Chapter 5.
Series Solutions of Linear Differential Equations

Chuan-Ching Sue

Dept. of Computer Science and Information Engineering,
National Cheng Kung University

Series Solutions

ex: $x^2y'' + \left(x^2 + \frac{5}{36}\right)y = 0$於$x = 0$的級數解

$x = 0$為規則異點,存在$y=\sum_{n=0}^{\infty} a_n x^{n+r}$

$$y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \left(x^2 + \frac{5}{36}\right)\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}a_n x^{n+r+2} + \frac{5}{36}\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$n \to n+2$

$$\sum_{n=-2}^{\infty}(n+r+2)(n+r+1)a_{n+2}x^{n+r+2} + \sum_{n=0}^{\infty}a_n xn^{n+r+2} + \frac{5}{36}\sum_{n=-2}^{\infty}a_{n+2}x^{n+r+2} = 0$$

Series Solutions

提前兩項

$$\left[r(r-1)a_0x^r+\frac{5}{36}a_0x^r\right]+\left[(r+1)ra_1x^{r+1}+\frac{5}{36}a_1x^{r+1}\right]$$

$$+\sum_{n=0}^{\infty}\left\{\left[(n+r+2)(n+r+1)+\frac{5}{36}\right]a_{n+2}+a_n\right\}x^{n+r+2}=0$$

$$\left.\begin{array}{l}
1.\left[r(r-1)a_0+\dfrac{5}{36}a_0\right]=0 \\[2ex]
2.\left[(r+1)ra_1+\dfrac{5}{36}a_1\right]=0 \\[2ex]
3.\left[\underbrace{(n+r+2)(n+r+1)+\dfrac{5}{36}}_{A(n,r)}\right]a_{n+2}+a_n=0
\end{array}\right\}\text{挑一個設不為0來微}$$

$$a_{n+2}=\frac{-a_n}{A(n,r)},n\geq 0$$

Series Solutions

$a \neq 0$, 帶入2確認 $a_1 = 0$

$$\therefore r\left(r-1\right)+\frac{5}{36}=0$$

$$r=\frac{1}{6},\frac{5}{6}$$

$$r=\frac{1}{6}$$

$$a_{n+2}=\frac{-1}{\left(n+\frac{1}{6}+2\right)\left(n+\frac{1}{6}+1\right)+\frac{5}{36}}a_n$$

$$=\frac{-1}{\left(n+\frac{13}{6}\right)\left(n+\frac{7}{6}\right)+\frac{5}{36}}a_n$$

$$a_2=\frac{-1}{\left(\frac{13}{6}\right)\left(\frac{7}{6}\right)+\frac{5}{36}}a_0=-\frac{3}{8}a_0$$

Series Solutions

$$a_3 = 0$$

$$a_4 = \frac{-1}{\frac{25}{6}\cdot\frac{19}{6}+\frac{5}{36}} a_2 = ?\, a_2$$

$$y_1 = \left(\left[a_0 x^{\frac{1}{6}} + a_2 x^{2+\frac{1}{6}} + a_4 x^{4+\frac{1}{6}} + \ldots + \frac{5}{36}\right] a_{n+2}\right) x^{n+r+2}$$

$$= a_0 x^{\frac{1}{6}} + \left(-\frac{3}{8}\right) a_0 x^{2+\frac{1}{6}} + \frac{9}{320} a_2 x^{4+\frac{1}{6}} + \ldots$$

$$y_2 = a_0 x^{\frac{1}{6}}\left(1-\frac{3}{8}x^2+\ldots\right) \Rightarrow \text{代} y = \frac{5}{6} \text{求得}$$

$$\frac{y_1}{y_2} \neq k_1,\ y_1, y_2 \text{獨立}$$

Series Solutions

ex: $x\left(x-1\right)y''+\left(3x-1\right)y'+y=0$於的級數解

$$y=\sum_{n=0}^{\infty}a_n x^{n+r},\left|x\right|<1$$

$$y'=\sum_{n=0}^{\infty}\left(n+r\right)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}\left(n+r\right)\left(n+r-1\right)a_n x^{n+r-2}$$

