

Chapter 2

Probability

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2.6 Conditional Probability

- Conditional probability: $P(B|A)$
 - The probability of an event B occurring when it is known that some event A has occurred.
 - “the probability that B occurs given that A occurs”
 - “the probability of B , given A ”
- E.g.P.62:
 - $S = \{1, 2, 3, 4, 5, 6\}$, $A = \{4, 5, 6\}$, $B = \{1, 3, 5\}$,
 $\Rightarrow P(B|A)$?
- Definition 2.10:
$$P(B | A) = \frac{P(A \cap B)}{P(A)}$$

Conditional Probability

- Example P.63:

- Our sample space S is the population of adults in a small town who have completed the requirements for a college degree.
- To investigate the advantage of establishing new industries in the town.

- The concerned events

- M : a man is chosen
- E : the one chosen is employed

	Employed	Unemployed	Total
Male	460	40	500
Female	140	260	400
Total	600	300	900

- $$P(M | E) = \frac{460}{600} = \frac{23}{30}$$

$$P(M | E) = \frac{n(E \cap M) / n(S)}{n(E) / n(S)} = \frac{P(E \cap M)}{P(E)}$$

$$P(E) = \frac{600}{900} = \frac{2}{3}, \quad P(E \cap M) = \frac{460}{900} = \frac{23}{45}, \quad P(M | E) = \frac{23/45}{2/3} = \frac{23}{30}$$

Conditional Probability

- Example P.65
 - 2 cards are drawn in succession, **with replacement**
 - A : the first card is an ace
 - B : the second card is a spade
 - $P(B|A) = \frac{13}{52} = \frac{1}{4}$ and $P(B) = \frac{13}{52} = \frac{1}{4}$
- Definition 2.11:
 - Two events A and B are said to be **independent** if and only if $P(B|A) = P(B)$.
 - Otherwise, A and B are **dependent**.
- The notion of conditional probability provides the capability of reevaluating the idea of probability of an event in light of **additional information**.

Multiplicative Rules

- Theorem 2.10: If in an experiment the events A and B can both occur, then

$$P(A \cap B) = P(A)P(B | A) = P(B)P(A | B)$$

- Example 2.36

- A fuse box contains 20 fuses, of which 5 are defective.
- If 2 fuses are selected at random and removed from the box in succession without replacing the first.
- What is the probability that both fuses are defective?
 - Let A be the event that the first fuse is defective
 - B be the event that the second fuse is defective

$$P(A \cap B) = P(A)P(B | A) = \left(\frac{5}{20}\right)\left(\frac{4}{19}\right) = \frac{1}{19}$$

Multiplicative Rules

- Example 2.37
 - One bag contains 4 white balls and 3 black balls.
 - A second bag contains 3 white balls and 5 black balls.
 - One ball is drawn from the first bag and placed unseen in the second bag.
 - What is the probability that a ball now drawn from the second bag is black?
 - Let B_1 , B_2 , and W_1 represent, respectively, the drawing of a black ball from bag 1, a black ball from bag 2, and a white ball from bag 1.
 - $$\begin{aligned} P[(B_1 \cap B_2) \cup (W_1 \cap B_2)] &= P(B_1 \cap B_2) + P(W_1 \cap B_2) \\ &= P(B_1)P(B_2 | B_1) + P(W_1)P(B_2 | W_1) \\ &= \left(\frac{3}{7}\right)\left(\frac{6}{9}\right) + \left(\frac{4}{7}\right)\left(\frac{5}{9}\right) = \frac{38}{63} \end{aligned}$$

