Chapter 4.
Laplace Transform

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Laplace Transform

例: $3e^{2t} \xrightarrow{\mathscr{L}} \dfrac{3}{s-2}$

$$2\cos 3t \xrightarrow{\mathscr{L}} \frac{2s}{s^2+9}$$

$$3\sin 2t \xrightarrow{\mathscr{L}} \frac{6}{s^2+4}$$

$$t \xrightarrow{\mathscr{L}} \frac{1}{s^2}$$

$$H(t) \xrightarrow{\mathscr{L}} \frac{1}{s}$$

$$\delta(t) \xrightarrow{\mathscr{L}} 1$$

Laplace Transform

- Properties：

1. Linear
2. 第一移位

$$f(t) \to F(s)$$

$$e^{at} f(t) \to F(s-a)$$

$$EX:$$

$$e^{2t}\cos 3t \xrightarrow{\mathscr{L}} \left.\frac{s}{s^2+9}\right|_{s=s-2} = \frac{s-2}{(s-2)^2+9}$$

Laplace Transform

3. 第二移位

$$f(t) \to F(s)$$

$$H(t-a)f(t-a) \to e^{-as}F(s)$$

$$EX:$$

$$F(s) = \frac{2}{s^2+4}e^{-2s}, f(t) = H(t-2)\sin 2(t-2)$$

Laplace Transform

4. $f(t) \rightarrow F(s)$

$$tf(t) \rightarrow \frac{-dF(s)}{ds}$$

$$t^2 f(t) \rightarrow \frac{-d}{ds}\left(\frac{-dF(s)}{ds}\right)$$

5. $f(t) \rightarrow F(s)$

$$\frac{1}{t} f(t) \rightarrow \int_s^\infty F(s)ds$$

$$\frac{1}{t^2} f(t) \rightarrow \int_s^\infty \int_s^\infty F(s)dsds$$

Laplace Transform

EX: $\frac{1}{t}\sin t \xrightarrow{\mathscr{L}} ?$

$$\sin t \xrightarrow{\mathscr{L}} \frac{1}{s^2+1}$$

$$\frac{1}{t}\sin t \xrightarrow{\mathscr{L}} \int_s^\infty \frac{1}{s^2+1} ds = \tan^{-1} s \Big|_s^\infty = \frac{\pi}{2} - \tan^{-1} s$$

微積分內瑕積分

Laplace Transform

$$\int_0^\infty \frac{\sin t}{t} dt$$

$$= \lim_{t\to\infty} \int_0^t \frac{\sin t}{t} dt$$

$$\lim_{t\to 0} \int_0^\infty \frac{\sin t}{t} e^{-st} dt$$

$$= \lim_{s\to 0} \mathscr{L}\{\frac{\sin t}{t}\} = \lim_{s\to 0}\left(\frac{\pi}{2} - \tan^{-1} s\right) = \frac{\pi}{2}$$

$$\int_0^\infty \lim_{s\to 0} \frac{\sin t}{t} e^{-st} dt$$

$$= \int_0^\infty \frac{\sin t}{t} dt = \text{原式}$$

Laplace Transform

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結合性質4.和性質5.

$$\begin{tikzcd}
\mathrm{G(s)} \arrow[d, "\mathscr{L}^{-1}"'] & F(s) \arrow[l, "\int_s^\infty F(s)ds"'] \arrow[r, "\frac{-dF(s)}{ds}"] \arrow[d, "\mathscr{L}^{-1}"] & G(s) \arrow[d, "\mathscr{L}^{-1}"] \\
g(t) \arrow[r, "\times t"] & f(t) & g(t) \arrow[l, "\div t"']
\end{tikzcd}$$

$$\mathscr{L}^{-1}\left(\int_s^\infty \mathscr{L}\left\{g(t)\cdot t\right\} ds\right) = g(t)$$