$$\sum_{n=0}^{\infty}\left(n+r\right)\left(n+r-1\right)a_n x^{n+r}-\sum_{n=0}^{\infty}\left(n+r\right)\left(n+r-1\right)a_n x^{n+r-1}+3\sum_{n=0}^{\infty}\left(n+r\right)a_n x^{n+r}$$

$$-\sum_{n=0}^{\infty}\left(n+r\right)a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

Series Solutions

$$n \to n+1$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=-1}^{\infty}(n+r+1)(n+r)a_{n+1}x^{n+r} + 3\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}$$

$$-\sum_{n=-1}^{\infty}(n+r+1)a_{n+1}x^{n+r} + \sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$n=-1$$

$$-r(r-1)a_0 x^{r-1} - \left[ra_0 x^{r-1}\right]$$

$$+\sum_{n=0}^{\infty}\left\{\left[-(n+r+1)(n+r)-(n+r+1)\right]a_{n+1} + \left[(n+r)(n+r-1)+3(n+r)+1\right]a_n\right\}x^{n+r} = 0$$

Series Solutions

1. $\left(-r\left(r-1\right)-r\right)a_0=0$
2. $a_{n+1}=a_n$

$\because a_0\neq 0$

$r^2=0,\qquad r=0,0$

另一獨立解改用參數變異法

$$y_2=\phi y_1$$

$$y_2{}'=\phi y_1'+\phi' y_1$$

$$y_2''=\phi y_1''+\phi' y_1'+\phi' y'+\phi'' y_1=\phi y_1''+2\phi' y_1'+\phi'' y_1$$

$$x\left(x-1\right)\left(\phi y_1''+2\phi' y_1'+\phi'' y_1\right)+\left((3x-1)\left(\phi y_1'+\phi' y_1\right)\right)+\phi y_1=0$$

$$\phi\left[x\left(x-1\right)y_1''+(3x-1)y_1'+y_1\right]+x\left(x-1\right)\left(2\phi' y_1'+\phi'' y_1\right)+\left(3x-1\right)\phi' y_1=0$$

Series Solutions

$$y_1 = \frac{1}{1-x}, y_1' = \frac{1}{(1-x)^2}$$

$$\Rightarrow x(x-1)\left[2\phi'' \tfrac{1}{(x-1)^2} + \phi'' \tfrac{-1}{x-1}\right] + (3x-1)\phi' \frac{-1}{x-1} = 0$$

$$2\phi' \frac{x}{x-1} + \phi''(-x) + \phi'\left(\frac{-(3x-1)}{x-1}\right) = 0$$

$$(2x-3x+1)\phi' + (-x^2+x)\phi'' = 0$$

$$\phi'' + \tfrac{1}{x}\phi' = 0$$

$$\frac{d\phi'}{\phi'} = -\frac{1}{x}dx$$

$$\ln|\phi'| = -\ln|x|$$

$$\phi' = \frac{1}{x} \qquad \therefore \phi = \ln x$$

Series Solutions

summary: $p(x)y'' + q(x)y' + r(x)y = 0,\ p,q,r$不能再消去項

$x = a$的級數解

1. $p(a) \neq 0 \Rightarrow$ 常數

$\Rightarrow$ 存在Taylor級數

$$y(x) = \sum_{n=0}^{\infty} a_n (x-a)^n, \quad |x-a| < L, \quad L : x = a\text{到最近異點的距離}$$

2. $p(a) = 0 \Rightarrow$ singular point

若$(x-a)\cdot\frac{q}{p},\ (x-a)\cdot\frac{r}{p}$這兩項於$x = a$均可微分

$\Rightarrow x = a$為規則異點

$\Rightarrow$ 存在一Frobenius級數解

$$y(x) = \sum_{n=0}^{\infty} a_n (x-a)^{n+r}, \quad |x-a| < L, \quad L : x - a\text{到另一點異點的距離}$$

