Chapter 1. Introduction to differential equations

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Linear Differential Equations

- Linear D.E. 線性微分方程式，不具有下列任何一項

  1.因變數的自乘項

  2.因變數導數的自乘項

  3.因變數及其導數的互乘項

  則稱為線性微分方程式；反之，若一微分方程式，具有上述1,2,3中任何一項，即稱為非線性D.E.。

Linear Differential Equations

例:

(1) $y'(x)+5y''(x)=e^{x} \Rightarrow$ 線性2階1次O.D.E

(2) $\dfrac{\partial u(x,y)}{\partial x}+u(x,y)\dfrac{\partial u(x,y)}{\partial y}+3\mu(x,y)=0$

$\Rightarrow$ 非線性(Disobey#3)，1階1次P.D.E.

(3) $\dfrac{\partial^{2} u}{\partial x^{2}}+x\dfrac{\partial u}{\partial y}=0 \qquad (u=u(x,y))$

$\Rightarrow$ 線性(x是自變數)，2階1次P.D.E.

(4) $y'''(x)+4y''(x)+2y(x)=x^{2} \Rightarrow$ 線性3階1次O.D.E.

Linear Differential Equations

- 一線性微分方程式會滿足重疊定理(Superposition)

$$V_{i1} \longrightarrow \boxed{F(x)} \longrightarrow V_{C1} \qquad V_{i2} \longrightarrow \boxed{F(x)} \longrightarrow V_{C2}$$

$$V_{C1} = F(V_{i1}) \qquad V_{C2} = F(V_{i2})$$

若滿足

$$\alpha V_{i1} + \beta V_{i2} \longrightarrow \boxed{F(x)} \longrightarrow \alpha V_{C1} + \beta V_{C2}$$

$$\alpha V_{C1} + \beta V_{C2} = F(\alpha V_{i1} + \beta V_{i2}) \qquad \alpha、\beta\text{為任意純量}$$

則為線性。

Differential Equations

- 目的： 分析微分方程式的解

  $\Rightarrow$ 了解微分方程式由何而來

例:

(1) $y(x)=C$ $C\in$ 常數 (Constant)

如何消去C?

$\Rightarrow$ 應用微分，才可以消去C

$$\frac{dy(x)}{dx}=0$$, 1階1次O.D.E.

Differential Equations

(2) $y(x) = C_1 x + C_2 \qquad \mathrm{C}_1, C_2 \in \mathrm{Constant}$

$$\Rightarrow \frac{dy(x)}{dx} = C_1$$

$$\frac{d^2 y(x)}{dx} = 0 \qquad \Rightarrow \text{線性}, 2\text{ 階 }1\text{ 次 O.D.E.}$$

(3) $y(x) = Cx + C^2 \qquad \mathrm{C} \in \mathrm{constant}$

$$\Rightarrow \frac{dy(x)}{dx} = C$$

一個常數，只能微分一次

($\because$ 一次不定積分, 只會產生一個常數)

$$\Rightarrow y(x) = \frac{dy(x)}{dx} x + \left(\frac{dy(x)}{dx}\right)^2 \qquad \Rightarrow \text{非線性,1 階 2 次 O.D.E.}$$

