Electrostatic Energy (Electric Field Energy) & The Capacitor

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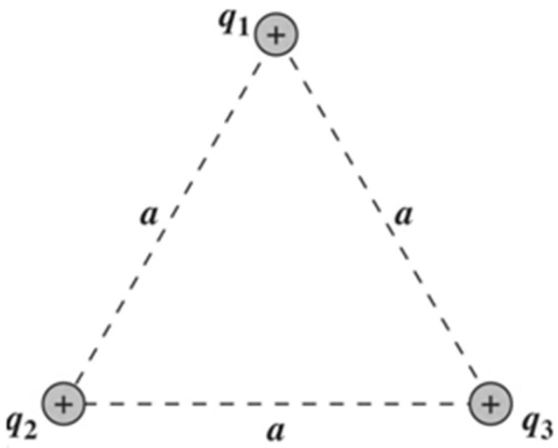
Electrostatic Energy

Electrostatic Energy = work done to assemble the charge configuration of a system.

Reference (0 energy):
when all component charges are widely separated.



Total electrostatic energy

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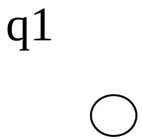

Bringing q_1 in place takes no work! ?

Total electrostatic energy

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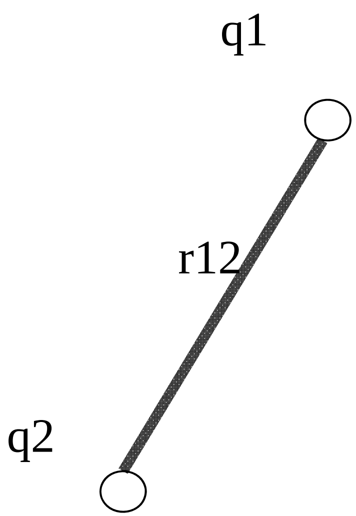

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Bringing in q_2 takes

$$W_2 = q_2 \phi_1(\mathbf{r}_2) = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}}$$

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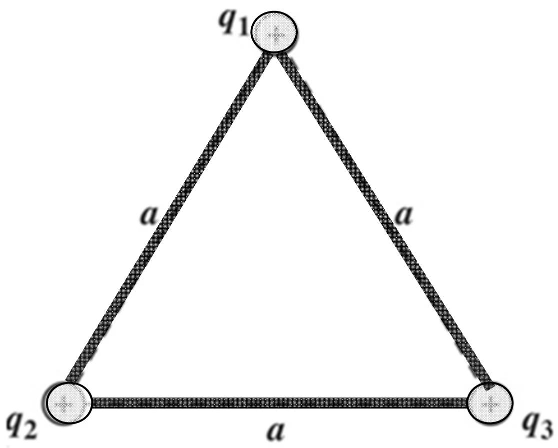


Bringing q_1 in place takes no work.

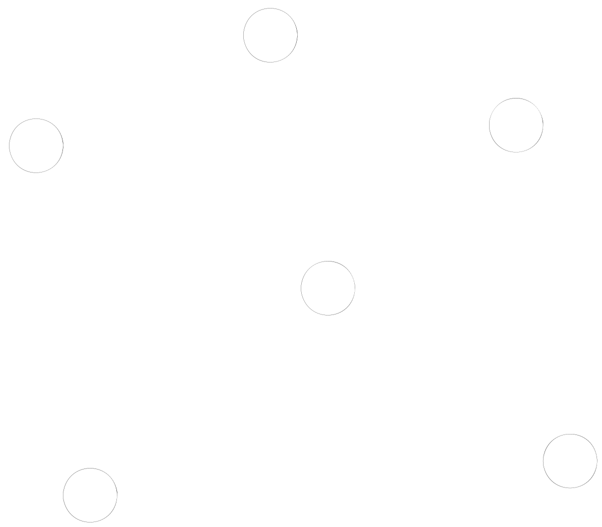
Bringing in q_2 takes $$W_2 = q_2\phi_1(\mathbf{r}_2) = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r_{12}}$$

Bringing in q_3 takes

$$W_3 = q_2[\phi_1(\mathbf{r}_3) + \phi_2(\mathbf{r}_3)] = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1q_3}{r_{13}} + \frac{q_2q_3}{r_{23}}\right)$$

Total electrostatic energy $$U_E = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1q_2}{r_{12}} + \frac{q_1q_3}{r_{13}} + \frac{q_2q_3}{r_{23}}\right)$$

Electrostatic Energy of a n-particle system



$$U_E = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1;j\neq i}^{N}\frac{q_i q_j}{4\pi\varepsilon_0 r_{ij}}$$

