

Chapter 4. Laplace Transform

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Laplace Transform

EX : $F(s) = \frac{s}{(s+1)(s-2)^2}$

$$= \frac{k_1}{s+1} + \frac{k_2}{s-2} + \frac{k_3}{(s-2)^2}$$

$$f(t) = k_1 e^{-t} + k_2 e^{2t} + k_3 t e^{2t}$$

$$k_1 = \frac{-1}{9}, k_2 = \frac{1}{9}, k_3 = \frac{2}{3}$$

Ex:

$$L^{-1} \left\{ \frac{1}{(s-2)^3} \right\} = \frac{1}{2} t^2 e^{2t}$$

$$L^{-1} \left\{ \frac{1}{(s-2)^n} \right\} = \frac{1}{(n-1)!} t^{(n-1)} e^{2t}$$

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Laplace Transform

EX : $F(s) = \frac{3s+2}{(s+3)(s-1)^2}$

$$= \frac{k_1}{s+3} + \frac{k_2}{s-1} + \frac{k_3}{(s-1)^2}$$

$$k_1 = \frac{-7}{16}, k_2 = \frac{7}{16}, k_3 = \frac{5}{4}$$

$$f(t) = \frac{-7}{16}e^{-3t} + \frac{7}{16}e^t + \frac{5}{4}te^t$$

Laplace Transform

EX : $F(s) = \frac{s^2 + s + 1}{(s-1)(s-3)^2}$

$$= \frac{k_1}{s-1} + \frac{k_2}{s-3} + \frac{k_3}{(s-3)^2}$$

$$k_1 = \frac{3}{4}, k_2 = \frac{1}{4}, k_3 = \frac{13}{2}$$

$$f(t) = \frac{3}{4}e^t + \frac{1}{4}e^{3t} + \frac{13}{2}te^{3t}$$

Laplace Transform

EX : $F(s) = \frac{s}{(s-1)(s^2+4s+13)}$

$= \frac{k_1}{(s-1)} + \frac{k_2s+k_3}{(s^2+4s+13)}$ 共軛複數根，假設 $k_2s + k_3$

$k_1 = \frac{1}{18}$

$\frac{1}{18} \frac{(s^2+4s+13) + (k_2s+k_3)(s-1)}{(s-1)(s^2+4s+13)}$

$\frac{1}{18} + k_2 = 0, k_2 = -\frac{1}{18}$

$\frac{13}{18} - k_3 = 0, k_3 = \frac{13}{18}$

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Laplace Transform

$F(s) = \frac{1}{18} \frac{1}{(s-1)} + \frac{-\frac{1}{18}s + \frac{13}{18}}{(s^2+4s+13)}$

$= \frac{1}{18} \frac{1}{(s-1)} + \frac{-\frac{1}{18}(s+2) + \frac{15}{18}}{(s+2)^2+3^2}$ 配方

$f(t) = \frac{1}{18} e^t - \frac{1}{18} \cos(3t) e^{-2t} + \frac{5}{18} \sin(3t) e^{-2t}$

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Laplace Transform

EX : $y'' - 5y' + 6y = e^{-t}, y(0) = 0, y'(0) = 2$

$$\mathcal{L}\{y'' - 5y' + 6y\} = \mathcal{L}\{e^{-t}\}$$

$$s^2 Y(s) - sy(0) - y'(0) - 5(sY(s) - y(0)) + 6Y(s) = \frac{1}{s+1}$$

$$s^2 Y(s) - 2 - 5sY(s) + 6Y(s) = \frac{1}{s+1}$$

$$Y(s)(s^2 - 5s + 6) = \frac{1}{s+1} + 2 = \frac{2s+3}{s+1}$$

$$Y(s) = \frac{2s+3}{(s+1)(s-2)(s-3)}$$

$$= \frac{k_1}{s-2} + \frac{k_2}{s-3} + \frac{k_3}{s+1}$$

$$k_1 = \frac{-7}{3}, k_2 = \frac{9}{4}, k_3 = \frac{1}{12}$$

$$y(t) = \frac{-7}{3}e^{2t} + \frac{9}{4}e^{3t} + \frac{1}{12}e^{-t}$$

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Laplace Transform

EX : $x'' + 4x' + 4x = 4, x(0) = 0, x'(0) = 0$

$$s^2 X(s) - sx(0) - x'(0) + 4(sX(s) - x(0)) + 4X(s) = \frac{4}{s}$$

$$(s^2 + 4s + 4)X(s) = \frac{4}{s}$$

$$X(s) = \frac{4}{s(s+2)^2} = \frac{k_1}{s} + \frac{k_2}{s+2} + \frac{k_3}{(s+2)^2}$$

