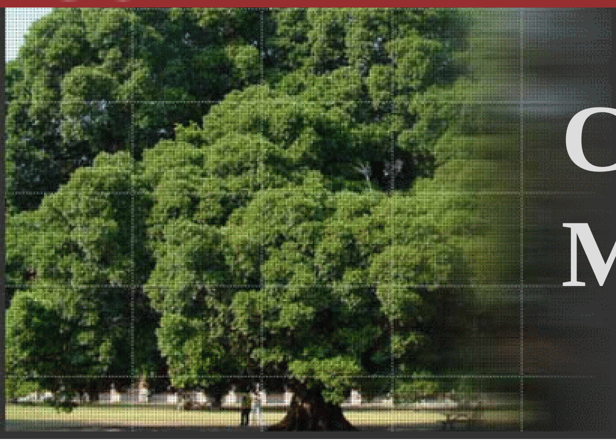
National Cheng Kung University



Chapter 23
Minimum Spanning Trees

Sun-Yuan Hsieh

謝孫源 教授

成功大學資訊工程學系

Overview

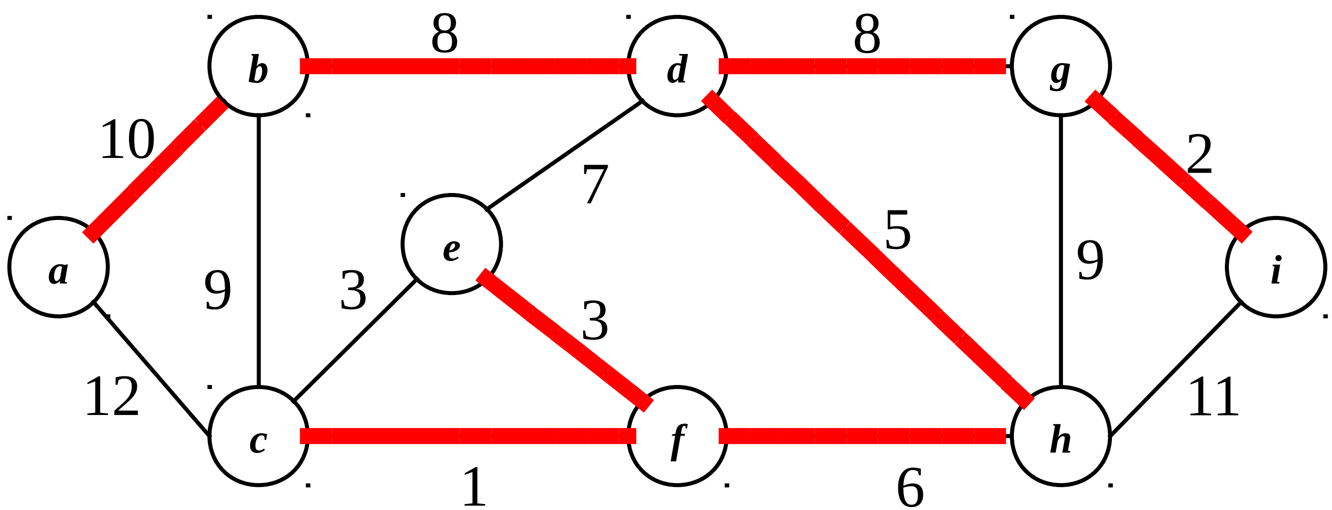
- **Problem**
  - A town has a set of houses and a set of roads.
  - A road connects 2 and only 2 houses.
  - A road connecting houses u and v has a repair cost $w(u, v)$.
  - ***Goal***: Repair enough (and no more) roads such that
    1. everyone stays connected: can reach every house from all other houses, and
    2. total repair cost is minimum.

Overview

- Model as a graph:
  - Undirected graph $G = (V, E)$
  - **Weight** $w(u, v)$ on each edge $(u, v) \in E$
  - Find $T \subseteq E$ such that
    1. T connects all vertices (T is a ***spanning tree***), and
    2. $w(T) = \sum_{(u,v)\in T} w(u,v)$ is minimized.

