Chapter 3.
Higher-Order Differential Equations

Chuan-Ching Sue

Dept. of Computer Science and Information Engineering,
National Cheng Kung University

Bernoulli Equation

- 非線性→線性

  型式(1)

$$f'(y)\frac{dy}{dx}+p(x)f(y)=g(x)$$

令$z=f(y)$

$$\frac{dz}{dx}=\frac{dz}{dy}\cdot\frac{dy}{dx}$$

$$=\frac{df(y)}{dy}\cdot\frac{dy}{dx}=f'(y)\cdot\frac{dy}{dx}$$

原式 $:\frac{dz}{dx}+p(x)z=q(x)$

Bernoulli Equation

例: $x^2 \cos y \dfrac{dy}{dx} = 2x \sin y - 1$

令 $z = \sin y$ 則 $\dfrac{dz}{dx} = \cos y \cdot \dfrac{dy}{dx}$

$$x^2 \frac{dz}{dx} - \frac{2}{x} z = \frac{-1}{x^2} \qquad 1$$

$$z = CI^{-1} + I^{-1} \int Ir dx$$

對於型式(1)

$f(y)$同常等於 $y^2, y^3, \cdots, \sin(y) \cdots e^y$

Bernoulli Equation

型式(2)

Bernoulli

型式(3)

Riccati

$$\frac{dy}{dx} + p(x)y = q(x) + y^2 r(x)$$

若 y_1 為上式之一特解

則令 $y = y_1 + \frac{1}{z}$ 得Z的線性D.E.

Bernoulli Equation

- 例: $y' + (2x-1)y = x^2 - x + 1 + y^2$

$$p = 2x - 1$$

$$q = x^2 - x + 1$$

$$r = 1$$

$$find\ \ y_1 \Rightarrow guess?\ \ 1, x, x^2, \sin x, \cos x, \cdots$$

$$\therefore y_1 = x$$

令$y = y_1 + \dfrac{1}{z}$代入原D.E.得 $z' + z = -1$

$$z = Ce^{-x} - 1$$

$$y = x + \frac{1}{ce^{-x} - 1}$$

Review

Chuan-Ching Sue

Dept. of Computer Science and Information Engineering,
National Cheng Kung University

Review

$$M(x,y)dx + N(x,y)dy = 0$$

$$\text{正合}：\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\text{非正合}：\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\underset{\mathrm{M}_1}{\underline{\underline{I(x,y)M(x,y)dx}}} + \underset{N_1}{\underline{\underline{I(x,y)N(x,y)dy}}} = 0$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

$$M(x,y)\frac{\partial I(x,y)}{\partial y} + I(x,y)\frac{\partial \mathrm{M}(x,y)}{\partial y} = \mathrm{N}(x,y)\frac{\partial I(x,y)}{\partial x} + I(x,y)\frac{\partial \mathrm{N}(x,y)}{\partial x}$$

$$M(x,y)\frac{\partial I(x,y)}{\partial y} - \mathrm{N}(x,y)\frac{\partial I(x,y)}{\partial x} = +I(x,y)\frac{\partial \mathrm{N}(x,y)}{\partial x} - I(x,y)\frac{\partial \mathrm{M}(x,y)}{\partial y}$$

Review

一階P.D.E.

$$P(x,y,z)\frac{\partial z}{\partial x}+Q(x,y,z)\frac{\partial z}{\partial y}=R(x,y,z)$$

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}\text{(輔助)}$$

$$u(x,y,z)=\alpha$$

$$\upsilon(x,y,z)=\beta$$

$$\varphi(u,\upsilon)=0 or C$$

$$\upsilon=f(u)$$

$$\frac{dx}{-N(x,y)}=\frac{dy}{M(x,y)}=\frac{dI}{I\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}$$

Review

(i)

希望 I(x)

$$\frac{dx}{-N(x,y)}=\frac{dI}{I\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}$$

$$\frac{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}{-N}dx=\frac{dI}{I}\quad if\ \ f(x)=\frac{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}{-N}$$

$$I=e^{\int f(x)dx}=I(x)$$

(ii)

希望 I(y)

$$\frac{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}{M}dy=\frac{dI}{I}\qquad f(y)dy=\frac{dI}{I}$$

$$I=e^{\int f(y)dy}$$

Review

(iii)

