

Chapter 3. Higher-Order Differential Equations

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1

Particular Solution

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = r(x)$$

其中 $a_0, a_1, a_2, \dots, a_n \in \text{Const}$, $r(x) \neq 0$ 常係數非齊項 O.D.E.

特解求法 $\begin{cases} \text{法一: undetermined coefficient} \\ \text{法二: 降階法} \rightarrow \text{聯想一階 O.D.E.} \end{cases}$

$$y = y_h + y_p$$

$\Rightarrow \begin{cases} \text{法一: } e^{\lambda x} \rightarrow \text{找 } \lambda \text{ 求特性方程式} \\ \text{法二: } D \rightarrow \text{降階(重根)} \end{cases}$

2

Differential Operator Method

- Method3:微分運算子法

$$D \equiv \frac{d}{dx} \quad D^n \equiv \frac{d^n}{dx^n} \quad D^{-1} \equiv \int dx$$

1. $y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = r(x)$
~~~~~特性方程式

$$\lambda^{(n)} + a_1 \lambda^{(n-1)} + \dots + a_n = 0 \text{ 求出 } \lambda_1, \lambda_2, \dots, \lambda_n$$

可以得到  $y_p = ?$

$$(D^n + a_1 D^{n-1} + \dots + a_n) y = r(x) \dots (*)$$

$y_p$  一定滿足(\*)

3

# Differential Operator Method

$$(D^n + a_1 D^{n-1} + \dots + a_n) y_p = r(x)$$

其中  $(D^n + a_1 D^{n-1} + \dots + a_n)$

Linear Differential Operator (線性微分運算子)

定義為  $L(D)$

$$\Rightarrow L(D) = D^n + a_1 D^{n-1} + \dots + a_n$$

4

# Differential Operator Method

- 特性1.  $L(D)e^{ax} = L(a)e^{ax}$

例:  $y''' + 6y'' + 11y' + 6y = e^x$

$$y_p = \frac{1}{24} e^x$$

$$(D^3 + 6D^2 + 11D + 6)y_p = e^x$$

$$\Rightarrow y_p = \frac{e^x}{D^3 + 6D^2 + 11D + 6}$$

$$= L(D)e^x$$

$$= L(1)e^x = \frac{e^x}{1 + 6 + 11 + 6} = \frac{1}{24} e^x$$

5

# Differential Operator Method

- Pf:

$$L(D)e^{ax}$$

$$= (D^n + a_1 D^{n-1} + \dots + a_n) e^{ax}$$

$$= D^n e^{ax} + a_1 D^{n-1} e^{ax} + \dots + a_n e^{ax}$$

$$= a^n e^{ax} + a_1 a^{n-1} e^{ax} + \dots + a_n e^{ax}$$

$$(\because D^n = \frac{d^n}{dx^n} \therefore D^n e^{ax} = \frac{d^n e^{ax}}{dx^n} = a \frac{d^{n-1} e^{ax}}{dx^{n-1}} = \dots = a_n e^{ax})$$

$$= (a^n + a_1 a^{n-1} + \dots + a_n) e^{ax}$$

$$= L(a)e^{ax}$$

6

# Differential Operator Method

例:  $y'' + 3y' + 2y = e^{2x}$

$$y = y_h + y_p$$

$$y_h: \lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda + 1)(\lambda + 2) = 0$$

$$\lambda = -1, -2$$

$$y_h = C_1 e^{-x} + C_2 e^{-2x}$$

7

# Differential Operator Method

• 求  $y_p$ :

法一: 降階法

$$\Rightarrow e^{-2x} \int e^{2x} [e^{-x} \int e^x e^{2x} dx] dx$$

$$\frac{1}{3} e^{3x}$$

$$\frac{1}{3} e^{4x}$$

$$\frac{1}{12} e^{4x}$$

$$= \frac{1}{12} e^{2x}$$

法二: 微分運算子法

$$(D^2 + 3D + 2)y_p = e^{2x}$$

$$\Rightarrow y_p = \frac{1}{D^2 + 3D + 2} e^{2x}$$

$$= \frac{1}{2^2 + 3 \cdot 2 + 2} e^{2x} = \frac{1}{12} e^{2x}$$

法三: 未定係數法

8

# Differential Operator Method

例:  $y' - 2y = e^{2x}$

$$y' - 2y = e^{2x}$$

$$y_h : \lambda = 2$$

$$y_h = Ce^{2x}$$

$$y_p = (D - 2)y_p = e^{2x}$$

$$y_p = \frac{e^{2x}}{D - 2} = \frac{e^{2x}}{2 - 2} = ?$$

