Discrete Mathematics (2014 Spring) Midterm II

1. (24 points) For each of the following statements, determine and explain (required) whether it is correct or not.
   (a). If (A, R) is a lattice, then it is a total order.
   (b). Suppose that R is an equivalence relation on {1, 2, 3, 4, 5, 6} and the equivalence classes induced by R are {1, 5}, {2, 4, 6}, {3}. The size of R is 13.
   (c). The least upper bound of {1, 2, 5, 10, 15} in the poset(Z^+, |) does not exist.
   (d). Only two of (Z, =), (Z, ≠), (Z, ≥), (Z, ∤) are posets.
   (e). Let f: A→B and g: B→C, $g \circ f$ is one-to-one if and only if f and g are one-to-one.
   (f). If A is a language, the $(A^*)^+=A^+$.

   a. False, False. Let $\mathcal{U} = \{1,2\}, A = \mathcal{P}(\mathcal{U})$, and $\mathcal{R}$ be the inclusion relation. Then $(A, \mathcal{R})$ is a lattice where for all $S, T \in A$, $lub\{S,T\} = S \cup T$ and $glb\{S,T\} = S \cap T$. However, $\{1\}$ and $\{2\}$ are not related, so $(A, \mathcal{R})$ is not a total order.
   b. False, R = { (1, 1), (1, 5), (2, 2), (2, 4), (2, 6), (4, 2), (3, 3), (4, 6), (4, 4), (5, 1), (5, 5), (6, 2), (6, 2), (6, 6)}, the size of R is 14
   c. False, lub is 30
   d. True, (Z, =), (Z, ≥)
   e. False, If f and g are one-to-one, then $g \circ f$ is one-to-one (true)
      If $g \circ f$ is one-to-one, then f and g are one-to-one (false)
   f. False, $(A^*)^+ = A^*$

2. (4,4,8 points) (a) How many selections from the set {1, 3, 5, 7, 9, 11, 13, 15} to guarantee that at least one pair of these selected numbers add up to 16. (b) How many selections from the set {1, 2, 3, 4, …, 29, 30} to guarantee that at least exist two integers x, y from our selection that $\gcd(x, y) \geq 2$. (c) Show that, in a group of five persons, there are at least two of them have the same number of friends in this group.

   (a) 將{1, 3, 5, 7, 9, 11, 13, 15}分為{1,15}{3,13}{5,11}{7,9}四組，根據pigeon-hole theorem, 至少要選5個數字
   (b) {1,2,3,5,7,11,13,17,19,23,29}間兩兩互質，至少選12個數字(pigeon-hole theorem)
   (c) 五個人有可能的朋友數: {0}{1}{2}{3}{4}
   其中{0}{1}不可能同時存在，所以朋友數只有可能為{0}{1}{2}{3}或{1}{2}{3}{4}，根據pigeon-hole theorem, 5 pigeons in 4 pigeon holes，因此至少有兩個人有相同朋友數

3. (9 points) Determine the following relations are reflexive, symmetric, antisymmetric, or transitive. (a) Let $x, y \in \boldsymbol{R}, and\ xRy\ if\ and\ only\ if\ x = 2y$. (b) $a, b \in \boldsymbol{Z}, and\ aRb\ if\ and\ only\ if\ |a - b| \leq 1$. (c) $a, b \in \boldsymbol{Z}, and\ aRb\ if\ and\ only\ if\ ab \leq 0$.

   (a) antisymmetric
   (b) reflexive、symmetric
   (c) symmetric

4. (5,10 points) (a) Find the number of ways to totally order the partial order of all positive-integer divisors of 75. (b) Let p, q be distinct primes. How many edges are there in the Hasse diagram of all positive divisors of p^4q^2 for the relation "|".

   (a) $75 = 3 \times 5^2$, There are $(1+1)\times(2+1) = 6$ divisors for this partial order. So, the number of ways to totally order are $\frac{1}{4}\binom{6}{3} = 5$ ways.
   (b) $(4+1)\times 2 + (2+1)\times 4 = 10 + 12 = 22$

$$\begin{array}{ccccccccc}
p^4q^2 & — & p^3q^2 & — & p^2q^2 & — & p^1q^2 & — & p^0q^2 \\
| & & | & & | & & | & & | \\
p^4q^1 & — & p^3q^1 & — & p^2q^1 & — & p^1q^1 & — & p^0q^1 \\
| & & | & & | & & | & & | \\
p^4q^0 & — & p^3q^0 & — & p^2q^0 & — & p^1q^0 & — & p^0q^0
\end{array}$$

5. (2, 3, 5 points) Let A={a, b, c, d}, B={1, 2, 3, 4, 5, 6, 7}. (a) How many one-to-one functions are there from A to B? (b) How many functions from A to B are nondecreasing? (c) How many onto functions from B to A satisfying $f(1)$=a?

