Homework 7

6-1 50

For $0 \le t \le 10, b(t) > d(t)$, so the area between the curves is given by

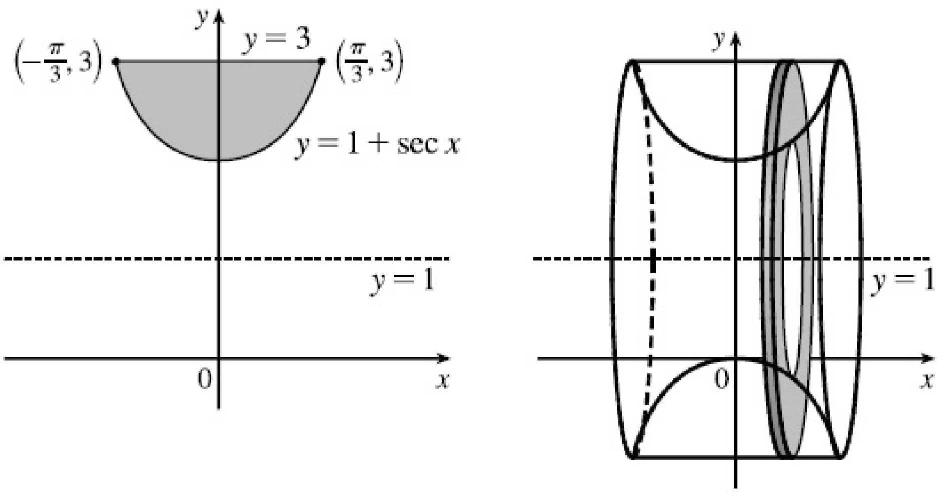
$$\int_0^{10} [b(t) - d(t)]\, dt = \int_0^{10} (2200e^{0.0024t} - 1460e^{0.018t})dt$$
$$= \left[\frac{2200}{0.024}e^{0.024t} - \frac{1460}{0.018}e^{0.018t}\right]_0^{10} \approx 8868 \,\text{people}$$

This area A represents the increase in population over a 10-year period.

6-2 13

A cross-section is a washer with inner radius $(1+\sec x)-1 = \sec x$ and outer radius $3-1 = 2$, so its area is $A(x) = \pi[2^2 - (\sec x)^2] = \pi(4 - \sec^2 x)$.

$$V = \int_{-\pi/3}^{\pi/3} A(x)\, dx = \int_{-\pi/3}^{\pi/3} \pi(4 - \sec^2 x)dx$$
$$= 2\pi \int_0^{\pi/3} (4 - \sec^2 x)dx \quad [bysymmetry]$$
$$= 2\pi \left[4x - \tan x\right]_0^{\pi/3} = 2\pi(\frac{4\pi}{3} - \sqrt{3})$$



6-2 42

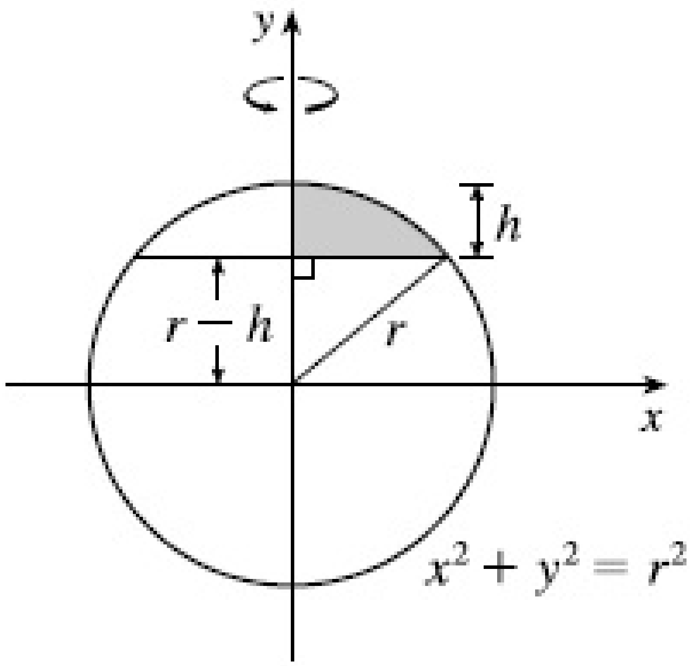
$\pi \displaystyle\int_1^4 \left[3^2 - (3 - \sqrt{x})^2\right] dx$ describes the volume of the solid obtained by rotating the region $\mathcal{R} = \{(x, y)|1 \le x \le 4,\ 3 - \sqrt{x} \le y \le 3\}$ of the xy-plane about the x-axis.

