Homework 3

Section 4.4 Indeterminate Forms and l'Hospital's Rule

EX.22

This limit has the form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty}\frac{\ln\ln x}{x} \overset{H}{=} \lim_{x\to\infty}\frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1} = \lim_{x\to\infty}\frac{1}{x\ln x} = 0$$

EX.25

This limit has the form $\frac{0}{0}$.

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{1+2x}-\sqrt{1-4x}}{x} &\overset{H}{=} \lim_{x\to 0}\frac{\frac{1}{2}(1+2x)^{-1/2}\cdot 2-\frac{1}{2}(1-4x)^{-1/2}(-4)}{1}\\ &= \lim_{x\to 0}\left(\frac{1}{\sqrt{1+2x}}+\frac{2}{\sqrt{1-4x}}\right) = \frac{1}{\sqrt{1}}+\frac{2}{\sqrt{1}} = 3\end{aligned}$$

EX.27

This limit has the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2} \overset{H}{=} \lim_{x\to 0}\frac{e^x-1}{2x} \overset{H}{=} \lim_{x\to 0}\frac{e^x}{2} = \frac{1}{2}$$

EX.28

This limit has the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{\sinh x - x}{x^3} \overset{H}{=} \lim_{x\to 0}\frac{\cosh x-1}{3x^2} \overset{H}{=} \lim_{x\to 0}\frac{\sinh x}{6x} \overset{H}{=} \lim_{x\to 0}\frac{\cosh x}{6} = \frac{1}{6}$$

EX.88

$L = \lim_{x\to 0}\left(\frac{\sin 2x}{x^3}+a+\frac{b}{x^2}\right) = \lim_{x\to 0}\frac{\sin 2x + ax^3 + bx}{x^3} \overset{H}{=} \lim_{x\to 0}\frac{2\cos 2x + 3ax^2 + b}{3x^2}$.
As $x\to 0$, $(2\cos 2x+3ax^2+b)\to b+2$, so the last limit exists only if $b+2=0$, that is, $b=-2$. Thus,
$\lim_{x\to 0}\frac{2\cos 2x+3ax^2+b}{3x^2} \overset{H}{=} \lim_{x\to 0}\frac{-4\sin 2x + 6ax}{6x} \overset{H}{=} \lim_{x\to 0}\frac{-8\cos 2x+6a}{6} = \frac{6a-8}{6}$,
which is equal to 0 if and only if $a=\frac{4}{3}$. Hence, $L=0$ if and only if $b=-2$ and $a=\frac{4}{3}$.

EX.90

Since $\lim_{h\to 0}[f(x+h)-2f(x)+f(x-h)] = f(x)-2f(x)+f(x) = 0$ [f is differentiable and hence continuous] and $\lim_{h\to 0} h^2 = 0$, we can apply l'Hospital's Rule:

$$\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2} \overset{H}{=} \lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h} = f''(x)$$

At the last step, we have applied the result of Exercise 89 to $f'(x)$.

EX.52

$y = f(x) = \frac{lnx}{x^2}$

A. $D = (0, \infty)$

B. y-intercept : none; x-intercept : $f(x) = 0 \Leftrightarrow lnx = 0 \Leftrightarrow x = 1$

C. No symmetry

D. $\lim_{x\to 0^+} f(x) = -\infty$, so $x = 0$ is a VA; $\lim_{x\to\infty}\frac{lnx}{x^2} \overset{H}{} \lim_{x\to\infty}\frac{1/x}{2x} = 0$, so $y = 0$ is a HA.

E. $f'(x) = \frac{x^2(1/x)-(lnx)(2x)}{(x^2)^2} = \frac{x(1-2lnx)}{x^4} = \frac{1-2lnx}{x^3}$. $f'(x) > 0 \Leftrightarrow 1-2lnx > 0 \Leftrightarrow lnx < \frac{1}{2} \Rightarrow 0 < x < e^{1/2}$ and $f'(x) < 0 \Rightarrow x > e^{1/2}$, so f is increasing on $(0, \sqrt{e})$ and decreasing on $(\sqrt{e}, \infty)$.