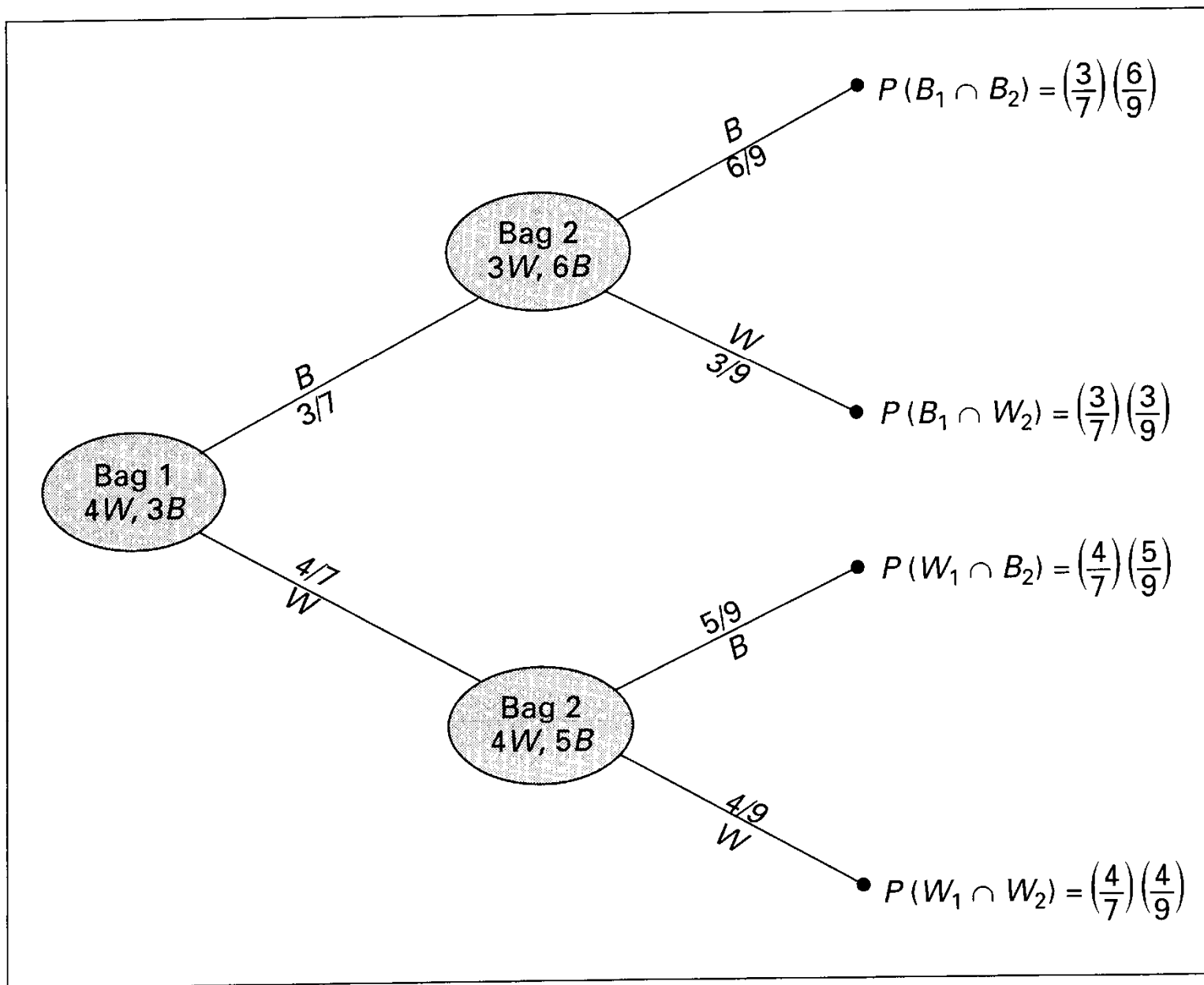


Figure 2.8 Tree diagram for Example 2.37 (2.36)

Multiplicative Rules

- Theorem 2.11: Two events A and B are independent if and only if $P(A \cap B) = P(A)P(B)$
- Example 2.38
 - A small town has one fire engine and one ambulance available for emergencies.
 - The probability that the fire engine is available is 0.98.
 - The probability that the ambulance is available is 0.92.
 - Find the probability that both the fire engine and the ambulance will be available when an event of an injury resulting from a burning building.
 - $P(A \cap B) = P(A)P(B) = 0.98 \times 0.92 = 0.9016$

