

HW4

HW4 總分 30 分

題目為設計一組題目，這一個題目無法被一般數學歸納法證明，並使用強數學歸納法嚴謹的證明設計正確。

1. 證明設計的這組題目無法被一般數學歸納法證明(5 分)
2. 強數學歸納法 (20 分)
 - Basis step <10 分>
 - $n=n_0$ 時，命題成立 <5 分>
 - 假設 $n_0 \leq n \leq K$ 命題成立時， $n=k+1$ 命題亦成立 <5 分>
3. 其他細節 (5 分)

→ Please design a proposition and prove it by strong mathematics induction. (Cannot be proved by general mathematics induction).

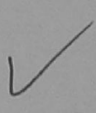
→ Proposition: if $n \in \mathbb{N}$, then $12 \mid (n^4 - n^2)$.

→ General mathematics induction:

When $n=1$, $12 \mid (n^4 - n^2) = 0 \rightarrow \text{true}$.

Assume that $12 \mid (k^4 - k^2)$ is true, such that $(k^4 - k^2) = 12a$ for some $a \in \mathbb{N}$

When $n=k+1$,

$$\begin{aligned}(k+1)^4 - (k+1)^2 &= k^4 + 4k^3 + 6k^2 + 4k + 1 - (k^2 + 2k + 1) = k^4 + 4k^3 + 5k^2 + 2k \\&= (k^4 - k^2) + 4k^3 + 6k^2 + 2k \\&= 12a + 2(2k^3 + 3k^2 + k)\end{aligned}$$


→ leads to nothing

→ Strong mathematical induction.

Base case:

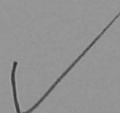
$$n=1 \rightarrow 12 \mid (1^4 - 1^2) = 0 = 12 \times 0$$

$$n=2 \rightarrow 12 \mid (2^4 - 2^2) = 12 = 12 \times 1$$

$$n=3 \rightarrow 12 \mid (3^4 - 3^2) = 72 = 12 \times 6$$

$$n=4 \rightarrow 12 \mid (4^4 - 4^2) = 240 = 12 \times 20$$

$$n=5 \rightarrow 12 \mid (5^4 - 5^2) = 600 = 12 \times 50$$

$$n=6 \rightarrow 12 \mid (6^4 - 6^2) = 1260 = 12 \times 105$$


↓ 归纳

Induction step:

let $k \geq 6 \in \mathbb{N}$ and assume that $12 \mid (m^4 - m^2)$ for $1 \leq m \leq k$

when $n = (k+1)$. \rightarrow prove $12 \mid [(k+1)^4 - (k+1)^2]$ is true.

Define $m = k-5$, according to the proposition,

$$m^4 - m^2 = 12a \text{ for some value of } a.$$

Therefore, I only have to prove $12 \mid (m+6)^4 - (m+6)^2$ is true.

$$(m+6)^4 - (m+6)^2$$

$$= m^4 + 24m^3 + 180m^2 + 864m + 1296 - (m^2 + 12m + 36)$$

$$= m^4 + 24m^3 + 179m^2 + 852m + 1260$$

$$= (m^4 - m^2) + 24m^3 + 180m^2 + 852m + 1260$$

$$= 12a + 12(2m^3 + 15m^2 + 71m + 105)$$

\hookrightarrow is clearly divisible by 12.

Proposition: if $n \in \mathbb{N}$, then $12 \mid (n^4 - n^2)$

If we use weak induction.....

(Base) when $n=1$ $12 \mid 1^4 - 1^2 = 12 \mid 0 = 0$, true

(Induction) Assume $12 \mid (k^4 - k^2)$ is true, which means $(k^4 - k^2) = 12a$ for some $a \in \mathbb{N}$

$$\text{For } k+1, (k+1)^4 - (k+1)^2 = k^4 + 4k^3 + 6k^2 + 4k + 1 - k^2 - 2k - 1 = (k^4 - k^2) + 4k^3 + 6k^2 + 2k = 12a + 4k^3 + 6k^2 + 2k$$

↳ 無法證明是12的倍數

If we use strong induction.....

(Base) $n=1$: $12 \mid (1^4 - 1^2) = 12 \mid (1 - 1) = 0$

$$n=2: 12 \mid (2^4 - 2^2) = 12 \mid (16 - 4) = 12$$

$$n=3: 12 \mid (3^4 - 3^2) = 12 \mid (81 - 9) = 72 = 6 \times 12$$

$$n=4: 12 \mid (4^4 - 4^2) = 12 \mid (256 - 16) = 240 = 20 \times 12$$

$$n=5: 12 \mid (5^4 - 5^2) = 12 \mid (625 - 25) = 600 = 50 \times 12$$

$$n=6: 12 \mid (6^4 - 6^2) = 12 \mid (1296 - 36) = 1260 = 105 \times 12$$

(Induction)

let $k \geq 6$ and $k \in \mathbb{N}$, and we assume that $12 \mid (m^4 - m^2)$ for $1 \leq m \leq k$

We define $r = k - 5$, from our assumption we say $(r^4 - r^2) = 12a$ for some $a \in \mathbb{N}$

Now we need to prove that $12 \mid (k+1)^4 - (k+1)^2$ is true ($k+1$ 的情況)

$$\text{and } k+1 = r+6$$

$$\begin{aligned} k+1 \Rightarrow (r+6)^4 - (r+6)^2 &= r^4 + 24r^3 + 216r^2 + 864r + 1296 - (r^2 + 12r + 36) = (r^4 - r^2) + 24r^3 + 216r^2 + 852r + 1260 \\ &= 12a + 12(2r^3 + 18r^2 + 71r + 105) \end{aligned}$$

So we prove the proposition.