Chapter 5.
Series Solutions of Linear Differential Equations

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Series Solutions

- 微分方程式的級數

$$P(x)y'' + q(x)y' + r(x)y = 0$$

在 $x = a$ 處的級數解是什麼?

定義：如果在 $x = a$, $y(x)$ 的任意階導數均存在

則在 $x = a$ 處, $y(x)$ 存在一Taylor級數解

可表成

$$y(x) = y(a) + y'(a)(x-a) + \frac{y''(a)}{2!}(x-a)^2 + \cdots + \frac{y^{(n)}(a)}{n!}(x-a)^n + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{y^{(n)}(a)}{n!}(x-a)^n$$

Series Solutions

把這些係數另外表成

$$\sum_{n=0}^{\infty} a_n (x-a)^n, \text{其中} a_n = \frac{y^{(n)}(a)}{n!}$$

把原本決定y(x)在x=a的任意階導數 $\dfrac{y^{(n)}(a)}{n!}$
改為決定 $(x-a)^n$ 的係數 a_n

Series Solutions

EX: $y' + 2y = 0, x = 0$的級數解

$$\Rightarrow \mathrm{y(x)} = \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow \mathrm{y'(x)} = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

帶入原式 $y' + 2y = 0$

$$\Rightarrow \sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n$$

想法：令 $k = n - 1, n = k + 1$

$$\Rightarrow \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k \quad 令 k = n$$

Series Solutions

答案： $\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_n x^n$$

$$=\sum_{n=0}^{\infty}[(n+1)a_{n+1}+2a_n]x^n=0$$

$$\Rightarrow (n+1)a_{n+1}+2a_n=0$$

$$a_{n+1}=\frac{-2}{n+1}a_n(n\geq 0)$$

$$n=0\Rightarrow a_1=\frac{-2}{1}a_0$$

$$n=1\Rightarrow a_2=\frac{-2}{2}a_1=\frac{-2}{2}\frac{-2}{1}a_0=\frac{(-2)^2}{2!}a_0$$

$$n=2\Rightarrow a_3=\frac{-2}{3}a_2=\frac{-2}{3}\frac{-2}{2}a_1=\frac{-2}{3}\frac{-2}{2}\frac{-2}{1}a_0=\frac{(-2)^3}{3!}a_0$$

$$\vdots$$

$$a_n=\frac{(-2)^n}{n!}a_0$$

Series Solutions

$$\Rightarrow y(x) = a_0 + \frac{(-2)}{1!}a_0 x + \frac{(-2)^2}{2!}a_0 x^2 + \cdots + \frac{(-2)^n}{n!}a_0 x^n + \cdots$$

$$= a_0\left(1 + \frac{(-2)}{1!}x + \frac{(-2)^2}{2!}x^2 + \cdots + \frac{(-2)^n}{n!}x^n + \cdots\right)$$

$$= a_0 e^{-2x}$$

$$\begin{cases} e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots \\ \cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \\ \sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \end{cases}$$

Series Solutions

- $P(x)y'' + q(x)y' + r(x)y = 0\ldots\ldots(1)$

若在 $x = a$ ，y存在Taylor級數解(ie 找的到任意階導數)
則稱 $x = a$ ，為方程式(1)的常點(Ordinary Point)
否則稱 $x = a$ 為方程式(1)的奇異點(Singular Point)

Series Solutions

EX:$(x-1)y'+2y=0$請問 $x=1$ 是常點或奇異點

假設$y(1)=C$ ，若$y'(1), y''(1),\ldots, y^{(n)}(1)$ 找的到就是常點

$$y'(x)=-\frac{2}{x-1}y(x)$$

$$y'(1)=\frac{2}{1-1}y(1)=\infty$$

EX: $(x-1)^2y''+2xy'+3y=0, y(1)=a, y'(1)=b, x=1$是什麼點?

$\because y''(1)=\infty$

$\therefore x=1$ singular point

Series Solutions

- 歸納

$$y''+\frac{q(x)}{p(x)}y'+\frac{r(x)}{p(x)}y=0$$

$p(a)=0 \Rightarrow x=a$ is singular point

令$\mathrm{p(x)=0}$ 的點, 皆為異點

Series Solutions

EX: $(x+1)(x-2)y''+3xy'+4y=0$

異點何在?

$x=-1,2$

EX：$(x+1)(x-2)y''+3(x-2)y'+4(x-2)y=0$

異點何在?

$x=-1$

Series Solutions

- Note: p(x).q(x).r(x)沒有公因式

設 $y(x)=\sum_{n=0}^{\infty}\frac{y^{(n)}(a)}{n!}(x-a)^{n}$

讓上述y(x)級數解存在的收斂區間為$|x-a|<L$ (L:收斂半徑)
L:由x=a處到最近異點的距離

Series Solutions

EX: $(x-1)y' + 2y = 1$

$x = 1$異點

$x = 0$常點

$$\therefore y(n) = \sum_{n=0}^{\infty} a_n x^n, |x-0| < L = 1$$

EX:

