Engineering Mathematics

Midterm Exam, Fall 2013/11/18

請詳細列出計算過程，如用到公式, 請列出公式的通式。請記得在答案卷上簽名。

1. (20%) Determine whether the given differential equation is exact, If it is exact, solve it, if not, explain why)

$$(1).(2x+y)dx-(x+6y)dy=0$$

$$(2).(\sin y - y\sin x)dx+(\cos x + x\cos y - y)dy=0$$

$$(3).(2y-\frac{1}{x}+\cos 3x)\frac{dy}{dx}+\frac{y}{x^2}-4x^3+3y\sin x=0$$

$$(4).\ 3x^2ydx+(x^3-5)dy=0$$

Sol:

$(1).\ M=2x+y, N=-x-6y$

$M_y=1, N_x=-1$

$not\ exact$

$(2).M=\sin y - y\sin x,\ N=\cos x + x\cos y - y$

$M_y=\cos y-\sin x=N_X$

$solution\ =\ x\sin y + y\cos x - \frac{1}{2}y^2=c$

$(3).M=-4x^3+3y\sin 3x+\frac{y}{x^2}, N=2y-\frac{1}{x}+\cos 3x$

$\mathrm{M}_y=3\sin 3x+\frac{1}{x^2}, N_x=\frac{1}{x^2}-3\sin 3x$

not exact

$(4).M=3x^2y, N=x^3-5$

$\frac{\partial M}{\partial y}=3x^2=\frac{\partial N}{\partial y}$ $\Rightarrow$ 正合

$\frac{\partial u(x,y)}{\partial x}=3x^2y$ $\int\partial u(x,y)=\int 3x^2ydx+f(y)$

$\frac{\partial u(x,y)}{\partial y}=x^3-5$ $\int\partial u(x,y)=\int(x^3-5)dy+g(x)$

$$u=\begin{cases} x^3y+f(y)\ \cdots\cdots\cdots\cdots(1) \\ x^3y-5y+g(x)\cdots\cdots\cdots(2)\end{cases}$$

$f(y)=-5y$

$g(x)=0$

$u(x,y)=x^3y-5y=C$

2. (14%) Classify each differential equation as separable, exact, linear, homogeneous, or Bernoulli. Some equations may be more than one kind

$$(a)\quad \frac{dy}{dx}=\frac{x-y}{x}$$

$$(b)\quad (x+1)\frac{dy}{dx}=-y+10$$

$$(c)\quad \frac{dy}{dx}=\frac{y^2+y}{x^2+x}$$

$$(d)\quad ydx=(y-xy^2)dy$$

$$(e)\quad xyy'+y^2=2x$$

$$(f)\quad \frac{dy}{dx}=\frac{x}{y}+\frac{y}{x}+1$$

$$(g)\quad \frac{y}{x^2}\frac{dy}{dx}+e^{2x^3+y^2}=0$$

Sol:

(a) Linear. Homogeneous, exact
(b) Separable, exact, linear
(c) Separable
(d) Linear
(e) Bernoulli
(f) Homogeneous
(g) Separable

3. (10%) Solve the given differential equation by undetermined coefficients

$$y''+4y=6\sin(x)\cos(x)$$

Sol:

Homogeneous solution:

Characteristic function: $m^2+4=0\rightarrow m=\pm 2i$

$\Rightarrow y_h=c_1\cos(2x)+c_2\sin(2x)$

Particular solution:

$y''+4y=6\sin(x)\cos(x)\Rightarrow y''+4y=3\sin(2x)$

Let $y_p=ax\sin(2x)+bx\cos(2x)$, then y_p substitute to $y''+4y=3\sin(2x)$

$\Rightarrow y_p''+4y_p=3\sin(2x)$

$\rightarrow 4a\cos(2x)-4ax\sin(2x)-4b\sin(2x)+4bx\cos(2x)+4ax\sin(2x)+4bx\cos(2x)=3\sin(2x)$

$\rightarrow 4a\cos(2x)-4b\sin(2x)=3\sin(2x)\rightarrow a=0,\ b=\dfrac{-3}{4}\rightarrow y_p=\dfrac{-3}{4}x\cos(2x)$