Series Solutions

$r = r_1 \rightarrow y_1(x)$

$r = r_2 \rightarrow y_2(x)$

(I) $r_1 \neq r_2 \quad , \quad |r_1 - r_2| \notin N$

y_1, y_2一定線性獨立, 構成一組基底解

$y = c_1 y_1 + c_2 y_2$

(II) $r_1 \neq r_2 \quad , \quad |r_1 - r_2| \in N$

(A) y_1, y_2獨立解

$\therefore y = c_1 y_1 + c_2 y_2$

(B) y_1, y_2線性相依, 另一個獨立解利用參數變異法求解$\bar{y}_2$

$\therefore y = c_1 y_1 + c_2 y_2$

Series Solutions

(III) $r_1 = r_2 = r \quad \Rightarrow y_1$

另一個獨立解也是由參數變異法求得

3. $x = a$為不規則異點則方程式於$x = a$處無級數解

ex: $x^2 y'' + \left(x^2 + \frac{5}{36}\right) y = 0$,其indicial e.g.為何?

$$x^2 \qquad x^{n+r-2} \qquad\qquad x^2 + \frac{5}{36} \qquad x^{n+r}$$

$$x^{n+r} \qquad\qquad\qquad x^{n+r+2} \; x^{n+r}$$

Series Solutions

ex: $2x\left(1-x\right)y''+\left(1+x\right)y'-y=0$

$n+r-1 \quad n+r \qquad n+r-1 \quad n+r \quad n+r$

$2r\left(r-1\right) \qquad +r=0$

ex: $x\left(1+x\right)y''+4\left(x+3\right)y'+5y=0$

$n+r-1 \quad n+r \quad 4n+r \quad 12n+r\text{-}1 \quad n+r$

$r\left(r-1\right)+12r=0$為指標方程式

$r^2+11r=0$

$r=0,-11$

$x=0,-1$

Series Solutions

$x = 0$, 規則異點, 指標方程式如上所示

$x = -1$, 規則異點

$$y = \sum_{n=0}^{\infty} a_n (x+1)^{n+r}, |x+1| < 1$$

$$y' = \sum_{n=0}^{\infty} a_n (n+r)(x+1)^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)(x+1)^{n+r-2}$$

$$(x+1-1)(x+1)\sum_{n=0}^{\infty} a_n (n+r)(n+r-1)(x+1)^{n+r-2} + 4(x+3)$$

$r = 0,\ 9$

Series Solutions

ex: $(x+1)(x-2)y''+4(3x+1)y'+6y=0$

(1) 於$x=1$級數解

$$y(x)=\sum_{n=0}^{\infty}a_n(x+1)^{n+r},\ 收斂區間:|x+1|<3$$

指標方程式$:-3r(r-1)-8r=0\Rightarrow -3r^2-5r=0$

(2) 於$x=2$級數解

$$y(x)=\sum_{n=0}^{\infty}a_n(x-2)^{n+r},\ 收斂區間:|x-2|<3$$

指標方程式$:(x-2+3)(x-2)y''+4(3(x-2)+7)y^2+6y=0$

$$3r(r-1)+28r=0$$

$$3r^2+25r=0$$

Series Solutions

Special case: 科西尤拉D.E.

