Chapter 4
Mathematical Expectation

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4.1 Mean of a Random Variable

期望值

- Definition 4.1: Mean (Expected value), Let *X* be a random variable with probability distribution *f*(*x*).

$$\begin{cases} \mu = E(X) = \sum_x x f(x), \text{ if } X \text{ is discrete,} \\ \mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx, \text{ if } X \text{ is continuous} \end{cases}$$

- Example 4.1
  - A lot contain 4 good components and 3 defective components.
  - A sample of 3 is taken by a quality inspector.
  - Find the expected value of the number of good components in this sample.
  - **Solution**

$$X \text{ represents the number of good components, } f(x) = \frac{\binom{4}{x}\binom{3}{3-x}}{\binom{7}{3}}, x = 0,1,2,3$$

$$\mu = E(X) = 0 \cdot f(0) + 1 \cdot f(1) + 2 \cdot f(2) + 3 \cdot f(3) = \frac{12}{7}$$

Mean of a Random Variable

- Example 4.3: Let *X* be the random variable that denotes the life in hours of a certain electronic device. The probability density function is as the following. Find the expected life of this type of device.

$$f(x) = \begin{cases} \dfrac{20{,}000}{x^3}, & x > 100 \\ 0, & \text{elsewhere.} \end{cases}$$

  - **Solution**

$$\mu = E(X) = \int_{100}^{\infty} x \cdot \frac{20{,}000}{x^3}\, dx = \int_{100}^{\infty} \frac{20{,}000}{x^2} dx = -\left.\frac{20{,}000}{x}\right|_{100}^{\infty} = 200.$$

Mean of a Random Variable

x函式的期望值

- Theorem 4.1: Let X be a random variable with probability distribution $f(x)$. The mean of the random variable $g(X)$ is

$$\begin{cases} \mu_{g(X)} = E[g(X)] = \sum_{x} g(x) f(x), & \text{if } X \text{ is discrete,} \\ \mu_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx, & \text{if } X \text{ is continuous.} \end{cases}$$

- Example 4.5
  - Let X be a random variable with density function $f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{elsewhere.} \end{cases}$
  - Find the expected value of $g(X) = 4X + 3$.
  - **Solution**

$$E[g(X)] = E(4X + 3)$$

$$= \int_{-1}^{2} \frac{(4x+3)x^2}{3} dx = \frac{1}{3} \int_{-1}^{2} (4x^3 + 3x^2) dx = 8.$$

Mean of a Random Variable

- Definition 4.2: Let *X* and *Y* be random variables with joint probability function *f*(*x* ,*y*). The mean of the random variable g(*X*, *Y*) is
$$\begin{cases} \mu_{g(X,Y)} = E[g(X,Y)] = \sum_x \sum_y g(x,y) f(x,y) & \text{if } X \text{ and } Y \text{ are discrete} \\ \mu_{g(X,Y)} = E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f(x,y)\,dx\,dy & \text{if } X \text{ and } Y \text{ are continuous} \end{cases}$$

- Example 4.7: Find $E\left(\frac{Y}{X}\right)$ for the density function

$$f(x,y) = \begin{cases} \frac{x(1+3y^2)}{4}, & 0 < x < 2, 0 < y < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

  – Solution

$$E\left(\frac{Y}{X}\right) = \int_0^1 \int_0^2 \frac{y}{x} \cdot \frac{x(1+3y^2)}{4} dx dy = \int_0^1 \frac{y+3y^3}{2} dy = \frac{5}{8}$$

Mean of a Random Variable

joint distribution

- If g(*X*, *Y*) = *X* is

$$E(X) = \begin{cases} \sum_x \sum_y x f(x,y) = \sum_x x g(x) \text{ (discrete case)} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x,y)\,dx\,dy = \int_{-\infty}^{\infty} x g(x)\,dx \text{ (continuous case)} \end{cases}$$

where $g(x)$ is the marginal distribution of X.

