Chapter 2.
First-Order Ordinary Differential Equations

Chuan-Ching Sue

Dept. of Computer Science and Information Engineering,
National Cheng Kung University

Exact Examples

例: $\underset{\text{M}}{\left(e^x y + 6x + 5y\right)} dx + \underset{\text{N}}{\left(e^x + 5x\right)} dy = 0$

$$\frac{\partial \mathrm{M}}{\partial y} = e^x + 5 \quad \Leftrightarrow \quad \frac{\partial N}{\partial x} = e^x + 5$$

(相等代表為正合)

$$\mathrm{M} = \frac{\partial u}{\partial x} \qquad\qquad \mathrm{N} = \frac{\partial u}{\partial y}$$

$$\partial u = \left(e^x y + 6x + 5y\right)\partial x \quad \Leftrightarrow \quad \partial u = \left(e^x + 5x\right)\partial y$$

Exact Examples

$$u=\int\left(e^{x}y+6x+5y\right)\partial x+f\left(y\right) \Leftrightarrow \mathrm{u}=\int\left(\mathrm{e}^{\mathrm{x}}+5x\right)\partial y+g\left(x\right)$$

$$=\mathrm{e}^{\mathrm{x}}y+3x^{2}+5xy+f\left(y\right) \qquad =\mathrm{e}^{\mathrm{x}}y+5xy+g\left(x\right)$$

因此 $g\left(x\right)=3x^{2}, f\left(y\right)=0$

$\mathrm{u}\left(\mathrm{x,y}\right)=\mathrm{e}^{x}y+5xy+3x^{2}=C$

Exact Examples

例: $\underbrace{\left(\cos y+8x\right)}_{\mathrm{M}}dx+\underbrace{\left(-x\sin y+3y^{2}\right)}_{\mathrm{N}}dy=0$

$$\frac{\partial M}{\partial y}=-\sin y \quad \Leftrightarrow \quad \frac{\partial N}{\partial x}=-\sin y$$

(相等代表為正合)

$$M=\frac{\partial u}{\partial x} \qquad \mathrm{N}=\frac{\partial u}{\partial y}$$

$$\partial u=\left(\cos y+8x\right)\partial x \qquad \partial u=\left(-x\sin y+3y^{2}\right)\partial y$$

$$u=\int\left(\cos y+8x\right)\partial x \quad \Leftrightarrow \quad \mathrm{u}=\int\left(-x\sin y+3y^{2}\right)\partial y$$

Exact Examples

$$u = \int(\cos y + 8x)\,\partial x \qquad\qquad \Leftrightarrow \mathrm{u} = \int(-x\sin y + 3y^2)\,\partial y$$

$$= x\cos y + 4x^2 + f(y) \text{ —(1)} \qquad = x\cos y + y^3 + g(x)\text{—(2)}$$

比較(1)(2)

得 $f(y) = y^3, g(x) = 4x^2$

$\therefore u(x,y) = x\cos y + y^3 + 4x^2 = C$

Non-Exact

- 若不為正合情況

$u(x,y) = C\ldots\ldots\ldots(\mathrm{A})$

$\downarrow$

$\mathrm{M}(x,y)dx + N(x,y)dy = 0\ldots\ldots(B)$

若有乘法消去項,怎麼辦？

想法：

還它

Non-Exact

方法：

解(B)式,應先將消去項,歸還回,使得(B)成為正合。

怎麼知道消去哪些項？

(1)假設消去 $I(x,y)$

$$I(x,y)M(x,y)dx + I(x,y)N(x,y)dy = 0$$

若 $\dfrac{\partial M_1(x,y)}{\partial y} = \dfrac{\partial N_1(x,y)}{\partial x}$, 則 $M_1dx + N_1dy = 0$ 為正合

Non-Exact

(2) $\dfrac{\partial}{\partial y}\big(I(x,y)M(x,y)\big) = \dfrac{\partial}{\partial x}\big(I(x,y)N(x,y)\big)$

$$M(x,y)\frac{\partial I(x,y)}{\partial y} + I(x,y)\frac{\partial M(x,y)}{\partial y} = \mathrm{N}(\mathrm{x,y})\frac{\partial I(x,y)}{\partial x} + I(x,y)\frac{\partial N(x,y)}{\partial x}$$

$$-N(x,y)\frac{\partial I(x,y)}{\partial x} + M(x,y)\frac{\partial I(x,y)}{\partial y} = I(x,y)\left[\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}\right]$$

Integrating Factor

- 目的：

解$I(x,y)$所形成一階P.D.E.；$I(x,y)$稱為積分因子(Integrating Factor)。

考慮一階P.D.E.

