$$\Rightarrow \frac{1}{x+y}(xy+y^2+1)\,dx + \frac{1}{x+y}(xy+x^2+1)\,dy = 0$$

$\vdots$

$$u = xy + \ln(x+y) = C \quad \#.$$

②. $\dfrac{x-y}{-y\cdot N + x\cdot M} = \dfrac{x-y}{-y(xy+x^2+1) + x(xy+y^2+1)} = 1$

$\Rightarrow I = e^{\int 1\, d(xy)} = e^{xy}.$ ⇒ 積分因子可以不唯一

$\Rightarrow \underbrace{e^{xy}(xy+y^2+1)}_{M = \frac{\partial u}{\partial x}}dx + e^{xy}(xy+x+1)\,dy = 0.$

$\Rightarrow u = \int e^{xy}(xy+y^2+1)\,dx + f_1(y).$

$= \underbrace{\int e^{xy}\cdot xy\cdot dx}_{ⓐ} + \underbrace{\int e^{xy}\cdot y^2 dx}_{ⓑ} + \boxed{\int e^{xy}\cdot 1\cdot dx}_{ⓒ} + f_1(y).$

$= y\int e^{xy}\cdot x\,dx = y\int \underbrace{u}_{x}\,\underbrace{dv}_{e^{xy}dx} = y(uv - \int v\,du)$ 抵消

$= y(x\cdot \frac{1}{y}e^{xy} - \int \frac{1}{y}e^{xy}dx) = xe^{xy} \boxed{-\int e^{xy}dx}$

$\vdots$

$$u = xe^{xy} + e^{xy} = C \quad \#.$$

③. 原式: $x(y\,dx + x\,dy) + y(y\,dx + x\,dy) + dx + dy = 0.$

$\Rightarrow (x+y)\,d(xy) + d(x+y) = 0.$

$\Rightarrow d(xy) + \frac{1}{x+y}d(x+y) = 0.$

積分..

$\Rightarrow xy + \ln(x+y) = C \quad \#.$

ex. $(y\cos x - \sin 2x)\,dx + dy = 0$

$\frac{\partial M}{\partial y} = \cos x. \qquad \frac{\partial N}{\partial x} = 0.$

$\Rightarrow \dfrac{0-\cos x}{-N} = \cos x.$

$\Rightarrow \cos x\,dx = \dfrac{dI}{I} \Rightarrow I = e^{\sin x}.$

$\Rightarrow \dfrac{\partial u}{\partial x} = y\cos x \cdot e^{\sin x} - \sin 2x \cdot e^{\sin x}$

$\Rightarrow u = \int (\underbrace{y\cos x \cdot e^{\sin x}}_{①} - \underbrace{\sin 2x \cdot e^{\sin x}}_{②})\,\partial x + f_1(y).$

$①: \int y\cos x \cdot e^{\sin x}\,\partial x = y e^{\sin x}.$

$②: \int \sin 2x \cdot e^{\sin x}\,dx = \int 2\sin x \cos x\, e^{\sin x}\,dx.$

令 $t = \sin x \Rightarrow dt = \cos x\,dx.$

$\Rightarrow dx = \dfrac{dt}{\cos x}.$

$\Rightarrow$ 原式: $2\int t \cancel{\cos x} \cdot e^t \cdot \dfrac{dt}{\cancel{\cos x}} = 2(te^t - e^t)$

$= 2(\sin x \cdot e^{\sin x} - e^{\sin x}).$

$\Rightarrow u = y\cdot e^{\sin x} - 2\sin x \cdot e^{\sin x} + 2e^{\sin x} + f_1(y).$

又 $\dfrac{\partial u}{\partial y} = e^{\sin x}.$

$\Rightarrow u = \int e^{\sin x}\,dy + f_2(x) = y e^{\sin x} + f_2(x).$

$\vdots$

$\Rightarrow u = ye^{\sin x} - 2\sin x \cdot e^{\sin x} + 2e^{\sin x} = C.$ #

ex. $\dfrac{dy}{dx} = 3x^2 - 3x^2 y.$

$\dfrac{\partial M}{\partial y} = -3x^2, \quad \dfrac{\partial N}{\partial x} = 0$