Overview

- A spanning tree whose weight is minimum over all spanning trees is called a ***minimum spanning tree***, or ***MST***.
  - Example of such a graph *[edges in MST are shaded]* :



    - In this example, there is more than one MST. Replace edge (e, f) by (c, e).
    - Get a different spanning tree with the same weight.

Growing a minimum spanning tree

- Some properties of an MST:
  - It has $|V|$-1 edges.
  - It has no cycles.
  - It might not be unique.

- **Building up the solution**
  - We will build a set *A* of edges.
  - Initially, *A* has no edges.
  - As we add edges to *A*, maintain a loop invariant:

    Loop invariant: *A* is a subset of some MST.

Growing a minimum spanning tree

- Add only edges that maintain the invariant. If A is a subset of some MST, an edge (u, v) is **safe** for A if and only if $A \cup \{(u, v)\}$ is also a subset of some MST. So we will add only safe edges.

Generic MST algorithm

GENERIC-MST(G, w)

$A \leftarrow \emptyset$

while A is not a spanning tree

  do find an edge (u, v) that is safe for A

    $A \leftarrow A \cup \{(u, v)\}$

return A

- Use the loop invariant to show that this generic algorithm works.
  - **Initialization**: The empty set trivially satisfies the loop invariant.
  - **Maintenance**: Since we add only safe edges, *A* remains a subset of some MST.
  - **Termination**: All edges added to *A* are in an MST, so when we stop, *A* is spanning tree that is also an MST.

Finding a safe edge

How do we find safe edges?

- Let's look at the example. Edge (c, f) has the lowest weight of any edge in the Graph. Is it safe for $A = \emptyset$?

- Intuitively: Let $S \subset V$ be any set of vertices that includes c but not f (so that f is in $V - S$). In any MST, there has to be one edge (at least) that connects S with $V - S$. Why not choose the edge with minimum weight? (Which would be (c, f) in this case.)

- Some definitions: Let $S \subset V$ and $A \subseteq E$.
  - A ***cut*** $(S, V\text{-}S)$ is a partition of vertices into disjoint sets S and $V\text{-}S$.
  - Edge $(u, v) \in E$ ***crosses*** cut $(S, V\text{-}S)$ if one endpoint is in S and the other is in $V\text{-}S$.
  - A cut ***respects*** A if and only if no edge in A crosses the cut.
  - An edge is a ***light edge*** crossing a cut if and only if its weight is minimum over all edges crossing the cut. For a given cut, there can be > 1 light edge crossing it.

- ***Theorem***

Let A be a subset of some MST, $(S, V\text{-}S)$ be a cut that respects A, and (u, v) be a light edge crossing $(S, V\text{-}S)$. Then (u, v) is safe for A.

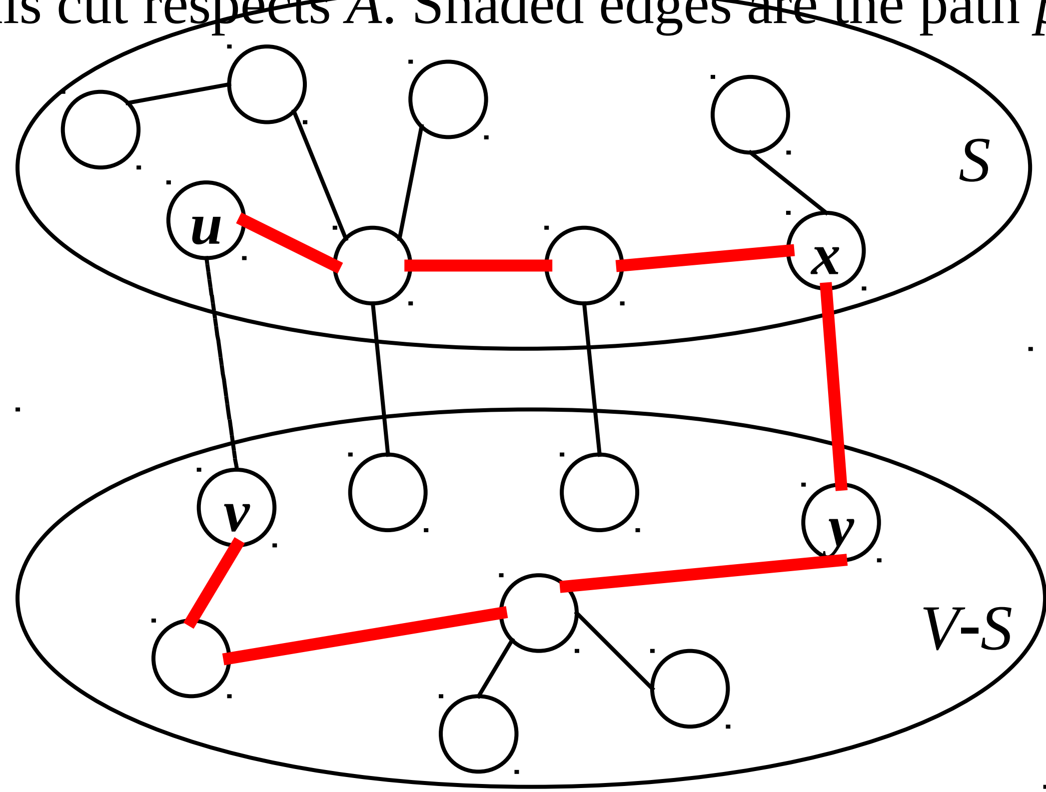
Proof Let T be an MST that includes A.