- If g(*X*, *Y*) = *Y* is

$$E(Y) = \begin{cases} \sum_x \sum_y y f(x,y) = \sum_y y h(y) \text{ (discrete case)} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(x,y)\,dx\,dy = \int_{-\infty}^{\infty} y h(y)\,dy \text{ (continuous case)} \end{cases}$$

where $h(y)$ is the marginal distribution of Y.

4.2 Variance and Covariance

- A mean does not give adequate description of the shape of a a random variable (probability distribution).
- We need to characterize the variability in the distribution.
- Definition 4.3: Let *X* be a random variable with probability distribution *f*(*x*) and mean μ. The variance of *X* is

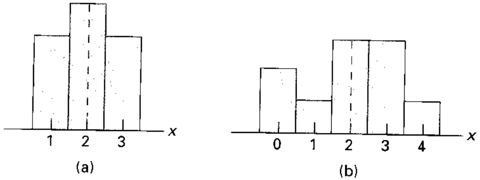

Figure 4.1 Distributions with equal means and different dispersions.

變異數越大：資料分佈越分散
變異數越小：資料分佈越集中

$$\begin{cases} \sigma^2 = E(X-\mu)^2 = \sum_x (x-\mu)^2 \cdot f(x) \text{ if } X \text{ is discrete} \\ \sigma^2 = E(X-\mu)^2 = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)dx \text{ if } X \text{ is continuous} \end{cases}$$

- σ is called the standard deviation of *X*.

Variance and Covariance

- Example 4.8: Let the random variable *X* represent the number of automobiles that are used for official business purposes on any given workday. The probability distribution for company *A* and *B* is as follows. Show that the variance of the probability distribution for company *B* is greater than that of company *A.*

A

x	1	2	3
$f(x)$	0.3	0.4	0.3

B

x	0	1	2	3	4
$f(x)$	0.2	0.1	0.3	0.3	0.1

  - **Solution**

$$\mu_A = E(X) = 1\cdot 0.3 + 2\cdot 0.4 + 3\cdot 0.3 = 2.0$$

$$\sigma_A^2 = \sum_{x=1}^{3}(x-2)^2 f(x) = (1-2)^2\cdot 0.3 + (2-2)^2\cdot 0.4 + (3-2)^2\cdot 0.3 = 0.6$$

$$\mu_B = E(X) = 0\cdot 0.2 + 1\cdot 0.1 + 2\cdot 0.3 + 3\cdot 0.3 + 4\cdot 0.1 = 2.0$$

$$\sigma_B^2 = \sum_{x=0}^{4}(x-2)^2 f(x) = (0-2)^2\cdot 0.2 + (1-2)^2\cdot 0.1 + (2-2)^2\cdot 0.3$$

$$+ (3-2)^2\cdot 0.3 + (4-2)^2\cdot 0.1 = 1.6$$

Variance and Covariance

- Theorem 4.2: The variance of a random variable X is

$$\sigma^2 = E(X^2) - \mu^2$$

  – Proof

$$\begin{aligned}
\sigma^2 &= \sum_x (x-\mu)^2 \cdot f(x) = \sum_x (x^2 - 2\mu x + \mu^2) f(x) \\
&= \sum_x x^2 f(x) - 2\mu \overbrace{\boxed{\sum_x xf(x)}}^{\mu} + \mu^2 \overbrace{\boxed{\sum_x f(x)}}^{1} \\
&= \sum_x x^2 f(x) - 2\mu \cdot \mu + \mu^2 \\
&= E(X^2) - \mu^2
\end{aligned}$$

Variance and Covariance

- Example 4.9: Let the random variable *X* represent the number of defective parts for a machine when 3 parts are sampled from a production line and tested. Calculate σ^2 using the following probability distribution.

