NCKU CSIE Discrete Mathematics (2015 Spring) Midterm I *(total 110 pts)*

[ch1:10+4+15+10 ch2:15+4 ch3:6+4+4 ch4:10,15 ch5:9+4]

1. (*20 pts*) For each of the following statements, **determine** and **explain** whether it is correct or not.

   (1). (F) Suppose A={1, 2, 3, 4, 5}. Two of the following statements are false: $(a)\{\{3\}\} \subseteq P(A), (b)\emptyset \subseteq A, (c)\{\emptyset\} \subseteq P(A), (d)\emptyset \subseteq P(A), (e)\{2,4\} \in AXA$

   (2). (F) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$ is the power set of some set.

   (3). $2\binom{n}{0} + \binom{n}{1} + 2\binom{n}{2} + \binom{n}{3} + 2\binom{n}{4} + \binom{n}{5} + \cdots + 2\binom{n}{n-2} + \binom{n}{n-1} + 2\binom{n}{n} = 2^{n-1} + 2^n$

   (4). $\neg(p \leftrightarrow q) \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$.

   (5). $f: \boldsymbol{R} \to \boldsymbol{R^2}, f(x) = (2x+1, x^2)$ is an one-to-one function.

(1)False , (e):{(2,4)} ∈ AXA

(2)False , $\{\{\emptyset\}, \{a\}, \{\emptyset, a\}\}$ is the power set of some set.

(3)True

(4)False, $\neg(p \leftrightarrow q)$

$\Leftrightarrow \neg[(p \to q) \wedge (q \to p)]$

$\Leftrightarrow \neg[(\neg p \vee q) \wedge (\neg q \vee p)]$

$\Leftrightarrow (p \wedge \neg q) \vee (q \wedge \neg p)$

(5)True

2. [ch1] (*15:10,5 pts*) Solve the equation $x_1+x_2+x_3+x_4 < 9$. (a) Find the integer solutions where x_1, $x_2 > 0$, $x_3 > 2$, $x_4 > -2$. (b) in (a), if $x_1, x_2, x_3 \in N, x_4 \in Z$.

(a)

$x_1>0$, $x_2>0$,$x_3>2$, $x_4>-2$,

Let $y_1=x_1-1$, $y_2=x_2-1$,$y_3=x_3-3$,$y_4=x_4+1$

Hence, $y_1+y_2+y_3+y_4<5 \implies y_1+y_2+y_3+y_4+y_5=4$, $y_5 \geq 0$

$H^5_4 = C^8_4 = 70$

(b)

The answer is the same as (a) . $\mathbb{Z}$ means integer: $\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$,and the question(a) set a condition x4>-2 is in that range.

3. [3.1-12, 5] *(15 pts)* Let A = {1, 2, 3, 4, 5, 7, 8, 10, 11, 14, 17, 18}. (a) How many 6-element subsets of A contain four even integers and two odd integers? (b) How many 5-element subsets of A that has the smallest element less than 4? (c) How many binary relations on A? (d) How many functions *f*: A→A? (e) in (d), how many one-to-one functions?

(a) C(6,4) * C(6,2) = 225

(b) C(12,5) - C(9,5) = 666

(c) A*A = 12^2 = 144 elements of binary relation on A

$2^{|A*A|} = 2^{144}$ subsets of binary relation on A

(d) 12^{12}

(e) 12!

4. [4.4-10] *(10 pts)* If a, b are relatively prime and a > b, prove that gcd(a-b, a+b) = 1 or 2. [Hint: if w = gcd(x, y), w|px+qy for all p, q $\in Z$]

If $c = \gcd(a-b, a+b)$ then $c|[(a-b)x + (a+b)y]$ for all $x, y \in \mathbf{Z}$. In particular, for $x = y = 1, c|2a$, and for $x = -1, y = 1, c|2b$. From Exercise 4, $\gcd(2a, 2b) = 2\gcd(a, b) = 2$, so $c|2$ and $c = 1$ or 2.

5. [4Supp-16] (*15 pts*) Frances spends \$6.20 on candy for prizes in a contest. If a 10-ounce box of this candy costs \$.50 and a 3-ounce box costs \$.20, how many boxes of each size did she purchase?

$0.5x + 0.2y = 6.2 \Rightarrow 5x + 2y = 62$

$\gcd(5,2) = 1$

$1 = 5*1 + 2*(-2)$

$62 = 5*62 + 2*(-124) = 5*(62-2k) + 2*(-124+5k)$

$x = 62-2k \geq 0$

$y = -124+5k \geq 0$

$31 \geq k \geq 24.8$

Solutions: x = 12, y = 1, x = 10, y = 6, x = 8, y = 11,

x = 6, y = 16, x = 4, y = 21, x = 2, y = 26, x = 0, y = 31

6. [2.2-16] *(15 pts)* Define the connective "Nor" by $(p \downarrow q) \Leftrightarrow \neg(p \vee q)$, for any statements p, q. Represent the following using only this connective. (a) $\neg p$ (b) $p \wedge q$, (c) $p \rightarrow q$.

(a) $\neg p \Leftrightarrow p \downarrow p$

(b) $p \wedge q \Leftrightarrow \neg(\neg p \vee \neg q) \Leftrightarrow (p \downarrow p) \downarrow (q \downarrow q)$

(c) $p \rightarrow q \Leftrightarrow \neg p \vee q \Leftrightarrow (\neg p \downarrow q) \downarrow (\neg p \downarrow q) \Leftrightarrow ((p \downarrow p) \downarrow q) \downarrow ((p \downarrow p) \downarrow q)$

7. [ch1] (*10:2,2,2,4 pts*) For the complete expansion of $(2x - y + 3z^{-1} + 1)^5$, determine the following value (a) the coefficient of xyz^{-2} (b) the number of the distinct terms (c) the sum of all coefficients, and (d) if we change the constant term '1' to '1+x^{-1}',what's the coefficient of xyz^{-1}.

(a) $\frac{5!}{1!1!2!1!} * 2 * (-1) * 3^2 * 1 = -1080$

(b) $C_5^{5+4-1} = C_5^8 = 56$

(c) $let\ x = y = z = 1\ \ => \ (2 - 1 + 3 + 1)^5 = 5^5 = 3125$

(d) $(2x - y + 3z^{-1} + 1 + x^{-1})^5$

Case1 $xyz^{-1} => \frac{5!}{1!1!1!2!0!} * 2^1 * (-1)^1 * 3^1 * 1^2 * 1^0 = -360$

Case2 $x^2yz^{-1}x^{-1}\ => \frac{5!}{2!1!1!0!1!} * 2^2 * (-1)^1 * 3^1 * 1^0 * 1^1 = -720$

8. [ch1] *(10 pts)* Use a combinatorial argument to show that $\binom{3n}{3} = 6n\binom{n}{2} + 3\binom{n}{3} + n^3$

從 3n 裡取 3 個 $\boxed{n}\boxed{n}\boxed{n}$

Case1 從 3 個 n 裡各取 1 個 => $C_1^nC_1^nC_1^n = n^3$

Case2 從 1 個 n 裡取 3 個 => $C_1^3C_3^n = 3\binom{n}{3}$

Case3 從 3 個 n 裡取 2 個，從一個 n 中取 1 個，再從另一個 n 中取 2 個，其中這兩個 n 可以會相對調 => $C_2^3C_2^nC_1^n * 2 = 6n\binom{n}{2}$

所以 $\binom{3n}{3} = 6n\binom{n}{2} + 3\binom{n}{3} + n^3$