ex. $u(x,y) = xy^2 + 3x + 5y = C$.

$\Rightarrow du(x,y) = \frac{\partial u}{\partial x}\cdot dx + \frac{\partial u}{\partial y}\cdot dy = 0$.

$\Rightarrow (y^2+3)dx + (2xy+5)dy = 0$. ⇒ 給定題目為此式.

如何解?

Sol: 令 $M(x,y) = y^2+3$, $N(x,y) = 2xy+5$.

先判斷是否正合 ⇒ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2y$.

又 $\frac{\partial u}{\partial x} = M(x,y)$. $\frac{\partial u}{\partial y} = 2xy+5$.

$\Rightarrow \partial u = (y^2+3)\partial x$ $\Rightarrow \partial u = (2xy+5)\partial y$.

$\Rightarrow \int \partial u = \int (y^2+3)dx + f_1(y)$. $\Rightarrow \int \partial u = \int (2xy+5)dy + f_2(x)$.

$\Rightarrow u = xy^2 + 3x + f_1(y)$. $\Rightarrow u = xy^2 + 5y + f_2(x)$.

此2式必相等 (∵正合).

$\Rightarrow f_1(y) = 5y$, $f_2(x) = 3x$

$\Rightarrow u(x,y) = xy^2 + 3x + 5y = C$.

ex. $(e^x y + 6x + 5y)dx + (e^x + 5x)dy = 0$.

$\Rightarrow \frac{\partial M}{\partial y} = e^x + 5$ 正合 $\frac{\partial N}{\partial x} = e^x + 5$

$\Rightarrow M = \frac{\partial u}{\partial x}$ $N = \frac{\partial u}{\partial y}$

$\Rightarrow \partial u = (e^x y + 6x + 5y)\partial x$ $\partial u = (e^x + 5x)\partial y$.

$\Rightarrow u = \int (e^x y + 6x + 5y)dx + f_1(y)$ $u = \int (e^x + 5x)dy + f_2(x)$.

$= e^x y + 3x^2 + 5xy + f_1(y)$ $= e^x y + 5xy + f_2(x)$.

$\Rightarrow f_1(y) = 0$, $f_2(x) = 3x^2$.

$\Rightarrow u = e^x y + 3x^2 + 5xy = C$ #.

ex. $(\cos y + 8x)dx + (-x\sin y + 3y^2)dy = 0$.

$\Rightarrow \frac{\partial M}{\partial y} = -\sin y$ 正合 $\frac{\partial N}{\partial x} = -\sin y$.

$\Rightarrow M = \frac{\partial u}{\partial x}$ $\qquad$ $N = \frac{\partial u}{\partial y}$

$\Rightarrow \partial u = M\,\partial x$ $\qquad$ $\partial u = N\,\partial y$

$\Rightarrow u = \int(\cos y + 8x)\,dx + f_1(y)$ $\qquad$ $u = \int(-x\sin y + 3y^2)\,dy + f_2(x)$.

$= x\cos y + 4x^2 + f_1(y)$ $\qquad$ $= x\cos y + y^3 + f_2(x)$.

$\Rightarrow f_1(y) = y^3$, $f_2(x) = 4x^2$

$\Rightarrow u = x\cos y + 4x^2 + y^3 = C$ #.

2.4 例1. 例2. 例3. 例4.

若 $u(x,y) = c$ —— (A).

$\vdots$

$M(x,y)dx + N(x,y)dy = 0$ —— (B).

有消去項，怎麼辦？

方法：解(B)，先將消去項歸还，使得(B)式成為正合

那，如何知道消去哪些項？

假設消去 $I(x,y)$.

$\Rightarrow \underbrace{I(x,y)\cdot M(x,y)}_{M_1(x,y)}dx + \underbrace{I(x,y)\cdot N(x,y)}_{N_1(x,y)}dy = 0$.

若 $\frac{\partial M_1(x,y)}{\partial y} = \frac{\partial N_1(x,y)}{\partial x}$，那称 $M_1dx + N_1dy = 0$ 為正合.

$\Rightarrow \frac{\partial}{\partial y}(I(x,y)\cdot M(x,y)) = \frac{\partial}{\partial x}(I(x,y)\cdot N(x,y))$

$\Rightarrow M(x,y)\cdot\frac{\partial I(x,y)}{\partial y} + I(x,y)\cdot\frac{\partial M(x,y)}{\partial y}$

$= N(x,y)\cdot\frac{\partial I(x,y)}{\partial x} + I(x,y)\cdot\frac{\partial N(x,y)}{\partial x}$

$\Rightarrow -N(x,y)\cdot\frac{\partial I(x,y)}{\partial x} + M(x,y)\cdot\frac{\partial I(x,y)}{\partial y} = I(x,y)\left[\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}\right]$

目的：解 $I(x,y)$ 所形成的一階 P.D.E.

