Chapter 4.
Laplace Transform

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Laplace Transform

- Review：

$$f(t) = e^{at} \xrightarrow{\mathscr{L}} \frac{1}{s-a}$$

$$f(t) = \cos at \xrightarrow{\mathscr{L}} \frac{s}{s^2+a^2}$$

$$f(t) = \sin at \xrightarrow{\mathscr{L}} \frac{a}{s^2+a^2}$$

$$f(t) = t \xrightarrow{\mathscr{L}} \frac{1}{s^2}$$

$$f(t) = H(t) \xrightarrow{\mathscr{L}} \frac{1}{s}$$

$$f(t) = \delta(t) \xrightarrow{\mathscr{L}} 1$$

$$f(t) = t^n \xrightarrow{\mathscr{L}} \frac{n!}{s^{n+1}}$$

Laplace Transform

- Laplace Transform之基本性質：

設$\mathscr{L}\{f(t)\} = F(s), \mathscr{L}\{g(t)\} = G(s)$已知

1. $\mathscr{L}\{k_1 f(t) + k_2 g(t)\} = k_1 \mathscr{L}\{f(t)\} + k_2 \mathscr{L}\{g(t)\} = k_1 F(s) + k_2 G(s)$

   線性轉換 $k_1, k_2 \in const$

   $pf: \mathscr{L}\{k_1 f(t) + k_2 g(t)\} = \int_0^\infty (k_1 f(t) + k_2 g(t)) e^{-st} dt$

   $= \int_0^\infty k_1 f(t) e^{-st} dt + \int_0^\infty k_2 g(t) e^{-st} dt$

   $= k_1 F(s) + k_2 G(s)$

Laplace Transform

2. First shifting Thm.(第一移位定理)

   $f(t) \xrightarrow{\mathscr{L}} F(s) = \mathscr{L}\{f(t)\}$

   $e^{at} f(t) \xrightarrow{\mathscr{L}} F(s-a)$

   $pf: \mathscr{L}\{e^{at} f(t)\} = \int_0^\infty e^{at} f(t) e^{-st} dt$

   $= \int_0^\infty f(t) e^{-(s-a)t} dt$

   $= \int_0^\infty f(t) e^{-s't} dt = F(s') = F(s-a)$

- 例：$H(t) \xrightarrow{\mathscr{L}} \dfrac{1}{s}, e^{at} \xrightarrow{\mathscr{L}} \dfrac{1}{s-a}$

  $\Rightarrow e^{at} H(t) \xrightarrow{\mathscr{L}} H(s-a) = \dfrac{1}{s}\Big|_{s \to s-a} = \dfrac{1}{(s-a)}$

Laplace Transform

- 例： $\mathscr{L}\{e^{-2t}t\}$

$\Rightarrow$ 已知 $f(t) \to F(s) = \dfrac{1}{s^2}$

$\Rightarrow F(s-(-2)) = F(s+2) = \dfrac{1}{s^2}\Big|_{s\to s+2} = \dfrac{1}{(s+2)^2}$

- 例： $\mathscr{L}\{e^{t}\cos 2t\}$

$= F(s-1)$ 其中 $F(s) = \dfrac{s}{s^2+4}$

$= \dfrac{s}{s^2+4}\Big|_{s\to s-1}$

$= \dfrac{s-1}{(s-1)^2+4} = \dfrac{s-1}{s-2s+5}$

Laplace Transform

- 例： $\mathscr{L}\{e^{3t}\sin 5t\}$

$\mathscr{L}\{e^{3t}\sin 5t\}$

$= F(s-3) = \dfrac{5}{s^2+25}\Big|_{s\to s-3}$

$= \dfrac{5}{(s-3)^2+25} = \dfrac{5}{s^2-6s+34}$

- 例： $\mathscr{L}^{-1}\{\dfrac{2}{s^2+2s+5}\}$

$= \mathscr{L}^{-1}\{\dfrac{2}{(s+1)^2+2^2}\}$

$= \sin(2t)e^{-t}$

Laplace Transform

- 例：$\mathscr{L}^{-1}\{\frac{s}{(s+2)^2+4}\}$

$= \mathscr{L}^{-1}\{\frac{s+2-2}{(s+2)^2+4}\}$

$= \mathscr{L}^{-1}\{\frac{s+2}{(s+2)^2+4}\} + \mathscr{L}^{-1}\{\frac{-2}{(s+2)^2+4}\}$

