Algorithm Hw1 Solution

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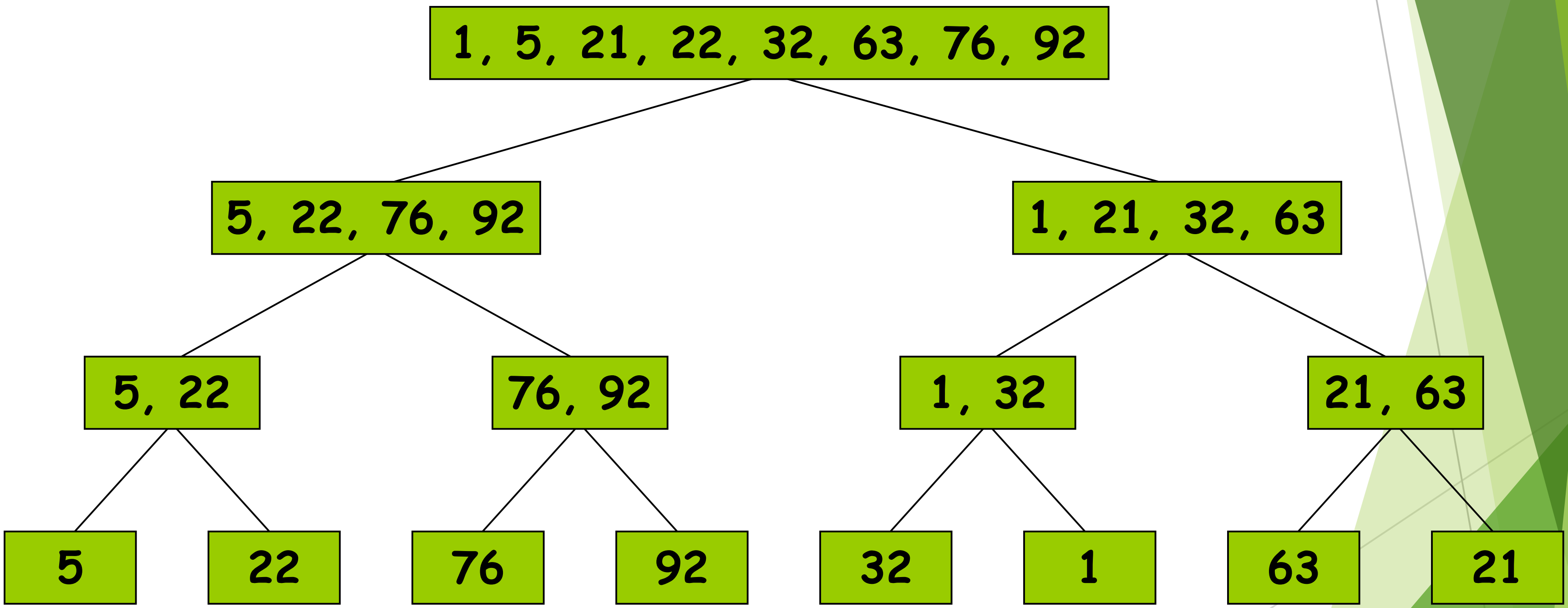
Question 1

解答:

