DISCRETE MATHEMATICS – CH8 Homework8

8.1

6. (10 pts) Determine how many integer solutions there are to $x_1 + x_2 + x_3 + x_4 = 20$, if

(a) $0 \le x_i$ for all $1 \le i \le 4$

(b) $0 \le x_i < 8$ for all $1 \le i \le 4$

(c) $0 \le x_1 \le 5$, $0 \le x_2 \le 6$, $3 \le x_3 \le 7$, $3 \le x_4 \le 8$

a. $\binom{4+20-1}{20} = \binom{23}{20}$

b. For $1 \le i \le 4$, let $c_i : x_i \ge 8$.

$N(c_i) : x_1 + x_2 + x_3 + x_4 = 12 : \binom{4+12-1}{12} = \binom{15}{12}$, $1 \le i \le 4$

$N(c_i\, c_j) : x_1 + x_2 + x_3 + x_4 = 4 : \binom{4+4-1}{4} = \binom{7}{4}$, $1 \le i \le 4$

$N - S_1 + S_2 = \binom{23}{20} - 4\binom{15}{12} + 6\binom{7}{4} = 1771 - 1820 + 210 = 161$

c. The number of solutions equals the number of solutions for $x_1 + x_2 + x_3 + x_4 =$ 14 with $0 \le x_1 \le 5$, $0 \le x_2 \le 6$, $0 \le x_3 \le 4$, $0 \le x_4 \le 5$

$N = \binom{4+14-1}{14} = \binom{17}{14}$

$x_1 \ge 6, x_4 \ge 6 : x_1 + x_2 + x_3 + x_4 = 8 : \binom{4+8-1}{8} = \binom{11}{8}$

$x_2 \ge 7 : x_1 + x_2 + x_3 + x_4 = 7 : \binom{4+7-1}{7} = \binom{10}{7}$

$x_3 \ge 5 : x_1 + x_2 + x_3 + x_4 = 9 : \binom{4+9-1}{9} = \binom{12}{9}$

$x_1 \ge 6$ and $x_2 \ge 7 : x_1 + x_2 + x_3 + x_4 = 1 : \binom{4+1-1}{1} = \binom{4}{1}$

$x_1 \ge 6$ and $x_3 \ge 5 : x_1 + x_2 + x_3 + x_4 = 3 : \binom{4+3-1}{3} = \binom{6}{3}$

$x_1 \ge 6$ and $x_4 \ge 6 : x_1 + x_2 + x_3 + x_4 = 2 : \binom{4+2-1}{2} = \binom{5}{2}$

$x_2 \ge 7$ and $x_3 \ge 5 : x_1 + x_2 + x_3 + x_4 = 2 : \binom{5}{2}$

$x_2 \ge 7$ and $x_4 \ge 6 : x_1 + x_2 + x_3 + x_4 = 1 : \binom{4}{1}$

$x_3 \ge 5$ and $x_4 \ge 6 : x_1 + x_2 + x_3 + x_4 = 3 : \binom{6}{3}$

Ans: $\binom{17}{14} - [2\binom{11}{8} + \binom{10}{7} + \binom{12}{9}] + 2[\binom{4}{1} + \binom{6}{3} + \binom{5}{2}]$

13. (10 pts) Find the number of permutations of $a, b, c, \dots, x, y, z$, in which none of the patterns spin, game, path, or net occurs. (Random select four English words as patterns, same with other students will share the score.)

Let c_1 denote that the arrangement contains the pattern *spin*. Likewise, let c_2, c_3, c_4 denote this for the patterns *game*, *path*, and *net*, respectively. $N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = 26! - [3(23!) + 24!] - (20! + 21!)$

8.2

4. (10 pts) Let $A = \{1, 2, 3, \ldots, 10\}$, and $B = \{1, 2, 3, \ldots, 7\}$. How many functions $f: A \rightarrow B$ satisfy $|f(A)| = 4$? How many have $|f(A)| \leq 4$?

For $1 \leq i \leq 7$ let c_i denote the condition that i is not in the range of f. Then the number of functions $f: A \rightarrow B$ where $|f(A)| = 4$ is $E_3 = S_3 - \binom{4}{1}S_4 + \binom{5}{2}S_5 - \binom{6}{3}S_6 + \binom{7}{4}S_7 = \binom{7}{3}4^{10} - \binom{4}{1}\binom{7}{4}3^{10} + \binom{5}{2}\binom{7}{5}2^{10} - \binom{6}{3}\binom{7}{6}1^{10} + \binom{7}{4}\binom{7}{7}0^{10} = 28648200$.

