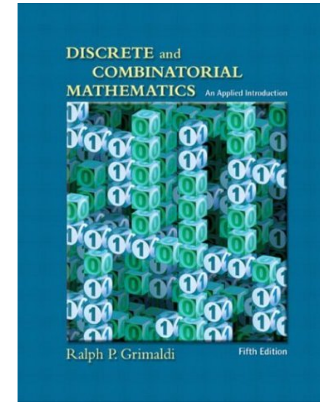
Discrete Mathematics
-- Chapter 1: Fundamental Principles of Counting



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Outline

- Sum & Product
- Permutations
- Combinations: The Binomial Theorem (二項式定理)
- Combinations with Repetition
- The Catalan Numbers
- Summary

Permutation / Combination

How many bit strings of length 8 do not contain three consecutive 1s?

In how many ways can be the integers 1, 2, 3, …, n be arranged in a line so that none of the patterns 12, 23, 34, …, (n-1)n occurs?

Prove that $\binom{3n}{3} = n^3 + 6n\binom{n}{2} + 3\binom{n}{3}$

$$\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \cdots + 2^n\binom{n}{n} = ?$$

1.1 The Rules of Sum and Product

- **The Rule of Sum**
  - If a first task can be performed in m ways, while a second task can be performed in n ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any one of $m+n$ ways.

- **The Rule of Product**
  - If a procedure can be broken down into first and second stages, and if there are m possible outcomes for the first stage and if, for each of these outcomes, there are n possible outcomes for the second stage, then the total procedure can be carried out, in the designated order, in mn ways.
    - Ex 1.6: If a license plate consists of two letters followed by four digits, how many different plates are there? $26\times26\times10\times10\times10\times10$

Rule combination

1.2 Permutations

- Permutation: counting linear arrangements of (**distinct**) objects
- If there are n distinct objects and r is an integer, with $1 \le r \le n$, then **by the rule of product**,
  - The number of permutations of size r for the n objects is

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$$

$$= n(n-1)(n-2)\cdots(n-r+1) \times \frac{(n-r)(n-r-1)\cdots(3)(2)(1)}{(n-r)(n-r-1)\cdots(3)(2)(1)}$$

$$= \frac{n!}{(n-r)!} \longleftarrow \textit{n factorial}$$

- **Ex 1.9**: Given 10 students, *three* are to be chosen and seated in a row. How many such linear arrangements are possible?

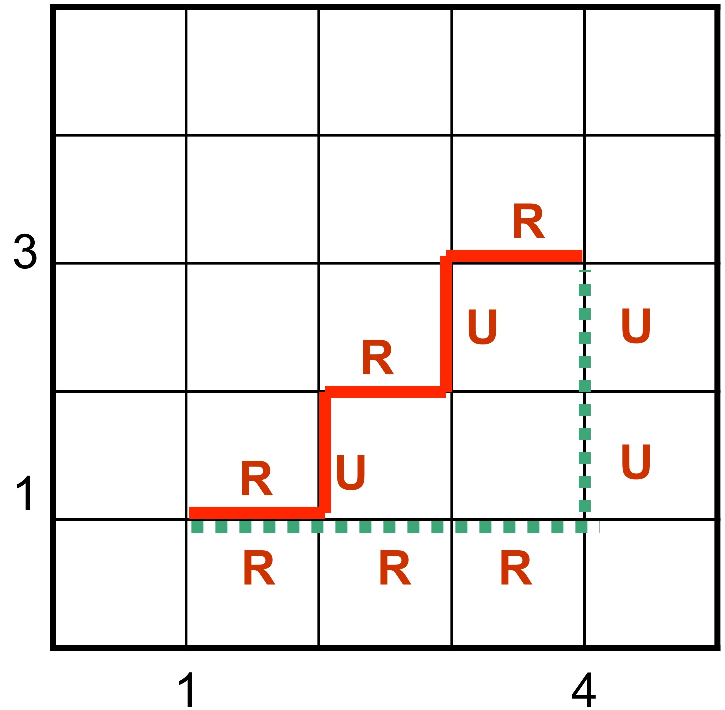
Permutations with Repeated Objects

- If there are n objects with n_1 **indistinguishable** objects of a first type, n_2 indistinguishable objects of a second type, …, and n_r indistinguishable objects of an rth type, where $n_1 + n_2 + \ldots + n_r = n$.
  - → the number of (linear) arrangements of the given n objects $= \dfrac{n!}{n_1!n_2!\ldots n_r!}$

- Ex 1.13: Arranging all of the letters in MASSASAUGA, we find there are $\dfrac{10!}{4!\,3!\,1!\,1!\,1!}$ possible arrangements, $\dfrac{7!}{3!\,1!\,1!\,1!}$ arrangements while all four A's are together.

