

Chapter 2.

First-Order Ordinary Differential Equations

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Review

- 上周 $Mdx + Ndy = 0$

若發現 $\frac{\partial N}{\partial x} \neq \frac{\partial M}{\partial y}$ 則檢查 $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

除以 $-N \rightarrow I(x)$

除以 $M \rightarrow I(y)$

除以 $-N + M \rightarrow I(x + y)$

除以 $-y \times N + x \times M \rightarrow I(x \times y)$

Integrating Factor Examples

- 今天

例: $(xy + y^2 + 1)dx + (xy + x^2 + 1)dy = 0$

$$\Rightarrow x(ydx + xdy) + y(ydx + xdy) + dx + dy = 0$$

$$xd(xy) + yd(xy) + d(x + y) = 0$$

$$(x + y)d(xy) + d(x + y) = 0 \Rightarrow d(xy) + \frac{1}{x + y}d(x + y) = 0$$

$$\int d(xy) + \int \frac{1}{x + y}d(x + y) = \int 0$$

$$xy + \ln(x + y) = C$$

Integrating Factor Examples

例: $(y \cos x - \sin 2x)dx + dy = 0$

$$\frac{\partial M}{\partial y} = \cos x, \frac{\partial N}{\partial x} = 0$$

$$\frac{0 - \cos x}{-N} = \frac{-\cos x}{-1} = \cos x$$

$$\cos x dx = \frac{dI}{I}$$

$$I = e^{\sin x}$$

$$(y(\cos x)e^{\sin x} - (\sin 2x)e^{\sin x})dx + e^{\sin x}dy = 0$$

Integrating Factor Examples

$$\frac{\partial u}{\partial x} = y(\cos x)e^{\sin x} - (\sin 2x)e^{\sin x}$$

$$\partial u = (y(\cos x)e^{\sin x} - (\sin 2x)e^{\sin x})dx$$

$$\int \partial u = \int y(\cos x)e^{\sin x}dx - \int (\sin 2x)e^{\sin x}dx$$

$$u = \int y(\cos x)e^{\sin x}dx - \int (\sin 2x)e^{\sin x}dx$$

↑
(1)

↑
(2)

Integrating Factor Examples

$$\begin{aligned}(1): \int y(\cos x)e^{\sin x} dx \\&= y \int (\cos x)e^{\sin x} dx \\&= ye^{\sin x}\end{aligned}$$

$$\begin{aligned}(2): \int (\sin 2x)e^{\sin x} dx \\&= \int (2\sin x \cos x)e^{\sin x} dx \\&= 2 \int te^t dt \quad (\text{令 } t=\sin x) \\&= 2(te^t - e^t)\end{aligned}$$

$$u = ye^{\sin x} - 2(\sin x)e^{\sin x} + 2e^{\sin x} + f(y)$$

$$\frac{\partial u}{\partial y} = e^{\sin x} \quad \partial u = e^{\sin x} dy$$

$$\int \partial u = \int e^{\sin x} dy$$

Integrating Factor Examples

$$u = \int e^{\sin x} dy + g(x)$$

$$= ye^{\sin x} + g(x)$$

$$f(y) = 0, g(x) = -2(\sin x)e^{\sin x} + 2e^{\sin x}$$

$$u = ye^{\sin x} - 2(\sin x)e^{\sin x} + 2e^{\sin x} = C$$

Integrating Factor Examples

例: $\frac{dy}{dx} = 3x^2 - 3x^2y$ (分離變數法可解)

$$(3x^2 - 3x^2y)dx - dy = 0$$

$$\frac{\partial M}{\partial y} = -3x^2, \frac{\partial N}{\partial x} = 0$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3x^2$$

除以-N $\frac{3x^2}{-(-1)} = 3x^2$

Integrating Factor Examples

$$3x^2 dx = \frac{dI}{I}$$

$$I = e^{x^3}$$

$$e^{x^3} (3x^2 - 3x^2 y) dx - e^{x^3} dy = 0$$

$$\frac{\partial u}{\partial x} = e^{x^3} (3x^2 - 3x^2 y)$$

$$u = \int e^{x^3} (3x^2 - 3x^2 y) dx + f(y)$$

$$u = e^{x^3} - ye^{x^3} + f(y)$$

Integrating Factor Examples

$$\frac{\partial u}{\partial y} = -e^{x^3}$$

$$u = -ye^{x^3} + g(x)$$

$$g(x) = e^{x^3}, f(y) = 0$$

$$u = e^{x^3} - ye^{x^3} = C$$

Separation of Variables

- 2.2 分離變數法(補充)