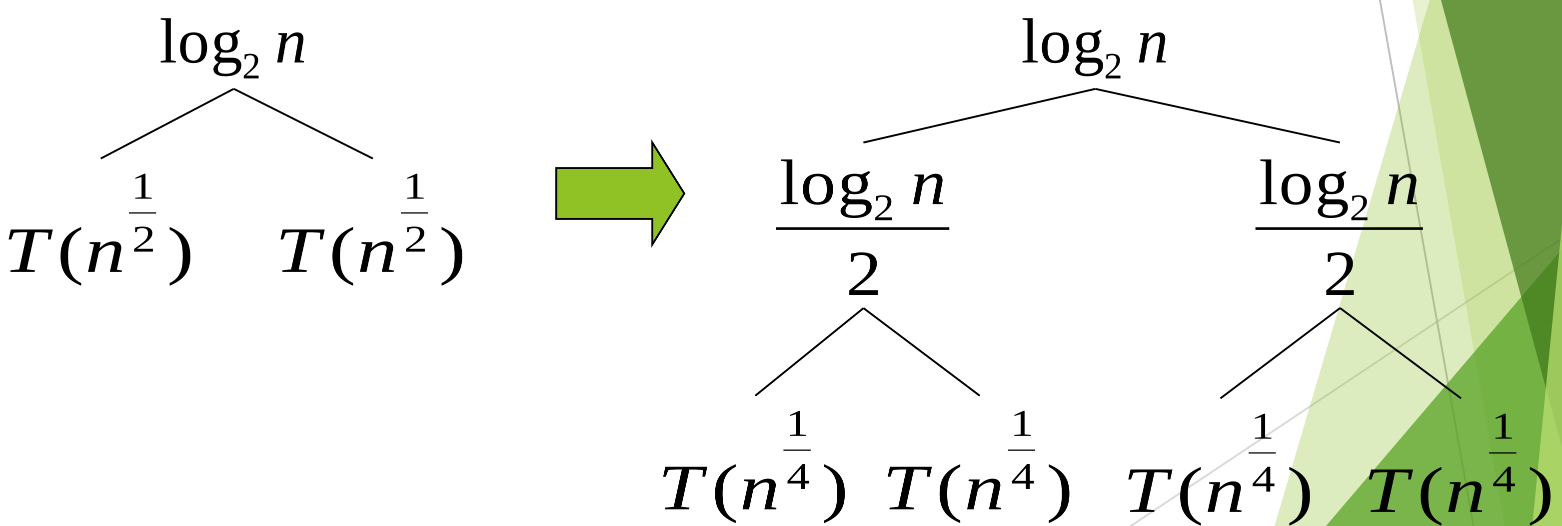


Question 2

解答:

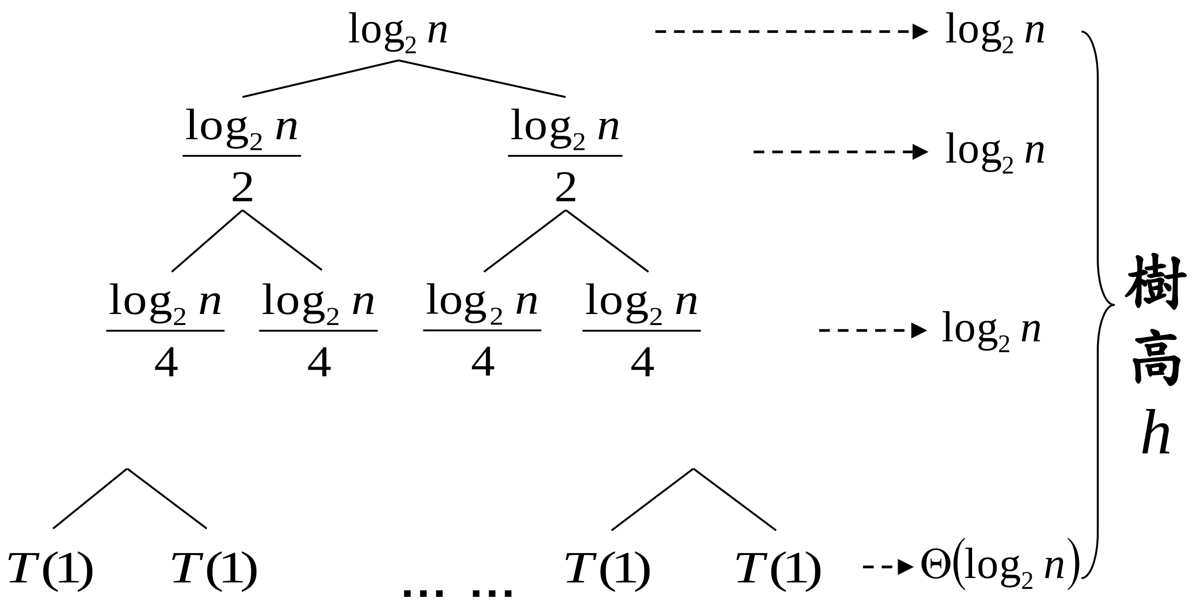
$$T(n) = 2T(\sqrt{n}) + \log_2 n = 2T(n^{\frac{1}{2}}) + \log_2 n$$

