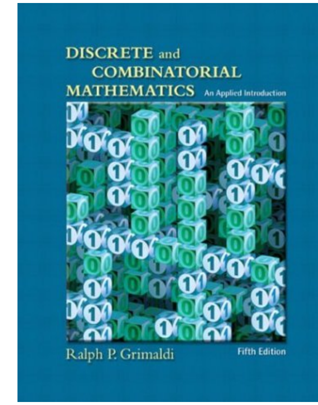
Discrete Mathematics
-- Chapter 2: Fundamentals of Logic



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Outline

- Basic Connectives and Truth Tables
- Logical Equivalence: The Law of Logic
- Logical Implication: Rule of Inference
- The Use of Quantifiers
- Quantifiers, Definitions, and the Proofs of Theorems

The Way to Proof

- 證明"A君是萬能"這句話是錯的 $\rightarrow p$
  (1) A君可以搬動任何石頭 $\rightarrow q$
  (2) A君可以製造出他無法搬動的石頭 $\rightarrow r$, $\rightarrow \neg q$
  (3) (1), (2) 相互矛盾, 原命題為假

p

(1) $p \rightarrow q$

(2) $p \rightarrow r$

$r \rightarrow \neg q$

A logical sequence of statements.

p
$p \rightarrow q$
q
$p \rightarrow r$
r
$r \rightarrow \neg q$
$\neg q$

2.1 Basic Connectives and Truth Tables

- Statement 敘述 (Proposition 命題): are declarative sentences that are either true or false, but not both.
- Primitive Statement (原始命題)
  - *Examples*
    - ***p***: 'Discrete Mathematics' is a required course for sophomores.
    - ***q***: Margaret Mitchell wrote '*Gone with the Wind*'.
    - ***r***: 2+3=5.
    - "What a beautiful evening!" (not a statement)
    - "Get up and do your exercises." (not a statement)
  - No way to make them simpler

"The number x is an integer." is a statement ?

2.1 Basic Connectives and Truth Tables

- New statements can be obtained from primitive statements in two ways
  - Transform a given statement p into the statement $\neg p$, which denotes its *negation* and is read "Not p". 非p (***Negation statements***)
  - Combine two or more statements into a *compound* statement, using *logical connectives*. (***Compound statements複合敘述***)

2.1 Basic Connectives and Truth Tables

- <u>Compound Statement (Logical Connectives)</u>
  - Conjunction: $p \wedge q$ (read "p and q")
  - Disjunction : $p \vee q$ (read "p or q")
  - Exclusive or : $p \underline{\vee} q$. $(p \oplus q)$
  - Implication: $p \rightarrow q$, "p implies q".

    p: hypothesis
    q: conclusion
  - Biconditional: $p \leftrightarrow q$
    - "p if and only if q" (若且為若)
    - "p iff q"
    - "p is necessary and sufficient for q".

$p \rightarrow q$ is also called,
- If p, then q
- p is sufficient for q
- p is a sufficient condition for q
- q is necessary for p
- q is a necessary condition for p
- p only if q
- q whenever p

2.1 Basic Connectives and Truth Tables

- The truth and falsity of the compound statements based on the truth values of their components (primitive statements).

We do not want a true statement to lead us into believing something that is false.

p	$\neg p$
0	1
1	0

p	q	$p \wedge q$	$p \vee q$	$p \veebar q$	$p \rightarrow q$	$p \leftrightarrow q$
0	0	0	0	0	1	**1**
0	1	0	1	1	1	**0**
1	0	0	1	1	**0**	**0**
1	1	1	1	0	**1**	**1**

Truth Tables

Writing Down Truth Tables

- "True, False" is preferred (less ambiguous) than "1, 0"
- List the elementary truth values in a consistent way (from F…F to T…T or from T…T to F…F).
- They get long: with n variables, the length is 2^n.
- In the end we want to be able to reason about logic without having to write down such tables.

2.1 Basic Connectives and Truth Tables

- $p \rightarrow q$ is equivalent with $(\neg p \vee q)$
- It's not relative about the causal relationship
  - *If 'Discrete Mathematics' is a required course for sophomores, then Margaret Mitchell wrote 'Gone with the Wind'.* **is true**
  - *If "2+3=5", then "4+2=6"* **is true**
  - *If "2+3=6", then "2+4=7"* **is true**

- "Margaret Mitchell wrote '*Gone with the Wind*' *(q)*, and $2+3 \neq 5$ *(not r)*, the 'Discrete Mathematics' is a required course for sophomores *(p)*.
  - $q \wedge (\neg r \rightarrow p)$

Table 2.3

p	q	r	$\neg r$ (1)	$\neg r \rightarrow p$ (2)	$q \wedge (\neg r \rightarrow p)$ (3)
0	0	0	1	0	0
0	0	1	0	1	0
0	1	0	1	0	0
0	1	1	0	1	1
1	0	0	1	1	0
1	0	1	0	1	0
1	1	0	1	1	1
1	1	1	0	1	1

Implications examples

p: *If it is sunny today, then we will go to the beach.*

q: *If today is Friday, then 2+3=5.*

r: *If today is Friday, then 2+3=6.*

r is true every day except Friday, even though 2+3=6 is false.

The Mathematical concept of an implication is independent of a cause-and-effect relationship between hypothesis and conclusion.

2.1 Basic Connectives and Truth Tables

- **Ex 2.1:** Let s, t, and u denote the primitive statements.
  - s: Phyllis goes out for a walk.
  - t: The moon is out.
  - u: It is snowing.

- English sentences for compound statements.
  - $(t \wedge \neg u) \rightarrow s$:
    - **If** the moon is out **and** it is not snowing, **then** Phyllis goes out for a walk.
  - $t \rightarrow (\neg u \rightarrow s)$:
    - **If** the moon is out, **then if** it is not snowing Phyllis goes out for a walk.
  - $\neg (s \leftrightarrow (u \vee t))$:
    - **It is not the case** that Phyllis goes out for a walk **if and only if** it is snowing **or** the moon is out.