例: $(1+x)dy - ydx = 0$

$$\Rightarrow -ydx + (1+x)dy = 0$$

$$\frac{\partial M}{\partial y} = -1, \frac{\partial N}{\partial x} = 1$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2$$

$$\text{除以 } M \Rightarrow \frac{2}{-y} dy = \frac{dI}{I}$$

Separation of Variables

$$I = y^{-2}$$

$$-y^{-1}dx + (1+x)y^{-2}dy = 0$$

$$\frac{\partial u}{\partial x} = -y^{-1}$$

$$\frac{\partial u}{\partial y} = (1+x)y^{-2}$$

$$\begin{aligned} u &= \int -y^{-1}dx + f(y) \\ &= -xy^{-1} + f(y) \end{aligned}$$

$$\begin{aligned} u &= \int (1+x)y^{-2}dy + g(x) \\ &= -(1+x)y^{-1} + g(x) \end{aligned}$$

$$f(y) = -y^{-1}, g(x) = 0$$

$$u = -(1+x)y^{-1} = C$$

$$y = C(1+x) \quad \Leftarrow \quad \text{課本答案}$$

Separation of Variables

$$(1+x)dy - ydx = 0$$

除以 $(1+x)y$

$$\frac{dy}{y} - \frac{dx}{1+x} = 0 \qquad \int \frac{dy}{y} - \int \frac{dx}{1+x} = \int 0$$

$$\ln y = \ln(1+x) + C$$

$$y = (1+x)e^C$$

$$= (1+x)C$$

Separation of Variables Example

例: $\frac{dy}{dx} = y^2 - 4$

$$\frac{dy}{y^2 - 4} = dx$$

$$\left(\frac{1}{y-2} - \frac{1}{y+2}\right)dy = dx$$

$$\frac{1}{4}\ln(y-2) - \frac{1}{4}\ln(y+2) = x + C$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4x + C$$

$$\frac{y-2}{y+2} = \frac{e^{4x+C}}{1}$$

$$\frac{2y}{4} = \frac{e^{4x+C} + 1}{1 - e^{4x+C}}$$

$$y = 2\frac{1 + e^{4x+C}}{1 - e^{4x+C}}, \quad y = \pm 2$$

First-Order O.D.E.

- 2.3 複習: 一階 O.D.E. (線性 OR 非線性)
一階線性 O.D.E. (常微分)

$$y'(x) + p(x)y(x) = r(x)$$

(1) $r(x) = 0$ Homogeneous 齊性

(2) $r(x) \neq 0$ Non-Homogeneous 非齊性

First-Order O.D.E. (homogeneous)

Case (1):

$$r(x) = 0 \Rightarrow y'(x) + p(x)y(x) = 0$$

$$\frac{dy(x)}{dx} = -p(x)y(x)$$

$$\frac{dy(x)}{y(x)} = -p(x)dx$$

$$\int \frac{dy(x)}{y(x)} = \int -p(x)dx$$

$$\ln y(x) = -\int p(x)dx + k$$

$$y(x) = e^{-\int p(x)dx} e^k$$

$$= Ce^{-\int p(x)dx} \quad (\text{令 } C = e^k)$$

First-Order O.D.E.(Non-homogeneous)

Case (2):

$$r(x) \neq 0 \Rightarrow y'(x) + p(x)y(x) = r(x)$$

$$\frac{dy}{dx} + p(x)y(x) - r(x) = 0$$

$$(p(x)y(x) - r(x))dx + dy = 0$$

$$\frac{\partial M}{\partial y} = p(x), \quad \frac{\partial N}{\partial x} = 0$$

$$\frac{0 - p(x)}{-N} = p(x) \quad p(x)dx = \frac{dI}{I}$$

First-Order O.D.E.(Non-homogeneous)