Permutations with Repeated Objects

- **Ex 1.14**: Determine the number of (staircase) paths in the xy-plane from (1, 1) to (4, 3), where each such path is made up of individual steps going one unit to the right or one unit upward.
  - As for xyz-space, from (1,1,1) to (4, 3, 2)?



Permutations with Repeated Objects

(Exercise in textbook P45)

29. **a)** In how many ways can a particle move in the xy-plane from the origin to the point (7, 4) if the moves that are allowed are of the form:

$$(\text{R}): (x, y) \rightarrow (x+1, y); \quad (\text{U}): (x, y) \rightarrow (x, y+1)?$$

b) How many of the paths in part (a) do not use the path from (2, 2) to (3, 2) to (4, 2) to (4, 3) shown in Fig. 1.12?

c) Answer parts (a) and (b) if a third type of move

$$(\text{D}): (x, y) \rightarrow (x+1, y+1)$$

is also allowed.

$$a)\ \frac{11!}{7!*4!}$$

$$b)\ \frac{11!}{7!*4!} - \frac{4!}{2!*2!} * \frac{4!}{3!*1!}$$

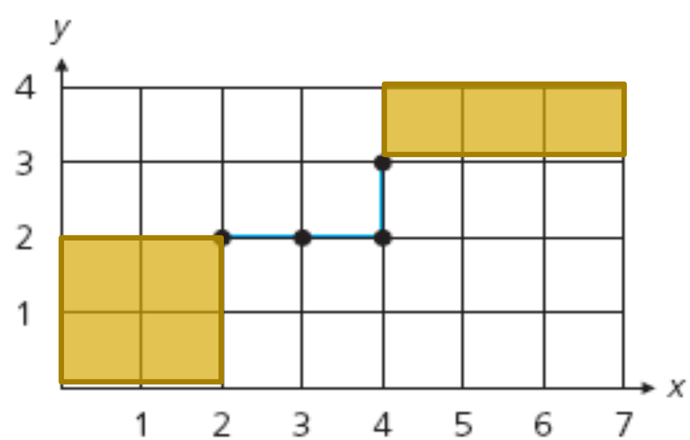


Figure 1.12

1D=1R+1U

d) How many of the paths in part(a) do not pass the points (0,1), (1,2), (2,3), (3,4)?

Combinatorial Proofs

- Prove that $\frac{(2k)!}{2^k}$ is an integer.
  - Consider $2k$ symbols $x_1, x_1, x_2, x_2, \ldots, x_k, x_k$.
  - The number of ways they can be arranged is

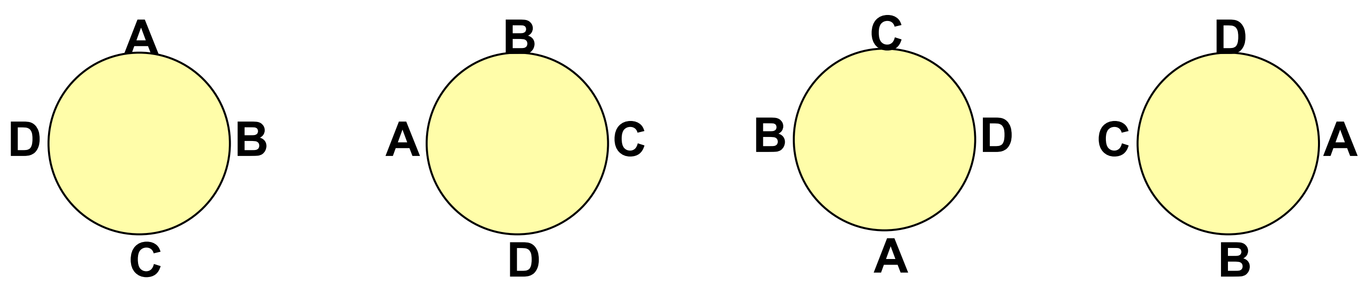
$$\frac{(2k)!}{2^k} = \frac{(2k)!}{2!2!\cdots 2!}$$

  - It must be an integer.