Recursion tree method



Question 2

解答:



$$\frac{\log_2 n}{2^h} = 1 \quad \Rightarrow \quad h = \log_2 \log_2 n$$

$$\begin{aligned}\text{Total cost} &= \log_2 n \cdot (h+1) = \log_2 n \cdot (\log_2 \log_2 n + 1) \\ &= \log_2 n \cdot \log_2 \log_2 n + \log_2 n \\ &= \Theta(\log_2 n \cdot \log_2 \log_2 n)\end{aligned}$$

Question 2

解答:

Substitution method

Guess $$T(n) \le d\log_2 n \cdot \log_2 \log_2 n$$

$$
\begin{aligned}
T(n) &= 2T(n^{\frac{1}{2}}) + \log_2 n \\
&\le 2d\log_2 n^{\frac{1}{2}} \cdot \log_2 \log_2 n^{\frac{1}{2}} + \log_2 n \\
&\le d\log_2 n \cdot \log_2\left(\frac{\log_2 n}{2}\right) + \log_2 n \\
&\le d\log_2 n \cdot \left(\log_2 \log_2 n - 1\right) + \log_2 n \\
&\le d\log_2 n \cdot \log_2 \log_2 n - d\log_2 n + \log_2 n \\
&\le d\log_2 n \cdot \log_2 \log_2 n \qquad for \quad d \ge 1
\end{aligned}
$$

$$\Rightarrow T(n) = O\left(\log_2 n \cdot \log_2 \log_2 n\right)$$

Question 2

解答:

Guess $T(n) \ge d\log_2 n \cdot \log_2 \log_2 n$

$$
\begin{aligned}
T(n) &= 2T(n^{\frac{1}{2}}) + \log_2 n \\
&\ge 2d\log_2 n^{\frac{1}{2}} \cdot \log_2 \log_2 n^{\frac{1}{2}} + \log_2 n \\
&\ge d\log_2 n \cdot \log_2\left(\frac{\log_2 n}{2}\right) + \log_2 n \\
&\ge d\log_2 n \cdot \left(\log_2 \log_2 n - 1\right) + \log_2 n \\
&\ge d\log_2 n \cdot \log_2 \log_2 n - d\log_2 n + \log_2 n \\
&\ge d\log_2 n \cdot \log_2 \log_2 n \qquad for \quad d = 1 > 0
\end{aligned}
$$