If T contains (u, v) , done.

So now assume that T does not contain (u, v) . We'll construct a different MST T' that includes $A \cup \{(u, v)\}$.

Recall: a tree has unique path between each pair of vertices. Since T is an MST, it contains a unique path p between u and v. Path p must cross the cut $(S, V\text{-}S)$ at least once. Let (x, y) be an edge of p that crosses the cut. From how We chose (u, v) , must have $w(u, v) \le w(x, y)$

[Except for the dashed edge (u, v), all edges shown are in T. A is some subset of the edges of T, but A cannot contain any edges that cross the cut (S, V-S), since this cut respects A. Shaded edges are the path p.]



Since the cut respects A, edge (x, y) is not in A.

To from T' from T:

- Remove (x, y) . Breaks T into two components.
- Add (u, v) . Reconnects.

So $T' = T - \{(x, y)\} \cup \{(u, v)\}$.

T' is a spanning tree.

$$w(T') = w(T) - w(x, y) + w(u, v)$$
$$\leq w(T)$$

since $w(u, v) \leq w(x, y)$.

Since T' is a spanning tree, $w(T') \leq w(T)$, and T is an MST, then T' must be an MST.

Need to show that (u, v) is safe for A:

- $A \subseteq T$ and $(x, y) \notin A \Rightarrow A \subseteq T'$
- $A \cup \{(u, v)\} \subseteq T'$
- Since T' is an MST, (u, v) is safe for A.

GENERIC-MST:

- *A* is a forest containing connected components. Initially, each component is a single vertex.
- Any safe edge merge two of these components into one. Each component is a tree.
- Since an MST has exactly |*V*|-1 edges, the **for** loop iterates |*V*|-1 times.

  Equivalently, after adding |*V*|-1 safe edges, we're down to just one component.

- ***Corollary***

  If $C = (V_C, E_C)$ is a connected component in the forest $G_A = (V, A)$ and (u, v) is a light edge connecting C to some other component in G_A (i.e., (u, v) is a Light edge crossing the cut $(V_C, V\text{-}V_C)$), then (u, v) is safe for A.

Proof Set $S = V_C$ in the theorem.

- This naturally leads to the algorithm called Kruskal's algorithm to solve the Minimum-spanning-tree problem.

