# Chapter 4.
# Laplace Transform



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# Laplace Transform

EX： $F(s)=\dfrac{s}{(s+1)(s-2)^2}$

$$=\frac{k_1}{s+1}+\frac{k_2}{s-2}+\frac{k_3}{(s-2)^2}$$

$$f(t)=k_1e^{-t}+k_2e^{2t}+k_3te^{2t}$$

$$k_1=\frac{-1}{9},k_2=\frac{1}{9},k_3=\frac{2}{3}$$

# Laplace Transform

EX： $F(s)=\dfrac{3s+2}{(s+3)(s-1)^2}$

$$=\frac{k_1}{s+3}+\frac{k_2}{s-1}+\frac{k_3}{(s-1)^2}$$

$$k_1=\frac{-7}{16},k_2=\frac{7}{16},k_3=\frac{5}{4}$$

$$f(t)=\frac{-7}{16}e^{-3t}+\frac{7}{16}e^{t}+\frac{5}{4}te^{t}$$

# Laplace Transform

EX： $F(s) = \dfrac{s^2 + s + 1}{(s-1)(s-3)^2}$

$$= \frac{k_1}{s-1} + \frac{k_2}{s-3} + \frac{k_3}{(s-3)^2}$$

$$k_1 = \frac{3}{4}, k_2 = \frac{1}{4}, k_3 = \frac{13}{2}$$

$$f(t) = \frac{3}{4}e^t + \frac{1}{4}e^{3t} + \frac{13}{2}te^{3t}$$

# Laplace Transform

EX： $$F(s)=\frac{s}{(s-1)(s^2+4s+13)}$$

$$=\frac{k_1}{(s-1)}+\frac{k_2 s+k_3}{(s^2+4s+13)}$$

$$k_1=\frac{1}{18}$$

$$\frac{\frac{1}{18}(s^2+s+13)+(k_2 s+k_3)(s-1)}{(s-1)(s^2+4s+13)}$$

$$\frac{1}{18}+k_2=0,k_2=\frac{-1}{18}$$

$$\frac{13}{18}-k_3=0,k_3=\frac{13}{18}$$

# Laplace Transform

$$F(s)=\frac{\frac{1}{18}}{(s-1)}+\frac{\frac{-1}{18}s+\frac{13}{18}}{(s^2+4s+13)}$$

$$=\frac{\frac{1}{18}}{(s-1)}+\frac{\frac{-1}{18}(s+2)+\frac{15}{18}}{(s+2)^2+3^2}$$

$$f(t)=\frac{1}{18}e^{t}-\frac{1}{18}\cos(3t)e^{-2t}+\frac{5}{18}\sin(3t)e^{-2t}$$

# Laplace Transform

EX： $y'' - 5y' + 6y = e^{-t}, y(0) = 0, y'(0) = 2$

$$\mathscr{L}\left\{y'' - 5y' + 6y\right\} = \mathscr{L}\left\{e^{-t}\right\}$$

$$s^2Y(s) - sy(0) - y'(0) - 5(sY(s) - y(0)) + 6Y(s) = \frac{1}{s+1}$$

$$s^2Y(s) - 2 - 5sY(s) + 6Y(s) = \frac{1}{s+1}$$

$$Y(s)(s^2 - 5s + 6) = \frac{1}{s+1} + 2 = \frac{2s+3}{s+1}$$

$$Y(s) = \frac{2s+3}{(s+1)(s-2)(s-3)}$$

$$= \frac{k_1}{s-2} + \frac{k_2}{s-3} + \frac{k_3}{s+1}$$

$$k_1 = \frac{-7}{3}, k_2 = \frac{9}{4}, k_3 = \frac{1}{12}$$

$$y(t) = \frac{-7}{3}e^{2t} + \frac{9}{4}e^{3t} + \frac{1}{12}e^{-t}$$

# Laplace Transform

EX：$x'' + 4x' + 4x = 4, x(0) = 0, x'(0) = 0$

$$s^2 X(s) - sx(0) - x'(0) + 4(sX(s) - x(0)) + 4X(s) = \frac{4}{s}$$

$$(s^2 + 4s + 4)X(s) = \frac{4}{s}$$

$$X(s) = \frac{4}{s(s+2)^2} = \frac{k_1}{s} + \frac{k_2}{s+2} + \frac{k_3}{(s+2)^2}$$