(類似於特定係數法.當 $r(x)$ 與 $e^{\lambda x}$ 相同時獲重根時的問題)

# Differential Operator Method

- 特性2.  $L(D)[e^{ax} f(x)] = e^{ax} L(D + a)[f(x)]$

$$y_p = \frac{1}{D - 2} e^{2x}$$

$$= e^{2x} \frac{1}{D - 2} \Big|_{D = D + 2} \times 1$$

$$= e^{2x} \frac{1}{D + 2 - 2} \times 1$$

$$= e^{2x} \frac{1}{D} \times 1$$

$$= e^{2x} \int 1 dx$$

$$= e^{2x} x$$

# Differential Operator Method

例:  $y'' + 4y' + 4y = e^{-2x}$

$$\begin{aligned} y_p &= \frac{1}{D^2 + 4D + 4} e^{-2x} \\ &= \frac{1}{(D+2)^2} e^{-2x} \times 1 \\ &= e^{-2x} \frac{1}{(D + (-2) + 2)^2} \times 1 \\ &= e^{-2x} \frac{1}{D^2} \times 1 \\ &= e^{-2x} \int \int 1 dx dx = \frac{1}{2} x^2 e^{-2x} \end{aligned}$$

$$y = y_p + y_h$$

$$y_h: \lambda^2 + 4\lambda + 4 = 0$$

$$(\lambda + 2)^2 = 0$$

$$\lambda = \pm 2$$

$$y_h = C_1 e^{-2x} + C_2 x e^{-2x}$$

11

# Differential Operator Method

## • Pf 特性2:

$$\begin{aligned} L(D)e^{ax} f(x) &= (D^n + a_1 D^{n-1} + \dots + a_n) e^{ax} f(x) \\ &= D^n e^{ax} f(x) + a_1 D^{n-1} e^{ax} f(x) + \dots + a_n e^{ax} f(x) \\ De^{ax} f(x) &= e^{ax} Df(x) + ae^{ax} f(x) \\ &= e^{ax} (D + a) f(x) \\ D^2 e^{ax} f(x) &= D(De^{ax} f(x)) \\ &= D(e^{ax} Df(x) + ae^{ax} f(x)) \\ &= e^{ax} D^2 f(x) + ae^{ax} Df(x) + ae^{ax} Df(x) + a^2 e^{ax} f(x) \\ &= e^{ax} (D^2 + 2aD + a^2) f(x) \\ &= e^{ax} (D + a)^2 f(x) \end{aligned}$$

...

12

# Differential Operator Method

(用數學歸納法或類推法)

$$D^n e^{ax} f(x) = e^{ax} (D + a)^n f(x)$$

$$L(D) e^{ax} f(x)$$

$$= e^{ax} (D + a)^n f(x) + a_1 e^{ax} (D + a)^{n-1} f(x) + \cdots + a_{n-1} e^{ax} (D + a) f(x) + a_n e^{ax} f(x)$$

$$= e^{ax} \left[ (D + a)^n + a_1 (D + a)^{n-1} + \cdots + a_{n-1} (D + a) + a_n \right] f(x)$$

$$= e^{ax} L(D + a) f(x)$$

13

# Differential Operator Method

- 特性3.  $L(D^2) \sin ax = L(-a^2) \sin ax$

$$L(D^2) \cos ax = L(-a^2) \cos ax$$

例:  $y'' + 4y = \cos 3x$

$$\lambda^2 + 4 = 0$$

$$\lambda = \pm 2i$$

$$y_h = C_1 \cos 2x + C_2 \sin 2x$$

$$(D^2 + 4)y_p = \cos 3x$$

$$y_p = \frac{1}{D^2 + 4} \cos 3x \quad (\because a = 3)$$

$$= \frac{1}{-3^2 + 4} \cos 3x$$

$$= -\frac{1}{5} \cos 3x$$

14

# Differential Operator Method

Pf特性3 :

$$D \cos ax = -a \sin ax$$

$$\begin{aligned} D^2 \cos ax &= D(D \cos ax) \\ &= D(-a \sin ax) \\ &= -a^2 \cos ax \end{aligned}$$

$$\therefore D^2 \equiv -a^2$$

$$L(D^2) \equiv L(-a^2)$$

# Differential Operator Method

例:  $y'' + a^2 y = \cos ax$

$$\lambda^2 + a^2 = 0 \quad \lambda = \pm ai$$

$$y_h = C_1 \cos ax + C_2 \sin ax$$

$$(D^2 + a^2)y_p = \cos ax$$

$$\Rightarrow y_p = \frac{1}{D^2 + a^2} \cos ax$$

$$= \frac{1}{-a^2 + a^2} \cos ax$$

$$=? \text{ (因此用極限)}$$



# Differential Operator Method

$$\begin{aligned}\Rightarrow y_p &= \frac{1}{D^2 + a^2} \cos ax \\ &= \lim_{\Delta \rightarrow 0} \frac{1}{-(a + \Delta)^2 + a^2} \cos(a + \Delta)x \\ &= \lim_{\Delta \rightarrow 0} \frac{1}{-2a\Delta - \Delta^2} \cos(a + \Delta)x\end{aligned}$$