$(P,Q,R)\cdot\left(\frac{\partial z}{\partial x},\frac{\partial z}{\partial y},-1\right)=0$

$$P(x,y,z)\frac{\partial z}{\partial x}+Q(x,y,z)\frac{\partial z}{\partial y}=R(x,y,z)\text{......}(*)$$

$\Rightarrow$由下列等式決定出兩個獨立解

$(dx,dy,dz)\left(\frac{\partial z}{dx},\frac{\partial z}{\partial y},-1\right)=0$

$=\nabla G$

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

稱輔助方程組、lagrange方程組

Integrating Factor

其中，(*)的通解可以是

$\varphi(u,v)=0$ 隱函數表示法

$v=f(u)$ 顯函數表示法

比較前式

$$-N\frac{\partial I}{\partial x}+M\frac{\partial I}{\partial y}=I\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$$

當 $\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=0$ 則 $I=1$

$$\boxed{\frac{dx}{-N}=\frac{dy}{M}=\frac{dI}{I\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}}$$

$\Rightarrow$ 可以利用此式來求解 I，而非要求 (*) 的通解

如何求 $\Longrightarrow$ TRY !!

Integrating Factor

- 招數：

(1)猜 I 是X 的函數，看看有沒有解。i.e. I (x)

$$\frac{dx}{-N}=\frac{dI}{I\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)} \Rightarrow \frac{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx}{-N}=\frac{dI}{I}$$

預達到希望(猜對)，則需 $\dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{-N}=f(x)$

Integrating Factor

$$f(x)dx=\frac{dI}{I}$$

$$(\text{兩邊積分})\Rightarrow \int f(x)dx=\ln I$$

$$\Rightarrow I=e^{\int f(x)dx}$$

(2) 猜 I 是 y 的函數。 i.e. I (y)

$$\frac{dy}{M}=\frac{dI}{I\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)} \Rightarrow \frac{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}{M}dy=\frac{dI}{I}$$

Integrating Factor

預達到希望(猜對)，則需 $\dfrac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}=g(\mathrm{y})$

$$g(y)dy=\frac{dI}{I}$$

$$(\text{兩邊積分})\Rightarrow\int g(y)dy=\ln I$$

$$\Rightarrow I=e^{\int g(y)dy}$$

Integrating Factor

(3)猜I是的 $(x+y)$ 函數

$$\frac{d}{dx}(x+y)=1+\frac{dy}{dx}$$

同乘dx $\Rightarrow d(x+y)=dx+dy$

Hint: (和分比概念)

$$\frac{1}{2}=\frac{2}{4}=\frac{1+2}{2+4}=\frac{1\times2}{2\times2}=\frac{2\times3}{4\times3}=\frac{1\times2+2\times3}{2\times2+4\times3}$$

Integrating Factor

$$\frac{dx+dy}{-N+M}=\frac{dI}{I\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)} \Rightarrow \frac{d(x+y)}{-N+M}=\frac{dI}{I\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}$$

預達到希望(猜對)，則需 $\dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{-N+M}=f(x+y)$

$$f(x+y)d(x+y)=\frac{dI}{I}$$

(兩邊積分) $\Rightarrow \int f(x+y)d(x+y)=\ln I$

$$\Rightarrow I=e^{\int f(x+y)d(x+y)}$$

Integrating Factor

(4)猜 I 是 xy 的函數。

其中：

$$d(xy)=?$$

$$\frac{d(xy)}{dx}=y\frac{dx}{dx}+x\frac{dy}{dx}$$

$$\frac{ydx+xdy}{y(-N)+xM}=\frac{dI}{\mathrm{I}\left(\frac{\partial \mathrm{N}}{\partial x}-\frac{\partial M}{\partial y}\right)}$$

Integrating Factor

$$\frac{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)d(xy)}{y(-N)+xM}=\frac{dI}{I}$$