$$k_1 = 1, k_2 = -1, k_3 = -2$$

$$x(t) = 1 - e^{-2t} - 2te^{-2t}$$

# Laplace Transform

EX： $y'' + 4y' + 5y = e^{-t}, y(0) = 0, y'(0) = 1$

$$s^2Y(s) - 1 + 4sY(s) + 5Y(s) = \frac{1}{s+1}$$

$$(s^2 + 4s + 5)Y(s) = \frac{1}{s+1}$$

$$Y(s) = \frac{s+2}{(s+1)(s^2+4s+5)}$$

$$= \frac{k_1}{s+1} + \frac{As+B}{s^2+4s+5}$$

$$k_1 = \frac{1}{2}, A = -\frac{1}{2}, B = -\frac{1}{2}$$

# Laplace Transform

$$Y(s) = \frac{\frac{1}{2}}{s+1} + \frac{-\frac{1}{2}s - \frac{1}{2}}{s^2 + 4s + 5}$$

$$= \frac{\frac{1}{2}}{s+1} + \frac{-\frac{1}{2}(s+2) + \frac{1}{2}}{(s+2)^2 + 1}$$

$$y(t) = \frac{1}{2}e^{-t} - \frac{1}{2}\cos(t)e^{-2t} + \frac{1}{2}\sin(t)e^{-2t}$$

# Laplace Transform

EX：Solve the following integral differential equation

$$y^{'} - 2y = \int_0^t e^{2(t-\tau)} \cos(3\tau) d\tau = \int_0^t g(\tau) f(t-\tau) d\tau$$

$$y^{'} - 2y = f(t) \otimes g(t) = e^{2t} \otimes \cos 3t$$

$$\mathscr{L}\left\{y^{'} - 2y\right\} = \mathscr{L}\left\{e^{2t} \otimes \cos 3t\right\}$$

$$sY(s) - y(0) - 2Y(s) = F(s)G(s) = \frac{1}{s-2}\frac{s}{s^2+9}$$

$$(s-2)Y(s) = \frac{s}{(s-2)(s^2+9)}$$

$$Y(s) = \frac{s}{(s-2)^2(s^2+9)} = \frac{k_1}{s-2} + \frac{k_2}{(s-2)^2} + \frac{As+B}{s^2+9}$$

$$k_2 = \frac{2}{13}$$

$$k_1 = \frac{d}{ds}\left(\frac{s}{s^2+9}\right)\Big|_{s=2} = \frac{s^2+9-s(2s)}{(s^2+9)^2} = \frac{5}{169}$$

# Laplace Transform

$$Y(s)=\frac{\frac{5}{169}(s^2+9)(s-2)+\frac{2}{13}(s^2+9)+(As+B)(s-2)^2}{(s-2)^2(s^2+9)}$$

$$\frac{5}{169}+A=0, A=-\frac{5}{169}$$

$$\frac{-90}{169}+\frac{18}{13}+4B=0, B=-\frac{36}{169}$$

$$y(t)=\frac{5}{169}e^{2t}+\frac{2}{13}te^{2t}-\frac{5}{169}\cos(3t)-\frac{12}{169}\sin(3t)$$

