Chapter 5.
Series Solutions of Linear Differential Equations

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Series Solutions

ex: $x^2y''+\left(x^2+\frac{5}{36}\right)y=0$於$x=0$的級數解

$x=0$為規則異點,存在$y=\sum_{n=0}^{\infty}a_nx^{n+r}$

$$y'=\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}$$

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}+\left(x^2+\frac{5}{36}\right)\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

$$\Rightarrow\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r}+\sum_{n=0}^{\infty}a_nx^{n+r+2}+\frac{5}{36}\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

$n\rightarrow n+2$

$$\sum_{n=-2}^{\infty}(n+r+2)(n+r+1)a_{n+2}x^{n+r+2}+\sum_{n=0}^{\infty}a_nxn^{n+r+2}+\frac{5}{36}\sum_{n=-2}^{\infty}a_{n+2}x^{n+r+2}=0$$

Series Solutions

提前兩項

$$\left[r(r-1)a_0x^r+\frac{5}{36}a_0x^r\right]+\left[(r+1)ra_1x^{r+1}+\frac{5}{36}a_1x^{r+1}\right]$$

$$+\sum_{n=0}^{\infty}\left\{\left[(n+r+2)(n+r+1)+\frac{5}{36}\right]a_{n+2}+a_n\right\}x^{n+r+2}=0$$

$$\left.\begin{array}{l}
1.\left[r(r-1)\quad x^r+\frac{5}{36}\quad x^r\right]a_0=0\\
2.\left[(r+1)r\quad x^{r+1}+\frac{5}{36}\quad x^{r+1}\right]a_1=0\\
3.\left[\underbrace{(n+r+2)(n+r+1)+\frac{5}{36}}_{A(n,r)}\right]a_{n+2}+a_n=0
\end{array}\right\}\text{挑一個設不為0來微}$$

$$a_{n+2}=\frac{-a_n}{A(n,r)},n\geq 0$$

Series Solutions

$a_0\neq 0$, 帶入2確認 $a_1=0$

$$\therefore r(r-1)+\frac{5}{36}=0$$

$$r=\frac{1}{6},\frac{5}{6}$$

$$r=\frac{1}{6}$$

$$a_{n+2}=\frac{-1}{\left(n+\frac{1}{6}+2\right)\left(n+\frac{1}{6}+1\right)+\frac{5}{36}}a_n$$

$$=\frac{-1}{\left(n+\frac{13}{6}\right)\left(n+\frac{7}{6}\right)+\frac{5}{36}}a_n$$

$$a_2=\frac{-1}{\left(\frac{13}{6}\right)\left(\frac{7}{6}\right)+\frac{5}{36}}a_0=-\frac{3}{8}a_0$$

Series Solutions

$$a_3 = 0$$

$$a_4 = \frac{-1}{\frac{25}{6}\cdot\frac{19}{6}+\frac{5}{36}} a_2 = \frac{9}{320} a_0$$

$$y_1 = \Big(\Big[a_0 x^{\frac{1}{6}} + a_2 x^{2+\frac{1}{6}} + a_4 x^{4+\frac{1}{6}} + \ldots$$

$$= a_0 x^{\frac{1}{6}} + \left(-\tfrac{3}{8}\right) a_0 x^{2+\frac{1}{6}} + \frac{9}{320} a_0 x^{4+\frac{1}{6}} + \ldots$$

$$y_2 = a_0 x^{\frac{1}{6}} \left(1 - \tfrac{3}{8} x^2 + \ldots\right) \Rightarrow \text{代} y = \tfrac{5}{6} \text{求得}$$

$$\frac{y_1}{y_2} \neq k_1,\ y_1, y_2 \text{獨立}$$

Series Solutions

ex: $x(x-1)y'' + (3x-1)y' + y = 0$ 於x = 0 級數解

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, |x| < 1$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} + 3\sum_{n=0}^{\infty} (n+r) a_n x^{n+r}$$

$$-\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Series Solutions

$n \to n+1$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=-1}^{\infty}(n+r+1)(n+r)a_{n+1}x^{n+r} + 3\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}$$
$$-\sum_{n=-1}^{\infty}(n+r+1)a_{n+1}x^{n+r} + \sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$n=-1$

$$-r(r-1)a_0x^{r-1} - \left[ra_0x^{r-1}\right]$$
$$+\sum_{n=0}^{\infty}\left\{\left[-(n+r+1)(n+r)-(n+r+1)\right]a_{n+1} + \left[(n+r)(n+r-1)+3(n+r)+1\right]a_n\right\}x^{n+r} = 0$$

