DISCRETE MATHEMATICS – CH4 Homework4

4.1

18. Consider the following four equations: **(10 pts)**

1) $1 = 1$
2) $2 + 3 + 4 = 1 + 8$
3) $5 + 6 + 7 + 8 + 9 = 8 + 27$
4) $10 + 11 + 12 + 13 + 14 + 15 + 16 = 27 + 64$

Conjecture the general formula suggested by these four equations, and prove your conjecture. (write down how do you make your conjecture)

S(n): $\sum_{i=1}^{2*n-1}((n-1)^2+i) = (n-1)^3+n^3$

S(1) 1=1

S(k) Suppose $\sum_{i=1}^{2*n-1}((n-1)^2+i) = (n-1)^3+n^3$ true.

S(k+1): $\sum_{i=1}^{2k+1}(k^2+i) = k^3+(k+1)^3$

$$k^2\sum_{i=1}^{2k+1}1+\sum_{i=1}^{2k+1}i = (2k+1)k^2+\frac{2k+1}{2}(2*1+(2k+1-1)*1)$$

$$= k^2(2k+1)+\frac{(2k+1)(2k+2)}{2}$$

$$= 2k^3+k^2+(2k+1)(k+1) =$$

$$= 2k^3+3k^2+3k+1 = k^3+k^3+3k^2+3k+1 = k^3+(k+1)^3$$

It is proved that $\sum_{i=1}^{2k+1}(k^2+i) = k^3+(k+1)^3$

4.2

12. For $n \geq 0$ let Fn denote the nth Fibonacci number. Prove that **(10 pts)**

$$F_0+F_1+F_2+\cdots+F_n = \sum_{i=0}^{n}F_i = F_{n+2}-1.$$

1. $n=0,\quad F_0=\sum_{i=0}^{0}F_i=F_{n+2}-1=1$

2. Assuming $n=k,\quad F_0+F_1+\ldots+F_k=\sum_{i=0}^{k}F_i=F_{k+2}-1$

   when $n=k+1,\quad F_0+F_1+\ldots+F_k+F_{k+1}=\sum_{i=0}^{k}F_i+F_{k+1}=F_{k+2}+F_{k+1}-1=F_{((k+1)+2)}-1$

   By the Mathematical Induction, RT is true for all $n \geq 0$

18. Consider the permutations of 1, 2, 3, 4. The permutation 1432, for instance, is said to have one ascent—namely, 14 (since $1 < 4$). This same permutation also has two descents—namely, 43 (since $4 > 3$) and 32 (since $3 > 2$). The permutation 1423, on the other hand, has two ascents, at 14 and 23—and the one descent 42. **(20 pts)**

(b) How many permutations of 1, 2, 3, 4 have k ascents, for k = 0, 1, 2, 3?

(d) Suppose a permutation of 1, 2, 3, …, m has k ascents, for $0 \leq k \leq m-1$. How many descents does the permutation have?

f) Let $\pi_{m,k}$ denote the number of permutations of 1, 2, 3, $\ldots, m$ with k ascents. Note how $\pi_{4,2} = 11 = 2(4) + 3(1) = (4-2)\pi_{3,1} + (2+1)\pi_{3,2}$. How is $\pi_{m,k}$ related to $\pi_{m-1,k-1}$ and $\pi_{m-1,k}$?

(b)	$k=0:$	1	4321
	$k=1:$	11	1432, 2143, 2431, 3142, 3214, 3241, 3421, 4132, 4213, 4231, 4312
	$k=2:$	11	1243, 1324, 1342, 1423, 2134, 2314, 2341, 2413, 3124, 3412, 4123
	$k=3:$	1	1 2 3 4

(d) $(m-1)-k = m-k-1$ descents.

(f) $\pi_{m,k} = (k+1)\pi_{m-1,k} + (m-k)\pi_{m-1,k-1}$.

Let $x: x_1, x_2, \ldots, x_m$ denote a permutation of $1, 2, 3, \ldots, m$ with k ascents (and $m-k-1$ descents). (1) If $m = x_m$ or if m occurs in $x_i m x_{i+2}, 1 \leq i \leq m-2$, with $x_i > x_{i+2}$ then the removal of m results in a permutation of $1, 2, 3, \ldots, m-1$ with $k-1$ ascents – for a total of $[1+(m-k-1)]\pi_{m-1,k-1} = (m-k)\pi_{m-1,k-1}$ permutations. (2) If $m = x_1$ or if m occurs in $x_i m x_{i+2}$, $1 \leq i \leq m-2$, with $x_i < x_{i+2}$, then the removal of m results in a permutation of $1, 2, 3, \ldots, m-1$ with k ascents – for a total of $(k+1)\pi_{m-1,k}$ permutations.

Since cases (1) and (2) have nothing in common and account for all possibilities the recursive formula for $\pi_{m,k}$ follows. [Note: These are the Eulerian numbers $a_{m,k}$ of Example 4.21.]

4.4

15. After a weekend at the Mohegan Sun Casino, Gary finds that he has won $1020—in $20 and $50 chips. If he has more $50 chips than $20 chips, how many chips of each denomination could he possibly have? **(10 pts)**

Let 20x + 50y = 1020, x < y

⇨ 2x+5y = 102

⇨ 2(-204) + 5(102) = 102 [2(-2) + 5(1) = 1]

⇨ 2(-204 + 5k) + 5(102 - 2k) = 102, -204 + 5k > 0, 102 - 2k > 0 → 41 <= k < 51

⇨ Since x < y → k = 41(1,20)、42(6,18)、43(11,16)

(16+2k)*50 + (11-5k)*20, k=0, 1, 2

4.5

19. Howmany different products can one obtain by multiplying any two (distinct) integers in the set **(10 pts)**

(c) {4, 8, 9, 16, 27, 32, 64, 81, 243}?

The set here may also be represented as A∪B

A=$\{2^n | n \in Z^+, 2 \leq n \leq 6\}$

B=$\{3^n | n \in Z^+, 2 \le n \le 5\}$

(e) $\{p^2, p^3, p^4, p^5, p^6, q^2, q^3, q^4, q^5, q^6, r^2, r^3, r^4, r^5\}$, where p, q, and r are distinct primes?

Consider the set given here as $A \cup B \cup C$

A = $\{ p^2, p^3, p^4, p^5, p^6\}$

B = $\{ q^2, q^3, q^4, q^5, q^6\}$

C = $\{ r^2, r^3, r^4, r^5\}$

Both element from A : 7 possibilities

Both element from B : 7 possibilities

Both element from C : 5 possibilities

One element form each of A, B : $5 \times 5 = 25$ possibilities

One element form each of A, C : $5 \times 4 = 20$ possibilities

One element form each of B, C : $5 \times 4 = 20$ possibilities

⇨ 84 possible products

Advanced assignment

- For n>=1, show that if n>=64, then n can be written as a sum of 5's and/or 17's. (92,95 nthu.cs) **(20 pts)**

  induction on n,

  $n = 64$ 時, $64 = 5 \times 6 + 17 \times 2$ 故命題成立.

  $n = 65$ 時, $65 = 5 \times 13$,

  $n = 66$ 時, $66 = 5 \times 3 + 17 \times 3$,

  $n = 67$ 時, $67 = 5 \times 10 + 17$,

  $n = 68$ 時, $68 = 17 \times 4$,

  設對 64 到 $k-1$ 的整數都.可表達為 5 與 17 的線性組合, $k \ge 69$,

  則 $n = k$ 時, $\because k = (k-5) + 5$, 而 $k-5$ 可表達為 5 與 17 的線性組合,

  故知 k 亦可表為 5 與 17 的線性組合.