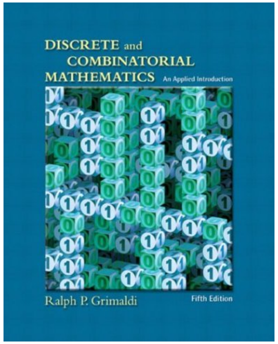
Discrete Mathematics
-- Chapter 6: Languages: Finite State Machine



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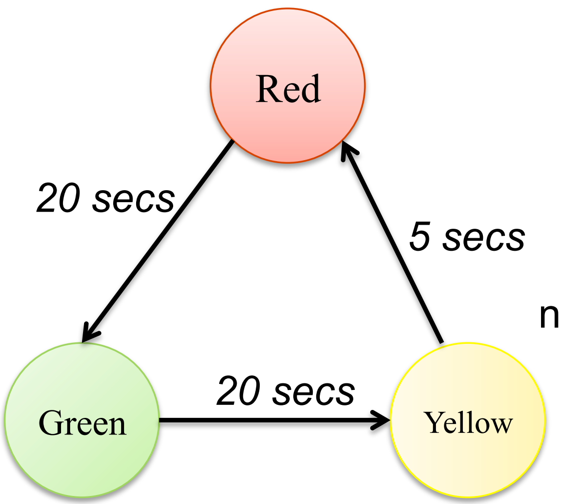
Outline

- 6.1 Language: The Set Theory of Strings
- 6.2 Finite State Machines: A First Encounter
- 6.3 Finite State Machines: A Second Encounter

有限狀態機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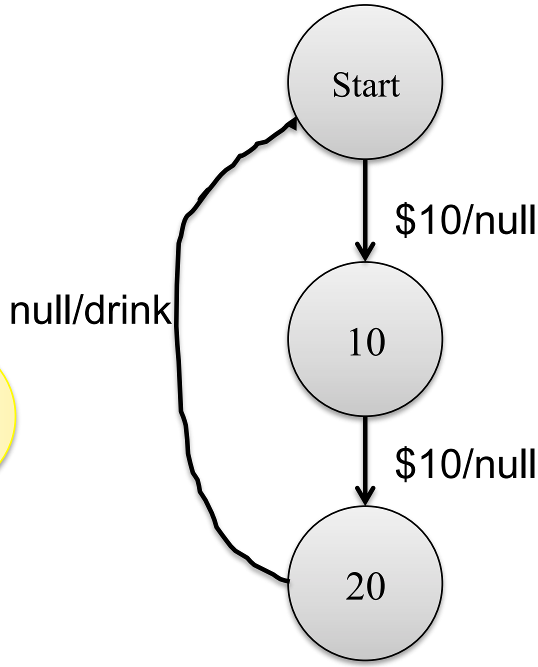
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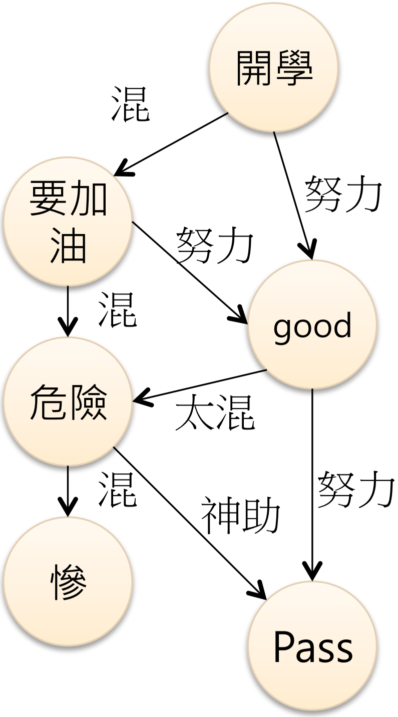
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Language: The Set Theory of Strings

- A finite state machine (FSM), which is an abstract model, has a finite number of internal states where the machine remembers certain information when it is in a particular state.
- Strings: Sequence of symbols (characters) play a key role in the processing of information by a computer.
- Σ denote a nonempty finite set of symbols, collectively called an alphabet. E.g., $\Sigma = \{0, 1\}$, $\Sigma = \{a, b, c, d, e\}$.
- Definition 6.1: If Σ is an alphabet and $n \in \mathbf{Z}^+$, we define the powers of Σ recursively as follows:

1) $\Sigma^1 = \Sigma$ 長度為1

2) $\Sigma^{n+1} = \{xy \mid x \in \Sigma, y \in \Sigma^n\}$, where xy denotes the juxtaposition of x and y

Language: The Set Theory of Strings

- **Ex 6.1** : Let $\sum$ an alphabet.

  If $n = 2, \sum^2 = \{xy \mid x \in \sum, y \in \sum\}$, e.g., $\sum = \{0,1\}, \sum^2 = \{00,01,10,11\}$

  $\sum = \{a,b,c,d,e\}, \sum^3$ would contain 5^3 three-symbol strings, e.g., aaa, acb, cdd, etc.

- Definition 6.2: For an alphabet $\sum$ we define $\sum^0 = \{\lambda\}$, where λ denotes the empty string, i.e., the string consisting of no symbols taken from $\sum$.

