# Chapter 7 Functions of Random Variables

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# Introduction

辨認何種機率分佈（計算E(x)）

- Moment-generating function: helpful in learning about distributions of linear functions of random variables
- Statistical hypothesis testing, estimation, or even statistical graphics involve functions of one or more random variables
  - The use of averages of random variables
  - The distribution of sums of squares of random variables

# Transformations of Variables

- Theorem 7.1: Suppose that $X$ is a discrete random variable with probability distribution $f(x)$. Let $Y = u(X)$ define a one-to-one transformation between the values of $X$ and $Y$ so that the equation $y = u(x)$ can be uniquely solved for $x$ in terms of $y$, say $x = w(y)$. Then the probability distribution of $Y$ is $g(y) = f[w(y)]$.
- Example 7.1: Let X is a geometric random variable with probability distribution $f(x) = \frac{3}{4}\left(\frac{1}{4}\right)^{x-1},\ x = 1,2,3,\ldots$ Find the probability distribution of the random variable $Y = X^2$.
  - **solution** $\because X$ are all positive (number of trials)

    $\therefore$ the transformation defines a one-to-one correpondance between $x$ and $y$

    $$y = x^2 \Rightarrow x = \sqrt{y}$$

    $$g(y) = \begin{cases} f(\sqrt{y}) = \frac{3}{4}\left(\frac{1}{4}\right)^{\sqrt{y}-1}, & y = 1,4,9,\ldots \\ 0, & \text{elsewhere} \end{cases}$$

# Transformations of Variables

- Theorem 7.2: Suppose that $X_1$ and $X_2$ are discrete random variables with joint probability distribution $f(x_1, x_2)$. Let $Y_1 = u_1(X_1, X_2)$ and $Y_2 = u_2(X_1, X_2)$ define a one-to-one transformation between the points $(x_1, x_2)$ and $(y_1, y_2)$ so that the equations $y_1 = u_1(x_1, x_2)$ , $y_2 = u_2(x_1, x_2)$ may be uniquely solved for $x_1$ and $x_2$ in terms of $y_1$ and $y_2$, say $x_1 = w_1(y_1, y_2)$ and $x_2 = w_2(y_1, y_2)$ Then the joint probability distribution of $Y_1$ and $Y_2$ is $g(y_1, y_2) = f[w_1(y_1, y_2), w_2(y_1, y_2)]$.

tions with parameters $\mu_1$ and $\mu_2$, respectively. Find the distribution of the random variable $Y_1 = X_1 + X_2$.

**Example 7.2**

**Solution** Since $X_1$ and $X_2$ are independent, we can write

$$f(x_1,x_2) = f(x_1)f(x_2) = \frac{e^{-\mu_1}\mu_1^{x_1}}{x_1!}\frac{e^{-\mu_2}\mu_2^{x_2}}{x_2!} = \frac{e^{-(\mu_1+\mu_2)}\mu_1^{x_1}\mu_2^{x_2}}{x_1!x_2!},$$

where $x_1 = 0, 1, 2, \ldots$ and $x_2 = 0, 1, 2, \ldots$. Let us now define a second random variable, say $Y_2 = X_2$. The inverse functions are given by $x_1 = y_1 - y_2$ and $x_2 = y_2$. Using Theorem 7.2, we find the joint probability distribution of $Y_1$ and $Y_2$ to be

$$g(y_1, y_2) = \frac{e^{-(\mu_1+\mu_2)}\mu_1^{y_1-y_2}\mu_2^{y_2}}{(y_1 - y_2)!y_2!},$$

where $y_1 = 0, 1, 2, \ldots$ and $y_2 = 0, 1, 2, \ldots, y_1$. Note that since $x_1 > 0$, the transformation $x_1 = y_1 - x_2$ implies that $x_2$ and hence $y_2$ must always be less than or equal to $y_1$. Consequently, the marginal probability distribution of $Y_1$ is

