Solution of

106-Calculus-CSIE
Midterm I

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October 27th, Fall 2017

Name: ______________________________

Student ID $\sharp$: ______________________________

Instructions:

1. This exam consists of 8 Problems with total of **110** points.
2. The maximum of the midterm is **100** points.
3. Put away books, notes, calculators, cell phones, and other electronic devices. No discussion during the exam.
4. It might be a good idea to finish the simpler questions first. Good luck!

1	2	3	4
5	6	7	8

1. Evaluate the following limits.

(a) **(7 points)**

$$\lim_{x\to-\infty}\left(\sqrt{x^2-x+1}+x+1\right)$$

$$\begin{aligned}
\lim_{x\to-\infty}\sqrt{x^2-x+1}+x+1 &= \lim_{x\to-\infty}\left(\sqrt{x^2-x+1}+x+1\right)\cdot\frac{\sqrt{x^2-x+1}-(x+1)}{\sqrt{x^2-x+1}-(x+1)}\\
&= \lim_{x\to-\infty}\frac{(x^2-x+1)-(x^2+2x+1)}{\sqrt{x^2-x+1}-(x+1)}\\
&= \lim_{x\to-\infty}\frac{-3x}{\sqrt{x^2-x+1}-(x+1)} \qquad \text{(2 points)}\\
&= \lim_{x\to-\infty}\frac{-3}{\frac{1}{x}\sqrt{x^2-x+1}-1-\frac{1}{x}}\\
&= \lim_{x\to-\infty}\frac{-3}{-\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}-1-\frac{1}{x}} \qquad \text{(3 points)}\\
&= \frac{-3}{-2}=\frac{3}{2} \qquad \text{(2 points)}
\end{aligned}$$

(b) **(8 points)**

$$\lim_{x \to 0} \frac{|2x-1| - |2[\![x]\!]+1|}{x}$$

Here $[\![x]\!]$ = the largest integer that is less than or equal to x is the **greatest integer function**.

$$\lim_{x \to 0^+} \frac{|2x-1| - |2[\![x]\!]+1|}{x} = \lim_{x \to 0^+} \frac{-(2x-1) - |0+1|}{x} = -2 \quad \text{(3 points)}$$

$$\lim_{x \to 0^-} \frac{|2x-1| - |2[\![x]\!]+1|}{x} = \lim_{x \to 0^-} \frac{-(2x-1) - |-2+1|}{x} = -2 \quad \text{(3 points)}$$

$$\because \lim_{x \to 0^+} \frac{|2x-1| - |2[\![x]\!]+1|}{x} = \lim_{x \to 0^-} \frac{|2x-1| - |2[\![x]\!]+1|}{x} = -2$$

$$\therefore \lim_{x \to 0} \frac{|2x-1| - |2[\![x]\!]+1|}{x} = -2 \quad \text{(2 points)}$$

2. Consider the function,

$$H(x) = \begin{cases} ax^2 + bx + c & \text{if } x \geq 0 \\ e^{x^2} & \text{if } x < 0 \end{cases}.$$

(a) **(5 points)** Find a, b, c so that $H(x)$ is continuous.
(b) **(5 points)** Find a, b, c so that $H(x)$ is differentiable everywhere and compute $H'(x)$.
(c) **(5 points)** Find a, b, c so that $H'(x)$ is continuous.

(a) $\lim_{x\to 0^-} H(x) = \lim_{x\to 0^-} e^{x^2} = e^0 = 1$

$\lim_{x\to 0^+} H(x) = \lim_{x\to 0^+} ax^2 + bx + c = c$

$\because \lim_{x\to 0^-} H(x) = \lim_{x\to 0^+} H(x) \quad \therefore c = 1$ (2 points)

$\Rightarrow a, b \in \mathbb{R},\ c = 1$ (3 points)