Differential Equations

(4) $y(x) = C_1 e^{2x} + C_2 e^{x} \qquad -(a)$, $C_1, C_2 \in \text{Constant}$

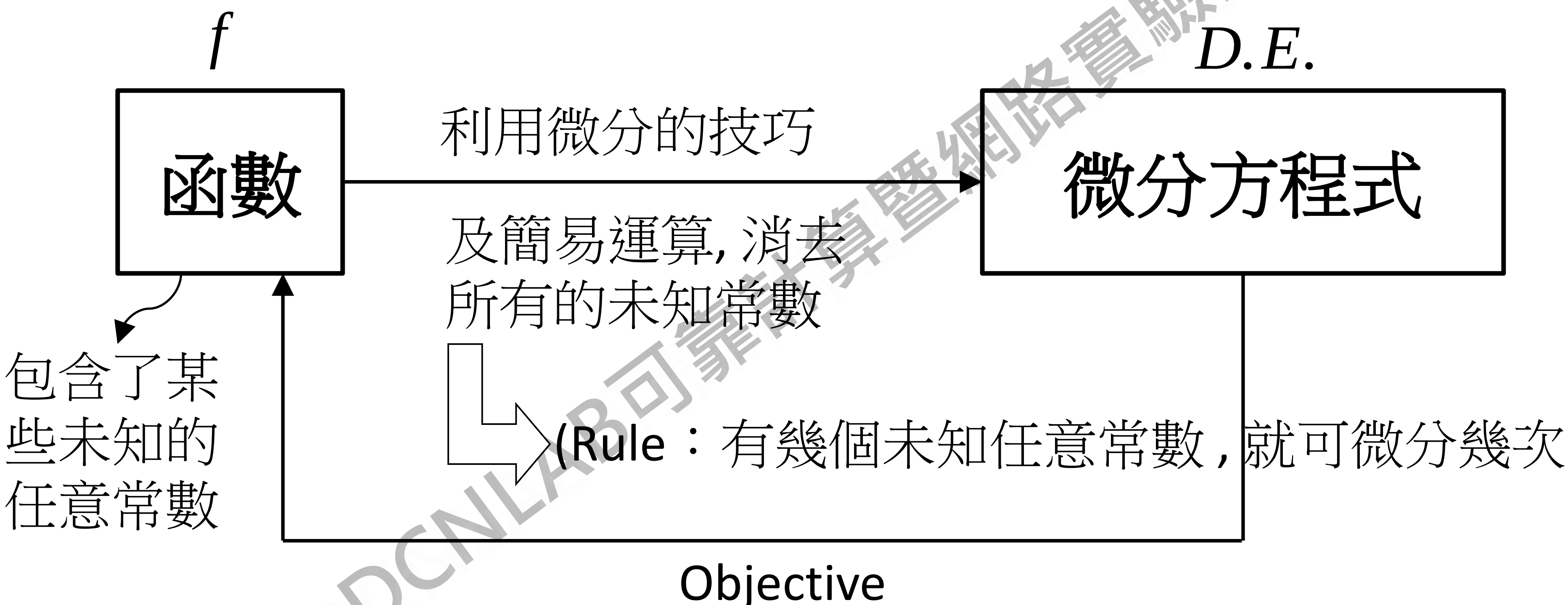
$$\frac{dy(x)}{dx} = 2C_1 e^{2x} + C_2 e^{x} \quad -(b)$$

$$\frac{d^2 y(x)}{dx^2} = 4C_1 e^{2x} + C_2 e^{x} \quad -(c)$$

$(a) \times 2 + (b) \times -3 + (c) \times 1$

$\Rightarrow \; y''(x) - 3y'(x) + 2y(x) = 0$

Differential Equations



$\Rightarrow$ Apply for O.D.E. Only

f 稱為 $D.E.$ 的通解(general solution) or 原函數

Differential Equations

例:

$$y(x) = C_1 \cos 3x + C_2 \sin 3x \qquad , C_1, C_2 \in \text{Constant}$$

$$\Rightarrow D.E. = ?$$

Sol:

$$\frac{dy}{dx} = -3C_1 \sin 3x + 3C_2 \cos 3x$$

$$\frac{d^2 y}{dx^2} = -9C_1 \cos 3x - 9C_2 \sin 3x$$

$$\Rightarrow y''(x) + 9y(x) = 0$$

Chapter 2.
First-Order Ordinary Differential Equations

First-Order Differential Equation

- 一般一階O.D.E.，可表成

1. $M(x,y)dx + N(x,y)dy = 0$
2. $y'(x) = f(x,y) \Rightarrow y'(x) = \dfrac{dy}{dx} = f(x,y)$

$$f(x,y)dx = dy \Rightarrow f(x,y)dx - dy = 0$$

$$\Rightarrow M(x,y)dx + N(x,y)dy = 0$$

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$$

$$y' = f(x,y)$$

First-Order Differential Equation

- 定理 P14 §1.2.1

  對 $y' = f(x,y)$ 之D.E.有一個Initial condition(I.C.)

  $y(x_0) = y_0$

  若 $f(x,y)$，$\dfrac{\partial f(x,y)}{\partial y}$ 於 (x_0, y_0)之鄰域為連續，

  則存在 $\varepsilon > 0$, 使得$y(x)$於 $(x_0 - \varepsilon, x_0 + \varepsilon)$

  間有唯一解

First-Order Differential Equation

例: 下列各問題, 何者有唯一解

(1) $y' = e^{xy^2}$, $y(0) = 1$

(2) $y' = \sqrt{y}$, $y(0) = 0$

(3) $y' = \sqrt{y}$, $y(0) = 1$

(4) $y' = -\sqrt{1 - y^2}$, $y(0) = 0$

(5) $y' = -\sqrt{1 - y^2}$, $y(0) = 1$

First-Order Differential Equation

- Sol:

(1) $y' = e^{xy^2}$, $y(0) = 1$

$f(x, y) = e^{xy^2}$ (0,1)

$\dfrac{\partial f(x, y)}{\partial y} = 2xye^{xy^2}$ (0,1) $\Rightarrow$ 具唯一解

(2)

$y' = \sqrt{y}$, $y(0) = 0$

$f(x, y) = \sqrt{y}$ (0,0)

$\dfrac{\partial f(x, y)}{\partial y} = \dfrac{1}{2\sqrt{y}}$ (0,0) $\Rightarrow$ 不具唯一解