Note: Using Stirling numbers of the second kind the result is $\binom{7}{4}4!S(10,4) = 28648200$.

$L_3 = S_3 - \binom{3}{2}S_4 + \binom{4}{2}S_5 - \binom{5}{2}S_6 + \binom{6}{2}S_7 = \binom{7}{3}4^{10} - \binom{3}{2}\binom{7}{4}3^{10} + \binom{4}{2}\binom{7}{5}2^{10} - \binom{5}{2}\binom{7}{6}1^{10}$.

8.3

6. (10 pts) How many derangements of 1, 2, 3, 4, 5, 6, 7, 8 start with (a) 1, 2, 3, and 4, in some order? (b) 5, 6, 7, and 8, in some order?

(a) There are $(d_4)^2 = 9^2 = 81$

(b) In such case we get $(4!)^2 = 24^2 = 576$ derangements

8.4 & 8.5

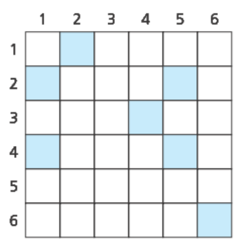
12. (10 pts) (f(4)≠x only, 學號奇數 add f(5)≠z, 學號偶數 add f(5)≠y)

12. For $A = \{1, 2, 3, 4, 5\}$ and $B = \{u, v, w, x, y, z\}$, determine the number of one-to-one functions $f: A \rightarrow B$ where $f(1) \neq v, w$; $f(2) \neq u, w$; $f(3) \neq x$; and $f(4) \neq v, x, y$.

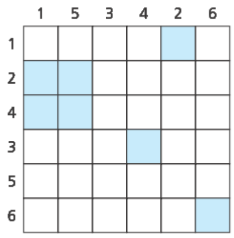
Advanced assignment

1. (20 pts) explain why the principle of inclusion and exclusion and rook polynomials suit for solving the problems 8.16, and 8.17 respectively **(參考答案)**

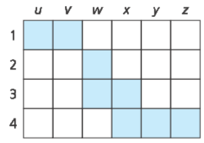
Arrangement with forbidden positions 問題通常是以全部的情況減去不合的情況（inclusion and exclusion principle），當限制變多，由排容原理知道，必須加上兩個不合的情況，然後減掉三個不合的情形（以此類推）...，而這樣的作法在計算上雖然比較麻煩，還是可以用排容原理解，此外可以利用 rook polynomial 來驗證這類問題的正確性。

Example 8.16 可將問題想成是 

，row 是紅骰子、column 是綠骰子，棋盤中的每一格表示投擲兩個骰子的狀況，藍色格子表示不能出現的狀況([(1, 2),(2, 1),(2, 5),(3, 4),(4, 1),(4, 5) ,(6, 6)])，這個題目的目的是要

