# Engineering Mathematics

Midterm Exam, Fall 2014/11/10

請詳細列出計算過程，如用到公式, 請列出公式的通式。請記得在答案卷上簽名。

1. (5%) Solve $y''+4y'+4y=3e^{-2x}$

Sol: $y = C_1 e^{-2x} + C_2 x e^{-2x} + \frac{3}{2}x^2 e^{-2x}$

2. (10%) Solve the given differential equation.
   $y^{(4)} + 5y^{(2)} + 4y = \cos(3x+3) + \sin(4x+1)$

Sol:

$$y = y_h + y_p = c_1\cos x + c_2\sin x + c_3\cos 2x + c_4\sin 2x + \frac{1}{40}\cos(3x+3) + \frac{\sin(4x+1)}{180}$$

3. (10%) Solve the given initial-value problem and give the interval $I$ over which the solution is defined.

   $y' + (\cot x)y = \csc x \cot x \quad y(\frac{\pi}{2}) = 1$

Sol:

A solution of the initial-value problem is $y = \frac{\ln(\sin x)+1}{\sin x}$ for interval $I$: $n\pi < x < (n+1)\pi, n \in \mathrm{N}$

4. (10%) Solve $(2y^2+3xy)dx - (3xy+4x^2)dy = 0$
   (i) Find the integrating factor equation
   (ii) Find the solution of the given differential equation

Sol:

(i)

The integrating factor: $xy^{-5}$

(ii)

A solution of the differential equation is $\frac{x^2}{y^3} + \frac{x^3}{y^4} = c$

5. (10%) Solve the given differential equation $x^2 y'' + xy' - y = \frac{1}{x+1}$

   Sol: $y = c_1 x^{-1} + c_2 x - \frac{1}{2} + \frac{1}{2}x\ln(1+\frac{1}{x}) - \frac{1}{2x}\ln(x+1), \quad x > 0$

6. (10%) Solve the given differential equation by undetermined coefficient

$$y''-8y'+20y=100x^2-26xe^x$$

Sol:

$$y=y_c+y_p$$

$$=e^{4x}(c_1\cos 2x+c_2\sin 2x)+5x^2+4x+\frac{11}{10}+(-2x-\frac{12}{13})e^x$$

7. (10%) Solve the differential equation by variation of parameters subject to the initial conditions

$$2y''+y'-y=x+1 \qquad y(0)=1,\quad y'(0)=0$$

Sol:

$$y=\frac{8}{3}e^{x/2}+\frac{1}{3}e^{-x}-x-2$$

8. (10%) Find a Cauchy-equation differential equation $ax^2y''+bxy'+cy=0$

Where a,b,c are real constants, if it is known that

$(a)$ $m_1=3$ *and* $m_2=-1$ *are roots of it auxiliary equation*

$(b)$ $m = i$ *is the root of it auxilary equation*

Sol:

$(a)$

*the differential equation is* $x^2y''-xy'-3y=0$

$(b)$

*the differential equation is* $x^2y''+xy'+y=0$

9. (10%) Solve the given differential equation by using an appropriate substitution

$$-ydx+(x+\sqrt{xy})dy=0$$

Sol:

$$y(\ln|y|-c)^2=4x$$

10. (5%) Determine whether the given differential equation is exact, if it is exact, solve it.

$$(x-y^3+y^2\sin x)dx=(3xy^2+2y\cos x)dy$$

Sol:

$$xy^3+y^2\cos x-\frac{1}{2}x^2=c$$

11. (5%) Classify which differential equations are separable

$(a)\ \dfrac{dy}{dx}=\dfrac{x-y}{x}$

$(b)\ (x+1)\dfrac{dy}{dx}=-y+10$

$(c)\ \dfrac{dy}{dx}=\dfrac{y^2+y}{x^2+x}$

$(d)\ ydx\ =(y-xy^2)dy$

$(e)\ xyy'+y^2=2x$

$(f)\ \dfrac{dy}{dx}=\dfrac{x}{y}+\dfrac{y}{x}+1$

$(g)\ \dfrac{y}{x^2}\dfrac{dy}{dx}+e^{2x^3+y^2}\ =0$

Sol:

b c g

12. (5%) Which of the following Differential Equations have a unique solution.

i. $y'=\ln\left(\frac{x+y}{x-y}\right)\ y(2)=1$

ii. $y'=\frac{e^y+e^{-y}}{2}\ y(0)=0$

iii. $y'=\sqrt{4-y^2}\ y(0)=2$

iv. $y'=\sin^{-1}\left(\frac{x+y}{x-y}\right)\ y(0)=1$

Sol:

i ii

Reference: Differentiation Table

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx} \qquad \frac{d}{dx}a^u = (\ln a)a^u\frac{du}{dx} \qquad \frac{d}{dx}\mathrm{e}^u = \frac{du}{dx}$$

$$\frac{d}{dx}\log_a \mathrm{u} = \frac{1}{(\ln a)u}\frac{du}{dx} \qquad \frac{d}{dx}\ln(\mathrm{u}) = \frac{1}{u}\frac{du}{dx} \qquad \frac{d}{dx}\sin u = \cos u\frac{du}{dx}$$

$$\frac{d}{dx}\cos u = -\sin u\frac{du}{dx} \qquad \frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx} \qquad \frac{d}{dx}\cot u = -\csc^2 u\frac{du}{dx}$$

$$\frac{d}{dx}\sec u = \sec u\tan u\frac{du}{dx} \qquad \frac{d}{dx}\csc u = -\csc u\cot u\frac{du}{dx} \qquad \frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\cos^{-1} u = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad \frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx} \qquad \frac{d}{dx}\cot^{-1} u = \frac{-1}{1+u^2}\frac{du}{dx}$$