$\Rightarrow \dfrac{3x^2}{-N} = 3x^2 \Rightarrow 3x^2\,dx = \dfrac{dI}{I} \Rightarrow I = e^{x^3}$

$\Rightarrow (3x^2 - 3x^2 y)e^{x^3}\,dx - e^{x^3}\,dy = 0.$

$\Rightarrow \dfrac{\partial u}{\partial x} = e^{x^3}(3x^2 - 3x^2 y)$

$\Rightarrow u = e^{x^3} - y e^{x^3} + f_1(y).$

$\frac{\partial u}{\partial y} = -e^{x^3}$

$\Rightarrow u = -y e^{x^3} + f_2(x)$

$\vdots$

$u = e^{x^3} - y e^{x^3} = C$ #.

◎ 分離變數法.

ex. $(1+x)dy - y\,dx = 0$.

① $\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = 1$

$\Rightarrow \frac{2}{M} = \frac{2}{-y} \Rightarrow \frac{2}{-y}dy = \frac{dI}{I} \Rightarrow I = y^{-2}$.

⇒原式 $-y^{-1}dx + (1+x)y^{-2}dy = 0$.

$\frac{\partial u}{\partial x} = -y^{-1} \Rightarrow u = \int -y^{-1}dx + f_1(y) = -xy^{-1} + f_1(y)$.

$\frac{\partial u}{\partial y} = (1+x)y^{-2} \Rightarrow u = -(1+x)y^{-1} + f_2(x)$

$\Rightarrow u = -(1+x)y^{-1} = C$ #.

② $\frac{dy}{y} - \frac{dx}{1+x} = 0 \Rightarrow \frac{dy}{y} = \frac{dx}{1+x}$

積分.

$\Rightarrow \ln y = \ln(1+x) + C$

$\Rightarrow y = (1+x)e^{c}$ #.

ex. $\frac{dy}{dx} = y^2 - 4$

$\Rightarrow \frac{dy}{y^2-4} = dx$ 　在將y^2-4移過去時就已將它視為"不為0".

$\Rightarrow \left(\frac{a}{y-2} + \frac{b}{y+2} \right) dy = dx$, $a = \frac{1}{4}$, $b = -\frac{1}{4}$

積分.

$\Rightarrow \frac{1}{4} \ln|y-2| - \frac{1}{4} \ln|y+2| = x + c$

$\Rightarrow \ln\left|\frac{y-2}{y+2}\right| = 4x + 4c$

$\Rightarrow \frac{y-2}{y+2} = e^{4x+4c}$

又 合分比

$\Rightarrow \frac{2y}{4} = \frac{e^{4x+4c}+1}{1-e^{4x+4c}}$, 令 $e^{4c} = c'$

$\Rightarrow y = 2 \cdot \frac{1+c'e^{4x}}{1-c'e^{4x}}$

$\Rightarrow$ 当 $\frac{dy}{dx} = 0$ 時. $y = \pm 2$.

◎ 一階線性 O.D.E.

$y'(x) + p(x) \cdot y(x) = r(x)$.

① $r(x) = 0$ homogeneous 齐性

② $r(x) \neq 0$ nonhomogeneous 非齐性

case ① $\Rightarrow y'(x) + p(x) \cdot y(x) = 0$

$\Rightarrow \frac{dy(x)}{dx} = -p(x) \cdot y(x)$

$\Rightarrow \frac{dy(x)}{dy} = -p(x)\, dx$

積分.

$\Rightarrow \ln y(x) = -\int p(x)\, dx + k$.

$$\Rightarrow y(x) = e^{-\int p(x)dx} \cdot \underbrace{(e^k)}_{C} \ \#.$$

ex. $y'(x) + 2x\,y(x) = 0$

$$\Rightarrow y(x) = e^{-\int 2x\,dx} \cdot C = Ce^{-x^2} \ \#.$$

case ②. $\Rightarrow y'(x) + p(x)y(x) = r(x)$

$$\Rightarrow \frac{dy}{dx} + p(x)\cdot y(x) - r(x) = 0$$

$$\Rightarrow \underbrace{(p(x)y(x) - r(x))}_{M}dx + \underbrace{dy}_{N} = 0$$

$$\frac{\partial M}{\partial y} = p(x) \qquad \frac{\partial N}{\partial x} = 0$$