6-2 49

$x^2 + y^2 = r^2 \Leftrightarrow x^2 = r^2 - y^2$

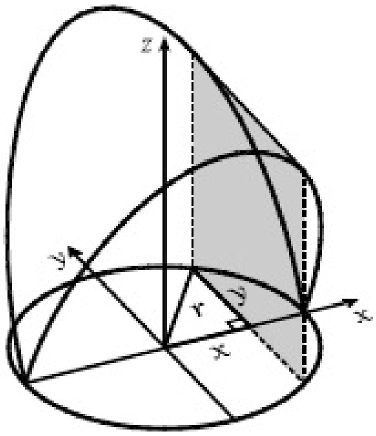
$$\begin{aligned} V &= \pi \int_{r-h}^{r} (r^2 - y^2)dy = \pi \left[r^2 y - \frac{y^3}{3} \right]_{r-h}^{r} \\ &= \pi \left\{ \left(r^3 - \frac{r^3}{3} \right) - \left[r^2(r-h) - \frac{(r-h)^3}{3} \right] \right\} \\ &= \pi \left\{ \frac{2}{3}r^3 - \frac{1}{3}(r-h)\left[3r^2 - (r-h)^2\right] \right\} \\ &= \frac{1}{3}\pi\{2r^3 - (r-h)[3r^2 - (r^2 - 2rh + h^2)]\} \\ &= \frac{1}{3}\pi\{2r^3 - (r-h)(2r^2 + 2rh - h^2)\} \\ &= \frac{1}{3}\pi(2r^3 - 2r^3 - 2r^2h + rh^2 + 2r^2h + 2rh^2 - h^3) \\ &= \frac{1}{3}\pi(3rh^2 - h^3) = \frac{1}{3}\pi h^2(3r - h) \end{aligned}$$

or, equivalently, $\pi h^2(r - \frac{h}{3})$



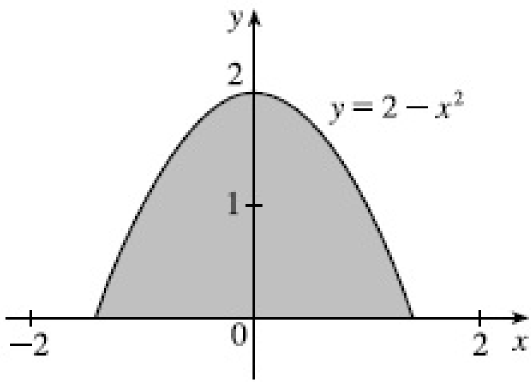
6-2 54

A cross-section is shaded in the diagram. $A(x) = (2y)^2 = (\sqrt{r^2 - x^2})^2$, so
$V = \displaystyle\int_{-r}^{r} A(x)dx = 2\int_0^r 4(r^2 - x^2)dx = 8\left[r^2x - \frac{1}{3}x^3\right]_0^r = \frac{16}{3}r^3$



6-2 60

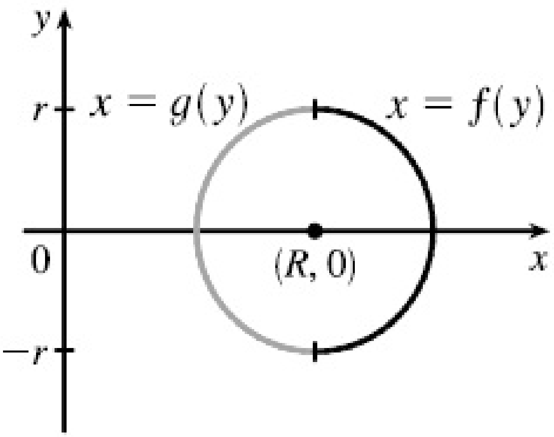
The cross-section of the base corresponding to the coordinate y has length $2x = 2\sqrt{2-y}$.
$[y = 2 - x^2 \Leftrightarrow x = \pm\sqrt{2-y}]$
The corresponding cross-section of the solid S is a quarter-circle with radius $2\sqrt{2-y}$ and area $A(y) = \frac{1}{4}\pi(2\sqrt{2-y})^2 = \pi(2-y)$. Therefore,
$V = \displaystyle\int_0^2 A(y)dy = \int_0^2 \pi(2-y)dy = \pi\left[2y - \frac{1}{2}y^2\right]_0^2 = 2\pi$



