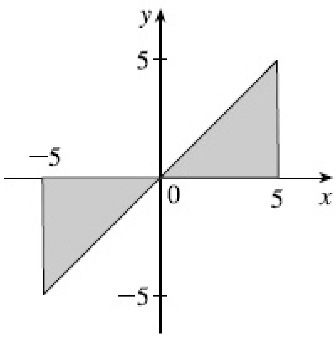
Homework 6

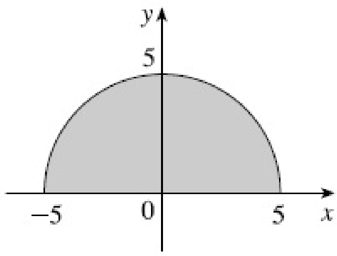
Section 5.2 38, 58
Section 5.3 4, 44, 57, 60, 62, 72
Section 5.4 16, 50
Section 5.5 18, 36, 38, 43, 73
Problem Plus 2, 3, 5, 7, 12, 15, 16, 19

Section 5.2 The Definite Integral

EX.38

$\int_{-5}^{5}(x - \sqrt{25 - x^2})\,dx = \int_{-5}^{5} x\,dx - \int_{-5}^{5}\sqrt{25 - x^2}\,dx$. By symmetry, the value of the first integral is 0 since the shaded area above the x-axis equals the shaded area below the x-axis. The second integral can be interpreted as one half the area of a circle with radius 5; that is, $\frac{1}{2}\pi(5)^2 = \frac{25}{2}\pi$. Thus, the value of the original integral is $0 - \frac{25}{2}\pi = -\frac{25}{2}\pi$.





EX.58

If $\dfrac{\pi}{6} \le x \le \dfrac{\pi}{3}$, then $\dfrac{1}{2} \le \sin x \le \dfrac{\sqrt{3}}{2}$ ($\sin x$ is increasing on $[\dfrac{\pi}{6}, \dfrac{\pi}{3}]$),

so $\dfrac{1}{2}(\dfrac{\pi}{3} - \dfrac{\pi}{6}) \le \displaystyle\int_{\pi/6}^{\pi/3} \sin x\,dx \le \dfrac{\sqrt{3}}{2}(\dfrac{\pi}{3} - \dfrac{\pi}{6})$ [Property 8];

that is, $\dfrac{\pi}{12} \le \displaystyle\int_{\pi/6}^{\pi/3} \sin x\,dx \le \dfrac{\sqrt{3}\pi}{12}$.

Section 5.3 The Fundamental Theorem of Calculus

EX.4

(a) $g(x) = \int_0^x f(t)\,dt$, so $g(0) = 0$ since the limits of integration are equal and $g(6) = 0$ since the areas above and below the t-axis are equal.

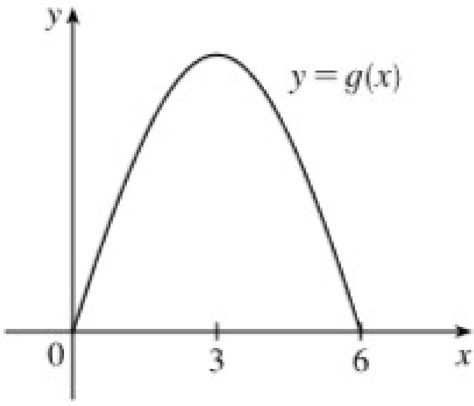
(b) $g(1)$ is the area under the curve from 0 to 1, which includes two unit squares and about 80% to 90% of a third unit square, so $g(1) \approx 2.8$. Similarly, $g(2) \approx 4.9$ and $g(3) \approx 5.7$. Now $g(3) - g(2) \approx 0.8$, so $g(4) \approx g(3) - 0.8 \approx 4.9$ by the symmetry of f about $x = 3$. Likewise, $g(5) \approx 2.8$.

(c) As we go from $x = 0$ to $x = 3$, we are adding area, so g increases on the interval $(0, 3)$.

(d) g increases on $(0, 3)$ and decreases on $(3, 6)$ [where we are subtracting area], so g has a maximum value at $x = 3$.

(e) A graph of g must have a maximum at $x = 3$, be symmetric about $x = 3$, and have zeros at $x = 0$ and $x = 6$.

(f) If we sketch the graph of g' by estimating slopes on the graph of g (as in Section 2.8), we get a graph that looks like f (as indicated by FTC1).



