pf: $F(s) = \int_0^\infty f(t)\cdot e^{-st}\,dt$ —— (1).

$$\int_s^\infty (1)\,ds = \int_s^\infty \int_s^\infty f(t)\, e^{-st}\,dt\,ds$$

$$= \int_0^\infty \int_s^\infty f(t)\cdot e^{-st}\,ds\,dt$$

(積分次序可対調 ∵ s, t 在範圍内独立)

$$= \int_0^\infty f(t) \int_s^\infty e^{-st}\,ds\cdot dt$$

$$= \int_0^\infty f(t)\left[-\frac{1}{t} e^{-st}\Big|_s^\infty\right] dt$$

$$= \int_0^\infty f(t)\left[0-\left(-\frac{1}{t}e^{-st}\right)\right] dt$$

$$= \int_0^\infty f(t)\cdot\frac{1}{t} e^{-st}\,dt = \mathcal{L}\left\{\frac{1}{t} f(t)\right\}. \quad \#$$

$1 \xrightarrow{\mathcal{L}} \frac{1}{s}$

$t\cdot 1 \xrightarrow{\mathcal{L}} \frac{1}{s^2} = -\frac{d}{ds}\left(\frac{1}{s}\right)$

$\frac{1}{t}\cdot 1 \xrightarrow{\mathcal{L}} \int_s^\infty \frac{1}{s}\,ds$

$\frac{1}{t^2} \xrightarrow{\mathcal{L}} \int_s^\infty\int_s^\infty \frac{1}{s}\,ds\,ds$

$\Rightarrow \mathcal{L}\left\{\frac{1}{t}\sin t\right\} = \int_s^\infty \frac{s}{s^2+1}\,ds$

$= \tan^{-1}s\Big|_s^\infty = \frac{\pi}{2} - \tan^{-1}s \quad \#$

Review.

ex. $3e^{2t} \xrightarrow{\mathcal{L}} \frac{3}{s-2}$ $\qquad$ $H(t) \xrightarrow{\mathcal{L}} \frac{1}{s}$

$2\cos 3t \xrightarrow{\mathcal{L}} \frac{2s}{s^2+9}$ $\qquad$ $\delta(t) \xrightarrow{\mathcal{L}} 1$

$3\sin 2t \xrightarrow{\mathcal{L}} \frac{3\times 2}{s^2+4}$

$t \xrightarrow{\mathcal{L}} \frac{1}{s^2}$

properties.

1. linear.

2. 第一移位

$f(t) \to F(s)$

$e^{at}f(t) \to F(s-a)$

ex. $e^{2t}\cos 3t \to \left.\frac{s}{s^2+9}\right|_{s=s-2}$

$= \frac{s-2}{(s-2)^2+9}$

3. 第二移位

$f(t) \to F(s)$

$H(t-a)f(t-a) \to e^{-as}F(s)$

ex. $F(s) = \frac{2}{s^2+4}e^{-2s}$

$\Rightarrow f(t) = \sin 2(t-2)\,H(t-2)$.

4. $f(t) \to F(s)$

$t f(t) \to -\frac{dF(s)}{ds}$

$t^2 f(t) \to -\frac{d}{ds}\left(-\frac{dF(s)}{ds}\right)$

5. $f(t) \to F(s)$

$\frac{1}{t}f(t) \to \int_s^\infty F(s)\,ds$.

$\frac{1}{t^2}f(t) \to \int_s^\infty\int_s^\infty F(s)\,ds\,ds$.

ex. $\frac{1}{t}\sin t \xrightarrow{L} ?$

我們知道 $\sin t \xrightarrow{L} \frac{1}{s^2+1}$

$\Rightarrow \frac{1}{t}\sin t \to \int_s^\infty \frac{1}{s^2+1}ds = \tan^{-1}s\Big|_s^\infty = \frac{\pi}{2} - \tan^{-1}s$

▲ 微積分内瑕積分.

$\int_0^\infty \frac{\sin t}{t}dt = \lim_{t\to\infty}\int_0^t \frac{\sin t}{t}dt$

$\lim_{s\to 0}\int_0^\infty \frac{\sin t}{t}e^{-st}dt = \lim_{s\to 0} L\left\{\frac{1}{t}\sin t\right\} = \lim_{s\to 0}\left(\frac{\pi}{2} - \tan^{-1}s\right)$

$= \frac{\pi}{2}$

$$\Rightarrow \lim_{s\to 0}\int_0^\infty \frac{\sin t}{t} e^{-st}\,dt = \int_0^\infty \lim_{s\to 0}\frac{\sin t}{t}e^{-st}\,dt = \int_0^\infty \frac{\sin t}{t}\,dt$$

現在，結合性質 4，5.