6-2 63

(a) The torus is obtained by rotating the circle $(x-R)^2+y^2=r^2$ about the y-axis. Solving for x, we see that the right half of the circle is given by $x=R+\sqrt{r^2-y^2}=f(y)$ and the left half by $x=R-\sqrt{r^2-y^2}=g(y)$. So

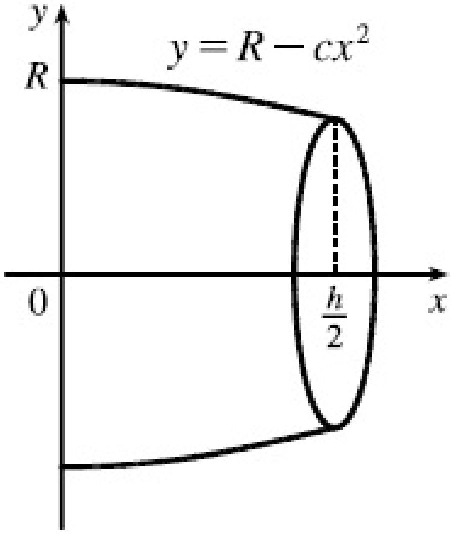
$$\begin{aligned} V &= \pi\int_{-r}^{r}\{[f(y)]^2-[g(y)]^2\}dy \\ &= 2\pi\int_0^r\left[(R^2+2R\sqrt{r^2-y^2}+r^2-y^2)-(R^2-2R\sqrt{r^2-y^2}+r^2-y^2)\right]dy \\ &= 2\pi\int_0^r 4R\sqrt{r^2-y^2}dy = 8\pi R\int_0^r\sqrt{r^2-y^2}dy \end{aligned}$$



(b) Observe that the integral represents a quarter of the area of a circle with radius r, so $8\pi R\int_0^r\sqrt{r^2-y^2}dy = 8\pi R\cdot\frac{1}{4}\pi r^2 = 2\pi^2r^2R$.

6-2 71

(a) The radius of the barrel is the same at each end by symmetry, since the function $y = R - cx^2$ is even. Since the barrel is obtained by rotating the graph of the function y about the x-axis, this radius is equal to the value of y at $x = \frac{1}{2}h$, which is $R - c(\frac{1}{2}h)^2 = R - d = r$.



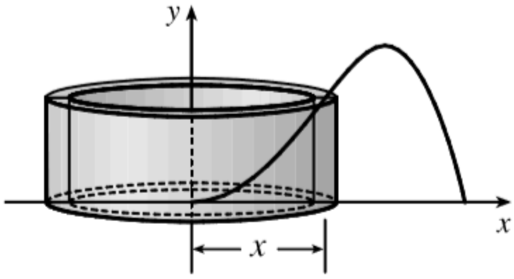
(b) The barrel is symmetric about the y-axis, so its volume is twice the volume of that part of the barrel for $x > 0$. Also, the barrel is a volume of rotation, so

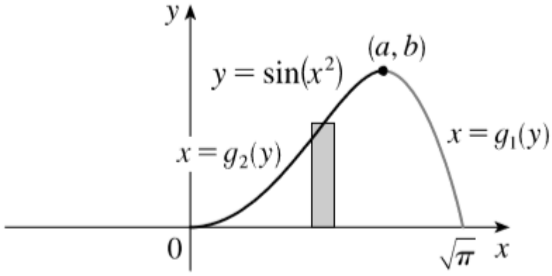
$$V = 2\int_0^{h/2} \pi y^2\, dx = 2\pi \int_0^{h/2} (R - cx^2)^2 dx = 2\pi \left[R^2 x - \frac{2}{3}Rcx^3 + \frac{1}{5}c^2x^5\right]_0^{h/2}$$
$$= 2\pi(\frac{1}{2}R^2h - \frac{1}{12}Rch^3 + \frac{1}{160}c^2h^5)$$