$$I = e^{\int p(x)dx}$$

$$e^{\int p(x)dx} (p(x)y(x) - r(x))dx + e^{\int p(x)dx} dy = 0$$

$$\frac{\partial u}{\partial x} = e^{\int p(x)dx} (p(x)y(x) - r(x))$$

$$\partial u = e^{\int p(x)dx} (p(x)y(x) - r(x)) \partial x$$

(兩邊同積分)

$$u = \int e^{\int p(x)dx} (p(x)y(x) - r(x))dx + f(y)$$

First-Order O.D.E.(Non-homogeneous)

$$\text{令 } t = \int p(x) dx$$

$$dt = p(x) dx, \quad dx = \frac{dt}{p(x)}$$

$$u = \int e^t y(x) dt - \int e^{\int p(x) dx} r(x) dx + f(y)$$

$$= y(x) e^t - \int e^{\int p(x) dx} r(x) dx + f(y)$$

$$= y(x) e^{\int p(x) dx} - \int e^{\int p(x) dx} r(x) dx + f(y)$$

First-Order O.D.E.(Non-homogeneous)

$$\frac{\partial u}{\partial y} = e^{\int p(x)dx}$$

$$\partial u = e^{\int p(x)dx} \partial y + g(x)$$

(兩邊同積分)

$$u = \int e^{\int p(x)dx} dy + g(x)$$

$$= ye^{\int p(x)dx} + g(x)$$

$$g(x) = -\int e^{\int p(x)dx} r(x) dx, \quad f(y) = 0$$

$$u = ye^{\int p(x)dx} - \int e^{\int p(x)dx} r(x) dx = C \quad \#$$

First-Order O.D.E.(Non-homogeneous)

$$y = Ce^{-\int p(x)dx} + e^{-\int p(x)dx} \int e^{\int p(x)dx} r(x) dx$$
$$= y_h + y_p$$

y_h : $r(x) = 0$ homogeneous solution

y_p : 特解(*particular solution*),

互補解(*complementary solution*)

記法 : $y' + p(x)y(x) = r(x)$ 之解

$$y = CI$$

First-Order O.D.E.(Non-homogeneous)

例: $y' + 2xy = 3x$

$$y = Ce^{-x^2} + e^{-x^2} \int e^{x^2} 3x dx$$

$$= Ce^{-x^2} + \frac{3}{2} \quad \#$$

$$y_h \quad y_p$$

$$(1) \quad y'_h + p(x)y_h = 0$$

(2) Theorem:

$$\text{又 } y_p(x) \text{ 滿足非齊性方程式 } \Leftrightarrow y'_p + p(x)y_p(x) = r(x)$$

First-Order O.D.E.(Non-homogeneous)

Proof:

$$\because y = y_h + y_p \quad \text{代入原方程式}$$

$$y' + Py = r$$

$$\left(y_h + y_p \right)' + P \left(y_h + y_p \right) = r$$

$$y_h' + y_p' + Py_h + Py_p = r$$

$$\because y_h' + Py_h = 0$$

$$\therefore y_p' + Py_p = r$$

Non-homogeneous Example

例: $y' + 2xy = x$

$$I = e^{\int 2x dx} = e^{x^2}$$

$$y = Ce^{-x^2} + e^{-x^2} \int e^{x^2} x dx$$

$$= Ce^{-x^2} + \frac{1}{2}$$

Chapter 3.

Higher-Order Differential Equations

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First-Order O.D.E.

$$y' + ay = r(x) \quad a \in \text{const.}$$

Case (1):

$$r(x) = 0 \quad \text{Homogeneous}$$

$$y' + ay = 0$$

$$y = Ce^{-\int a dx}$$

$$= Ce^{-ax}$$

$$= y_h(x)$$

常係數 $\Rightarrow y_h(x)$ 的部分必為指數函數 $e^{\lambda x} \quad \lambda \in \text{const.}$

First-Order O.D.E.

例: $y' + 2y = 0$

$$y = Ce^{-2x}$$

apply $y = e^{\lambda x}$ 代入 $y' + 2y = 0$

$$\lambda e^{\lambda x} + 2e^{\lambda x} = 0$$

$$(\lambda + 2)e^{\lambda x} = 0, \lambda + 2 = 0 \Rightarrow \text{特性方程式}$$

$$\lambda = -2$$

$$y = Ce^{-2x}$$

Higher-Order O.D.E.