- Prove that $\frac{(mk)!}{(m!)^k}$ is also an integer.

Arrangement around a Circle

- Consider n distinct objects
- Two arrangements are considered the same when one can be obtained from the other by rotation.
- How many different circular arrangements?
  - Thinking distinct linear arrangements for 4 objects, e.g., ABCD, BCDA, CDAB, and …
  - So, the number of circular arrangements is? **4!/4 = 3!**



Arrangement around a Circle

- **Ex 1.17**: arrange six people (3 males, 3 females) around the table so that the sexes alternate.

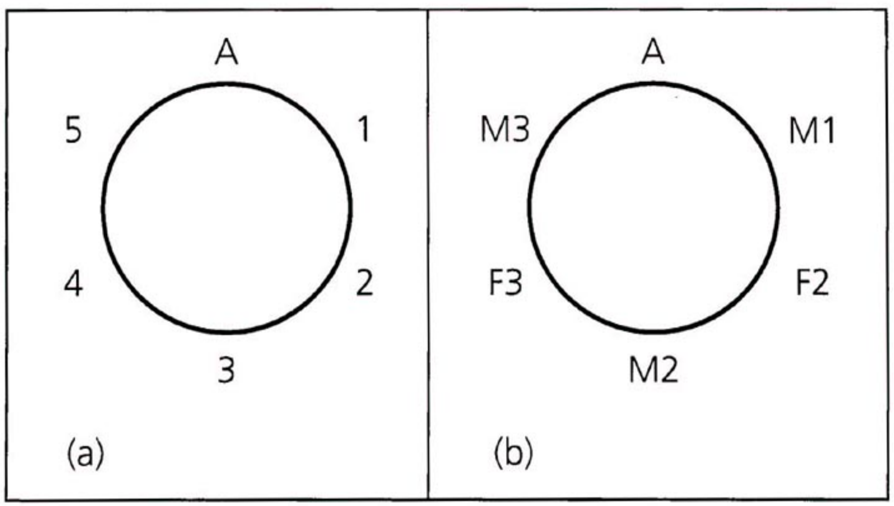


Figure 1.3

No constraint 6!/6 = 5! = 120

Sexes alternate
3 x 2 x 2 x 1 x 1 = 12

1.3 Combinations: The Binomial Theorem

- If there are n distinct objects and r is an integer, with $1 \le r \le n$
- The number of combinations (***selections without reference to order***) of size r for the n objects is

$$C(n,r) = \boxed{\binom{n}{r}} = \frac{P(n,r)}{r!} = \frac{n(n-1)\cdots(n-r+1)}{r(r-1)\cdots 1} = \frac{n!}{r!(n-r)!}$$

  - are sometimes read "n choose r".

- $C(n, 0) = C(n, n) = 1$, for all $n \ge 0$.
- $C(n, r) = C(n, n-r)$, for all $n \ge 0$.

Combinations

- <u>Ex 1.19</u>
  - To win the grand prize for PowerBall one must match <u>five</u> numbers selected from <u>1 to 49</u> **inclusive** and then must also match the powerball, an integer from <u>1 to 42</u> inclusive.
  - Consequently, by the rule of product, they can select the six numbers in ways.

$$\binom{49}{5}\binom{42}{1} = 80{,}089{,}128$$

How about the case in Taiwan?

$$\binom{38}{6}\binom{8}{1} = 22{,}085{,}448$$

Combinations

- **Ex 1.20**
  - A student taking a history examination is directed to answer any seven of 10 essay questions. There is no concern about order here.
    - She can answer the examination in $\binom{10}{7} = 120$ ways
  - If the student must answer three questions from the first five and four questions from the last five.

$$\binom{5}{3}\binom{5}{4} = 5 \times 10 = 50$$

  - If the student must answer **at least** three questions from the first five.