Same?

2.1 Basic Connectives and Truth Tables

- Let s, t, and u denote the primitive statements.
  - s: Phyllis goes out for a walk.
  - t: The moon is out.
  - u: It is snowing.

- Reversely, examine the logical form for given English sentences.
  - "Phyllis will go out walking if and only if the moon is out."
    - $s \leftrightarrow t$
  - If it is snowing and the moon is not out, then Phyllis will not go out for a walk."
    - $(u \wedge \neg t) \rightarrow \neg s$
  - It is snowing but Phyllis will still go out for a walk.
    - $u \wedge s$

2.1 Basic Connectives and Truth Tables

- **Ex 2.3:**
  - Decision (selection) structure
    - In computer science, the **if-then** and **if-then-else** decision structure arise in high-level programming languages such as Java and C++.
    - E.g., “if p then q else r,” q is executed when p is true and r is executed when p is false.

2.1 Basic Connectives and Truth Tables

- **Tautology**, T_0: If a compound statement is true for all truth value assignments for its component statements.
  - Example: $p \vee \neg p$
- **Contradiction**, F_0: If a compound statement is false for all truth value assignments for its component statements.
  - Example: $p \wedge \neg p$

- Examples
  - "2 = 3–1" is **not** a tautology, but "2=1 or 2≠1" is;
  - "1+1=3" is **not** a contradiction, but "1=1 and 1≠1" is.

2.1 Basic Connectives and Truth Tables

- An **argument** starts with a list of given statements called premises (*hypothesis*) and a statement called the conclusion of the argument.
  - $(p_1 \land p_2 \land \dots \land p_n) \rightarrow q$

2.2 Logical Equivalence: The Laws of Logic

- **Logically equivalent**, $s_1 \Leftrightarrow s_2$: When the statement s_1 is true (false) if and only if s_2 is true (false).

$$\neg p \vee q \Leftrightarrow p \rightarrow q$$

$$(p \rightarrow q) \wedge (q \rightarrow p) \Leftrightarrow p \leftrightarrow q$$

the same truth tables

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$
0	0	1	1	1	1	1	1
0	1	1	1	1	0	0	0
1	0	0	0	0	1	0	0
1	1	0	1	1	1	1	1

2.2 Logical Equivalence: The Laws of Logic

- Negation, $\underline{\vee}$ (exclusive)

Table 2.8

p	q	$p \veebar q$	$p \vee q$	$p \wedge q$	$\neg(p \wedge q)$	$(p \vee q) \wedge \neg(p \wedge q)$
0	0	0	0	0	1	0
0	1	1	1	0	1	1
1	0	1	1	0	1	1
1	1	0	1	1	0	0

2.2 Logical Equivalence: The Laws of Logic

- Logically equivalent examples
  - $p \Leftrightarrow (p \vee p)$
    - "1+1=2" $\Leftrightarrow$ "1+1=2 or 1+1=2"
  - $\neg\neg p \Leftrightarrow p$
    - "He did not not do it" $\Leftrightarrow$ "He did it"
  - $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
  - "If p then not p" $\Leftrightarrow$ "not p"

The Laws of Logic (1/2)

1) Law of Double Negation: $\neg\neg p \Leftrightarrow p$

2) DeMorgan's Laws: $\begin{cases} \neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q \\ \neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q \end{cases}$

3) Commutative Laws: $\begin{cases} p \vee q \Leftrightarrow q \vee p \\ p \wedge q \Leftrightarrow q \wedge p \end{cases}$

4) Associative Laws: $\begin{cases} p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r \\ p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r \end{cases}$

5) Distributive Laws: $\begin{cases} p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r) \\ p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r) \end{cases}$

The Laws of Logic (2/2)

$$6)\ \text{Idempotent Laws}: \begin{cases} p \vee p \Leftrightarrow p \\ p \wedge p \Leftrightarrow p \end{cases}$$

$$7)\ \text{Identity Laws}: \begin{cases} p \vee F_0 \Leftrightarrow p \\ p \wedge T_0 \Leftrightarrow p \end{cases}$$

$$8)\ \text{Inverse Laws}: \begin{cases} p \vee \neg p \Leftrightarrow T_0 \\ p \wedge \neg p \Leftrightarrow F_0 \end{cases}$$

$$9)\ \text{Domination Laws}: \begin{cases} p \vee T_0 \Leftrightarrow T_0 \\ p \wedge F_0 \Leftrightarrow F_0 \end{cases}$$

$$10)\ \text{Absorption Laws}: \begin{cases} p \vee (p \wedge q) \Leftrightarrow p \\ p \wedge (p \vee q) \Leftrightarrow p \end{cases}$$

2.2 Logical Equivalence: The Laws of Logic

- <u>DeMorgan's Laws</u>:
$$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$$
$$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$$

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
0	0	0	1	1	1	1	0	1	1
0	1	0	1	1	0	1	1	0	0
1	0	0	1	0	1	1	1	0	0
1	1	1	0	0	0	0	1	0	0

The Principle of Duality

- **Definition 2.3**: Let s be a statement. Dual of s, denoted s^d, is the statement obtained from s by replacing each occurrence of $\wedge$ and $\vee$ by $\vee$ and $\wedge$, respectively, and each occurrence of T_0 and F_0 by F_0 and T_0, respectively.
  - E.g. $s: \quad (p \wedge \neg q) \vee (r \wedge T_0)$

    $s^d: \quad (p \vee \neg q) \wedge (r \vee F_0)$ *Keep negation!*

- **The Principle of Duality**: Let s and t be statements that contain no logical connectives other than $\wedge$ and $\vee$.
  If $s \Leftrightarrow t$, then $s^d \Leftrightarrow t^d$.

Ex:
$$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$$
$$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$$

Substitution Rules

- **Rule 1**: Suppose that the compound statement P is a **tautology**.
  - If p is a primitive statement that appears in P and we replace **each** occurrence of p by the same statement q, then the resulting compound statement P_1 is also a tautology.