  (1) $\{\lambda\} \not\subseteq \sum$ since $\lambda \notin \sum$

  (2) $\{\lambda\} \neq \phi$ because $|\{\lambda\}| = 1 \neq 0 = |\phi|$

- Definition 6.3: If $\sum$ is an alphabet, then

  (1) $\sum^+ = \bigcup_{n=1}^{\infty} \sum^n = \bigcup_{n \in \mathbf{Z}^+} \sum^n$

  (2) $\sum^* = \bigcup_{n=0}^{\infty} \sum^n$

Language: The Set Theory of Strings

- **Ex 6.2** : Let $\sum = \{0,1\}$ the set $\sum^*$ consists of all finite strings of 0's and 1's together with the empty string. (how about $\sum^+$?)
- If $\sum = \{ß,0,1,\ldots,9,+,-,/,\}$, where ß denotes the blank (space). Here in $\sum^*$ we find familiar arithmetic expression such as (7+5)/(2-3).

- Definition 6.4:

  If $w_1, w_2 \in \sum^+$, $w_1 = x_1 x_2 \cdots x_m$, $w_2 = y_1 y_2 \cdots y_n$

  and $x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n \in \sum$,

  then we say w_1 and w_2 are equal $(w_1 = w_2)$ if $m = n$ and $x_i = y_i$ for all i.

Language: The Set Theory of Strings

- Definition 6.5:

  Let $w = x_1 x_2 \cdots x_n \in \sum^+$, where $x_i \in \sum$ for $1 \le i \le n$.

  The length of w is n, donated by $\| w \|$, and $\| \lambda \| = 0$.

- Definition 6.6:

  Let $x, y \in \sum^+$, $x = x_1 x_2 \cdots x_m$, $y = y_1 y_2 \cdots y_n$,

  and $x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n \in \sum$.

  The concatenation of x and y: $xy = x_1 x_2 \cdots x_m y_1 y_2 \cdots y_n$.

  串聯

  The concatenation of x and λ: $x\lambda = x, \lambda x = x$.

  The concatenation of λ and λ: $\lambda\lambda = \lambda$.

Language: The Set Theory of Strings

- Definition 6.7:

  $x \in \sum^*$, we define the powers of x by $x^0 = \lambda, x^1 = x, x^2 = xx,$

  $x^3 = xx^2, \cdots, x^{n+1} = xx^n, \cdots,$ where $n \in N$.

- Definition 6.8:

  If $x, y \in \sum^*$ and $w = xy$,

  then x is called a <u>prefix</u> of w, and if $y \neq \lambda$, then x is to be a proper prefix.

  Similarly, y is called a <u>suffix</u> of w, it is a proper suffix when $x \neq \lambda$.

- Definition 6.9:

  If $x, y, z \in \sum^*$ and $w = xyz$, then y is called a <u>substring</u> of w.

  When at least one of x and z is different from λ, we call y a proper substring.

Language: The Set Theory of Strings

- Definition 6.10:

  For a given alphabet $\sum$, any subset of $\sum^*$ is called a language over $\sum$.

  This includes the subset ϕ, which we call the empty language.

- Ex 6.9 :
  - With $\sum$ the alphabet of 26 letters, 10 digits, and the special symbols used in a given implementation of C++, the collection of executable programs for that implementation constitutes a language.
  - In the same situation, **each** executable program could be considered a language, as could a particular set of such programs.

- Since **languages are sets**, we can form the union, intersection, and symmetric difference of two languages.

Language: The Set Theory of Strings

- Definition 6.11:

  For an alphabet $\sum$ any languages $A, B \subseteq \sum^*$, the concatenation of A and B, denoted AB, is $\{ab \mid a \in A, b \in B\}$.

- **Ex 6.10** :

  無交換律，順序很重要

  Let $\sum = \{x, y, z\}$, and let A, B be the finite languages $A = \{x, xy, z\}, B = \{\lambda, y\}$.
  Then $AB = \{x, xy, z, xyy, zy\}$ and $BA = \{x, xy, z, yx, yxy, yz\}$

  1) $|AB| = 5 \neq 6 = |BA|$

  2) $|AB| = 5 \neq 6 = 3 \cdot 2 = |A||B|$

Language: The Set Theory of Strings

- Theorem 6.1:

  For an alphabet $\sum$, let $A, B, C \subseteq \sum^*$.

  a) $A\{\lambda\} = \{\lambda\}A = A$ b) $(AB)C = A(BC)$

  c) $A(B \cup C) = AB \cup AC$ d) $(B \cup C)A = BA \cup CA$

  e) $A(B \cap C) \subseteq AB \cap AC$ f) $(B \cap C)A \subseteq BA \cap CA$

  Proof :

  (f) For $x \in \sum^*$,

  $x \in (B \cap C)A \Rightarrow x = yz$ where $y \in B \cap C$ and $z \in A$

  $\Rightarrow (x = yz$ for $y \in B$ and $z \in A)$ and $(x = yz$ for $y \in C$ and $z \in A)$

  $\Rightarrow x \in BA$ and $x \in CA$

  $\Rightarrow x \in BA \cap CA$

  $\therefore (B \cap C)A \subseteq BA \cap CA$

  $B = \{x, xx, y\}, C = \{y, xy\}, A = \{y, yy\}$

  $xyy \in BA \cap CA, xyy \notin (B \cap C)A$

Language: The Set Theory of Strings

- Definition 6.12:

  For a given language $A \subseteq \sum^*$ we can construct other languages as follows:

  a) $A^0 = \{\lambda\}$, $A^1 = A$, and $n \in \mathbf{Z}^+$, $A^{n+1} = \{ab | a \in A, b \in A^n\}$

  b) $A^+ = \bigcup_{n \in \mathbf{Z}^+} A^n$, the positive closure of A.

  c) $A^* = A^+ \cup \{\lambda\}$. The language A * is called the Kleene closure of A.