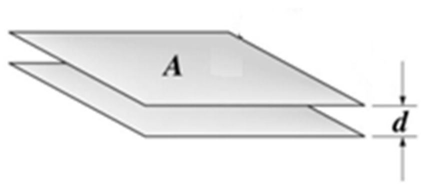
$$U_E = \frac{1}{2}\sum_{i=1}^{N} q_i \cdot \left(\sum_{j=1;j\neq i}^{N}\frac{q_j}{4\pi\varepsilon_0 r_{ij}}\right)$$

$$U_E = \frac{1}{2}\sum_{i=1}^{N} q_i \cdot \phi_i$$

Capacitors to store the energy

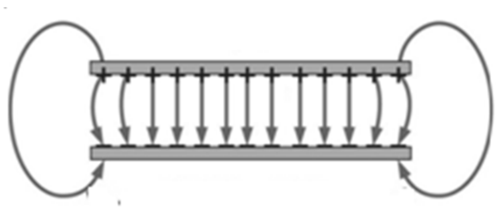
Capacitor: pair of conductors carrying equal but opposite charges.

Usage: store electrical energy



Parallel-Plate Capacitor:

2 conducting plates of area *A* separated by a small distance *d* .



Plates are initially neutral.

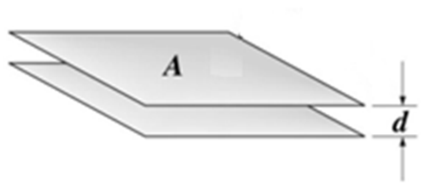
They're charged by connecting to a battery.

Charge transfer $\rightarrow$ plates are equal but oppositely charged.

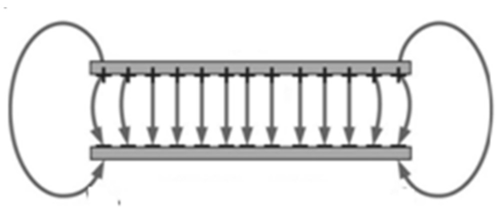
Large *A*, small *d* $\rightarrow$ $\mathbf{E} \approx 0$ outside.

Capacitors

Capacitor: pair of conductors carrying equal but opposite charges.



Far from the edges



$$\mathbf{E}_{inside} = -\frac{\sigma}{\varepsilon_0}\hat{\mathbf{z}} \quad = -\frac{Q}{\varepsilon_0 A}\hat{\mathbf{z}}$$

$$V \equiv \Delta\phi = \phi_{upper} - \phi_{lower}$$
$$= -E_{inside} \cdot d\hat{z}$$
$$= \frac{Q}{\epsilon_0 A}\, d$$

Capacitance

Parallel-plate capacitor: $V = \frac{Q}{\varepsilon_0 A} d \quad \rightarrow \quad Q = \frac{\varepsilon_0 A}{d} V \quad = C\,V$

C = Q / V = capacitance $\quad C = \frac{dQ}{dV}$

$$C = \frac{\varepsilon_0 A}{d}$$

Unit: $[C] = \frac{C}{V} = farad = F$

Practical capacitor ~ μF (10^{-6} F) or pF (10^{-12} F) $\quad [\varepsilon_0] = \left[\frac{C\,d}{A}\right] = F\,/\,m$

Charging / Discharging

$$dV = \frac{1}{C} dQ$$

Energy Stored in Capacitors

When potential difference between capacitor plates is *V*,

work required to move charge *dQ* from – to + plate is

$$dW = -dQ \int_{-plate}^{+plate} \mathbf{E} \cdot d\mathbf{r} \;= dQ\left(V_{+plate} - V_{-plate}\right) \;= V\,C\,dV$$

$\mathbf{E} \cdot d\mathbf{r} < 0$

Work required to charge the capacitor from 0 to V is

$$W = C \int_0^V V\,dV \;= \frac{1}{2} C\,V^2$$ = *U* = energy stored in capacitor

Note:

In a “charged” capacitor, *Q* is the charge on the + plate.

The total charge of the capacitor is always zero.

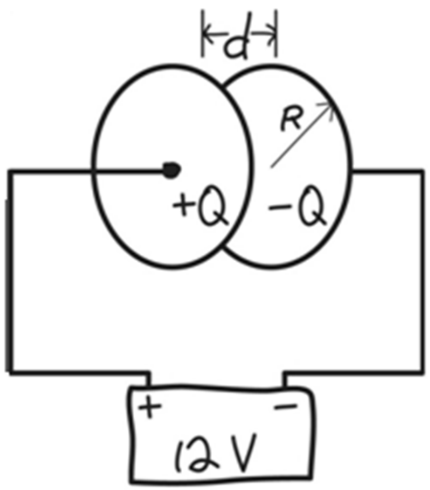
Example Parallel-Plate Capacitor

A capacitor consists of two circular metal plates of radius R = 12 cm, separated by d = 5.0 mm. Find

(a) Its capacitance,

(b) the charge on the plates, and

(c) the stored energy when the capacitor is connected to a 12-V battery.