- **Rule 2**: Let P be a compound statement where p is an arbitrary statement that appears in P, and let q be a statement such that $q \Leftrightarrow p$.
  - Suppose that in P we replace **one or more** occurrences of p by q. Then this replacement yields the compound statement P_1. Under these circumstances $P_1 \Leftrightarrow P$.

Simplification of Compound Statements

- **Ex 2.10:**
  - From the first of DeMorgan's Laws

    $P: \neg (p \vee q) \leftrightarrow (\neg p \wedge \neg q)$ is a tautology
  - from the first substitution rule

    $P_1: \neg [(r \wedge s) \vee q] \leftrightarrow [\neg(r \wedge s) \wedge \neg q]$ is also a tautology
  - replace each occurrence of q by $t \rightarrow u$

    $P_2: \neg [(r \wedge s) \vee (t \rightarrow u)] \leftrightarrow [\neg (r \wedge s) \wedge \neg (t \rightarrow u)]$

Simplification of Compound Statements

- **Ex 2.11:**
  - P: $(p \rightarrow q) \rightarrow r$, and because $(p \rightarrow q) \Leftrightarrow \neg p \vee q$
  - from the second substitution rule

    if P_1: $(\neg p \vee q) \rightarrow r$, *then* $P_1 \Leftrightarrow P$

  - Check 2.11 (b) for another example using the second rule
    - $p \rightarrow (p \vee q)$, $\neg\neg p \Leftrightarrow p$
      - $P_1 : p \rightarrow (\neg\neg p \vee q)$, $P_1 \Leftrightarrow P$
      - $P_2 : \neg\neg p \rightarrow (\neg\neg p \vee q)$, $P_2 \Leftrightarrow P$

Simplification of Compound Statements

- **Ex 2.12:**
  - Negate and simplify the compound statement $(p \vee q) \rightarrow r$

1. $(p \vee q) \rightarrow r \Leftrightarrow \neg(p \vee q) \vee r$ (First substitution rule)
2. $\neg[(p \vee q) \rightarrow r] \Leftrightarrow \neg[\neg(p \vee q) \vee r]$ (Negating)
3. $\neg[\neg(p \vee q) \vee r] \Leftrightarrow \neg\neg(p \vee q) \wedge \neg r$ (DeMorgan's Laws)
4. $\neg\neg(p \vee q) \wedge \neg r \Leftrightarrow (p \vee q) \wedge \neg r$ (Law of double Negation)
5. $\neg[(p \vee q) \rightarrow r] \Leftrightarrow (p \vee q) \wedge \neg r$

Simplification of Compound Statements

- **Ex 2.13:**
- p: Joan goes to Lake George
- q: Mary pays for Joan's shopping spree.
- $p \rightarrow q$: If Joan goes to Lake George, then Mary will pay for Joan's shopping spree.
- The negation of $p \rightarrow q$:
  - One way: $\neg (p \rightarrow q)$. *It is not the case that if Joan goes to Lake George, then Mary will pay for Joan's shopping spree.*
  - Another way: $p \wedge \neg q$. Joan goes to Lake George, but Mary does not pay for Joan's shopping spree.

$$
\begin{aligned}
& \neg(p \rightarrow q) \\
\Leftrightarrow\ & \neg(\neg p \vee q) \\
\Leftrightarrow\ & \neg\neg p \wedge \neg q \\
\Leftrightarrow\ & p \wedge \neg q
\end{aligned}
$$

Relevant Statements to Implication Statement

- **Ex 2.15:**

p	q	$p \to q$	$\neg q \to \neg p$	$q \to p$	$\neg p \to \neg q$
0	0	1	1	1	1
0	1	1	1	0	0
1	0	0	0	1	1
1	1	1	1	1	1

  - $p \to q \Leftrightarrow (\neg q \to \neg p)$
  - $q \to p \Leftrightarrow (\neg p \to \neg q)$
- Contrapositive of $p \to q$: $\underline{\neg q \to \neg p}$
- Converse of $p \to q$: $\underline{q \to p}$
- Inverse of $p \to q$: $\underline{\neg p \to \neg q}$