$$h(y_1) = \sum_{y_2=0}^{y_1} g(y_1, y_2) = e^{-(\mu_1+\mu_2)}\sum_{y_2=0}^{y_1}\frac{\mu_1^{y_1-y_2}\mu_2^{y_2}}{(y_1 - y_2)!y_2!}$$

$$= \frac{e^{-(\mu_1+\mu_2)}}{y_1!}\sum_{y_2=0}^{y_1}\frac{y_1!}{y_2!(y_1 - y_2)!}\mu_1^{y_1-y_2}\mu_2^{y_2}$$

$$= \frac{e^{-(\mu_1+\mu_2)}}{y_1!}\sum_{y_2=0}^{y_1}\binom{y_1}{y_2}\mu_1^{y_1-y_2}\mu_2^{y_2}.$$

Recognizing this sum as the binomial expansion of $(\mu_1 + \mu_2)^{y_1}$, we obtain

$$h(y_1) = \frac{e^{-(\mu_1+\mu_2)}(\mu_1 + \mu_2)^{y_1}}{y_1!}, \qquad y_1 = 0, 1, 2, \ldots,$$

from which we conclude that the sum of the two independent random variables having Poisson distributions, with parameters $\mu_1$ and $\mu_2$, has a Poisson distribution with parameter $\mu_1 + \mu_2$.

# Transformations of Variables

- Theorem 7.3: Suppose that $X$ is a continuous random variable with probability distribution $f(x)$. Let $Y = u(X)$ define a one-to-one transformation between the values of $X$ and $Y$ so that the equation $y = u(x)$ can be uniquely solved for $x$ in terms of $y$, say $x = w(y)$. Then the probability distribution of Y is $g(y) = f[w(y)]|J|$, where $J = w'(y)$ and is called the Jacobian of the transformation.
  - **solution**

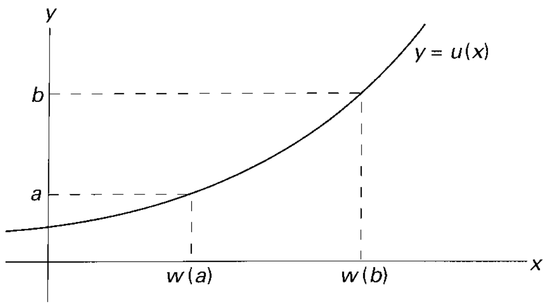


**Figure 7.1** Increasing function.

# Transformations of Variables

(skipped in 9th ed.)

– **Proof**

Suppose that $y = u(x)$ is an increasing function

$P(a < Y < b) = P[w(a) < X < w(b)] = \int_{w(a)}^{w(b)} f(x)dx$

$x = w(y) \Rightarrow dx = w'(y)dy$

$P(a < Y < b) = \int_{a}^{b} f[w(y)]w'(y)dy$

$g(y) = f[w(y)]w'(y) = f[w(y)]J$

If we recognize $J = w'(y)$ as the reciprocal of the slope of tangent line to the curve of the increasing function $y = u(x)$, it is then obvious that $J = |J|$. (Fig. 7.1)

In the case the slope of the curve is negative and $J = -|J|$ (Fig. 7.2)

$\therefore g(y) = f[w(y)]|J|$

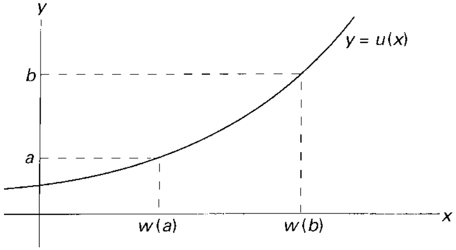


**Figure 7.1** Increasing function.

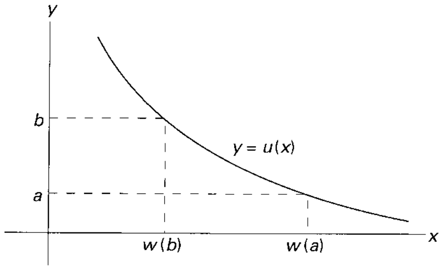


**Figure 7.2** Decreasing function.

# Transformations of Variables

- Example 7.3: Let *X* be a continuous random variable with probability distribution

$$f(x) = \begin{cases} \frac{x}{12}, & 1 < x < 5 \\ 0, & \text{elsewhere} \end{cases}$$

Find the probability distribution of $Y = 2X - 3$.