(a)

(b)

(c)

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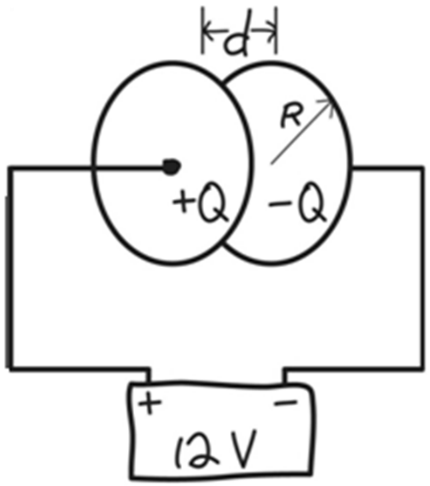


(a) $$C = \varepsilon_0 \frac{A}{d} = \frac{1}{4\pi \times 9 \times 10^9 \; Vm/C} \frac{\pi \left(12 \times 10^{-2} m\right)^2}{5.0 \times 10^{-3} m}$$

$$= 0.8 \times 10^{-10} F \quad = 80 \; pF \qquad\qquad F = \frac{C}{V}$$

(b) $$Q = C\,V \quad = \left(80 \; pF\right) \times \left(12\,V\right) \quad = 960 \; pC$$

(c) $$U = \frac{1}{2} C\,V^2 \quad = \frac{1}{2}\left(80 \; pF\right) \times \left(12\,V\right)^2 \quad = 5760 \; pJ$$

$$= 5.76 \, nJ$$

Example

- A parallel-plate capacitor with a plate separation of 1mm has a capacitance of 1F. What is the area of each plate?

Example

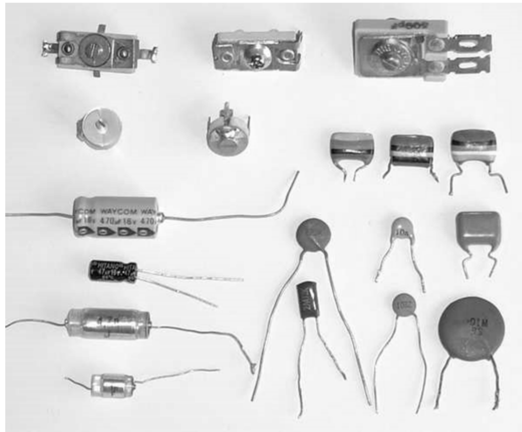
- What is the capacitance of an isolated sphere of radius R?

Using Capacitors

Computer memories: billions of 25 fF capacitors.

Rectifiers: mF

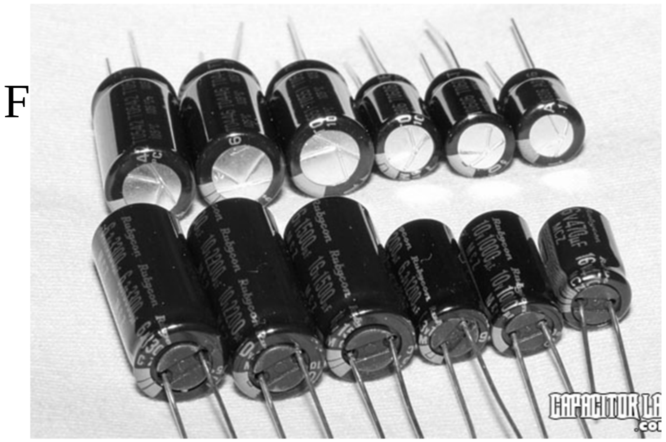
Fuel-cells: 10^2 F



220-mF electrolytic capacitor

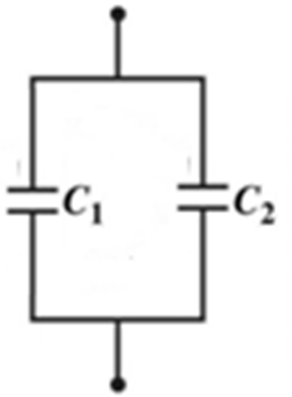
1 F

43 pF to 2.2 mF



Connecting Capacitors

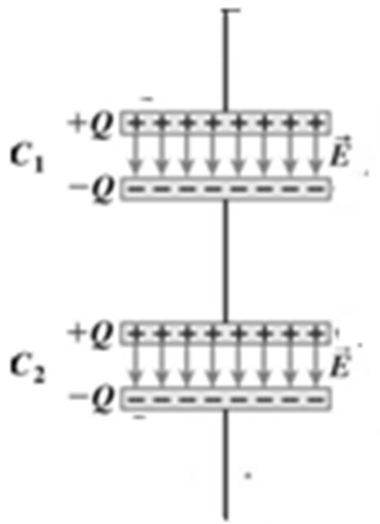
Two ways to connect 2 electronic components: parallel & series