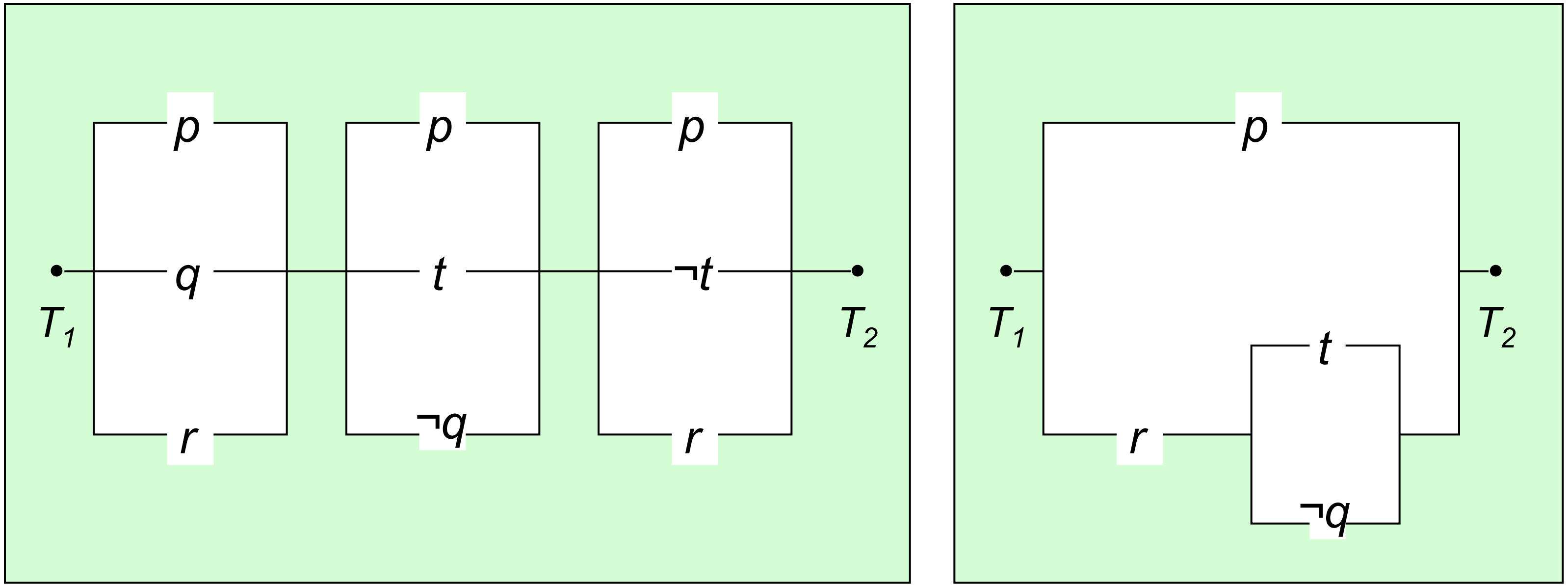
Simplification of Compound Statements

- **Ex 2.16:** How to simply express the compound statement $(p \vee q) \wedge \neg(\neg p \wedge q)$?

$(p \vee q) \wedge \neg(\neg p \wedge q)$	Reasons
$\Leftrightarrow (p \vee q) \wedge (\neg\neg p \vee \neg q)$	(DeMorgan's Law)
$\Leftrightarrow (p \vee q) \wedge (p \vee \neg q)$	(Law of Double Negation)
$\Leftrightarrow p \vee (q \wedge \neg q)$	(Distributive Law of $\vee$ over $\wedge$)
$\Leftrightarrow p \vee F_0$	(Inverse Law)
$\Leftrightarrow p$	(Identity Law)

Simplification of Compound Statements

- **Ex 2.18:** A switching network is made up of wires and switches connecting two terminals T_1 and T_2



$$(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r) \Leftrightarrow p \vee [r \wedge (t \vee \neg q)]$$

Simplification of Compound Statements

$$(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r)$$

$$\Leftrightarrow p \vee [(q \vee r) \wedge (t \vee \neg q) \wedge (\neg t \vee r)]$$

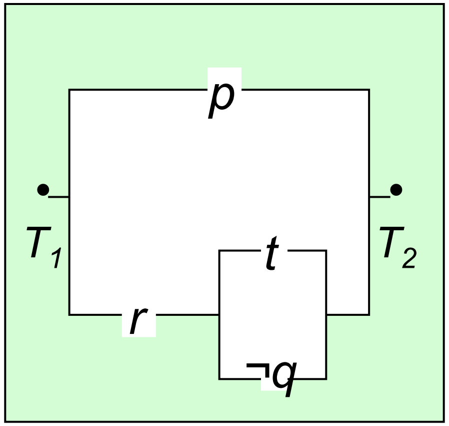
$$\Leftrightarrow p \vee [((q \wedge \neg t\,) \vee r) \wedge (t \vee \neg q)]$$

$$\Leftrightarrow p \vee [((q \wedge \neg t\,) \vee r) \wedge \neg(\neg t \wedge q)]$$

$$\Leftrightarrow p \vee [((q \wedge \neg t\,) \wedge \neg(\neg t \wedge q)) \vee (\neg(\neg t \wedge q) \wedge r)]$$

$$\Leftrightarrow p \vee [F_0 \vee (\neg(\neg t \wedge q) \wedge r)]$$

$$\Leftrightarrow p \vee [r \wedge (t \vee \neg q)]$$



Ex 2.2-15

- Define "Nand" as $(p \uparrow q) \Leftrightarrow \neg(p \wedge q)$, represent the following using only this connective.
  - $\neg p$
    - $\Leftrightarrow \neg(p \wedge p) \Leftrightarrow (p \uparrow p)$
  - $p \vee q$
    - $\Leftrightarrow \neg(\neg p \wedge \neg q) \Leftrightarrow \neg p \uparrow \neg q \Leftrightarrow (p \uparrow p) \uparrow (q \uparrow q)$
  - $p \wedge q$
  - $p \rightarrow q$
    - $\Leftrightarrow \neg p \vee q \Leftrightarrow (\neg p \uparrow \neg p) \uparrow (q \uparrow q) \Leftrightarrow \neg\neg p \uparrow (q \uparrow q) \Leftrightarrow p \uparrow (q \uparrow q)$
  - $p \leftrightarrow q$

2.3 Logic Implication: Rules of Inference

- Argument: $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$
  Premiss: $p_1, p_2, \ldots, p_n$
  Conclusion: q

- **Ex 2.19:**
  - p: Roger studies.
  - q: Roger plays racketball.
  - r: Roger passes discrete mathematics.

  - p_1: Roger studies, then he will pass discrete mathematics.
  - p_2: If Roger don't play racketball, then he'll study.
  - p_3: Roger failed discrete mathematics.
  - Determine whether the argument $(p_1 \wedge p_2 \wedge p_3) \rightarrow q$ is valid.

$p_1 : p \rightarrow r$

$p_2 : \neg q \rightarrow p$

$p_3 : \neg r$

Logic Implication: Rules of Inference

- Examine the truth table for the implication

  $[(p \rightarrow r) \wedge (\neg q \rightarrow p) \wedge \neg r] \rightarrow q$

			p_1	p_2	p_3	$(p_1 \wedge p_2 \wedge p_3) \rightarrow q$
p	q	r	$p \rightarrow r$	$\neg q \rightarrow p$	$\neg r$	$[(p \rightarrow r) \wedge (\neg q \rightarrow p) \wedge \neg r] \rightarrow q$
0	0	0	1	0	1	1
0	0	1	1	0	0	1
0	1	0	1	1	1	1
0	1	1	1	1	0	1
1	0	0	0	1	1	1
1	0	1	1	1	0	1
1	1	0	0	1	1	1
1	1	1	1	1	0	1

$p_1 : p \rightarrow r$

$p_2 : \neg q \rightarrow p$

$p_3 : \neg r$

- $(p_1 \wedge p_2 \wedge p_3) \rightarrow q$ is a valid argument.

