上週提到 $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y^{0} = r(x)$.

其中 $a_0 \sim a_n \in$ const, $r(x) \neq 0$ (常係數非齊性 O.D.E)

我們說其解 $y = y_h + y_p$.

y_h: { 代 $e^{\lambda x}$ 解 λ
D (降階尋根)

y_p: { 未定係數法
降階法 $\Rightarrow y_p = \cdots I_2^{-1} \int I_2 [I_1^{-1} \int I_1 r\, dx]\, dx \cdots$,

↳ I_1, I_2 其實就是由

$(D+\lambda_1)(D+\lambda_2) y_p = r$

$\Rightarrow$ $I_1 = e^{\int \lambda_1 dx} = e^{\lambda_1 x}$, $I_2 = e^{\int \lambda_2 dx} = e^{\lambda_2 x}$ 其中 I_1, I_2 可互換(結果同).

Method 3. 線性微分運算子法

($D \equiv \frac{d}{dx}$, $D^n = \frac{d^n}{dx^n}$, $D^{-1} \equiv \int dx$)

$1 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y^{0} = r(x)$.

其特性方程式

$$\lambda^n + a_1 \lambda^{(n-1)} + \cdots + a_n = 0$$

$\Rightarrow$ 求出 $\lambda_1 \sim \lambda_n$ 即可得到 y_h.

当改寫特性方程式

$$\Rightarrow (D^n + a_1 D^{n-1} + \cdots + a_n)\, y = r(x)$$

其中 y_p 必能滿足上述式子.

$$\Rightarrow \underbrace{(D^n + a_1 D^{n-1} + \cdots + a_n)}\, y_p = r(x)$$

└ Linear differential operator 線性微分運算子

定义為 $L(D)$

其特性:

① $L(D) e^{ax} = L(a) e^{ax}$

ex. $y''' + 6y'' + 11y' + 6y = e^{x}$

之前解出 $y = \frac{1}{24} e^{x}$

用現在的方法.

$\Rightarrow (D^3+6D^2+11D+6)\, y_p = e^x$

$\Rightarrow y_p = \dfrac{1}{D^3+6D^2+11D+6}\cdot e^x$

$= L(D)\cdot e^x = L(1)\cdot e^x = \frac{1}{24}e^x$ #.

pf: 特性①.

$L(D)e^{ax}$

$= (D^n + a_1 D^{n-1} + \cdots + a_n)\, e^{ax}$

$= D^n e^{ax} + a_1 D^{n-1} e^{ax} + \cdots + a_n e^{ax}$

$= a^n e^{ax} + a_1 D^{n-1} e^{ax} + \cdots + a_n e^{ax}$

$= L(a)e^{ax}$.

依D的定义. $D^n \equiv \frac{d^n}{dx^n}$

$\Rightarrow D^n \cdot e^{ax} = \frac{d^n}{dx^n} e^{ax} = a^n e^{ax}$.

ex. $y'' + 3y' + 2y = e^{2x}$.

$y_h = c_1 e^{-x} + c_2 e^{-2x}$.

$y_p = \dfrac{1}{D^2+3D+2} e^{2x}$.

$= L(D)e^{2x} = L(2)e^{2x} = \frac{1}{12}e^{2x}$

ex. $y' - 2y = e^{2x}$

$y_h = c_1 e^{2x}$

$y_p = \dfrac{1}{D-2} e^{2x} = \dfrac{1}{2-2} e^{2x}$?

$\Rightarrow$ 当 $r(x)$ 與 $e^{\lambda_i x}$ 相同時 或 重根時. 都要乘个 x^n 之类的使其不相等.

$\Rightarrow$ ② $L(D)[e^{ax}\cdot f(x)] = e^{ax}\cdot L(D+a)[f(x)]$.

$\Rightarrow$ 回到上面的 ex.

$\Rightarrow y_p = \frac{1}{D-2} e^{2x}$

$= \frac{1}{D-2} [e^{2x} \cdot 1] \rightarrow$ 扮演 $f(x)$ 的角色.