Kruskal's algorithm

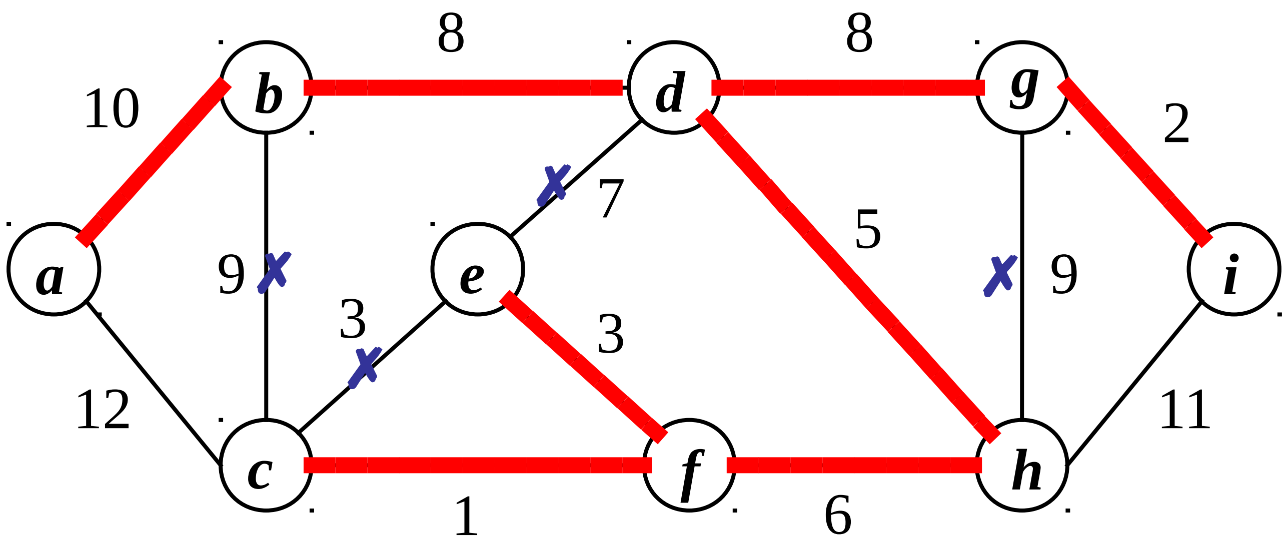
- $G = (V,E)$ is a connected, undirected, weighted graph.
  $w: E \to \Re$
  - Starts with each vertex being its own component.
  - Repeatedly merges two components into one by choosing the light edge that connects them (i.e., the light edge crossing the cut between them).
  - Scans the set of edges in monotonically increasing order by weight.
  - Uses a disjoint-set data structure to determine whether an edge connects vertices in different components.

- KRUSKAL(V, E, w)

$A \leftarrow \emptyset$
for each vertex $v \in V$[G]
    do MAKE-SET(v)
sort E into nondecreasing order by weight w
for each (u, v) taken from the sorted list
    do if FIND-SET(u) $\neq$ FIND-SET(v)
        then $A \leftarrow A \cup \{(u, v)\}$
            UNION(u,v)
return A

Run through the above example to see how Kruskal's algorithm works on it:

(c, f) : chosen
(g, i) : chosen
(e, f) : chosen
(c, e) : reject
(d, h) : chosen
(f, h) : chosen
(e, d) : reject
(b, d) : chosen
(d, g) : chosen
(b, c) : reject
(g, h) : reject
(a, b) : chosen



- At this point, we have only one component, so all other edges will be rejected.
  - [We could add a test to the main loop of KRUSKAL to stop once $|V|$-1 edges have been added to A.]

- Get the shaded edges shown in the figure.

- Suppose we had examined (c, e) before (e, f). Then, would have found (c, e) safe and would have rejected (e, f).