Series Solutions

1. $\left(-r(r-1)-r\right)a_0 = 0$
2. $a_{n+1} = a_n$

$\because a_0 \neq 0$

$r^2 = 0, \qquad r = 0,0$

$$y_1 = a_0 + a_1x + a_2x^2 + ... + a_nx^n + ...$$
$$y_1 = a_0(1 + x + x^2 + x^3 + ... + x^n + ...)$$
$$y_1 = \left(\frac{1}{1-x}\right)a_0$$

另一獨立解改用參數變異法

$$y_2 = \phi y_1$$
$$y_2' = \phi y_1' + \phi' y_1$$
$$y_2'' = \phi y_1'' + \phi' y_1' + \phi' y' + \phi'' y_1 = \phi y_1'' + 2\phi' y_1' + \phi'' y_1$$

$$x(x-1)(\phi y_1'' + 2\phi' y_1' + \phi'' y_1) + \left((3x-1)\ (\phi y_1' + \phi' y_1)\right) + \phi y_1 = 0$$
$$\phi\Big[\underbrace{x(x-1)y_1'' + (3x-1)\ \ y_1' + y_1}_{湊成題目\ =0}\Big] + x(x-1)(2\phi' y_1' + \phi'' y_1) + (3x-1)\phi' y_1 = 0$$

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Series Solutions

$$y_1 = \frac{1}{1-x}, y_1' = \frac{1}{(1-x)^2}$$

$$\Rightarrow x(x-1)\left[2\phi' \tfrac{1}{(x-1)^2} + \phi'' \tfrac{-1}{x-1}\right] + (3x - 1)\ \phi' \frac{-1}{x-1} = 0$$

$$2\phi' \frac{x}{x-1} + \phi''(-x) + \phi'\left(\frac{-(3x-1)}{x-1}\right) = 0$$

$$(2x - 3x + 1)\phi' + (-x^2 + x)\phi'' = 0$$

$$\phi'' + \tfrac{1}{x}\phi' = 0 \quad \rightarrow \Phi'' = (\Phi')'$$

$$\rightarrow \Phi' = \varphi$$

$$\frac{d\phi'}{\phi'} = -\frac{1}{x}dx \qquad \varphi' + \frac{1}{x}\varphi = 0$$

$$\ln|\phi'| = -\ln|x| \qquad y_2 = \phi y_1 \qquad y = c_1\left(\frac{1}{1-x}\right) + c_2\left(\frac{\ln x}{1-x}\right)$$

$$\phi' = \frac{1}{x} \qquad \therefore \phi = \ln x \qquad y_2 = \left(\frac{\ln x}{1-x}\right)a_0$$

Series Solutions

summary: $p(x)y'' + q(x)y' + r(x)y = 0$, p,q,r不能再消去項

$x = a$的級數解

1. $p(a) \neq 0 \Rightarrow$ 常數

$\Rightarrow$ 存在Taylor級數 在x = a 處

$$y(x) = \sum_{n=0}^{\infty} a_n (x-a)^n, \quad |x-a| < L, \quad L: x = a\text{到最近異點的距離}$$

2. $p(a) = 0 \Rightarrow$ singular point

若$(x-a)\cdot\frac{q}{p}$, $(x-a)^2\cdot\frac{r}{p}$這兩項於$x = a$均可微分

$\Rightarrow x = a$為規則異點

$\Rightarrow$ 存在一Frobenius級數解 在x = a處

$$y(x) = \sum_{n=0}^{\infty} a_n (x-a)^{n+r}, \quad |x-a| < L, \quad L: x - a\text{到另一點異點的距離}$$

Series Solutions

$r = r_1 \rightarrow y_1(x)$

$r = r_2 \rightarrow y_2(x)$

(I) $r_1 \neq r_2$, $|r_1 - r_2| \notin N$

y_1, y_2一定線性獨立,構成一組基底解

$y = c_1 y_1 + c_2 y_2$

(II) $r_1 \neq r_2$, $|r_1 - r_2| \in N$

(A) y_1, y_2獨立解

$\therefore y = c_1 y_1 + c_2 y_2$

(B) y_1, y_2線性相依, 另一個獨立解利用參數變異法求解y_2

$\therefore y = c_1 y_1 + c_2 y_2$ $y_2 = \phi y_1$

Series Solutions

(III) $r_1 = r_2 = r \Rightarrow y_1$

另一個獨立解也是由參數變異法求得

3. a為不規則異點則方程式於$x = a$處無級數解

ex: $x^2 y'' + \left(x^2 + \frac{5}{36}\right) y = 0$, 其indicial e.g.為何?