$\Rightarrow f(t) = e^{-2t}\cos 2t - e^{-2t}\sin 2t$

Laplace Transform

3. Second shifting Thm.(第二移位定理)

$f(t) \xrightarrow{\mathscr{L}} F(s)$

$f(t-a)H(t-a) \xrightarrow{\mathscr{L}} F(s)e^{-as}$

只有在f(t-a) != σ(t-a) 時
H(t-a)均需保留

$pf: \mathscr{L}\{f(t-a)H(t-a)\}$

$= \int_0^\infty f(t-a)H(t-a)e^{-st}dt \qquad \because H(t-a) = \begin{cases} 1, t > a \\ 0, t < a \end{cases}$

$= \int_a^\infty f(t-a)e^{-st}dt \qquad$ 令$x = t-a, dx = dt$

$= \int_0^\infty f(x)e^{-s \cdot x}e^{-as}dx$

$= e^{-as}\int_0^\infty f(x)e^{-sx}dx$

$= e^{-as}F(s)$

Laplace Transform

- 例：$f(t) = e^{2t} \xrightarrow{\mathscr{L}} \dfrac{1}{s-2}$

  H (t-3)

  $\mathscr{L}\{e^{2(t-3)}\} = F(s)e^{-as} = \dfrac{1}{s-2}e^{-3s}$

- 例：$\mathscr{L}\{\cos(t-2)H(t-2)\}$

  $\because \mathscr{L}\{\cos t\} = \dfrac{s}{s^2+1}$

  $= \dfrac{s}{s^2+1}e^{-2s}$

- 例：$G(s) = e^{-3s}\dfrac{s+1}{(s+1)^2+1}$

  $\Rightarrow g(t) = e^{-(t-3)}\cos(t-3)H(t-3)$

Laplace Transform

- 例：$f(t) = t, \mathscr{L}\{f(t)\} = \dfrac{1}{s^2}$

  $\Rightarrow \mathscr{L}\{f(t-2)\} = \mathscr{L}\{t-2\} = \mathscr{L}\{t\} - \mathscr{L}\{2\} = \dfrac{1}{s^2} - \dfrac{2}{s}$

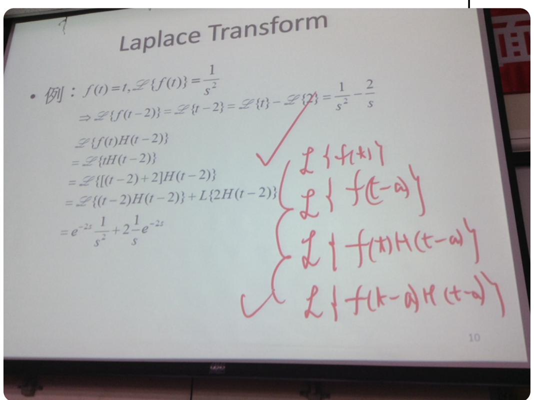
  $\mathscr{L}\{f(t)H(t-2)\}$

  $= \mathscr{L}\{tH(t-2)\}$

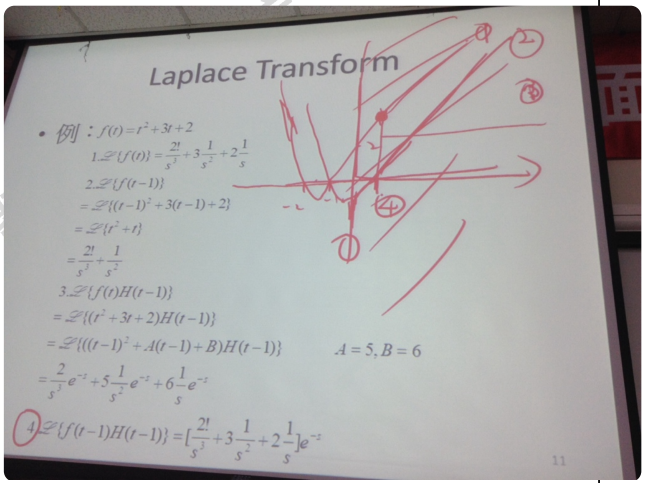
  $= \mathscr{L}\{[(t-2)+2]H(t-2)\}$

  $= \mathscr{L}\{(t-2)H(t-2)\} + L\{2H(t-2)\}$

  $= e^{-2s}\dfrac{1}{s^2} + 2\dfrac{1}{s}e^{-2s}$



Laplace Transform



- 例：$f(t)=t^2+3t+2$

$$1.\mathscr{L}\{f(t)\}=\frac{2!}{s^3}+3\frac{1}{s^2}+2\frac{1}{s}$$

$$2.\mathscr{L}\{f(t-1)\}$$

$$=\mathscr{L}\{(t-1)^2+3(t-1)+2\}$$

$$=\mathscr{L}\{t^2+t\}$$

$$=\frac{2!}{s^3}+\frac{1}{s^2}$$

$$3.\mathscr{L}\{f(t)H(t-1)\}$$

$$=\mathscr{L}\{(t^2+3t+2)H(t-1)\}$$

$$=\mathscr{L}\{((t-1)^2+A(t-1)+B)H(t-1)\} \qquad A=5, B=6$$

$$=\frac{2}{s^3}e^{-s}+5\frac{1}{s^2}e^{-s}+6\frac{1}{s}e^{-s}$$

$$4.\mathscr{L}\{f(t-1)H(t-1)\}=[\frac{2!}{s^3}+3\frac{1}{s^2}+2\frac{1}{s}]e^{-s}$$