找出每個 row 選出一個 column，不能重複 column 並避開藍色的 forbidden zone（

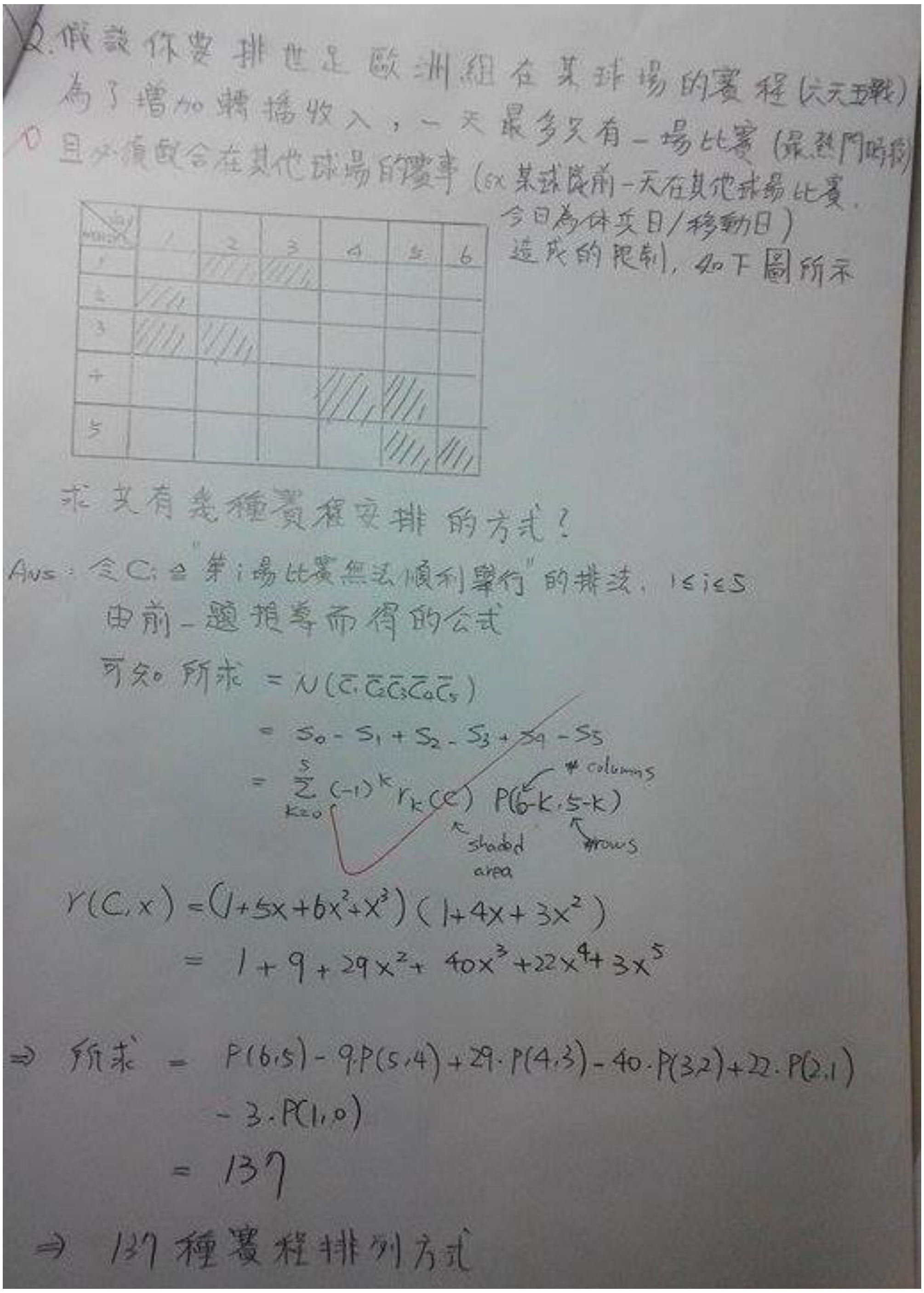
）。這個題目的設定符合 arrangement with forbidden positions 的題型。

Example 8.17 這個問題可想成是

，row 是 input{ 1, 2, 3, 4}，column 是 output{ u, v, w, x, y, z}，棋盤每一格表示 function 有某 input，所對應的 output，藍色格子表示不能出現的情況，這個題目的目的是要找出每個 row 選出一個 column 而不能重複且避開藍色的 forbidden zone，這個題目的設定也與 arrangement with forbidden postions 的題型相符合。

因為 Example 8.16、8.17 都是屬於這種 Arrangement with forbidden positions 的題型，可以將問題轉換成：每個 row 選一個 column，不能重複且需要避開 forbidden zone 的問題，因此可以用上述的方法解。

2. (30 pts) design a counting problem and find its solution that the principle of inclusion and exclusion and rook polynomials are helpful for solving it. (same with other students will share the score.) **(優解共賞)**



Q. 假設你要排世足歐洲組在某球場的賽程(六天五戰)
為了增加轉播收入，一天最多只有一場比賽(最熱門時段)
且必須配合在其他球場的賽事(ex 某球隊前一天在其他球場比賽，今日為休息日/移動日)
造成的限制，如下圖所示

求只有幾種賽程安排的方式?

Ans: 令 $C_i \triangleq$ "第 i 場比賽無法順利舉行" 的排法，$1 \le i \le 5$
由前一題推導而得的公式
可知 所求 $= N(\bar{C_1}\bar{C_2}\bar{C_3}\bar{C_4}\bar{C_5})$

$$= S_0 - S_1 + S_2 - S_3 + S_4 - S_5$$

$$= \sum_{k=0}^{5} (-1)^k \, r_k(C) \, P(6-k, 5-k)$$

shaded area　# columns　# rows

$$r(C,x) = (1+5x+6x^2+x^3)(1+4x+3x^2)$$
$$= 1 + 9 + 29x^2 + 40x^3 + 22x^4 + 3x^5$$

$\Rightarrow$ 所求 $= P(6,5) - 9 \cdot P(5,4) + 29 \cdot P(4,3) - 40 \cdot P(3,2) + 22 \cdot P(2,1) - 3 \cdot P(1,0)$
$= 137$

$\Rightarrow$ 137 種賽程排列方式