  (in honor of the American logician Stephen Cole Kleene, 1909 - 1994)

- **Ex 6.11** :

  If $\sum = \{x, y, z\}$, and $A = \{x\}$, then (1) $A^0 = \{\lambda\}$; (2) $A^n = \{x^n\}, n \in N$;

  (3) $A^+ = \{x^n \mid n \geq 1\}$; and (4) $A^* = \{x^n \mid n \geq 0\}$.

Language: The Set Theory of Strings

- Lemma 6.1:

Let $\sum$ be an alphabet, with languages $A, B \subseteq \sum^*$. If $A \subseteq B$, then $A^n \subseteq B^n$

Proof : (i) $n = 1$, $A^1 = A \subseteq B = B^1$

(ii) Assuming the truth for $n = k$, $A \subseteq B \Rightarrow A^k \subseteq B^k$

(iii) If $x = x_1 x_k \in A^{k+1}$, i.e., $x_1 \in A, x_k \in A^k$.

$\because A \subseteq B \Rightarrow A^k \subseteq B^k$ (induction hypothesis), $\therefore x_1 \in B, x_k \in B^k$

$\Rightarrow x = x_1 x_k \in BB^k = B^{k+1}$

$\Rightarrow A^{k+1} \subseteq B^{k+1}$

Language: The Set Theory of Strings

- Theorem 6.2:

  For an alphabet $\sum$ and languages $A, B \subseteq \sum^*$,

  a) $A \subseteq AB^*$ b) $A \subseteq B^* A$

  c) $A \subseteq B \Rightarrow A^+ \subseteq B^+$ d) $A \subseteq B \Rightarrow A^* \subseteq B^*$

  e) $AA^* = A^* A = A^+$ f) $A^* A^* = A^* = (A^*)^* = (A^*)^+ = (A^+)^*$

  g) $(A \cup B)^* = (A^* \cup B^*)^* = (A^* B^*)^*$

  Proof :

  (g) $[(A \cup B)^* = (A^* \cup B^*)^*]$

  (i) $A \subseteq A^*, B \subseteq B^* \Rightarrow (A \cup B) \subseteq (A^* \cup B^*)$

  $\Rightarrow (A \cup B)^* \subseteq (A^* \cup B^*)^*$ [by (d)]

  (ii) $A, B \subseteq A \cup B \Rightarrow A^*, B^* \subseteq (A \cup B)^*$ [by (d)]

  $\Rightarrow (A^* \cup B^*) \subseteq (A \cup B)^*$

  $\Rightarrow (A^* \cup B^*)^* \subseteq ((A \cup B)^*)^* = (A \cup B)^*$ [by (d) and (f)]

Language: The Set Theory of Strings

- **Ex 6.14** :

For an alphabet $\sum = \{0,1\}$ consider the languages $A \subseteq \sum^*$, where each word in A contains exactly one occurrence of the symbol 0, e.g., 0, 01, 10, 0111, etc. We can define this language A recursively as follows:

1) Our base step tells us that $0 \in A$
2) For the recursive process we want to include in A the words $1x$ and $x1$, for $x \in A$.

Language: The Set Theory of Strings

- Definition (palindrome):

  Given an alphabet $\sum$, consider $x = x_1x_2 \cdots x_n$ in $\sum^*$.

  The reversal of x is denoted $x^R = x_n x_{n-1} \cdots x_1$.

  We can define the reversal of a string recursively as follows:

  1) $\lambda^R = \lambda$

  2) For $n \in \mathbf{N}$, if $x \in \sum^{n+1}$, then we can write $x = zy$ where $z \in \sum$ and $y \in \sum^n$

     --- here, we define $x^R = (zy)^R = (y^R)z$.

- **Ex 6.16** : Prove that $x_1, x_2 \in \sum^* \Rightarrow (x_1x_2)^R = x_2{}^R x_1{}^R$.