(b) $\lim_{x\to 0^+} \dfrac{H(x) - H(0)}{x - 0} = 2ax + b|_{x=0} = b$

Since e^{x^2} is differentiable at $x = 0$,

$$\lim_{x\to 0^-} \frac{H(x) - H(0)}{x - 0} = 2xe^{x^2}|_{x=0} = 0.$$

$H(x)$ is differentiable $\Rightarrow$ $H(x)$ is continuous $\Rightarrow$ $c = 1$

$\therefore a \in \mathbb{R},\ b = 0,\ c = 1$ (3 points)

$$H'(x) = \begin{cases} 2ax & \text{if } x \geq 0 \\ 2xe^{x^2} & \text{if } x < 0 \end{cases}$$ (2 points)

(c) Since

$$\lim_{x\to 0^-} H'(x) = 0 = \lim_{x\to 0^+} H'(x),$$ (2 points)

$H'(x)$ is always continuous $\Rightarrow$ $a \in \mathbb{R},\ b = 0,\ c = 1$ (3 points)

3. **(10 points)** Find all the asymptotes of the function,

$$y = f(x) = \frac{\sqrt{x^6+3} - x^3 - x^2}{x^2 - x}.$$

Vertical Asymptotes: We look at those x so that the denominator goes to 0, i.e., $x = 0$ or 1.

At $x = 0$, we have

$$\lim_{x \to 0^+} y(x) = -\infty \Rightarrow x = 0 \text{ is a vertical asymptote.}$$

or

$$\lim_{x \to 0^-} y(x) = \infty \Rightarrow x = 0 \text{ is a vertical asymptote.}$$

Hence $x = 0$ is a vertical asymptote. (2 point)

At $x = 1$, we have

$$\begin{aligned}\lim_{x \to 1} y(x) &= \lim_{x \to 1} \frac{(x-1)\left(-2x^4 - 3(x+1)(x^2+1)\right)}{x(x-1)\left(\sqrt{x^6+3} + x^3 + x^2\right)} \\ &= \lim_{x \to 1} \frac{-2x^4 - 3(x+1)(x^2+1)}{x(\sqrt{x^6+3} + x^3 + x^2)} = \frac{-14}{4} \text{ (1 point)}\end{aligned}$$

Hence $x = 1$ is not a vertical asymptote. (1 point)

Slant/Horizontal Asymptote: Let $y = mx + b$ be the oblique asymptote. We look for slant/horizontal asymptote as $x \to \infty$ and $x \to -\infty$.

Suppose that $x \to \infty$, then

$$m_\infty = \lim_{x \to \infty} \frac{y(x)}{x} = \frac{\sqrt{x^6+3} - x^3 - x^2}{x^3 - x^2} = 0 \text{ (1 point)}$$

and

$$b_\infty = \lim_{x \to \infty} (y - 0x) = \lim_{x \to \infty} = \frac{\sqrt{x^6+3} - x^3 - x^2}{x^2 - x} = -1 \text{ (1 point)}$$

Hence $y = m_\infty x + b_\infty = 0x - 1 = -1$ is the slant asymptote as $x \to \infty$. In fact, $y = -1$ **is a horizontal asymptote.** (1 point)

For $x \to -\infty$, we have

$$m_{-\infty} = \lim_{x \to -\infty} \frac{y(x)}{x} = \frac{\sqrt{x^6+3} - x^3 - x^2}{x^3 - x^2} = -2 \text{ (1 point)}$$

and

$$b_{-\infty} = \lim_{x \to -\infty} [y - (-2x)] = \lim_{x \to -\infty} \frac{\sqrt{x^6+3} - x^3 - x^2 + 2x^3 - 2x^2}{x^2 - x} = -3 \text{ (1 point)}$$

Hence the slant asymptote for $x \to -\infty$ is given by $y = -2x - 3$. (1 point)

4. Consider the function $f(x) = \sin(\tan^{-1}(\sqrt{x}))$.

(a) **(5 points)** What is the domain of $f(x)$? Simplify $f(x)$ into a rational expression of x.