EX.44

$$\text{If } f(x) = \begin{cases} 2,\text{if } -2 \le x \le 0 \\ 4 - x^2,\text{if } 0 < x \le 2 \end{cases}$$

then

$$\begin{aligned} \int_{-2}^{2} f(x)dx &= \int_{-2}^{0} 2dx + \int_{0}^{2} (4 - x^2)dx = [2x]_{-2}^{0} + [4x - \frac{1}{3}x^3]_0^2 \\ &= [0 - (-4)] + (\frac{16}{3} - 0) = \frac{28}{3} \end{aligned}$$

EX.57

$f(\theta) = \sec\theta\tan\theta$ is not continuous on the interval $[\pi/3, \pi]$, so FCT2 cannot be applied. In fact, f has an infinite discontinuity at $x = \pi/2$, so $\int_{\pi/3}^{\pi} \sec\theta\tan\theta\, d\theta$ does not exist.

EX.60

$$\begin{aligned} g(x) &= \int_{1-2x}^{1+2x} t\sin t\, dt = \int_{1-2x}^{0} t\sin t\, dt + \int_{0}^{1+2x} t\sin t\, dt \\ &= -\int_{0}^{1-2x} t\sin t\, dt + \int_{0}^{1+2x} t\sin t\, dt \end{aligned}$$

$$\begin{aligned} g'(x) &= -(1-2x)\sin(1-2x) \cdot \frac{d}{dx}(1-2x) + (1+2x)\sin(1+2x) \cdot \frac{d}{dx}(1+2x) \\ &= 2(1-2x)\sin(1-2x) + 2(1+2x)\sin(1+2x) \end{aligned}$$

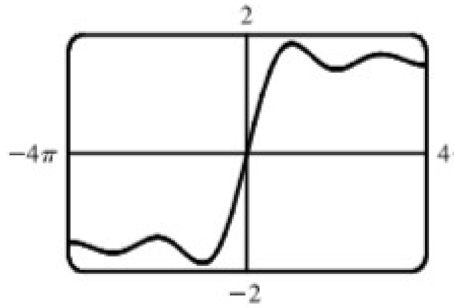
EX.62

$$\begin{aligned} F(x) &= \int_{\sqrt{x}}^{2x} \arctan t\, dt = \int_{\sqrt{x}}^{0} \arctan t\, dt + \int_{0}^{2x} \arctan t\, dt \\ &= -\int_{0}^{\sqrt{x}} \arctan t\, dt + \int_{0}^{2x} \arctan t\, dt \end{aligned}$$

$$\begin{aligned} \Rightarrow F'(x) &= -\arctan\sqrt{x} \cdot \frac{d}{dx}(\sqrt{x}) + \arctan 2x \cdot \frac{d}{dx}(2x) \\ &= -\frac{1}{2\sqrt{x}}\arctan\sqrt{x} + 2\arctan 2x \end{aligned}$$

EX.72

(a) In Maple, we should start by setting $si := int(sin(t)/t, t = 0..x)$. In Mathematica, the command is $si = Integrate[Sin[t]/t, (t, 0, x)]$. Note that both systems recognize this function; Maple calls it $Si(x)$ and Mathematica calls it $SinIntegral[x]$. In Maple, the command to generate the graph is $plot(si, x = -4^*Pi..4^*Pi)$. In Mathematica, it is $Plot[si, (x, -4^*Pi, 4^*Pi)]$. In Derive, we load the utility file EXP_INT and plot $SI(x)$.



(b) $Si(x)$ has local maximum values where $Si'(x)$ changes from positive to negative, passing through 0. From the Fundamental Theorem we know that $Si'(x) = \frac{d}{dx}\int_0^x \frac{\sin t}{t}dt = \frac{\sin x}{x}$, so we must have $\sin x = 0$ for a maximum, and for $x > 0$ we must have $x = (2n-1)\pi$, n any positive integer, for Si' to be changing from positive to negative at x.
For $x < 0$, we must have $x = 2n\pi$, n any positive integer, for a maximum, since the denominator of $Si'(x)$ is negative for $x < 0$. Thus, the local maxima occur at $x = \pi, -2\pi, 3\pi, -4\pi, 5\pi, -6\pi, ...$

(c) To find the first inflection point, we solve $Si''(x) = \frac{\cos x}{x} - \frac{\sin x}{x^2} = 0$. We can see from the graph that the first inflection point lies somewhere between $x = 3$ and $x = 5$. Using a rootfinder gives the value $x \approx 4.4934$. To find the $y-$coordinate of the inflection point, we evaluate $Si(4.4934) \approx 1.6556$. So the coordinates of the first inflection point to the right of the origin are about (4.4934,1.6556). Alternatively, we could graph $Si''(x)$ and estimate the first positive x-value at which it changes sign.