Parallel: Same V for both components

$$Q = C\,V \;= Q_1 + Q_2 \;= C_1 V + C_2 V$$

$$\rightarrow \qquad \boxed{C = C_1 + C_2} \qquad C > C_1 \;\; or\; C_2$$



Series: Same I (Q) for both components

$$V = \frac{Q}{C} \;= V_1 + V_2 \;= \frac{Q}{C_1} + \frac{Q}{C_2}$$

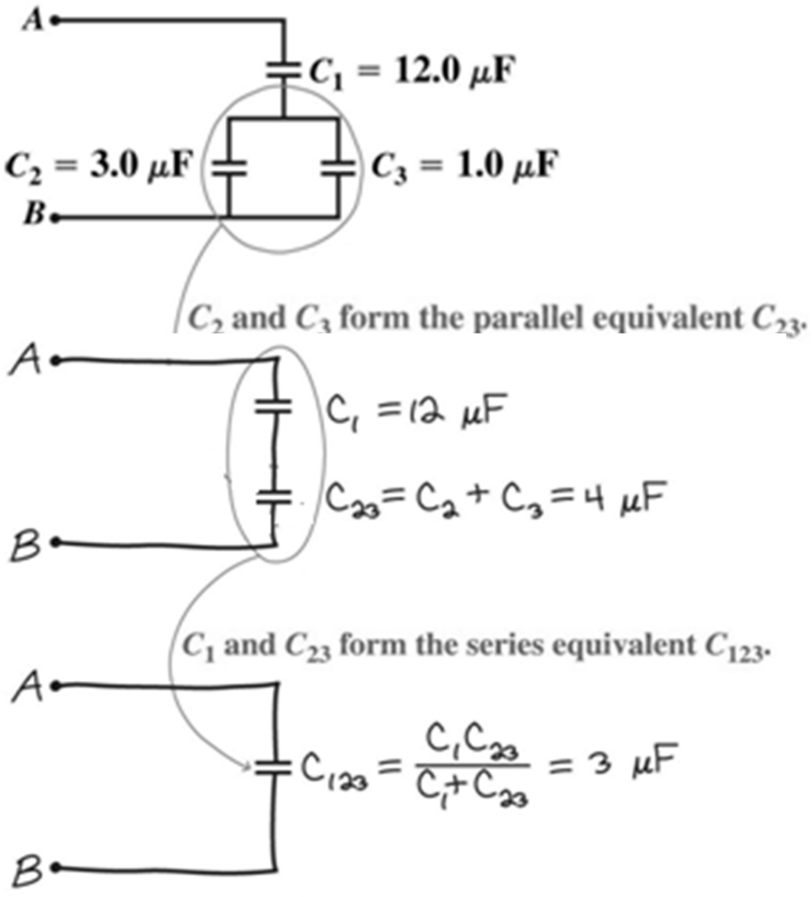
$$\rightarrow \qquad \boxed{\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}} \qquad C < C_1 \;\; or\; C_2$$

Example Connecting Capacitors

Find the equivalent capacitance of the combinations shown in the Figure.

If the maximum voltage to be applied between points *A* and *B* is 100 *V*,

what should be the working voltage of C_1 ?



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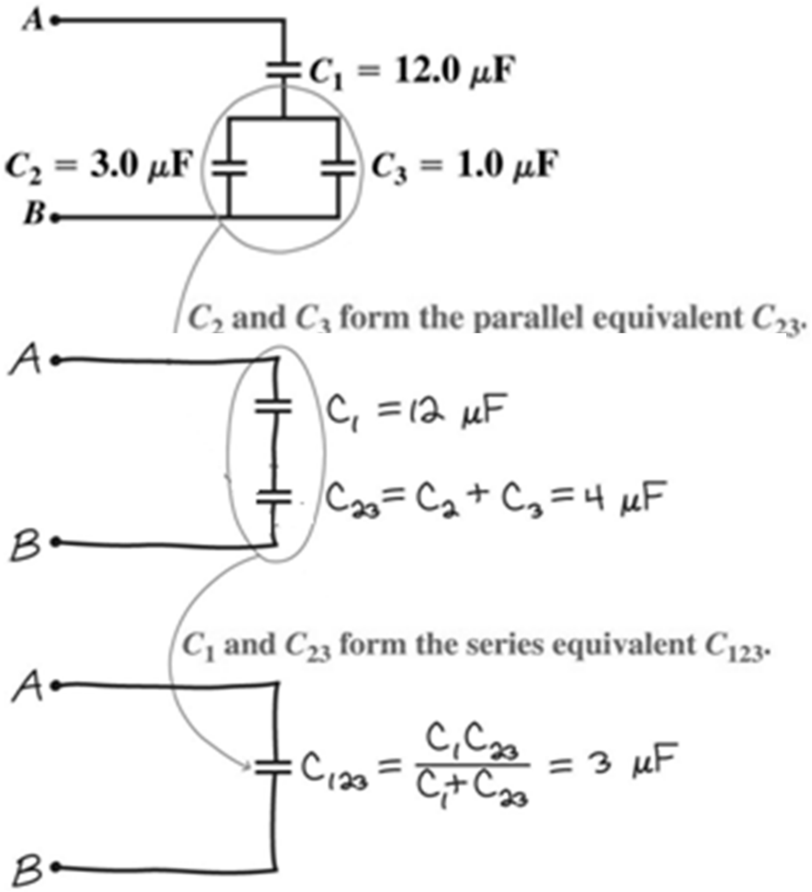