Logic Implication: Rules of Inference

- Definition 2.4: p logically implies q, $p \Rightarrow q$.
  - If p, q are arbitrary statements such that $p \rightarrow q$ is a tautology.
- Rules of inference ← *Use instead of constructing the huge truth table*
  - Using these techniques will enable us to consider only the cases wherein all the premises are true.
  - Development of a step-by-step validation of how the conclusion q logically follows from the premises $p_1, p_2, \ldots, p_n$ in an implication of the form

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$$

Logic Implication: Rules of Inference

- Rule of Detachment (分離)
  - Modus Ponens: the method of affirming
- Example: $[p \wedge (p \rightarrow q)] \rightarrow q$

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$[p \wedge (p \rightarrow q)] \rightarrow q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

- Tabular form:

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array} \qquad \begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

therefore

Logic Implication: Rules of Inference

- Law of the Syllogism (演繹推理)
  - Tabular form:

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

- Modus Tollens: Method of Denying
  - Tabular form:

$$\begin{array}{l} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Rules of Inference

Table 2.19

Rule of Inference	Related Logical Implication	Name of Rule
1) p, $p \rightarrow q$, $\therefore q$	$[p \wedge (p \rightarrow q)] \rightarrow q$	Rule of Detachment (Modus Ponens)
2) $p \rightarrow q$, $q \rightarrow r$, $\therefore p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Law of the Syllogism
3) $p \rightarrow q$, $\neg q$, $\therefore \neg p$	$[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$	Modus Tollens
4) p, q, $\therefore p \wedge q$		Rule of Conjunction
5) $p \vee q$, $\neg p$, $\therefore q$	$[(p \vee q) \wedge \neg p] \rightarrow q$	Rule of Disjunctive Syllogism
6) $\neg p \rightarrow F_0$, $\therefore p$	$(\neg p \rightarrow F_0) \rightarrow p$	Rule of Contradiction
7) $p \wedge q$, $\therefore p$	$(p \wedge q) \rightarrow p$	Rule of Conjunctive Simplification
8) p, $\therefore p \vee q$	$p \rightarrow p \vee q$	Rule of Disjunctive Amplification
9) $p \wedge q$, $p \rightarrow (q \rightarrow r)$, $\therefore r$	$[(p \wedge q) \wedge [p \rightarrow (q \rightarrow r)]] \rightarrow r$	Rule of Conditional Proof
10) $p \rightarrow r$, $q \rightarrow r$, $\therefore (p \vee q) \rightarrow r$	$[(p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow [(p \vee q) \rightarrow r]$	Rule for Proof by Cases
11) $p \rightarrow q$, $r \rightarrow s$, $p \vee r$, $\therefore q \vee s$	$[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r)] \rightarrow (q \vee s)$	Rule of the Constructive Dilemma
12) $p \rightarrow q$, $r \rightarrow s$, $\neg q \vee \neg s$, $\therefore \neg p \vee \neg r$	$[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (\neg q \vee \neg s)] \rightarrow (\neg p \vee \neg r)$	Rule of the Destructive Dilemma

Logic Implication: Rules of Inference

Ex 2.30:

$$
\begin{array}{l}
p \rightarrow q \\
q \rightarrow (r \wedge s) \\
\neg r \vee (\neg t \vee u) \\
p \wedge t \\
\hline
\therefore u
\end{array}
$$

Steps		Reasons
1)	$p \rightarrow q$	Premise
2)	$q \rightarrow (r \wedge s)$	Premise
3)	$p \rightarrow (r \wedge s)$	Steps (1) and (2) and the Law of the Syllogism
4)	$p \wedge t$	Premise
5)	p	Step (4) and the Rule of Conjunctive Simplification
6)	$r \wedge s$	Steps (5) and (3) and the Rule of Detachment
7)	r	Step (6) and the Rule of Conjunctive Simplification
8)	$\neg r \vee (\neg t \vee u)$	Premise
9)	$\neg(r \wedge t) \vee u$	Step (8), the Associative Law of $\vee$, and DeMorgan's Laws
10)	t	Step (4) and the Rule of Conjunctive Simplification
11)	$r \wedge t$	Steps (7) and (10) and the Rule of Conjunction
12)	$\therefore u$	Steps (9) and (11), the Law of Double Negation, and the Rule of Disjunctive Syllogism

Logic Implication: Rules of Inference

- Ex 2.31:
  - If the band could not play rock music or the refreshments were not delivered on time, then the New Year's party would have been cancelled and Alicia would have been angry.
  - If the party were cancelled, then refunds would have had to be made.
  - No refunds were made.
  - Therefore the band could play rock music.

p: The band could play rock music.
q: The refreshments were delivered on time
r: The New Year's party was cancelled .
s: Alicia was angry.
t: Refunds had to be made.

$$\begin{array}{l} (\neg p \vee \neg q) \rightarrow (r \wedge s) \\ r \rightarrow t \\ \neg t \\ \hline \therefore p \end{array}$$

Logic Implication: Rules of Inference

$$
\begin{array}{l}
(\neg p \vee \neg q) \rightarrow (r \wedge s) \\
r \rightarrow t \\
\neg t \\
\hline
\therefore p
\end{array}
$$

Steps	**Reasons**
(1) $r \rightarrow t$	Premise
(2) $\neg t$	Premise
(3) $\neg r$	(1) and (2), and Method of Denying
(4) $\neg r \vee \neg s$	(3) and Rule of Disjunctive Amplification
(5) $\neg(r \wedge s)$	(4) and DeMorgan's Laws
(6) $(\neg p \vee \neg q) \rightarrow (r \wedge s)$	Premise
(7) $\neg(\neg p \vee \neg q)$	(6) and (5), and Method of Denying
(8) $p \wedge q$	(7), DeMorgan's Laws, and Law of Double Negation
(9) $\therefore p$	(8) and Rule of Conjunctive Simplification