  Proof : By mathematical induction

  (i) $\| x_1 \| = 0$ $\quad x_1 = \lambda$ and $(x_1x_2)^R = (\lambda x_2)^R = x_2^R = x_2^R\lambda = x_2^R\lambda^R = x_2^R x_1^R$ because $\lambda^R = \lambda$

  (ii) $\| x_1 \| = k$,

  (iii) $\| x_1 \| = k+1, x_1 = zy_1, \| z \| = 1, \| y_1 \| = k$,

  $$(x_1x_2)^R = (zy_1x_2)^R = (y_1x_2)^R z = x_2{}^R y_1{}^R z = x_2{}^R (zy_1)^R = x_2{}^R x_1{}^R.$$

6.2 Finite State Machines: A First Encounter

- Example: A vending machine dispenses two flavors of chewing gum: peppermint (P) and spearmint (S).
  - The cost of either flavor is 20c. The machine accepts nickels; dimes, and quarters and returns the necessary change.
  - Mary Jo inserts two nickels and a dime, and press the white button (W) for a package of peppermint-flavored chewing gum.

	t_0	t_1	t_2	t_3	t_4
State	(1) s_0	(4) s_1 (5c)	(7) s_2 (10c)	(10) s_3 (20c)	(13) s_0
Input	(2) 5c	(5) 5c	(8) 10c	(11) W	
Output	(3) Nothing	(6) Nothing	(9) Nothing	(12) P	

Finite State Machines: A First Encounter

- Definition 6.13

  A finite state machine is five-tuple $M = (S, I, O, v, w)$, where

  S = the set of internal states for M;

  I = the input alphabet for M;

  O = the output alphabet for M;

  $v : S \times I \to S$ is the next state function;

  $w : S \times I \to O$ is the output function;

	v		w	
	0	1	0	1
s_0	s_0	s_1	0	0
s_1	s_2	s_1	0	0
s_2	s_0	s_1	0	1

- **Ex 6.17**

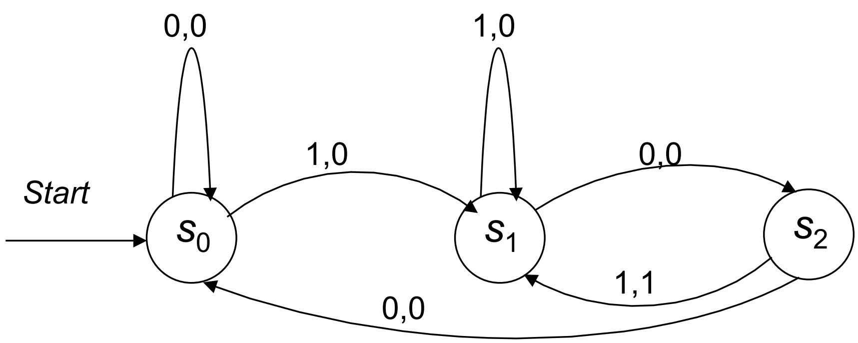
  Consider the finite state machine $M = (S, I, O, v, w)$, where

  $S = \{s_0, s_1, s_2\}, I = O = \{0,1\}$, and v, w are given by the state table.

Finite State Machines: A First Encounter

- Another representation for finite state machine
  - State diagram
- What is the output string for the input string 1100101101?
  - Input (output) string is an element of I^* (O^*), the Kleene closure of I (O).

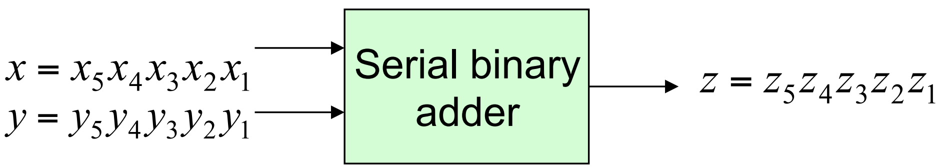
State	s_0	$v(s_0, 1) = s_1$	$v(s_1, 0) = s_2$	$v(s_2, 1) = s_1$	$v(s_1, 0) = s_2$
Input	1	0	1	0	
Output	$w(s_0, 1) = 0$	$w(s_1, 1) = 0$	$w(s_2, 1) = 1$	$w(s_1, 0) = 0$	



	v		w	
	0	1	0	1
s_0	s_0	s_1	0	0
s_1	s_2	s_1	0	0
s_2	s_0	s_1	0	1

Finite State Machines: A First Encounter

- **Ex 6.19** : A serial binary adder is a finite state machine that we can use to obtain $x + y$.
  - E.g., $x = x_5x_4x_3x_2x_1 = 00111, y = y_5y_4y_3y_2y_1 = 01101$

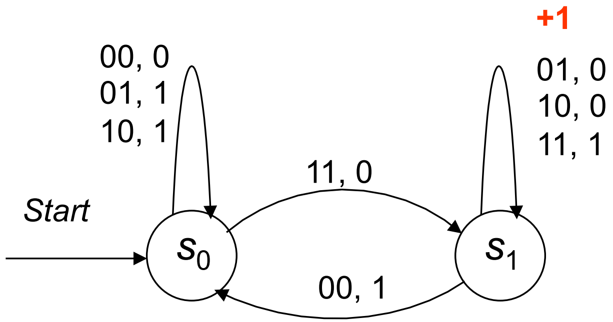


$$
\begin{array}{rcccccc}
 & & +1 & +1 & +1 & +1 & \\
x & = & 0 & 0 & 1 & 1 & 1 \\
+\ y & = & 0 & 1 & 1 & 0 & 1 \\
\hline
z & = & 1 & 0 & 1 & 0 & 0 \\
 & & & & \uparrow & & \uparrow \\
 & & & & \text{third addition} & & \text{first addition}
\end{array}
$$

Finite State Machines: A First Encounter

- The serial binary adder is modeled by a finite state machine $M = (S, I, O, v, w)$. $S = \{s_0, s_1\}$, where s_i indicates a carry of i; $I = \{00, 01, 10, 11\}$; $O = \{0, 1\}$; and v, w are given in the following state table.