$I(x,y)$ 称為積分因子(Integrating factor)

* 考慮一階 P.D.E.

$P(x,y,z)\frac{\partial z}{\partial x}+Q(x,y,z)\cdot\frac{\partial z}{\partial y}=R(x,y,z)$.

⇒ 由下列等式,決定出2個獨立解

$$\left(\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}\right)$$ 輔助方程組(Lagrange方程組)

$\Rightarrow u(x,y,z)=\alpha,\quad v(x,y,z)=\beta$

其通解可以是 $\phi(u,v)=0$(隱函數表示法)或是 $v=f(u)$(顯函數表示法).

用上述*的輔助方程組解前頁的式子.

(即 $-N\frac{\partial I}{\partial x}+M\frac{\partial I}{\partial y}=I\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$)

$$\Rightarrow \frac{dx}{-N}=\frac{dy}{M}=\frac{dI}{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}$$ 用此式求解I即可

(1) 猜I是x的函數(即 $I(x)$)

$$\Rightarrow \frac{dx}{-N}=\frac{dI}{I\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}\Rightarrow\left(\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{-N}\right)dx=\frac{dI}{I}$$

欲達到所求 ⇒ $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{-N}=f(x)$

$\Rightarrow f(x)dx=\frac{dI}{I}$

$\Rightarrow \int f(x)dx=\int\frac{dI}{I}=\ln I\quad\Rightarrow I=e^{\int f(x)dx}$.

(2) 猜I是y的函數(即 $I(y)$)

$$\Rightarrow \frac{dy}{M}=\frac{dI}{I\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}\Rightarrow\frac{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}{M}\cdot dy=\frac{dI}{I}$$

欲達到所求 ⇒ $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=g(y)$

$\Rightarrow g(y)dy=\frac{dI}{I}$

$\Rightarrow \int g(y)\,dy = \ln I \qquad \Rightarrow I = e^{\int g(y)\,dy}$

(3) 猜 I 是 $(x+y)$ 的函數 (即 $I(x+y)$)

$\left(\frac{d}{dx}(x+y) = 1 + \frac{dy}{dx} \Rightarrow d(x+y) = dx + dy \right)$

↳ 同乘 dx.

又合分比 $\left(\frac{1}{2} = \frac{2}{4} = \frac{1+2}{2+4}\right)$

$\Rightarrow \frac{dx + dy}{-N+M} = \frac{dI}{I\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}$

欲達到所求 $\Rightarrow \frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}{-N+M} = f(x+y)$

$\Rightarrow f(x+y)\,d(x+y) = \frac{dI}{I}$

$\Rightarrow \int f(x+y)\,d(x+y) = \ln I \qquad \Rightarrow I = e^{\int f(x+y)\,d(x+y)}$

(4). 猜 I 是 xy 的函數 (即 $I(xy)$)

$\left(\frac{d(xy)}{dx} = y + x\frac{dy}{dx} \Rightarrow d(xy) = y\,dx + x\,dy \right)$

↳ 同乘 dx.

$\Rightarrow \frac{y\,dx + x\,dy}{y\cdot(-N) + x\cdot M} = \frac{dI}{I\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}$

$\Rightarrow \frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) d(xy)}{y\cdot(-N) + x\cdot M} = \frac{dI}{I}$

欲達到所求 $\Rightarrow \frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}{y\cdot(-N)+x\cdot M} = f(xy)$

$\Rightarrow f(xy)\,d(xy) = \frac{dI}{I}$

$\Rightarrow \int f(xy)\,d(xy) = \ln I \Rightarrow I = e^{\int f(xy)\,d(xy)}$

summery : (1) ~ (4) 分子都是 $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

若發現 $\frac{\partial N}{\partial x} \neq \frac{\partial M}{\partial y}$

則檢查 $\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

$\Rightarrow$

除以 $-N \longrightarrow I(x)$

〃 $M \longrightarrow I(y)$

〃 $-N+M \longrightarrow I(x+y)$

〃 $-y\cdot N + x\cdot M \longrightarrow I(xy)$

ex. $\underbrace{(x^2+y^2+x)}_{M}dx + \underbrace{xy}_{N}dy = 0$.

$\Rightarrow \frac{\partial M}{\partial y} = 2y$ $\qquad \frac{\partial N}{\partial x} = y$.