Proof by Contradiction

- <u>**Ex 2.35:**</u>
  - Is the right argument valid or invalid?

$$
\begin{array}{l}
p \rightarrow q \\
q \rightarrow s \\
r \rightarrow \neg s \\
\neg p \veebar r \\
\hline
\therefore \neg p
\end{array}
$$

  - Let conclusion $\neg p$ be false (the argument invalid)
    $\Rightarrow p$ is true
    $\Rightarrow q$ is true
    $\Rightarrow s$ is true
    $\Rightarrow r$ is false
    $\Rightarrow r$ is true ($\neg p \veebar r$, now p is true)
    $\Rightarrow$ contradiction
    $\Rightarrow$ argument is valid

Interesting examples

- A. 考前猜題 (question selection)
  - 習題A-1, A-2兩題中必出一題, 但不會兩題都出
  - 習題A-2和A-3兩題要嘛都出, 要嘛都不出
  - 如果不出A-1, 也不會出A-3
  - 請問哪一題會出哪一題不會出?

p: A-1 is selected, q: A-2 is selected, r: A-3 is selected

(p v q) and not(p and q)
(q and r) v not (q v r)
Not p → not r
Test p (success) , not p (fail)

Ans: A-1 selected, A-2 not selected, A-3 not selected

Interesting examples

- B. 誰是兇手
  - 角頭老大A死了, 警方請來他幫派裡兩位二哥級人物B和C來協助瞭解死因。
    - B說: 如果A被謀殺, 那肯定是C幹的
    - C說: 如果A不是自殺, 那就是被謀殺
  - 警方不知B,C供詞真偽, 但可以確定的是
    - A死因只有三種: 意外, 自殺和謀殺
    - 如果B和C都沒有說謊, 那A就死於一場意外
    - 如果B和C兩人中有一人說謊, 那麼A就不是死於意外
  - 請問A的死因為何?

Ans: 謀殺, B說謊

Interesting examples

p:A 被謀殺, q:A 自殺, r:A 意外死

他只有可能是一種死法:

$$\left.\begin{array}{l}(\neg p \wedge \neg q) \leftrightarrow r \\ (\neg p \wedge \neg r) \leftrightarrow q \\ (\neg q \wedge \neg r) \leftrightarrow p\end{array}\right\} \qquad (p \veebar q \veebar r) \wedge [\neg (p \wedge q \wedge r)] \Leftrightarrow T_0$$

s: B 說真話 t: C 說真話 u: C 幹的

B 說: $s \Leftrightarrow p \rightarrow u$ $\qquad (u \rightarrow p) \Leftrightarrow p \leftrightarrow u$

C 說: $t \Leftrightarrow \neg q \rightarrow p$

警察認為:

$(s \wedge t) \rightarrow r$

$(s \wedge \neg t) \vee (\neg s \wedge t) \rightarrow p \Leftrightarrow (s \veebar t) \rightarrow p$

Test
(1) s and t
(2) s and not t
(3) not s and t

2.4 The Use of Quantifiers

- Definition 2.5: A declarative sentence is an **open statement** if
  - it contains one or more variables, and
  - it is not a statement, but
  - it becomes a statement when *the variables in it are replaced by certain allowable choices*.
  - E.g.,
    - $p(x)$: The number $x+2$ is an even integer.
    - $q(x, y)$: The number $y+2$, $x - y$, and $x+2y$ are even integers.
    - For some x, $p(x)$. For some x, y, $q(x, y)$.
    - For some x, $\neg p(x)$. For some x, y, $\neg q(x, y)$.

- **Existential quantifier**: $\exists x$, "for some x", "for at least one x" or "there exists an x such that"
- **Universal quantifier**: $\forall x$, "for all x", "for any x" "for every x" or "for each x such that"

The Use of Quantifiers

- **Ex 2.36:**
  - Given the open statements $p(x): \; x \geq 0, \quad q(x): \; x^2 \geq 0$
    $r(x): \; x^2 - 3x - 4 = 0, s(x): x^2 - 3 > 0$
  - The statement $\forall x\,[p(x) \rightarrow q(x)]$ is true
    - For every real number x, if $x \geq 0$, then $x^2 \geq 0$.
    - Every nonnegative real number has a nonnegative square.
    - The square of any nonnegative real number is a nonnegative real number.
    - All nonnegative real numbers have nonnegative squares.
  - The statement $\exists x\,[p(x) \wedge r(x)]$ is true.
  - The statement $\forall x\,[q(x) \rightarrow s(x)]$ is false. ← *Find counterexample!*

Summarization

Table 2.21

Statement	When Is It True?	When Is It False?
$\exists x\ p(x)$	For some (at least one) a in the universe, $p(a)$ is true.	For every a in the universe, $p(a)$ is false.
$\forall x\ p(x)$	For every replacement a from the universe, $p(a)$ is true.	There is at least one replacement a from the universe for which $p(a)$ is false.
$\exists x\ \neg p(x)$	For at least one choice a in the universe, $p(a)$ is false, so its negation $\neg p(a)$ is true.	For every replacement a in the universe, $p(a)$ is true.
$\forall x\ \neg p(x)$	For every replacement a from the universe, $p(a)$ is false and its negation $\neg p(a)$ is true.	There is at least one replacement a from the universe for which $\neg p(a)$ is false and $p(a)$ is true.