x	0	1	2	3
$f(x)$	0.51	0.38	0.10	0.01

  - **Solution**

$$\mu = E(X) = 0 \cdot 0.51 + 1 \cdot 0.38 + 2 \cdot 0.10 + 3 \cdot 0.01 = 0.61$$

$$E(X^2) = \sum_{x=0}^{3} x^2 f(x) = 0^2 \cdot 0.51 + 1^2 \cdot 0.38 + 2^2 \cdot 0.10 + 3^2 \cdot 0.01$$

$$= 0.87$$

$$\sigma^2 = E(X^2) - \mu^2 = 0.87 - 0.61^2 = 0.4979$$

Variance and Covariance

- Theorem 4.3: Let *X* be a random variable with probability distribution *f*(*x*). The variance of the random variable *g*(*X*) is

$$\begin{cases} \sigma^2_{g(X)} = E\left\{[g(X) - \mu_{g(X)}]^2\right\} = \sum_x [g(X) - \mu_{g(X)}]^2 \cdot f(x) \text{ if } X \text{ is discrete} \\ \sigma^2_{g(X)} = E\left\{[g(X) - \mu_{g(X)}]^2\right\} = \int_{-\infty}^{\infty} [g(X) - \mu_{g(X)}]^2 \cdot f(x)dx \text{ if } X \text{ is continuous} \end{cases}$$

- Example 4.11: Calculate the variance of *g*(*X*)=2*X*+3, where *X* is a random variable with probability distribution.
  - **Solution**

x	0	1	2	3
f(*x*)	1/4	1/8	1/2	1/8

$$\mu_{2X+3} = E(2X+3) = \sum_{x=0}^{3} (2x+3) f(x) = 6$$

$$\sigma^2_{2X+3} = E\left\{[(2X+3) - \mu_{2X+3}]^2\right\} = E\left\{[2X+3-6]^2\right\}$$

$$= E(4X^2 - 12X + 9) = \sum_{x=0}^{3} (4x^2 - 12x + 9) f(x) = 4$$

Variance and Covariance

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- Definition 4.4: Let X and Y be random variables with joint probability distribution $f(x,y)$. The **covariance** of X and Y is

$$\begin{cases} \sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = \sum_x \sum_y (x-\mu_X)(y-\mu_Y) f(x,y) \text{ if } X \text{ and } Y \text{ are discrete} \\ \sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y) f(x,y)\,dx\,dy \text{ if } X \text{ and } Y \text{ are continuous} \end{cases}$$

- The covariance between two random variables is a measurement of the nature of the **association** between the two.
- The sign of the covariance indicates whether the relationship between two dependent random variables is positive or negative.
- When X and Y are statistically independent, it can be shown that the covariance is zero. The converse, however, is not generally true.

Variance and Covariance

- Theorem 4.4: The covariance of two random variables *X* and *Y* with means μ_X and μ_Y respectively, is given by

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y .$$

  – Proof

$$\sigma_{XY} = \sum_x \sum_y (x - \mu_X)(y - \mu_Y) f(x, y)$$

$$= \sum_x \sum_y (xy - \mu_X y - \mu_Y x + \mu_X \mu_Y) f(x, y)$$

$$= \sum_x \sum_y xyf(x, y) - \mu_X \sum_x \sum_y yf(x, y) - \mu_Y \sum_x \sum_y xf(x, y) + \mu_X \mu_Y \sum_x \sum_y f(x, y)$$

$$\because \mu_X = \sum_x \sum_y xf(x, y), \mu_Y = \sum_x \sum_y yf(x, y), \text{and} \sum_x \sum_y f(x, y) = 1$$

$$\therefore \sigma_{XY} = E(XY) - \mu_X \mu_Y - \mu_Y \mu_X + \mu_X \mu_Y$$

$$= E(XY) - \mu_X \mu_Y$$

$$\mu_y \underbrace{\sum_x x \overbrace{\sum_y f(x,y)}^{g(x)}}_{\mu_x}$$

$$\sum_y y \sum_x f(x,y)$$

$$= \sum_y y\, h(y)$$

$$= E(Y) = \mu_y$$

Variance and Covariance

- Definition 4.5: Let X and Y be random variables with covariance σ_{XY} and standard deviations σ_X and σ_Y. The correlation coefficient of X and Y is