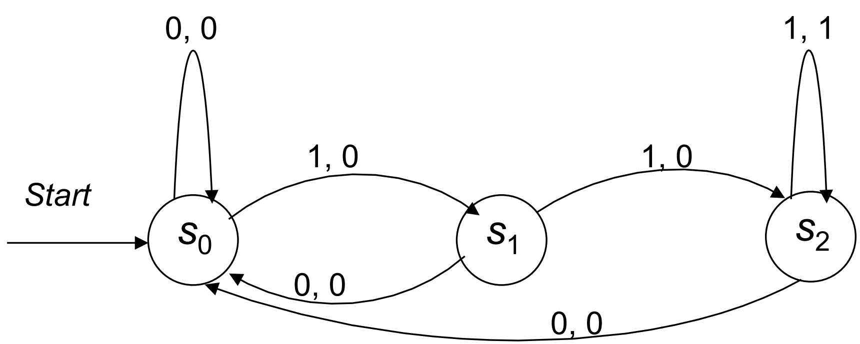
	v				w			
	00	01	10	11	00	01	10	11
s_0	s_0	s_0	s_0	s_1	0	1	1	0
s_1	s_0	s_1	s_1	s_1	1	0	0	1



6.3 Finite State Machines: A Second Encounter

- Some additional machines relevant to the design of computer hardware, e.g., <u>sequence recognizer</u>.
- <u>Ex 6.20</u> : Construct a machine that recognizes each occurrence of the sequence 111.
  - Input: 1110101111, output: 0010000011

$\{0, 1\}^*\{111\}$



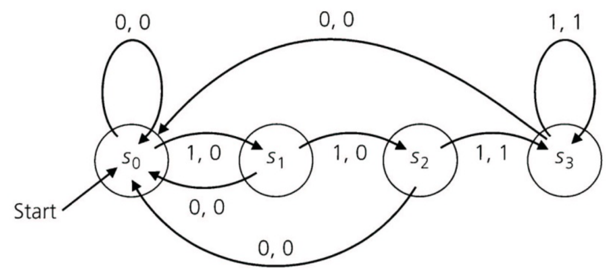
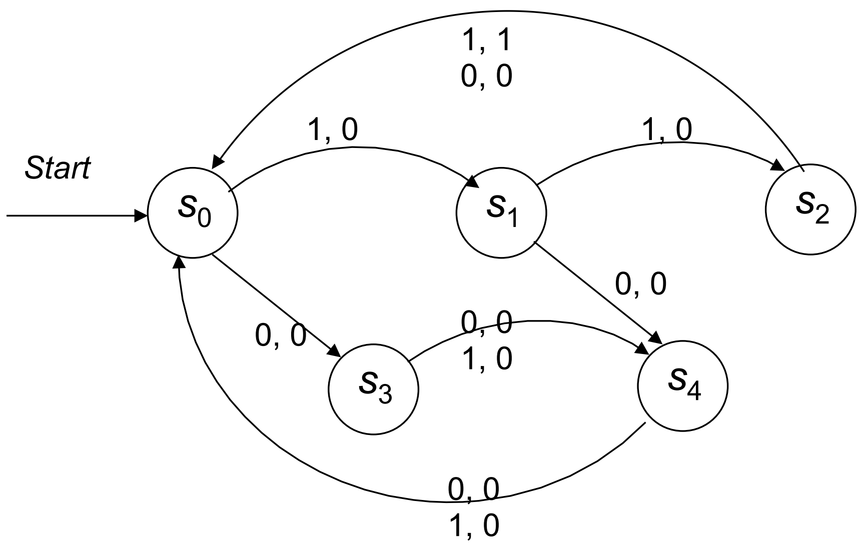


Figure 6.10

equivalent

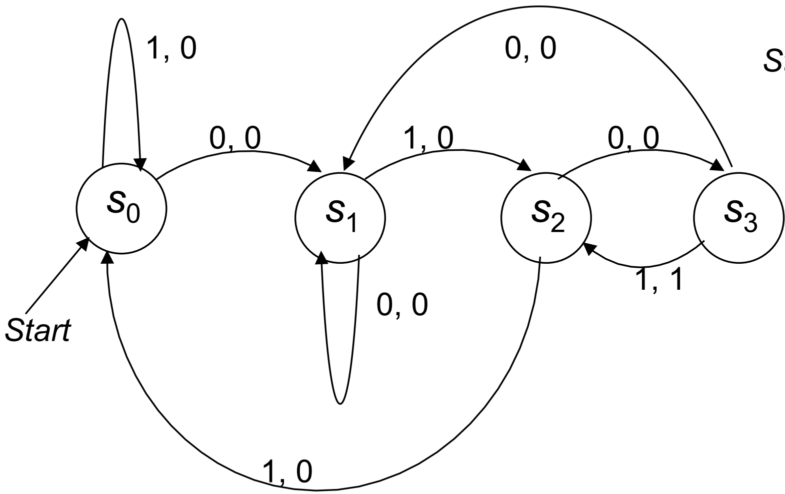
Finite State Machines: A Second Encounter