$\neg\exists x: p(x)$ v.s. $\exists x: \neg p(x)$

$\neg\forall x: p(x)$ v.s. $\forall x: \neg p(x)$

The Use of Quantifiers

- For open statements $p(x)$, $q(x)$, the universally quantified statement $\forall x[p(x) \rightarrow q(x)]$, we define
  - Contrapositive: $\forall x[\neg q(x) \rightarrow \neg p(x)]$
  - Converse: $\forall x[q(x) \rightarrow p(x)]$
  - Inverse: $\forall x[\neg p(x) \rightarrow \neg q(x)]$

Logical Equivalence and Implication for Quantified Statements

$$\exists x\,[p(x) \wedge q(x)] \Rightarrow [\exists x\, p(x) \wedge \exists x\, q(x)]$$

$$\exists x\,[p(x) \vee q(x)] \Leftrightarrow [\exists x\, p(x) \vee \exists x\, q(x)]$$

$$\forall x\,[p(x) \wedge q(x)] \Leftrightarrow [\forall x\, p(x) \wedge \forall x\, q(x)]$$

$$[\forall x\, p(x) \vee \forall x\, q(x)] \Rightarrow \forall x\,[p(x) \vee q(x)]$$

The Use of Quantifiers

- Rules for negating statements with one quantifier

a) $\forall x \neg\neg p(x) \Leftrightarrow \forall x\, p(x)$
b) $\forall x \neg[p(x) \wedge q(x)] \Leftrightarrow \forall x\, [\neg p(x) \vee \neg q(x)]$
c) $\forall x \neg[p(x) \vee q(x)] \Leftrightarrow \forall x\, [\neg p(x) \wedge \neg q(x)]$

$$\neg[\forall x\, p(x)] \Leftrightarrow \exists x \neg p(x)$$

$$\neg[\exists x\, p(x)] \Leftrightarrow \forall x \neg p(x)$$

$$\neg[\forall x\, \neg p(x)] \Leftrightarrow \exists x \neg\neg p(x) \Leftrightarrow \exists x\, p(x)$$

$$\neg[\exists x\, \neg p(x)] \Leftrightarrow \forall x \neg\neg p(x) \Leftrightarrow \forall x\, p(x)$$

Commuting Quantifiers

Identical quantifiers commute:

$\exists x \exists y: P(x,y) \Leftrightarrow \exists y \exists x: P(x,y)$ and

$\forall x \forall y: P(x,y) \Leftrightarrow \forall y \forall x: P(x,y)$

But non-identical ones do not, see:

$\exists x \forall y: x=y$ v.s. $\forall y \exists x: x=y$

The Use of Quantifiers

- **Ex 2.49:**
  - What is the negation of the following statement?

$$\forall x\, \exists y\, [(p(x,y) \wedge q(x,y)) \rightarrow r(x,y)]$$

$$\neg[\forall x\, \exists y\, [(p(x,y) \wedge q(x,y)) \rightarrow r(x,y)]]$$

$$\Leftrightarrow \exists x\, [\neg \exists y\, [(p(x,y) \wedge q(x,y)) \rightarrow r(x,y)]]$$

$$\Leftrightarrow \exists x\, \forall y \neg\, [(p(x,y) \wedge q(x,y)) \rightarrow r(x,y)]$$

$$\Leftrightarrow \exists x\, \forall y \neg\, [\neg[p(x,y) \wedge q(x,y)] \vee r(x,y)]$$

$$\Leftrightarrow \exists x\, \forall y [\neg\, \neg[p(x,y) \wedge q(x,y)] \wedge \neg r(x,y)]$$

$$\Leftrightarrow \exists x\, \forall y [p(x,y) \wedge q(x,y)] \wedge \neg r(x,y)]$$

2.5 Quantifiers, Definitions, and the Proofs of Theorems

- **Ex 2.52:**
  - show that for all n, n=2, 4, 6,…,24, 26, we can write n as the sum of at most three perfect squares.
  - Exhaustion method

2=1+1	10=9+1	20=16+4
4=4	12=4+4++4	22=9+9+4
6=4+1+1	14=9+4+1	24=16+4+4
8=4+4	16=16	26=25+1
	18=16+1+1	

Quantifiers, Definitions, and the Proofs of Theorems

- **The Rule of Universal Specification**: If an open statement becomes true for all replacements by the members in a given universe, then that open statement is true for <u>each specific individual member</u> in that universe.

$$\text{if } \forall x\ p(\mathrm{x}) \text{ is true, then } p(x) \text{ is true for each } a \text{ in the universe.}$$

- E.g.

$$\begin{array}{l} \forall x\ [m(x) \to c(x)] \\ \underline{m(a) \qquad\qquad\qquad} \\ \therefore c(a) \end{array}$$

Steps	**Reasons**
(1) $\forall x[m(x) \to c(x)]$	Premise
(2) $m(\mathrm{a})$	Premise
(3) $m(\mathrm{a}) \to \mathrm{c(a)}$	(1) and the rule of Universal Specification
(4) $\therefore \mathrm{c(a)}$	(2) and (3) and the Rule of Detachment

Quantifiers, Definitions, and the Proofs of Theorems

- **The Rule of Universal Generalization**: If an open statement $p(x)$ is proved to be true when x is replaced by **any arbitrarily chosen** element c from our universe, then the universally quantified statement $\forall x\, p(x)$ is true. Furthermore, the rule extends beyond a single variable.
  - **Ex 2.56:**

$$\begin{array}{l} \forall x\ [p(x) \vee q(x)] \\ \forall x\ [(\neg p(x) \wedge q(x)) \rightarrow r(x)] \\ \hline \therefore \forall x\ [\neg r(x) \rightarrow p(x)] \end{array}$$

$$\underline{[\neg r(x) \rightarrow \neg(\neg p(x) \wedge q(x))]}$$

$$\downarrow$$

$$\underline{[\neg r(x) \rightarrow p(x) \vee \neg q(x))]}$$

Quantifiers, Definitions, and the Proofs of Theorems

$$\begin{array}{l} \forall x\ [p(x) \vee q(x)] \\ \forall x\ [(\neg p(x) \wedge q(x)) \rightarrow r(x)] \\ \hline \therefore \forall x\ [\neg r(x) \rightarrow p(x)] \end{array}$$