$$\Rightarrow \frac{-p(x)}{-N} = p(x) \quad \Rightarrow p(x)dx = \frac{dI}{I} \quad \Rightarrow I = e^{\int p(x)dx}$$

$$\Rightarrow e^{\int p(x)dx}(p(x)y(x) - r(x))dx + e^{\int p(x)dx}dy = 0$$

$$\frac{\partial u}{\partial x} = e^{\int p(x)dx}(p(x)y(x) - r(x))$$

$$\Rightarrow u = \int e^{\int p(x)dx}(p(x)y(x) - r(x))\,dx + f_1(y)$$

令 $t = \int p(x)dx$, $\frac{dt}{dx} = p(x)$

$$\Rightarrow u = \int e^{t}\left(\frac{dt}{dx}y(x) - r(x)\right)dx + f_1(y).$$

$$= \int e^{t}y(x)\,dt - \int e^{t}\,r(x)\,dx + f_1(y)$$

$$= y(x)\,e^{\int p(x)dx} - \int e^{\int p(x)dx} r(x)\cdot dx + f_1(y)$$

$$\frac{\partial u}{\partial y} = e^{\int p(x)dx} \Rightarrow u = \int e^{\int p(x)dx}dy + f_2(x)$$

$$= y\cdot e^{\int p(x)dx} + f_2(x)$$

$\vdots$

$\Rightarrow u = y \cdot e^{\int P(x)dx}\ r(x)\,dx = C$ #.

$\Rightarrow y = c \cdot e^{-\int P(x)dx} + e^{-\int P(x)dx} \cdot \int e^{\int P(x)dx}\ r(x)\,dx$

$= (y_h) + (y_p)$

homogeneous sol. ($r(x)=0$)

particular sol - 特解

記法:

$y' + p(x)y(x) = r(x)$ 之解.

$y = c \cdot I^{-1} + I^{-1}\int I \cdot r(x)\,dx \quad (I = e^{\int P(x)dx})$

ex. $y' + 2xy = 3x$

$y = cI^{-1} + I^{-1}\int I \cdot 3x\,dx$

$= ce^{-x^2} + e^{-x^2}\int e^{x^2} \cdot 3x\,dx$

$= ce^{-x^2} + e^{-x^2} \cdot \frac{3}{2} e^{x^2} = ce^{-x^2} + \frac{3}{2}$

◎ note.

① $y_h' + P(x)y_h = 0$

② theorm

$y_p(x)$ 滿足 非奇性方程式

$\Rightarrow y_p' + P(x)y_p(x) = r(x)$

pf.

$\because y = y_h + y_p$ 代入原方程式

$\Rightarrow (y_h + y_p)' + P(y_h + y_p) = r$

$\Rightarrow \underbrace{y_h' + Py_h}_{=0} + y_p' + Py_p = r$

$\Rightarrow y_p' + Py_p = r$ #

ex. $y' + 2xy = x$

$\Rightarrow I = e^{\int p(x)dx} = e^{x^2}$

$\Rightarrow y = ce^{-x^2} + e^{-x^2} \cdot \int e^{x^2} \cdot x\,dx = ce^{-x^2} + \frac{1}{2}$

Ch 3

◎ 一階線性(常係數) O.D.E.

$y' + ay = r(x)$, $a \in$ const

case ① $r(x) = 0$ homogeneous

$\Rightarrow y' + ay = 0$

$y = ce^{-ax} = y_h(x)$

常係數 $\Rightarrow y_h(x)$ 的部分一定為指數函數 ($e^{\lambda x}$)
└ λ: const.

ex. $y' + 2y = 0$

$\Rightarrow y = ce^{-2x}$.

如果 apply 上述特性 ($e^{\lambda x}$ 必為解).

$(e^{\lambda x})' + 2(e^{\lambda x}) = 0$

$\Rightarrow \lambda e^{\lambda x} + 2e^{\lambda x} = 0$

$\Rightarrow e^{\lambda x}(\lambda + 2) = 0$ $\because e^{\lambda x} \neq 0$ (為解)

$\Rightarrow \lambda = -2$ —— characteristic equation. 特性方程式

$\Rightarrow y = ce^{-2x}$

ex. $y' - 3y = 0$

$\Rightarrow y = ce^{3x}$.