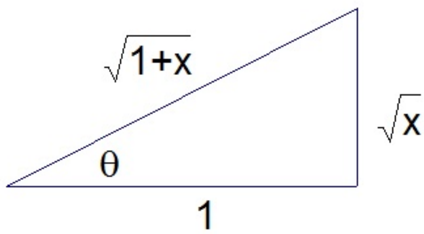
(b) **(5 points)** Use the expression in (a) to find the derivative $f'(x)$.

(c) **(5 points)** Compute $\dfrac{d}{dx}\sin(\tan^{-1}(\sqrt{x}))$ by using the **chain rule**. You don't have to simplify your answer into a rational expression of x. (But by doing so, you can check if you answers in (b) and (c) are compatible.)

(a) Dom(f)=$\{x \geq 0\}$ (1 point)

Let $\theta = \tan^{-1}\left(\sqrt{x}\right) \quad \Rightarrow \quad \tan\theta = \sqrt{x}$ (2 points)

$\Rightarrow \quad \sin(\tan^{-1}(\sqrt{x})) = \sin\theta = \sqrt{\dfrac{x}{x+1}} = f(x)$ (2 points)



(b) $f'(x) = \dfrac{1}{2}\left(\dfrac{x}{x+1}\right)^{-\frac{1}{2}} \cdot \dfrac{1}{(x+1)^2}$ (5 points)

(c)

$$\frac{d}{dx}f(x) = \cos(\tan^{-1}(\sqrt{x})) \cdot \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} \quad \text{(5 points)}$$

$$= \frac{1}{\sqrt{x+1}} \cdot \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}}$$

5. (10 points) The inverse tangent function $\tan^{-1} x$ is very useful for determining the shooting angle of artillery. Use linear approximation to estimate $\tan^{-1}\dfrac{3}{5}$ with the data $\tan(\dfrac{\pi}{6}) = \dfrac{1}{\sqrt{3}} \approx 0.577$.

We use $\tan^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$ to approximate $\tan^{-1}\frac{3}{5}$.
For $y = f(x) = \tan^{-1} x$, we have

$$\begin{aligned} y = f(x) &= \tan^{-1} x \\ &\approx \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + f'\left(\frac{1}{\sqrt{3}}\right)\left(x - \frac{1}{\sqrt{3}}\right) \quad \text{(5 points)} \end{aligned}$$

Therefore the linear approximation is given by

$$\begin{aligned} L\left(\frac{3}{5}\right) &= \frac{\pi}{6} + \frac{1}{1+(\frac{1}{\sqrt{3}})^2}\left(\frac{3}{5} - \frac{1}{\sqrt{3}}\right) \\ &= \frac{\pi}{6} + \frac{3}{4}\left(\frac{3}{5} - \frac{1}{\sqrt{3}}\right). \quad \text{(5 points)} \end{aligned}$$

6.

(a) **(7 points)** Consider the curve with equation $x^2 + xy + y^2 = 1$. Find those points on the curve with horizontal tangent line.

We have

$$\begin{aligned} & x^2 + xy + y^2 = 1 \\ \Rightarrow\ & \frac{d}{dx}(x^2 + xy + y^2) = \frac{d}{dx}(1) \\ \Rightarrow\ & 2x + y + xy' + 2yy' = 0 \\ \Rightarrow\ & y' = \frac{-(2x+y)}{x+2y} \quad \text{(4 points)} \end{aligned}$$

To find horizontal tangent line, set $2x + y = 0$ with $x + 2y \neq 0$. The condition $y = -2x$ implies that

$$\begin{aligned} & x^2 + xy + y^2 = 1 \\ \Rightarrow\ & x^2 + x(-2x) + (-2x)^2 = 1 \\ \Rightarrow\ & 3x^2 = 1. \end{aligned}$$

So $x = \pm\frac{1}{\sqrt{3}}$ (1 point)

$\Rightarrow (x, y) = \left(\frac{1}{\sqrt{3}}, \frac{-2}{\sqrt{3}}\right)$ (1 point)

or $(x, y) = \left(\frac{-1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ (1 point)

(b) **(8 points)** Suppose that $y = f(x)$ is twice differentiable and has an inverse function $f^{-1}(x)$. Suppose that $f(2) = 1$, $f'(2) = 3$, $f''(2) = e$, and $f^{-1}(x)$ is twice differentiable. Find $(f^{-1})'(1)$ and $(f^{-1})''(1)$.