  – **solution**

$$y = 2x - 3 \Rightarrow x = (y+3)/2$$

$$J = w'(y) = dx/dy = \tfrac{1}{2}$$

$$g(y) = \begin{cases} \dfrac{(y+3)/2}{12} \cdot \dfrac{1}{2} = \dfrac{y+3}{48}, & -1 < y < 7 \\ 0, & \text{elsewhere} \end{cases}$$

# Transformations of Variables

- Theorem 7.4: Suppose that $X_1$ and $X_2$ are continuous random variables with joint probability distribution $f(x_1, x_2)$. Let $Y_1 = u_1(X_1, X_2)$ and $Y_2 = u_2(X_1, X_2)$ define a one-to-one transformation between the points $(x_1, x_2)$ and $(y_1, y_2)$ so that the equations $y_1 = u_1(x_1, x_2)$ , $y_2 = u_2(x_1, x_2)$ may be uniquely solved for $x_1$ and $x_2$ in terms of $y_1$ and $y_2$, say $x_1 = w_1(y_1, y_2)$ and $x_2 = w_2(y_1, y_2)$ Then the joint probability distribution of $Y_1$ and $Y_2$ is

$$g(y_1, y_2) = f[w_1(y_1, y_2), w_2(y_1, y_2)]\underline{|J|},$$

where the Jacobian is the 2×2 determinant

$$J = \begin{vmatrix} \partial x_1/\partial y_1 & \partial x_1/\partial y_2 \\ \partial x_2/\partial y_1 & \partial x_2/\partial y_2 \end{vmatrix}$$

and $\partial x_1/\partial y_1$ is simply the derivative of $x_1 = w_1(y_1, y_2)$ with respect to $y_1$ with $y_2$ held constant, referred to in calculus as the partial derivative of $x_1$ with respect to $y_1$. The other partial derivatives are defined in a similar manner.

# Transformations of Variables

邊界轉換

$x_1 = 0 \Rightarrow y_1 = 0 = y_2$

$x_2 = 0 \Rightarrow y_2 = 0, y_1 = x_1^2 \Rightarrow 0 < y_1 < 1$

$x_1 = 1 \Rightarrow y_1 = 1, y_2 = x_2 \Rightarrow 0 < y_2 < 1$

$x_2 = 1 \Rightarrow y_1 = x_1^2 = y_2^2, y_2 = x_1 \Rightarrow 0 < y_1 = y_2^2 = x_1^2 < 1$

- Example 7.4: Let $X_1$ and $X_2$ be two continuous random variables with joint probability distribution

$$f(x_1, x_2) = \begin{cases} 4x_1x_2, & 0 < x_1 < 1, 0 < x_2 < 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find the joint probability distribution of $Y_1 = X_1^2$ and $Y_2 = X_1X_2$.

– **Solution**

$$\because y_1 = x_1^2, y_2 = x_1x_2 \Rightarrow x_1 = \sqrt{y_1}, x_2 = y_2 / \sqrt{y_1}$$

$$J = \begin{vmatrix} 1/(2\sqrt{y_1}) & 0 \\ -y_2 / 2y_1^{3/2} & 1/\sqrt{y_1} \end{vmatrix} = \frac{1}{2y_1}$$

To determine the set of $B$ of points in the $y_1y_2$ − plane

$$A = \{(x_1, x_2) \mid 0 < x_1 < 1, 0 < x_2 < 1\} \Rightarrow B = \{(y_1, y_2) \mid y_2^2 < y_1 < 1, 0 < y_2 < 1\}$$

$$g(y_1, y_2) = 4(\sqrt{y_1})\frac{y_2}{\sqrt{y_1}}\frac{1}{2y_1} = \begin{cases} \dfrac{2y_2}{y_1}, & y_2^2 < y_1 < 1, 0 < y_2 < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