17

# Differential Operator Method

$$\left( \begin{aligned} & f(t) \text{ 於 } t = a \text{ 之 Taylor 級數展開式} \\ & f(t) = f(a) + f'(a)(t-a) + \frac{1}{2!} f''(a)(t-a)^2 + \cdots + \\ & \quad \frac{1}{n!} f^{(n)}(a)(t-a)^n + \cdots \\ & \cos t \text{ 於 } t = ax \text{ 之 Taylor 展開} \\ & \cos t = \cos ax - \sin ax(t-ax) - \frac{1}{2!} \cos ax(t-ax)^2 + \frac{1}{3!} \sin ax(t-ax)^3 + \cdots \end{aligned} \right)$$

18

# Differential Operator Method

其中 令  $t = (a + \Delta)x$

$$\cos((a + \Delta)x) = \cos ax - \sin ax((a + \Delta)x - ax) - \frac{1}{2!} \cos ax((a + \Delta)x - ax)^2 +$$

$$\frac{1}{3!} \sin ax((a + \Delta)x - ax)^3 + \dots$$

$$= \cos ax - \sin ax \Delta x - \frac{1}{2!} \cos ax (\Delta x)^2 + \frac{1}{3!} \sin ax (\Delta x)^3 + \dots$$

19

# Differential Operator Method

技巧：遇到cos儘量有理化

$$\Rightarrow y_p = \lim_{\Delta \rightarrow 0} \frac{1}{-(2a + \Delta)} \left[ \cos ax - \Delta x \sin ax - \frac{1}{2!} (\Delta x)^2 \cos ax + \frac{1}{3!} (\Delta x)^3 \sin ax + \dots \right]$$

(因解不下去,想一想cos(ax)是否可以不考慮?)  
(YES 因為  $y_h$  已含 cos(ax), 可以消去)

$$= \lim_{\Delta \rightarrow 0} \frac{1}{-(2a + \Delta)} \left[ -x \sin ax - \frac{1}{2!} \Delta x^2 \cos ax + \frac{1}{3!} \Delta^2 x^3 \sin ax + \dots \right]$$

$$= \frac{1}{-2a} - x \sin ax = \frac{1}{2a} x \sin ax$$

$$\Rightarrow y'' + a^2 y = \cos ax \quad \text{ps. } y'' + a^2 y = \sin ax$$

$$y = c_1 \cos ax + c_2 \sin ax - \frac{1}{2a} x \cos ax$$

$$y = C_1 \cos ax + C_2 \sin ax + \frac{1}{2a} x \sin ax$$

20

## Example Practice

例:  $y'' + 6y' + 9y = x^2$

$$\lambda^2 + 6\lambda + 9 = 0$$

$$(\lambda + 3)^2 = 0$$

$$\lambda = -3, -3$$

$$y_h = C_1 e^{-3x} + C_2 x e^{-3x}$$

21

## Example Practice

- [法1] 未定係數法

$$y_p = K_1 x^2 + K_2 x + K_3$$

代入  $y'_p = ?$

$$y''_p = ?$$

代入原式求  $K_1 K_2 K_3$

22

## Example Practice

- [法2] 降階法

$$(D^2 + 6D + 9)y_p = x^2$$

$$\Rightarrow (D + 3)(D + 3)y_p = x^2$$

$$\Rightarrow e^{-3x} \int e^{3x} \left[ e^{-3x} \int e^{3x} x^2 dx \right] dx = ?$$

23

## Example Practice

- [法3] 微分運算子法

$$(D^2 + 6D + 9)y_p = x^2$$

$$y_p = \frac{1}{(D^2 + 6D + 9)} x^2$$

$$= \frac{1}{9(1 + \frac{D^2 + 6D}{9})} x^2$$

技巧：換成無窮等比（才會收斂）

$$\frac{1}{1+x} = (1 - x + x^2 - x^3 + \dots)$$

$$= \frac{1}{9} \left[ 1 - \frac{D^2 + 6D}{9} + \left( \frac{D^2 + 6D}{9} \right)^2 - \left( \frac{D^2 + 6D}{9} \right)^3 + \dots \right] x^2$$

24

# Example Practice

$$\begin{aligned} &= \frac{1}{9} \left[ 1 - \frac{D^2 + 6D}{9} + \frac{D^4 + 12D^3 + 36D^2}{81} + \dots \right] x^2 \\ &= \frac{1}{9} \left[ x^2 - \frac{1}{9} (2 + 12x) + \frac{1}{81} \cdot 36 \cdot 2 \right] \\ &= \frac{1}{9} \left( x^2 - \frac{2}{9} - \frac{12}{9}x + \frac{8}{9} \right) \\ &= \frac{1}{9} \left( x^2 - \frac{4}{3}x + \frac{2}{3} \right) \end{aligned}$$