Multiplicative Rules

- Example 2.39
 - Find the probability that
 - a) the entire system works
 - b) the component C does not work, given that the entire system works

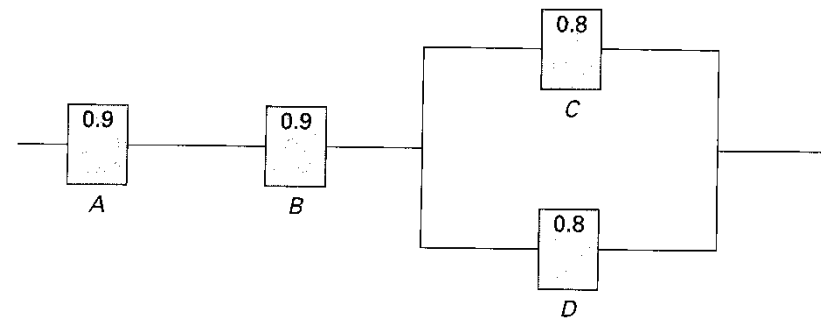


Figure 2.9 An electrical system for Example 2.35.

Multiplicative Rules

- Example 2.39

- Find the probability that
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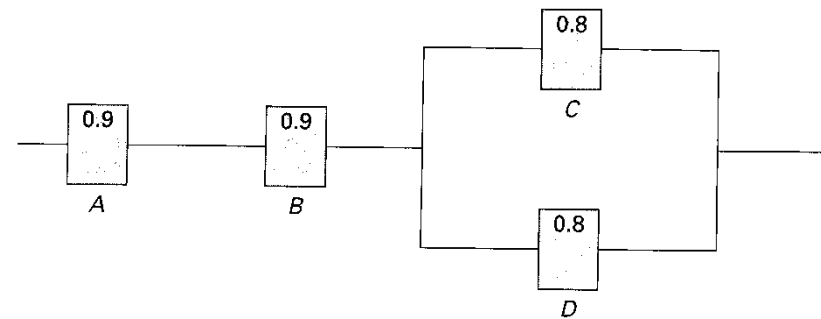


Figure 2.9 An electrical system for Example 2.35.

- a) $P(A \cap B \cap (C \cup D)) = P(A)P(B)P(C \cup D) = P(A)P(B)(1 - P(C' \cap D'))$
 $= P(A)P(B)(1 - P(C')P(D')) = 0.9 \times 0.9 \times (1 - (1 - 0.8)(1 - 0.8)) = 0.7776$
- b) $P = \frac{P(\text{the system works but } C \text{ does not work})}{P(\text{the system works})}$
 $= \frac{P(A \cap B \cap C' \cap D)}{P(A \cap B \cap (C \cup D))} = \frac{0.9 \times 0.9 \times (1 - 0.8) \times 0.8}{0.7776} = 0.1667$

Multiplicative Rules

- Theorem 2.12 : If the events $A_1, A_2, A_3, \dots, A_k$ can occur, then
$$P(A_1 \cap A_2 \cap \dots \cap A_k)$$
- $$= P(A_1)P(A_2 | A_1)P(A_3 | A_1 \cap A_2) \dots P(A_k | A_1 \cap A_2 \cap \dots \cap A_{k-1}).$$
 - If the events $A_1, A_2, A_3, \dots, A_k$ are independent, then
$$P(A_1 \cap A_2 \cap \dots \cap A_k) = P(A_1)P(A_2) \dots P(A_k).$$
- Example 2.40
 - Three cards are drawn in succession without replacement. Find the probability that the event $A_1 \cap A_2 \cap A_3$ occurs
 - A_1 : the first card is red ace
 - A_2 : the second card is a 10 or jack
 - A_3 : the third card is greater than 3 but less than 7