$$\binom{5}{3}\binom{5}{4} + \binom{5}{4}\binom{5}{3} + \binom{5}{5}\binom{5}{2} = 50 + 50 + 10 = 110$$

$$also, \sum_{i=3}^{5}\binom{5}{i}\binom{5}{7-i}$$

- 窮舉
- 反例

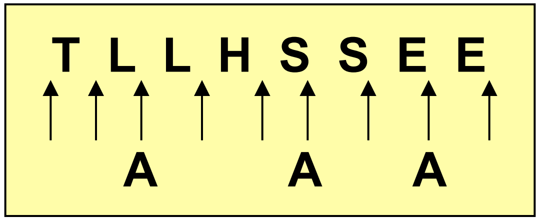
Combinations

- <u>**Ex 1.23**</u>
  - The number of arrangements of the letters in TALLAHASSEE is?

$$\frac{11!}{3!\,2!\,2!\,2!\,1!\,1!} = 831{,}600 \quad \textit{Permutations with Repeated Objects}$$

  - How many of these arrangements have no adjacent A's?

$$\left(\frac{8!}{2!\,2!\,2!\,1!\,1!}\right)\binom{9}{3} = 5040 \times 84 = 423{,}360$$



Overcounting

- **Ex 1.25**
  - How many ways to draw five cards from a standard deck of 52 cards with no clubs?

$$\binom{39}{5}$$

  - How many ways to draw five cards with at least one club?

$$\binom{13}{1}\binom{51}{4} = 3{,}248{,}700$$ ✗

What are the repetition cases ?

C2 – C5 D6 C9 H3
C5 – C9 H3 C2 D6
C9 – D6 C2 H3 C5

反例 $$\binom{52}{5} - \binom{39}{5} = 2{,}023{,}203$$

窮舉 $$\binom{13}{1}\binom{39}{4} + \binom{13}{2}\binom{39}{3} + \binom{13}{3}\binom{39}{2} + \binom{13}{4}\binom{39}{1} + \binom{13}{5}\binom{39}{0} = \sum_{1}^{5}\binom{13}{i}\binom{39}{5-i} = 2{,}023{,}203$$

Theorem 1.1: The Binomial Theorem

- $$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \ldots + \binom{n}{n}x^ny^0 = \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}$$
- There are $C(n, k)$ ways to choose k x's and n-k y's.
- $C(n, k)$ is often referred to as a binomial coefficient.
- In case $(x+y)^2$

$$\begin{array}{rrr} & x & + & y \\ & x & + & y \\ \hline & xy & + & y^2 \\ x^2 + xy & & & \\ \hline x^2y^0 + 2x^1y^1 + x^0y^2 \end{array}$$

Theorem 1.1: The Binomial Theorem

- the coefficient of x^2y^2 in the expansion of $(x+y)^4$ is $\binom{4}{2} = 6$

$$x + y$$

$$x + y$$

$$x + y$$

$$x + y$$

Table 1.5

Factors Selected for x		Factors Selected for y	
(1)	1, 2	(1)	3, 4
(2)	1, 3	(2)	2, 4
(3)	1, 4	(3)	2, 3
(4)	2, 3	(4)	1, 4
(5)	2, 4	(5)	1, 3
(6)	3, 4	(6)	1, 2

The Binomial Theorem

- $$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \ldots + \binom{n}{n}x^ny^0 = \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}$$
- **Ex 1.26**
  - a) What is the coefficient of x^5y^2 in the expansion of $(x + y)^7$?
  - b) What is the coefficient of a^5b^2 in the expansion of $(2a - 3b)^7$?

a) the coefficient of x^5y^2 in $(x+y)^7$ is $\binom{7}{5}$

$b)$ *Set* $x = 2a,\ y = -3b$

$$\binom{7}{5}x^5y^2 = \binom{7}{5}(2a)^5(-3b)^2 = \binom{7}{5}(2)^5(-3)^2a^5b^2 = 6048a^5b^2$$