Trying to make this look more like the expression we want, we rewrite it as $V = \frac{1}{3}\pi h[2R^2 + (R^2 - \frac{1}{2}Rch^2 + \frac{3}{80}c^2h^4)]$. But $R^2 - \frac{1}{2}Rch^2 + \frac{3}{80}c^2h^4 = (R - \frac{1}{4}ch^2)^2 - \frac{1}{40}c^2h^4 = (R-d)^2 - \frac{2}{5}(\frac{1}{4}ch^2)^2 = r^2 - \frac{2}{5}d^2$. Substituting this back into V, we see that $V = \frac{1}{3}\pi h(2R^2 + r^2 - \frac{2}{5}d^2)$, as required.

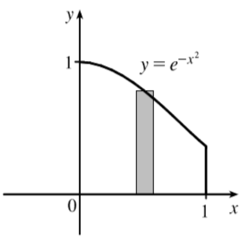
6-3 2

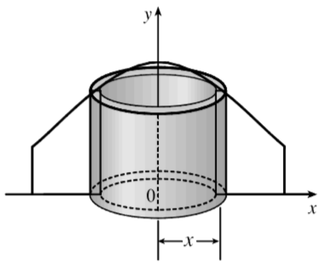
A typical cylindrical shell has circumference $2\pi x$ and height $\sin(x^2)$. $V = \int_0^{\sqrt{\pi}} 2\pi x \sin(x^2)\, dx$. Let $u = x^2$. Then $du = 2xdx$, so $V = \pi \int_0^{\pi} \sin u\, du = \pi\big[-\cos u\big]_0^{\pi} = 2\pi$. For slicing, we would first have to locate the local maximum point (a, b) of $y = \sin(x^2)$ using the methods of Chapter 4. Then we would have to solve the equation $y = \sin(x^2)$ for $x$in terms of $y$to obtain the functions $x = g_1(y)$ and $x = g_2(y)$ shown in the second figure. Finally we would find the volume using $V = \pi \int_0^b [g_1(y)]^2 - g_2(y)]^2 dy$. Using shells is definitely preferable to slicing.





6-3 5
$V = \int_0^1 2\pi x e^{-x^2}\,dx$. Let $u = x^2$. Thus, $du = 2x dx$, so
$V = \pi\int_0^1 e^{-u}\,du = \pi\big[-e^{-u}\big]_0^1 = \pi(1 - 1/e)$.

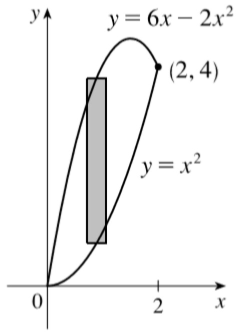


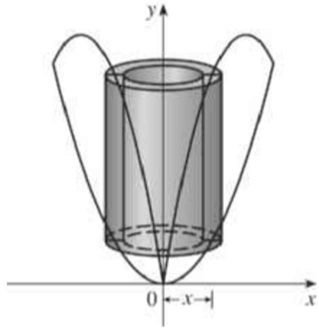


6-3 7
$x^2 = 6x - 2x^2 \Leftrightarrow 3x(x-2) = 0 \Leftrightarrow x = 0$ or 2.
$V = \int_0^2 2\pi x[(6x - 2x^2) - x^2]\,dx = 2\pi\int_0^2 (-3x^3 + 6x^2)\,dx$
$= 2\pi\big[-\frac{3}{4}x^4 + 2x^3\big]_0^2 = 8\pi$

Problem Plus 2

The total area of the region bounded by the parabola $y = x - x^2 = x(1-x)$ and the x-axis is $\int_0^1 (x - x^2)\,dx = \big[\frac{1}{2}x^2 - \frac{1}{3}x^3\big]_0^1 = \frac{1}{6}$. Let the slope of the line we are looking for be m. Then the area above this line but below the parabola is $\int_0^a (x - x^2) - mx\,dx$, where a is the x-coordinate of the point of the intersection of the line and the parabola. We find the point

