Chapter 4.
Laplace Transform

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Laplace Transform

性質9:

Initial Value Theorem 初值定理

Final Value Theorem 終值定理

$F(s) = L\{f(t)\} \Rightarrow f(t) = L^{-1}\{F(s)\}$

能不能只看 $F(s)$ 而不看 $f(t)$ 就知道 $f(0)$ ，$f(\infty)$ 之值

If $F(s)$ is given $f(0) = \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$ 初值定理

$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$ 終值定理

Laplace Transform

EX： $F(s) = \dfrac{2s}{(s+1)(s+2)}$

Verify： $f(t) = ?$

$$F(0) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{2s^2}{s^2+3s+2} = 2$$

$$F(\infty) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{2s^2}{s^2+3s+2} = 0$$

$$F(s) = \frac{2s}{(s+1)(s+2)} = \frac{a}{s+1} + \frac{b}{s+2}$$

$$\Rightarrow a = \frac{2s}{s+2}\Big|_{s=-1} = \frac{-2}{1} = -2$$

$$b = \frac{2s}{s+1}\Big|_{s=-2} = \frac{-4}{-1} = 4$$

$$\Rightarrow f(t) = -2e^{-t} + 4e^{-2t},\ f(0) = 2$$

Laplace Transform

pf：聯想$sF(s)$ 在 $\mathscr{L}\{f'\}$ 出現過(性質6)

$$\mathscr{L}\left\{f'(t)\right\} = \int_0^\infty f'(t)e^{-st}dt = sF(s) - f(0)$$

$$\lim_{s\to\infty}[\int_0^\infty f'(t)e^{-st}dt] = \lim_{s\to\infty}(sF(s) - f(0))$$

$$\int_0^\infty f'(t)\lim_{s\to\infty}[e^{-st}]dt = \lim_{s\to\infty} sF(s) - f(0)$$

$$\Rightarrow 0 = \lim_{s\to\infty} sF(s) - f(0)$$

$$\Rightarrow \lim_{s\to\infty} sF(s) = f(0)$$

Laplace Transform

$$\mathscr{L}\left\{f'(t)\right\} = \int_0^\infty f'(t)e^{-st}dt = sF(s) - f(0)$$

$$\lim_{s\to 0}[\int_0^\infty f'(t)e^{-st}dt] = \lim_{s\to 0}(sF(s) - f(0))$$

$$\int_0^\infty f'(t)dt = \lim_{s\to 0} sF(s) - f(0)$$

$$f(t)\Big|_0^\infty = \lim_{s\to 0} sF(s) - f(0)$$

$$f(\infty) - f(0) = \lim_{s\to 0} sF(s) - f(0)$$

$$\lim_{s\to 0} sF(s) = f(\infty)$$

Laplace Transform

EX：$F(s)=\dfrac{2s}{(s-1)(s+2)}$

$$f(0)=\lim_{s\to\infty}\frac{2s^2}{(s-1)(s+2)}=2$$

$$f(\infty)=\lim_{s\to 0}\frac{2s^2}{(s-1)(s+2)}=0$$

但$F(s)=\dfrac{\frac{2}{3}}{s-1}+\dfrac{\frac{4}{3}}{s+2}$

$$f(t)=\frac{2}{3}e^{t}+\frac{4}{3}e^{-2t}$$

$$f(0)=\frac{2}{3}+\frac{4}{3}=2$$

$f(\infty)=\infty$ 矛盾

Laplace Transform

終值定理要成立，必須滿足某些條件才行

$$F(s)=\frac{N(s)}{D(s)}$$

$D(s)$ 的根不能在S-平面的右平面i.e. $\mathrm{Re}(s)>0$

若在虛軸上不可有重根不能有共軛虛根

Laplace Transform

EX： $F(s)=\dfrac{2s}{(s^2+4)^2}, s^2+4=0, s=\pm 2i$

$$f(\infty)=\lim_{s\to 0} sF(s)=0$$

$$f(t)=\frac{1}{2}t\sin 2t \ \text{(如何得到?)}$$

$$\Rightarrow \int_s^{\infty}\frac{2s}{(s^2+4)^2}ds,\ \text{令}\ u=s^2+4$$

$$=\int_{s^2+4}^{\infty}\frac{1}{u^2}du$$

$$=\frac{-1}{u}\Big|_{s^2+4}^{\infty}=\frac{1}{s^2+4}$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+4}\right\}=\frac{1}{2}\sin 2t$$

$$f(t)=\frac{1}{2}t\sin 2t$$

Laplace Transform

1.基本*function* 1~6

2.基本性質 1~9

3.週期函數

性質11. 週期函數的 Laplace transform

考慮函數f(t)若滿足f(t+nT)=f(t)