Corollaries of The Binomial Theorem

- Corollary 1.1: a) $\binom{n}{0}+\binom{n}{1}+\ldots+\binom{n}{n}=?$ **2^n**

  b) $\binom{n}{0}-\binom{n}{1}+\ldots+(-1)^n\binom{n}{n}=?$ **0**

  - Proof
    - Part (a) set $x=y=1$
    - Part (b) set x=-1 and y=1

$$(x+y)^n=\binom{n}{0}x^0y^n+\binom{n}{1}x^1y^{n-1}+\ldots+\binom{n}{n}x^ny^0=\sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}$$

how about $\binom{n}{0}+2\binom{n}{1}+2^2\binom{n}{2}+\ldots+2^k\binom{n}{k}+\ldots+2^n\binom{n}{n}=?$

Theorem 1.2 : The Multinomial Theorem

- The coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$ in the expansion of

  $(x_1 + x_2 + \cdots + x_t)^n$ is $\dfrac{n!}{n_1! n_2! \ldots n_t!} = \dbinom{n}{n_1, n_2, \cdots, n_t}$

  where $0 \le n_i \le n$, and $n_1 + n_2 + \ldots + n_t = n$.

- Proof
  - The number of ways we can select x_1 from n_1 of the n factors, x_2 from n_2 of the $n - n_1$ remaining factors, …

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\ldots-n_{t-1}}{n_t} = \frac{n!}{n_1! n_2! \cdots n_t!} = \binom{n}{n_1, n_2, \cdots, n_t}$$

Multinomial coefficient

t=2 → binomal coefficient

The Multinomial Theorem

- **Ex 1.27**
  - What is the coefficient of x^5y^2 in the expansion of $(x + y+z)^7$?

$$\binom{7}{5\ 2\ 0} = \frac{7!}{5!\,2!\,0!} = 21$$

  - What is the coefficient of $a^2b^3c^2d^5$ in the expansion of $(a+2b\text{-}3c+2d+5)^{16}$?

$$Set\ \ \mathrm{v} = a,\ w = 2b,\ x = -3c,\ y = 2d,\ z = 5$$

$$\text{the coefficient of } v^2w^3x^2y^5z^4 \text{ in } (v+w+x+y+z)^{16} \text{ is } \binom{16}{2,3,2,5,4}$$

$$\binom{16}{2,3,2,5,4}(a)^2(2b)^3(\text{-}3\mathrm{c})^2(2\mathrm{d})^5(5)^4 = \binom{16}{2,3,2,5,4}(1)^2(2)^3(\text{-}3)^2(2)^5(5)^4a^2b^3\mathrm{c}^2\mathrm{d}^5$$

$$= 435{,}891{,}456{,}000{,}00a^2b^3\mathrm{c}^2\mathrm{d}^5$$

1.4 Combination with Repetition

- **Ex 1.28**
  - How many different purchases are possible for **seven** students each having one of the following, a cheeseburger, a hot dog, a taco, or a fish sandwich?

Possible way	Another way
c,c,h,h,t,t,f	xx|xx|xx|x
c,c,c,c,h,t,f	xxxx|x|x|x
c,c,c,c,c,c,f	xxxxxx|||x
h,t,t,f,f,f,f	|x|xx|xxxx

7 x's + 3 |'s $\binom{10}{7} = \frac{10!}{7!3!} = \frac{(4+7-1)!}{7!(4-1)!}$

- The number of combinations of n objects taken r at a time, with repetition, is

(foods) (students)

$$H_r^n = C(n+r-1,r) = \frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r}$$

1.4 Combination with Repetition

- **Ex 1.31**
  - In how many ways can we distribute seven bananas and six oranges among four children so that each child receives at least one banana?
  - Remaining bananas: 7-4=3
  - 3 bananas was distributed 4 children: (n=4, r=3) C(4+3-1, 3) = C(6, 3) =20
  - 6 oranges was distributed 4 children: (n=4, r=6) C(4+6-1, 6)= C(9, 6)=84
  - Thus, 20×84=1680

Distribute 3 bananas to 4 children	
c_1, c_2, c_3	b|b|b|
c_1, c_3, c_3	b||bb|
c_3, c_4, c_4	||b|bb
c_4, c_4, c_4	|||bbb

3 b's + 3 |'s

1.4 Combination with Repetition

- **Ex 1.33**
  - Determine all integer solutions to the equation $x_1+x_2+x_3+x_4=7$, where $x_i \geq 0$ for all $1 \leq i \leq 4$.
  - $n=4$, $r=7 \rightarrow C(4+7-1, 7)$

3' "+", 7' "1" linear permutation

- Equivalence: $C(n+r-1, r)$
  - The number of integer solutions of the equation $x_1+x_2+_{\dots}+x_n=r$, where $x_i \geq 0$ for all $1 \leq i \leq n$.
  - The number of selections, with repetition, of size r from a collection of size n.
  - The number of ways r identical objects can be distributed among n distinct containers.