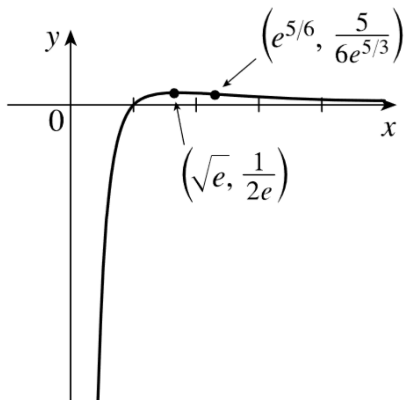
F. Local maximum value $f(e^{1/2}) = \frac{1/2}{e} = \frac{1}{2e}$

G. $f''(x) = \frac{x^3(-2/x)-(1-2lnx)(3x^2)}{(x^3)^2} = \frac{x^2[-2-3(1-2lnx)]}{x^6} = \frac{-5+6lnx}{x^4}$

$f''(x) > 0 \Leftrightarrow -5+6lnx > 0 \Leftrightarrow lnx > \frac{5}{6} \Leftrightarrow x > e^{5/6}$ [f is CU]

and $f''(x) > 0 \Leftrightarrow 0 < x < e^{5/6}$ [f is CD]. IP at $(e^{5/6}, 5/(6^{5/3}))$

H.



EX.44

$f''(x) = x^3 + \sinh x \Leftrightarrow f'(x) = \frac{1}{4}x^4 + \cosh x + C \Rightarrow f(x) = \frac{1}{20}x^5 + \sinh x + Cx + D.$

$f(0) = D$ and $f(0) = 1 \Rightarrow D = 1$, so $f(x) = \frac{1}{20}x^5 + \sinh x + Cx + 1.$

$f(2) = \frac{32}{20} + \sinh 2 + 2C{+}1$ and $f(2) = 2.6 \Rightarrow \sinh 2 + 2C = 0 \Rightarrow$

$C = -\frac{1}{2}\sinh 2$, so $f(x) = \frac{1}{20}x^5 + \sinh x - \frac{1}{2}(\sinh 2)x + 1.$

EX.48

$f'''(x) = \cos x \Rightarrow f''(x) = \sin x + C$. $f''(0) = C$ and $f''(0) = 3 \Rightarrow C = 3$. $f''(x) = \sin x + 3 \Rightarrow f'(x) = -\cos x + 3x + D$. $f'(0) = -1 + D$ and $f'(0) = 2 \Rightarrow D = 3$. $f'(x) = -\cos x + 3x + 3 \Rightarrow f(x) = -\sin x + \frac{3}{2}x^2 + 3x + E$. $f(0) = E$ and $f(0) = 1 \Rightarrow E = 1$. Thus, $f(x) = -\sin x + \frac{3}{2}x^2 + 3x + 1$.

Problem Plus 5.

$y = \frac{\sin x}{x} \Rightarrow y' = \frac{x\cos x - \sin x}{x^2} \Rightarrow y'' = \frac{-x^2\sin x - 2x\cos x + 2\sin x}{x^3}$.If (x, y) is an inflection point, then $y'' = 0 \Rightarrow (2 - x^2)\sin x = 2x\cos x \Rightarrow (2 - x^2)^2\sin^2 x = 4x^2\cos^2 x$

$\Rightarrow (2 - x^2)^2\sin^2 x = 4x^2(1 - \sin^2 x) \Rightarrow (4 - 4x^2 + x^4)\sin^2 x = 4x^2 - 4x^2\sin^2 x$

$\Rightarrow (4 + x^4)\sin^2 x = 4x^2 \Rightarrow (x^4 + 4)\frac{\sin^2 x}{x^2} = 4 \Rightarrow y^2(x^4 + 4) = 4$ since $y = \frac{\sin x}{x}$.

Problem Plus 7.

Let $L = \lim\limits_{x\to 0} \frac{ax^2 + \sin bx + \sin cx + \sin dx}{3x^2 + 5x^4 + 7x^6}$. Now L has the indeterminate form of type $\frac{0}{0}$, so we can apply L'Hospital's Rule. $L = \lim\limits_{x\to 0} \frac{2ax + b\cos bx + c\cos cx + d\cos dx}{6x + 20x^3 + 42x^5}$. The denominator approaches 0 as $x \to 0$, so the numerator must also again. $L = \lim\limits_{x\to 0} \frac{2x - b^2\sin bx - c^2\sin cx - d^2\sin dx}{6 + 60x^2 + 210x^4} = \frac{2a - 0}{6 + 0} = \frac{2a}{6}$, which must equal 8.

$\frac{2a}{6} = 8 \Rightarrow a = 24$. Thus, $a + b + c + d = a + (b + c + d) = 24 + 0 + 24$.