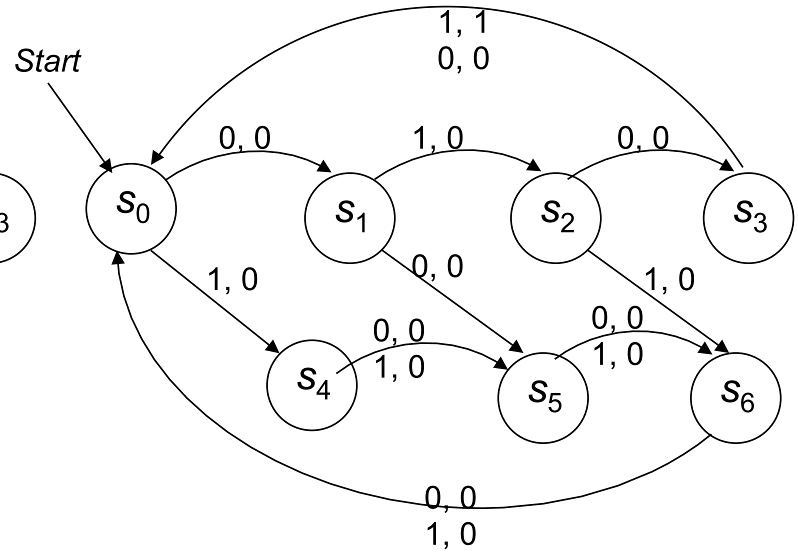
- **Ex 6.21** : Construct a machine that not only recognizes the occurrence of 111, but also recognizes those occurrences that end in a position that is a **multiple of three**.
  - Input: 1110111, output: 0010000, not 0010001
  - Input: 111100111, output: 001000001



Finite State Machines: A Second Encounter

- **Ex 6.22**: Construct a machine that not only recognizes (a) the occurrence of 0101, (b) those occurrences that end in a position that is a multiple of four.
  - Input: 01010100101, output: (a) 00010100001 (b)00010000000





Finite State Machines: A Second Encounter

- **Ex 6.23** : Can we construct a finite state machine that recognizes precisely those strings in the language $A = \{01, 0011, 000111, \ldots\} = \{0^i 1^i \mid i \in \mathbf{Z}^+\}$?
  - Apply the pigeonhole principle

State	s_0	s_1	s_2		s_i	s_{i+1}		s_j	s_{j+1}		s_n	s_{n+1}		s_{2n}	s_{2n+1}
Input	0	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	1	$\cdots$	1	1
Output	0	0	0		0	0		0	0		0	1		1	1

State	s_0	s_1	s_2		s_i	s_{j+1}		s_n	s_{n+1}		s_{2n}	s_{2n+1}
Input	0	0	0	$\cdots$	0	0	$\cdots$	0	1	$\cdots$	1	1
Output	0	0	0		0	0		0	1		1	1

Finite State Machines: A Second Encounter

- **Proof**

$M = \{S, I, O, v, w\}$ can recognize strings in A, let $|S| = n \geq 1$.

Consider $0^{n+1}1^{n+1}$ in the language A

$\Rightarrow M$ will process $\underline{n+1}$ 0's by $n+1$ states $s_0, s_1, \cdots, s_n$

$\because |S| = n$, by pigeonhole principle, exists two states $s_i = s_j$

$\therefore$ We can remove $j - i$ columns (see the tables)

$\Rightarrow M$ recognizes $x = 0^{(n+1)-(j-i)}1^{n+1}$, where $(n+1) - (j-i) < n+1$

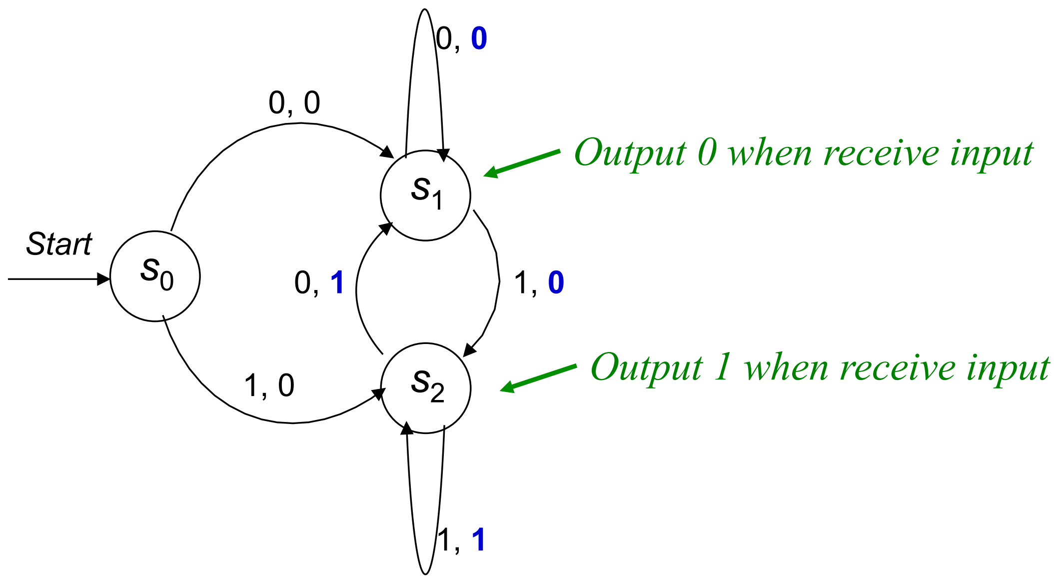
In fact, $x \notin A \Rightarrow M$ cannot recognize x

$\therefore$ contradict

$\therefore$ We cannot construct a finite state machine recognizing strings in A.