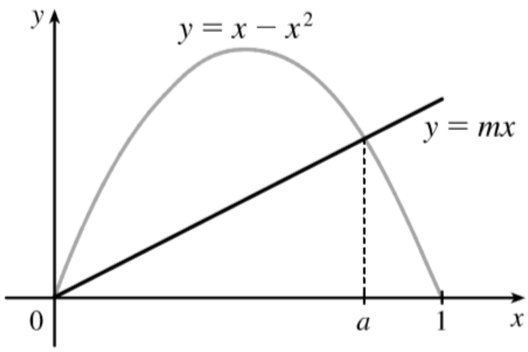




of intersection by solving the equation $x - x^2 = mx \Leftrightarrow 1 - x = m \Leftrightarrow x = 1 - m$. So the value of a is $1 - m$, and

$$\int_0^{1-m} (x - x^2) - mx\,dx = \int_0^{1-m} ((1-m)x - x^2) - mx\,dx = \Big[\frac{1}{2}(1-m)x^2 - \frac{1}{3}x^3\Big]_0^{1-m}$$

$= \frac{1}{6}(1-m)^3$. We want this to be half of $\frac{1}{6}$, so $\frac{1}{6}(1-m^3) = \frac{1}{12} \Leftrightarrow m = 1 - \frac{1}{\sqrt[3]{2}}$.So the slope of the required line is $1 - \frac{1}{\sqrt[3]{2}} \approx 0.206$.



Problem Plus 3

Let a and b be the x-coordinates of the points where the line intersects the curve. From the figure, $R_1 = R_2 \Rightarrow$

$\int_0^a c - ([8x - 27x^3)]\,dx = \int_a^b [(8x - 27x^3) - c]\,dx$

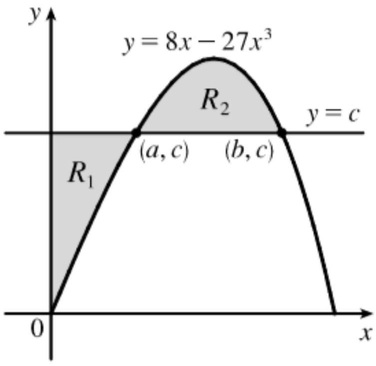
$\left[cx - 4x^2 + \frac{27}{4}x^4\right]_0^a = \left[4x^2 - \frac{27}{4}x^4 - cx\right]$

$0 = b^2(\frac{81}{4}b^2 - 4)$

So for $b > 0$, $b^2 = \frac{16}{81} \Rightarrow b = \frac{4}{9}$. Thus, $c = 8b - 27b^3 = \frac{32}{27}$

Problem Plus 7

We are given that the rate of change of the volume of water is $\frac{dV}{dt} = -kA(x)$, where k is some positive constant and A(x) is the area of the surface when the water has depth



x. Now we are concerned with the rate of change of the depth of the water repect to time, that is, $\frac{dx}{dt}$. But by the chain rule, $\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt}$, so the first equation can be written $\frac{dV}{dx}\frac{dx}{dt} = -kA(x)(*)$. Also, we know that the total volume of water up to a depth x is $V(x) = \int_0^x A(s)ds$, where A(s) is the area of a cross-section of the water at a depth s. Differentiating this equation with respect to x, we get $dV/dx = A(x)$. Substituting this into equation $*$, we get $A(x)(dx/dt) = -kA(x) \Rightarrow dx/dt = -k$, a constant.

Problem Plus 9

We must find expressions for the areas A and B, and then set them equal and see what this says about the curve C. If $P = (a, 2a^2)$, then area A is just $\int_0^a (2x^2 - x^2)dx = \frac{1}{3}a^3$. To find area B, we use y as the variable of integration. So we find the equation of the middle curve as a function of $y : y = 2x^2 \Leftrightarrow x = \sqrt{y/2}$, since we are concerned with the first quadrant only. We can express area B as

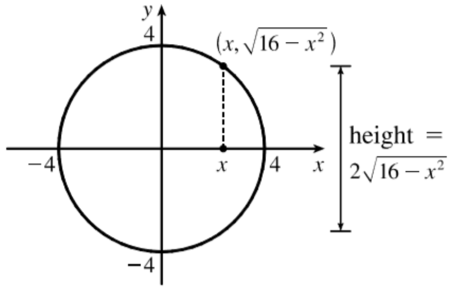
$$\int_0^{2a^2} [\sqrt{y/2} - C(y)]\, dy = \big[\frac{4}{3}(y/2)^{3/2}\big]_0^{2a^2} - \int_0^{2a^2} C(y)\, dy = \frac{4}{3}a^3 - \int_0^{2a^2} C(y)\, dy$$