By above $y=y_h+y_p=c_1\cos(2x)+c_2\sin(2x)-\dfrac{3}{4}x\cos(2x)$

4. (10%) Solve the given differential equation by variation of parameters. $4y'' - 4y' + y = e^{\frac{x}{2}}\sqrt{1-x^2}$

Sol:

Characteristic function: $4\lambda^2 - 4\lambda + 1 = (2\lambda - 1)^2 = 0 \Rightarrow \lambda = \frac{1}{2} \quad \therefore y_h = c_1 e^{\frac{x}{2}} + c_2 x e^{\frac{x}{2}}$

Particular solution:

Let $y_p = y_{p_1} + y_{p_2}$, and identify $4y'' - 4y' + y = e^{\frac{x}{2}}\sqrt{1-x^2} \Rightarrow y'' - y' + \frac{1}{4}y = \frac{1}{4}e^{\frac{x}{2}}\sqrt{1-x^2}$

$$w = \begin{vmatrix} e^{\frac{x}{2}} & xe^{\frac{x}{2}} \\ \frac{1}{2}e^{\frac{x}{2}} & \frac{1}{2}xe^{\frac{x}{2}} + e^{\frac{x}{2}} \end{vmatrix} = e^x \quad \mathrm{w}_1 = \begin{vmatrix} 0 & xe^{\frac{x}{2}} \\ \frac{1}{4}e^{\frac{x}{2}}\sqrt{1-x^2} & \frac{1}{2}xe^{\frac{x}{2}} + e^{\frac{x}{2}} \end{vmatrix} = -xe^{\frac{x}{2}} \cdot \frac{1}{4}e^{\frac{x}{2}}\sqrt{1-x^2}$$

$$w_2 = \begin{vmatrix} e^{\frac{x}{2}} & 0 \\ \frac{1}{2}e^{\frac{x}{2}} & \frac{1}{4}e^{\frac{x}{2}}\sqrt{1-x^2} \end{vmatrix} = e^{\frac{x}{2}}\frac{1}{4}e^{\frac{x}{2}}\sqrt{1-x^2}$$

$$\Rightarrow u_1' = \frac{-xe^{\frac{x}{2}} \cdot \frac{1}{4}e^{\frac{x}{2}}\sqrt{1-x^2}}{e^x} = -\frac{1}{4}x\sqrt{1-x^2} \qquad u_2' = \frac{e^{\frac{x}{2}}\frac{1}{4}e^{\frac{x}{2}}\sqrt{1-x^2}}{e^x} = \frac{1}{4}\sqrt{1-x^2}$$

$$\Rightarrow u_1 = \int -\frac{1}{4}x\sqrt{1-x^2}\,dx = \frac{1}{12}(1-x^2)^{\frac{3}{2}} \qquad u_2 = \int \frac{1}{4}\sqrt{1-x^2}\,dx = \frac{1}{8}(x\sqrt{1-x^2} + \sin^{-1}x)$$

$$\Rightarrow y_{p_1} = e^{\frac{x}{2}}\frac{1}{12}(1-x^2)^{\frac{3}{2}}, \quad y_{p_2} = xe^{\frac{x}{2}}\frac{1}{8}(x\sqrt{1-x^2} + \sin^{-1}x)$$

$$\therefore y = y_h + y_p = c_1 e^{\frac{x}{2}} + c_2 x e^{\frac{x}{2}} + \frac{1}{12}e^{\frac{x}{2}}(1-x^2)^{\frac{3}{2}} + \frac{1}{8}xe^{\frac{x}{2}}(x\sqrt{1-x^2} + \sin^{-1}x)$$

5. (10%) Use the substitution $x = e^t$ to transform the given Cauchy-Euler equation to a differential equation with constant coefficients and solve it. $x^3 y''' - 3x^2 y'' + 6xy' - 6y = 3 + \ln x^3$

Sol:

Let $x = e^t$, $\wp \equiv \frac{d}{dt}$

Homogenous solution:

$$\wp(\wp-1)(\wp-2)y - 3\wp(\wp-1)y + 6\wp y - 6y = 0 \Rightarrow (\wp^3 - 6\wp^2 + 11\wp - 6)y = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \Rightarrow (\lambda-1)(\lambda-2)(\lambda-3) = 0 \Rightarrow \lambda = 1, 2, 3$$

$$\therefore y_h = c_1 e^t + c_2 e^{2t} + c_3 e^{3t} = c_1 x + c_2 x^2 + c_3 x^3$$

Particular solution:

$$(\wp^3 - 6\wp^2 + 11\wp - 6)y_p = 3 + \ln e^{3t} = 3 + 3t \Rightarrow y_p = \frac{1}{\wp^3 - 6\wp^2 + 11\wp - 6}(3+3t)$$

$$= \frac{1}{-6[1 - (\frac{\wp^3}{6} - \wp^2 + \frac{11}{6}\wp)]}(3+3t) = \frac{-1}{6}[1 + (\frac{\wp^3}{6} - \wp^2 + \frac{11}{6}\wp) + \cdots](3+3t)$$

$$= \frac{-1}{6}[(3+3t) + \frac{11}{2}] = \frac{-17}{12} - \frac{1}{2}t = \frac{-17}{12} - \frac{1}{2}\ln x$$

By above $y = c_1 x + c_2 x^2 + c_3 x^3 - \frac{17}{12} - \frac{1}{2}\ln x$

6. (10%) Find the **particular solution** of the given high-order differential equation.

$$2\frac{d^5x}{ds^5}-7\frac{d^4x}{ds^4}+12\frac{d^3x}{ds^3}+8\frac{d^2x}{ds^2}=e^s(1+e^s(1+e^s(1+e^s)))$$

Sol:

Particular solution:

$$(2m^5-7m^4+12m^3+8m^2)y_p=e^s+e^{2s}+e^{3s}+e^{4s}$$

$$y_p=\frac{e^s+e^{2s}+e^{3s}+e^{4s}}{2m^5-7m^4+12m^3+8m^2}=\frac{e^s}{15}+\frac{e^{2s}}{80}+\frac{e^{3s}}{315}+\frac{e^{4s}}{1152}$$

7. (6%) Solve the given differential equation.

$$(2x-5y+3)dx-(2x+4y-6)dy=0$$

Sol:

$2x-5y+3=0$, 2x+4y-6=0

Intersect at x=1, y=1 $\Rightarrow 2(x-1)-5(y-1)=0$, $2(x-1)+4(y-1)=0$

Let $t=x-1,\ s=y-1 \Rightarrow dt=dx,\ ds=dy$

$\Rightarrow (2t-5s)dt-(2t+4s)ds=0$

Let $z=\dfrac{t}{s}$, then $sdz+zds=dt$

$\rightarrow (2z-5)(sdz+zds)-(2z+4)ds=0 \rightarrow (2z-5)sdz+[(2z-5)z-(2z+4)]ds=0$

$\rightarrow (2z-5)sdz+(2z^2-7z-4)ds=0 \rightarrow \dfrac{(2z-5)}{(2z^2-7z-4)}dz+\dfrac{1}{s}ds=0$

$\rightarrow \left(\dfrac{\frac{4}{3}}{2z+1}+\dfrac{\frac{1}{3}}{z-4}\right)dz+\dfrac{1}{s}ds=0$

$\Rightarrow \dfrac{2}{3}\ln|2z+1|+\dfrac{1}{3}\ln|z-4|+\ln|s|=c \Rightarrow (2z+1)^{\frac{2}{3}}(z-4)^{\frac{1}{3}}s=c'$

$\because z=\dfrac{t}{s}\ \ \therefore (2z+1)^{\frac{2}{3}}(z-4)^{\frac{1}{3}}s=c' \Rightarrow (2\dfrac{t}{s}+1)^{\frac{2}{3}}(\dfrac{t}{s}-4)^{\frac{1}{3}}s=c'$

$\because t=x-1\ ,\ s=y-1\ \ \therefore (2\dfrac{t}{s}+1)^{\frac{2}{3}}(\dfrac{t}{s}-4)^{\frac{1}{3}}s=c' \Rightarrow (2\dfrac{x-1}{y-1}+1)^{\frac{2}{3}}(\dfrac{x-1}{y-1}-4)^{\frac{1}{3}}(y-1)=c'$