$$\begin{tikzcd}
G(s) \arrow[d, "L^{-1}"'] & F(s) \arrow[l, "\int_s^\infty F(s)ds"'] \arrow[r, "-\frac{dF(s)}{dt}"] \arrow[d, "L^{-1}"] & G(s) \arrow[d, "L^{-1}"] \\
g(t) \arrow[r, "\times t"] & f(t) & g(t) \arrow[l, "\div t"']
\end{tikzcd}$$

$$\Rightarrow L^{-1}\left\{\left(\int_s^\infty L\{g(t)\cdot t\}\cdot ds\right)\right\} = g(t)$$

ex. $F(s) = \ln\dfrac{s+1}{s+2}$, $f(t) = ?$

$$-\frac{dF(s)}{ds} = -\frac{d}{ds}\ln\frac{s+1}{s+2} = -\frac{s+2}{s+1}\left(\frac{d}{ds}\frac{s+1}{s+2}\right) = \frac{-1}{(s+1)(s+2)}$$

$$= -\left(\frac{a}{s+1}+\frac{b}{s+2}\right) \Rightarrow a=-1,\ b=1$$

$$\Rightarrow g(t) \xrightarrow{L} -\left(\frac{-1}{s+1}+\frac{1}{s+2}\right)$$

$$\Rightarrow g(t) = -\left(-e^{-2t}+e^{-t}\right) = e^{-2t}-e^{-t}$$

$$\Rightarrow f(t) = \frac{1}{t}\left(e^{-2t}-e^{-t}\right) \quad \#$$

*再接下去做驗算的話.

$$\int_0^\infty \frac{1}{t}\left(e^{-2t}-e^{-t}\right)dt$$

$$= \int_0^\infty \lim_{s\to 0}\frac{1}{t}\left(e^{-2t}-e^{-t}\right)e^{-st}\,dt$$

$$= \lim_{s\to 0}\underbrace{\int_0^\infty \frac{1}{t}\left(e^{-2t}-e^{-t}\right)e^{-st}\,dt}_{L\left\{\frac{1}{t}\left(e^{-2t}-e^{-t}\right)\right\}}$$

$= \lim_{s \to 0} \ln \frac{s+1}{s+2} = -\ln 2 = -0.693$

ex. $F(s) = \frac{2s}{(s^2+4)^2}$, $f(t) = ?$

$\int_s^\infty F(s)\,ds = \int_{s^2+4}^\infty \frac{1}{t^2}\,dt = -\frac{1}{t}\Big|_{s^2+4}^\infty$

$= 0 - (-\frac{1}{s^2+4}) = \frac{1}{s^2+4}$

$\Rightarrow g(t) = \frac{1}{2}\sin 2t$. $\Rightarrow f(t) = \frac{1}{2}t\sin 2t$.

*. $\int_0^\infty \frac{1}{2}t \sin 2t\,dt = 0$

property 6.

$f(t) \xrightarrow{L} F(s)$

$f'(t) \xrightarrow{L} sF(s) - f(0)$

ex. $y' + 2y = e^t$, $y(0) = 0$

$\Rightarrow L\{y' + 2y\} = L\{e^t\}$

$\Rightarrow L\{y'\} + L\{2y\} = L\{e^t\}$

$\Rightarrow sY(s) - y(0) + 2Y(s) = \frac{1}{s-1}$

$\Rightarrow Y(s) = \frac{-\frac{1}{3}}{s+2} + \frac{\frac{1}{3}}{s-1}$

$\Rightarrow y(t) = -\frac{1}{3}e^{-2t} + \frac{1}{3}e^t$

pf: $L\{f'(t)\} = \int_0^\infty f'(t)e^{-st}\,dt = \int_0^\infty e^{-st}\,df(t)$

$= e^{-st}f(t)\Big|_0^\infty - \int_0^\infty f(t)(-se^{-st})\,dt$

$= 0 - f(0) + s\underbrace{\int_0^\infty f(t)e^{-st}\,dt}_{L\,f(s)} = sF(s) - f(0)$

要在 $f(t)$ 為指數階時才成立

6.1 $L\{f''(t)\}$

$= L\{(f'(t))'\}$

$= L\{g(t)'\}$

$= s\,L\{g(t)\} - g(0)$

$= s(sF(s) - f(0)) - f'(0) = s^2F(s) - sf(0) - f'(0)$ #.

6.2. $L\{f^{(n)}(t)\}$

$= s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) \;\cdots\; - f^{(n-1)}(0)$

ex. $y'' + 3y' + 2y = e^{t}$, $y(0) = 0$, $y'(0) = 0$.

$L\{y''\} = s^2Y(s) - sy(0) - y'(0)$

$L\{y'\} = sY(s) - y(0)$

$\Rightarrow s^2Y(s) + 3sY(s) + 2Y(s) = \frac{1}{s-1}$

$\Rightarrow Y(s) = \frac{1}{(s-1)(s+1)(s+2)} = \frac{a}{s-1} + \frac{b}{s+1} + \frac{c}{s+2}$

$a = \frac{1}{6}$, $b = \frac{-1}{2}$, $c = \frac{1}{3}$.

$\Rightarrow y(t) = \frac{1}{6}e^{t} - \frac{1}{2}e^{-t} + \frac{1}{3}e^{-2t}$ #.

property 7.