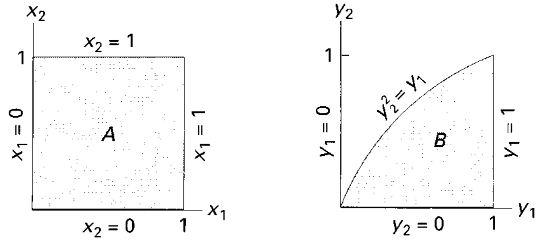


**Figure 7.3** Mapping set $A$ into set $B$.

# Transformations of Variables

- Theorem 7.5: Suppose that $X$ is a continuous random variable with probability distribution $f(x)$. Let $Y = u(X)$ define a transformation between the values of $X$ and $Y$ that is not one-to-one. If the interval over which X is defined can be partitioned into k mutually disjoint sets such that each of the inverse functions 分成k段，每段為one-to-one mapping

  $$x_1 = w_1(y), \quad x_2 = w_2(y), \quad \ldots \quad , \quad x_k = w_k(y)$$

  of $y = u(x)$ define a one-to-one correspondence, then the probability distribution of $Y$ is

  $$g(y) = \sum_{i=1}^{k} f[w_i(y)]\,|J_i|,$$

  where $J_i = w_i'(y), i = 1,2,\ldots,k.$

# Transformations of Variables

- Example 7.5: Show that $Y = (X - \mu)^2/\sigma^2$ has a chi-squared distribution with 1 degree of freedom when $X$ has a normal distribution with mean $\mu$ and variance $\sigma^2$.
  - Solution

Let $Z = (X - \mu)/\sigma$, where $Z$ has the standard normal distrinution

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \qquad -\infty < z < \infty.$$

not one-to-one

Now $Y = Z^2, y = z^2 \Rightarrow z = \pm\sqrt{y}$, if $z_1 = -\sqrt{y}$ and $z_2 = \sqrt{y}$

then $J_1 = -\dfrac{1}{2\sqrt{y}}$ and $J_2 = \dfrac{1}{2\sqrt{y}}$

# Transformations of Variables

– Solution

By Theorem 7.5, $g(y) = \frac{1}{\sqrt{2\pi}} e^{-y/2} \left| \frac{-1}{2\sqrt{y}} \right| + \frac{1}{\sqrt{2\pi}} e^{-y/2} \left| \frac{1}{2\sqrt{y}} \right| = \frac{1}{2^{1/2}\sqrt{\pi}} y^{1/2-1} e^{-y/2}$

$\because g(y)$ is a density function, it follows that

$$1 = \frac{1}{2^{1/2}\sqrt{\pi}} \int_0^\infty y^{1/2-1} e^{-y/2} dy = \frac{\Gamma(1/2)}{\sqrt{\pi}} \int_0^\infty \frac{1}{2^{1/2}\Gamma(1/2)} y^{1/2-1} e^{-y/2} dy = \frac{\Gamma(1/2)}{\sqrt{\pi}}$$

the integral being the area under a gamma probability curve

with parameters $\alpha = 1/2$ and $\beta = 2$

$\therefore \Gamma(1/2) = \sqrt{\pi}$ and the probability distribution of $Y$

$$g(y) = \begin{cases} \frac{1}{2^{1/2}\Gamma(1/2)} y^{1/2-1} e^{-y/2}, & y > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

which is seen to be a chi – squared distribution with 1 degree of freedom.

$$f(x) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$.

# Moments and Moment-Generating Functions

- Definition 7.1: The *r*th moment about the origin of the random variable $X$ is given by

$$\mu_r' = E(X^r) = \begin{cases} \sum_x x^r f(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x^r f(x)dx & \text{if } X \text{ is continuous} \end{cases}$$

  - The first and second moments about the origin are given by $\mu_1' = E(X)$ and $\mu_2' = E(X^2)$, so mean $\mu = \mu_1'$ and variance $\sigma^2 = \mu_2' - \mu^2$.