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$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}, \quad -1 \le \rho_{XY} \le 1.$$

- Exact linear dependency: $Y = a + bX$

$$\begin{cases} \rho_{XY} = 1 & \text{if } b > 0 \\ \rho_{XY} = -1 & \text{if } b < 0 \end{cases}$$

4.3 Means and Variance of Linear Combinations of Random Variables

- Theorem 4.5: If *a* and *b* are constants, then

  $E(aX + b) = aE(X) + b.$

  – Proof

  $$E(aX+b) = \int_{-\infty}^{\infty}(ax+b)f(x)dx = a\int_{-\infty}^{\infty}xf(x)dx + b\int_{-\infty}^{\infty}f(x)dx$$
  $$= aE(X) + b$$

- Corollary 4.1: $E(b) = b.$
- Corollary 4.2: $E(aX) = aE(X).$
- Example 4.18(4.16): Applying Theorem 4.5 to the continuous random variable $g(X) = 4X+3$, rework Example 4.5 (Find the expected value of $g(X)$).

  – the density function of X is : $f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{elsewhere.} \end{cases}$

  – Solution

  $$E(4X+3) = 4E(X) + 3$$
  $$E(X) = \int_{-1}^{2} x \cdot \frac{x^2}{3} dx = \int_{-1}^{2} \frac{x^3}{3} dx = \frac{5}{4}$$
  $$E(4X+3) = 4 \cdot \frac{5}{4} + 3 = 8$$

Means and Variance of Linear Combinations of Random Variables

- Theorem 4.6: $E[g(X) \pm h(X)] = E[g(X)] \pm E[h(X)]$
  - **Proof**

$$E[g(X) \pm h(X)] = \int_{-\infty}^{\infty} [g(x) \pm h(x)]\, f(x)\, dx$$

$$= \int_{-\infty}^{\infty} g(x) f(x) dx \pm \int_{-\infty}^{\infty} h(x) f(x) dx$$

$$= E[g(X)] \pm E[h(X)]$$

- Example 4.19: Let X be a random variable with probability distribution as follows:

x	0	1	2	3
$f(x)$	1/3	1/2	0	1/6

Find the expected value of $Y = (X - 1)^2$

Means and Variance of Linear Combinations of Random Variables

- Example 4.19:
  - **Solution**

$$E[(X-1)^2] = E(X^2 - 2X + 1)$$
$$= E(X^2) - 2E(X) + E(1).$$

From Corollary 4.1, E(1) = 1, and by direct computation,

$$E[X] = (0)(\frac{1}{3}) + (1)(\frac{1}{2}) + (2)(0) + (3)(\frac{1}{6}) = 1$$

and

$$E[X^2] = (0)(\frac{1}{3}) + (1)(\frac{1}{2}) + (4)(0) + (9)(\frac{1}{6}) = 2$$

Hence

$$E[(X-1)^2] = 2 - (2)(1) + 1 = 1$$

Means and Variance of Linear Combinations of Random Variables

- Theorem 4.7: $E[g(X,Y) \pm h(X,Y)] = E[g(X,Y)] \pm E[h(X,Y)]$.
  - **Proof**

$$E[g(X,Y) \pm h(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[g(x,y) \pm h(x,y)]f(x,y)dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)dxdy \pm \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)f(x,y)dxdy$$

$$= E[g(X,Y)] \pm E[h(X,Y)]$$

- Corollary 4.3: Setting $g(X, Y) = g(X)$ and $h(X, Y) = h(Y)$.