則稱T為f(t)的週期，而f(t)為週期函數 $h \in Z$

$$\cos x \xrightarrow{T} 2\pi$$

$$\cos(x+2n\pi)=\cos x$$

$$\cos 2x \xrightarrow{T} \pi$$

$$\cos(2x+n\pi)=\cos 2x$$

Laplace Transform

$$\mathscr{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

$$= \int_0^T f(t)e^{-st}dt + \int_T^{2T} f(t)e^{-st}dt + \int_{2T}^{3T} f(t)e^{-st}dt + \cdots$$

$$= \int_0^T f(t)e^{-st}dt + \int_T^{2T} f(t)e^{-st}dt \quad 令 \quad x = t - T \quad dt = dx$$

其中所以 $\Rightarrow \int_0^T f(t)e^{-s(\mathrm{x}+\mathrm{T})}dx$

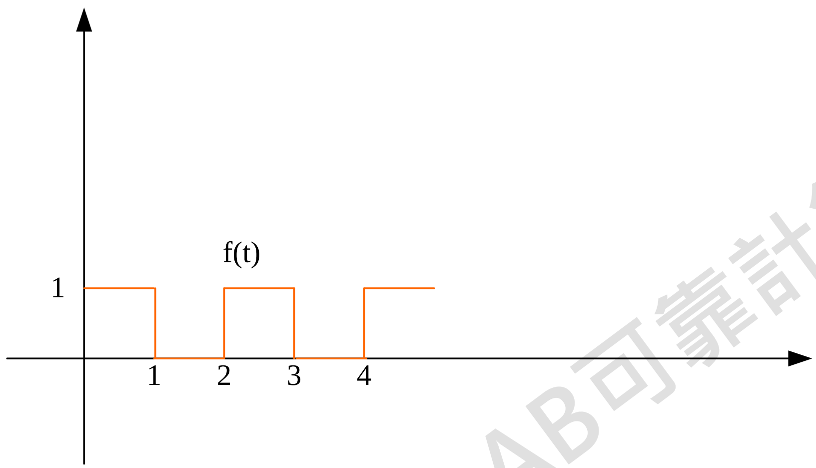
$$= e^{-s\mathrm{T}} \int_0^T f(t)e^{-s\mathrm{x}}dx$$

$$\Rightarrow \mathscr{L}\{f(t)\} = \int_0^T f(t)e^{-st}dt(1 + e^{-s\mathrm{T}} + e^{-2s\mathrm{T}} + \cdots)$$

$$= \int_0^T f(t)e^{-st}dt \frac{1}{1 - e^{-s\mathrm{T}}}$$

Laplace Transform

EX： $f(t), T=2$



$$L\{f(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 f(t)e^{-st}dt$$

$$= \frac{1}{1-e^{-2s}}\int_0^2 1e^{-st}dt$$

$$= \frac{1}{1-e^{-2s}}(-\frac{1}{s}e^{-st})\Big|_0^1$$

$$= \frac{1}{1-e^{-2s}}(-\frac{1}{s}e^{-s}-(-\frac{1}{s}))$$

$$= \frac{1}{1-e^{-2s}}\frac{1}{s}(1-e^{-s})$$

$$= \frac{1}{s(1+e^{-s})}$$

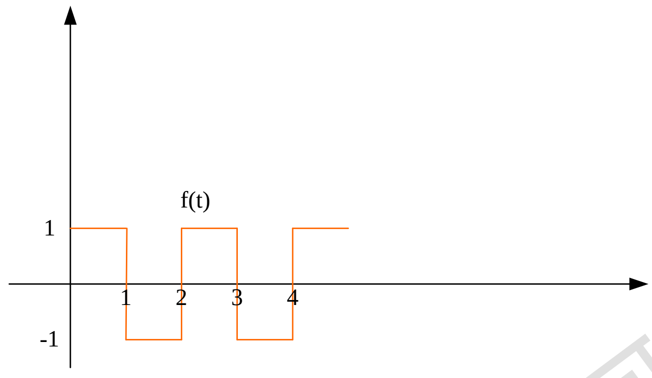
已知$\mathscr{L}^{-1}\left\{\frac{1}{s(1-e^{-s})}\right\} = g(t)$

求 $g(0)=1$

$$g(0) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\frac{1}{1-e^{-s}} = 1$$