x^2 x^{n+r-2} $x^2 + \dfrac{5}{36}$ x^{n+r}

x^{n+r} x^{n+r+2} x^{n+r}

Series Solutions

ex: $2x(1-x)y''+(1+x)y'-y=0$

$n+r-1$ $n+r$ $n+r-1$ $n+r$ $n+r$

$2r(r-1)$ $+r=0$

ex: $x(1+x)y''+4(x+3)y'+5y=0$ x = 0 的指標方程式

$n+r-1$ $n+r$ $4n+r$ $12n+r\text{-}1$ $n+r$

$r(r-1)+12r=0$為指標方程式

$r^2+11r=0$

$r=0,-11$

Series Solutions

$x=0$,規則異點,指標方程式如上所示

$x=-1$,規則異點

$$y=\sum_{n=0}^{\infty}a_n(x+1)^{n+r},|x+1|<1$$

$$y'=\sum_{n=0}^{\infty}a_n(n+r)(x+1)^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}a_n(n+r)(n+r-1)(x+1)^{n+r-2}$$

$$(x+1-1)(x+1)\sum_{n=0}^{\infty}a_n(n+r)(n+r-1)(x+1)^{n+r-2}+4(x+3)$$

$r=0,\ 9$

Series Solutions

((x + 1) - 3) (3(x + 1) - 2)

ex: $(x+1)(\underline{x-2})y''+4(\underline{3x+1})y'+6y=0$

先判斷何種異點（常點）
再求其解、收斂區間、指標方程式

(1) 於$x=-1$級數解

$$y(x)=\sum_{n=0}^{\infty}a_n(x+1)^{n+r}, \text{收斂區間}:|x+1|<3$$

指標方程式: $-3r(r-1)-8r=0 \Rightarrow -3r^2-5r=0$

(2) 於$x=2$級數解

$$y(x)=\sum_{n=0}^{\infty}a_n(x-2)^{n+r}, \text{收斂區間}:|x-2|<3$$

指標方程式: $(x-2+3)(x-2)y''+4(3(x-2)+7)y^2+6y=0$

$$3r(r-1)+28r=0$$

$$3r^2+25r=0$$

(3) 於x = 0 級數解 $y=\sum_{n=0}^{\infty}a_nx^n$, 收斂區間：｜x｜< 1 指標方程式：不存在

Series Solutions

Special case: 科西尤拉D.E.

ex: $x^2y''+4xy'+2y=0$

(1) 令$x=e^t$, $xy'=\dfrac{dy}{dt}$, $x^2y''=\dfrac{d^2y}{dt^2}-\dfrac{dy}{dt}$

$$\frac{d^2y}{dt^2}-\frac{dy}{dt}+4\frac{dy}{dt}+2y=0$$

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=0$$

$$\lambda^2+3\lambda+2=0$$

$$\lambda=-1,-2$$

$$y(x)=c_1e^{-t}+c_2e^{-2t}$$

$$=c_1x^{-1}+c_2x^{-2}$$

Series Solutions

(2) 利用於$x=0$的級數解, 驗證(1)的結果

$\because x=0$為規則異點 $\Rightarrow$ 存在Frobenius級數

$$y(x)=\sum_{n=0}^{\infty}a_n x^{n+r}, |x|<\infty$$

$$y'(x)=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''(x)=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+4x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+2\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left(a_n(n+r)(n+r-1)+4a_n(n+r)+2a_n\right)x^{n+r}=0$$