$f(t) \xrightarrow{L} F(s)$

$\int_0^t f(t)\,dt \xrightarrow{L} \frac{1}{s}F(s)$

$1 \xrightarrow{L} \frac{1}{s}$

$H(t)$

$\int_0^t 1\,dt \xrightarrow{L} \frac{\frac{1}{s}}{s} = \frac{1}{s^2}$

推廣. $L\{\underbrace{\int_0^t\int_0^t\int_0^t \cdots}_{k\text{個}} f(x)\,\underbrace{dt\,dt\cdots}_{k\text{個}}\} = \frac{1}{s^k}\cdot F(s)$

pf: $L\{\int_0^t f(t)\,dt\}$

$= \int_0^\infty \int_0^t f(t)\,dt\; e^{-st}\,dt$

$= \int_0^\infty \underbrace{\int_0^t f(\lambda)\,d\lambda}_{u}\; \underbrace{e^{-st}\,dt}_{dv} \quad = uv - \int v\,du$

$= \left(\int_0^t f(\lambda)\,d\lambda\right)\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}\cdot f(t)\cdot dt$

↳ 令為0.

$= 0 - 0 + \frac{1}{s}\int f(t)\,e^{-st}\,dt = \frac{1}{s}F(s)$ #

ex. $L\{e^{2t}\int_0^t e^{3t}\cdot t\cdot \sin t\,dt\}$

$\sin t \to \frac{1}{s^2+1}$

$\to -\frac{d}{ds}\frac{1}{s^2+1}$

$\to \frac{2(s-3)}{((s-3)^2+1)^2}\Big|_{s=s-2}$

$\to \frac{1}{s}$

$\Rightarrow \frac{2(s-5)}{(s-2)[(s-5)^2+1]^2}$

property 8. convolution thm.

1. $f(t) * g(t) \triangleq \int_0^t f(\lambda)\,g(t-\lambda)\,d\lambda$

ex. $f(t) = e^{2t}$, $g(t) = t$

$f(t) * g(t) = \int_0^t f(\lambda)\,g(t-\lambda)\,d\lambda$

$= \int_0^t e^{2\lambda}(t-\lambda)\,d\lambda$

$\Rightarrow L\{f*g\} = F(s)\cdot G(s)$

2. $f(t) * g(t) \triangleq \int_0^t f(\lambda) g(t-\lambda) d\lambda$

令 $\underset{t \to 0}{x} = \underset{0 \to t}{t - \lambda}$, $dx = -d\lambda$

$= \int_t^0 f(t-x) g(x) - dx$

$= \int_0^t f(t-x) g(x) dx$

$x \to \lambda$

$\Rightarrow \int_0^t g(\lambda) \cdot f(t-\lambda) d\lambda = g(t) * f(t)$

ex. $\int_0^t \cos\lambda \, e^{2(t-\lambda)} d\lambda$

convolution thm $= \cos t * e^{2t}$

$f(t) \to F(s)$

$g(t) \to G(s)$

$f(t) * g(t) \to F(s) \cdot G(s)$

ex. $L\{\int_0^t e^{2\lambda} (t-\lambda) d\lambda\} = \frac{1}{s-2} \cdot \frac{1}{s^2}$

ex. $f(t) = e^{2t}$, $g(t) = e^{3t}$

$f(t) * g(t) = \int_0^t e^{2\lambda} \cdot e^{3(t-\lambda)} d\lambda$

$= e^{3t} \int_0^t e^{-\lambda} d\lambda$

$= e^{3t} (-e^{-\lambda} |_0^t) = e^{3t}(-e^{-t} - (-1))$

$= e^{3t} - e^{2t}$

$\Rightarrow L\{e^{3t} - e^{2t}\} = \frac{1}{s-3} - \frac{1}{s-2}$

pf: $L\{f(t) * g(t)\}$.

$= \int_0^\infty [f(t) * g(t)] \cdot e^{-st} dt$

$= \int_0^\infty \int_0^t f(\lambda) g(t-\lambda) d\lambda \, e^{-st} dt$

$= \int_0^\infty \int_0^t f(\lambda) g(t-\lambda) e^{-st} dt \, d\lambda$

若想对調 $d\lambda$, dt, 則積分区域要相同.

$$= \int_0^\infty \int_\lambda^\infty f(\lambda)\, g(t-\lambda)\, e^{-st}\, dt\, d\lambda$$

$$= \int_0^\infty f(\lambda) \int_\lambda^\infty g(t-\lambda)\, e^{-st}\, dt\, d\lambda$$

令 $x = t-\lambda$, $dx = dt$

$$= \int_0^\infty f(\lambda) \int_0^\infty g(x)\, e^{-s(x+\lambda)}\, dx\, d\lambda.$$

$$= \int_0^\infty f(\lambda)\, e^{-s\lambda} \underbrace{\left(\int_0^\infty g(x)\, e^{-sx}\, dx\right)}_{G(s)} d\lambda.$$

$$= G(s) \int_0^\infty f(\lambda)\, e^{-s\lambda}\, d\lambda = G(s)\, F(s) \quad \#$$