ex: $x^2 y'' + 4xy' + 2y = 0$

(1) 令$x = e^t$, $xy' = \dfrac{dy}{dt}$, $x^2 y'' = \dfrac{d^2 y}{dt^2} - \dfrac{dy}{dt}$

$$\frac{d^2 y}{dt^2} - \frac{dy}{dt} + 4\frac{dy}{dt} + 2y = 0$$

$$\frac{d^2 y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\lambda = -1, -2$$

$$y(x) = c_1 e^{-t} + c_2 e^{-2t}$$

$$= c_1 x^{-1} + c_2 x^{-2}$$

Series Solutions

(2) 利用於$x=0$的級數解, 驗證(1)的結果

$\because\ x=0$為規則異點 $\Rightarrow$ 存在Frobenius級數

$$y(x)=\sum_{n=0}^{\infty}a_n x^{n+r}, |x|<\infty$$

$$y'(x)=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''(x)=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+4x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+2\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left(a_n(n+r)(n+r-1)+4a_n(n+r)+2a_n\right)x^{n+r}=0$$

$n=0$最低次, $a_0\left(r(r-1)+4r+2\right)x^r=0$

Series Solutions

$$\because a_0 \neq 0$$

$$\therefore r(r-1)+4r+2=0$$

$$r^2+3r+2=0$$

$$(r+1)(r+2)=0$$

$$r=-1,-2$$

$$(\mathrm{i})\, r=-1$$

$$n \geq 1$$

$$\{(n+r)(n+r-1)+4(n+r)+2\}a_n=0$$

$$\{(n-1)(n-2)+4(n-1)+2\}a_n=0$$

$$\left(n^2-3n+2+4n-4+2\right)a_n=0$$

Series Solutions

$$\left(n^2+n\right)a_n=0$$

$$n\left(n+1\right)a_n=0,\quad n\left(n+1\right)\neq 0$$

$$n\geq 1,\quad a_n=0$$

$$y_1\left(x\right)=a_0x^{-1}+a_1x^0+a_2=a_0x^{-1}$$

$$\left(\text{ii}\right)r=-2$$

$$n\geq 1$$

$$\left[\left(n+r\right)\left(n+r-1\right)+4\left(n+r\right)+2\right]a_n^*=0$$

$$\left(n-2\right)\left(n-3\right)+4\left(n-2\right)+2$$

$$n^2-5n+6+4n-8+2$$

$$\left(n^2-n\right)a_n^*=0$$

$$n\left(n-1\right)a_n^*=0$$

Series Solutions

$n=1, 0\cdot a_1^*=0, a_1^*$可以不為0

$n\geq 2, a_n^*=0$

$$y_2(x)=\sum_{n=0}^{\infty}a_n x^{n-2}$$

$$=a_0^* x^{-2}+a_1^* x^{-1}+a_2^* x^0+\ldots$$

$$=a_0^* x^{-2}+a_1^* x^{-1}$$

$$\therefore y=k_1 y_1(x)+k_2 y_2(x)$$

$$=k_1 a_0 x^{-1}+k_2\left(a_0^* x^{-2}+a_1^* x^{-1}\right)$$

$=c_1 x^{-1}+c_2 x^{-2}$ 其中 $c_1=k_1 a_0^*+k_2 a_1^*$

$c_2=k_2 a_0^*$

Series Solutions

補充:若題目要求$a_3 \neq 0$

則$n = 3$

$\left((3+r)(2+r)+4(3+r)+2\right)a_n = 0$成為新的指標方程式

ex: $(x-2)^2 y'' + 4(x-2)y' + 2y = 0$

(i) 令 $u = x-2$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{dy}{dx}\right)\cdot\frac{du}{dx} = \frac{d}{du}\left(\frac{dy}{du}\right)$$

$$u^2 y'' + 4uy' + 2y = 0$$

$$y(u) = c_1 u^{-1} + c_2 u^{-2}$$

$$= c_1 (x-2)^{-1} + c_2 (x-2)^{-2}$$

Series Solutions

(ii) 利用於$x=2$的級數解,驗證(i)的結果

$$y' = \sum_{n=0}^{\infty} a_n (n+r)(x-2)^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)(x-2)^{n+r-2}$$

$$(x-2)^2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)(x-2)^{n+r-2} + 4(x-2)\sum_{n=0}^{\infty} a_n (n+r)(x-2)^{n+r-1}$$