$n=0$最低次, $a_0\left(r(r-1)+4r+2\right)x^r=0$

Series Solutions

$\because a_0\neq 0$

$\therefore r(r-1)+4r+2=0$

$r^2+3r+2=0$

$(r+1)(r+2)=0$

$r=-1,-2$

(i) $r=-1$

$n\geq 1$

$$\left\{(n+r)(n+r-1)+4(n+r)+2\right\}a_n=0$$

$$\left\{(n-1)(n-2)+4(n-1)+2\right\}a_n=0$$

$$\left(n^2-3n+2+4n-4+2\right)a_n=0$$

Series Solutions

$$\left(n^2+n\right)a_n = 0$$
$$n\left(n+1\right)a_n = 0,\quad n\left(n+1\right)\neq 0$$
$$n \geq 1,\quad a_n = 0$$
$$y_1\left(x\right) = a_0 x^{-1}$$
$$\left(\text{ii}\right) r = -2$$
$$n \geq 1$$
$$\left[\left(n+r\right)\left(n+r-1\right)+4\left(n+r\right)+2\right]a_n^* = 0$$
$$\left(n-2\right)\left(n-3\right)+4\left(n-2\right)+2$$
$$n^2-5n+6+4n-8+2$$
$$\left(n^2-n\right)a_n^* = 0$$
$$n\left(n-1\right)a_n^* = 0$$

Series Solutions

$n = 1,0$　　a_0^*, a_1^*可以不為0

$$n \geq 2, a_n^* = 0$$
$$y_2\left(x\right) = \sum_{n=0}^{\infty} a_n x^{n-2}$$
$$= a_0^* x^{-2} + a_1^* x^{-1} + a_2^* x^0 + \ldots$$
$$= a_0^* x^{-2} + a_1^* x^{-1}$$
$$\therefore y = k_1 y_1\left(x\right) + k_2 y_2\left(x\right)$$
$$= k_1 a_0 x^{-1} + k_2\left(a_0^* x^{-2} + a_1^* x^{-1}\right)$$

$= c_1 x^{-1} + c_2 x^{-2}$　其中 $c_1 = k_1 a_0 + k_2 a_1^*$

$c_1 = k_2 a_0^*$

Series Solutions

補充:若題目要求$a_3 \neq 0$

則$n = 3$

$\left((3+r)(2+r)+4(3+r)+2\right)a_n = 0$成為新的指標方程式

ex: $(x-2)^2 y'' + 4(x-2)y' + 2y = 0$

(i) 令 $u = x-2$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{y}{dx}\right)\cdot\frac{du}{dx}1 = \frac{d}{du}\left(\frac{dy}{du}\right)$$

$$u^2y'' + 4uy' + 2y = 0$$

$$y(u) = c_1u^{-1} + c_2u^{-2}$$

$$= c_1(x-2)^{-1} + c_2(x-2)^{-2}$$

Series Solutions

(ii) 利用於$x = 2$的級數解,驗證(i)的結果

$$y' = \sum_{n=0}^{\infty} a_n(n+r)(x-2)^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)(x-2)^{n+r-2}$$

$$(x-2)^2\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)(x-2)^{n+r-2} + 4(x-2)\sum_{n=0}^{\infty} a_n(n+r)(x-2)^{n+r-1}$$

$$+2\sum_{n=0}^{\infty} a_n(x-2)^{n+r} = 0$$

Series Solutions

$$\sum_{n=0}^{\infty}\left[a_n(n+r)(n+r-1)+4a_n(n+r)+2a_n\right](x-2)^{n+r}=0$$

$n=0,\quad r(r-1)+4r+2=0$ 指標方程式

$r=-1,-2$

r = $-1,\quad y_1(x)=a_0(x-2)^{-1}$

r = $-2,\quad y_2(x)=a_0^*(x-2)^{-2}+a_1^*(x-2)^{-1}$

$\therefore y=k_1y_1+k_2y_2=c_1(x-2)^{-1}+c_2(x-2)^{-2}$

Series Solutions

ex: Find the indicial equation of e^x可換sin(x)、cos(x)考

$x^2y''+xe^xy'+(x^2-1)y=0$

if the solution is required near $x=0$

sol: $x^2=0,\quad x=0$異點

但$x\dfrac{xe^x}{x^2},x^2\dfrac{x^2-1}{x^2}$二者皆可微分

$\therefore x$=0為規則異點 $\Rightarrow$ 存在Frobenius級數

Series Solutions

$$y(x)=\sum_{n=0}^{\infty}a_n x^{n+r}, |x|<\infty$$

$$y'(x)=\sum_{n=0}^{\infty}a_n(n+r)x^{n+r-1}$$

$$y''(x)=\sum_{n=0}^{\infty}a_n(n+r)(n+r-1)x^{n+r-2}$$

代入$x^2y''+x\underbrace{\left(1+x+\frac{1}{2}x^2+\ldots\right)}_{e^x}y'+\left(x^2-1\right)y=0$

$n+r$次的係數

$r(r-1)+r-1=0$

$r^2-r+r-1=0$

$r^2-1=0\quad\rightarrow$ 指標方程式

$r=1,-1$

Series Solutions

§ Legendre differential equation

$\left(1-x^2\right)y''-2xy'+\lambda y=0\quad\rightarrow$ 出現未知數λ

in which $-1\le x\le 1$, and λ is a real constant

$x=0$的級數解 $|x|<1$

$<$分析$>$ $1-x^2=(1-x)(1+x)$

$\therefore x=1,-1$為方程式異點

而$x=0$為常點(O.D.P.)