- Definition 7.2: The moment-generating function of the random variable $X$ is given by $E(e^{tX})$ and is denoted by $M_X(t)$.

$$M_X(t) = E(e^{tX}) = \begin{cases} \sum_x e^{tx} f(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x)dx & \text{if } X \text{ is continuous} \end{cases}$$

# Moments and Moment-Generating Functions

- Theorem 7.6: Let $X$ is a random variable with moment-generating function $M_x(t)$. Then $\left.\frac{d^r M_X(t)}{dt^r}\right|_{t=0} = \mu'_r$.  微分x次，x moment
  - Proof

Assume we can differentiate inside summation and integral signs, we obtain

$$\frac{d^r M_X(t)}{dt^r} = \begin{cases} \sum_x x^r e^{tx} f(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x^r e^{tX} f(x)dx & \text{if } X \text{ is continuous} \end{cases}$$

Setting $t = 0$, we see that both cases reduce to $E(X^r) = \mu_r'$

$r=1$, $\frac{d M_x(t)}{dt} = \sum_x x e^{tx} f(x)$

$= \sum_x x f(x) = E(x)$

# Moments and Moment-Generating Functions

- Example 7.6: Find the moment-generating function of the binomial random variable *X* and then use it to verify that $\mu = np$ and $\sigma^2 = npq$.
  - Proof

$$M_X(t) = \sum_{x=0}^{n} e^{tx} f(x) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x} = (pe^t + q)^n$$

$$\frac{dM_X(t)}{dt} = n(pe^t + q)^{n-1} pe^t$$

$$\frac{d^2 M_X(t)}{dt^2} = np[e^t (n-1)(pe^t + q)^{n-2} pe^t + (pe^t + q)^{n-1} e^t].$$

Setting $t = 0$, we get $\mu_1' = np$ and $\mu_2' = np[(n-1)p + 1]$

$\therefore \mu = \mu_1' = np$ and $\sigma^2 = \mu_2' - \mu^2 = np(1-p) = npq$

# Moments and Moment-Generating Functions

- Example 7.7: Show that the moment-generating function of the random variable $X$ having a normal probability distribution with mean $\mu$ and variance $\sigma^2$ is given by $M_X(t) = \exp(\mu t + \frac{1}{2}\sigma^2 t^2).$
  - Proof

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[ -\frac{1}{2}\left( \frac{x-\mu}{\sigma} \right)^2 \right] dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ \frac{-[x^2 - 2(\mu + t\sigma^2)x + \mu^2]}{2\sigma^2} \right\} dx$$

$$x^2 - 2(\mu + t\sigma^2)x + \mu^2 = [x - (\mu + t\sigma^2)]^2 - 2\mu t\sigma^2 - t^2\sigma^4$$

$$M_X(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ \frac{-\{[x-(\mu + t\sigma^2)]^2 - 2\mu t\sigma^2 - t^2\sigma^4\}}{2\sigma^2} \right\} dx$$

$$= \exp\left( \frac{2\mu t + t^2\sigma^2}{2} \right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ \frac{-[x - (\mu + t\sigma^2)]^2}{2\sigma^2} \right\} dx$$

$$\text{Let } w = \frac{[x - (\mu + t\sigma^2)]}{\sigma}, \text{ then } dx = \sigma \cdot dw$$

$$M_X(t) = \exp\left( \mu t + \tfrac{1}{2}\sigma^2 t^2 \right) \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw}_{1} = \exp\left( \mu t + \tfrac{1}{2}\sigma^2 t^2 \right)$$

# Moments and Moment-Generating Functions

(skipped in 9th ed.)