$$+2\sum_{n=0}^{\infty} a_n (x-2)^{n+r} = 0$$

Series Solutions

$$\sum_{n=0}^{\infty}\left[a_n(n+r)(n+r-1)+4a_n(n+r)+2a_n\right](x-2)^{n+r}=0$$

$n=0,\quad r(r-1)+4r+2=0$ 指標方程式

$r=-1,-2$

$n=-1,\quad y_1(x)=a_0(x-2)^{-1}$

$n=-2,\quad y_2(x)=a_0^*(x-2)^{-2}+a_1^*(x-2)^{-1}$

$\therefore y=k_1y_1+k_2y_2=c_1(x-2)^{-1}+c_2(x-2)^{-1}$

Series Solutions

ex: Find the indicial equation of

$$x^2 y'' + xe^x y' + \left(x^2 - 1\right) y = 0$$

if the solution is required near $x = 0$

sol: $x^2 = 0,\ \ x = 0$異點

但$x\dfrac{xe^x}{x^2}, x^2\dfrac{x^2-1}{x^2}$二者皆可微分

$\therefore$ x=0為規則異點 $\Rightarrow$ 存在Frobenius級數

Series Solutions

$$y(x)=\sum_{n=0}^{\infty}a_n x^{n+r}, |x|<\infty$$

$$y'(x)=\sum_{n=0}^{\infty}a_n(n+r)x^{n+r-1}$$

$$y''(x)=\sum_{n=0}^{\infty}a_n(n+r)(n+r-1)x^{n+r-2}$$

代入$x^2y''+x\left(1+x+\frac{1}{2}x^2+\ldots\right)y'+\left(x^2-1\right)y=0$

$n+r$次的係數

$r(r-1)+r-1=0$

$r^2-r+r-1=0$

$r^2-1=0 \quad \rightarrow$ 指標方程式

$r=1,-1$

Series Solutions

§ Legendre differential equation

$\left(1+x^2\right)y''-2xy'+\lambda y=0$ $\rightarrow$ 出現未知數λ

in which $-1\le x\le 1$, and λ is a real constant

$x=0$的級數解

$<$分析$>$ $1-x^2=\left(1-x\right)\left(1+x\right)$

$\therefore x=1,-1$為方程式異點

而$x=0$為常點$\left(\text{O.D.P.}\right)$

對一個Taylor級數

Series Solutions

$$y(x)=\sum_{n=0}^{\infty}a_n x^n,|x|<1$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}$$

$$(1-x^2)\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}-2x\sum_{n=1}^{\infty}na_n x^{n-1}+\lambda\sum_{n=0}^{\infty}a_n x^n=0$$

$$\Downarrow\ n\to n+2$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_n x^n-\sum_{n=2}^{\infty}n(n-1)a_n x^n-2\sum_{n=1}^{\infty}na_n x^n+\lambda\sum_{n=0}^{\infty}a_n x^n=0$$

$n\geq 2$有共同項

$n=0$

$$(2a_2+\lambda a_0)+(6a_3x-2a_1x+\lambda a_1x)=0$$

Series Solutions

1. $2a_2 + \lambda a_0 = 0 \Rightarrow a_2 = -\frac{\lambda}{2} a_0 ......(1)$
2. $6a_3 - 2a_1 + \lambda a_1 = 0 \Rightarrow a_3 = \frac{2-\lambda}{6} a_1 ....(2)$
3. $(n+2)(n+1)a_{n+2} + \left(-(n)(n-1) - 2n + \lambda\right)a_n = 0$

由3循環公式

$$a_{n+2} = \frac{n(n+1)-\lambda}{(n+1)(n+2)} a_n, \quad n \geq 2$$

Series Solutions

<分析>

$n=2 \qquad a_4=\frac{2\cdot 3-\lambda}{3\cdot 4}a_2=\frac{6-\lambda}{4\cdot 3}\left(-\frac{\lambda}{2}\right)a_0=\frac{(6-\lambda)(-\lambda)}{4!}a_0$