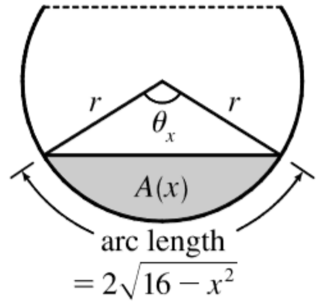
where C(y) is the function with graph C. Setting $A = B$, we get $\frac{1}{3}a^3 = \frac{4}{3}a^3 - \int_0^{2a^2} C(y)\, dy \Leftrightarrow \int_0^{2a^2} C(y)\, dy = a^3$. Now we differentiate this equation with respect to a using the Chian Rule and the Fundamental Thorem: $C(2a^2)(4a) = 3a^2 \Rightarrow C(y) = \frac{3}{4}\sqrt{y/2}$, where $y = 2a^2$. Now we can solve for $y : x = \frac{3}{4}\sqrt{y/2} \Rightarrow x^2 = \frac{9}{16}(y/2) \Rightarrow y = \frac{32}{9}x^2$.

Problem Plus 14

(a) Place the round flat torilla on an xy-coordinate system as shown in the first figure. An equation of circle is $x^2 + y^2 = 4^2$ and the height of a cross-sections is $2\sqrt{16 - x^2}$.
Now look at a cross-section with central angle θ_x as shown in the second figure (r is the radius of the circular cylinder). The filled area A(x) is equal to the area $A_1(x)$ of the sector minus the area $A_2(x)$ of the triangle.

$A(x) = A_1(x) - A_2(x) = \frac{1}{2}r^2\theta_x - \frac{1}{2}r^2\sin\theta_x = \frac{1}{2}r(r\theta_x) - \frac{1}{2}r^2\sin(\frac{s}{r}) = r\sqrt{16 - x^2} - \frac{1}{2}r^2\sin(\frac{2}{r}\sqrt{16 - x^2})(*)$.





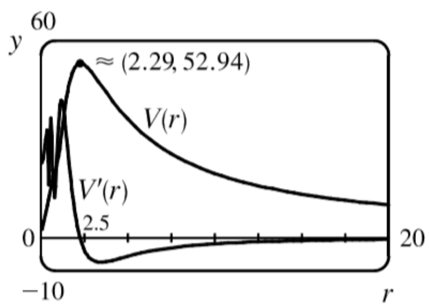
Note that the central angle θ_x will be small near the ends of tortilla; that is, when $|x| \approx 4$. But near the center of the torilla (when $|x| \approx 0$), the central angle θ_x may exceed $180°$. Thus, the sine of θ_x will be negative and the second term in $(*)$ will be positive (actually addind area to the area of the sector). The volume of the taco can be found by integrating the cross-sectional areas from $x = -4$ to $x = 4$. Thus,

$$V(x) = \int_{-4}^{4} A(x)dx = \int_{-4}^{4} [r\sqrt{16 - x^2} - \frac{1}{2}r^2 \sin(\frac{2}{r}\sqrt{16 - x^2})]\, dx$$

(b) To find the value of r that maximizes the volume of the taco, we can define the function

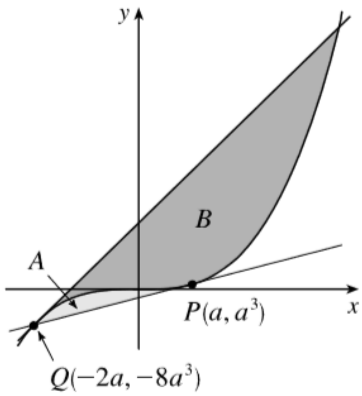
$$V(r) = \int_{-4}^{4} [r\sqrt{16 - x^2} - \frac{1}{2}r^2 \sin(\frac{2}{r}\sqrt{16 - x^2})]\, dx$$

The figure shows a graph of $y = V(r)$ and $y = V'(r)$. The maximum volume of about 52.94 occurs when $r \approx 2.2912$.