8. (10%) Solve the given differential equation.

$$y^{(4)}+10y^{(2)}+9y=\cos(2x+3)+\sin(3x+1)$$

Sol:

$(D^4+10D^2+9)y_p=\cos(2x+3)+\sin(3x+1)$

Let $y_p=y_{p_1}+y_{p_2}$

$y_{p_1}=\dfrac{\cos(2x+3)}{D^4+10D^2+9}=\dfrac{\cos(2x+3)}{(D^2+1)(D^2+9)}=\dfrac{\cos(2x+3)}{(-2^2+1)(-2^2+9)}=\dfrac{\cos(2x+3)}{(-3)(5)}=\dfrac{-1}{15}\cos(2x+3)$

Let $y_{p_2}=A\cos(3x+1)+B\sin(3x+1)+Cx\cos(3x+1)+Dx\sin(3x+1)$

$y_{p_2}^{'}=-3A\sin(3x+1)+3B\cos(3x+1)+(D-3Cx)\sin(3x+1)+(C+3Dx)\cos(3x+1)$

$y_{p_2}^{''}=-9A\cos(3x+1)-9B\sin(3x+1)-(6C+9Dx)\sin(3x+1)+(6D-9Cx)\cos(3x+1)$

$y_{p_2}^{'''}=27A\sin(3x+1)-27B\cos(3x+1)-27[(D-Cx)\sin(3x+1)+(C+Dx)\cos(3x+1)]$

$y_{p_2}^{(4)}=81A\cos(3x+1)+81B\sin(3x+1)+27[(4C+3Dx)(\sin(3x+1)+(3Cx-4D)\cos(3x+1)]$

$y_{p_2}^{(4)}+10y_{p_2}^{''}+9y_{p_2}=\sin(3x+1)\Rightarrow$

$[81A\cos(3x+1)+81B\sin(3x+1)]+10[-9A\cos(3x+1)-9B\sin(3x+1)]+9[A\cos(3x+1)+B\sin(3x+1)]=0$

$[108C+10(-6C)+9*0]\sin(3x+1)=\sin(3x+1)\rightarrow C=\dfrac{1}{48}$

$[-108D+10(6D)+9*0]\cos(3x+1)=0\rightarrow D=0$

$\Rightarrow y_{p_2}^{''}=\dfrac{1}{48}x\cos(3x+1)$

另解:

$(D^4+10D^2+9)y_{p_2}=\sin(3x+1)$

$y_{p_2}=\dfrac{1}{(\mathrm{D}^2+1)(\mathrm{D}^2+9)}\sin(3x+1)=\dfrac{1}{8}[\dfrac{1}{(D^2+1)}-\dfrac{1}{(D^2+9)}]\sin(3x+1)$

$\dfrac{1}{(D^2+1)}\sin(3x+1)$ 此特解會包含在$\mathrm{c}_3\cos 3\mathrm{x} + \mathrm{c}_4\sin 3\mathrm{x}$

所以y_{p_2}只計算$\dfrac{1}{8}\dfrac{-1}{(D^2+9)}\sin(3x+1)\ \Rightarrow\ \dfrac{1}{8}[\lim\limits_{\Delta\to 0}\dfrac{-1}{-(3+\Delta)^2+9}\sin((3+\Delta)x+1)]$

$=\dfrac{1}{8}[\lim\limits_{\Delta\to 0}\dfrac{-1}{-6\Delta-\Delta^2}\sin((3+\Delta)x+1)]=\dfrac{1}{8}[\lim\limits_{\Delta\to 0}\dfrac{-1}{-6\Delta-\Delta^2}[\sin(3x+1)+\Delta x\cos(3x+1)-\dfrac{1}{2!}(\Delta x)^2\sin(3x+1)+\cdots]$

$=\dfrac{1}{8}[\lim\limits_{\Delta\to 0}\dfrac{-1}{-6-\Delta}[x\cos(3x+1)-\dfrac{1}{2!}(\Delta x^2)\sin(3x+1)+\cdots]=\dfrac{1}{48}x\cos(3x+1)$

characteristic function: $\mathrm{m}^4+10m^2+9=0\Rightarrow m=\pm i,\ \pm 3\mathrm{i}$