$$k_1 = 1, k_2 = -1, k_3 = -2$$

$$x(t) = 1 - e^{-2t} - 2te^{-2t}$$

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Laplace Transform

EX : $y'' + 4y' + 5y = e^{-t}, y(0) = 0, y'(0) = 1$

$$s^2 Y(s) - 1 + 4sY(s) + 5Y(s) = \frac{1}{s+1}$$

$$(s^2 + 4s + 5)Y(s) = \frac{1}{s+1} + 1$$

$$Y(s) = \frac{s+2}{(s+1)(s^2 + 4s + 5)}$$

$$= \frac{k_1}{s+1} + \frac{As+B}{s^2 + 4s + 5}$$

$$k_1 = \frac{1}{2}, A = -\frac{1}{2}, B = -\frac{1}{2}$$

Laplace Transform

$$Y(s) = \frac{\frac{1}{2}}{s+1} + \frac{-\frac{1}{2}s - \frac{1}{2}}{s^2 + 4s + 5}$$

$$= \frac{\frac{1}{2}}{s+1} + \frac{-\frac{1}{2}(s+2) + \frac{1}{2}}{(s+2)^2 + 1}$$

$$y(t) = \frac{1}{2}e^{-t} - \frac{1}{2}\cos(t)e^{-2t} + \frac{1}{2}\sin(t)e^{-2t}$$

Laplace Transform

EX : Solve the following integral differential equation

$$y' - 2y = \int_0^t e^{2(t-\tau)} \cos(3\tau) d\tau = \int_0^t g(\tau) f(t-\tau) d\tau, \quad y(0) = 0$$

$$y' - 2y = f(t) \otimes g(t) = e^{2t} \otimes \cos 3t$$

$$\mathcal{L}\{y' - 2y\} = \mathcal{L}\{e^{2t} \otimes \cos 3t\}$$

$$sY(s) - y(0) - 2Y(s) = F(s)G(s) = \frac{1}{s-2} \frac{s}{s^2+9}$$

$$(s-2)Y(s) = \frac{s}{(s-2)(s^2+9)}$$

$$Y(s) = \frac{s}{(s-2)^2(s^2+9)} = \frac{k_1}{s-2} + \frac{k_2}{(s-2)^2} + \frac{As+B}{s^2+9}$$

$$k_2 = \frac{2}{13}$$

$$k_1 = \frac{d}{ds} \left(\frac{s}{s^2+9} \right) \Big|_{s=2} = \frac{s^2+9-s(2s)}{(s^2+9)^2} \Big|_{s=2} = \frac{5}{169}$$

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Laplace Transform

$$Y(s) = \frac{\frac{5}{169}(s^2+9)(s-2) + \frac{2}{13}(s^2+9) + (As+B)(s-2)^2}{(s-2)^2 + (s^2+9)}$$

$$\frac{5}{169} + A = 0, A = -\frac{5}{169}$$

$$\frac{-90}{169} + \frac{18}{13} + 4B = 0, B = -\frac{36}{169}$$

$$y(t) = \frac{5}{169} e^{2t} + \frac{2}{13} t e^{2t} - \frac{5}{169} \cos(3t) - \frac{12}{169} \sin(3t)$$