=the number of ways r distinct objects be distributed among n identical containers ?

- Difference
  - r distinct objects can be distributed among n distinct containers in n^r ways.

1.4 Combination with Repetition

*the number of ways **r** objects be distributed among **n** containers*

	r distinct	r identical
n distinct	n^r	$C(n+r-1, r)$
n identical	$\cancel{n^r/n!}$ See Chapter 5	See Chapter 9

$$\sum_{i=1}^{n} S(r,i)$$

Partitions of integers

r=3, distinct, n=3, distinct → 3x3x3=27

n_1	n_2	n_3	n_1	n_2	n_3	n_1	n_2	n_3
ABC			BC	A		BC		A
AB	C		B	AC		B	C	A
AB		C	B	A	C	B		AC
AC	B		C	AB		C	B	A
A	BC			ABC			BC	A
A	B	C		AB	C		B	AC
AC		B	C	A	B	C		AB
A	C	B		AC	B		C	AB
A		BC		A	BC			ABC

r distinct, n identical

=3x2=6

=3x2x1=6

Ans: 5 (=S(3,1)+S(3,2)+S(3,3))

{{ABC},{AB,C},{AC,B},{BC,A},{A,B,C}}

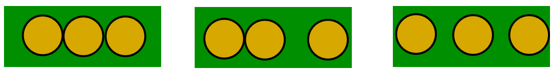
r identical, n distinct

=C(3,2)=3

Ans: C(3+3-1, 3)=10

r identical, n identical

Ans: 3



1.4 Combination with Repetition

- **Ex 1.35**
  - How many nonnegative integer solutions to the inequality $x_1 + x_2 + _{\dots} + x_6 < 10$?
  - Transform the problem to $x_1 + x_2 + _{\dots} + x_6 + x_7 = 10$, $x_i \geq 0$ for all $1 \leq i \leq 6$, but $x_7 > 0$.
  - $y_1 + y_2 + _{\dots} + y_6 + y_7 = 9$, where $y_1 = x_i$ for all $1 \leq i \leq 6$, and $y_7 = x_7 - 1$.
  - $C(7+9-1, 9) = 5005$.

1.4 Combination with Repetition

- **Ex 1.36**
  - In the binomial expansion for $(x + y)^n$, each term is of the form $C(n, k)x^k y^{n-k}$
  - The total number of terms in the expansion is the number of nonnegative integer solutions of $n_1 + n_2 = n$.
  - $C(2 + n - 1, n) = C(n+1, n) = n+1$.
  - How many terms are there in the expansion of $(w+x+y+z)^{10}$? $\underline{C(4+10-1, 10)}$

1.4 Combination with Repetition

- **Ex 1.39**

Consider the following program segment, where i, j, and k are integer variables.

```
for i := 1 to 20 do
  for j := 1 to i do
    for k := 1 to j do
      print (i * j + k)
```

How many times is the **print** statement executed in this program segment?

k j i

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

(i, j, k) = (13, 8, 6)

(I, j, k)=(8, 4, 4)

(i, j, k) = (13, 8, 11) X

|||xx||||x|||||||||||, 3 x's 19|'s combinations = C(22, 3)=C(22, 19)=1540

1.4 Combination with Repetition

- **Ex 40**
  - Summation formula
  - *counter* = C(n+2-1, 2)=C(n+1, 2)
  - Also, *counter* = $\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \binom{n+1}{2} = \frac{n(n+1)}{2}$

```
counter := 0
for i := 1 to n do
   for j := 1 to i do
      counter := counter + 1
```

1.5 Catalan Number

- Count paths from (0,0)→(5,5) but never rise over the line y=x
- No constraint C(10,5)
- With constraint
  - C(10,5)-**C(10,4)**
- Exchange R,U after the first "crossing" U