預達到希望(猜對)，則需 $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{y(-N)+xM}=f(xy)$

$$f(xy)d(xy)=\frac{dI}{I}$$

(兩邊積分) $\Rightarrow \int f(xy)d(xy)=\ln I$

$$\Rightarrow I=e^{\int f(xy)d(xy)}$$

Integrating Factor

- Summary：

(1)(2)(3)(4)分子都是 $\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}$

若發現 $\frac{\partial N}{\partial x}\neq\frac{\partial M}{\partial y}$

則算出 $(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y})$

除以 –N → I(x)　　除以 –N+M → I(x+y)

除以 M → I(y)　　除以 –Y*N+X*M → I(x*y)

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Integrating Factor Examples

例: $\underset{M}{(x^2+y^2+x)}dx + \underset{N}{xy}dy = 0$

$$\frac{\partial M}{\partial y} = 2y \longleftarrow \text{不相等} \longrightarrow \frac{\partial N}{\partial x} = y$$

$$\frac{dx}{x} = \frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = -y}{-N}dx = \frac{dI}{I}$$

$$I = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

Integrating Factor Examples

原式=> $(x^3 + xy^2 + x^2)dx + x^2 y dy = 0$

$$\frac{\partial M}{\partial y} = 2xy \quad = \quad \frac{\partial N}{\partial x} = 2xy$$

正合

$$M = \frac{\partial u}{\partial x} \qquad N = \frac{\partial u}{\partial y}$$

$$\partial u = (x^3 + xy^2 + x^2)\partial x$$

$$u = \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + \frac{1}{3}x^3 + f(y)$$

Integrating Factor Examples

$$\partial u = (x^2 y)\partial y \qquad u = \frac{1}{2}x^2 y^2 + g(x)$$

$$\therefore f(y) = 0$$

$$g(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3$$

$$\mu(x,y) = \frac{1}{2}x^2 y^2 + \frac{1}{4}x^4 + \frac{1}{3}x^3 = C$$

Integrating Factor Examples

Sol2(用微積分解) 不好解，但可以用來當驗算題: $(x^2 + y^2 + x)dx + xydy = 0$

$$y^2 dx + xydy + (x^2 + x)dx = 0$$

$$y(ydx + xdy) + (x^2 + x)dx = 0$$

$$yd(xy) + (x^2 + x)dx = 0$$

同乘x => $xyd(xy) + (x^3 + x^2)dx = 0$

=> $\frac{1}{2}x^2 y^2 + \frac{1}{4}x^4 + \frac{1}{3}x^3 = C$

Integrating Factor Examples

例: $2\sin(y^2)dx + xy\cos(y^2)dy = 0$

$$\frac{\partial M}{\partial y} = 2\cos(y^2)2y = 4y\cos(y^2) \qquad \frac{\partial N}{\partial x} = y\cos(y^2)$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -3y\cos(y^2)$$

$$\Rightarrow \frac{-3y\cos(y^2)}{-N}dx = \frac{-3y\cos(y^2)}{-xy\cos(y^2)}dx = \frac{3}{x}dx = \frac{dI}{I}$$

$$\Rightarrow I = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$$

Integrating Factor Examples

得 $2x^3\sin(y^2)dx + x^4y\cos(y^2)dy = 0$

$$M = \frac{\partial u}{\partial x} \qquad N = \frac{\partial u}{\partial y}$$

$$u = \int 2x^3\sin(y^2)dx + f(y) \qquad u = \int x^4y\cos(y^2)dy + g(x)$$

$$= \frac{1}{2}x^4\sin(y^2) + f(y) \qquad t = y^2, dt = 2ydy, dy = \frac{dt}{2y}$$

$$= \frac{1}{2}x^4\sin(y^2) + g(x)$$

Integrating Factor Examples

$\because g(x)=0, f(y)=0 \qquad \therefore u=\frac{1}{2}x^4\sin(y^2)=C$

其中在 u中沒有 x 的式

dy^2

$$\int y\cos(y^2)dy=\int y\cos(t)\frac{dt}{2y}=\int\frac{1}{2}\cos(t)dt$$

$$=\frac{1}{2}\sin(t)$$

$$=\frac{1}{2}\sin(y^2)$$

Integrating Factor Examples

例: $xy\cos(x^2)dx+2\sin(x^2)dy=0$

$$\frac{\partial M}{\partial y}=x\cos(x^2)\neq\frac{\partial N}{\partial x}=2\cos(x^2)2x=4x\cos(x^2)$$

$$\therefore\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=3x\cos(x^2)$$

$$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=3x\cos(x^2)}{xy\cos(x^2)}dy=\frac{3}{y}dy \quad I=e^{\int\frac{3}{y}dy}=y^3$$

