Review.
Differential and Integral

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Differential

- 極限

  定義：

  若 $\forall \varepsilon > 0, \exists \delta > 0$

  st. $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$

  則稱 $f(x)$ 於 x_0 的 極限為 L

  記為 $\lim_{x \to x_0} f(x) = L$

Differential

- 連續 <=> 極限 ＝ 函數值

  (1) $f(x)$有意義

  (2) $\lim_{x \to x_0} f(x) = L$ 存在

  (3) $L = f(x_0)$

  則稱 $f(x_0)$於 x_0 為連續

Differential

- 有界

  定義：

  若 $f(x)$ 於 $[a,b]$為連續，則 $f(x)$ 於 $[a,b]$為有界，若為開區間 (a,b)則不一定有界。

Differential

- 分段(片段)連續 (piecewise continuous)

  若 $\exists a < x_1 < x_2 < \ldots < x_n < b$

  (1) $f(x)$於 $(a,x_1)(x_2,x_3)\ldots(x_n,b)$為連續

  (2) $f(a^+)$、$f(x_1^-)$、$f(x_2^+)$、…、$f(x_n^+)$、$f(b^-)$ 均存在

  則稱 $f(x)$於(a,b) 分段連續

Differential

- f： V → W　V、W：集合

  定義域　對應域

$f \in C_p([a,b])$

$C_p(V) \cong \{f \mid f$ 於V為分段連續$\}$

$C(V) \cong \{ f \mid f$ 於為連續 $\}$

Differential

- 導數定義

$$f'(x_0) \cong \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h},$$

$f'(x)$稱為 $f(x)$ 於 x_0的導數

若 $f'(x_0)$存在，則稱 $f(x)$ 於 x_0 為可微分

$$df = f'(x)dx$$

Differential

- 對數微分法

$$\frac{df(x)}{dx} = f(x)\frac{d}{dx}\ln f(x)$$

$$\frac{d\ln f(x)}{dx} = \frac{1}{f(x)}\cdot\frac{df(x)}{dx}$$

$$\therefore \frac{df(x)}{dx} = f(x)\cdot\frac{d\ln f(x)}{dx}$$

例：

$$f(x) = \frac{(x-3)^2}{(x-2)^3(x+1)}$$

$$\Rightarrow \ln f(x) = 2\ln(x-3) - 3\ln(x-2) - \ln(x+1)$$

$$\Rightarrow \frac{d\ln f(x)}{dx} = \frac{2}{x-3} - \frac{3}{x-2} - \frac{1}{x+1}$$

$$\Rightarrow \frac{df(x)}{dx} = \frac{(x-3)^2}{(x-2)^3(x+1)}\left(\frac{2}{x-3} - \frac{3}{x-2} - \frac{1}{x+1}\right)$$

Differential

若 $f(x)=\dfrac{\mathrm{v}_1(x)\mathrm{v}_2(x)}{\mathrm{u}_1(x)\mathrm{u}_2(x)}$

$$\Rightarrow \ln f(x)=\ln \mathrm{v}_1(x)+\ln \mathrm{v}_2(x)-\ln \mathrm{u}_1(x)-\ln \mathrm{u}_2(x)$$

$$\Rightarrow \frac{d\ln f(x)}{dx}=\frac{\mathrm{v}_1'}{\mathrm{v}_1}+\frac{\mathrm{v}_2'}{\mathrm{v}_2}-\frac{\mathrm{u}_1'}{\mathrm{u}_1}-\frac{\mathrm{u}_2'}{\mathrm{u}_1}$$

例：

$$f(x)=\frac{x^2+4x+13}{(x^2+3x+3)(x+1)^4}$$

$$\Rightarrow \ln f(x)=\ln(x^2+4x+13)-\ln(x^2+3x+3)-4\ln(x+1)$$

$$\Rightarrow \frac{df(x)}{dx}=f(x)\left(\frac{2x+4}{x^2+4x+13}-\frac{2x+3}{x^2+3x+3}-\frac{4(x+1)^3}{(x+1)^4}\right)$$

Differential

- Formula

$$f,g\in C^n([a,b])$$

$$\Rightarrow \frac{d^n}{dx^n}(f\bullet g)=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}f^{(n-k)}g^{(k)}$$

Integral

- 不定積分＝反(逆)導數

若存在 $F(x)$ 使得 $F'(x) = f(x)$

則 $\int f(x)dx \triangleq F(x) + c$

例：$\int \cos x dx = \sin x + c$

Integral

- 微分運算子

$$D(f(x)) = \frac{d}{dx} f(x)$$

$$\Rightarrow \frac{1}{D} f(x) = D^{-1} f(x) = \int f(x)dx$$

Integral

- 分部積分法

$d(uv) = vdu + udv$　　同取積分

$\Rightarrow \int d(uv) = \int (vdu + udv)$

$\Rightarrow uv = \int vdu + \int udv$

$\Rightarrow \int udv = uv - \int vdu$

Integral

$\int uv'dx = uv - \int u'vdx$　　$u, v \rightarrow u(x), v(x)$

若定義　$f^{(1)} = \dfrac{d}{dx} f(x)$　　$f_{(1)} = \int f(x)dx$

$f^{(2)} = \dfrac{d^2}{dx^2} f(x)$　　$f_{(2)} = \int\int f(x)dxdx$

$u \rightarrow f, v' \rightarrow g$

$\Rightarrow \int fgdx = fg_{(1)} - \int f^{(1)} g_{(1)} dx$

$$\begin{array}{ccc} f & \oplus & g \\ f^{(1)} & \ominus & g_{(1)} \end{array}$$

DCNLAB可靠計算暨網路實驗室

Integral

$$\int f^{(1)} g_{(1)} dx = f^{(1)} g_{(2)} - \int f^{(2)} g_{(2)} dx$$

$$\begin{array}{ccc} f^{(1)} & \oplus & g_{(1)} \\ f^{(2)} & \ominus & g_{(2)} \end{array}$$