對一個Taylor級數

Series Solutions

$$y(x)=\sum_{n=0}^{\infty}a_n x^n, |x|<1$$

$$y'(x)=\sum_{n=0}^{\infty}na_n x^{n-1}$$

$$y''(x)=\sum_{n=0}^{\infty}n(n-1)a_n x^{n-2}$$

$$(1-x^2)\sum_{n=0}^{\infty}n(n-1)a_n x^{n-2}-2x\sum_{n=0}^{\infty}na_n x^{n-1}+\lambda\sum_{n=0}^{\infty}a_n x^n=0$$

$$\Downarrow\ n\to n+2$$

$$\sum_{n=-2}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}n(n-1)a_n x^n\quad-2\sum_{n=0}^{\infty}na_n x^n+\lambda\sum_{n=0}^{\infty}a_n x^n=0$$

n = 0 $\quad 2a_2+\lambda a_0=0$

n = 1 $\quad 6a_3x-2a_1x+\lambda a_1x=0$

n >= 2 $\quad \{[(n+2)(n+1)]a_{n+2}+[-n(n-1)-2n+\lambda]a_n\}x^n=0$

Series Solutions

1. $2a_2+\lambda\ a_0=0\Rightarrow a_2=-\frac{\lambda}{2}a_0......(1)$
2. $6a_3-2a_1+\lambda a_1=0\Rightarrow a_3=\frac{2-\lambda}{6}a_1.......(2)$
3. $(n+2)(n+1)a_{n+2}+(-(n)(n-1)-2n+\lambda)a_n=0$

由3循環公式

$$a_{n+2}=\frac{n(n+1)-\lambda}{(n+1)(n+2)}a_n,\quad n\geq 2$$

Series Solutions

λ可控制級數，一個λ只會對應一特定級數

<分析>

$n=2$ $$a_4=\frac{2\cdot3-\lambda}{3\cdot4}a_2=\frac{6-\lambda}{4\cdot3}\left(-\frac{\lambda}{2}\right)a_0=\frac{(6-\lambda)(-\lambda)}{4!}a_0$$

$n=3$ $$a_5=\frac{3\cdot4-\lambda}{4\cdot5}a_3=\frac{12-\lambda}{4\cdot5}\left(-\frac{2\lambda}{2\cdot3\cdot6}\right)a_1=\frac{(12-\lambda)(2-\lambda)}{5!}a_1$$

$n=4$ $$a_6=\frac{4\cdot5-\lambda}{5\cdot6}a_4=\frac{(-\lambda)(6-\lambda)(20-\lambda)}{6!}a_0$$

$n=5$ $$a_7=\frac{5\cdot6-\lambda}{6\cdot7}a_5=\frac{(2-\lambda)(12-\lambda)(30-\lambda)}{7!}a_1$$

Series Solutions

$$\left(1-x^2\right)y''-2xy'+\lambda y=0$$

$$y(x)=\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+\ldots$$

$$=a_0+a_1x+\left(\tfrac{-\lambda}{2}\right)a_0x^2+\left(\tfrac{2-\lambda}{6}\right)a_1x^3+\tfrac{-\lambda(6-\lambda)}{4!}a_0x^4+\ldots$$

$$=a_0[\qquad \text{偶} \qquad]+a_1[\qquad \text{奇} \qquad]$$

$$y(x)=a_0\left(1+\left(\tfrac{-\lambda}{2}\right)x^2+\tfrac{-\lambda(6-\lambda)}{4!}x^4+\tfrac{-\lambda(6-\lambda)(20-\lambda)}{6!}x^6+\ldots\right)$$

$$+a_1\left(x+\tfrac{(2-\lambda)}{6}x^3+\tfrac{(2-\lambda)(12-\lambda)}{5!}x^5+\tfrac{(2-\lambda)(12-\lambda)(30-\lambda)}{7!}x^7+\ldots\right)$$

$$=a_0y_e(x)+a_1y_0(x)$$

$$a_{n+2}=\frac{n(n+1)-\lambda}{(n+2)(n+1)}a_n$$ 循環公式

Series Solutions

若λ=N$\left(N+1\right)$

$$a_{n+2}=\frac{n\left(n+1\right)-N\left(N+1\right)}{\left(n+2\right)\left(n+1\right)}a_n,\quad n\geq 2$$