$= e^{2x} \cdot \left(\frac{1}{D-2}\right) \cdot 1$

└→ 此時 $D = D+2$.

$= e^{2x} \cdot \frac{1}{D} \cdot 1$

(又 $\frac{1}{D} = D^{-1} \equiv \int dx$.)

$= e^{2x} \int 1 \, dx \cdot 1 = e^{2x} \cdot x$ #.

ex. $y'' + 4y' + 4y = e^{-2x}$

$y_h = c_1 e^{-2x} + c_2 e^{-2x} \cdot x$

$y_p = \frac{1}{D^2+4D+4} e^{-2x}$

$= \frac{1}{(D+2)^2} e^{-2x} = e^{-2x} \cdot \frac{1}{(D+2)^2} \cdot 1 \rightarrow f(x)$

$= e^{-2x} \cdot \left(\frac{1}{(D+2)^2}\right) \cdot 1$

└→ 此時 $D = D-2$

$= e^{-2x} \cdot D^{-2} \cdot 1 = e^{-2x} \iint 1 \, dx\, dx = \frac{1}{2} x^2 e^{-2x}$ #.

pf: 特性②.

$L(D)[e^{ax} \cdot f(x)] = (D^n + a_1 D^{n-1} + \cdots + a_n) e^{ax} \cdot f(x)$

$= D^n e^{ax} f(x) + a_1 D^{n-1} e^{ax} f(x) + \cdots + a_n e^{ax} f(x)$

又 $D(e^{ax} \cdot f(x)) = e^{ax} \cdot D f(x) + a \cdot e^{ax} f(x)$

$= e^{ax} (D+a) \cdot f(x)$.

$D^2 (e^{ax} \cdot f(x)) = D[D(e^{ax} \cdot f(x))]$

$= e^{ax} (D+a)^2 \cdot f(x)$.

$\vdots$

$$\Rightarrow D^n e^{ax} f(x) = e^{ax}(D+a)^n \cdot f(x).$$

$$\Rightarrow L(D)\, e^{ax} f(x) = e^{ax}(D+a)^n \cdot f(x)$$
$$+ a_1 e^{ax}(D+a)^{n-1} \cdot f(x)$$
$$+ \quad \vdots$$
$$+ a_{n-1} e^{ax}(D+a) \cdot f(x)$$
$$+ a_n e^{ax} \cdot f(x).$$
$$= e^{ax}\left[(D+a)^n + a_1(D+a)^{n-1} + \dots + a_{n-1}(D+a) + a_n\right] \cdot f(x)$$
$$= e^{ax} L(D+a) \cdot f(x). \quad \#.$$

③ $L(D^2)\cos ax = L(-a^2)\cos ax.$

④ $L(D^2)\sin ax = L(-a^2)\sin ax.$

ex. $y'' + 4y = \cos 3x.$

$y_h = c_1 \cos 2x + c_2 \sin 2x$

$y_p = \dfrac{1}{D^2+4} \cos 3x$

$= L(-3^2)\cos 3x = -\frac{1}{5}\cos 3x.$

pf: 特性 ③. ④.

$D \cos ax = -a \sin ax$

$D^2 \cos ax = D(D \cos ax)$

$= -a^2 \cos ax$

$D^2 \equiv -a^2$

$\Rightarrow L(D^2) \equiv L(-a^2)$

ex. $y'' + a^2 y = \cos ax$

$y_h = c_1 \cos ax + c_2 \sin ax$

$$y_p = \frac{1}{D^2+a^2}\cos ax = \frac{1}{-a^2+a^2}\cos ax \quad ?$$

又遇到類似狀況，這次引進 limit.

$$\Rightarrow y_p = \lim_{\Delta \to 0} \frac{1}{-(a+\Delta)^2+a^2}\cos(a+\Delta)x$$

$$= \lim_{\Delta \to 0} \frac{1}{\Delta(-2a-\Delta)}\cos(a+\Delta)x$$

$f(t)$ 於 $t=a$ 之 Taylor 級數展開式

$$f(t) = f(a) + f'(a)(t-a) + \frac{1}{2!}f''(a)(t-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(t-a)^n + \cdots$$