Laplace Transform

EX: $F(s) = \ln\frac{s+1}{s+2}, f(t) = ?$

$$\frac{dF(s)}{ds} = \frac{s+2}{s+1}\left(\frac{d}{ds}\cdot\frac{s+1}{s+2}\right)$$

$$= \frac{s+2}{s+1}\cdot\frac{(s+2)-(s+1)}{(s+2)^2} = \frac{1}{(s+2)(s+1)} = \frac{a}{s+1} + \frac{b}{s+2}$$

$$= \frac{1}{s+1} + \frac{-1}{s+2}$$

$$= e^{-t} - e^{-2t}$$

$$\Rightarrow \frac{-dF(s)}{ds} = e^{-2t} - e^{-t}$$

$$f(t) = \frac{1}{t}\left(e^{-2t} - e^{-t}\right)$$

Laplace Transform

$$\int_0^\infty \frac{1}{t}\left(e^{-2t} - e^{-t}\right)dt$$

$$= \int_0^\infty \lim_{s\to 0}\frac{1}{t}\left(e^{-2t} - e^{-t}\right)e^{-st}dt = \lim_{s\to 0}\int_0^\infty \frac{1}{t}\left(e^{-2t} - e^{-t}\right)e^{-st}dt$$

$$= \lim_{s\to 0}\int_0^\infty \frac{1}{t}\left(e^{-2t} - e^{-t}\right)e^{-st}dt$$

$$= \lim_{s\to 0}\mathscr{L}\{\frac{1}{t}\left(e^{-2t} - e^{-t}\right)\} = \lim_{s\to 0}\ln\frac{s+1}{s+2} = -\ln 2$$

Laplace Transform

EX: $F(s)=\dfrac{2s}{(s^2+4)^2}, f(t)=?$

越微分越複雜

$$\int_s^\infty F(s)ds=\int_s^\infty \frac{2s}{(s^2+4)^2}ds$$

令$t=s^2+4, dt=2sds, ds=\dfrac{1}{2s}dt$

$$=\int_{s^2+4}^\infty \frac{2s}{t^2}\frac{1}{2s}dt$$

$$=\int_{s^2+4}^\infty \frac{2s}{t^2}\frac{1}{2s}dt$$

$$=\int_{s^2+4}^\infty \frac{1}{t^2}dt$$

$$=-\frac{1}{t}\Big|_{s^2+4}^\infty$$

Laplace Transform

$$=-\frac{1}{t}\Big|_{s^2+4}^\infty$$

$$=0-\left(-\frac{1}{s^2+4}\right)$$

$$=\frac{1}{s^2+4}$$

$$\Rightarrow g(t)=\frac{1}{2}\sin 2t$$

$$\Rightarrow f(t)=\frac{1}{2}t\sin 2t$$

$$\int_0^\infty \frac{1}{2}t\sin 2tdt=0$$

Laplace Transform

6.

$$f(t) \xrightarrow{\mathscr{L}} F(s)$$

$$f'(t) \xrightarrow{\mathscr{L}} sF(s) - f(0)$$

Laplace Transform

EX:

$$y' + 2y = e^t, y(0) = 0$$

$$\mathscr{L}\{y' + 2y\} = \mathscr{L}\{e^t\}$$

$$\mathscr{L}\{y'\} + \mathscr{L}\{2y\} = \mathscr{L}\{e^t\}$$

$$SY(s) - y(0) + 2Y(s) = \frac{1}{s-1}$$

$$(s+2)Y(s) = \frac{1}{s-1}$$

$$Y(s) = \frac{1}{(s-1)(s+2)} = \frac{\frac{1}{3}}{s-1} + \frac{-\frac{1}{3}}{s+2}$$

$$y(t) = \frac{-1}{3}e^{-2t} + \frac{1}{3}e^t$$

Laplace Transform

Pf:

$$\mathscr{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}dt$$

$$= \int_0^\infty e^{-st}df(t)$$

$$= e^{-st}f(t)\Big|_0^\infty - \int_0^\infty f(t)(-se^{-st})dt$$

$$= (0 - f(0)) + s\int_0^\infty f(t)e^{-st}dt$$

$$= sF(s) - f(0)$$

Laplace Transform

性質6.1

$$\mathscr{L}\left\{f''(t)\right\} = L\left\{(f'(t))'\right\} \text{ 令 } g(t) = f'(t)$$

$$= \mathscr{L}\left\{g'(t)\right\}$$

$$= s\mathscr{L}\left\{g(t)\right\} - g(0)$$

$$= s(sF(s) - f(0)) - f'(0)$$

$$= s^2F(s) - sf(0) - f'(0)$$

Laplace Transform

性質6.2

$\mathscr{L}\left\{f^{(n)}(t)\right\}$

$= s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$

$\mathscr{L}\{y'''\}$

*用數學歸納法可證

$y'' + 3y' + 2y = e^t, y(0) = 0, y'(0) = 0$

$\mathscr{L}\left\{y''\right\} = s^2Y(s) - sy(0) - y'(0) \qquad \mathscr{L}\left\{y'\right\} = sY(s) - y(0)$

