)a_0=0$$

$$2.a_{n+1}=a_n$$

$$\because a_0\neq 0,\ r^2=0,\ \ r=0,0$$

Use Variation of Variables

$$y_2 = \phi y_1$$

$$y_2{}' = \phi y_1' + \phi' y_1$$

$$y_2'' = \phi y_1'' + \phi' y_1' + \phi' y_1' + \phi'' y_1 = \phi y_1'' + 2\phi' y_1' + \phi'' y_1$$

$$x(x-1)(\phi y_1'' + 2\phi' y_1' + \phi'' y_1) + ((3x-1)(\phi y_1' + \phi' y_1)) + \phi y_1 = 0$$

$$\phi\left[x(x-1)y_1'' + (3x-1)\, y_1' + y_1\right] + x(x-1)(2\phi' y_1' + \phi'' y_1) + (3x-1)\phi' y_1 = 0$$

$$y_1 = \frac{1}{1-x},\ y_1' = \frac{1}{(1-x)^2}$$

$$\Rightarrow x(x-1)\left[2\phi' \frac{1}{(x-1)^2} + \phi'' \frac{-1}{x-1}\right] + (3x-1)\ \phi' \frac{-1}{x-1} = 0$$

$$2\phi' \frac{x}{x-1} + \phi''(-x) + \phi'\left(\frac{-(3x-1)}{x-1}\right) = 0$$

$$(2x-3x+1)\phi' + (-x^2+x)\phi'' = 0$$

$$\phi'' + \frac{1}{x}\phi' = 0$$

$$\frac{d\phi'}{\phi'} = -\frac{1}{x}dx$$

$$\ln|\phi'| = -\ln|x|$$

$$\phi' = \frac{1}{x} \qquad \therefore \phi = \ln x$$

$$y_2 = \phi y_1 = \frac{\ln x}{1-x}$$

$$y(x) = C_1 \frac{1}{1-x} + C_2 \frac{\ln x}{1-x}$$