$a_{n+2}=0,\quad \forall 2\leq n\leq N$

$\therefore N$可以為奇數或偶數

$\therefore a_{n+2}=0$

$\therefore a_{n+4}=0$

$\therefore y_e\left(x\right)$或$y_o(x)$有一個會有有限項

$\Rightarrow$針對有限項的解,若選擇當$x=1$時,讓$y_e(1)=1$或$y_o(1)=1$的有限項解,

則此解稱為Legendre's polynornail,記為$P_n(x)$

Series Solutions

$P_n(x)=1$

$$a_0=(-1)^{\frac{n}{2}}\frac{1\times3\times...\times(n-1)}{2\times4\times...\times n}$$

$$a_1=(-1)^{\frac{n-1}{2}}\frac{1\times3\times...\times n}{2\times4\times...\times(n-1)}$$

$\lambda=0$ $\quad P_0(x)=1$ $\qquad\qquad \lambda=n(n+1)$

$\lambda=2$ $\quad P_1(x)=x$

$\lambda=6$ $\quad P_2(x)=(-1)^{\frac{2}{2}}\frac{1}{2}[1-3\chi^2]=\frac{1}{2}(3x^2-1)$

$\lambda=12$ $\quad P_3(x)=\frac{1}{2}\left(5x^2-3x\right)$

$\lambda=20$ $\quad P_4(x)=\frac{1}{8}\left(35x^4-30x^2+3\right)$

$\lambda=30$ $\quad P_5(x)=\frac{1}{8}\left(63x^5-70x^3+15x\right)$

Series Solutions

補充:

$\because \lambda$是變數，會有special function產生

p.674

$$\left(1-x^2\right)y'' - 2xy' + \lambda y = 0$$

原式可改寫為

$\left(\left(1-x^2\right)y'\right)' + \lambda y = 0$, 產生之非零解的稱為eigenvalue，

其對應的解叫做eigenfunction，而且eigenfunction在收斂區間內是正交的。

另一個special function：Bessel differential equation

$$x^2 y'' + xy' + \left(x^2 - \upsilon^2\right)y = 0$$

$x = 0$規則異點 $\Rightarrow$ Frobenious級數解

Series Solutions

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad |x| < \infty$$

$$y'(x) = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

代入原式

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \left(x^2 - \upsilon^2\right) \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$n \to n+2$代入

$$\sum_{n=-2}^{\infty} (n+r+2)(n+r+1) a_{n+2} x^{n+r+2} + \sum_{n=-2}^{\infty} (n+r+2) a_{n+2} x^{n+r+2} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - \upsilon^2 \sum_{n=-2}^{\infty} a_{n+2} x^{n+r+2} = 0$$

Series Solutions

$n=-2$

① $r(r-1)a_0x^r + ra_0x^r - \upsilon^2 a_0 x^r = 0$

$n=-1$

② $(r+1)ra_1x^{r+1} + (r+1)a_1x^{r+1} - \upsilon^2 a_1 x^{r+1} = 0$

③ $\left\{\left[(n+r+2)(n+r+1)+(n+r+2)-\upsilon^2\right]a_{n+2}+a_n\right\}x^{n+r+2}=0$

Series Solutions

① $\left(r(r-1)+r-\upsilon^2\right)a_0=0$

② $\left[r(r+1)+(r+1)-\upsilon^2\right]a_1=0$

③ $\left[(n+r+2)^2-\upsilon^2\right]a_{n+2}+a_n=0$

$\because a_0 \neq 0, \Rightarrow$ 指標方程式 $r^2-\upsilon^2=$ $r=\upsilon,-\upsilon$