We start with the equation

$$f(f^{-1}(x)) = x \text{ (1 point)}$$

Take derivative, we have

$$f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$$

$$\Rightarrow (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}. \text{ (2 points)}$$

So

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(2)} = \frac{1}{3} \text{ (1 point)}$$

Differentiate $f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$ again, we get

$$f''(f^{-1}(x)) \cdot ((f^{-1})'(x))^2 + f'(f^{-1}(x)) \cdot (f^{-1})''(x) = 0$$

(2 points)

With $x = 1$, we have

$$f''(f^{-1}(1)) \cdot ((f^{-1})'(1))^2 + f'(f^{-1}(1)) \cdot (f^{-1})''(1) = 0$$

(1 point)

and hence

$$\Rightarrow \frac{e}{9} + 3(f^{-1})''(1) = 0$$

$$\Rightarrow (f^{-1})''(1) = -\frac{e}{27} \text{ (1 point)}$$

7. (10 points) For what values of c does the equation $\ln x = cx^2$ have exactly one solution? (**Hint**: Look at tangent lines of these two equations.)

The two equations $y = \ln x$ and $y = cx^2$ intersect at points (t, ct^2) (2 point) for some $t > 0$. We consider a differentiable function $g(x) = cx^2 - \ln x$, the intersections satisfy $g(x) = 0$.

- For $c > 0$, observing from the graph, we can find that the only intersection happens at these two curves tangent to each other. (Since the unique maximum happens at $g'(x) = 0$)

$$\frac{dy}{dx} \Rightarrow \left.\frac{1}{x}\right|_{x=t} = \left.2cx\right|_{x=t} \text{ (2 point)} \Rightarrow 2ct^2 = 1$$

$$\ln t = \frac{1}{2} \Rightarrow t = e^{\frac{1}{2}} \text{ (2 point)}, c = \frac{1}{2}e^{-1} \text{ (1 points)}$$

- For $c \leq 0$, using intermediate value theorem, we know $\exists x \in \mathbb{R}$ such that $g(x) = 0$, that is, every curves $y = cx^2$ will intersect $\ln x$. (3 points)

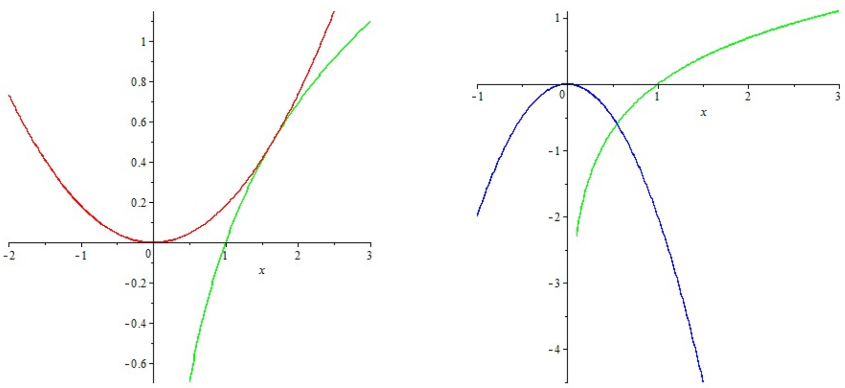


FIGURE 1. left : $c > 0$, right: $c < 0$

Because $g'(x) = 2cx - \ln x < 0$, $\forall x > 0$, f is strictly decreasing, so $f(x) = 0$ has exactly one point.