# Laplace Transform



*Vatera積分方程式

EX： $f(t) = -1 + \int_0^t f(t-\tau)e^{-3\tau}d\tau$

求$f(t) = ?$

$$F(s) = \frac{-1}{s} + F(s)\frac{1}{s+3}$$

$$F(s) = \frac{-(s+3)}{(s+2)s} = \frac{k_1}{s} + \frac{k_2}{s+2}$$

$$k_1 = \frac{-3}{2}, k_2 = \frac{1}{2}$$

$$f(t) = \frac{-3}{2} + \frac{1}{2}e^{-2t}$$

# Laplace Transform

EX： $f(t) = e^{-2t} - 3e^{-3t}\int_0^t f(\tau)e^{3\tau}d\tau$

$$f(t) = e^{-2t} - 3\int_0^t f(\tau)e^{-3(t-\tau)}d\tau$$

$$F(s) = \frac{1}{s+2} - 3F(s)\frac{1}{s+3}$$

$$F(s) = \frac{s+3}{(s+6)(s+2)} = \frac{k_1}{s+6} + \frac{k_2}{s+2}$$

$$k_1 = \frac{3}{4}, k_2 = \frac{1}{4}$$

$$f(s) = \frac{3}{4}e^{-6t} + \frac{1}{4}e^{-2t}$$

# Laplace Transform

- 聯立方程式

EX：$\begin{cases} \dfrac{dx_1(t)}{dt} = x_2(t) \\ \dfrac{dx_2(t)}{dt} = -2x_1(t) - 3x_2(t) \end{cases}$ $\quad x_1(0) = 1, x_2(0) = 1$

偶合 *Coupled*

$$\begin{cases} sX_1(s) - x_1(0) = X_2(s) \\ sX_2(s) - x_2(0) = -2X_1(s) - 3X_2(s) \end{cases}$$

$$\Rightarrow \begin{cases} sX_1(s) - X_2(s) = 1 \\ 2X_1(s) + (s+3)X_2(s) = 1 \end{cases}$$

$$X_1(s) = \frac{\begin{vmatrix} 1 & -1 \\ 1 & s+3 \end{vmatrix}}{\begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix}} = \frac{s+3+1}{s^2+3s+2} = \frac{3}{s+1} + \frac{-2}{s+2}$$

# Laplace Transform

$$X_2(s) = \frac{\begin{vmatrix} s & 1 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix}} = \frac{s-2}{s^2+3s+2} = \frac{-3}{s+1} + \frac{4}{s+2}$$

$$x_1(t) = 3e^{-t} - 2e^{-2t}$$

$$x_2(t) = -3e^{-t} + 4e^{-2t}$$

# Laplace Transform

*線性代數

$$\begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}\begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{s(s+3)+2}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2+3s+2}\begin{bmatrix} s+4 \\ s-2 \end{bmatrix} = \begin{bmatrix} \dfrac{s+4}{s^2+3s+2} \\ \dfrac{s-2}{s^2+3s+2} \end{bmatrix}$$

# Laplace Transform

EX： $y'' + 2ty' - 4y = 6, \quad y(0) = 0, y'(0) = 0$

$$\Rightarrow S^2Y(s) + 2\left(-\frac{d(SY(s))}{ds}\right) - 4Y(s) = \frac{6}{s}$$

$$S^2Y(s) - 2Y(s) - 2SY'(s) - 4Y(s) = \frac{6}{s}$$

$$-2SY'(s) + (s^2 - 6)Y(s) = \frac{6}{s}$$

$$y' + py = r$$

$$y = CI^{-1} + I^{-1}\int Irdx$$

$$I = e^{\int pdx}$$

$$Y'(s) + \frac{s^2 - 6}{-2s}Y(s) = \frac{6}{s(-2s)}$$

$$I = e^{\int \frac{s^2-6}{-2s}ds} = e^{\int\left(\frac{-s}{2}+\frac{3}{s}\right)ds} = e^{\frac{-1}{4}s^2 + 3\ln s} = e^{\frac{-1}{4}s^2} \cdot s^3$$

# Laplace Transform

$$Y(s) = Ce^{\frac{1}{4}s^2} \cdot s^{-3} + e^{\frac{1}{4}s^2} \cdot s^{-3} \int e^{\frac{-1}{4}s^2} \cdot s^3 \cdot \frac{6}{s(-2s)} ds$$

$$\left( 令\ u = \frac{-s^2}{4}, du = \frac{-1}{2} sds \right)$$

$$= Ce^{\frac{1}{4}s^2} \cdot s^{-3} + e^{\frac{1}{4}s^2} \cdot s^{-3} \int 6e^u du$$

$$= Ce^{\frac{1}{4}s^2} \cdot s^{-3} + 6s^{-3}$$

如何解C，利用初值定理

$$y(0) = \lim_{s \to \infty} SY(s) = \lim_{s \to \infty} \left( Ce^{\frac{s^2}{4}} s^{-2} + \frac{6}{s^2} \right) = 0$$