$s^2Y(s) + 3sY(s) + 2Y(s) = \dfrac{1}{s-1}$

$Y(s) = \dfrac{1}{(s-1)(s+1)(s+2)} = \dfrac{\frac{1}{6}}{s-1} + \dfrac{\frac{-1}{2}}{s+1} + \dfrac{\frac{1}{3}}{s+2}$

$y(t) = \frac{1}{6}e^t - \frac{1}{2}e^{-t} + \frac{1}{3}e^{-2t}$

Laplace Transform

性質7

$f(t) \xrightarrow{\mathscr{L}} F(s)$

$\int_0^t f(t)dt \xrightarrow{\mathscr{L}} \dfrac{F(s)}{s}$

$1 \xrightarrow{\mathscr{L}} \dfrac{1}{s}$

xt

$t \xrightarrow{\mathscr{L}} \dfrac{1}{s^2}$

$\int_0^t 1dt \xrightarrow{\mathscr{L}} \dfrac{1}{s^2}$

$\mathscr{L}\left\{\int_0^t \cdots \int_0^t f(t)dt \cdots dt\right\} = \dfrac{1}{s^k}F(s)$

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Laplace Transform

pf：

$$\mathscr{L}\left\{\int_0^t f(t)\right\}$$
$$= \int_0^\infty \int_0^t f(t)dte^{-st}dt$$
$$= \int_0^\infty \int_0^t f(\lambda)d\lambda e^{-st}dt \qquad u = \int_0^t f(\lambda)d\lambda \;\; du = f(t)dt \;\; dv = e^{-st}dt \;\; v = \frac{-1}{s}e^{-st}$$
$$= uv\Big|_0^\infty - \int_0^\infty vdu$$
$$= \left(\int_0^t f(\lambda)d\lambda\right)\left(\frac{-1}{s}e^{-st}\right)\Big|_0^\infty - \int_0^\infty \frac{-1}{s}e^{-st}f(t)dt$$
$$= (0-0) + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt$$
$$= \frac{1}{s}F(s)$$

Laplace Transform

EX： $\mathscr{L}\left\{e^{2t}\int_0^t e^{3t}(t)(\sin t)dt\right\}$

$$\mathscr{L}\{\sin t\} = \frac{1}{s^2+1}$$
$$\mathscr{L}\{t\sin t\} = \frac{-d}{ds}\frac{1}{s^2+1} = \frac{2s}{(s^2+1)^2}$$
$$\mathscr{L}\{e^{3t}t\sin t\} = \frac{2s}{(s^2+1)^2}\Big|_{s=s-3} = \frac{2(s-3)}{((s-3)^2+1)^2} = \frac{2s-6}{(s^2-6s+10)^2}$$
$$\mathscr{L}\left\{\int_0^t e^{3t}\times(t)\times(\sin t)dt\right\} = \frac{1}{s}\frac{2s-6}{(s^2-6s+10)^2}$$
$$\mathscr{L}\left\{e^{2t}\int_0^t e^{3t}\times(t)\times(\sin t)dt\right\} = \frac{2s-6}{s(s^2-6s+10)^2}\Big|_{s=s-2}$$
$$= \frac{2(s-2)-6}{(s-2)((s-2)^2-6(s-2)+10)^2}$$
$$= \frac{2s-10}{(s-2)(s^2-10s+26)^2}$$