$\Rightarrow \cos t$ 於 $t=ax$ 之 Taylor 展開.

$$\cos t = \cos ax - \sin ax\,(t-ax) - \frac{1}{2!}\cos ax\,(t-ax)^2 + \frac{1}{3!}\sin ax\,(t-ax)^3 + \cdots$$

令 $t = (a+\Delta)x$.

$$\Rightarrow \cos(a+\Delta)x = \cos ax - \sin ax\,((a+\Delta)x - ax) - \frac{1}{2!}\cos ax\,((a+\Delta)x-ax)^2 + \frac{1}{3!}\sin ax\,((a+\Delta)x-ax)^3 \; \vdots$$

$$= \cos ax - \sin ax \cdot \Delta x - \frac{1}{2!}\cos ax \cdot (\Delta x)^2 + \frac{1}{3!}\sin ax \cdot (\Delta x)^3 \; \vdots$$

$$\Rightarrow y_p = \lim_{\Delta \to 0} \frac{1}{-\Delta(2a+\Delta)}\left[\cos ax - \sin ax \cdot \Delta x - \frac{1}{2!}\cos ax \cdot (\Delta x)^2 + \frac{1}{3!}\sin ax \cdot (\Delta x)^3 \; \vdots \right]$$

$\because y_h$ 已含有 $\cos ax \Rightarrow$ 可忽略

$$\Rightarrow y_p = \lim_{\Delta \to 0} \frac{1}{-\cancel{\Delta}(2a+\Delta)} \cdot \cancel{\Delta}\left[-\sin ax \cdot x - \frac{1}{2!}\cos ax \cdot \Delta \cdot x^2 + \frac{1}{3!}\sin ax \cdot \Delta^2 \cdot x^3 + \cdots \right]$$

$= \frac{1}{2a}(\sin ax)\cdot x$ #.

ex. $y''+4y=\cos 2x$

$y_h = c_1\cos 2x + c_2 \sin 2x$

$y_p = \frac{1}{2a} x \sin ax = \frac{1}{4} x \sin 2x$ #.

ex. $y''+a^2 y = \sin ax$

⋮ 自己練習.

$y_p = \frac{1}{-2a} x \cdot \cos ax$

ex. $y''+9y=\cos 3x$.

$y = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{6} x \sin 3x$.

ex. $y''+6y'+9y = x^2$ → 多項式.

$y_h = c_1 e^{-3x} + c_2 \cdot x \cdot e^{-3x}$

y_p:

① $= k_1 x^2 + k_2 x + k_3$ } 一起代回原式.

$\Rightarrow y_p', y_p''$

$\Rightarrow$ 得 k_1, k_2, k_3 $\Rightarrow$ 得 y_p.

② $(D^2+6D+9)y_p = x^2$

$\Rightarrow (D+3)(D+3)y_p = x^2$

$\Rightarrow y_p = e^{-3x}\int e^{3x}\left[e^{-3x}\int e^{3x}\cdot x^2\cdot dx\right]dx$

③ $y_p = \frac{1}{D^2+6D+9} x^2 = \frac{1}{9\left(1+\frac{D^2+6D}{9}\right)} x^2$

引進無窮等比級數表示法.

$$= \frac{1}{9}\left[1 - \frac{D^2+6D}{9} + \left(\frac{D^2+6D}{9}\right)^2 - \cdots\right]x^2$$

$$= \frac{1}{9}\left[1 - \frac{D^2+6D}{9} + \frac{D^4+12D^3+36D^2}{81} - \cdots\right]x^2$$

只需到第三项是因为之後的微分次数已大过 x^2 项 ⇒ 都是 0.

$$= \frac{1}{9}\left[x^2 - \frac{1}{9}(2+6\cdot 2x) + \frac{1}{81}(36\cdot 2)\right]$$

$$= \frac{1}{9}\left(x^2 - \frac{4}{3}x + \frac{2}{3}\right) \quad \#.$$