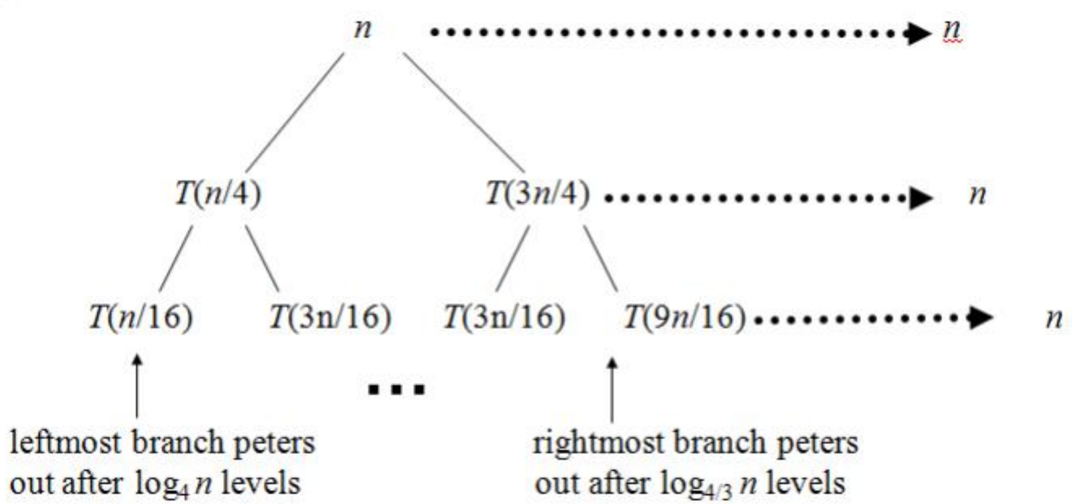
$$\Rightarrow T(n) = \Omega\left(\log_2 n \cdot \log_2 \log_2 n\right)$$

$$\Rightarrow T(n) = \Theta\left(\log_2 n \cdot \log_2 \log_2 n\right)$$

Question 3

解答:

- Given tight asymptotic bounds for $T\left(\frac{n}{4}\right) + T(\frac{3}{4}n) + n$



Question 3

解答:

- **Upper bound:**
  - Guess: $T(n) \leq dn \log n$
  - Substitution:
    - T(n) ≤ T(n/4) + T(3n/4) +n

      ≤ d (n/4) lg (n/4) + d(3n/4) lg (3n/4) + n

      = (d (n/4) lg n – d (n/4)lg4) + (d (3n/4) lgn – d (3n/4) lg (4/3)) + n

      = d n lg n – d ((n/4) lg4 + (3n/4) lg (4/3)) + n

      = d n lg n – d ((n/4) lg4 + (3n/4) lg4 – (3n/4) lg3) cn

      = d n lg n – d n (lg4 – 3/4lg3) + n

      = d n lg n +d n (3/4lg3-2) + n

      ≤ d n lg n

      if d n (3/4lg3-2) + n ≤ 0
  - $d \geq \frac{c}{2-\frac{3}{4 \lg 3}}$
  - Therefore , T(n) = O(n lg n)

Question 3

解答:

- **Lower bound:**
- Guess: $T(n) \geq dn \log n$
  - Substitution:
  - Same as for the upper bound, but replacing ≤ by ≥. End up needing
  - $0 \leq d \leq \frac{c}{2 - \frac{3}{4 \lg 3}}$
  - Therefore , T(n) = Ω(n lg n)
  - Since T(n) = O(n lg n) and T(n) = Ω(n lg n)
  - We conclude that T(n) = Θ(n lg n)

Question 4

解答:

- By Master Theorem:
- $T(n) = 9T\left(\frac{n}{3}\right) + n$
- $a = 9, b = 3, f(n) = n$
- $n^{log_b a} = n^{log_3 9} = n^2$
- 取$\varepsilon = 1$，則f(n) $= n = O(n^{2-1}) = O(n)$
- $T(n) = \theta(n^2)$

Question 5

解答:

- 5.1 $n^2, 2n^2$
- 5.2 $n^2, 2n^2$

Question 6

解答:

- 先算O:
  - $\log(n!) = \log(n) + \log(n-1) + \cdots + \log(1)$
  - $\leq \log(n) + \log(n) + \cdots + \log(n) = n\log(n)$
  - 所以$\log(n!) = O(n\log(n))$
- 再算Ω:
  - $\log(n!) = \log(n) + \log(n-1) + \cdots + \log\left(\frac{n}{2}\right) + \cdots + \log(1)$
  - $\geq \log\left(\frac{n}{2}\right) + \log\left(\frac{n}{2}\right) + \cdots + \log\left(\frac{n}{2}\right) = n\log\left(\frac{n}{2}\right)$
  - 所以$\log(n!) = \Omega(n\log(n))$
- 有上述可得 $\log(n!) = \theta(n\log(n))$

Question 7

解答:

- **a.** For the recurrence,
- $T(n) = 4T(n/2) + n$,
- we have $a = 4$, $b = 2$, $f(n) = n$, and thus $n^{\log_b a} = n^{\log_2 4} = Q(n^2)$.