Laplace Transform

EX： $T = 2$



$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 f(t)e^{-st}dt$$

$$= \frac{1}{1-e^{-2s}} [\int_0^1 e^{-st}dt - \int_1^2 e^{-st}dt]$$

$$= \frac{1}{1-e^{-2s}} [\frac{-1}{s}e^{-s} + \frac{1}{s} - (-\frac{1}{s}e^{-2s} + \frac{1}{s}e^{-s})$$

$$= \frac{1}{1-e^{-2s}} \frac{1}{s} [-2e^{-s} + 1 + e^{-2s}]$$

$$= \frac{1}{1-e^{-2s}} \frac{1}{s} (1-e^{-s})^2$$

$$= \frac{1-e^{-s}}{1+e^{-s}} \frac{1}{s}$$

Laplace Transform

- Inverse Laplace transform

設 $F(s)=\dfrac{N(s)}{D(s)}$

$Case\ I:$

$D(s)=(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)$

$\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n \in \Re$ ，相異($Distinsct$)實根

$$F(s)=\frac{N(s)}{(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)}=\frac{k_1}{s-\lambda_1}+\frac{k_2}{s-\lambda_2}+\cdots+\frac{k_n}{s-\lambda_n}$$

$$\mathscr{L}^{-1} \Rightarrow f(t)=k_1e^{\lambda_1 t}+k_2e^{\lambda_2 t}+\cdots+k_ne^{\lambda_n t}$$

其中 $k_1=\left.\dfrac{N(s)}{(s-\lambda_2)\cdots(s-\lambda_n)}\right|_{s=\lambda_1}$

Laplace Transform

EX:

$$F(s)=\frac{2s+1}{(s+1)(s+2)(s+3)}=\frac{a}{(s+1)}+\frac{b}{(s+2)}+\frac{c}{(s+3)}$$

$$a=\frac{-1}{2},b=\frac{-3}{-1}=3,c=\frac{-5}{(-2)(-1)}=\frac{-5}{2}$$

$$f(t)=-\frac{1}{2}e^{-t}+3e^{-2t}-\frac{5}{2}e^{-3t}$$

Laplace Transform

EX:

$$F(s)=\frac{2s+3}{(s-1)(s+5)(s-3)}=\frac{a}{(s-1)}+\frac{b}{(s+5)}+\frac{c}{(s-3)}$$

$$a=\frac{-5}{12},b=\frac{-7}{48},c=\frac{9}{16}$$

$$f(t)=\frac{-5}{12}e^{t}+\frac{-7}{48}e^{-5t}+\frac{9}{16}e^{3t}$$

Laplace Transform

Case II

$$F(s) = \frac{N(s)}{D_1(s)(s-\lambda)^n},\text{其中 } D_1(s)\text{為單根}$$

Laplace Transform

EX:

$$F(s)=\frac{s}{(s+1)(s-2)^2}=\frac{N_1(s)}{(s+1)}+\frac{N_2(s)}{(s-2)^2}=\frac{a}{s+1}+\frac{k_1}{(s-2)}+\frac{k_1}{(s-2)^2}$$

$$\Rightarrow a=F(s)(s+1)\Big|_{s=-1}=\frac{-1}{9}$$

$$k_1=\frac{d}{ds}\left(F(s)(s-2)^2\right)\Big|_{s=2}=\frac{d}{ds}\left(\frac{s}{s+1}\right)\Big|_{s=2}=\frac{1}{(s+1)^2}\Big|_{s=2}=\frac{1}{9}$$

$$k_2=F(s)(s-2)^2\Big|_{s=2}=\frac{2}{3}$$

$$f(t)=\frac{-1}{9}e^{-t}+\frac{1}{9}e^{2t}+\frac{2}{3}te^{2t}$$

Laplace Transform

Case II

$$F(s)=\frac{N(s)}{D_1(s)(s-\lambda)^n}=\frac{N_1(s)}{D_1(s)}+\frac{N_2(s)}{(s-\lambda)^n}$$

$$=\frac{N_1(s)}{D_1(s)}+\frac{k_1}{s-\lambda}+\frac{k_2}{(s-\lambda)^2}+\cdots+\frac{k_n}{(s-\lambda)^n}$$

$$\Rightarrow k_n=(s-\lambda)^n F(s)\Big|_{s=\lambda}=\frac{N(s)}{D_1(s)}\Big|_{s=\lambda}$$

$$\vdots$$

$$k_1=\frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}((s-\lambda)^n F(s))$$