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Laplace Transform

*Vaterra積分方程式

$$\text{EX : } f(t) = -1 + \int_0^t f(t-\tau)e^{-3\tau}d\tau$$

$$\text{求 } f(t) = ?$$

$$F(s) = \frac{-1}{s} + F(s) \frac{1}{s+3}$$

$$F(s) = \frac{-(s+3)}{(s+2)s} = \frac{k_1}{s} + \frac{k_2}{s+2}$$

$$k_1 = \frac{-3}{2}, k_2 = \frac{1}{2}$$

$$f(t) = \frac{-3}{2} + \frac{1}{2}e^{-2t}$$

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Laplace Transform

$$\text{EX : } f(t) = e^{-2t} - 3e^{-3t} \int_0^t f(\tau)e^{3\tau}d\tau$$

$$f(t) = e^{-2t} - 3 \int_0^t f(\tau)e^{-3(t-\tau)}d\tau$$

$$F(s) = \frac{1}{s+2} - 3F(s) \frac{1}{s+3}$$

$$F(s) = \frac{s+3}{(s+6)(s+2)} = \frac{k_1}{s+6} + \frac{k_2}{s+2}$$

$$k_1 = \frac{3}{4}, k_2 = \frac{1}{4}$$

$$f(s) = \frac{3}{4}e^{-6t} + \frac{1}{4}e^{-2t}$$

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Laplace Transform

- 聯立方程式

$$\text{EX : } \begin{cases} \frac{dx_1(t)}{dt} = x_2(t) \\ \frac{dx_2(t)}{dt} = -2x_1(t) - 3x_2(t) \end{cases} \quad x_1(0) = 1, x_2(0) = 1$$

耦合 *Coupled*

$$\begin{cases} sX_1(s) - x_1(0) = X_2(s) \\ sX_2(s) - x_2(0) = -2X_1(s) - 3X_2(s) \end{cases}$$

$$\Rightarrow \begin{cases} sX_1(s) - X_2(s) = 1 \\ 2X_1(s) + (s+3)X_2(s) = 1 \end{cases}$$

$$X_1(s) = \frac{\begin{vmatrix} 1 & -1 \\ 1 & s+3 \end{vmatrix}}{\begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix}} = \frac{s+3+1}{s^2+3s+2} = \frac{3}{s+1} + \frac{-2}{s+2}$$

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Laplace Transform

$$X_2(s) = \frac{\begin{vmatrix} s & 1 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix}} = \frac{s-2}{s^2+3s+2} = \frac{-3}{s+1} + \frac{4}{s+2}$$

$$x_1(t) = 3e^{-t} - 2e^{-2t}$$

$$x_2(t) = -3e^{-t} + 4e^{-2t}$$

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Laplace Transform

*線性代數

$$\begin{aligned} \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} &= \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{s(s+3)+2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{s^2+3s+2} \begin{bmatrix} s+4 \\ s-2 \end{bmatrix} = \begin{bmatrix} \frac{s+4}{s^2+3s+2} \\ \frac{s-2}{s^2+3s+2} \end{bmatrix} \end{aligned}$$

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Laplace Transform

EX : $y'' + 2ty' - 4y = 6, \quad y(0) = 0, \quad y'(0) = 0$

$$\Rightarrow S^2 Y(s) + 2 \left(-\frac{d(SY(s))}{ds} \right) - 4Y(s) = \frac{6}{s}$$

$$S^2 Y(s) - 2Y(s) - 2SY'(s) - 4Y(s) = \frac{6}{s}$$

$$-2SY'(s) + (s^2 - 6)Y(s) = \frac{6}{s}$$

$$y' + py = r$$

$$y = CI^{-1} + I^{-1} \int I r dx$$

$$I = e^{\int p dx}$$

$$Y'(s) + \frac{s^2 - 6}{-2s} Y(s) = \frac{6}{s(-2s)}$$

$$I = e^{\int \frac{s^2 - 6}{-2s} ds} = e^{\int \left(\frac{-s}{2} + \frac{3}{s} \right) ds} = e^{\frac{-1}{4}s^2 + 3 \ln s} = e^{\frac{-1}{4}s^2} \cdot s^3$$

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Laplace Transform

$$Y(s) = Ce^{\frac{1}{4}s^2} \cdot s^{-3} + e^{\frac{1}{4}s^2} \cdot s^{-3} \int e^{-\frac{1}{4}s^2} \cdot s^3 \cdot \frac{6}{s(-2s)} ds$$

$$\left(\text{令 } u = \frac{-s^2}{4}, du = \frac{-1}{2} s ds \right)$$

$$= Ce^{\frac{1}{4}s^2} \cdot s^{-3} + e^{\frac{1}{4}s^2} \cdot s^{-3} \int 6e^u du$$

