1.

(a)

$O(g(n)) = \{f(n)$: there exist positive constants c and n_0 such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0\}$.

$\Omega(g(n)) = \{ f(n)$: there exist positive constants c and n_0 such that $0 \le cg(n) \le f(n)$ for all $n \ge n_0\}$.

$\Theta(g(n)) = \{ f(n)$: there exist positive constants c_1, c_2, and n_0 such that $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0\}$.

(b)

Let's define the function $h(n) = \max(f(n), g(n))$. Then

$$h(n) = \begin{cases} f(n) & \text{if } f(n) \ge g(n) , \\ g(n) & \text{if } f(n) < g(n) . \end{cases}$$

Since $f(n)$ and $g(n)$ are asymptotically nonnegative, there exists n_0 such that $f(n) \ge 0$ and $g(n) \ge 0$ for all $n \ge n_0$. Thus for $n \ge n_0$, $f(n) + g(n) \ge f(n) \ge 0$ and $f(n)+g(n) \ge g(n) \ge 0$. We have $f(n) + g(n) \ge h(n) \ge 0$, which shows that $h(n) = \max(f(n), g(n)) \le c_2(f(n) + g(n))$ for all $n \ge n_0$ (with $c_2 = 1$ in the definition of Θ).

Similarly, since for any particular n, $h(n)$ is the larger of $f(n)$ and $g(n)$, we have for all $n \ge n_0$, $0 \le f(n) \le h(n)$ and $0 \le g(n) \le h(n)$. Adding these two inequalities yields $0 \le f(n) + g(n) \le 2h(n)$, or equivalently $0 \le (f(n) + g(n))/2 \le h(n)$, which shows that $h(n) = \max(f(n), g(n)) \ge c_1(f(n)+g(n))$ for all $n \ge n_0$ (with $c_1 = 1/2$ in the definition of Θ).

2.

(1)

We can know $\lg(f(n)) = O(\lg n)$, then f is polynomially bounded.

$$\begin{aligned} \lg(\lceil \lg n \rceil !) &= \Theta(\lceil \lg n \rceil \lg \lceil \lg n \rceil) \\ &= \Theta(\lg n \lg \lg n) \\ &= \omega(\lg n). \end{aligned}$$

Therefore, $\lg(\lceil \lg n \rceil !) \ne O(\lg n)$, and so $\lceil \lg n \rceil !$ is not polynomially bounded

(2)

$\lg(\lceil \lg \lg n \rceil !) = \Theta(\lceil \lg \lg n \rceil \lg \lceil \lg \lg n \rceil)$

$= \Theta(\lg \lg n \lg \lg \lg n)$

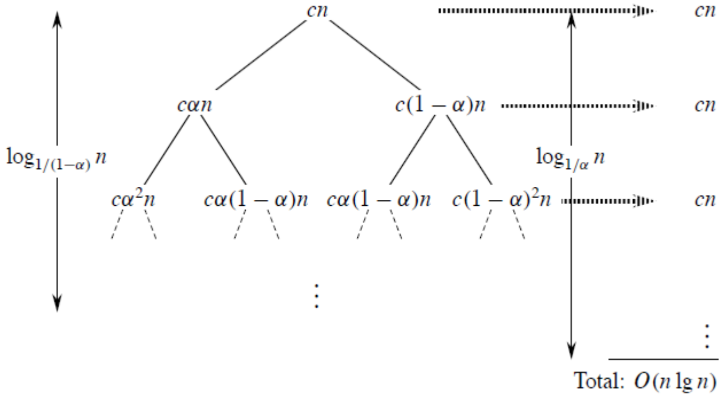
$= o((\lg \lg n)^2)$

$= o(\lg^2(\lg n))$

$= o(\lg n)$.

$\lg(\lceil \lg \lg n \rceil !) = O(\lg n)$, and so $\lceil \lg \lg n \rceil !$ is polynomially bounded.

3.