  RUUURRRUUR

→ RUURUUURRU

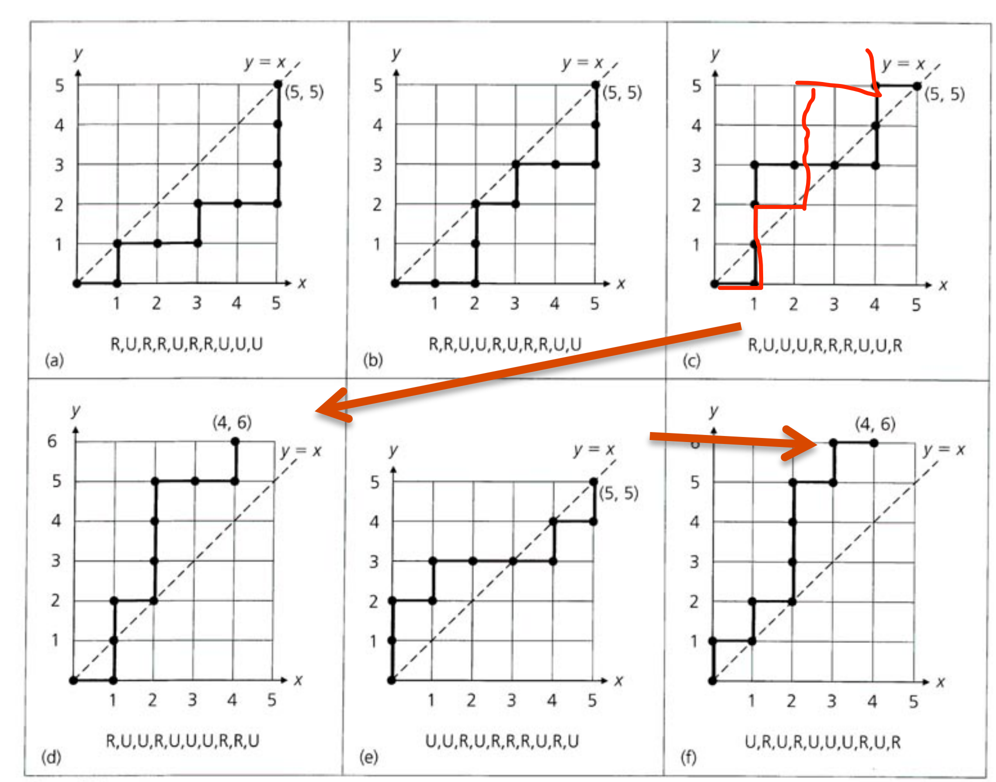


Figure 1.9

1.5 Catalan Number

- From (0,0)→(n,n)

$$\# path = b_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$$

- which can determine the number of ways to parenthesize the product $x_1 x_2 x_3 x_4 \cdots x_n$.
  - E.g., n=4, #parenthesis = b3 = ¼*C(6,3) = 5

$$(((x_1 x_2) x_3) x_4), ((x_1 (x_2 x_3) x_4)), ((x_1 x_2)(x_3 x_4)),$$
$$(x_1 ((x_2 x_3) x_4)), (x_1 (x_2 (x_3 x_4)))$$

1.6 Summary

- Fundamental techniques in counting:
  - ***Top-down approach***: Divide the problems into subproblems suitable for discrete and combinatorial mathematics.

Order Is Relevant	Repetitions Are Allowed	Type of Result	Formula	Location in Text
Yes	Yes	Arrangement	n^{r}	Page 7
Yes	No	Permutation	$P(n,r)=\dfrac{n!}{(n-r)!}$	Page 7
No	No	Combination	$C(n,r)=\dbinom{n}{r}=\dfrac{n!}{r!(n-r)!}$	Page 15
No	Yes	Combination with repetition	$C(n+r-1,r)=\dbinom{n+r-1}{r}=\dfrac{(n+r-1)!}{r!(n-1)!}$	Page 27

Observation & Tricks

- The previous 4 cases are not answers by themselves. Instead they are tools that you have to learn to use.
- It is often not immediately clear which tool to use.

- Experiences
  - Try small examples and solve them by “brute force”.
  - Verify your answers in the special cases (e.g., $k=0$ or $k=n$).
  - More Practice, more experiences