Problem Plus 15

We assume that P lies in the region of positive x. Since $y = x^3$ is an odd function, this assumption will not effect the result of the calculation. Let $P = (a, a^3)$. The slope of the tangent to the curve $y = x^3$ at P is $3a^2$, and so the equation of the tangent is $y - a^3 = 3a^2(x-a) \Leftrightarrow y = 3a^2x-2a^3$. We solve this simultaneously wiyh $y = x^3$ to find the other point of intersection: $x^3 = 3a^2x - 2a^3 \Leftrightarrow (x-a)^2(x+2a) = 0$. So $Q = (-2a, -8a^3)$ is the other point of intersection. The equation of the tangent at Q is $y-(-8a^3) = 12a^2[x-(-2a)] \Leftrightarrow y = 12a^2x + 16a^3$. By symmetry, this tangent will intersect the curve again at $x = -2(-2a) = 4a$. The curve lies above the first tangent, and below the second, so we are looking for a relationship between $A = \int_{-2a}^{a} [x^3 - (3a^2 - 2a^3)]\, dx$ and $B = \int_{-2a}^{4a} [(12a^2 + 16a^3) - x^3]\, dx$. We calculate $A = \frac{27}{4}a^4$, and $B = 108a^4$. We see that $B = 16A = 2^4A$. This is because our calculation of area B was essentially the same as that of area A, with a replaced by $-2a$, so if we replace a with $-2a$ in our expression for A, we get $\frac{27}{4}(-2a)^4 = 108a^4 = B$.



7-1 8

Let $u = t, dv = \sec^2 2tdt \Leftrightarrow du = dt, v = \frac{1}{2}\tan 2t$.

Then $\int tsec^2 2tdt = \frac{1}{2}t\tan 2t - \frac{1}{2}\int \tan 2tdt = \frac{1}{2}t\tan 2t - \frac{1}{4}\ln|\sec 2t| + C$.

7-1 9

Let $u = cos^{-1}x, dv = dx \Leftrightarrow du = \frac{-1}{\sqrt{1-x^2}}dx, v = x$. Then by Equation 2,

$\int cos^{-1}xdx = x\cos^{-1}x - \int \frac{-x}{\sqrt{1-x^2}}dx = x\cos^{-1}x - \int \frac{1}{\sqrt{t}}(\frac{1}{2}dt)$

$= x\cos^{-1}x - \frac{1}{2}\cdot 2t^{1/2} + C = x\cos^{-1}x - \sqrt{1-x^2} + C$

7-1 25

Let $u = y, dv = \sinh ydy \Leftrightarrow du = dy, v = \cosh y$. By (6),

$\int_0^2 y\sinh ydy = [y\cosh y]_0^2 - \int_0^2 \cosh ydy = 2\cosh 2 - 0 - [\sinh y]_0^2 = 2\cosh 2 - \sinh 2.$

7-1 51

Let $u = (\ln x)^n, dv = dx \Leftrightarrow du = n(\ln x)^{n-1}(dx/x), v = x$. By Equation 2,

$\int (\ln x)^n dx = x(\ln x)^n - \int nx(\ln x)^{n-1}(dx/x) = x(\ln x)^n - n\int (\ln x)^(n-1)dx$.

7-1 52

Let $u = (x)^n, dv = e^x dx \Leftrightarrow du = n(x)^{n-1}dx, v = e^x$. By Equation 2,

$\int (x)^n e^x dx = (x)^n e^x - n\int (x)^(n-1)e^x dx$.

7-1.53

$\int \tan^n x dx = \int \tan^{n-2} x \tan^2 x dx = \int \tan^{n-2} x(\sec^2 x - 1)dx = \int \tan^{n-2} x \sec^2 x dx - \int \tan^{x-2} x dx = I - \int \tan^{n-2} x dx$. Let $u = \tan^{n-2} x, dv = sec^2 x dx \Leftrightarrow du = (n-2)\tan^{n-3} x \sec^2 x dx, v = tanx$. By Equation 2,

$\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx$

7-1 54

Let $u = \sec x^n - 2x, dv = \sec^2 x dx \Leftrightarrow du = (n-2)\sec^{n-3} \tan x dx, v = \tan x$. By Equation 2,

$=\int \sec^n x dx = \tan x \sec^{n-2} x - (n-2)\int \sec^{n-2} x \tan^2 x dx.$

$=\int \sec^n x dx = \tan x \sec^{n-2} x - (n-2)\int \sec^{n-2} x(\sec^2 - 1)x dx$

$=\int \sec^n x dx = \tan x \sec^{n-2} x - (n-2)\int \sec^n x dx + (n-2)\int \sec^{n-2} x dx$

so $(n-1)\int \sec^n x dx = \tan x \sec^{n-2} x + (n-2)\int \sec^{n-2} x dx$. If $n - 1 \neq 0$,

then $\int \sec^n x dx = \frac{\tan x sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x dx$.