First-Order Differential Equation

(3) $y' = \sqrt{y}\ \ , y(0) = 1$

$(0,1)\ \ \Rightarrow$具唯一解

(4) $y' = -\sqrt{1-y^2}\ \ , y(0) = 0$

$f(x,y) = -\sqrt{1-y^2}\qquad (0,0)$

$$\frac{\partial f(x,y)}{\partial y} = \frac{y}{\sqrt{1-y^2}}\qquad (0,0)$$

$\Rightarrow$具唯一解

(5) $y' = -\sqrt{1-y^2}\ \ , y(0) = 1$

$(0,1)\ \ \Rightarrow$ 不具唯一解

$\Rightarrow$ (1) (3) (4) 具唯一解

Exact Equation

- 如何解$M(x,y)dx+N(x,y)dy=0\cdots\cdots(B)$

  令$u(x,y)=C\cdots\cdots(A)\,,C\in\text{const.}$

$$\Delta u(x,y)=u(x+\Delta x,y+\Delta y)-u(x,y)$$

$$=u(x+\Delta x,y+\Delta y)-u(x,y+\Delta y)+u(x,y+\Delta y)-u(x,y)$$

$$=\frac{\partial u(x,y)}{\partial x}(x+\Delta x-x)+\frac{\partial u(x,y)}{\partial y}(y+\Delta y-y)=0$$

★ Mean-Value-Theorem 均值定理

$f(x)$ ， $a\le x\le b$ ， 一定存在一個C，$a\le C\le b$

St. $f'(C)=\dfrac{f(b)-f(a)}{b-a}\Rightarrow f'(C)(b-a)=f(b)-f(a)$

Exact Equation

$$\Rightarrow \Delta u(x,y) = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y = 0$$

$$\Rightarrow du(x,y) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

$$= M(x,y)dx + N(x,y)dy = 0$$

$$\Rightarrow M(x,y) = \frac{\partial u}{\partial x} \qquad N(x,y) = \frac{\partial u}{\partial y}$$

$$\partial u = M(x,y)\partial x \qquad \partial u = N(x,y)\partial y$$

$$\int \partial u = \int M(x,y)\partial x + f(y) \qquad \int \partial u = \int N(x,y)\partial y + g(x)$$

Exact Equation

$$\Rightarrow u = \begin{cases} \int M(x,y)\partial x + f(y) \quad \cdots\cdots\cdots(1) \\ \int N(x,y)\partial y + g(x)\cdots\cdots\cdots(2) \end{cases}$$

經過比較(1)及(2)式，決定 $f(x)$ 和 $g(x)$

$$\Rightarrow u(x,y) = C$$

Exact Equation

- Q : 如何知道 (A) → (B) 只有微分而已 ?

$$M(x,y)=\frac{\partial u}{\partial x} \qquad N(x,y)=\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{M(x,y)}{\partial y}=\frac{\partial^2 u}{\partial x \partial y} = \frac{N(x,y)}{\partial x}=\frac{\partial^2 u}{\partial x \partial y}$$

假設 $u(x,y)$ 具有連續二階偏導數

$$\Rightarrow M(x,y)dx+N(x,y)dy=0$$

若 $$\frac{\partial M(x,y)}{\partial y}=\frac{\partial N(x,y)}{\partial x}$$

則稱此微分方程式為”正合”(Exact)

Exact Equation

例:

$$u(x, y) = x^2 y^3 = C$$

$$du(x, y) = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

$$\Rightarrow 2xy^3 dx + x^2 3y^2 dy = 0$$

Check: $\dfrac{\partial M(x, y)}{\partial y} = 2x3y^2 = \dfrac{\partial N(x, y)}{\partial x}$

Exact Equation

例:

$$u(x,y) = xy^2 + 3x + 5y = C$$

$$du(x,y) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

$$\Rightarrow (y^2+3)dx + (2xy+5)dy = 0$$

$$\frac{\partial M(x,y)}{\partial y} = 2y = \frac{\partial N(x,y)}{\partial x} \Rightarrow \text{正合}$$

$$\Rightarrow \frac{\partial u(x,y)}{\partial x} = y^2 + 3$$

$$\int \partial u(x,y) = \int (y^2+3)dx + f(y)$$

Exact Equation

$$\Rightarrow \frac{\partial u(x,y)}{\partial y} = 2xy + 5$$

$$\int \partial u(x,y) = \int (2xy+5)dy + g(x)$$

$$\Rightarrow u = \begin{cases} xy^2 + 3x + f(y) \\ xy^2 + 5y + g(x) \end{cases}$$

$$\Rightarrow f(y) = 5y \quad , \quad g(x) = 3x$$

$$\Rightarrow u(x,y) = xy^2 + 3x + 5y = C$$