$$C_{23} = C_2 + C_3 = 3.0\ \mu F + 1.0\ \mu F = 4.0\ \mu F$$

$$\frac{1}{C_{123}} = \frac{1}{C_1} + \frac{1}{C_{23}} = \frac{1}{12.0\ \mu F} + \frac{1}{4.0\ \mu F} = \frac{1}{3.0\ \mu F}$$

$$C_{123} = 3.0\ \mu F$$

$$V_1 = \frac{Q_1}{C_1}$$

$$Q_1 = Q_{123} = C_{123}\, V_{AB}$$

$$= (3.0\ \mu F)(100\ V) = 300\ \mu C$$

$$V_1 = \frac{300\ \mu C}{12.0\ \mu F} = 25\ V$$

(min. working voltage)

Energy Stored in Capacitors

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$\mathbf{E} \cdot d\mathbf{r} < 0$

Work required to charge the capacitor from 0 to V is

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= *U* = energy stored in capacitor

Note:

In a “charged” capacitor, *Q* is the charge on the + plate.

The total charge of the capacitor is always zero.

Example:

- Two capacitors, $C_1 = 5\mu F \; and \; C_2 = 3\mu F,$ are initially in parallel in 12V battery. After charged, the capacitors are reconnected. Find the charges, potential differences and energies of the two configuration

Field Energy Density

Charging a capacitor rearranges charges $\rightarrow$ energy stored in **E**

Energy density = energy per unit volume

Parallel-plate capacitor: $$U = \frac{Q^2}{2C} \quad = \frac{Q^2}{2\varepsilon_0 \frac{A}{d}}$$

Energy density : $$u_E = \frac{U}{A\,d} \quad = \frac{Q^2}{2\varepsilon_0\ A^2} \quad = \frac{\sigma^2}{2\varepsilon_0} \quad = \frac{1}{2}\varepsilon_0\ E^2 \qquad\qquad E = \frac{\sigma}{\varepsilon_0}$$

$$\boxed{u_E = \frac{1}{2}\varepsilon_0\ \mathbf{E}^2}$$ is universal $$\left[u_E\right] = J/m^3$$

$$U = \frac{1}{2}\varepsilon_0 \int \left|\mathbf{E}\right|^2 dV$$

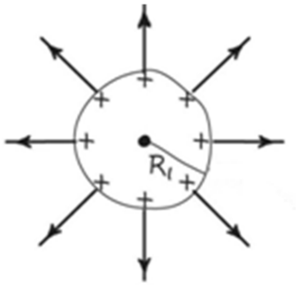
Example

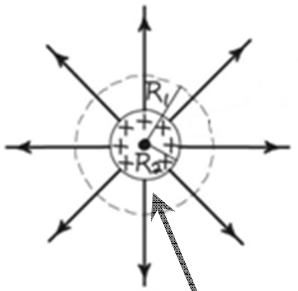
- The “breakdown field strength at which dry air loses its insulation ability and allows discharge to pass through it is about $3 \times 10^6 V/m$. What is the energy density at the field strength

Example A Shrinking Sphere

A sphere of radius R_1 carries charge Q distributed uniformly over its surface.