8. Suppose that $f(x)$ is differentiable on $(-1,1)$ with
$$\lim_{x\to 0}\frac{f(x)}{x^2}=L$$
for some real number L. Define a new function $g(x)$ with $\mathrm{Dom}(g)=(-1,1)$ by the formula,
$$g(x)=\begin{cases} f(x)\sin(\frac{1}{x}) & \text{if } 0<|x|<1 \\ A & \text{if } x=0 \end{cases}.$$
(a) **(4 points)** Is $f(x)$ continuous? Why?
(b) **(4 points)** Find $f(0)$ and $f'(0)$.
(c) **(4 points)** If g is continuous at $x=0$, find the value of A and compute $g'(0)$.
(d) **(4 points)** Write down a formula of $g'(x)$ in terms of $f(x)$ and $f'(x)$ for $0<|x|<1$.
(e) **(4 points)** Suppose that $f'(x)$ and $g'(x)$ are both continuous at 0. Find the value of L.

(a) Since f is differentiable on $(-1,1)$, f is continuous on $(-1,1)$. (4 points)
(b) Because $\lim_{x\to 0}\frac{f(x)}{x^2}=L$, $\lim_{x\to 0}x=0$ and $\lim_{x\to 0}x^2=0$ exists, we can use the product rule of limits.
$f(0)=\lim_{x\to 0}f(x)=\left(\lim_{x\to 0}\frac{f(x)}{x^2}\right)\left(\lim_{x\to 0}x^2\right)=0$ (2 points)
$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{f(x)}{x}=\left(\lim_{x\to 0}\frac{f(x)}{x^2}\right)\left(\lim_{x\to 0}x\right)=0$ (2 points)
(c) g is continuous, so
$$\begin{aligned} A &= g(0) \\ &= \lim_{x\to 0}g(x)=\lim_{x\to 0}f(x)\sin(\frac{1}{x}) \\ &= \lim_{x\to 0}\frac{f(x)}{x^2}\cdot\left(x^2\sin(\frac{1}{x})\right) \\ &= 0 \text{ (2 points)} \end{aligned}$$
where we have used $\lim_{x\to 0}x^2\sin(\frac{1}{x})=0$ by the squeeze theorem as
$$-x^2\le x^2\sin(\frac{1}{x})\le x^2.$$
Similarly, by the squeez theorem we have
$$\lim_{x\to 0}x\sin(\frac{1}{x})=0$$

and

$$g'(0) = \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)\sin(\frac{1}{x})}{x}$$
$$= \lim_{x\to 0} \frac{f(x)}{x^2}\ x\sin(\frac{1}{x}) = 0 \qquad (2 \text{ points})$$

(d) The derivative of $g(x)$ is given by

$$g'(x) = \begin{cases} f'(x)\sin(\dfrac{1}{x}) - f(x)\cos(\dfrac{1}{x})\cdot\dfrac{1}{x^2} & \text{if } 0 < |x| < 1 \quad (3 \text{ points}) \\ 0 & \text{if } x = 0 \quad (1 \text{ point}) \end{cases}.$$

(e) Since g' and f' are continuous, and $g'(0) = \lim_{x\to 0} g'(x)$. So

$$g'(0) = \lim_{x\to 0^+} g'(x) = \lim_{x\to 0^+} \left(f'(x)\sin(\frac{1}{x}) - \frac{f(x)}{x^2}\cos(\frac{1}{x}) \right)$$

As

$$-|f'(x)| \le f'(x)\sin(\frac{1}{x}) \le |f'(x)|,$$

by Squeeze theorem

$$\lim_{x\to 0} f'(x)\sin(\frac{1}{x}) = 0 \qquad (1 \text{ point})$$

If $L \neq 0$, then since $\cos(\frac{1}{x})$ does not exist at $x \to 0$,

$$\lim_{x\to 0} g'(x) \to 0 + L\cdot\cos(\frac{1}{x})$$

fails to exist. (1 point)

Hence L can only be 0, and indeed,

$$\lim_{x\to 0} g'(x) = \lim_{x\to 0} -\frac{f(x)}{x^2}\cos(\frac{1}{x}) = 0 = g'(0) \qquad (2 \text{ points})$$

as expected.