Laplace Transform

$$4.f(t)\xrightarrow{\text{L}}F(s)$$

$$tf(t)\xrightarrow{\text{L}}\frac{-dF(s)}{ds}$$

- 例：$1\xrightarrow{\mathscr{L}}\frac{1}{s}$

$$t\xrightarrow{\mathscr{L}}\frac{1}{s^2}$$

$$t\xrightarrow{\mathscr{L}}\frac{-d}{ds}(\frac{1}{s})=\frac{-1}{-s^2}=\frac{1}{s^2}$$

Laplace Transform

- 例：$t^n \xrightarrow{\mathscr{L}} \dfrac{n!}{s^{n+1}}$

$$t \xrightarrow{\mathscr{L}} \frac{1}{s^2}$$

$$\text{t • t = t f(t)} = t^2 \xrightarrow{\mathscr{L}} -\left(\frac{d}{ds}\left(\frac{1}{s^2}\right)\right) = \frac{-(-2s)}{(s^2)^2} = \frac{2}{s^3}$$

$$\vdots$$

$$t^n \xrightarrow{\mathscr{L}} \frac{n!}{s^{(n+1)}}$$

Laplace Transform

證明：$\mathscr{L}\{tf(t)\} = \int_0^\infty tf(t)e^{-st}dt$

$$\because \frac{d}{ds}e^{-st} = -te^{-st}$$

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty f(t)e^{-st}dt$$

$$= \int_0^\infty \frac{\partial}{\partial s}(f(t)e^{-st})dt$$

$$= \int_0^\infty f(t)(-t)e^{-st}dt$$

$$= -\int_0^\infty f(t)te^{-st}dt$$

$$= -\mathscr{L}\{tf(t)\}$$

Laplace Transform

- 例：$\mathscr{L}\{t\sin 2t\}$

$$= -\frac{d}{ds}\left(\frac{2}{s^2+4}\right)$$

$$= -\frac{0-2(2s)}{(s^2+4)^2} = \frac{4s}{(s^2+4)^2}$$

- 例：$\mathscr{L}\{e^{2t}\cdot \underline{t\cdot\sin 2t}\} = F(s-2)$

$$\left.\frac{4s}{(s^2+4)^2}\right|_{s=s-2} = \frac{4\cdot(s-2)}{((s-2)^2+4)^2}$$

Laplace Transform

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

證明：$1 \xrightarrow{h} \frac{1}{s}$

利用數學歸納法 第一數學歸納法

n $=1$ $\mathscr{L}\{t\ \ \} = \frac{-d}{ds}\cdot\frac{1}{s} = \frac{1}{s^2}$

設 $n=k-1$為真

$$\Rightarrow \mathscr{L}\{t^{k-1}\} = \frac{(k-1)!}{s^k}$$

欲證

$n=k$時亦為真

$$\mathscr{L}\{t^k\} = \mathscr{L}\{t\cdot t^{k-1}\} = \frac{-d}{ds}\left(\frac{(k-1)!}{s^k}\right) = -(k-1)(-k)s^{-k-1} = \frac{k!}{s^{k+1}}$$ 故得證

Laplace Transform

推廣：$t^n f(t) \xrightarrow{\mathscr{L}} \frac{-d}{ds} \cdots (\frac{-d}{ds} F(s))$

4. $tf(t) \xrightarrow{\mathscr{L}} \frac{-d}{ds} F(s)$

$$\frac{1}{t} f(t) \xrightarrow{\mathscr{L}} \int_s^\infty \int_0^\infty f(t)e^{-st} dtds$$

$$\Rightarrow \int_0^\infty (1) ds = \int_s^\infty \int_0^\infty f(t)e^{-st} dtds$$

$$= \int_0^\infty \int_s^\infty f(t)e^{-st} dsdt \qquad \text{(積分次序對調，要S .T在積分範圍獨立)}$$

$$= \int_0^\infty f(t) \int_s^\infty e^{-st} dsdt$$

$$= \int_0^\infty f(t) \frac{1}{t} e^{-st} dt$$

$$= \mathscr{L}\{\frac{1}{t} f(t)\}$$