Discrete Mathematics (2016 Spring) Midterm I solution

<table>
<tr><td>1.</td><td>
1.
True (1) If $17 \mid 3a+4b$ then $17 \mid 5a+b$
$\Rightarrow 17 \mid 3a+4b \xrightarrow{\times 4} 17 \mid 12a+16b$ & $17 \mid 17a+17b$
$\therefore 17 \mid 17a+17b-(12a+16b) \Rightarrow 17 \mid 5a+b$

False (2) $|A| = 4.$ $|B| = 7.$
from $A \to B$ (All): 7^4
one-to-one : $7\cdot6\cdot5\cdot4$ $\Rightarrow$ $p = \frac{\cancel{7}\cdot6\cdot5\cdot4}{\cancel{7}\cdot7\cdot7\cdot7} > \frac{1}{3}$

False (3) 10 ⑪ ↷ $\underset{A}{\sqcup}\ \underset{B}{\sqcup}\ \underset{C}{\sqcup}\ \underset{D}{\sqcup}\ \underset{E}{\sqcup}$ (5)
i.e. $A+B+C+D+E=10$
All: $H^5_{10} = C^{14}_{10} = 1001$, 僅一桶空: $C^5_1 \cdot H^4_6 = 5 \cdot C^9_6 = 420$
$P = 420/1001 < \frac{1}{2}$

True (4) If "I am lying" → true: then means someone is lying → someone tell the false (矛盾)
→ false: then means someone tell the true → someone is lying (矛盾)
∴ "I am lying" can't be either true or false

True (5) $f: \mathbb{Z} \to \mathbb{Z}$. $f(x) = 2x+1$ is one-to-one funtion
任一個 x 皆可得唯一的 $f(x)$，且任二 $f(x)$ 不相同.
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<tr><td>2.</td><td>$p \to (q \vee \neg p) = \neg p \vee (q \vee \neg p) = \neg p \vee q = \neg(p \wedge \neg q)$</td></tr>
<tr><td>3.</td><td>
解法 1
$$[(p \to q) \wedge (\neg r \vee s) \wedge (p \vee r)] \to (\neg q \to s).$$
Steps   Reasons
1) $\neg(\neg q \to s)$
2) $\neg q \wedge \neg s$
3) $\neg s$
4) $\neg r \vee s$
5) $\neg r$
6) $p \to q$
7) $\neg q$
8) $\neg p$
9) $p \vee r$
10) r
11) $\neg r \wedge r$
12) $\therefore \neg q \to s$
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解法 2

3、 $[(p \to q) \wedge (\neg r \vee s) \wedge (p \vee r)] \to (\neg q \to s)$

① $p \to q$ 由題目

② $\neg q \to \neg p$ 由②

③ $\neg p$ 由②

④ $p \vee r$ 由題目

⑤ $\neg r$ 由⑤

⑥ $\neg r \vee s$ 由題目

⑦ s 由⑥

⑧ $\neg p \to s$ 由③及⑦

⑨ $\neg q \to s$ 由⑧及②.

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<tr><td>4.</td><td>

(a) $\{\emptyset\}$ (b) $\emptyset$ (c) $\{a, \emptyset, \{\emptyset\}\}$

(d) $\{\emptyset, \{\emptyset\}, \{a\}, \{\{a\}\}, \{\emptyset, a\}, \{\emptyset, \{a\}\}, \{a, \{a\}\}, \{\emptyset, a, \{a\}\}\}$

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5.

解 1.

5.

以二項式定理

$$(x+1)^n = C_0^n x^0 + C_1^n x^1 + C_2^n x^2 \dots C_n^n x^n$$

$$\xrightarrow{\text{微分}} n(x+1)^{n-1} = 1\cdot C_1^n \cdot x^0 + 2\cdot C_2^n \cdot x + \dots n\cdot C_n^n \cdot x^{n-1}$$

$$\xrightarrow{\text{微分}} n\cdot(n-1)(x+1)^{n-2} = 2\cdot C_2^n + 3\cdot C_3^n x + \dots n(n-1)C_n^n$$

x代1 : $n\cdot(n-1)2^{n-2} = 0+0+2\cdot C_2^n + 3\times C_3^n \dots (n)(n-1)C_n^n$

$$= \sum_{k=0}^{n} k(k-1)\binom{n}{k} \quad \text{when } n \geq 0$$

解 2.