Now we show that $T(n) = \Theta(n \lg n)$ by substitution. To prove the upper bound, we need to show that $T(n) \le dn \lg n$ for a suitable constant $d > 0$.

$$\begin{aligned} T(n) &= T(\alpha n) + T((1-\alpha)n) + cn \\ &\le d\alpha n \lg(\alpha n) + d(1-\alpha)n \lg((1-\alpha)n) + cn \\ &= d\alpha n \lg \alpha + d\alpha n \lg n + d(1-\alpha)n \lg(1-\alpha) + d(1-\alpha)n \lg n + cn \\ &= dn \lg n + dn(\alpha \lg \alpha + (1-\alpha)\lg(1-\alpha)) + cn \\ &\le dn \lg n , \end{aligned}$$

Since $1/2 \le \alpha < 1$ and $0 < 1-\alpha \le 1/2$, we have that $\lg\alpha < 0$ and $\lg(1-\alpha) < 0$. Thus, $\alpha \lg \alpha + (1-\alpha)\lg(1-\alpha) < 0$, so that when we multiply both sides of the inequality by this factor, we need to reverse the inequality:

$$d \ge \frac{-c}{\alpha \lg \alpha + (1-\alpha)\lg(1-\alpha)}$$

or

$$d \ge \frac{c}{-\alpha \lg \alpha + -(1-\alpha)\lg(1-\alpha)}$$

The fraction on the right-hand side is a positive constant, and so it sufÞces to pick any value of d that is greater than or equal to this fraction.

To prove the lower bound, we need to show that $T(n) \ge dn \lg n$ for a suitable constant $d > 0$. We can use the same proof as for the upper bound, substituting $\ge$ for $\le$, and we get the requirement that

$0 < d \le c$

$$0 < d \le \frac{c}{-\alpha \lg \alpha - (1-\alpha)\lg(1-\alpha)} .$$

Therefore, $T(n) = \Theta(n \lg n)$.

4.

(1) T(n)=9T(n/3) + n (Case 1)

a=9, b=3, f(n) = n

$n^{\log_b a} = n^{\log_3 9} = \Theta(n^2),\ f(n) = O(n^{\log_3 9-1})$

T(n) = θ(n2)

(2)T(n) = T(2n/3) + 1 (Case 2)

a=1, b=3/2, f(n)=1

$n^{\log_b a} = n^{\log_{3/2} 1} = 1,\ f(n) = 1 = \Theta(1)$

T(n) = θ(lg *n*)

(3) T(n) = 3T(n/4) + n lg n (Case 3)

a=3, b=4, f(n) = n lg n

$$n^{\log_b a} = n^{\log_4 3} = O(n^{0.793})\ f(n) = \Omega(n^{\log_4 3+\varepsilon})$$ (ε ≈0.2)

For sufficiently large n,

af(n/b) = 3(n/4)lg(n/4) ≤ (3/4)n lg n=cf(n) for c=3/4

T(n) = θ(n lg n)

5.
利用 decision tee 方法證明:
假設有 n 個元素作比較， l 為 decision tree 的 leaf 個數， h 為 decision tree 的樹高
(1) n 個元素作比較有 $n!$ 種排列結果，decision tree 的 leaf 個數 $n! \le l$ 。
(2)因為 decision tree 為一 binary tree， $l \le 2^h$
所以， $2^h \ge n! \Rightarrow h \ge \lg(n!)$

By stirling approximation: $n! \ge \left(\frac{n}{e}\right)^n$

So, $h \ge \lg(n!) \ge \lg\left(\frac{n}{e}\right)^n = n\lg n - n\lg e = \Omega(n\lg n)$

6.

Counting sort (A, *n*, *k*, *a*, *b*)

for *i* ← 0 to *k*

C[i] ← 0　　　//初始化　　　$\theta(k)$

for i ← 1 to n

C[A[i]] ← C[A[i]] + 1　//計算每個 integer 出現的次數　　　$\theta(n)$

for i ← 1 to k

C[i] ← C[i] + C[i – 1]　//累加 integer 次數，方便計算 range[a…b]次數和 $\theta(k)$

if(a = = 0)