- Example (7.8): Show that the moment-generating function of the random variable *X* having a chi-squared distribution with *v* degrees of freedom is $M_X(t) = (1-2t)^{-v/2}.$

Chi-squared distribution:

$$f(x) = \begin{cases} \dfrac{1}{2^{v/2}\Gamma(v/2)} x^{v/2-1} e^{-x/2}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

where $v$ is a positive integer.

  - Proof

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{2^{v/2}\Gamma(v/2)} x^{v/2-1} e^{-x/2} dx$$

$$= \frac{1}{2^{v/2}\Gamma(v/2)} \int_{-\infty}^{\infty} x^{v/2-1} e^{-x(1-2t)/2} dx$$

Setting $y = x(1-2t)/2$ and $dx = [2/(1-2t)]dy$, for $t < \frac{1}{2}$

$$M_X(t) = \frac{1}{2^{v/2}\Gamma(v/2)} \int_{-\infty}^{\infty} \left(\frac{2y}{1-2t}\right)^{v/2-1} e^{-y} \frac{2}{1-2t} dy$$

$$= \frac{1}{(1-2t)^{v/2}\Gamma(v/2)} \underline{\int_{-\infty}^{\infty} y^{v/2-1} e^{-y} dy}$$

$$\because \Gamma(v/2) = \int_{-\infty}^{\infty} y^{v/2-1} e^{-y} dy$$

$$\therefore M_X(t) = (1-2t)^{-v/2}$$

$$\boxed{\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx, \ \text{for } \alpha > 0}$$

# Moments and Moment-Generating Functions

- Theorem 7.7: (Uniqueness Theorem) Let *X* and *Y* be two random variables with moment-generating functions $M_x(t)$ and $M_Y(t)$, respectively. If $M_x(t) = M_Y(t)$ for all values of *t*, then *X* and *Y* have the same probability distribution.
- Theorem 7.8: $M_{X+a}(t) = e^{at} M_X(t).$
  - Proof $$M_{X+a}(t) = E[e^{t(X+a)}] = e^{at} E(e^{tX}) = e^{at} M_X(t).$$
- Theorem 7.9: $M_{aX}(t) = M_X(at).$
  - Proof $$M_{aX}(t) = E[e^{t(aX)}] = E[e^{(at)X}] = M_X(at).$$

# Moments and Moment-Generating Functions

- Theorem 7.10: If $X_1$, $X_2$ ,…, $X_n$ are independent random variables with moment-generating functions $M_{x_1}(t)$, $M_{x_2}(t)$,…, $M_{x_n}(t)$, respectively, and $Y = X_1 + X_2 + \ldots + X_n$, then

$$M_Y(t) = M_{X_1}(t) M_{X2}(t) \ldots M_{X_n}(t).$$

  – Proof

$$M_Y(t) = E[e^{tY}] = E[e^{t(X_1+X_2+\ldots+X_n)}]$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{t(X_1+X_2+\ldots+X_n)} f(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n$$

$$f(x_1, x_2, \cdots, x_n) = f_1(x_1) f_2(x_2) \cdots f_n(x_n) \quad (\because \text{independent})$$

$$M_Y(t) = \int_{-\infty}^{\infty} e^{tX_1} f_1(x_1) dx_1 \int_{-\infty}^{\infty} e^{tX_2} f_2(x_2) dx_2 \cdots \int_{-\infty}^{\infty} e^{tX_n} f_n(x_n) dx_n$$

$$= M_{X_1}(t) M_{X_2}(t) \cdots M_{X_n}(t).$$

# Moments and Moment-Generating Functions

- Example: The <u>sum</u> of two independent random variables having Poisson distributions with parameters $\mu_1$ and $\mu_2$, has a Poisson distribution with parameter $\mu_1 + \mu_2$.