$$E[g(X) \pm h(Y)] = E[g(X)] \pm E[h(Y)].$$

- Corollary 4.4: Setting $g(X, Y) = X$ and $h(X, Y) = Y$.

$$E(X \pm Y) = E(X) \pm E(Y).$$

Means and Variance of Linear Combinations of Random Variables

- Theorem 4.8: Let X and Y be two independent random variables. Then $E(XY) = E(X)E(Y)$.
  - **Proof** $E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy$

$$\because f(x,y) = g(x)h(y)$$

$$\therefore E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyg(x)h(y)dxdy = \int_{-\infty}^{\infty} xg(x)dx \int_{-\infty}^{\infty} yh(y)dy$$

$$= E(X)E(Y)$$

- Example 4.21: In producing gallium-arsenide microchips, it is known that the ratio between gallium and arsenide is independent of producing a high percentage of workable wafers, which are the main components of microchips. Let X denote the ratio of gallium to arsenide and Y denote the percentage of workable microwafers retrieved during a 1-hour period. X and Y are independent random variables with the joint density being known as

Means and Variance of Linear Combinations of Random Variables

$$f(x,y)=\begin{cases}\frac{x(1+3y^2)}{4}, 0<x<2, 0<y<1\\ 0, \qquad\qquad \text{elsewhere.}\end{cases}$$

Illustrate that $E(XY) = E(X)E(Y)$.

– **Solution**

$$E(XY)=\int_0^1\int_0^2 xyf(x,y)dxdy=\int_0^1\int_0^2\frac{x^2y(1+3y^2)}{4}dxdy$$

$$=\int_0^1\frac{x^3y(1+3y^2)}{12}\Big|_{x=0}^{x=2}dy=\int_0^1\frac{2y(1+3y^2)}{3}dy=\frac{5}{6}$$

$$E(X)=\int_0^1\int_0^2 xf(x,y)dxdy=\int_0^1\int_0^2\frac{x^2(1+3y^2)}{4}dxdy$$

$$=\int_0^1\frac{x^3(1+3y^2)}{12}\Big|_{x=0}^{x=2}dy=\int_0^1\frac{2(1+3y^2)}{3}dy=\frac{4}{3}$$

$$E(Y)=\int_0^1\int_0^2 yf(x,y)dxdy=\int_0^1\int_0^2\frac{xy(1+3y^2)}{4}dxdy$$

$$=\int_0^1\frac{x^2y(1+3y^2)}{8}\Big|_{x=0}^{x=2}dy=\int_0^1\frac{y(1+3y^2)}{2}dy=\frac{5}{8}$$

$$E(X)E(Y)=\frac{4}{3}\times\frac{5}{8}=\frac{5}{6}=E(XY)$$

Means and Variance of Linear Combinations of Random Variables

- Theorem 4.9: If *X* and *Y* are random variables with joint probability distribution *f*(*x*, *y*), then

$$\sigma^2_{aX+bY+c} = a^2\sigma^2_X + b^2\sigma^2_Y + 2ab\sigma_{XY}.$$

  - **Proof**

$$\sigma^2_{aX+bY+c} = E\{[(aX+bY+c)-\mu_{aX+bY+c}]^2\}$$

$$\because \mu_{aX+bY+c} = E(aX+bY+c) = aE(X)+bE(Y)+c = a\mu_X + b\mu_Y + c$$

$$\begin{aligned}
\therefore \sigma^2_{aX+bY+c} &= E\{[(aX+bY+c)-(a\mu_X+b\mu_Y+c)]^2\} \\
&= E\{[a(X-\mu_X)+b(Y-\mu_Y)]^2\} \\
&= a^2E[(X-\mu_X)^2]+b^2E[(Y-\mu_Y)^2]+2abE[(X-\mu_X)(Y-\mu_Y)] \\
&= a^2\sigma^2_X + b^2\sigma^2_Y + 2ab\sigma_{XY}
\end{aligned}$$