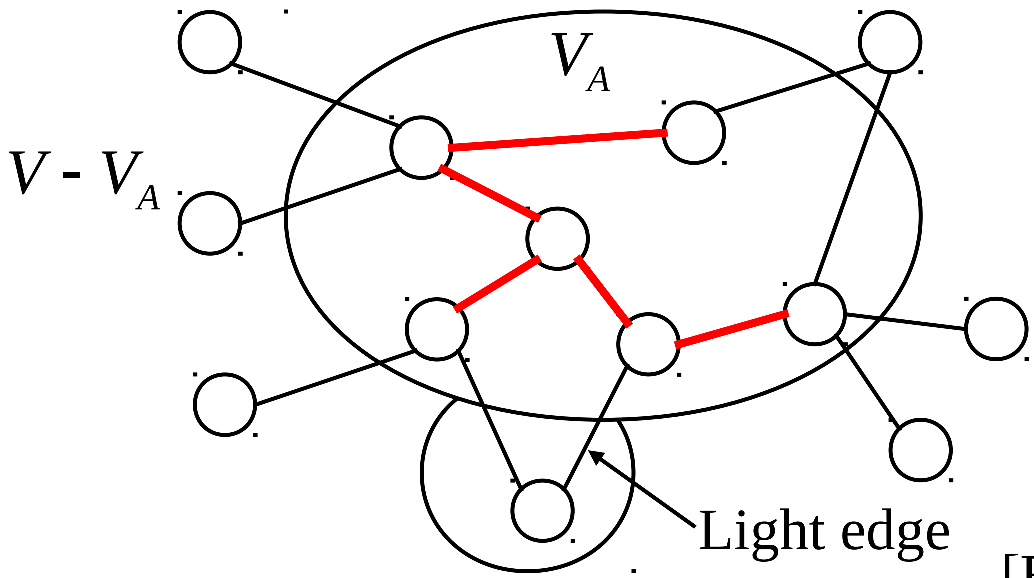
- **Analysis**

Initialize A:	$O(1)$
First **for** loop:	$\lvert V\rvert$ MAKE-SETS
Sort E:	$O(E \lg E)$
Second **for** loop:	$O(E)$ FIND-SETS and UNIONS

- Assuming the implementation of disjoint-set data structure, already seen in chapter21, that uses union by rank and path compression: $O((V + E)\alpha(V)) + O(E \lg E)$
- Since G is connected, $|E| \geq |V| - 1 \Rightarrow O(E\ \alpha(V)) + O(E \lg E)$.
- $\alpha(|V|) = O(\lg V) = O(\lg E)$
- Therefore, total time is $O(E \lg E)$.
- $|E| \leq |V|^2 \Rightarrow \lg |E| = O(2 \lg V) = O(\lg V)$
- Therefore, $O(E \lg V)$ time. (If edges are already sorted, $O(E\alpha(V))$, which is almost linear.)

Prim's algorithm

- Builds one tree, so *A* is always a tree.
- Starts from an arbitrary "root" *r*.
- At each step, find a light edge crossing cut (V_A, V-V_A), where V_A= vertices that *A* is incident on. Add this edge to *A*.



[Edges of *A* are shaded.]

- How to find the light edge quickly?

  Use a priority queue Q:

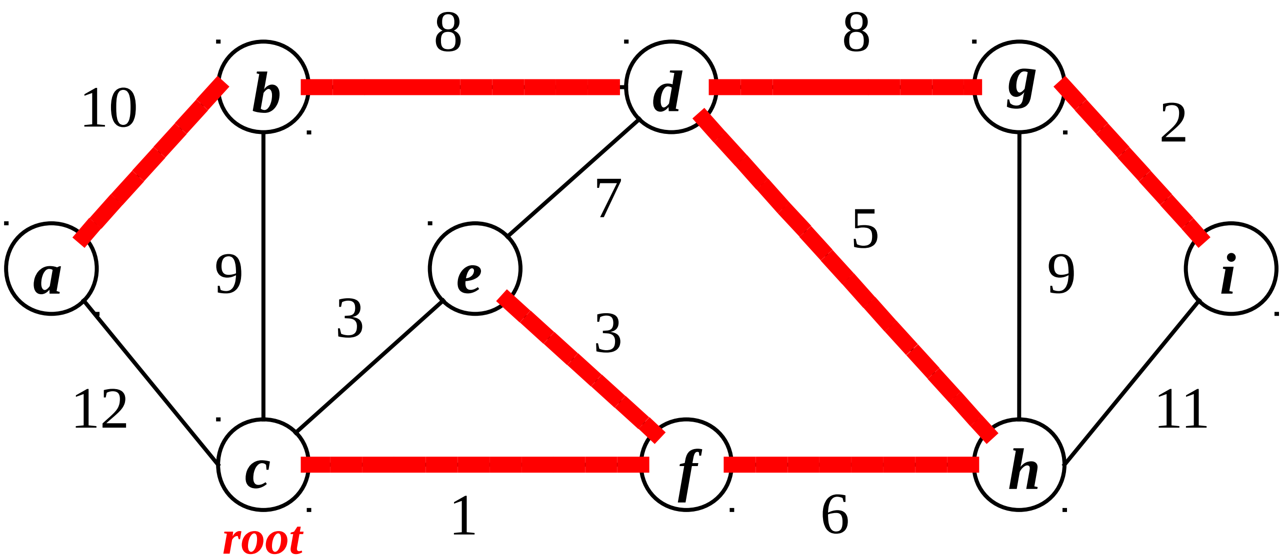
  - Each object is a vertex in V-V_A.
  - Key of v is minimum weight of any edge (u, v) , where $u \in V_A$.
  - Then the vertex returned by EXTRACT-MIN is v such that there exists $u \in V_A$ and (u, v) is light edge crossing $(V_A, V\text{-}V_A)$.
  - Key of v is ∞ if v is not adjacent to any vertices in V_A.