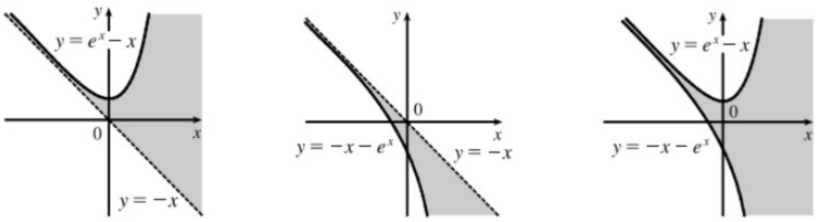
Problem Plus 10.

Case(i)(first graph): For $x + y \geq 0$, that is, $y \geq -x, \mid x + y \mid = x + y \leq e^x \Rightarrow y \leq e^x - x$.

Note that $y = e^x - x$ is always above the line $y = -x$ and that $y = -x$ is a slant asymptote. *Case(ii)(second graph)*: For $x + y < 0$, that is, $y < -x, \mid x + y \mid = -x - y \leq e^x \Rightarrow y \geq -x - e^x$.

Note that $-x - e^x$ is always below the line $y = -x$ and $y = -x$ is a slant asymptote.

Putting the two pieces together gives the third graph.



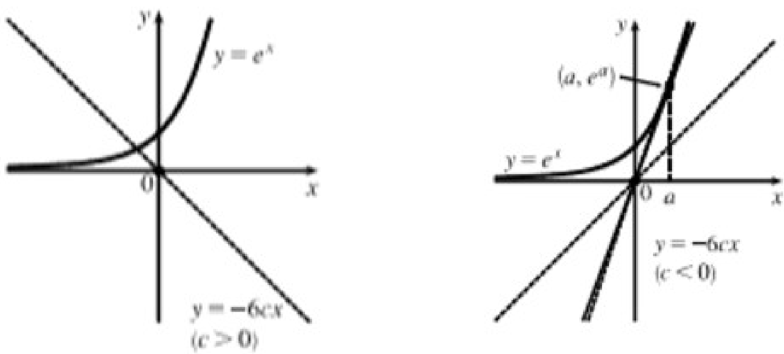
Problem Plus 12.

$y = cx^3 + e^x \Rightarrow y' = 3cx^2 + e^x \Rightarrow y'' = 6cx + e^x$. The curve will have inflection points when y'' changes sign. $y'' = 0 \Rightarrow -6cx = e^x$, so y'' will change sign when the line $y = -6cx$ intersects the curve $y = e^x$ (but is not tangent to it).

Note that if $c = 0$, the curve is just $y = e^x$, which has no inflection point.

The first figure shows that for $c > 0$, $y = -6cx$ will intersect $y = e^x$ once, so $y = cx^3 + e^x$ will have one inflection point.

The second figure shows that for $c < 0$, the line $y = -6cx$ can intersect the curve $y = e^x$ in two points (two inflection points), be tangent to it (no inflection point), or not intersect it (no inflection point). The tangent line at (a, e^a) has slope e^a, but from the diagram we see that the slope is $\frac{e^a}{a}$. So $\frac{e^a}{a} = e^a \Rightarrow a = 1$. Thus, the slope is e. The line $y = -6cx$ must have slope greater than e, so $-6c > e \Rightarrow c < \frac{-e}{6}$. Therefore, the curve $y = cx^3 + e^x$ will have one inflection point if $c > 0$ and two inflection points if $c < \frac{-e}{6}$.



Problem Plus 18.

If $L = \lim\limits_{x\to\infty} (\frac{x+a}{x-a})^x$, then L has the indeterminate form 1^∞, so

$\ln L = \lim\limits_{x\to\infty} \ln(\frac{x+a}{x-a})^x = \lim\limits_{x\to\infty} x \ln(\frac{x+a}{x-a}) = \lim\limits_{x\to\infty} \frac{\ln(x+a) - \ln(x-a)}{1/x} \stackrel{H}{=} \lim\limits_{x\to\infty} \frac{\frac{1}{x+a} - \frac{1}{x-a}}{-1/x^2}$

$= \lim\limits_{x\to\infty} [\frac{(x-a)-(x+a)}{(x+a)(x-a)} \cdot \frac{-x^2}{1}] = \lim\limits_{x\to\infty} \frac{2ax^2}{x^2 - a^2} = \lim\limits_{x\to\infty} \frac{2a}{1 - a^2/x^2} = 2a$

Hence, $\ln L = 2a$, so $L = e^{2a}$.

From the original equation, we want $L = e^1 \Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2}$.