不相等

$\Rightarrow$ 檢查 $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ (即 $y - 2y = -y$)

$\Rightarrow \frac{-y}{-N} = \frac{dI}{I} = \frac{1}{x} \Rightarrow I = e^{\int \frac{1}{x}dx} = x$.

$\Rightarrow$ 原式 $(x^3+xy^2+x^2)dx + x^2y\,dy = 0$

$\vdots$

$u = \frac{1}{3}x^3 + \frac{1}{2}x^2y^2 + \frac{1}{4}x^4 = C$ #.

ex. $(x^2+y^2+x)dx + xy\,dy = 0$. another sol.

$\Rightarrow (y^2dx + xy\,dy) + (x^2+x)dx = 0$

$\hookrightarrow y(y\,dx + x\,dy) = y\,d(xy)$

$\Rightarrow y\,d(xy) + (x^2+x)dx = 0$.

$\Rightarrow xy\,d(xy) + x(x^2+x)dx = x\cdot 0 = 0$.

$\Rightarrow \frac{1}{2}x^2y^2 + \frac{1}{4}x^4 + \frac{1}{3}x^3 = C$ #.

此法不甚妥 ∵不是每次都配得出來.

ex. $2\sin(y^2)dx + xy\cos(y^2)dy = 0$

$\Rightarrow \frac{\partial M}{\partial y} = 4y\cos(y^2)$ $\qquad \frac{\partial N}{\partial x} = y\cos(y^2)$

不相等

$\Rightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -3y\cos(y^2)$.

$\Rightarrow \left(\frac{-3y\cos(y^2)}{-N}\right)dx = \frac{3}{x}dx = \frac{dI}{I} \Rightarrow I = x^3$.

$\Rightarrow$ 原式: $2x^3\sin(y^2)dx + x^4y\cos(y^2)dy = 0$.

$\vdots$

$u = \frac{1}{2}x^4\sin(y^2) = C$ #

ex. $2\sin(x^2)dy + xy\cos(x^2)dx = 0$

$\Rightarrow \frac{\partial M}{\partial y} = x\cos(x^2)$ $\qquad \frac{\partial N}{\partial x} = 4x\cos(x^2)$

$\Rightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3x\cos(x^2)$.

$\Rightarrow \left(\frac{3x\cos(x^2)}{M}\right)dy = \frac{3}{y}dy = \frac{dI}{I} \Rightarrow I = y^3$.

$\Rightarrow$ 原式: $xy^4\cos(x^2)dx + 2y^3\sin(x^2)dy = 0$.

$\vdots$

$u = \frac{1}{2}y^4\sin(x^2) = C$ #.

ex. $(xy + y^2 + 1)dx + (xy + x^2 + 1)dy = 0$.

$\Rightarrow \frac{\partial M}{\partial y} = x + 2y$ $\qquad \frac{\partial N}{\partial x} = y + 2x$.

$\Rightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = x - y$

Φ. $\frac{x-y}{-N+M} = \frac{-1}{x+y}$

$\Rightarrow I = e^{\int \frac{-1}{x+y} d(x+y)} = \frac{1}{x+y}$

$$\Rightarrow \frac{1}{x+y}(xy+y^2+1)\,dx + \frac{1}{x+y}(xy+x^2+1)\,dy = 0$$

$$\vdots$$

$$u = xy + \ln(x+y) = C \quad \#.$$

②. $\dfrac{x-y}{-y\cdot N + x\cdot M} = \dfrac{x-y}{-y(xy+x^2+1) + x(xy+y^2+1)} = 1$

$\Rightarrow I = e^{\int 1\, d(xy)} = e^{xy}.$ ⇒ 積分因子可以不唯一

$$\Rightarrow \underbrace{e^{xy}(xy+y^2+1)}_{M = \frac{\partial u}{\partial x}}\,dx + e^{xy}(xy+x+1)\,dy = 0.$$

$$\Rightarrow u = \int e^{xy}(xy+y^2+1)\,dx + f_1(y).$$

$$= \int e^{xy}\cdot xy\cdot dx \text{ ⓐ} + \int e^{xy}\cdot y^2\,dx \text{ ⓑ} + \boxed{\int e^{xy}\cdot 1\cdot dx \text{ ⓒ}} + f_1(y).$$

$$= y\int e^{xy}\cdot x\,dx = y\int \underbrace{u}_{x}\,\underbrace{dv}_{e^{xy}dx} = y\left(uv - \int v\,du\right)$$

抵消.

$$= y\left(x\cdot\frac{1}{y}e^{xy} - \int \frac{1}{y}e^{xy}\,dx\right) = xe^{xy}\,\boxed{-\int e^{xy}\,dx}$$

$$\vdots$$

$$u = xe^{xy} + e^{xy} = C \quad \#.$$