若想对調 $d\lambda$, dt, 則積分区域要相同.

$$= \int_0^\infty \int_\lambda^\infty f(\lambda)\, g(t-\lambda)\, e^{-st}\, dt\, d\lambda$$

$$= \int_0^\infty f(\lambda) \int_\lambda^\infty g(t-\lambda)\, e^{-st}\, dt\, d\lambda$$

令 $x = t-\lambda$, $dx = dt$

$$= \int_0^\infty f(\lambda) \int_0^\infty g(x)\, e^{-s(x+\lambda)}\, dx\, d\lambda.$$

$$= \int_0^\infty f(\lambda)\, e^{-s\lambda} \underbrace{\left(\int_0^\infty g(x)\, e^{-sx}\, dx\right)}_{G(s)} d\lambda.$$

$$= G(s) \int_0^\infty f(\lambda)\, e^{-s\lambda}\, d\lambda = G(s)F(s) \quad \#$$

Property 9. Initial value thm 初值定理.

& Final value thm. 終值定理.

$F(s) = L\{f(t)\}$.

$\Rightarrow f(t) = L^{-1}\{F(s)\}$.

能不能只看 $F(s)$, 而不看 $f(t)$, 就知道 $f(0)$, $f(\infty)$ 之值

$\Rightarrow$ If $F(s)$ is given. $f(0) = \lim\limits_{t\to 0} f(t) = \lim\limits_{s\to\infty} s\cdot F(s)$

$f(\infty) = \lim\limits_{t\to\infty} f(t) = \lim\limits_{s\to 0} s\cdot F(s)$

ex. $F(s) = \dfrac{2s}{(s+1)(s+2)}$

$f(0) = \lim\limits_{s\to\infty} s\cdot F(s) = \lim\limits_{s\to\infty} \dfrac{2s^2}{s^2+3s+2} = 2.$

$f(\infty) = \lim\limits_{s\to 0} s\cdot F(s) = \lim\limits_{s\to 0} \dfrac{2s^2}{s^2+3s+2} = 0$

verify $f(t) = ?$

$F(s) = \dfrac{2s}{(s+1)(s+2)} = \dfrac{a}{s+1} + \dfrac{b}{s+2} \Rightarrow a=-2,\ b=4$

$\Rightarrow f(t) = -2e^{-t} + 4e^{-2t}$ #.

$\Rightarrow f(0) = 2$, $f(\infty) = 0$.

pf: $sF(s)$ 在 $L\{f'\}$ 出現过 (property 6).

$\Rightarrow L\{f'(t)\} = \int_0^\infty f'(t) e^{-st} dt = sF(s) - f(0)$

$\Rightarrow \lim_{s\to\infty} \left[\int_0^\infty f'(t) e^{-st} dt\right] = \lim_{s\to\infty} (sF(s) - f(0))$

$\Rightarrow \int_0^\infty f'(t) \underbrace{\lim_{s\to\infty}[e^{-st}]}_{=0} dt = \lim_{s\to\infty} sF(s) - f(0)$

$\Rightarrow 0 = \lim_{s\to\infty} s\cdot F(s) - f(0)$

$\Rightarrow \lim_{s\to\infty} sF(s) = f(0)$ #.

$L\{f'(t)\} = \int_0^\infty f'(t) e^{-st} dt = sF(s) - f(0)$

$\lim_{s\to 0} \left[\int_0^\infty f'(t) e^{-st} dt\right] = \lim_{s\to 0} (sF(s) - f(0))$

$\Rightarrow \int_0^\infty f'(t) \cdot \underbrace{\lim_{s\to 0}(e^{-st})}_{=1} dt = \lim_{s\to 0} sF(s) - f(0).$

$\Rightarrow f(t)\Big|_0^\infty = \lim_{s\to 0} sF(s) - f(0).$

$\Rightarrow f(\infty) - f(0) = \lim_{s\to 0} sF(s) - f(0)$

$\Rightarrow \lim_{s\to 0} sF(s) = f(\infty)$ #.

ex. $F(s) = \frac{2s}{(s-1)(s+2)}$

$\Rightarrow f(0) = \lim_{s\to\infty} \frac{2s}{(s-1)(s+2)} = 2.$

$f(\infty) = \lim_{s\to 0} \frac{2s}{(s-1)(s+2)} = 0.$

但 $F(s) = \frac{\frac{2}{3}}{s-1} + \frac{\frac{4}{3}}{s+2}$

$\Rightarrow f(t) = \frac{2}{3}e^{t} + \frac{4}{3}e^{-2t}$

$f(0) = \frac{2}{3} + \frac{4}{3} = 2$.

$f(\infty) = \infty + 0 = \infty$ → 與前面矛盾!!

