

# Chapter 5. Series Solutions of Linear Differential Equations

Chuan-Ching Sue

Dept. of Computer Science and Information Engineering,  
National Cheng Kung University

1

## Series Solutions

- 微分方程式的級數

$$P(x)y'' + q(x)y' + r(x)y = 0$$

在  $x = a$  處的級數解是什麼?

定義：如果在  $x = a$ ,  $y(x)$  的任意階導數均存在  
則在  $x = a$  處,  $y(x)$  存在一Taylor級數解  
可表成

$$\begin{aligned} y(x) &= y(a) + y'(a)(x-a) + \frac{y''(a)}{2!}(x-a)^2 + \cdots + \frac{y^{(n)}(a)}{n!}(x-a)^n + \cdots \\ &= \sum_{n=0}^{\infty} \frac{y^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

2

# Series Solutions

把這些係數另外表成

$$\sum_{n=0}^{\infty} a_n (x-a)^n, \text{ 其中 } a_n = \frac{y^{(n)}(a)}{n!}$$

把原本決定  $y(x)$  在  $x=a$  的任意階導數  $\frac{y^{(n)}(a)}{n!}$   
改為決定  $(x-a)^n$  的係數  $a_n$

# Series Solutions

EX:  $y' + 2y = 0, x = 0$  的級數解

$$\Rightarrow y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

帶入原式  $y' + 2y = 0$

$$\Rightarrow \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n$$

想法：令  $k = n - 1, n = k + 1$

$$\Rightarrow \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k \quad \text{令 } k = n$$

# Series Solutions

答案：  $\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} [(n+1)a_{n+1} + 2a_n]x^n = 0$$

$$\Rightarrow (n+1)a_{n+1} + 2a_n = 0$$

$$a_{n+1} = \frac{-2}{n+1}a_n \quad (n \geq 0)$$

$$n=0 \Rightarrow a_1 = \frac{-2}{1}a_0$$

$$n=1 \Rightarrow a_2 = \frac{-2}{2}a_1 = \frac{-2}{2} \frac{-2}{1}a_0 = \frac{(-2)^2}{2!}a_0$$

$$n=2 \Rightarrow a_3 = \frac{-2}{3}a_2 = \frac{-2}{3} \frac{-2}{2} \frac{-2}{1}a_0 = \frac{(-2)^3}{3!}a_0$$

$\vdots$

$$a_n = \frac{(-2)^n}{n!}a_0$$

5

# Series Solutions

$$\begin{aligned} \Rightarrow y(x) &= a_0 + \frac{(-2)}{1!}a_0x + \frac{(-2)^2}{2!}a_0x^2 + \cdots + \frac{(-2)^n}{n!}a_0x^n + \cdots \\ &= a_0 \left( 1 + \frac{(-2)}{1!}x + \frac{(-2)^2}{2!}x^2 + \cdots + \frac{(-2)^n}{n!}x^n + \cdots \right) \\ &= a_0 e^{-2x} \end{aligned}$$

$$\begin{cases} e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots \\ \cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \\ \sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \end{cases}$$

6

# Series Solutions

- $P(x)y'' + q(x)y' + r(x)y = 0 \dots (1)$

若在  $x = a$  ,  $y$  存在 Taylor 級數解 (ie 找到任意階導數)

則稱  $x = a$  , 為方程式(1)的常點 (Ordinary Point)

否則稱  $x = a$  為方程式(1)的奇異點 (Singular Point)

# Series Solutions

EX:  $(x-1)y' + 2y = 0$  請問  $x = 1$  是常點或奇異點

假設  $y(1) = C$  , 若  $y'(1), y''(1), \dots, y^{(n)}(1)$  找到就是常點

$$y'(x) = -\frac{2}{x-1}y(x)$$

$$y'(1) = \frac{2}{1-1}y(1) = \infty$$

EX:  $(x-1)^2 y'' + 2xy' + 3y = 0, y(1) = a, y'(1) = b, x = 1$  是什麼點?

$$\therefore y''(1) = \infty$$

$\therefore x = 1$  singular point

# Series Solutions

- 歸納

$$y'' + \frac{q(x)}{p(x)} y' + \frac{r(x)}{p(x)} y = 0$$

$$p(a) = 0 \Rightarrow x = a \text{ is singular point}$$

令  $p(x) = 0$  的點, 皆為異點  $p(x)$  為和  $q(x)$  化簡後

# Series Solutions

$$\text{EX: } (x+1)(x-2)y'' + 3xy' + 4y = 0$$

異點何在?

$$x = -1, 2$$

$$\text{EX : } (x+1)(x-2)y'' + 3(x-2)y' + 4(x-2)y = 0$$

異點何在?

$$x = -1$$

# Series Solutions

- Note:  $p(x).q(x).r(x)$  沒有公因式

$$\text{設 } y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(a)}{n!} (x-a)^n \quad p(a) \neq 0$$

讓上述  $y(x)$  級數解存在的收斂區間為  $|x-a| < L$  ( $L$ : 收斂半徑)

$L$ : 由  $x=a$  處到最近異點的距離

11

# Series Solutions

EX:  $(x-1)y' + 2y = 1$

$x = 1$  異點

$x = 0$  常點

$$\therefore y(x) = \sum_{n=0}^{\infty} a_n x^n, |x-0| < L = 1$$

EX:  $y'' + y = 0$

$x = 0$  常點

$$\therefore y(x) = \sum_{n=0}^{\infty} a_n x^n, |x-0| < L = \infty$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