#5. ~~For~~ Since the summands at $k=0$ & $k=1$ are zero, we could rewrite the LHS as $\sum_{k=2}^{n} k(k-1)\binom{n}{k}$.

$$= \sum_{k=2}^{n} k(k-1)\frac{n!}{k!(n-k)!} = \sum_{k=2}^{n} \frac{n!}{(k-2)!(n-k)!} = n(n-1)\sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!(n-k)!}$$

$$= n(n-1)\sum_{k=2}^{n} \binom{n-2}{k-2} = n(n-1)\,2^{n-2}.$$

6.

(6)

(a) $\gcd(n, n+2) = w$.

$w \mid n \cdot x + (n+2) \cdot y,\ x, y \in \mathbb{Z}$.

當 $x=-1,\ y=1$ 時,

$w \mid -n + (n+2) \cdot 1$

$\Rightarrow w \mid 2$

$\therefore$ $w = 1$ or 2 $\Rightarrow \gcd(n, n+2) = 1$ or 2 得證

(b) $\gcd(n, n+k) = w$.

$w \mid n \cdot x + (n+k) \cdot y,\ x, y \in \mathbb{Z}$.

當 $x=-1,\ y=1$ 時

$w \mid n \cdot (-1) + (n+k) \cdot 1$

$\Rightarrow w \mid k$

$\therefore$ w 是 k 的因數

$\Rightarrow \gcd(n, n+k)$ 的所有可能為 k 的因數.

7.	⑦ $n=64 \Rightarrow 64 = 5\times 6 + 17\times 2$ $n=65 \Rightarrow 65 = 5\times 13$ $n=66 \Rightarrow 66 = 5\times 3 + 17\times 3$ $n=67 \Rightarrow 67 = 5\times 10 + 17$ $n=68 \Rightarrow 68 = 17\times 4$ $n=69 \Rightarrow 69 = (69-5)+5$ $\because$ $69-5=64$ 時成立 $\therefore$ $n=69$ 成立 假設 $n=64, 65, \ldots, k-2, k-1, k$ $(k\leq 64)$ 時成立 當 $n=k+1$ $k+1 = (k-4)+5$ $\because$ $64 \leq k-4 \leq k$ $\therefore$ $n=k-4$ 時成立 $\Rightarrow (k-4)+5$ 成立 $\Rightarrow n=k+1$ 時成立 由數學歸納法知，當 $n\geq 64$ 時，n可以由5,17所組成
8.	⑧ −0 (a) $\frac{5!}{2!1!2!}\cdot(2)^2\cdot(-1)\cdot(3)^2 = \frac{120}{\cancel{4}}\cdot 4\cdot(-1)\cdot 9$ $= -1080$ (b) $H^4_5 = C^8_5 = \frac{8\cdot 7\cdot \cancel{6}}{\cancel{6}} = 56$ (c) $(2-1+3+4)^5 = 8^5 = 32768$ (d) $(2x - x^2 + 3z^{-1} + 4)^5$ x^4z^{-1} ① $\frac{5!}{4!1!}(2)^4\cdot(3)^1 = 5\cdot 16\cdot 3 = 240$ ② $\frac{5!}{2!1!2!}(-1)^2\cdot 3\cdot 4^2 = \frac{120}{4}\cdot 1\cdot 3\cdot \overset{4}{\cancel{16}} = 1440$

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<tr><td></td><td>③
$\frac{5!}{2!1!1!1!}(2)^2\cdot(-1)\cdot3\cdot4$
$=\frac{120}{2}\cdot4\cdot(-1)\cdot3\cdot4$
$=-2880$
Total: $240+1440-2880$
$=240-1440=\underline{-1200}$ #
(e)
No.
在(d)小題，因為多了 x^2 方了 x 項，因此在所有的可能中，當x的次方介於 2~5 次項可以進行合併，所以總項數會減少。原本的(b)小題，x不能和x合併，故總項數較多</td></tr>
<tr><td>9.</td><td>9. $n\binom{n-1}{r}$ 表示從 n 個東西裡先選出一個，再從剩下的東西選取 r 個的可能數
$(r+1)\binom{n}{r+1}$ 表示從 n 個東西裡先選出 $r+1$ 個，再從 $r+1$ 個裡選出一個的可能數</td></tr>
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