$n=3 \qquad a_5=\frac{3\cdot 4-\lambda}{4\cdot 5}a_3=\frac{12-\lambda}{4\cdot 5}\left(\frac{2-\lambda}{2\cdot 3}\right)a_1=\frac{(12-\lambda)(2-\lambda)}{5!}a_1$

$n=4 \qquad a_6=\frac{4\cdot 5-\lambda}{5\cdot 6}a_4=\frac{(-\lambda)(6-\lambda)(20-\lambda)}{6!}a_0$

$n=5 \qquad a_7=\frac{5\cdot 6-\lambda}{6\cdot 7}a_5=\frac{(2-\lambda)(12-\lambda)(30-\lambda)}{7!}a_1$

Series Solutions

$$\left(1-x^2\right)y''-2xy'+\lambda y=0$$

$$y\left(x\right)=\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+\ldots$$

$$=a_0+a_1x+\left(\tfrac{-\lambda}{2}\right)a_0x^2+\left(\tfrac{2-\lambda}{6}\right)a_1x^3+\frac{-\lambda(6-\lambda)}{4!}a_0x^4+\ldots$$

$$=a_0\left[\qquad\qquad\qquad\qquad\right]+a_1\left[\qquad\qquad\qquad\qquad\right]$$

$$y\left(x\right)=a_0\left(1+\left(\tfrac{-\lambda}{2}\right)x^2+\frac{-\lambda(6-\lambda)}{4!}x^4+\frac{-\lambda(6-\lambda)(20-\lambda)}{6!}x^6+\ldots\right)$$

$$+a_1\left(x+\frac{(2-\lambda)}{6}x^3+\frac{(2-\lambda)(12-\lambda)}{5!}x^5+\frac{(2-\lambda)(12-\lambda)(30-\lambda)}{7!}x^7+\ldots\right)$$

$$=a_0y_e\left(x\right)+a_1y_0\left(x\right)$$

$$a_{n+2}=\frac{n\left(n+1\right)-\lambda}{\left(n+2\right)\left(n+1\right)}a_n$$ 循環公式

Series Solutions

若λ=N$\left(N+1\right)$

$$a_{n+2}=\frac{n\left(n+1\right)-N\left(N+1\right)}{\left(n+2\right)\left(n+1\right)}a_n,\quad n\geq 2$$

$a_{n+2}=0,\quad \forall 2\leq n\leq N$

$\therefore N$可以為奇數或偶數

$\therefore a_{n+2}=0$

$\therefore a_{n+4}=0$

$\therefore y_e\left(x\right)$或$y_o(x)$有一個會有有限項

$\Rightarrow$ 針對有限項的解,若選擇當$x=1$時,讓$y_e(1)=1$或$y_o(1)=1$的有限項解,

則此解稱為Legendre's polynornail,記為$P_n(x)$

Series Solutions

$$P_n(x) = 1$$

$$y(x) = a_0\left(1+0\right) + (a_1 + \ldots)$$

$$a_0 = 1$$

$$P_1(x) = a_1 x = x$$

$$a_1 = 1$$

$$P_2(x) = a_2\left(1 + \tfrac{-6}{2}x\right) = a_2\left(1-3x\right)$$

$$a_0\left(1-3\right) = 1$$

$$P_2(x) = \tfrac{-1}{2}\left(1-3x\right) = \frac{3x-1}{2}$$

$$P_3(x) = \tfrac{1}{2}\left(5x^2 - 3x\right)$$

$$P_4(x) = \tfrac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$

$$P_5(x) = \tfrac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$$

Series Solutions

補充:

∵ λ是變數，會有special function產生

$$\left(1-x^{2}\right)y^{''}-2xy^{'}+\lambda y=0$$

原式可改寫為

$\left(\left(1-x^{2}\right)y^{'}\right)^{'}+\lambda y=0$, 產生之非零解的稱為eigenvalue，

其對應的解叫做eigenfunction，而且eigenfunction在收斂區間內是正交的。

另一個special function：Bassel's differential equation

$$x^{2}y^{''}+xy^{'}+\left(x^{2}-\upsilon^{2}\right)y=0$$

$\upsilon\geq 0,\ x=0$有級數解

$x=0$規則異點 $\Rightarrow$ Frobenious級數解

Series Solutions

$$y(x)=\sum_{n=0}^{\infty}a_n x^{n+r},?|x|<\infty$$

$$y^{'}(x)=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y^{''}(x)=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

代入原式

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+\left(x^2-\upsilon^2\right)\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$n\rightarrow n+2$代入

$$\sum_{n=-2}^{\infty}(n+r+2)(n+r+1)a_{n+2}x^{n+r+2}+\sum_{n=-2}^{\infty}(n+r+2)a_{n+2}x^{n+r+2}+\sum_{n=0}^{\infty}a_n x^{n+r+2}-\upsilon^2\sum_{n=-2}^{\infty}a_{n+2}x^{n+r+2}=0$$

Series Solutions

$n=-2$

$$①\, r\left(r-1\right)a_0x^r+ra_0x^r-\upsilon^2a_0x^r$$

$$+\sum_{n=0}^{\infty}\left\{\left[\left(n+r+2\right)\left(n+r+1\right)+\left(n+r+2\right)-\upsilon^2\right]a_{n+2}+a_n\right\}x^{n+r+2}=0$$

$n=-1$

$$②\left(r+1\right)ra_1x^{r+1}+(r+1)a_1x^{r+1}-\upsilon^2a_1x^{r+1}$$

$$+\sum_{n=0}^{\infty}\left\{\left[\left(n+r+2\right)\left(n+r+1\right)+\left(n+r+2\right)-\upsilon^2\right]a_{n+2}+a_n\right\}x^{n+r+2}=0$$

Series Solutions

$$①\left(r(r-1)+r-\upsilon^2\right)a_0=0$$

$$②\left[r(r+1)+(r+1)-\upsilon^2\right]a_1=0$$

$$③\left[(n+r+2)^2-\upsilon^2\right]a_{n+2}+a_n=0$$

$\because a_0 \neq 0 \Rightarrow$ 指標方程式 $r^2-\upsilon^2=0,\ r=\upsilon,-\upsilon$

$$② \rightarrow \left[(r+1)^2-\upsilon^2\right]a_1=0$$

$$\therefore a_1=0$$

$$③ \rightarrow a_{n+2}=\frac{-1}{(n+r+2)^{-2}-\upsilon^2}a_n$$

$$\because \left|r_1-r_2\right|=2\upsilon$$

若$2\upsilon \notin \mathcal{N}$，會有二個獨立解

$$r=\upsilon,\ a_{n+2}=\frac{-1}{(n+\upsilon+2)^2-\upsilon^2}a_n=\frac{-1}{(n+2)(n+2+2\upsilon)}a_n$$

Series Solutions

$$n=1, a_3=\frac{-1}{3(3+2\upsilon)}a_1=0$$

$$a_1=a_3=\cdots=a_{2n+1}=0$$

為了方便計算$n+2 \to n$

$$a_{n+2}=\frac{-1}{(n+2)(n+2+2\upsilon)}a_n,\ n\ge 0$$

$$n=2$$

$$a_2=\frac{-1}{2(2+2\upsilon)}a_0=\frac{-1}{2^2(1+\upsilon)}a_0$$

$$\mathrm{n}=4$$

$$a_4=\frac{-1}{4(4+2\upsilon)}a_2=\frac{-1}{2^3(2+\upsilon)}a_2=\frac{(-1)^2}{2^3\cdot 2^2(2+\upsilon)(1+\upsilon)}a_0$$