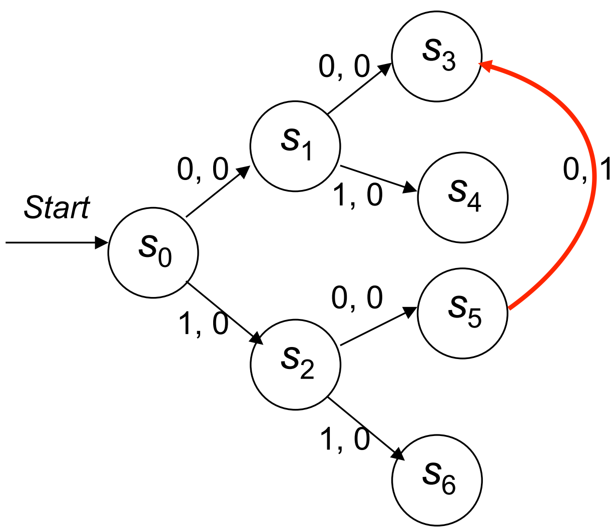
Finite State Machines: A Second Encounter

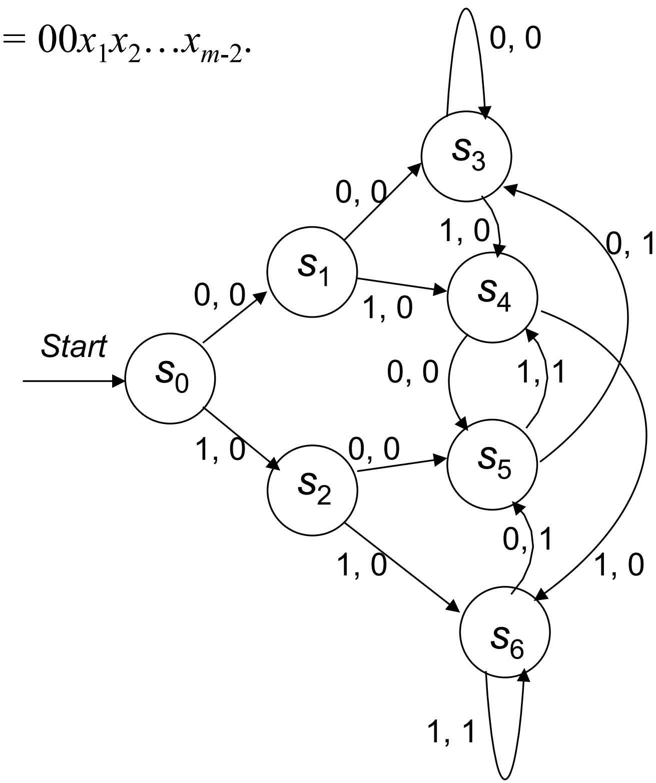
- One-unit delay machine
  - **Ex 6.24** : If $x = x_1x_2\ldots x_m$, then $w(s_0, x) = 0x_1x_2\ldots x_{m-1}$.



Finite State Machines: A Second Encounter

- Two-unit delay machine
  - **Ex 6.25** : If $x = x_1x_2\ldots x_m$, then $w(s_0, x) = 00x_1x_2\ldots x_{m-2}$.





Finite State Machines: A Second Encounter

- Definition 6.14

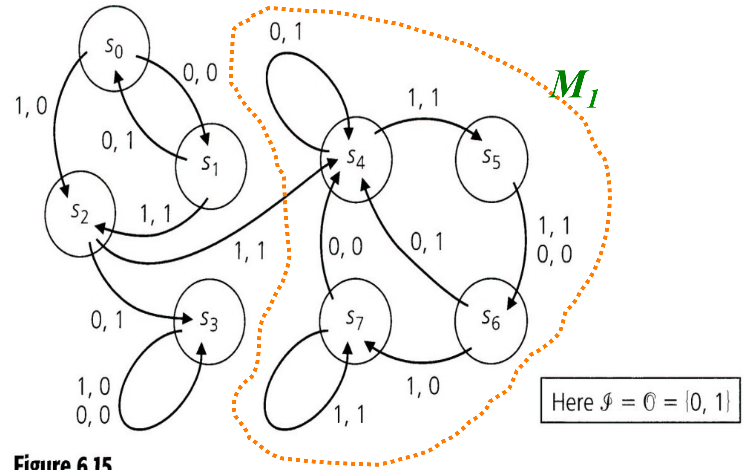


Figure 6.15

$M = \{S, I, O, v, w\}$ is a finite state machine

a) $s_i, s_j \in S$, s_j is *reachable* from s_i

if $s_i = s_j$ or $v(s_i, x) = s_j$ (e.g., s_3 is reachable from s_0, s_1, s_2 in the Fig.)