(d) It seems from the graph that the function has horizontal asymptotes at $y \approx 1.5$, with $\lim\limits_{x\to\pm\infty} Si(x) \approx \pm 1.5$ respectively. Using the limit command, we get $\lim\limits_{x\to\infty} Si(x) = \frac{\pi}{2}$. Since $Si(x)$ is an odd function, $\lim\limits_{x\to-\infty} Si(x) = -\frac{\pi}{2}$. So $Si(x)$ has the horizontal asymptotes $y = \pm\frac{\pi}{2}$.

(e) We use the **fsolve** command in Maple(or **Findroot** in Mathematica)to find that the solution is $x \approx 1.1$. Or, as in Exercise 65(c), we graph $y = Si(x)$ and $y = 1$ on the same screen to see where they intersect.

Section 5.4 Indefinite Integrals and the Net Change Theorem

EX.16

$\int \sec t(\sec t + \tan t)\, dt = \int (\sec^2 t + \sec t \tan t)\, dt = \tan t + \sec t + C.$

EX.50

$y = \sqrt[4]{x} \Rightarrow x = y^4$, so $A = \int_0^1 y^4\, dy = [\dfrac{1}{5}y^5]_0^1 = \dfrac{1}{5}.$

Section 5.5 The Substitution Rule

EX.18

Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}\, dx$ and $2du = \frac{1}{\sqrt{x}}\, dx$, so

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx = \int \sin u(2du) = -2\cos u + C = -2\cos\sqrt{x} + C$$

EX.36

Let $u = 2^t + 3$. Then $du = 2^t \ln 2\, dt$ and $2^t\, dt = \frac{1}{\ln 2}\, du$, so

$$\int \frac{2^t}{2^t + 3}\, dt = \int \frac{1}{u}\left(\frac{1}{\ln 2\, du}\right) = \frac{1}{\ln 2}\ln|u| + C = \frac{1}{\ln 2}\ln(2^t + 3) + C$$

EX.38

Let $u = 1 + \tan t$. Then $du = \sec^2 t\, dt$, so

$$\begin{aligned}\int \frac{dt}{\cos^2 t\sqrt{1 + \tan t}} &= \int \frac{\sec^2 t\, dt}{\sqrt{1 + \tan t}} = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\, du \\ &= \frac{u^{1/2}}{1/2} + C = 2\sqrt{1 + \tan t} + C\end{aligned}$$

EX.43

Let $u = \sin^{-1} x$. Then $du = \frac{1}{\sqrt{1-x^2}}\,dx$, so

$$\int \frac{dx}{\sqrt{1-x^2}\sin^{-1} x} = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\sin^{-1} x| + C$$

EX.73

Let $u = 1 + \sqrt{x}$, so $du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2\sqrt{x}\,du = dx \Rightarrow 2(u-1)du = dx$. When $x = 0, u = 1$; when $x = 1, u = 2$. Thus,

$$\int_0^1 \frac{dx}{(1+\sqrt{x})^4} = \int_1^2 \frac{1}{u^4} \cdot [2(u-1)du] = 2\int_1^2 \left(\frac{1}{u^3} - \frac{1}{u^4}\right) du$$
$$= 2\left[-\frac{1}{2u^2} + \frac{1}{3u^3}\right]_1^2 = 2[(-\frac{1}{8} + \frac{1}{24}) - (-\frac{1}{2} + \frac{1}{3})] = 2(\frac{1}{12}) = \frac{1}{6}$$

PROBLEMS PLUS

EX.2

The area A under the curve $y = x + 1/x$ from $x = a$ to $x = a + 1.5$ is given by $A(a) = \int_a^{a+1.5}(x + \frac{1}{x})\,dx$. To find the minimum value of A, we'll differentiate A using FTC1 and set the derivative equal to 0.