希望 $I(x+y)$

$$\frac{dx+dy}{-N+M}=\frac{dI}{I\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}$$

$$\frac{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}{\text{- N + M}}d(x+y)=\frac{dI}{I}$$

$$I=e^{\int f(x+y)d(x+y)}$$

(iv)

希望 $I(xy)$

$$\frac{ydx+xdy}{-yN+xM}d(x,y)=\frac{dI}{I}$$

$$I=e^{\int f(x,y)dxy}$$

Review

(v)

希望 $\mathrm{I}(x^a y^b)$

$$d(x^a y^b) = ?$$

$$\frac{\mathrm{d}(x^a y^b)}{\mathrm{dx}} = ax^{a-1}y^b + bx^a y^{b-1}\frac{dy}{dx}$$

$$d(x^a y^b) = ax^{a\text{-}1}y^b dx + bx^a y^{b-1}dy = x^{a-1}y^{b-1}(aydx + bxdy)$$

$$\frac{aydx + bxdy}{-ayN + bxM} = \frac{dI}{I\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}$$

$$\frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)d(x^a y^b)}{(-ayN + bxM)x^{a-1}y^{b-1}} = \frac{\partial I}{I}$$

Review

$$if \quad f(x^a y^b) = \frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}{(-ayN + bxM)x^{a-1}y^{b-1}}$$

$$I = e^{\int f\left(x^a y^b\right) d(x^a y^b)}$$

$$\frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}{\frac{(-ayN + bxM)}{xy}} \cdot \frac{1}{x^a y^b} d\left(x^a y^b\right) = \frac{dI}{I}$$

$$\frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}{\frac{(-ayN + bxM)}{xy}} = 1 \Rightarrow I = x^a y^b$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{(-ayN + bxM)}{xy} = \frac{-aN}{x} + \frac{bM}{y}$$

Review

- 例: $\left(4xy+6y^2\right)dx+\left(2x^2+6xy\right)dy=0$

$$\frac{\partial M}{\partial y}=4x+12y,\ \frac{\partial N}{\partial x}=4x+6y,\ \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=-6y$$

若 $I=x^a y^b$,則 $\dfrac{-aN}{x}+\dfrac{bM}{y}=-6y$

$$\frac{-a\left(2x^2+6xy\right)}{x}+\frac{b\left(4xy+6y^2\right)}{y}=-a\left(2x+6y\right)+b\left(4x+6y\right)$$

$$=(-2a+4b)x+(-6a+6b)y$$

$$\begin{cases}-2a+4b=0\\-6a+6b=-6\end{cases}\Rightarrow\begin{cases}a=2\\b=1\end{cases}$$

Review

$$\because I = x^2 y$$

$$x^2 y \ \left(4xy + 6y^2\right)dx + x^2 y \ \left(2x^2 + 6xy\right)dy = 0$$

$$\frac{\partial u}{\partial x} = 4x^3y^2 + 6x^2y^3, \frac{\partial u}{\partial y} = 2x^4y + 6x^3y^2$$

$$u = x^4y^2 + 2x^3y^3 + f(y), u = x^4y^2 + 2x^3y^3 + g(x)$$

$$f(y) = 0, g(x) = 0$$

$$\therefore u = x^4y^2 + 2x^3y^3 = 0$$

Review

- §2.2分離變數

例4：$\cos x(e^{2y}-y)\dfrac{dy}{dx}=e^{y}\sin 2x$

$$(e^{x}\sin 2x)dx-(\cos x)(e^{2y}-y)dy=0$$

$$\frac{\partial M}{\partial y}=e^{y}\sin 2x,\frac{\partial N}{\partial x}=\sin x(e^{2y}-y)$$

$$\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=\sin x(e^{2y}-y)-e^{y}\sin 2x$$

$$I=?$$

Review

$$e^{y} + ye^{-y} + e^{-y} + 2\cos x = C$$

$$\frac{\partial u}{\partial x} = -2\sin x, \frac{\partial u}{\partial y} = e^{y} + e^{-y} + y(\text{-}e^{-y}) + (\text{-}e^{-y}) = e^{y} - ye^{-y}$$