- **Ex 2.56:**

Steps	Reasons
1) $\forall x\, [p(x) \vee q(x)]$	Premise
2) $p(c) \vee q(c)$	Step (1) and the Rule of Universal Specification
3) $\forall x\, [(\neg p(x) \wedge q(x)) \rightarrow r(x)]$	Premise
4) $[\neg p(c) \wedge q(c)] \rightarrow r(c)$	Step (3) and the Rule of Universal Specification
5) $\neg r(c) \rightarrow \neg[\neg p(c) \wedge q(c)]$	Step (4) and $s \rightarrow t \Leftrightarrow \neg t \rightarrow \neg s$
6) $\neg r(c) \rightarrow [p(c) \vee \neg q(c)]$	Step (5), DeMorgan's Law, and the Law of Double Negation
7) $\neg r(c)$	Premise (assumed)
8) $p(c) \vee \neg q(c)$	Steps (7) and (6) and Modus Ponens
9) $[p(c) \vee q(c)] \wedge [p(c) \vee \neg q(c)]$	Steps (2) and (8) and the Rule of Conjunction
10) $p(c) \vee [q(c) \wedge \neg q(c)]$	Step (9) and the Distributive Law of $\vee$ over $\wedge$
11) $p(c)$	Step (10), $q(c) \wedge \neg q(c) \Leftrightarrow F_0$, and $p(c) \vee F_0 \Leftrightarrow p(c)$
12) $\therefore \forall x\, [\neg r(x) \rightarrow p(x)]$	Steps (7) and (11) and the Rule of Universal Generalization

Quantifiers, Definitions, and the Proofs of Theorems

- Argument

$$\forall x\ [(j(x) \vee s(x)) \rightarrow \neg p(x)]$$
$$p(m)$$
$$\therefore \neg s(m)$$

- Establish the validity of this argument

- No junior or senior is enrolled in a physical education class.
- Mary is enrolled in a physical education class.
- Thus Mary is not a senior.

Steps	**Reasons**
1) $\forall x\ [j(x) \vee s(x) \rightarrow \neg p(x)]$	Premise
2) $p(m)$	Premise
3) $j(m) \vee s(m) \rightarrow \neg p(m)$	(1) and Rule of Universal Specification
4) $p(m) \rightarrow \neg(j(m) \vee s(m))$	(3), $(q \rightarrow t) \Leftrightarrow (\neg t \rightarrow \neg q)$ and Law of Double Negation
5) $p(m) \rightarrow (\neg j(m) \wedge \neg s(m))$	(4) and DeMorgan's Law
6) $\neg j(m) \wedge \neg s(m)$	(2) and (5) and Rule of Detachment
7) $\therefore \neg s(m)$	(6) and Rule of Conjunctive Simplification

Quantifiers, Definitions, and the Proofs of Theorems

- Definition 2.8:
  - Let n be an integer. We call n even if n is divisible by 2, that is, if there exists an integer r so that $n = 2r$. If n is not even, then we call n odd and find for this case that there exists an integer s where $n = 2s + 1$.

- Theorem 2.2: For all integers k and l, if k and l are both odd, then $k + l$ is even.
  - Proof

  1. Since k and l are odd, we may write $k = 2a+1$ and $l = 2b+1$, for some integers a, b. *(the Rule of Universal Specification)*
  2. Then $k + l = (2a+1)+(2b+1) = 2(a+b+1)$, which hold for integers. (apply Commutative, Associative, and Distributive Laws)
  3. Since a, b are integers, $a+b+1 = c$ is an integer; with $k + l = 2c$, so $k + l$ is even.

Quantifiers, Definitions, and the Proofs of Theorems

- Theorem 2.3: For all integers k and l, if k and l are both odd, then kl is also odd.
  - Proof?

Quantifiers, Definitions, and the Proofs of Theorems

- Theorem 2.4: If m is an even integer, then $m + 7$ is odd.
  - Proof

1. Since m is even, we have $m = 2a$ for some integer a. Then $m + 7 = 2a + 7 = 2a + 6 + 1 = 2(a + 3) + 1$. since $a+3$ is an integer, we know that $m + 7$ is odd.
2. Suppose that $m + 7$ is not odd, hence even. Then $m + 7 = 2b$ for some integer b, and $m = 2b - 7 = 2b - 8 + 1 = 2(b - 4) + 1$, where $b - 4$ is an integer. Hence m is odd. (*contraposition method*)
3. Assume that m is even and $m + 7$ is also even. Then $m + 7$ even implies that $m + 7 = 2c$ for some integer c. Consequently, $m = 2c - 7 = 2c - 8 + 1 = 2(c - 4) + 1$ with $c - 4$ an integer, so m is odd. Now we have contradiction. So the assumption is false ($m + 7$ is even), and we have $m + 7$ odd. (*contradiction method*)

	Assumption	Result Derived
Contraposition	$\neg q(m)$	$\neg p(m)$
Contradiction	$p(m)$ and $\neg q(m)$	F_0

Proof

推論證明

推論證明有下列三種方式:

1. 正證法: 以$P \Rightarrow Q$的方式完成推論證明的過程, 稱之為正證法。
2. 反證法: 以$\sim Q \Rightarrow \sim P$的方式完成推論證明的過程, 稱之為反證法。
3. 矛盾證法: 以$\sim P \vee Q$的方式完成推論證明的過程, 稱之為矛盾證法。

Prove (p and ¬ q) is F_0 ➔ prove (¬p or q) is true

Quantifiers, Definitions, and the Proofs of Theorems

- Theorem 2.5:
  - For all positive real numbers x and y, if the product xy exceeds 25, then $x > 5$ or $y > 5$.
  - Proof: (*Contrapositive*)
    - Suppose that $0 < x \leq 5$ and $0 < y \leq 5$ $(\neg q(x, y))$
    - We find that $0 = 0 \cdot 0 < x \cdot y \leq 5 \cdot 5 = 25$ $(\neg p(x, y))$

Negation where?

- 糖果 / 我們 / 太多 / 牙醫 / 吃 / 告訴 / 不可以
- 牙醫告訴我們不可以吃太多糖果
- 糖果吃太多我們不可以告訴牙醫