Integrating Factor Examples

得 $xy^4\cos(x^2)dx + 2y^3\sin(x^2)dy = 0$

$$M = \frac{\partial u}{\partial x} \qquad N = \frac{\partial u}{\partial y}$$

$$u = \int xy^4\cos(x^2)dx + f(y) \qquad u = \int 2y^3\sin(x^2)dy + g(x)$$

$$= \frac{1}{2}y^4\sin(x^2) + f(y) \qquad = \frac{1}{2}y^4\sin(x^2) + g(x)$$

$$\because f(y) = 0, g(x) = 0 \qquad \therefore u = \frac{1}{2}y^4\sin(x^2) = C$$

Integrating Factor Examples

例: $(xy + y^2 + 1)dx + (xy + x^2 + 1)dy = 0$

$$\frac{\partial M}{\partial y} = x + 2y \qquad \frac{\partial N}{\partial x} = y + 2x$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = x - y \qquad \frac{x - y}{-N + M}d(x + y)$$

$$= \frac{x - y}{y^2 - x^2}d(x + y) = \frac{-1}{x + y}d(x + y)$$

$$I = e^{\int \frac{-1}{x+y}d(x+y)} = e^{-\ln(x+y)} = e^{\ln\frac{1}{(x+y)}} = \frac{1}{x + y}$$

$$\frac{1}{x + y}(xy + y^2 + 1)dx + \frac{1}{x + y}(xy + x^2 + 1)dy = 0$$

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Integrating Factor Examples

$$M = \frac{\partial u}{\partial x} \qquad N = \frac{\partial u}{\partial y}$$

$$u = \int \frac{xy + y^2 + 1}{x + y} dx + f(y) \qquad u = \int \frac{xy + x^2 + 1}{x + y} dy + g(x)$$

$$= \int (y + \frac{1}{x + y}) dx + f(y) \qquad = \int (x + \frac{1}{x + y}) dy + g(x)$$

$$= yx + \ln(x + y) + f(y) \qquad = xy + \ln(x + y) + g(x)$$

$$\because g(x) = 0, f(y) = 0 \quad \therefore u(x, y) = xy + \ln(x + y) = C$$

Integrating Factor Non-Unique

Note: 若取的不同，則積分因子可以不唯一，但答案一定相同。

如上題 $(xy + y^2 + 1)dx + (xy + x^2 + 1)dy = 0$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = x - y$$

$$\frac{x - y}{-yN + xM} d(xy) = \frac{x - y}{-y(xy + x^2 + 1) + x(xy + y^2 + 1)} d(xy)$$

$$= d(xy)$$

$$I = e^{\int 1 d(xy)} = e^{xy} \quad \Leftarrow$$ 積分因子不唯一

Integrating Factor Non-Unique

得 $e^{xy}(xy+y^2+1)dx+e^{xy}(xy+x^2+1)dy=0$

$M=\dfrac{\partial\mu}{\partial x} \quad u=\int e^{xy}(xy+y^2+1)dx+f(y)$

其中 $\int e^{xy}xydx=?......(1)$

$$y\int e^{xy}xdx=y(x\frac{1}{y}e^{xy}-\int\frac{1}{y}e^{xy}dx)=xe^{xy}-\int e^{xy}dx$$

$$\int e^{xy}y^2dx......(2)$$

$$=\frac{1}{y}e^{xy}y^2=ye^{xy}$$

$$\int e^{xy}dx......(3)$$

Integrating Factor Non-Unique

因此 $u=(1)+(2)+(3)+f(y)$

$$=xe^{xy}+ye^{xy}+f(y)$$

同理 $N=\dfrac{\partial\mu}{\partial y}$

$$u=xe^{xy}+ye^{xy}+g(x)$$

$\because f(y)=0, g(x)=0 \quad \therefore u=xe^{xy}+ye^{xy}=C$

$$\|$$

$$xy+\ln(x+y)=c$$