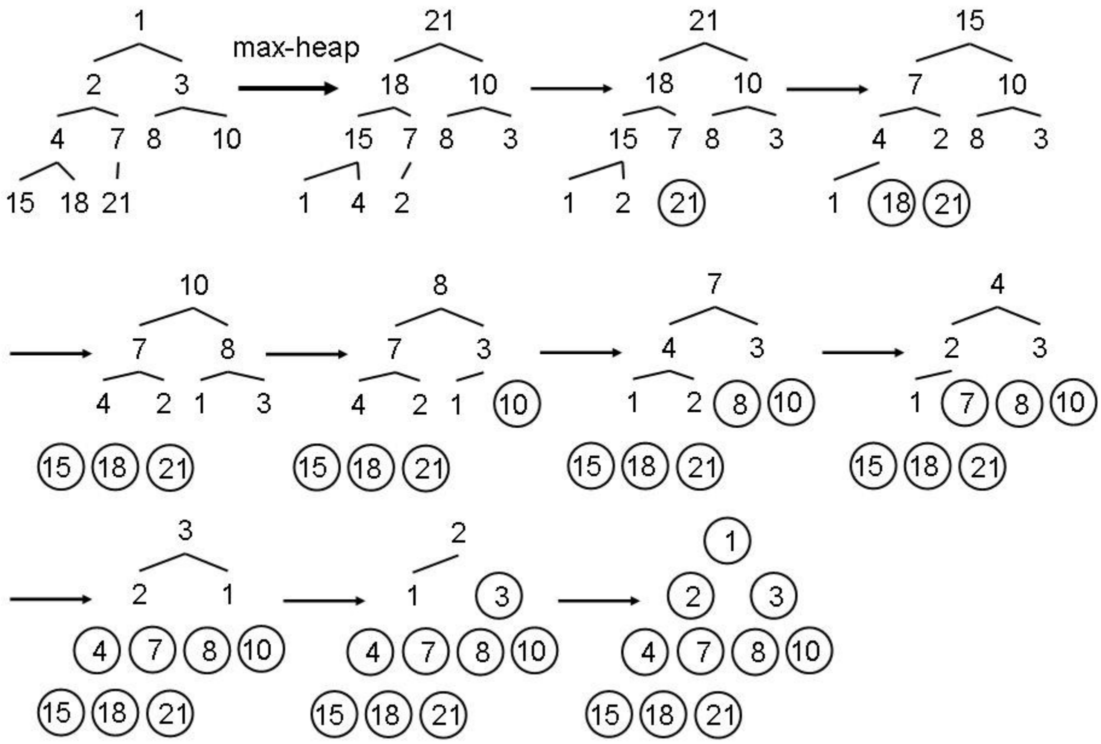
count ← C[b]

else

count ← C[b] – C[a – 1]　　　$\theta(1)$

return count

7.



→A

1	2	3	4	7	8	10	15	18	21

8.

若有 height 為 h 的 heap，則此 heap 最少有 2^h 個 elements

最多有 $2^{h+1}–1$ 個 elements

⇨ elements 數目為 n，$2^h \le n \le 2^{h+1}–1$

⇨ 同取 lg

⇨ $h \le n < h+1$

⇨ $h = \lfloor \lg n \rfloor$

9.

n 個元素 $z_1, z_2, \ldots, z_n$

Z_{ij} : $\{z_i, z_{i+1}, \ldots, z_j\}$: the set of elements between z_i and z_j

X_{ij} : $I\{z_i \text{ is compared to } z_j\}$

$$X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} .$$

$$\begin{aligned} \mathrm{E}[X] &= \mathrm{E}\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\right] \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathrm{E}[X_{ij}] \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\{z_i \text{ is compared to } z_j\} . \end{aligned}$$

$$\begin{aligned} \Pr\{z_i \text{ is compared to } z_j\} &= \Pr\{z_i \text{ or } z_j \text{ is first pivot chosen from } Z_{ij}\} \\ &= \Pr\{z_i \text{ is first pivot chosen from } Z_{ij}\} \\ &\quad + \Pr\{z_j \text{ is first pivot chosen from } Z_{ij}\} \\ &= \frac{1}{j-i+1} + \frac{1}{j-i+1} \\ &= \frac{2}{j-i+1} . \end{aligned}$$

$$\mathrm{E}[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} .$$

$$\begin{aligned} \mathrm{E}[X] &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\ &= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} \\ &< \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} \\ &= \sum_{i=1}^{n-1} O(\lg n) \\ &= O(n \lg n) . \end{aligned}$$

10.

$w[i, j] = w[i, j-1] + pj$

w	0	1	2	3	4	5	6

1	0	0.25	0.45	0.55	0.6	0.9	1
2		0	0.2	0.3	0.35	0.65	0.75
3			0	0.1	0.15	0.45	0.55
4				0	0.05	0.35	0.45
5					0	0.3	0.4
6						0	0.4
7							0

$e[i, j] = e[i, r-1] + e[r+1, j] + w(i, j)$. 取min

w	0	1	2	3	4	5	6
1	0	0.25	0.65	0.9	1.05	1.8	2.1
2		0	0.2	0.4	0.55	1.2	1.4
3			0	0.1	0.2	0.65	0.85
4				0	0.05	0.4	0.6
5					0	0.3	0.5
6						0	0.1
7							0

Root

root	1	2	3	4	5	6
1	1	1	2	2	2	2
2		2	2	2	5	5
3			3	3	5	5
4				4	5	5
5					5	5
6						6