$$\therefore C = 0$$

$$\therefore Y(s) = 6s^{-3}$$

$$Y(t) = 3t^2$$

# Laplace Transform

EX： $y'' + 3y' + 2y = \delta(t-2), y(0) = 0, y'(0) = 1$

$$\left(s^2 Y_{(s)} - 0 - 1\right) + 3\left(SY_{(s)} - 0\right) + 2Y(s) = e^{-2s}$$

$$Y(s) = \frac{1+e^{-2s}}{s^2+3s+2} = \frac{1}{s+1} + \frac{-1}{s+2} + \left[\frac{1}{s+1} + \frac{-1}{s+2}\right]e^{-2s}$$

$$y(t) = e^{-t} - e^{-2t} + \left[e^{-(t-2)} - e^{-2(t-2)}\right]H(t-2)$$

# Laplace Transform

EX: Solve the following problems

(i) $y'' + 4y' + 3y = 3\delta(t-2) + H(t-1), y(0) = y'(0) = 0$

$$S^2Y(s) - SY(0) - Y'(0) + 4\left(SY(0) - Y(0)\right) + 3Y(s) = 3e^{-2s} + \frac{1}{s}e^{-s}$$

$$\left(S^2 + 4S + 3\right)Y(s) = 3e^{-2s} + \frac{1}{s}e^{-s}$$

$$Y(s) = \frac{3e^{-2s}}{\left(S^2 + 4S + 3\right)} + \frac{e^{-s}}{s\left(S^2 + 4S + 3\right)}$$

$$= \left(\frac{\frac{3}{2}}{s+1} + \frac{\frac{-3}{2}}{s+3}\right)e^{-2s} + \left(\frac{\frac{1}{3}}{s} + \frac{\frac{-1}{2}}{s+1} + \frac{\frac{1}{6}}{s+3}\right)e^{-s}$$

$$y(t) = \left[\frac{3}{2}e^{-(t-2)} - \frac{3}{2}e^{-3(t-2)}\right]H(t-2) + \left[\frac{1}{3} - \frac{1}{2}e^{-(t-1)} + \frac{1}{6}e^{-3(t-1)}\right]H(t-1)$$

# Laplace Transform

(ii) $$y'' + y = f(t),\ f(t) = \begin{cases} 0, 0 \le t < \pi \\ 1, \pi \le t < 2\pi \\ 0, t \ge 2\pi \end{cases},\ y(0) = 0,\ y'(0) = 1$$

$$\left(S^2 Y(s) - SY(0) - Y'(0)\right) + Y = \frac{1}{s}e^{-\pi s} - \frac{1}{s}e^{-2\pi s}$$

$$\left(S^2 + 1\right)Y(s) = 1 + \frac{1}{s}\left[e^{-\pi s} - e^{-2\pi s}\right]$$

$$Y(s) = \frac{1}{\left(S^2+1\right)} + \frac{1}{S\left(S^2+1\right)}\left(e^{-\pi s} - e^{-2\pi s}\right)$$

$$= \frac{1}{\left(S^2+1\right)} + \left(\frac{1}{s} + \frac{-s}{\left(S^2+1\right)}\right)\left(e^{-\pi s} - e^{-2\pi s}\right)$$

$$y(t) = \sin t + \left[1 - \cos(t-\pi)\right]H(t-\pi) - [1 - \cos(t-2\pi)]H(t-2\pi)$$

# Laplace Transform

(iii) $f(t) = \int_0^t f(t-\tau)e^{-\tau}d\tau + 3t^5$

$$F(s) = F(s)\frac{1}{s+1} + 3\frac{5!}{s^{5+1}}$$

$$\frac{s}{s+1}F(s) = 3\frac{5!}{s^{5+1}}$$

$$F(s) = \frac{3\cdot 5!\cdot\left(s+1\right)}{s\cdot s^6}$$

$$= \frac{3\cdot 5!}{s^6} + \frac{3\cdot 5!\cdot\frac{6!}{6!}}{s^7}$$

$$f(t) = 3t^5 + \frac{1}{2}t^6$$