- The edges of A will form a rooted tree with root r:
  - r is given as an input to the algorithm, but it can be any vertex.
  - Each vertex knows its parent in the tree by the attribute $\pi[v]$= parent of v. $\pi[v]$ = NIL if $v = r$ or v has no parent.
  - As algorithm progresses, $A = \{(v, \pi[v]) : v \in V\text{-}\{r\}\text{-}Q\}$.
  - At termination,

    $V_A = V \Rightarrow Q = \emptyset$, so MST is $A = \{(v, \pi[v]) : v \in V - \{r\}\}$.

```
PRIM(V, E, w, r)
    Q ← Ø
    for each u ∈ V[G]
        do key[u] ← ∞
           π[u] ← NIL
           INSERT(Q, u)
    DECREASE-KEY(Q, r, 0)
    while Q ≠ Ø
        do u ← EXTRACT-MIN(Q)
           for each v ∈ Adj[u]
               do if v ∈ Q and w(u, v) < key[v]
                     then π[v] ← u
                          DECREASE-KEY(Q, v, w(u, v) )
```

Run through the above example to see how Prim's algorithm works on it: Vertex ***c*** has been chosen as a starting point.



(c, f) : chosen

(e, f) : chosen

(f, h) : chosen

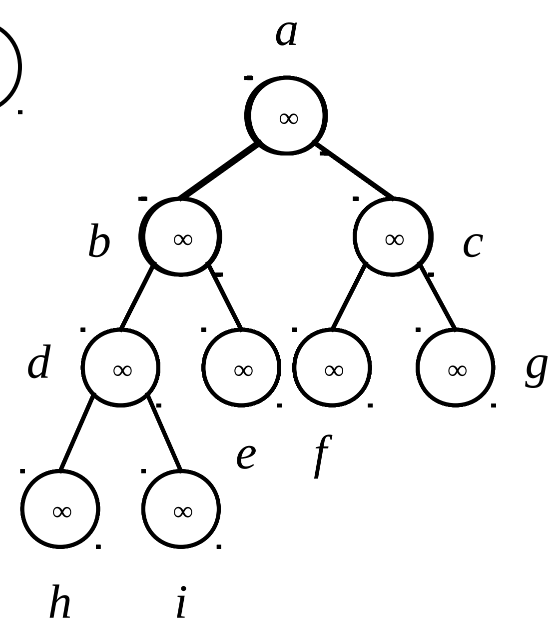
(d, h) : chosen

(d, g) : chosen

(g, i) : chosen

(b, d) : chosen

(a, b) : chosen



- **Analysis**
  - Depends on how the priority queue is implemented:
    - Suppose Q is a binary heap.
    - Initialize Q and first **for** loop: $O(V \lg V)$
    - Decrease key of r: $O(\lg V)$
    - **while** loop: $|V|$ EXTRACT-MIN calls $\Rightarrow O(V \lg V)$

      $\le |E|$ DECREASE-KEY calls $\Rightarrow O(E \lg E)$
    - Total: $O(E \lg V)$

- Suppose we could do DECREASE-KEY in $O(1)$ *amortized* time. Then $\leq |E|$ DECREASE-KEY calls take $O(E)$ time altogether

  $\Rightarrow$ total time becomes $O(V \lg V + E)$

  In fact, there is a way to do DECREASE-KEY in $O(1)$ amortized time: Fibonacci heaps, in chapter20.