例: $\int (x^2 + 3x + 3)\cos x dx = ?$

$sol:$

$$\begin{array}{ccc} x^2 + 3x + 3 & \oplus & \cos x \\ 2x + 3 & \ominus & \sin x \\ 2 & \oplus & -\cos x \\ 0 & \ominus & -\sin x \end{array}$$

$$\Rightarrow (x^2 + 3x + 3)\sin x - (2x+3)(-\cos x) + 2(-\sin x) + c$$

Integral

- Euler 公式

$$\begin{cases} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{cases}$$

$$\Rightarrow \begin{cases} \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} \end{cases}$$

Integral

$$\begin{cases} \int e^{ax}\cos(bx)dx = ?........(1) & 實 \\ \int e^{ax}\sin(bx)dx = ?..........(2) & 虛 \end{cases}$$

$$\Rightarrow (1) + i(2)$$

$$= \int e^{ax} * e^{ibx} dx$$

$$= \frac{1}{a+ib} e^{(a+ib)x} = \frac{a-ib}{a^2+b^2} e^{ax}(\cos bx + i\sin bx)$$

$$實 = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$虛 = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

Integral

- 定積分

$$\int_a^b f(x)dx$$

$$= \lim_{(\Delta x)_{\max}\to 0} \sum_k f(x_k^*) \times \Delta x_k$$

$$\Delta x_k = x_k - x_{k-1}$$

x_k^*：為任一個在 $[x_k, x_{k-1}]$ 之點

Integral

- Thm：微積分第一基本定理

若 $f(x)$於 $[a,b]$連續，則存在 $F(x) \in C'([a,b])$

使得 $F'(x) = f(x)$ ， $a \le x \le b$

$\Rightarrow \quad \int_a^b f(x)dx = F(b) - F(a)$

則 $\int_a^x f(u)du = F(x) - F(a)$

$\therefore \quad \frac{d}{dx}\int_a^x f(u)du = F'(x) = f(x)$

Integral

例: $\frac{d}{dx}\int_0^{x^3} e^{-u^2} du = ?$

Sol:

令$x^3 = \zeta$

$J(x) = \int_0^{\zeta} e^{-u^2} du$

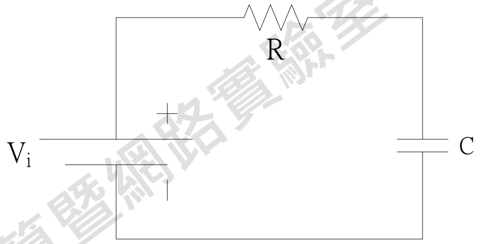
$\Rightarrow \frac{dJ(x)}{dx} = \frac{dJ(x)}{d\zeta} \times \frac{d\zeta}{dx}$

$= e^{-\zeta^2} \times 3x^2$

$= e^{-x^6} \times 3x^2$

Introduction to differential equations

- 例：RC電路



$$i(t) = C \times \frac{dV_C(t)}{dt}$$

歐姆定律 $V_R = iR$

$$= CR \times \frac{dV_C(t)}{dt}$$

$$V_i = V_R + V_C$$

$$\Rightarrow V_i(t) = CR \times \frac{dV_C(t)}{dt} + V_C(t)$$

$$V_i : given$$

$$\Rightarrow \; V_C(t) = ?$$

Definitions

- What is differential equations(微分方程式)？

一個方程式包含因變數及因變數之導數項

$$\frac{dy^2}{dt^2} + 3\frac{dy}{dt} + 5y = 3e^t \Rightarrow y(t)$$

$$CR \times \frac{dV_C(t)}{dt} + V_C(t) = V_i(t)$$

$\Rightarrow y(t), V_i(t)$ 這樣，只有一個自變數t，此微分方程式又稱**常微分方程式(Ordinary DE)**，若包含二項變數以上，則稱**偏微分方程式(Partial DE)**

Definitions

$$\text{D.E.}\begin{cases}\text{O.D.E.}\\ \text{P.D.E.}\end{cases}$$

例:

$$\frac{\partial^2 v(x,y)}{\partial x^2}+\frac{\partial^2 v(x,y)}{\partial y^2}=0 \Rightarrow \text{P.D.E.}$$

$$y''+4y'+8y=0 \qquad \Rightarrow \text{O.D.E.}$$

Terminology

- Terminology 術語
  1. 階數(Order):

一個微分方程式中，最高階導數項的微分次數即稱此微分方程式的階數

例:

$$\frac{\partial u^2(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}+5u(x,y)=0 \Rightarrow \text{二階P.D.E.}$$

$$\frac{d^3 y(x)}{dx^3}+4(y(x))^5=e^x \qquad \Rightarrow \text{三階O.D.E.}$$

Terminology

2.Degree(次數)：

Pre: 須將所有項有理化(ie：沒有 $x^{\frac{1}{2}}$ 項)

一個微分方程式中，每項均為有理項，最高階導數項的次數，即稱為此方程式之次數

例:

$$y' = \sqrt{y} + 5y$$

$$\Rightarrow (y' - 5y)^2 = y$$

$$\Rightarrow (y')^2 - 25yy' + 25y^2 = y$$

$\Rightarrow$ 一階二次O.D.E.

$$y'' + 3y^2 = e^x$$

$\Rightarrow$ 二階一次O.D.E.