② $\rightarrow \left[(r+1)^2-\upsilon^2\right]a_1=0$

$\therefore a_1=0$

③ $\rightarrow a_{n+2}=\dfrac{-1}{(n+r+2)^2-\upsilon^2}a_n$

$\because |r_1-r_2|=2\upsilon$

若$2\upsilon\notin\mathcal{N}$,會有二個獨立解

$r=\upsilon, a_{n+2}=\dfrac{-1}{(n+\upsilon+2)^2-\upsilon^2}a_n=\dfrac{-1}{(n+2)(n+2+2\upsilon)}a_n$

Series Solutions

$$n = 1, a_3 = \frac{-1}{3(3+2\upsilon)} a_1 = 0$$

$$a_1 = a_3 = \cdots = a_{2n+1} = 0$$

為了方便計算$n + 2 \to n$

$$a_{n+2} = \frac{-1}{(n+2)(n+2+2\upsilon)} a_n$$, n ≥ 2

$$n = 2$$

$$a_2 = \frac{-1}{2(2+2\upsilon)} a_0 = \frac{-1}{2^2(1+\upsilon)} a_0$$

n = 4

$$a_4 = \frac{-1}{4(4+4\upsilon)} a_2 = \frac{-1}{2^2 2(2+\upsilon)} a_2 = \frac{(-1)^2}{2^2 2(2+\upsilon) 2^2 (1+\upsilon)} a_0$$

Series Solutions

n = 6

$$a_6 = \frac{-1}{6(6+2\upsilon)} a_4 = \frac{(-1)^3}{2^6 \cdot 3 \cdot 2(3+\upsilon)} a_0$$

$$a_{2n} = \frac{(-1)^n}{2^{2n} \cdot n!(n+\upsilon)(n-1+\upsilon)(n-2+\upsilon)\cdots(1+\upsilon)} a_0$$

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \quad (\because a_{2n+1} = 0)$$

$$= \sum_{n=0}^{\infty} a_{2n} x^{2n+\upsilon}$$

$$a_0 = \frac{1}{2^{\nu} \Gamma(1+v)}$$

$$= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} \cdot n!(n+\upsilon)(n-1+\upsilon)(n-2+\upsilon)\cdots(1+\upsilon)} x^{2n+\upsilon}$$

Series Solutions

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} = \mathcal{J}_\nu(x)$$ $\Gamma(n+1) = n!$

$\mathcal{J}_\nu(x)$: Bessel function of the first kind

另外一個$r = -\upsilon$

$$y_2(x) = \mathcal{J}_{-\upsilon}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{2n-\upsilon}$$

$$\therefore y = c_1\mathcal{J}_\upsilon + c_2\mathcal{J}_{-\upsilon}$$

Series Solutions

Ex: $x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0$ Bessel function

$$r = \frac{1}{3}, \frac{-1}{3}$$

$$|\upsilon_1 - \upsilon_2| = \frac{2}{3} \notin \mathcal{N}$$

$$y = c_1\mathcal{J}_{\frac{1}{3}}(x) + c_2\mathcal{J}_{-\frac{1}{3}}(x)$$

另外需將$\mathcal{J}_\upsilon(x) = \sum_{n=0}^{\infty}\cdots$的形式寫出來，以及$\mathcal{J}_{-\upsilon}(x)$

Series Solutions

ex: $9x^2y'' - 27xy' + (9x^2 + 35)y = 0$

令$y = x^2u$ 解不出來令y=(x^a)u，慢慢猜出a，通常a=1, 1/2, 2

$$y' = x^2u' + 2xu$$

$$y'' = x^2u'' + 2xu' + 2xu' + 2u$$

$$= x^2u'' + 4xu' + 2u$$

$$9x^2\left(x^2u'' + 4xu' + 2u\right) - 27x\left(x^2u' + 2xu\right) + \left(9x^2 + 35\right)\left(x^2u\right) = 0$$

$$9x^4u'' + 36x^3u' + 18x^2u - 27x^3u' - 54x^2u + 9x^4u + 35x^2u = 0$$

$$9x^4u'' + 9x^3u' + \left(9x^4 - x^2\right)u = 0$$

同除$9x^2$

Series Solutions

$$x^2u'' + xu' + \left(x^2 - \frac{1}{9}\right)u = 0$$

上式為標準Bassel function

$$\therefore \mathrm{u}\left(x\right) = c_1J_{\frac{1}{3}}\left(x\right) + c_2J_{\frac{-1}{3}}(x)$$

$$\mathrm{y} = \mathrm{u}x^2 = x^2\left(c_1J_{\frac{1}{3}}\left(x\right) + c_2J_{\frac{-1}{3}}(x)\right)$$