$\Rightarrow y_h=\mathrm{c}_1\cos\mathrm{x} + \mathrm{c}_2\sin\mathrm{x} + \mathrm{c}_3\cos 3\mathrm{x} + \mathrm{c}_4\sin 3\mathrm{x}$

$\therefore$ the general solution :

$y=y_h+y_p=\mathrm{c}_1\cos\mathrm{x} + \mathrm{c}_2\sin\mathrm{x} + \mathrm{c}_3\cos 3\mathrm{x} + \mathrm{c}_4\sin 3\mathrm{x}-\dfrac{1}{15}\cos(2x+3)+\dfrac{1}{48}x\cos(3x+1)$

9. (10%) Solve $(4y^2+3xy)dx-(3xy+2x^2)dy=0$

(i) Find the integrating factor equation

(ii) Find the solution of the given differential

(Hint) $if \ \dfrac{\left(\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}\right)}{\dfrac{(-ayN+bxM)}{xy}}=1 \Longrightarrow I = x^a y^b$

Sol:

(i)

Let $M=4y^2+3xy$, $N=-(3xy+2x^2)$, then $\dfrac{\partial M}{\partial y}=8y+3x\neq -3y-4x=\dfrac{\partial N}{\partial x}$ (not exact)

Guess the integrating factor: $x^a y^b \Rightarrow M'=x^a y^b(4y^2+3xy)$, $N'=-x^a y^b(3xy+2x^2)$

For the new equation to be exact, we need $\dfrac{\partial M'}{\partial y}=\dfrac{\partial N'}{\partial x}$

So $\dfrac{\partial M'}{\partial y}=4(2+b)x^a y^{1+b}+3(b+1)x^{a+1}y^b$, $\dfrac{\partial N'}{\partial x}=-3(1+a)x^a y^{b+1}-2(2+a)x^{1+a}y^b$

$\because \dfrac{\partial M'}{\partial y}=\dfrac{\partial N'}{\partial x}$ $\therefore a=-5,\ b=1 \Rightarrow$ The integrating factor : $x^{-5}y$

(ii)

By (i): $M'=f_x=4x^{-5}y^3+3x^{-4}y^2 \Rightarrow f=-y^3x^{-4}-y^2x^{-3}+h(y)$

$N'=f_y=-3x^{-4}y^2-2x^{-3}y \Rightarrow f=-y^3x^{-4}-y^2x^{-3}+g(x)$

A solution of the differential equation is $\dfrac{y^3}{x^4}+\dfrac{y^2}{x^3}=c$

Reference: Differentiation Table

$$\frac{d}{dx}u^n=nu^{n-1}\frac{du}{dx} \qquad \frac{d}{dx}a^u=(\ln a)a^u\frac{du}{dx} \qquad \frac{d}{dx}\mathrm{e}^u=\frac{du}{dx}$$

$$\frac{d}{dx}\log_a \mathrm{u}=\frac{1}{(\ln a)u}\frac{du}{dx} \qquad \frac{d}{dx}\ln(\mathrm{u})=\frac{1}{u}\frac{du}{dx} \qquad \frac{d}{dx}\sin u=\cos u\frac{du}{dx}$$

$$\frac{d}{dx}\cos u=-\sin u\frac{du}{dx} \qquad \frac{d}{dx}\tan u=\sec^2 u\frac{du}{dx} \qquad \frac{d}{dx}\cot u=-\csc^2 u\frac{du}{dx}$$

$$\frac{d}{dx}\sec u=\sec u\tan u\frac{du}{dx} \qquad \frac{d}{dx}\csc u=-\csc u\cot u\frac{du}{dx} \qquad \frac{d}{dx}\sin^{-1}u=\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\cos^{-1}u=\frac{-1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad \frac{d}{dx}\tan^{-1}u=\frac{1}{1+u^2}\frac{du}{dx} \qquad \frac{d}{dx}\cot^{-1}u=\frac{-1}{1+u^2}\frac{du}{dx}$$