  Since $f(n) = O(n^{\log_2 4 - e})$, where $e = 1$, we can apply case 1 of the master theorem and conclude that the solution is $T(n) = Q(n^2)$.

Question 7

解答:

- **b.** For the recurrence,
- $T(n) = 4T(n/2) + n^2$,
- we have $a = 4$, $b = 2$, $f(n) = n^2$, and thus $n^{\log_b a} = n^{\log_2 4} = Q(n^2)$.
  Since $f(n) = Q(n^{\log_2 4})$, we can apply case 2 of the master theorem and conclude that the solution is $T(n) = Q(n^2 \lg n)$.

Question 7

解答:

- **c.** For the recurrence,
- $T(n) = 4T(n/2) + n^3$,
- we have $a = 4$, $b = 2$, $f(n) = n^3$, and thus $n^{\log_b a} =$
  $n^{\log_2 4} = Q(n^2)$.
  Since $f(n) = W(n^{\log_2 4 + e})$, where $e = 1/2$, case 3 of the master theorem applies if we can show the regularity condition holds for $f(n)$.
  For all n,
- $af(n/b) = 4(n/2)^3 = (1/2)\ (n/2)^3$ £ $(2/3)\ n^3 = cf(n)$ for $c = 2$.
- Consequently, by case 3, the solution to the recurrence is
- $T(n) = Q(f(n)) = Q(n^3)$.

Question 8

解答:

From "asymptotically nonnegative", we can assume that

$$\exists n_1, n_2: f(n) \geq 0, for\ n > n_1$$
$$g(n) \geq 0, for\ n > n_2$$

Let $n_0 = \max(n_1, n_2)$. Some obvious things for $n > n_0$

$$f(n) \leq \max\big(f(n), g(n)\big)$$
$$g(n) \leq \max\big(f(n), g(n)\big)$$
$$\frac{f(n) + g(n)}{2} \leq \max(f(n), g(n))$$
$$\max\big(f(n), g(n)\big) \leq f(n) + g(n)$$

From the last two inequalities, we get:

$$0 < \frac{1}{2}\big(f(n) + g(n)\big) \leq \min\big(f(n), g(n)\big) \leq f(n) + g(n) for\ n > n_0$$

Which is the definition of $\Theta\big(f(n) + g(n)\big)$ with $c_1 = \frac{1}{2}$, $c_2 = 1$

Question 9

解答:

$$a^{\log_b c} = a^{\frac{\log_a c}{\log_a b}} = (a^{\log_a c})^{\frac{1}{\log_a b}} = c^{\log_b a}$$

Question 10

解答:

$$
\begin{aligned}
T(n) &= n^{\frac{1}{2}} T\left(n^{\frac{1}{2}}\right) + n \\
&= n^{\frac{1}{2}}(n^{\frac{1}{4}} T(n^{\frac{1}{4}})) + n^{\frac{1}{2}}) + n \\
&= n^{\frac{1}{2}+\frac{1}{4}} T\left(n^{\frac{1}{4}}\right) + n + n \\
&= n^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}} T\left(n^{\frac{1}{8}}\right) + n + n + n \\
&= \cdots \\
&= n^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^i}} T\left(n^{\frac{1}{2^i}}\right) + i * n \\
&= kn^{(2^i-1)/2^i} + in
\end{aligned}
$$

$$
\because \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^i} = \frac{1}{2} * \frac{1 - \frac{1}{2^i}}{\frac{1}{2}} = \frac{2^i - 1}{2^i}, T\left(n^{\frac{1}{2^i}}\right)
$$

$$
= T(m) = k
$$