Multiplicative Rules

- Example 2.40
 - Three cards are drawn in succession without replacement. Find the probability that the event $A_1 \cap A_2 \cap A_3$ occurs
 - A_1 : the first card is red ace
 - A_2 : the second card is a 10 or jack
 - A_3 : the third card is greater than 3 but less than 7

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2 | A_1)P(A_3 | A_1 \cap A_2) = \left(\frac{2}{52}\right)\left(\frac{8}{51}\right)\left(\frac{12}{50}\right) = \frac{8}{5525}$$

Multiplicative Rules

- Example: Several Chinese characters, such as 機率統計 (微分方程, 線性代數...), are written down successively. Find the probability that the string occurs in your conversation, text book, Web,...

Multiplicative Rules

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- $(C_1 \cap C_2 \cap C_3 \cap C_4)$

- C_1 : 機, C_2 : 率, C_3 : 統, C_4 : 計

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$$P(C_1 \cap C_2 \cap C_3 \cap C_4) =$$

$$\begin{cases} P(C_1)P(C_2 | C_1)P(C_3 | C_1 \cap C_2)P(C_4 | C_1 \cap C_2 \cap C_3) = ? \\ P(C_1)P(C_2)P(C_3)P(C_4) = ? \end{cases}$$

Multiplicative Rules

- Definition 2.12: A collection of events $A = \{A_1, A_2, \dots, A_n\}$ are mutually independent if for any subset of A ($A_i = \{A_{i1}, A_{i2}, \dots, A_{ik}\}$), for $k \leq n$, we have

$$P(A_{i1} \cap A_{i2} \cap \dots \cap A_{ik}) = P(A_{i1})P(A_{i2}) \dots P(A_{ik})$$

2.7 Bayes' Rules

- In the example of employment status (Section 2.6).
 - Give the additional information that 36 of those employed and 12 of those unemployed are members of the Rotary Club.
 - Find the probability of the event A that individual selected is a member of the Rotary Club.

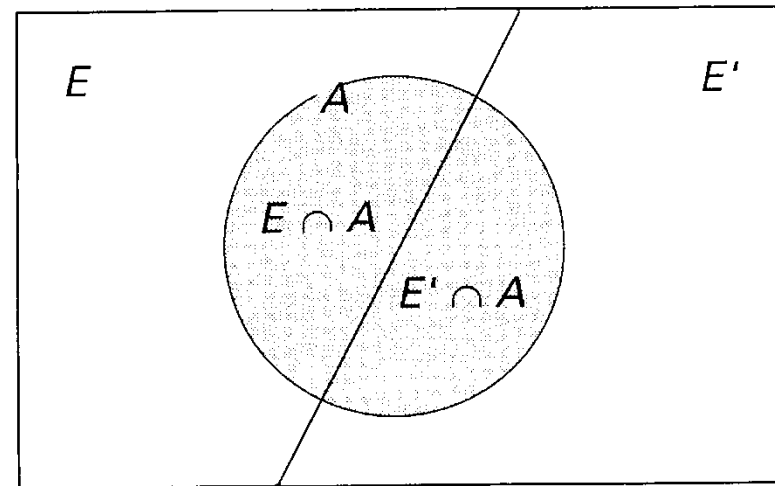


Figure 2.12 Venn diagram for the events A , E , and E' .