$y'' + y = 0$

$x = 0$常點

$$\therefore y(n) = \sum_{n=0}^{\infty} a_n x^n, |x-0| < L = \infty$$

$$y'(n) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Series Solutions

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$y'' + y = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + a_n] x^n = 0$$

$$\Rightarrow a_{n+2} = \frac{-a_n}{(n+2)(n+1)}$$ (循環公式.recurrence formula)

$$a_n = ?$$

Series Solutions

$\because$ 二階微分方程式本來就有兩個未知數

$$a_{2n} = \frac{(-1)^n}{(2n)!} a_0$$

$$a_{2n+1} = \frac{(-1)^n}{(2n+1)!} a_1$$

$$\therefore y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} a_0 x^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} a_{2n+1} x^{2n+1}$$

$$= a_0 \cos x + a_1 \sin x$$

Series Solutions

EX: y'+2y=1,x=0 的級數解

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, |x| < \infty$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\Rightarrow \sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 1$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 2\sum_{n=0}^{\infty} a_n x^n = 1$$

$$\Rightarrow \sum_{n=0}^{\infty} [(n+1) a_{n+1} + 2a_n] x^n = 1$$

Series Solutions

$$\begin{cases} n=0, a_1+2a_0=1 \\ n\geq 1, (n+1)a_{n+1}+2a_n=0 \end{cases}$$

$$\Rightarrow a_{n+1}=\frac{-2}{n+1}a_n$$

$$n=1, a_2=\frac{-2}{2}a_1$$

$$n=2, a_3=\frac{-2}{3}a_2=\frac{(-2)(-2)}{3\cdot 2}a_1$$

$$n=3, a_4=\frac{-2}{4}a_3=\frac{(-2)(-2)(-2)}{4\cdot 3\cdot 2}a_1$$

$$\vdots$$

$$a_n=\frac{(-2)^{n-1}}{n!}a_1$$

$$\begin{aligned} \mathrm{y(x)} &= \mathrm{a}_0+a_1x+a_2x^2+\ldots+a_nx^n+\cdots \\ &= \frac{1}{2}-\frac{1}{2}a_1+a_1x+\frac{-2}{2}a_1x^2+\cdots+\frac{(-2)^{n-1}}{n!}a_1x^n+\cdots \\ &= \frac{1}{2}-\frac{1}{2}a_1(1-2x+\frac{(-2)^2}{2!}x^2+\cdots+\frac{(-2)^n}{n!}x^n+\cdots) \\ &= \frac{1}{2}-\frac{1}{2}a_1e^{-2x} \end{aligned}$$

Series Solutions

EX: $y'+2y=x+1, x=1$的級數解

$\because x=1$是常點

$\therefore$存在Tayler級數解

$$y(x)=\sum_{n=0}^{\infty}a_n(x-1)^n, |x-1|<L=\infty$$

$$y'(x)=\sum_{n=1}^{\infty}na_n(x-1)^{n-1}$$

$$\Rightarrow \sum_{n=1}^{\infty}na_n(x-1)^{n-1}+2\sum_{n=0}^{\infty}a_n(x-1)^n=x+1$$

$$\Rightarrow \sum_{n=0}^{\infty}[(n+1)a_{n+1}+2a_n](x-1)^n=(x-1)+2$$

Series Solutions

$$\Rightarrow \begin{cases} n=0, a_1+2a_0=2 \\ n=1, 2a_2+2a_1=1 \\ n\geq 2, (n+1)a_{n+1}+2a_n=0 \end{cases}$$

$$\Rightarrow \begin{cases} a_0=\dfrac{2-a_1}{2}=1-\dfrac{1}{4}+\dfrac{a_2}{2}=\dfrac{3}{4}+\dfrac{a_2}{2} \\ a_1=\dfrac{1-2a_2}{2}=\dfrac{1}{2}-a_2 \end{cases}$$

$$n=2, a_3=\frac{-2}{3}a_2$$

$$n=3, a_4=\frac{-2}{4}a_3=\frac{(-2)(-2)}{4\cdot 3}a_2$$

$$n=4, a_5=\frac{-2}{5}a_4=\frac{(-2)(-2)(-2)}{5\cdot 4\cdot 3}a_2$$

$$\vdots$$

$$a_n=\frac{(-2)(-2)\cdots(-2)\cdot 2}{n(n-1)(n-2)\cdots 3\cdot 2}a_2=\frac{2(-2)^{n-2}}{n!}a_2$$

Series Solutions

$$\Rightarrow y(x) = \mathrm{a}_0 + \mathrm{a}_1(x-1) + \mathrm{a}_2(x-1)^2 + \cdots + \mathrm{a}_\mathrm{n}(x-1)^n + \cdots$$

$$= \frac{3}{4} + \frac{a_2}{2} + (\frac{1}{2} - a_2)(x-1) + \mathrm{a}_2(x-1)^2 + \cdots + \frac{2(-2)^{n-2}}{n!}\mathrm{a}_2(x-1)^n + \cdots$$

$$= \frac{3}{4} + \frac{1}{2}(x-1) + \frac{a_2}{2}[1 - 2(x-1) + 2(x-1)^2 + \frac{(-2)^3}{3!}(x-1)^3 + \cdots + \frac{(-2)^n}{n!}(x-1)^n + \cdots]$$

$$= \frac{3}{4} + \frac{1}{2}(x-1) + \frac{a_2}{2}e^{-2(x-1)}$$

Series Solutions

$verify$

$y^1 + 2y = x + 1 = (x-1) + 2$

令$t = x - 1, \dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt} = \dfrac{dy}{dx} \cdot 1$

$$\frac{dy}{dt} + 2y = t + 2$$

$$\Rightarrow y_h = c_1 e^{-2t}$$

$$y_p = \frac{1}{D+2}(t+2) = \frac{1}{2}\frac{1}{1+\frac{D}{2}}(t+2) = \frac{1}{2}(1 - \frac{D}{2} + \frac{D^2}{4} + \cdots)(t+2)$$

$$= \frac{1}{2}(t + 2 - \frac{1}{2}) = \frac{t}{2} + \frac{3}{4}$$

$$y = y_n + y_p$$

$$\Rightarrow y = c_1 e^{-2t} + \frac{t}{2} + \frac{3}{4}$$