7.2 28

Let $u = \sec x$,so $du = \sec x \tan x dx$. Thus,

$\int \tan^5 x \sec^3 x dx = \tan^4 x \sec^2 x(\sec x \tan x)dx = \int (\sec^2 x - 1)^2 \sec^2 x(\sec x \tan x dx)$

$= \int (u^2-1)^2 u^2 du = \int (u^6 - 2u^4 + u^2)du$

$= \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C = \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C$

7.2 70

$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx dx = \frac{1}{\pi}\int_{-\pi}^{\pi}[(\sum_{n=1}^{m} a_n \sin nx)\sin mx]dx = \sum_{n=1}^{m}\frac{a_n}{\pi}\int_{-\pi}^{\pi}\sin mx \sin nx dx.$

By Exercise 68, every term is zero except the mth one, and that term is $\frac{a_m}{\pi}\cdot\pi = a_m$.

7.3 24

$\int_0^1 \sqrt{x - x^2}dx = \int_0^1 \sqrt{\frac{1}{4} - (x^2 - x + \frac{1}{4})}dx = \int_0^1 \sqrt{\frac{1}{4} - (x - \frac{1}{2})^2}dx$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\frac{1}{4} - \frac{1}{4}\sin^2\theta}\cdot\frac{1}{2}\cos\theta d\theta \ (\substack{x-\frac{1}{2}=\frac{1}{2}\sin\theta \\ dx=\frac{1}{2}\cos\theta d\theta})$

$= 2\int_0^{\frac{\pi}{2}} \frac{1}{2}\cos\theta\frac{1}{2}\cos\theta d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}}\cos^2\theta d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{1}{2}(1+\cos 2\theta)d\theta$

$= \frac{1}{4}[\theta + \frac{1}{2}\sin 2\theta]_0^{\frac{\pi}{2}} = \frac{1}{4}(\frac{\pi}{2}) = \frac{\pi}{8}$

7.3 32

(a.) Let $x = a\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then,

$I = \int \frac{x^2}{(x^2+a^2)^{\frac{3}{2}}} dx = \int \frac{a^2\tan^2\theta}{a^3\sec^3\theta} \cdot a\sec^2\theta d\theta = \int \frac{\tan^2\theta}{\sec\theta} d\theta = \int \frac{\sec^2\theta - 1}{\sec\theta} d\theta$

$= \int (\sec\theta - \cos\theta) d\theta = \ln|\sec\theta + \tan\theta| - \sin\theta + C$

$= \ln|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}| - \frac{x}{\sqrt{x^2+a^2}} + C = \ln(x + \sqrt{x^2+a^2}) - \frac{x}{\sqrt{x^2+a^2}} + C_1$

(b.) Let $x = a\sinh t$. Then,

$I = \int \frac{a^2\sinh^2 t}{a^3\cosh^3 t} \cdot a\cosh t dt = \int \tanh^2 t dt = \int (1 - \sec h^2 t) dt = t - \tanh t + C$

$= \sinh^{-1}\frac{x}{a} - \frac{x}{\sqrt{a^2+x^2}} + C$

7.5 72

Use parts with $u = \ln(x+1)$, $dv = \frac{dx}{x^2}$:

$\int \frac{\ln(x+1)}{x^2} dx = -\frac{1}{x}\ln(x+1) + \int \frac{dx}{x(x+1)}$

$= -\frac{1}{x}\ln(x+1) + \int [\frac{1}{x} - \frac{1}{x+1}] dx$

$= -\frac{1}{x}\ln(x+1) + \ln|x| - \ln(x+1) + C$

$= -(1 + \frac{1}{x})\ln(x+1) + \ln|x| + C$