$$-2\sin x dx + e^{-y}(e^{2y} - y)dy = 0$$

$$I = -e^{y}\cos x$$

$$\frac{e^{2y} - y}{e^{y}} dy = \frac{\sin 2x}{\cos x} dx$$

$$\int (e^{y} - ye^{-y})dy = 2\int \sin x dx + C$$

$$e^{y} + ye^{-y} + e^{-y} = -2\cos x + C$$

$$u = C$$

Review

- §2.3線性微分方程式

例1：$\frac{dy}{dx}$-3y = 6

$$y = CI^{-1} + I^{-1}\int Irdx$$

$$I = e^{\int -3dx} = e^{-3x}$$

$$y = Ce^{3x} + e^{3x}\int e^{-3x}6dx$$

$$= Ce^{3x} - 2$$

Review

例2：$x\dfrac{dy}{dx}-4y=x^6e^x \Rightarrow \dfrac{dy}{dx}-4\dfrac{y}{x}=x^5e^x$

$$I=e^{\int \frac{-4}{x}dx}=e^{-4\ln x}=e^{\ln x^{-4}}=x^{-4}$$

$$y=CI^{-1}+I^{-1}\int Irdx=Cx^4+x^4\int x^{-4}x^5e^xdx$$

$$=Cx^4+x^5e^x-x^4e^x$$

例3：$(x^2-9)\dfrac{dy}{dx}+xy=0 \Rightarrow \dfrac{dy}{dx}+\dfrac{1}{x^2-9}xy=0$

$$I=e^{\int \frac{x}{x^2-9}dx}=e^{\int \frac{1}{2}\frac{x}{x^2-9}d(x^2-9)}=e^{\frac{1}{2}\ln\left|x^2-9\right|}=(x^2-9)^{\frac{1}{2}}$$

$$y=CI^{-1}=\frac{C}{\sqrt{x^2-9}}$$

Review

- §2.4正合方程式

  例1：$2xydx+(x^2\text{-}1)dy=0$

  $$\frac{\partial u}{\partial x}=2xy,\frac{\partial u}{\partial y}=x^2-1$$

  $$u=x^2y+f(y),u=x^2y-y+g(x)$$

  $$f(y)=-y,g(x)=0$$

  $$u=x^2y-y$$

Review

例4：$xydx + (2x^2 + 3y^2 - 20)dy = 0$

$$\frac{\partial M}{\partial y} = x, \frac{\partial N}{\partial x} = 4x$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3x$$

$$\frac{3xdy}{M} = \frac{dI}{I} \Rightarrow \frac{3dy}{y} = \frac{dI}{I}$$

$$I = e^{\int \frac{3}{y}dy} = e^{3\ln y} = y^3$$

Review

$$xy(y^3)dx + (2x^2 + 3y^2 - 20)(y^3)dy = 0$$

$$xy^4dx + (2x^2y^3 + 3y^5 - 20y^3)dy$$

$$\frac{\partial u}{\partial x} = xy^4, \frac{\partial u}{\partial y} = 2x^2y^3 + 3y^5 - 20y^3$$

$$u = \frac{1}{2}x^2y^4 + f(y), u = \frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 + g(x)$$