How much work does it take to compress the sphere to a smaller radius R_2 ?





$$U = \frac{1}{2}\varepsilon_0 \int |\mathbf{E}|^2 \, dV \qquad \mathbf{E} = k\frac{Q}{r^2}\hat{\mathbf{r}}$$

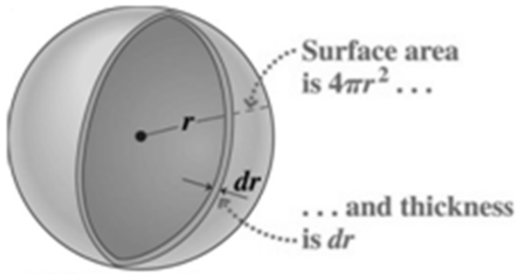
$$\Delta U = U_2 - U_1 \quad = \frac{1}{2}\varepsilon_0 \left(\int_{R_2}^{\infty} - \int_{R_1}^{\infty} \right) |\mathbf{E}|^2 \, 4\pi r^2 dr$$

$$= \frac{1}{8\pi k} \int_{R_2}^{R_1} \left(k\frac{Q}{r^2} \right)^2 4\pi r^2 dr \quad = \frac{1}{2} k\, Q^2 \int_{R_2}^{R_1} \frac{1}{r^2} dr$$

$$= \frac{1}{2} k\, Q^2 \left(\frac{1}{R_2} - \frac{1}{R_1} \right)$$

Work need be done to shrink sphere

$$\Delta U > 0 \quad for \;\; R_2 < R_1$$

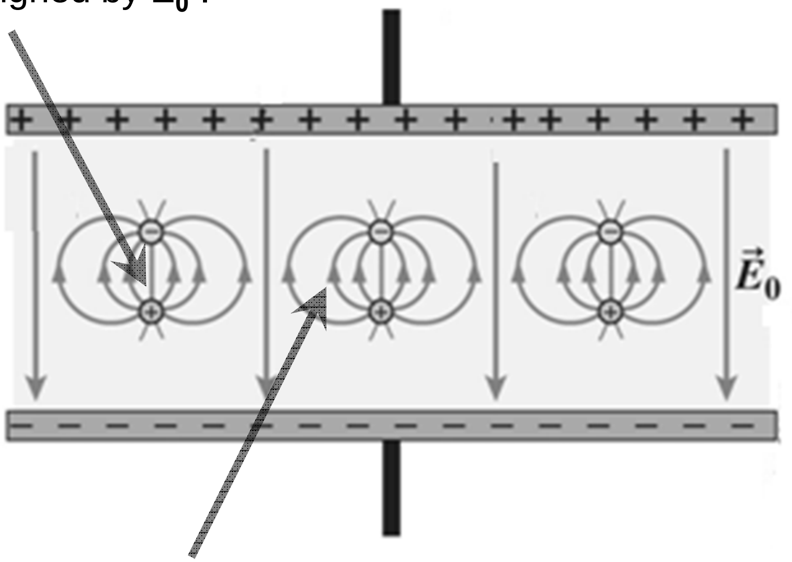


Extra energy stored here

Dielectrics

Dielectrics: insulators containing molecular dipoles but no free charges.

Molecular dipoles aligned by $\mathbf{E_0}$.



Dipole fields oppose $\mathbf{E_0}$.
Net field reduced to $\mathbf{E} = \mathbf{E_0} / \kappa$.

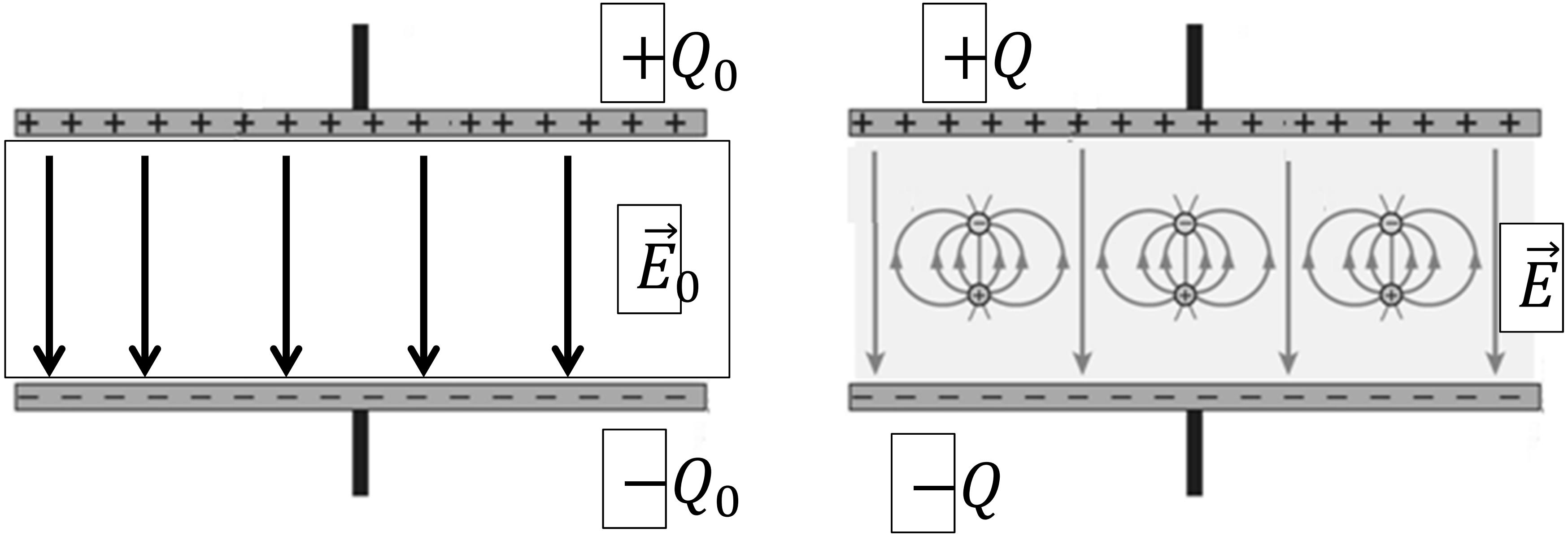
Hence $V = V_0 / \kappa$.
Q is unchanged, so $C = \kappa C_0$.