Means and Variance of Linear Combinations of Random Variables

- Theorem 4.9: If X and Y are random variables with joint probability distribution $f(x, y)$, then

$$\sigma^2_{aX+bY+c} = a^2\sigma^2_X + b^2\sigma^2_Y + 2ab\sigma_{XY}.$$

- Corollary 4.6:

$$\sigma^2_{aX+c} = a^2\sigma^2_X = a^2\sigma^2$$

- Corollary 4.7:

$$\sigma^2_{X+c} = \sigma^2_X = \sigma^2$$

- Corollary 4.8:

$$\sigma^2_{aX} = a^2\sigma^2_X = a^2\sigma^2$$

Means and Variance of Linear Combinations of Random Variables

- Corollary 4.6 and 4.7 state that the variance is unchanged if a constant is added to or subtracted from a random variable.
- The addition or subtraction of a constant simply shifts the values of X to the right/left but does not change their variability. 水平移動
- Corollary 4.6 and 4.8 state that the variance is multiplied or divided by the square of the constant.

Means and Variance of Linear Combinations of Random Variables

- Corollary 4.9: If *X* and *Y* are independent random variables, then $\sigma^2_{aX+bY} = a^2\sigma^2_X + b^2\sigma^2_Y$.
  - $\because E(XY) = E(X)E(Y)$ for independent variables
    $\therefore \sigma_{XY} = E(XY) - E(X)E(Y) = 0$.
- Corollary 4.10: If *X* and *Y* are independent random variables, then $\boxed{\sigma^2_{aX-bY} = a^2\sigma^2_X + b^2\sigma^2_Y.}$
- Corollary 4.11: If X_1, X_2, …, X_n are independent random variables, then

$$\sigma^2_{a_1X_1+a_2X_2+\cdots+a_nX_n} = a_1^2\sigma^2_{X_1} + a_2^2\sigma^2_{X_2} + \cdots + a_n^2\sigma^2_{X_n}.$$

Means and Variance of Linear Combinations of Random Variables

- Example 4.22: If X and Y are random variables with variances $\sigma_X^2 = 2, \sigma_Y^2 = 4,$ and covariance $\sigma_{XY} = -2,$ find the variance of the random variable $Z = 3X - 4Y + 8$.
  - Solution

$$\sigma_Z^2 = \sigma_{3X-4Y+8}^2 = \sigma_{3X-4Y}^2$$
$$= 9\sigma_X^2 + 16\sigma_Y^2 - 24\sigma_{XY}$$
$$= 9 \cdot 2 + 16 \cdot 4 - 24 \cdot (-2) = 130.$$

- Example 4.23: Let X and Y denote the amount of two different types of impurities in a batch of a certain chemical product. Suppose that X and Y are <u>independent</u> random variables with variances $\sigma_X^2 = 2, \sigma_Y^2 = 3.$ Find the variance of the random variable $Z = 3X - 2Y + 5$.
  - Solution

$$\sigma_Z^2 = \sigma_{3X-2Y+5}^2 = \sigma_{3X-2Y}^2 = 9\sigma_X^2 + 4\sigma_Y^2$$
$$= 9 \cdot 2 + 4 \cdot 3 = 30.$$

4.4 Chebyshev's Theorem

- If a random variable has a small variance or standard deviation, we would expect most of the values to be grouped around the mean. 越集中
- A large variance indicates a greater variability, so the area of distribution should be spread out more. 越分散

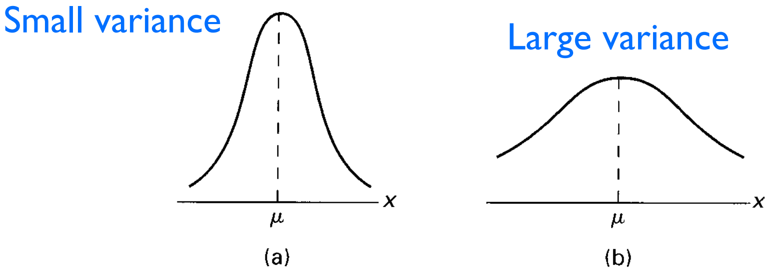

Figure 4.2 Variability of continuous observations about the mean.

