QUIZ 2

NAME:____________________________ **NCKU id:**____________________

1. Find the values of a and b so that f is continuous everywhere, where

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \le x < 3 \\ 2x - a + b & \text{if } x \ge 3 \end{cases}.$$

Answer :

At $x = 2$:

$$\lim_{x\to 2^-} \frac{x^2-4}{x-2} = \lim_{x\to 2^-} x + 2 = 4$$

$$\lim_{x\to 2^+} ax^2 - bx + 3 = 4a - 2b + 3$$

We must have $4a - 2b + 3 = 4$ ...**(1)**

At $x = 3$:

$$\lim_{x\to 3^-} ax^2 - bx + 3 = 9a - 3b + 3$$

$$\lim_{x\to 3^+} 2x - a + b = 6 - a + b$$

We must have $9a - 3b + 3 = 9a - 3b + 3$, or $10a - 4b = 3$ ...**(2)**

Solve equation**(1).(2)**, we get $a = \frac{1}{2}, b = \frac{1}{2}$. Thus, for f to be continuous on $(-\infty, \infty)$, $a = b = \frac{1}{2}$

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2. Use the Intermediate Value Theorem to show that the equation $x = 2\cos x + 1$ has at least one solution.

Answer :

Set $f(x) = 2\cos x + 1 - x$. Then $f(1) = 2\cos 1 > 0$ and $f(2) = 2\cos 2 - 1 < 0$. By the intermediate-value theorem, $f(x)$ as zero in [1,2].

3. Find all the asymptotes of $y = \dfrac{\sqrt{x^6+3} - x^3 - x^2}{x^2 - x}$.

Answer :

For vertical asymptotes, we look at those x so that denominator goes to 0, i.e., $x = 0$ or 1.

At $x = 0$, we have

$$\lim_{x\to 0^+} y(x) = -\infty \Rightarrow x = 0 \text{ is a vertical asymptote.}$$

$$\lim_{x\to 0^-} y(x) = \infty \Rightarrow x = 0 \text{ is a vertical asymptote.}$$

At $x = 1$, we have

$$\begin{aligned}\lim_{x\to 1} y(x) &= \lim_{x\to 1} \frac{(x-1)\left(-2x^4 - 3(x+1)(x^2+1)\right)}{x(x-1)\left(\sqrt{x^6+3} + x^3 + x^2\right)} \\ &= \lim_{x\to 1} \frac{-2x^4 - 3(x+1)(x^2+1)}{x(\sqrt{x^6+3} + x^3 + x^2)} = \frac{-14}{4}.\end{aligned}$$

Hence $x = 1$ is not a vertical asymptote.

For slant asymptote, we aim to find (m, b) so that

$$\lim_{x\to\infty} (y(x) - (mx+b)) = 0 \text{ or } \lim_{x\to-\infty} (y(x) - (mx+b)) = 0.$$

We see that as $x \to \infty$,

$$\begin{aligned}
&\lim_{x\to\pm\infty} (y(x) - (mx+b)) = 0 \\
\Rightarrow &\lim_{x\to\pm\infty} \left(\frac{y(x)}{x} - (m + \frac{b}{x})\right) = 0 \\
\Rightarrow &\lim_{x\to\pm\infty} \left(\frac{y(x)}{x} - m\right) = 0 \\
\Rightarrow &\lim_{x\to\pm\infty} \frac{y(x)}{x} = m
\end{aligned}$$

Hence to find m, we only need to find $\lim_{x\to\pm\infty} \frac{y(x)}{x}$. Once m is determined, we can use the implication (necessary condition)

$$\begin{aligned}
&\lim_{x\to\infty} (y(x) - (mx+b)) = 0 \\
\Rightarrow &\lim_{x\to\infty} (y(x) - mx) = b
\end{aligned}$$

to find b.

Let y = mx+b be the oblique asymptote.

$$m_1 = \lim_{x\to\infty} \frac{y(x)}{x} = \frac{\sqrt{x^6+3} - x^3 - x^2}{x^3 - x^2} = 0$$

$$b_1 = \lim_{x\to\infty} (y - 0x) = \lim_{x\to\ \infty} = \frac{\sqrt{x^6+3} - x^3 - x^2}{x^2 - x} = -2$$

Hence $y = m_1 x + b_1 = 0x - 2 = -2$ is the slant asymptote as $x \to \infty$. In fact, $y = -2$ is a horizontal asymptote.

For $x \to -\infty$, we have

$$m_2 = \lim_{x\to-\infty} \frac{y(x)}{x} = \frac{\sqrt{x^6+3} - x^3 - x^2}{x^3 - x^2} = -2$$

$$b_2 = \lim_{x\to-\infty} [y - (-2x)] = \lim_{x\to\ -\infty} \frac{\sqrt{x^6+3} - x^3 - x^2 + 2x^3 - 2x^2}{x^2 - x} = -3$$

Hence the slant asymptote for $x \to -\infty$ is given by $y = -2x - 3$.