Dielectric layer lowers V between capacitor plates by factor $1/\kappa$ ($\kappa > 1$).

$$C = \frac{Q}{V} = \kappa\, \varepsilon_0 \frac{A}{d}$$

κ = dielectric constant

Battery is not connected



$$\vec{E} = \vec{E}_0/\kappa$$
$$C = \kappa C_0$$

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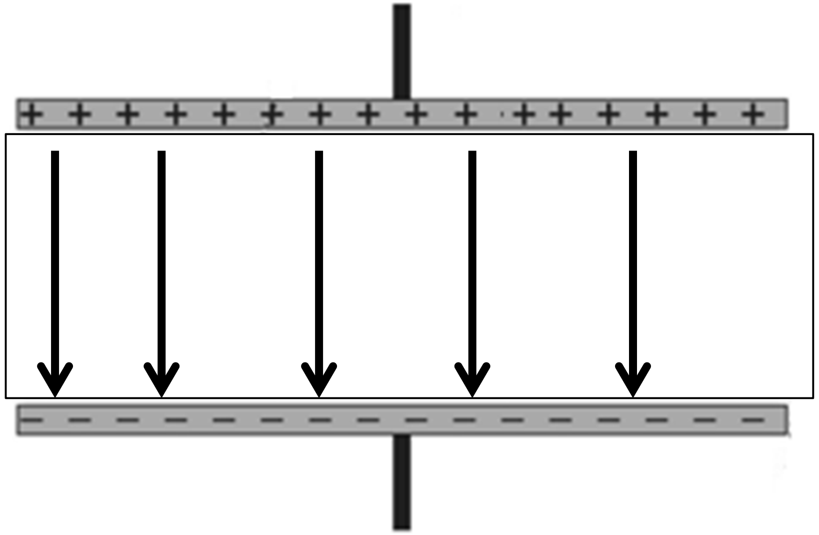