2.7 Bayes' Rules

- Event A is the union of the two mutually exclusive events $E \cap A$ and $E' \cap A$. Hence,

$$A = (E \cap A) \cup (E' \cap A)$$

$$\begin{aligned} P(A) &= P[(E \cap A) \cup (E' \cap A)] \\ &= P(E \cap A) + P(E' \cap A) \\ &= P(E)P(A|E) + P(E')P(A|E') \end{aligned}$$

$$P(E) = \frac{600}{900} = \frac{2}{3}, \quad P(A|E) = \frac{36}{600} = \frac{3}{50},$$

$$P(E') = \frac{1}{3}, \quad P(A|E') = \frac{12}{300} = \frac{1}{25}$$

$$P(A) = \left(\frac{2}{3}\right)\left(\frac{3}{50}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{25}\right) = \frac{4}{75}$$

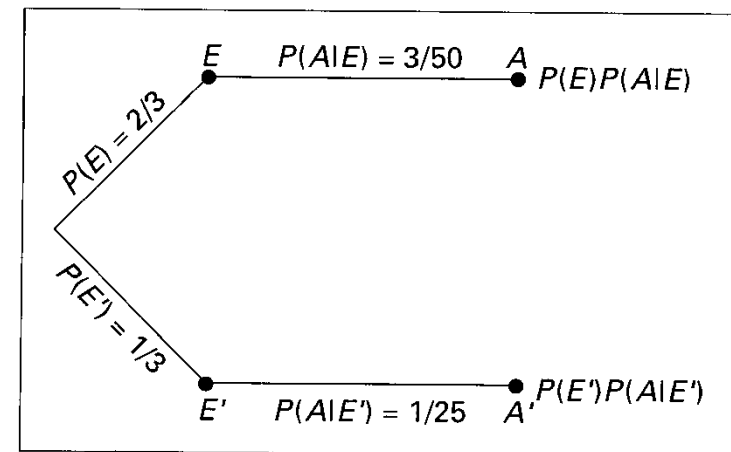


Figure 2.13 Tree diagram for the data on page 48.

Total Probability (Rule of Elimination)

- Theorem 2.13: If the events B_1, B_2, \dots, B_k constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \dots, k$, then
$$P(A) = \sum_{i=1}^k P(B_i \cap A) = \sum_{i=1}^k P(B_i)P(A|B_i).$$

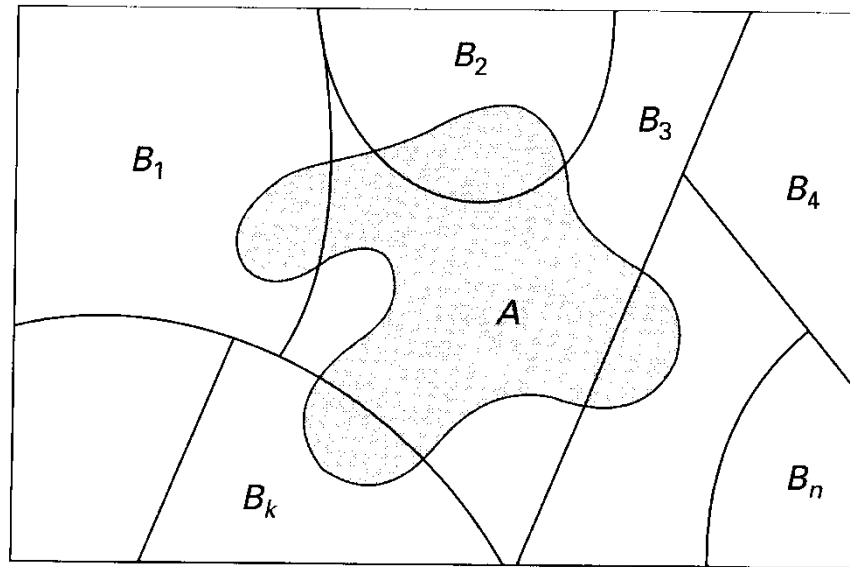


Figure 2.14 Partitioning the sample space S .