12

# Series Solutions

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$y'' + y = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + a_n] x^n = 0$$

$$\Rightarrow a_{n+2} = \frac{-a_n}{(n+2)(n+1)} \text{ (循環公式, recurrence formula)}$$

$$a_n = ?$$

13

# Series Solutions

∴ 二階微分方程式本來就有兩個未知數

$$\begin{cases} a_{2n} = \frac{(-1)^n}{(2n)!} a_0 \\ a_{2n+1} = \frac{(-1)^n}{(2n+1)!} a_1 \end{cases}$$

$$\therefore y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} a_0 x^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} a_1 x^{2n+1}$$

$$= a_0 \cos x + a_1 \sin x$$

14

# Series Solutions

EX:  $y' + 2y = 1, x=0$  的級數解

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, |x| < \infty$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\Rightarrow \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 1$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 1$$

$$\Rightarrow \sum_{n=0}^{\infty} [(n+1) a_{n+1} + 2a_n] x^n = 1$$

15

# Series Solutions

$$\begin{cases} n=0, a_1 + 2a_0 = 1 \\ n \geq 1, (n+1)a_{n+1} + 2a_n = 0 \end{cases}$$

$$\Rightarrow a_{n+1} = \frac{-2}{n+1} a_n$$

$$n=1, a_2 = \frac{-2}{2} a_1$$

$$n=2, a_3 = \frac{-2}{3} a_2 = \frac{(-2)(-2)}{3 \cdot 2} a_1$$

$$n=3, a_4 = \frac{-2}{4} a_3 = \frac{(-2)(-2)(-2)}{4 \cdot 3 \cdot 2} a_1$$

$\vdots$

$$a_n = \frac{(-2)^{n-1}}{n!} a_1$$

$$y(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

$$= \frac{1}{2} - \frac{1}{2} a_1 + a_1 x + \frac{-2}{2} a_1 x^2 + \dots + \frac{(-2)^{n-1}}{n!} a_1 x^n + \dots$$

$$= \frac{1}{2} - \frac{1}{2} a_1 (1 - 2x + \frac{(-2)^2}{2!} x^2 + \dots + \frac{(-2)^n}{n!} x^n + \dots)$$

$$= \frac{1}{2} - \frac{1}{2} a_1 e^{-2x}$$

16



# Series Solutions

EX:  $y' + 2y = x + 1$ ,  $x = 1$  的級數解

$\because x = 1$  是常點

$\therefore$  存在 Taylor 級數解

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n, |x-1| < L = \infty$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$$

$$\Rightarrow \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + 2 \sum_{n=0}^{\infty} a_n (x-1)^n = x + 1$$

$$\Rightarrow \sum_{n=0}^{\infty} [(n+1)a_{n+1} + 2a_n](x-1)^n = (x-1) + 2$$

17

# Series Solutions

$$\Rightarrow \begin{cases} n=0, a_1 + 2a_0 = 2 \\ n=1, 2a_2 + 2a_1 = 1 \\ n \geq 2, (n+1)a_{n+1} + 2a_n = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a_0 = \frac{2-a_1}{2} = 1 - \frac{1}{4} + \frac{a_2}{2} = \frac{3}{4} + \frac{a_2}{2} \\ a_1 = \frac{1-2a_2}{2} = \frac{1}{2} - a_2 \end{cases}$$

$$n=2, a_3 = \frac{-2}{3} a_2$$

$$n=3, a_4 = \frac{-2}{4} a_3 = \frac{(-2)(-2)}{4 \cdot 3} a_2$$

$$n=4, a_5 = \frac{-2}{5} a_4 = \frac{(-2)(-2)(-2)}{5 \cdot 4 \cdot 3} a_2$$

$\vdots$

$$a_n = \frac{(-2)(-2) \cdots (-2) \cdot 2}{n(n-1)(n-2) \cdots 3 \cdot 2} a_2 = \frac{2(-2)^{n-2}}{n!} a_2$$

18

# Series Solutions

$$\begin{aligned}
 \Rightarrow y(x) &= a_0 + a_1(x-1) + a_2(x-1)^2 + \cdots + a_n(x-1)^n + \cdots \\
 &= \frac{3}{4} + \frac{a_2}{2} + \left(\frac{1}{2} - a_2\right)(x-1) + a_2(x-1)^2 + \cdots + \frac{2(-2)^{n-2}}{n!} a_2(x-1)^n + \cdots \\
 &= \frac{3}{4} + \frac{1}{2}(x-1) + \frac{a_2}{2} [1 - 2(x-1) + 2(x-1)^2 + \frac{(-2)^3}{3!}(x-1)^3 + \cdots + \frac{(-2)^n}{n!}(x-1)^n + \cdots] \\
 &= \frac{3}{4} + \frac{1}{2}(x-1) + \frac{a_2}{2} e^{-2(x-1)}
 \end{aligned}$$

19

# Series Solutions

verify

$$y' + 2y = x + 1 = (x-1) + 2$$

$$\text{令 } t = x - 1, \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dy}{dx} \cdot 1$$

$$\frac{dy}{dt} + 2y = t + 2$$

$$\Rightarrow y_h = c_1 e^{-2t}$$

$$y_p = \frac{1}{D+2}(t+2) = \frac{1}{2} \frac{1}{1+\frac{D}{2}}(t+2) = \frac{1}{2} \left(1 - \frac{D}{2} + \frac{D^2}{4} + \cdots\right)(t+2)$$

$$= \frac{1}{2} \left(t+2 - \frac{1}{2}\right) = \frac{t}{2} + \frac{3}{4}$$

$$y = y_h + y_p$$

$$\Rightarrow y = c_1 e^{-2t} + \frac{t}{2} + \frac{3}{4}$$

20