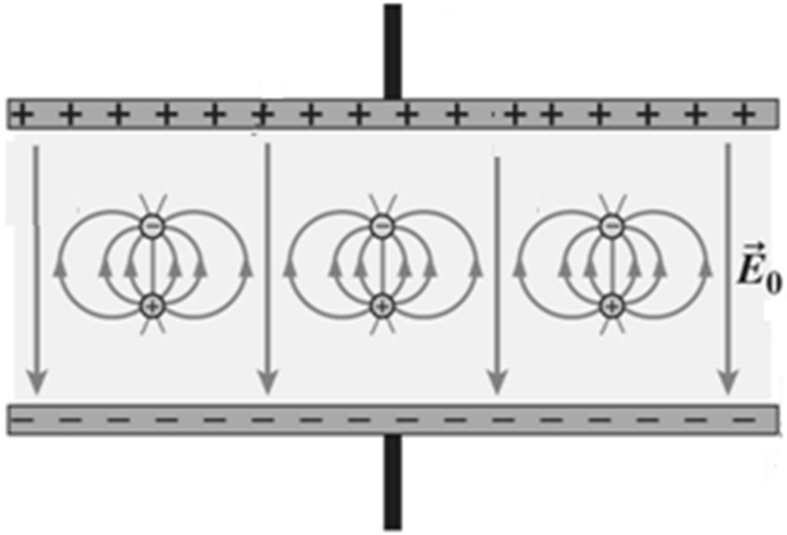

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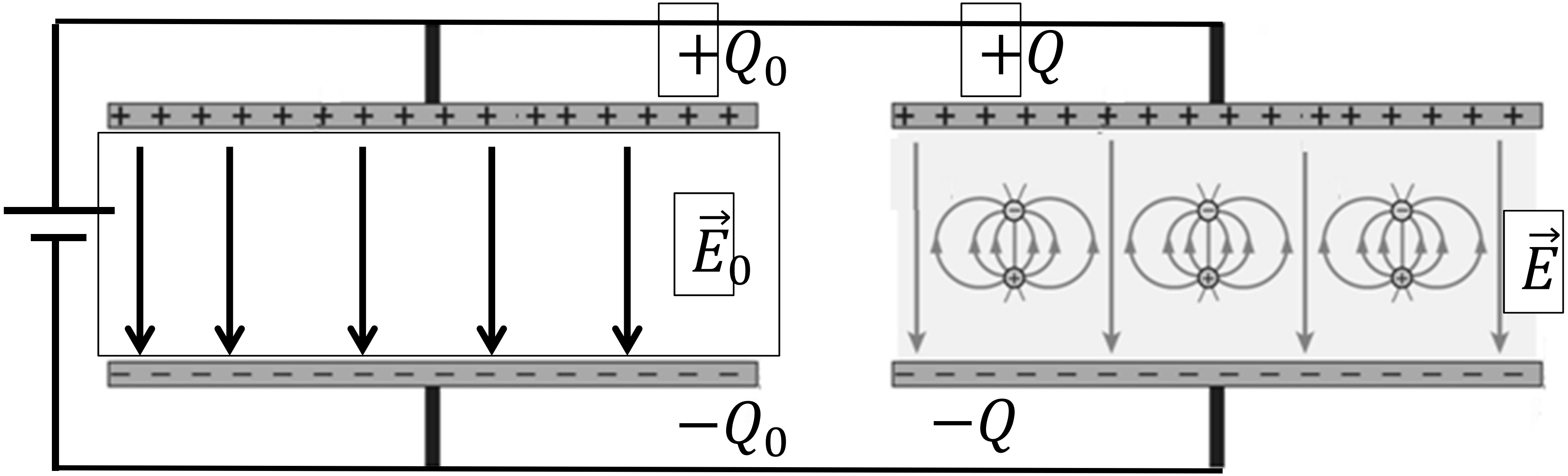
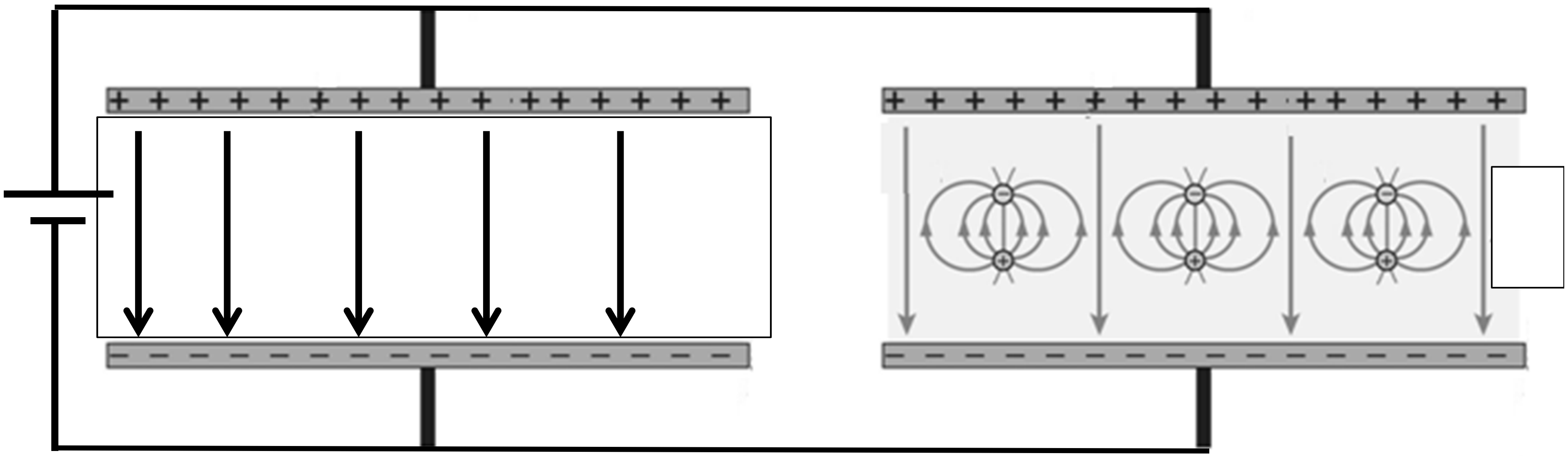


$$Q = Q_0\kappa$$
$$C = \kappa C_0$$

Battery is connected



Example

- A dielectric slab of thickness t and dielectric constant κ is inserted into a parallel plate capacitor of area A and separated by distance d. Assume that the batter is disconnected before the slab is inserted what is the capacitance.