⇒ 終值定理要成立，須滿足下列條件：

(1). $F(s) = \frac{N(s)}{D(s)}$，$D(s)$ 的根不能在 s—平面

的右半面（也就是 $Re(s) > 0$）

若在虛軸上，只能單根，不能重根 & 共軛虛根

ex. $F(s) = \frac{2s}{(s^2+4)^2}$，$f(t) = ?$

$\Rightarrow f(t) = \frac{1}{2}\sin 2t$（前面有）.

我們知道 $D(s)$ 的根 $s = \pm 2i$（重根）

⇒ property 9 の終值定理不適用.

⇒ 用之前的方法做.

* 到目前為止，介紹了

1. 基本 function 1～6.
2. 基本性質 1～9.

現在來介紹 3. 週期函數.

三. 週期函數的 Laplace transform.

考慮函數 $f(t)$.

若滿足 $f(t+nT) = f(t)$.

則稱 T 為 $f(t)$ 的週期. 而 $f(t)$ 稱為週期函數.

例如：$\cos x \xrightarrow{T} 2\pi$.

$\Rightarrow \cos(x + 2 \cdot n\pi) = \cos x$

$n \cdot 2\pi \rightarrow T$

$$L\{f(t)\} = \int_0^\infty f(t) e^{-st} dt$$

$$= \int_0^T f(t)e^{-st}dt + \int_T^{2T} f(t)e^{-st}dt + \int_{2T}^{3T} f(t)e^{-st}dt + \cdots$$

$$= \int_0^T f(t)e^{-st}dt + \left(\int_T^{2T} f(t-T)e^{-st}dt\right) + \left(\int_{2T}^{3T} f(t)e^{-st}dt\right) + \cdots$$

令 $x = t - T \Rightarrow dx = dt \Rightarrow$ $\quad = \int_0^T f(x) e^{-s(x+T)} dx$

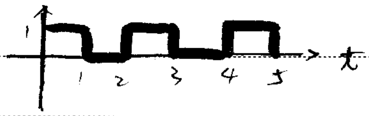
$$= e^{-st}\int_0^T f(x) e^{-sx} dx$$

$$e^{-2sT}\int_0^T f(x) e^{-sx} dx$$

$$= \int_0^T f(t)e^{-st}dt \left(1 + e^{-sT} + e^{-2sT} + \cdots\right)$$

$$= \int_0^T f(t)e^{-st}dt \ \frac{1}{1-e^{-sT}}$$ #.

ex. $f(t)$ $\quad T = 2$

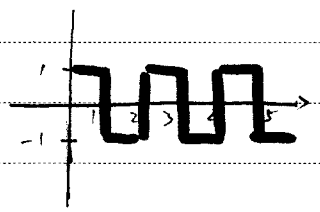


$$L\{f(t)\} = \frac{1}{1-e^{-s\cdot 2}} \cdot \int_0^2 f(t) e^{-st} dt$$

$$= \frac{1}{1-e^{-2s}} \cdot \int_0^1 1 \cdot e^{-st} dt$$

$$= \frac{1}{1-e^{-2s}} \cdot \left(-\frac{1}{s}e^{-st}\right)\Big|_0^1 = \frac{1}{s(1+e^{-s})}$$ #.

ex. $\quad f(t) \quad T = 2.$



$$L\{f(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 f(t) e^{-2t} dt$$

$$= \frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}dt - \int_1^2 e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2s}}\left[-\frac{1}{s}e^{-s} + \frac{1}{s} - \left(-\frac{1}{s}e^{-2s} + \frac{1}{s}e^{-s}\right)\right]$$

$$= \frac{1}{1-e^{-2s}} \cdot \frac{1}{s}\left(-2e^{-s} + 1 + e^{-2s}\right)$$

$$= \frac{1}{1-e^{-2s}} \cdot \frac{1}{s}\left(1-e^{-s}\right)^2 = \frac{1-e^{-s}}{1+e^{-s}} \cdot \frac{1}{s}$$

#.

§ Inverse Laplace transform

設. $F(s) = \frac{N(s)}{D(s)}$

case I: $D(s) = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)$

$\lambda_1 \neq \lambda_2 \cdots \neq \lambda_n \in R$. (相異实根)

$$\Rightarrow F(s) = \frac{N(s)}{(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)}$$

$$= \frac{k_1}{s-\lambda_1} + \frac{k_2}{s-\lambda_2} + \cdots + \frac{k_n}{s-\lambda_n}$$

$$\Big\downarrow L^{-1}$$

$$f(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} + \cdots + k_n e^{\lambda_n t}$$

$$k_1 = \left.\frac{N(s)}{(s-\lambda_2)\cdots(s-\lambda_n)}\right|_{s=\lambda_1}$$

$$\vdots$$

$$k_i = (s-\lambda_i)\cdot F(s)\Big|_{s=\lambda_i}$$

ex. $F(s) = \frac{2s+1}{(s+1)(s+2)(s+3)} = \frac{a}{s+1} + \frac{b}{s+2} + \frac{c}{s+3}$