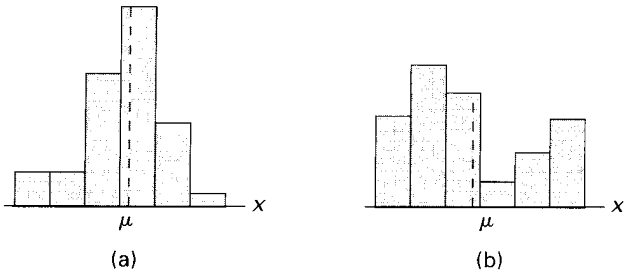

Figure 4.3 Variability of discrete observations about the mean.

Chebyshev's Theorem

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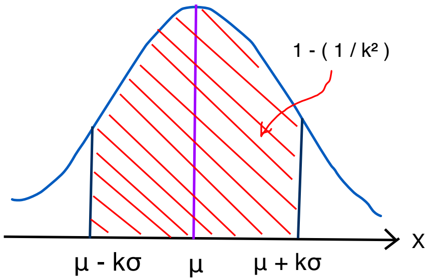
- Theorem 4.10: **Chebyshev's theorem**, the probability that any random variable *X* will assume a value within k standard deviation of the mean is at least **1-1/k²**.

  That is $P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \dfrac{1}{k^2}.$

  – Proof



$$\sigma^2 = E[(X-\mu)^2]$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

$$= \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)dx + \int_{\mu-k\sigma}^{\mu+k\sigma} (x-\mu)^2 f(x)dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)dx$$

$$\geq \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)dx$$

$$\text{now, if } |x-\mu| \geq k\sigma \therefore (x-\mu)^2 \geq k^2\sigma^2$$

$$\Rightarrow \sigma^2 \geq \int_{-\infty}^{\mu-k\sigma} k^2\sigma^2 f(x)dx + \int_{\mu+k\sigma}^{\infty} k^2\sigma^2 f(x)dx$$

$$\Rightarrow \int_{-\infty}^{\mu-k\sigma} f(x)dx + \int_{\mu+k\sigma}^{\infty} f(x)dx \leq \frac{1}{k^2}$$

$$\therefore P(\mu-k\sigma < X < \mu+k\sigma) = \int_{\mu-k\sigma}^{\mu+k\sigma} f(x)dx \geq 1 - \frac{1}{k^2}$$

Chebyshev's Theorem

- Example 4.27: A random variable *X* has a meanμ= 8, a variance σ^2 = 9, and an unknown probability distribution. Find

  (a) $P(-4 < X < 20)$

  (b) $P(|X-8| \geq 6)$

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

  – **Solution** (a) $P(-4 < X < 20) = P[8 - \overset{4}{k} \cdot 3 < X < 8 + \overset{4}{k} \cdot 3] \geq 1 - \frac{1}{4^2} = \frac{15}{16}$

$$\text{(b) } P(|X-8| \geq 6) = 1 - P(|X-8| < 6)$$

$$= 1 - P(-6 < X - 8 < 6)$$

$$= 1 - \underbrace{P(8 - 2\cdot 3 < X < 8 + 2\cdot 3)}_{\geq 1 - \frac{1}{2^2}}$$

$$\leq \frac{1}{2^2} = \frac{1}{4}$$

Chebyshev’s Theorem

- The use of Chebyshev’s theorem
  - Holds for any distribution of observations
  - Gives a lower bound only
  - Is called a distribution-free result
  - Is suitable to situations where the form of the distribution is unknown.

Exercise

- 4.23, 4.36, 4.77, 4.82