$$= Ce^{\frac{1}{4}s^2} \cdot s^{-3} + 6s^{-3}$$

如何解C，利用初值定理

$$y(0) = \lim_{s \rightarrow \infty} sY(s) = \lim_{s \rightarrow \infty} \left(Ce^{\frac{s^2}{4}} s^{-2} + \frac{6}{s^2} \right) = 0$$

$$\therefore C = 0$$

$$\therefore Y(s) = 6s^{-3}$$

$$Y(t) = 3t^2$$

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Laplace Transform

$\sigma(t-2)$: 只有在 $t=2$ 時有意義

$$\text{EX : } y'' + 3y' + 2y = \delta(t-2), y(0) = 0, y'(0) = 1$$

$$(s^2 Y(s) - 0 - 1) + 3(sY(s) - 0) + 2Y(s) = e^{-2s}$$

$$Y(s) = \frac{1 + e^{-2s}}{s^2 + 3s + 2} = \frac{1}{s+1} + \frac{-1}{s+2} + \left[\frac{1}{s+1} + \frac{-1}{s+2} \right] e^{-2s}$$

$$y(t) = e^{-t} - e^{-2t} + \left[e^{-(t-2)} - e^{-2(t-2)} \right] H(t-2)$$

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Laplace Transform

EX: Solve the following problems

(i) $y'' + 4y' + 3y = 3\delta(t - 2) + H(t - 1), y(0) = y'(0) = 0$

$$S^2Y(s) - SY(0) - Y'(0) + 4(SY(0) - Y(0)) + 3Y(s) = 3e^{-2s} + \frac{1}{s}e^{-s}$$

$$(S^2 + 4S + 3)Y(s) = 3e^{-2s} + \frac{1}{s}e^{-s}$$

$$Y(s) = \frac{3e^{-2s}}{(S^2 + 4S + 3)} + \frac{e^{-s}}{s(S^2 + 4S + 3)}$$

$$= \left(\frac{\frac{3}{2}}{s+1} + \frac{\frac{-3}{2}}{s+3} \right) e^{-2s} + \left(\frac{\frac{1}{3}}{s} + \frac{\frac{-1}{2}}{s+1} + \frac{\frac{1}{6}}{s+3} \right) e^{-s}$$

$$y(t) = \left[\frac{3}{2}e^{-(t-2)} - \frac{3}{2}e^{-3(t-2)} \right] H(t-2) + \left[\frac{1}{3} - \frac{1}{2}e^{-(t-1)} + \frac{1}{6}e^{-3(t-1)} \right] H(t-1)$$

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Laplace Transform

(ii) $y'' + y = f(t), f(t) = \begin{cases} 0, 0 \leq t < \pi \\ 1, \pi \leq t < 2\pi, y(0) = 0, y'(0) = 1 \\ 0, t \geq 2\pi \end{cases}$

$$(S^2Y(s) - SY(0) - Y'(0)) + Y = \frac{1}{s}e^{-\pi s} - \frac{1}{s}e^{-2\pi s}$$

$$(S^2 + 1)Y(s) = 1 + \frac{1}{s}[e^{-\pi s} - e^{-2\pi s}]$$

$$Y(s) = \frac{1}{(S^2 + 1)} + \frac{1}{S(S^2 + 1)}(e^{-\pi s} - e^{-2\pi s})$$

$$= \frac{1}{(S^2 + 1)} + \left(\frac{1}{s} + \frac{-s}{(S^2 + 1)} \right) (e^{-\pi s} - e^{-2\pi s})$$

$$y(t) = \sin t + [1 - \cos(t - \pi)]H(t - \pi) - [1 - \cos(t - 2\pi)]H(t - 2\pi)$$

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Laplace Transform

$$(iii) \quad f(t) = \int_0^t f(t-\tau) e^{-\tau} d\tau + 3t^5$$

$$F(s) = F(s) \frac{1}{s+1} + 3 \frac{5!}{s^{5+1}}$$

$$\frac{s}{s+1} F(s) = 3 \frac{5!}{s^{5+1}}$$

$$F(s) = \frac{3 \cdot 5! \cdot (s+1)}{s \cdot s^6}$$

$$= \frac{3 \cdot 5!}{s^6} + \frac{3 \cdot 5! \cdot \frac{6!}{6!}}{s^7}$$

$$f(t) = 3t^5 + \frac{1}{2}t^6$$