$$a = \left.\frac{2s+1}{(s+2)(s+3)}\right|_{s=-1} = -\frac{1}{2}$$

$$b = \frac{-3}{-1\cdot 1} = 3, \quad c = \frac{-5}{2}$$

$$\Rightarrow f(t) = -\frac{1}{2}e^{-t} + 3e^{-2t} - \frac{5}{2}e^{-3t} \quad \#$$

ex. $F(s) = \dfrac{2s+3}{(s-1)(s+5)(s-3)} = \dfrac{a}{s-1} + \dfrac{b}{s+5} + \dfrac{c}{s-3}$

$$a = \frac{2s+3}{(s+5)(s-3)}\bigg|_{s=1} = \frac{5}{-12}, \quad b = \frac{-7}{48}, \quad c = \frac{9}{16}$$

$$\Rightarrow f(t) = -\frac{5}{12}e^{t} - \frac{7}{48}e^{-5t} + \frac{9}{16}e^{3t} \quad \#$$

case II $F(s) = \dfrac{N(s)}{D_1(s)\cdot(s-\lambda)^n} = \dfrac{N_1(s)}{D_1(s)} + \left(\dfrac{N_2(s)}{(s-\lambda)^n}\right)$ └ 暫为 $F(s)$.

$$\frac{N_2(s)}{(s-\lambda)^n} = \frac{k_1}{(s-\lambda)} + \frac{k_2}{(s-\lambda)^2} + \cdots + \frac{k_n}{(s-\lambda)^n}$$

$$(s-\lambda)^n \cdot F(s) = \frac{N(s)}{D_1(s)} = k_1(s-\lambda)^{n-1} + k_2(s-\lambda)^{n-2} + \cdots + k_{n-1}(s-\lambda) + k_n$$

$$k_n = (s-\lambda)^n F(s)\Big|_{s=\lambda} = \frac{N(s)}{D(s)}\bigg|_{s=\lambda}$$

又 $\dfrac{d}{ds}\left((s-\lambda)^n F(s)\right) = (n-1)k_1(s-\lambda)^{n-2} + \cdots + k_{n-1}$

$$\Rightarrow k_{n-1} = \frac{d}{ds}\left((s-\lambda)^n F(s)\right)\Big|_{s=\lambda}$$

$$\vdots$$

$$k_1 = \frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}\left((s-\lambda)^n F(s)\right)\Big|_{s=\lambda} \quad \#$$

ex. $F(s) = \dfrac{s}{(s+1)(s-2)^2}$

case II.

先看 ex. $F(s) = \dfrac{s}{(s+1)(s-2)^2}$

$$= \frac{N_1(s)}{s+1} + \frac{N_2(s)}{(s-2)^2}$$

$$= \frac{a}{s+1} + \frac{k_1}{(s-2)} + \frac{k_2}{(s-2)^2}$$

$$\Rightarrow a = F(s)(s+1)\Big|_{s=-1} = -\frac{1}{9}$$

$$k_2 = F(s)\cdot(s-2)^2\Big|_{s=2} = \frac{2}{3}$$

$$k_1 = \frac{d}{ds}\left(F(s)\cdot(s-2)^2\right)\Big|_{s=2} = \frac{1}{9}$$

$$\Rightarrow f(t) = -\frac{1}{9}e^{-t} + \frac{1}{9}e^{2t} + \frac{2}{3}t\,e^{2t} \quad \#$$

$$F(s) = \frac{N(s)}{D_1(s)\cdot(s-\lambda)^n}$$

$$= \frac{N_1(s)}{D_1(s)} + \frac{N_2(s)}{(s-\lambda)^n}$$

$$= \frac{N_1(s)}{D_1(s)} + \frac{k_1}{s-\lambda} + \frac{k_2}{(s-\lambda)^2} + \cdots + \frac{k_n}{(s-\lambda)^n}$$

$$\Rightarrow k_n = (s-\lambda)^n F(s)\Big|_{s=\lambda}$$

$$= \frac{N(s)}{D_1(s)}\Big|_{s=\lambda}$$

$$\Rightarrow (s-\lambda)^n F(s)\Big|_{s=\lambda} = \frac{N_1(s)\cdot(s-\lambda)^n}{D_1(s)} + k_1(s-\lambda)^{n-1} + \cdots + k_{n-1}(s-\lambda) + k_n$$

$$\Rightarrow \frac{d}{ds}\left((s-\lambda)^n F(s)\right)\Big|_{s=\lambda} = f(s-\lambda) + k_1(n-1)(s-\lambda)^{n-2} + \cdots + k_{n-1}$$

$$= k_{n-1}$$

$$\vdots$$

$$\frac{d^{n-1}}{ds^{n-1}}\left((s-\lambda)^n F(s)\right)\Big|_{s=\lambda} = 0 + (n-1)!\,k_1$$