  – **Solution**

    Two independent Poisson random variables with moment-generating functions given by (Exercise 19)

$$M_{X_1}(t) = e^{\mu_1(e^t - 1)} \text{ and } M_{X_2}(t) = e^{\mu_2(e^t - 1)},$$

    respectively. According to Theorem 7.10, $Y_1 = X_1 + X_2$ is

$$M_{Y_1}(t) = M_{X_1}(t)M_{X_2}(t) = e^{\mu_1(e^t - 1)}e^{\mu_2(e^t - 1)} = e^{(\mu_1 + \mu_2)(e^t - 1)}$$

So, $Y_1$ have a Poisson distribution with parameter $\mu_1 + \mu_2$.

# Moments and Moment-Generating Functions

- Theorem 7.11: If $X_1, X_2, \ldots, X_n$ are independent random variables having normal distributions with means $\mu_1, \mu_2, \ldots, \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, respectively, then the random variable
$$Y = a_1X_1 + a_2X_2 + \ldots + a_nX_n$$
has a normal distribution with mean
$$\mu_Y = a_1\mu_1 + a_2\mu_2 + \ldots + a_n\mu_n$$
and variance
$$\sigma_Y^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \ldots + a_n^2\sigma_n^2.$$

$$Y = a_1X_1 + a_2X_2$$
$$M_Y(t) = M_{a_1X_1}(t)M_{a_2X_2}(t) \text{ [Theorem 7.10]}$$
$$= M_{X_1}(a_1t)M_{X_2}(a_2t) \text{ [Theorem 7.9]}$$
$$M_Y(t) = e^{(a_1\mu_1t + a_1^2\sigma_1^2t^2/2 + a_2\mu_2t + a_2^2\sigma_2^2t^2/2)}$$
$$= e^{[\underbrace{(a_1\mu_1 + a_2\mu_2)}_{\mu_Y}t + \underbrace{(a_1^2\sigma_1^2 + a_2^2\sigma_2^2)}_{\sigma_Y 2}t^2/2]} \text{ [Ex. 7.7]}$$

$$M_X(t) = \exp(\mu t + \tfrac{1}{2}\sigma^2t^2).$$

# Moments and Moment-Generating Functions

- Theorem 7.12: If $X_1, X_2, \ldots, X_n$ are mutually independent random variables that have, respectively, chi-squared distributions with $v_1, v_2, \ldots, v_n$ degrees of freedom, then the random variable

$$Y = X_1 + X_2 + \ldots + X_n$$

  has a chi-squared distribution with $v = v_1 + v_2 + \ldots + v_n$ degrees of freedom.

  – **Proof**

$$\text{By Theorem 7.10, } M_Y(t) = M_{X_1}(t) M_{X_2}(t) \cdots M_{Xn}(t)$$

$$\text{From Example 7.8, } M_{X_i}(t) = (1-2t)^{-v_i/2}, \quad i = 1,2,\cdots,n.$$

$$\text{Therefore, } M_Y(t) = (1-2t)^{-v_1/2}(1-2t)^{-v_2/2}\cdots(1-2t)^{-v_n/2}$$

$$= (1-2t)^{\underline{-(v_1+v_2+\ldots+v_n)/2}},$$

$$\therefore Y \text{ has } v = v_1 + v_2 + \ldots + v_n \text{ degrees of freedom.}$$

# Moments and Moment-Generating Functions

- Corollary: If $X_1, X_2, \ldots, X_n$ are independent random variables having identical normal distributions with mean $\mu$ and variances $\sigma^2$

$$Y = \sum_{i=1}^{n} \left( \frac{X_i - \mu}{\sigma} \right)^2$$

has a chi-squared distribution with $v = n$ degrees of freedom.

  - Example 7.5 states that each of the $n$ independent random variables $Y = [(X_i - \mu)/\sigma]^2$ has a chi-squared distribution with 1 degree of freedom.
  - It establishes a relationship between chi-squared distribution and the normal distribution.
  - If $Z_1, Z_2, \ldots, Z_n$ are independent standard normal random variables, then $\sum_{i=1}^{n} Z_i^2$ has a chi-square distribution and single parameter, $v$, the degrees of freedom, is $n$ , the number of standard normal variates.

# Exercise

- 7.2, 7.17, 7.19