Bayes' Rules

- Example 2.41: In a certain assembly plant, three machines, B_1 , B_2 , and B_3 , make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective?

$$\begin{aligned} P(A) &= P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3) \\ &= 0.3 \times 0.02 + 0.45 \times 0.03 + 0.25 \times 0.02 = 0.0245 \end{aligned}$$

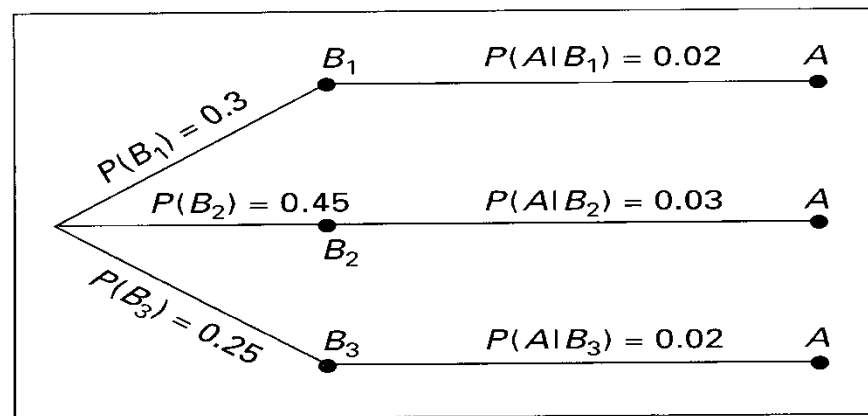


Figure 2.15 Tree diagram for Example 2.41.

Bayes' Rules

- Theorem 2.14: (Bayes's Rule) If the events B_1, B_2, \dots, B_k constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \dots, k$, then

$$P(B_r | A) = \frac{P(B_r \cap A)}{\sum_{i=1}^k P(B_i \cap A)} = \frac{P(B_r)P(A | B_r)}{\sum_{i=1}^k P(B_i)P(A | B_i)}.$$

for $r = 1, 2, \dots, k$

Bayes' Rules

- Example 2.42: With reference to Example 2.41, if a product were chosen randomly and found to be defective, what is the probability that it was made by machine B_3 ?

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Bayes' Rules

- Example 2.42: With reference to Example 2.41, if a product were chosen randomly and found to be defective, what is the probability that it was made by machine B_3 ?

$$P(B_3 | A) = \frac{P(B_3)P(A | B_3)}{P(B_1)P(A | B_1) + P(B_2)P(A | B_2) + P(B_3)P(A | B_3)}.$$

$$P(B_3 | A) = \frac{0.005}{0.006 + 0.0135 + 0.005} = \frac{0.005}{0.0245} = \frac{10}{49}.$$

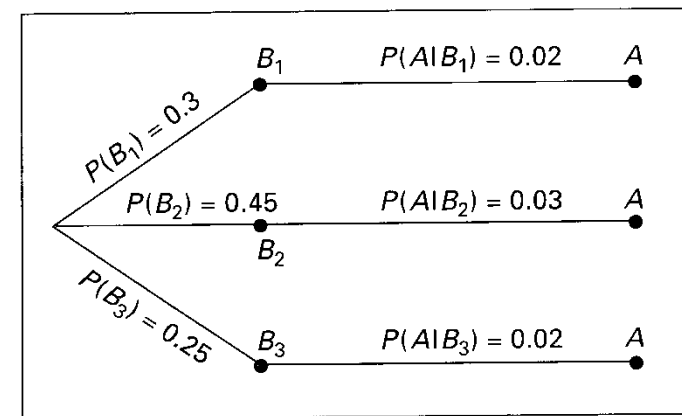


Figure 2.13 Tree diagram for Example 2.38.

Basic Formula

$$P(x) = \sum_y P(x, y) \quad P(y) = \sum_x P(x, y)$$

$$P(x | y) = \sum_h P(x, h | y)$$

$$P(x | y) = \sum_h P(h | y) P(x | y, h)$$

$$P(x, h | y) = P(h | y) P(x | y, h)$$

$$P(x | y) \cong \sum_h P(h | y) P(x | h)$$

$$P(x | y) = \frac{P(x, y)}{P(y)} \\ = \frac{P(x, y)}{\sum_x P(x, y)}$$