(a) $P_4^7 = 7\times6\times5\times4 = 840$

(b) $\binom{7+4-1}{4} = \binom{10}{4} = \frac{10!}{6!4!} = 210$

(c) $4!\times S(6,4) + 3!\times S(6,3) = 24\times(25+4\times10) + 6\times90 = 1560 + 540 = 2100$

(Note: $S(6, 4) = S(5, 3) + 4\times S(5,4)$)

6. (3,3,4 points) Let A={a, b, c, d, e} (a) how many closed binary operations f on A satisfy $f(a, b) \neq c$? (b) How many closed binary operations f on A have an identity and $f(a, b)$=c? (c) How many f in (b) are commutative?

(a) $4 * 5^{24}$

(b) a and b cannot be an identity and f(a,b) is fixed. For each c, d, e there are 5^{15} closed binary operations on A where c,d or e is identity. So Answer is $\mathbf{3 * 5^{15}}$

f	a	b	c	d	e
a					
b					
c					
d		✔			
e					

(c) $\binom{3}{1} 5^9 = 3 * 5^9$

f	a	b	c	d	e
a	●			●	●
b		●		●	●
c					
d				●	●
e					●

7. (2,2,3,3 points) If A = {a, b, c, d}, determine the number of relations on A that are (a) antisymmetric and do not contain (a, b), (b) reflexive and symmetric but not transitive, (c) equivalence relations, (d) equivalence relations that determine more than two (include two) equivalence classes.

(a) $2^{n+1}3^{\frac{n^2-n}{2}-1} = 2^5 3^5$

(b) $2^6 - \sum_{i=1}^{4} S(4,i) = 64 - (1+7+6+1) = 49$

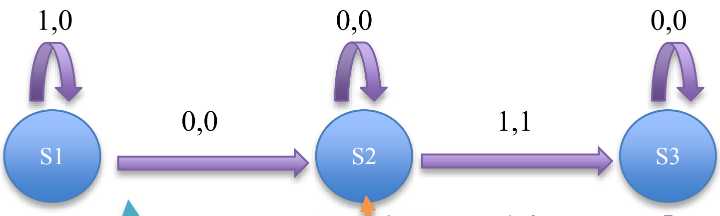
(c) $\sum_{i=1}^{4} S(4,i) = 1+7+6+1 = 15$

(d) $\sum_{i=2}^{4} S(4,i) = 7+6+1 = 14$

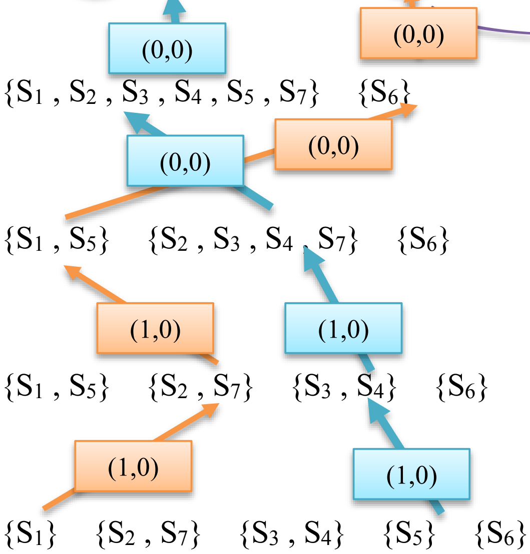
8. [7.5-1(b)] (6, 5, 5 points) (a) Construct a finite-state machine that recognizes the set of bit strings consisting of a 0 following by a string with even number of 1s (at least two 1s). (b) Apply the minimization process to the finite-state machine in the right table and draw the final reduced finite-state machine. (c) Find a distinguishing string for s_1 and s_5.

	ν		ω	
	0	1	0	1
s_1	s_6	s_3	0	0
s_2	s_3	s_1	0	0
s_3	s_2	s_4	0	0
s_4	s_7	s_4	0	0
s_5	s_6	s_7	0	0
s_6	s_5	s_2	1	0
s_7	s_4	s_1	0	0

(a)



(b)



(c)

1100

(Stirling number of the second kind: S(4, 2)=7, S(4, 3)=6, S(5, 2)=15, S(5, 3)=25, S(5, 4)=10, S(6, 2)=31, S(6, 3)=90)