$$\begin{aligned} A'(a) &= \frac{d}{da}\int_a^{a+1.5}(x + \frac{1}{x})\,dx \\ &= \frac{d}{da}\int_a^1 (x + \frac{1}{x})\,dx + \frac{d}{da}\int_1^{a+1.5}(x + \frac{1}{x})\,dx \\ &= -\frac{d}{da}\int_1^a (x + \frac{1}{x})\,dx + \frac{d}{da}\int_1^{a+1.5}(x + \frac{1}{x})\,dx \\ &= -(a + \frac{1}{a}) + (a + 1.5 + \frac{1}{a+1.5}) = 1.5 + \frac{1}{a+1.5} - \frac{1}{a} \end{aligned}$$

$A'(a) = 0 \Leftrightarrow 1.5 + \frac{1}{a+1.5} - \frac{1}{a} \Leftrightarrow 1.5a(a + 1.5) + a - (a + 1.5) = 0 \Leftrightarrow 1.5a^2 + 2.25a - 1.5 = 0$ [multiply by $\frac{4}{3}$] $\Leftrightarrow 2a^2 + 3a - 2 = 0 \Leftrightarrow (2a-1)(a+2) = 0 \Leftrightarrow a = \frac{1}{2}$ or $a = -2$. Since $a > 0, a = \frac{1}{2}.A''(a) = -\frac{1}{(a+1.5)^2} + \frac{1}{a^2} > 0$, so $A(\frac{1}{2}) = \int_{1/2}^2(x + \frac{1}{x})\,dx = [\frac{x^2}{2} + \ln|x|]_{1/2}^2 = (2 + \ln 2) - (\frac{1}{8} - \ln 2) = \frac{15}{8} + 2\ln 2$ is the minimum value of A.

EX.3

For $I = \int_0^4 xe^(x-2)^4\,dx$, let $u = x - 2$ so that $x = u + 2$ and $dx = du$. Then $I = \int_{-2}^{2}(u+2)e^{u^4}du = \int_{-2}^{2} ue^{u^4}\,du + \int_{-2}^{2} 2e^{u^4}\,du = 0\,[by 5.5.7(b)] + 2\int_0^4 e^{(x-2)^4}dx = 2k$.

EX.5

$f(x) = \int_0^{g(x)} \frac{1}{\sqrt{1+t^3}}dt$, where $g(x) = \int_0^{\cos x}[1+\sin(t^2)]\,dt$. Using FTC1 and the Chain Rule (twice) we have $f'(x) = \frac{1}{\sqrt{1+[g(x)]^3}}g'(x) = \frac{1}{\sqrt{1+[g(x)]^3}}[1 + \sin(\cos^2 x)](-\sin x)$. Now $g(\frac{\pi}{2}) = \int_0^0[1+\sin(t^2)]\,dt = 0$, so $f'(\frac{\pi}{2}) = \frac{1}{\sqrt{1+0}}(1 + \sin 0)(-1) = 1\cdot 1\cdot(-1) = -1$.

EX.7

By l'Hospital's Rule and the Fundamental Theorem, using the notation $\exp(y)=e^y$,

$$\lim_{x\to 0}\frac{\int_0^x (1-\tan 2t)^{\frac{1}{t}}}{x}\,dt \overset{H}{=} \lim_{x\to 0}\frac{(1-\tan 2x)^{\frac{1}{x}}}{1} = \exp\left(\lim_{x\to 0}\frac{\ln(1-\tan 2x)}{x}\right)$$

$$\overset{H}{=} \exp\left(\lim_{x\to 0}\frac{-2\sec^2 2x}{1-\tan 2x}\right) = \exp\left(\frac{-2\cdot 1^2}{1-0}\right) = e^{-2}$$

EX.12

By FTC1, $\frac{d}{dx}\int_0^x\left(\int_1^{\sin t}\sqrt{1+u^4}du\right)dt = \int_1^{\sin x}\sqrt{1+u^4}du$. Again using FTC1, $\frac{d^2}{dx^2}\int_0^x\left(\int_1^{\sin t}\sqrt{1+u^4}du\right)dt = \frac{d}{dx}\int_1^{\sin x}\sqrt{1+u^4}\,du = \sqrt{1+\sin^4 x}\cos x$.