推廣 $y'' + ay' + by = 0$, $a, b \in$ const.

猜 $e^{\lambda x}$ 為解

$\Rightarrow (e^{\lambda x})'' + a(e^{\lambda x})' + b(e^{\lambda x}) = 0$

$\Rightarrow \lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + b e^{\lambda x} = 0$

$\Rightarrow e^{\lambda x}(\lambda^2 + a\lambda + b) = 0$

$\because e^{\lambda x} \neq 0$

$\Rightarrow \lambda^2 + a\lambda + b = 0 \Rightarrow \lambda : \dfrac{-a \pm \sqrt{a^2-4b}}{2}$

1. $\lambda_1 \neq \lambda_2 \in R$ 相異實根.

$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

ex. $y'' + 3y' + 2y = 0$

$\Rightarrow \lambda^2 + 3\lambda + 2 = 0$

$\Rightarrow \lambda = -1 \vee -2.$

$\Rightarrow y = c_1 e^{-x} + c_2 e^{-2x}$ ※.

ex. $y'' + 6y' + 5y = 0$

$\Rightarrow y = C_1 e^{-5x} + C_2 e^{-x}$

2. $\lambda_1, \lambda_2 = \alpha \pm \beta i$ 共軛虛根.

$\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$

$\Rightarrow y = k_1 e^{(\alpha+\beta i)x} + k_2 e^{(\alpha-\beta i)x}.$

尤拉公式

$\Rightarrow e^{\alpha x}(k_1(\cos\beta x + i\sin\beta x) + k_2(\cos\beta x - i\sin\beta x))$

令 $k_2 = i k_2'$

$\Rightarrow e^{\alpha x}(k_1(\cos\beta x + i\sin\beta x) + ik_2'(\cos\beta x - i\sin\beta x))$

$= e^{\alpha x}((k_1 + ik_2')\cos\beta x + (k_2' + ik_1)\sin\beta x)$

令 $C_1 = k_1 + ik_2'$, $C_2 = k_2' + ik_1$

$\Rightarrow e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x)$

ex. $y'' + 2y' + 10y = 0$

$\Rightarrow \lambda = -1 \pm 3i$

$\Rightarrow y = e^{-x}(C_1\cos 3x + C_2\sin 3x)$

ex. $y'' + 4y = 0$

$\lambda = \pm 2i$

$\Rightarrow y = C_1\cos 2x + C_2\sin 2x$

*. $y'' + ay' + b = 0$

$\Rightarrow \lambda'' + a\lambda + b = 0$

$\Rightarrow (\lambda - \lambda_1)(\lambda - \lambda_2) = 0$

$\Rightarrow \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0$

$a = -(\lambda_1 + \lambda_2)$

$b = \lambda_1\lambda_2$

$\Rightarrow y'' - (\lambda_1 + \lambda_2)y' + \lambda_1\lambda_2 y = 0$

定义 $D \equiv \frac{d}{dx}$ differential operator 微分運算子

$Dx^2 = \frac{d}{dx}x^2 = 2x$ D只对右边函数做運算.

$\therefore Dx^2 \neq x^2 D$

$\Rightarrow D^k = \frac{d^k}{dx^k}$

$\Rightarrow D^2 y - (\lambda_1 + \lambda_2)Dy + \lambda_1\lambda_2 y = 0$

$\Rightarrow (D^2 - (\lambda_1 + \lambda_2)D + \lambda_1\lambda_2)y = 0$

$\Rightarrow (D - \lambda_1)(D - \lambda_2)y = 0$

令 $(D - \lambda_2)y = z(x)$

$\Rightarrow z'(x) - \lambda_1 z(x) = 0$

$\Rightarrow z(x) = e^{\lambda_1 x}$

$\Rightarrow (D - \lambda_2) y = z(x) = k_1 e^{\lambda_1 x}$

$y'' - \lambda_2 y = k_1 e^{\lambda_1 x}$

$\Rightarrow y = C I^{-1} + I^{-1} \int I r \, dx.$

$\Rightarrow I = e^{-\int \lambda_2 dx} = e^{-\lambda_2 x}$