Laplace Transform

迴旋積，捲積

性質8 Convolution theorem $\mathscr{L}\{f \otimes g\} = F(s)G(s)$

$f(t)\otimes g(t) \triangleq \int_0^t f(\lambda)g(t-\lambda)d\lambda = \int_0^t g(\lambda)f(t-\lambda)d\lambda$

pf:)

$$f(t) * g(t) = \int_0^t f(\lambda)g(t-\lambda)d\lambda$$

令 $t-\lambda = x$，$d\lambda = -dx$

$$\begin{array}{cc} 0 & t \\ | & | \\ t & 0 \end{array}$$

$$= \int_0^t f(t-x)g(x)dx$$

$$= \int_t^0 f(t-\lambda)g(x)(-dx)$$ 令 $x = \lambda$，$dx = d\lambda$

$$= \int_0^t g(\lambda)f(t-\lambda)d\lambda$$

Laplace Transform

EX： $\int_0^t \cos\lambda e^{2(t-\lambda)}d\lambda$

$$= \cos t \otimes e^{2t}$$

$$f(t) \xrightarrow{\mathscr{L}} F(s)$$

$$g(t) \xrightarrow{\mathscr{L}} G(s)$$

$$f(t)\otimes g(t) \xrightarrow{\mathscr{L}} F(s)G(s)$$

Laplace Transform

EX： $\mathscr{L}\left\{\int_0^t e^{2\lambda}(t-\lambda)d\lambda\right\}$

$$= \frac{1}{s-2}\frac{1}{s^2}$$

EX： $f(t) = e^{2t}, g(t) = e^{3t}$

$$f(t) \otimes g(t) = \int_0^t e^{2\lambda} e^{3(t-\lambda)} d\lambda$$

$$= e^{3t}\int_0^t e^{-\lambda} d\lambda$$

$$= e^{3t}(-e^{-\lambda}\Big|_0^t)$$

$$= e^{3t}(-e^{-t} - (-1))$$

$$= -e^{2t} + e^{3t}$$

$$\mathscr{L}\left\{e^{3t} - e^{2t}\right\}$$

Laplace Transform

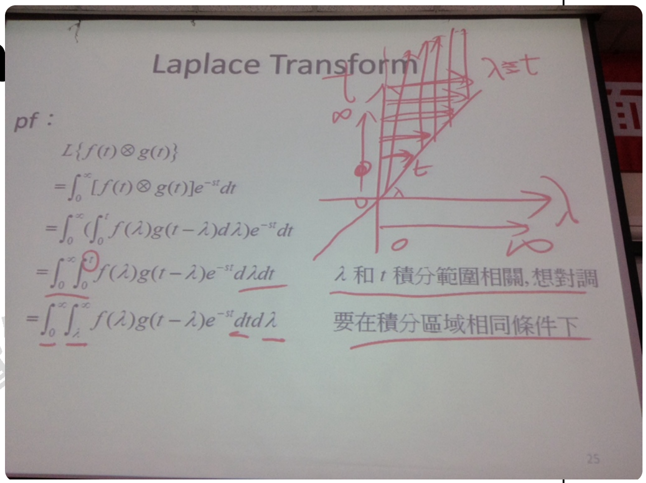
$$= \frac{1}{s-3} - \frac{1}{s-2}$$

$$= \frac{1}{(s-3)(s-2)}$$

$$= \frac{1}{s-3} \bullet \frac{1}{s-2}$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s-3} \bullet \frac{1}{s-2}\right\} = e^{3t} \otimes e^{2t}$$

Laplace Tran



pf：

$$L\{f(t)\otimes g(t)\}$$

$$=\int_0^\infty [f(t)\otimes g(t)]e^{-st}dt$$

$$=\int_0^\infty (\int_0^t f(\lambda)g(t-\lambda)d\lambda)e^{-st}dt$$

$$=\int_0^\infty \int_0^t f(\lambda)g(t-\lambda)e^{-st}d\lambda dt \qquad \lambda \text{ 和 } t \text{ 積分範圍相關,想對調}$$

$$=\int_0^\infty \int_\lambda^\infty f(\lambda)g(t-\lambda)e^{-st}dtd\lambda \qquad \text{要在積分區域相同條件下}$$

Laplace Transform

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$$=\int_0^\infty f(\lambda)\int_0^\infty g(t-\lambda)e^{-st}dtd\lambda \qquad \text{令} x=t-\lambda, dx=dt$$

$$=\int_0^\infty f(\lambda)\int_0^\infty g(x)e^{-s(\lambda+x)}dxd\lambda$$

$$=\int_0^\infty f(\lambda)e^{-s\lambda}\int_0^\infty g(x)e^{-st}dxd\lambda$$

$$=G(s)\int_0^\infty f(\lambda)e^{-s\lambda}d\lambda$$

$$=G(s)F(s)$$