b) s is *transient* if $v(s, x) = s$ for $x \in I^*$ implies $x = \lambda$ (e.g., s_2 in the Fig.) (出去回不來)

c) s is a sink if $v(s, x) = s$ for all $x \in I^*$ (e.g., s_3 in the Fig.)

d) $S_1 \subseteq S, I_1 \subseteq I$, if $v_1 = v|_{S_1 \times I_1} : S_1 \times I_1 \to S$,

$w_1 = w|_{S_1 \times I_1}, M_1 = \{S_1, I_1, O, v_1, w_1\}$ is a submachine of M

e) Strongly connected if for any $s_i, s_j \in S$, s_j is reachable from s_i

Finite State Machines: A Second Encounter

- Definition 6.15

  For a finite state machine M, let s_i, s_j be two distinct states.

  $x \in I^+$ is called a *transfer (transition) sequence* from s_i to s_j if

  a) $v(s_i, x) = s_j$

  b) $y \in I^+$ with $v(s_i, y) = s_j \Rightarrow \| y \| \geq \| x \|$

- **Ex 6.26** : Find a transfer sequence from s_0 to s_2 in the state table.
  - $x = 0000$

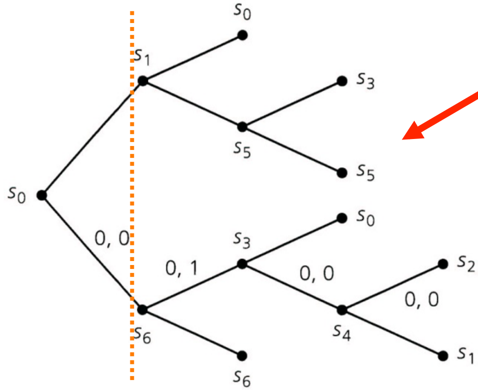
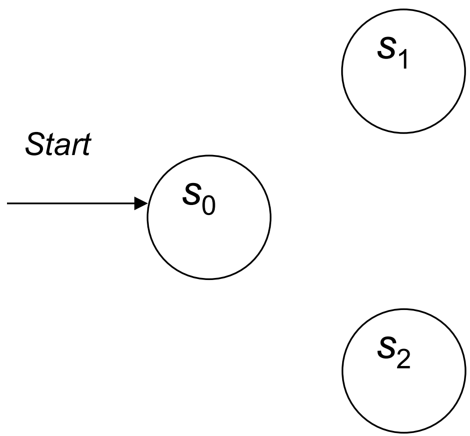


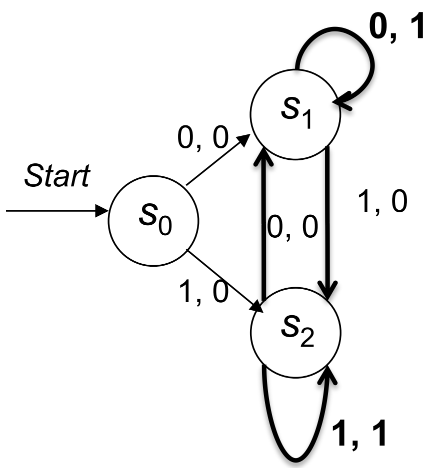
Figure 6.16

	v		w	
	0	1	0	1
s_0	s_6	s_1	0	1
s_1	s_5	s_0	0	1
s_2	s_1	s_2	0	1
s_3	s_4	s_0	0	1
s_4	s_2	s_1	0	1
s_5	s_3	s_5	1	1
s_6	s_3	s_6	1	1

Practice (6.3-3)

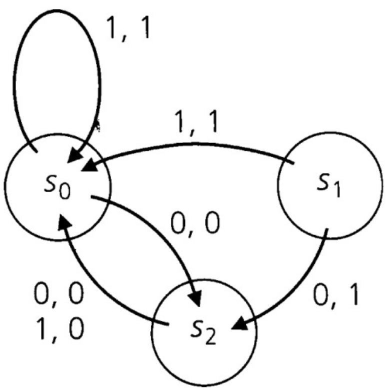
- Construct a state diagram for a finite state machine with I=O={0, 1} that recognizes all strings in the language {0, 1}*{00}U{0, 1}*{11}





Practice (Sppl.-5)

- Let M be the finite state machine in the following figure. For states S_i, S_j , where , let $\boldsymbol{O_{ij}}$ denote the set of all nonempty output strings that M can produce as it goes from state S_i, to state S_j, e.g., $\boldsymbol{O_{20}} = \{0\}\{1, 00\}^*$. Find $\boldsymbol{O_{02}}$, $\boldsymbol{O_{22}}$, $\boldsymbol{O_{11}}$, and $\boldsymbol{O_{10}}$.



$O_{00} = \{1, 00\}^*-\{\lambda\}$

$O_{02} = \{1, 00\}^*\{0\}$
$O_{22} = \{0\}\{1, 00\}^*\{0\}$
$O_{11} = \Phi$
$O_{10} = \{1\}\{1, 00\}^* \cup \{10\} \{1, 00\}^*$