例: $y' - 3y = 0$

$$y = Ce^{3x} \#$$

推廣 $y'' + ay' + by = 0$, $a, b \in \text{const.}$

猜 $e^{\lambda x}$ 是解

$$(e^{\lambda x})'' + a(e^{\lambda x})' + b(e^{\lambda x}) = 0$$

$$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = 0$$

$$e^{\lambda x} (\lambda^2 + a\lambda + b) = 0$$

$$\because \forall x, e^{\lambda x} \neq 0$$

$$\therefore \lambda^2 + a\lambda + b = 0$$

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Higher-Order O.D.E.

Case (1):

$\lambda_1 \neq \lambda_2 \in \mathfrak{R}$ 相異實根

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

例: $y'' + 3y' + 2y = 0$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\lambda = -1, -2$$

$$y(x) = C_1 e^{-x} + C_2 e^{-2x} \#$$

例: $y'' + 6y' + 5y = 0$

$$\lambda = -5, -1$$

$$y = C_1 e^{-x} + C_2 e^{-5x}$$

Higher-Order O.D.E.

Case (2):

$$\lambda_1 = \alpha + \beta i, \quad \lambda_2 = \alpha - \beta i \quad \text{共軛複數根}$$

$$y = k_1 e^{(\alpha + \beta i)x} + k_2 e^{(\alpha - \beta i)x}$$

$$= e^{\alpha x} (k_1 (\cos \beta x + i \sin \beta x) + k_2 (\cos \beta x - i \sin \beta x))$$

$$(\text{令 } k_2 = ik'_2)$$

$$= e^{\alpha x} (k_1 (\cos \beta x + i \sin \beta x) + ik'_2 (\cos \beta x - i \sin \beta x))$$

$$= e^{\alpha x} ((k_1 + ik'_2) \cos \beta x + (k'_2 + ik_1) \sin \beta x)$$

$$= e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$$

$$\Rightarrow C_1 = k_1 + ik'_2, \quad C_2 = k'_2 + ik_1$$

Higher-Order O.D.E. Examples

例: $y'' + 2y' + 10y = 0$

$$\lambda = -1 \pm 3i$$

$$y = e^{-x} (C_1 \cos 3x + C_2 \sin 3x) \#$$

例: $y'' + 4y = 0$

$$\lambda = \pm 2i$$

$$y = C_1 \cos 2x + C_2 \sin 2x \#$$

Higher-Order O.D.E.

- 補充：

$$y'' + ay' + by = 0$$

$$\Rightarrow \lambda^2 + a\lambda + b = 0$$

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

$$\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0$$

$$a = -(\lambda_1 + \lambda_2) \quad b = \lambda_1\lambda_2$$

$$y'' - (\lambda_1 + \lambda_2)y' + \lambda_1\lambda_2 y = 0$$

Higher-Order O.D.E.

- 定義:

$$D \equiv \frac{d}{dx} : \text{微分運算子}$$

$$Dx^2 = \frac{d}{dx} x^2 \quad D \text{只對右邊函式作運算} \quad \text{note: } Dx^2 \neq x^2 D$$

$$D^k = \frac{d^k}{dx^k}$$

Higher-Order O.D.E.

$$y'' - (\lambda_1 + \lambda_2) y' + \lambda_1 \lambda_2 y = 0$$

$$\Rightarrow D^2 y - (\lambda_1 + \lambda_2) D y + \lambda_1 \lambda_2 y = 0$$

$$(D^2 - (\lambda_1 + \lambda_2) D + \lambda_1 \lambda_2) y = 0$$

$$(D - \lambda_1)(D - \lambda_2) y = 0$$

$$\circ \text{令 } (D - \lambda_2) y = z(x)$$

$$\Rightarrow (D - \lambda_1) z(x) = 0$$

$$z'(x) - \lambda_1 z(x) = 0$$

$$z(x) = k_1 e^{-\lambda_1 x}$$

$$(D - \lambda_2) y = z(x) = k_1 e^{-\lambda_1 x}$$

$$y' - \lambda_2 y = k_1 e^{-\lambda_1 x}$$

$$y = C I^{-1} + I^{-1} \int I r dx$$

$$I = e^{-\int \lambda_2 dx} = e^{-\lambda_2 x}$$