$$f(y) = \frac{1}{2}y^6 - 5y^4, g(x) = 0$$

$$u = \frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = C$$

Review

- §2.5取代法

例1： $(\mathrm{x}^2+\mathrm{y}^2)dx+(x^2-xy)dy=0$

$$\frac{\partial M}{\partial y}=2y,\frac{\partial N}{\partial x}=2x-y$$

$$\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=2x-3y$$

令 $\frac{y}{x}=u, y=ux \Rightarrow dy=udx+xdu$

$$(x^2+u^2x^2)dx+(x^2-ux^2)(udx+xdu)=0$$

$$(x^2+u^2x^2+ux^2-u^2x^2)dx+(x^3-ux^3)du=0$$

$$x^2(1+u)dx+x^3(1-u)du=0$$

$$(1+u)dx+x(1-u)du=0$$

Review

$$\frac{\partial M}{\partial u} = 1, \frac{\partial N}{\partial x} = 1 - u$$

$$\frac{1-u}{1+u}du + \frac{1}{x}dx = 0$$

$$\frac{1}{x}dx = -(\frac{1-u}{1+u})du$$

$$\ln x = -(-1 + \frac{2}{1+u})du$$

$$\ln x = -(-u + 2\ln|1+u|) + C$$

Review

代$u = \frac{y}{x}$

$$\ln x = u - 2\ln\left|1 + \frac{y}{x}\right| + C$$

$$\ln x + 2\ln\left|1 + \frac{y}{x}\right| = \frac{y}{x} + \ln C^{'}$$

$$\frac{x(1 + \frac{y}{x})^2}{C^{'}} = e^{\frac{y}{x}}$$

$$(x + y)^2 = C^{'} x e^{\frac{y}{x}}$$

Review

- P.D.E

例： $x\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}-3z=0,\quad Initial\ \ Condition\ \ z(x,1)=x^2e^{x^2}$

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{3z}$$

$$\frac{dx}{x}=\frac{dy}{y}\Rightarrow\frac{y}{x}=\alpha=u$$

$$\frac{dx}{x}=\frac{dz}{3z}\Rightarrow\frac{z}{x^3}=\beta=v$$

通解$\Phi(u,v)=0,\frac{z}{x^3}=f(\frac{y}{x})$

$z(x,1)$ 帶入通解把 f 求出

$\because z(x,1)=x^2e^{x^2}$

$z=x^3f(\frac{y}{x})$

Review

$$z(x,1) = x^3 f(\frac{1}{x}) = x^2 e^{x^2}$$

$$f(\frac{1}{x}) = \frac{1}{x} e^{x^2}$$

$$令\mu = \frac{1}{x} 則 x = \frac{1}{\mu}$$

$$f(\mu) = \mu e^{\frac{1}{\mu^2}}$$

$$\therefore z = x^3 f(\frac{y}{x})$$

$$= x^3 \frac{y}{x} e^{\frac{1}{(\frac{y}{x})^2}}$$

$$= x^2 y e^{\frac{x^2}{y^2}}$$

Review

- Chap3

  常O.D.E

  法一U.C 3-3, 3-4

  法二R.O. 補充

  法三D.D. 補充

  法四VV 3-5

Review

3-3:

例9: $y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$

$y = y_h + y_p$

$\lambda^2 - 6\lambda + 9 = 0 \Rightarrow \lambda = 3,3$

$y_h = C_1 e^{3x} + C_2 x e^{3x}$

法四$: y_p = y_{p1}, y_{p2}$

$$w = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{6x} + 3xe^{6x} - 3xe^{6x} = e^{6x}$$

$$y_{p1} = e^{3x} \int \frac{-(6x^2+2)xe^{3x}}{e^{6x}} dx + xe^{3x} \int \frac{(6x^2+2)e^{3x}}{e^{6x}} dx$$

$$= e^{3x} \int -(6x^2+2)xe^{-3x} dx + xe^{3x} \int (6x^2+2)e^{-3x} dx$$

$=$ 換方法

Review

法三:

$$y_{p1} = \frac{1}{\mathrm{D}^2\text{-}6\mathrm{D}+9}(6x^2+2)$$

$$= \frac{1}{9(1+\dfrac{\mathrm{D}^2\text{-}6\mathrm{D}}{9})}(6x^2+2)$$

$$= \frac{1}{9}[1\text{-}\frac{\mathrm{D}^2\text{-}6\mathrm{D}}{9}+(\frac{\mathrm{D}^2\text{-}6\mathrm{D}}{9})^2\text{-}(\frac{\mathrm{D}^2\text{-}6\mathrm{D}}{9})^3+......](6x^2+2)$$

$$= \frac{1}{9}(6\mathrm{x}^2+2\text{-}\frac{12\text{-}72\mathrm{x}}{9}+\frac{36*12}{81})$$

$$= \frac{1}{9}(6\mathrm{x}^2+8x+6) = \frac{2}{3}\mathrm{x}^2+\frac{8}{9}x+\frac{2}{3}$$

Review

$$y_{p2} = \frac{1}{\mathrm{D}^2 - 6\mathrm{D} + 9}(-12e^{3x})$$

$$= e^{3x}\frac{1}{(D+3)^2 - 6(D+3) + 9}(-12)$$

$$= (-12)e^{3x}\frac{1}{D^2}\times 1$$

$$= e^{3x}(-6x^2)$$

$$= -6x^2 e^{3x}$$