EX.15

Note that $\frac{d}{dx}\left(\int_0^x\left[\int_0^u f(t)dt\right]du\right) = \int_0^x f(t)\,dt$ by FTC1, while

$$\frac{d}{dx}\left[\int_0^x f(u)(x-u)du\right] = \frac{d}{dx}\left[x\int_0^x f(u)du\right] - \frac{d}{dx}\left[\int_0^x f(u)(u)du\right]$$

$$= \int_0^x f(u)du + xf(x) - f(x)x = \int_0^x f(u)du$$

Hence, $\int_0^x f(u)(x-u)\,du = \int_0^x[\int_0^u f(t)dt]du + C$. Setting $x = 0$ gives $C = 0$.

EX.16

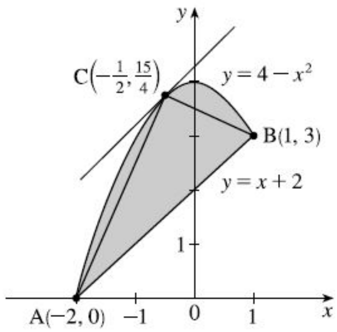
The parabola $y = 4 - x^2$ and the line $y = x + 2$ intersect when $4 - x^2 = x + 2 \Leftrightarrow x^2 + x - 2 = 0 \Leftrightarrow (x+2)(x-1) = 0 \Leftrightarrow x = -2$ or 1. So the point A is $(-2, 0)$ and B is $(1, 3)$. The slope of the line $y = x + 2$ is 1 and the slope of the parabola $y = 4 - x^2$ at x-coordinate x is $-2x$. These slopes are equal when $x = -\frac{1}{2}$, so the point C is $(-\frac{1}{2}, \frac{15}{4})$.
The Area A_1 of the parabolic segment is the area under the parabola from $x = -2$ to $x = 1$, minus the area under the line $y = x + 2$ from -2 to 1. Thus,

$$A_1 = \int_{-2}^{1} (4 - x^2)dx - \int_{-2}^{1} (x+2)dx = [4x - \frac{x^3}{3}]_{-2}^{1} - [\frac{x^2}{2} + 2x]_{-2}^{1}$$
$$= [(4 - \frac{1}{3}) - (-8 + \frac{8}{3})] - [(\frac{1}{2} + 2) - (2 - 4)] = 9 - \frac{9}{2} = \frac{9}{2}$$

The area A_2 of the inscribed triangle is the area under the line segment AC plus the area under the line segment CB minus the area under the line segment AB. The line through A and C has slope $\frac{15/4-0}{-1/2+2} = \frac{5}{2}$ and equation $y - 0 = \frac{5}{2}(x+2)$, or $y = \frac{5}{2}x + 5$. The line through C and B has slope $\frac{3-15/4}{1+1/2} = -\frac{1}{2}$ and equation $y - 3 = -\frac{1}{2}(x-1)$, or $y = -\frac{1}{2}x + \frac{7}{2}$. Thus,

$$A_2 = \int_{-2}^{-1/2} (\frac{5}{2}x + 5)dx + \int_{-1/2}^{1} (-\frac{1}{2}x + \frac{7}{2})dx - \int_{-2}^{1} (x+2)dx$$
$$= [\frac{5}{4}x^2 + 5x]_{-2}^{1/2} + [-\frac{1}{4}x^2 + \frac{7}{2}x]_{-1/2}^{1} - \frac{9}{2} = \frac{27}{8}$$

Archimedes' result states that $A_1 = \frac{4}{3}A_2$, which is verified in this case since $\frac{4}{3} \cdot \frac{27}{8} = \frac{9}{2}$.



EX.19

$$
\begin{aligned}
&\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\sqrt{n+1}}+\frac{1}{\sqrt{n}\sqrt{n+2}}+\cdots+\frac{1}{\sqrt{n}\sqrt{n+n}}\right)\\
&=\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{\frac{n}{n+1}}+\sqrt{\frac{n}{n+2}}+\cdots+\sqrt{\frac{n}{n+n}}\right)\\
&=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{\sqrt{1+1/n}}+\frac{1}{\sqrt{1+2/n}}+\cdots+\frac{1}{\sqrt{1+1}}\right)\\
&=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{i}{n}\right) \qquad \left[\text{where} f(x)=\frac{1}{\sqrt{1+x}}\right]\\
&=\int_0^1\frac{1}{\sqrt{1+x}}dx=[2\sqrt{1+x}]_0^1=2(\sqrt{2}-1)
\end{aligned}
$$