Series Solutions

$$\mathrm{n} = 6$$

$$a_6 = \frac{-1}{6\left(6+2\upsilon\right)} a_4 = \frac{\left(-1\right)^3}{2^6 \cdot 3 \cdot 2\left(3+\upsilon\right)(2+\upsilon)(1+\upsilon)} a_0$$

$$a_{2n} = \frac{\left(-1\right)^n}{2^{2n} \cdot n!\left(n+\upsilon\right)\left(n-1+\upsilon\right)\left(n-2+\upsilon\right)\cdots\left(1+\upsilon\right)} a_0$$

$$y_1\left(x\right) = \sum_{n=0}^{\infty} a_n x^{n+r} \quad \left(\because a_{2n+1} = 0\right)$$

$$= \sum_{n=0}^{\infty} a_{2n} x^{2n+\upsilon}$$

$$= a_0 \sum_{n=0}^{\infty} \frac{\left(-1\right)^n}{2^{2n} \cdot n!\left(n+\upsilon\right)\left(n-1+\upsilon\right)\left(n-2+\upsilon\right)\cdots\left(1+\upsilon\right)} x^{2n+\upsilon}$$

Series Solutions

$$y_1(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n \tau(\upsilon+1)}{2^{2n} \cdot n!\Gamma(n+\upsilon+1)} x^{2n+\upsilon} = \mathcal{J}_\upsilon(x)$$

$\mathcal{J}_\upsilon(x)$: Bessel function of the first kind

另外一個$r = -\upsilon$

$$y_2(x) = \mathcal{J}_{-\upsilon}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{(2n-\upsilon)} \cdot n!\Gamma(n-\upsilon+1)} x^{2n-\upsilon}$$

$$\therefore y = c_1\mathcal{J}_\upsilon + c_2\mathcal{J}_{-\upsilon}$$

Series Solutions

Ex: $x^2 y'' + xy' + \left(x^2 - \frac{1}{9}\right) y = 0$

$$r = \frac{1}{3}, \frac{-1}{3}$$

$$\left|\upsilon_1 - \upsilon_2\right| = \frac{2}{3} \notin \mathcal{N}$$

$$y = c_1 \mathcal{J}_{\frac{1}{3}}(x) + c_2 \mathcal{J}_{-\frac{1}{3}}(x)$$

另外需將$\mathcal{J}_{\upsilon}\left(x\right) = \sum_{n=0}^{\infty} \cdots$的形式寫出來，以及$\mathcal{J}_{-\upsilon}\left(x\right)$

Series Solutions

ex: $9x^2y'' - 27xy' + (9x^2+35)y = 0$

令 $\mathrm{y} = x^2u$

$$y' = x^2u' + 2xu$$

$$y'' = x^2u'' + 2xu' + 2xu' + 2u$$

$$= x^2u'' + 4xu' + 2u$$

$$9x^2\left(x^2u'' + 4xu' + 2u\right) - 27x\left(x^2u' + 2xu\right) + \left(9x^2+35\right)\left(x^2u\right) = 0$$

$$9x^4u'' + 36x^3u' + 18x^2u - 27x^3u' - 54x^2u + 9x^4u + 35x^2u = 0$$

$$9x^4u'' + 9x^3u' + \left(9x^4 - x^2\right)u = 0$$

同除 $9x^2$

Series Solutions

$$x^2u'' + xu' + \left(x^2 - \frac{1}{9}\right)u = 0$$

上式為標準Bassel function

$$\therefore \mathrm{u}(x) = c_1 J_{\frac{1}{3}}(x) + c_2 J_{\frac{-1}{3}}(x)$$

$$\mathrm{y} = \mathrm{u}x^2 = x^2\left(c_1 J_{\frac{1}{3}}(x) + c_2 J_{\frac{-1}{3}}(x)\right)$$