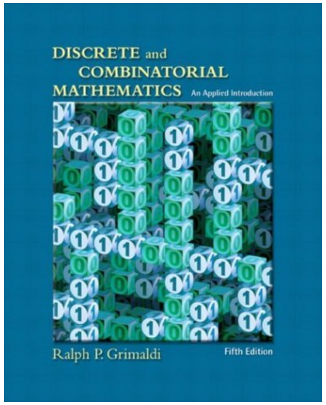
# Discrete Mathematics
## -- *Chapter 7: Relations: The Second Time Round*

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# Outline

- Relations Revisited: Properties of Relations
- Computer Recognition: Zero-One Matrices and Directed Graphs
- **Partial Orders**: Hasse Diagrams
- **Equivalence Relations** and Partitions
- Finite State Machine: The Minimization Process
  - Application of equivalence relation
  - Minimization process: find a machine with the same function but fewer internal states

# 7.1 Relations Revisited: Properties of Relations

- Definition 7.1: For sets A, B, any subset of A × B is called a (binary) relation from A to B. Any subset of A × A is called a (binary) relation on A .

- **Ex 7.1**
  - Define the relation $\Re$ on the set Z by $a\Re b$, if $a \le b$.
  - For x, $y \in Z$ and $n \in Z^+$, the modulo $n$ relation $\Re$ is defined by $x\Re y$ if x - y is a multiple of *n,* e.g., with n=7, $9\Re 2$, $-3\Re 11$, but $3 \not\Re 7$

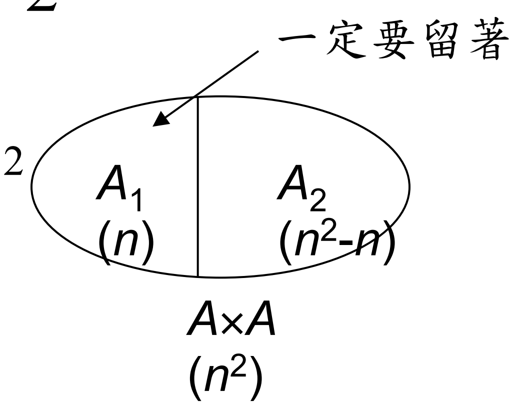
- **Ex 7.2** : Language $A \subseteq \Sigma^*$. For x, $y \in A$, define $x\Re y$ if x is a prefix of y.

# Relations Revisited: Properties of Relations

- Finite state machine M = ($S$, $I$, $O$, $v$, $w$)
  - Reachability
    - $s_1 \Re s_2$ if $v(s_1, x) = s_2$, $x \in I$. $\Re$ denotes the first level of reachability.
    - $s_1 \Re s_2$ if $v(s_1, x_1 x_2) = s_2$, $x_1 x_2 \in I^2$. $\Re$ denotes the second level of reachability.
  - Equivalence
    - 1-equivalence relation: $s_1 E_1 s_2$ if $w(s_1, x) = w(s_2, x)$ for $x \in I$.
    - $k$-equivalence relation: $s_1 E_k s_2$ if $w(s_1, y) = w(s_2, y)$ for $y \in I^k$.
    - If two states are $k$-equivalent for **all $k \in \mathbf{Z}^+$**, they are called equivalent.

# Reflexive

- Definition 7.2: A relation $\Re$ on a set $A$ is called **reflexive** if $(x, x)\in\Re$, for **all** $x\in A$.
- **Ex 7.4** : For $A = \{1, 2, 3, 4\}$, a relation $\Re \subseteq A\times A$ will be reflexive if and only if $\Re \subseteq \{(1, 1), (2, 2), (3, 3), (4, 4)\}$. But $\Re_1 = \{(1, 1), (2, 2), (3, 3)\}$ is not reflexive, $\Re_2 = \{(x, y)| x \le y, x, y\in A\}$ is reflexive.
- **Ex 7.5** : Given a finite set $A$ with $|A| = n$, we have $|A\times A| = n^2$, so there are $2^{n^2}$ relations on $A$. Among them $2^{(n^2-n)}$ are reflexive.
  - $A = \{a_1, a_2,\ldots, a_n\}$
  - $A\times A = \{(a_i, a_j)|1 \le i, j\le \text{n}\} = A_1\cup A_2$
  - $A_1 = \{(a_i, a_i)|1 \le i \le \text{n}\}$
  - $A_2 = \{(a_i, a_j)|i \ne j, 1 \le i, j\le \text{n}\}$



# Symmetric

- Definition 7.3: A relation $\Re$ on a set $A$ is called **symmetric** if for **all x, $y \in A$**, **if** $(x, y) \in \Re \Rightarrow (y, x) \in \Re$ .

- **Ex 7.6** : $A = \{1, 2, 3\}$
  - $\Re_1 = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$, symmetric, but not reflexive.
  - $\Re_2 = \{(1, 1), (2, 2), (3, 3), (2, 3)\}$, reflexive, but not symmetric.
  - $\Re_3 = \{(1, 1), (2, 2), (3, 3)\}$ and $\Re_4 = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$, both reflexive and symmetric.
  - $\Re_5 = \{(1, 1), (2, 3), (3, 3)\}$, neither reflexive nor symmetric.

# Symmetric

- To count the symmetric relations on $A = \{a_1, a_2, \ldots, a_n\}$.
  - $A \times A = A_1 \cup A_2$, $A_1 = \{(a_i, a_i) | 1 \le i \le \mathrm{n}\}$, $A_2 = \{(a_i, a_j) | i \ne j,\ 1 \le i, j \le n\}$
  - $A_1$ contains $n$ pairs, and $A_2$ contains $n^2$-$n$ pairs.
  - $A_2$ contains $(n^2$-$n)/2$ subsets $S_{i,j}$ of the form $\{(a_i, a_j), (a_j, a_i) \mid i < j\}$.
  - So, we have totally $2^n \times 2^{(1/2)(n^2-n)}$ symmetric relations on A.

- If the relations are both reflexive and symmetric, we have $2^{(1/2)(n^2-n)}$ choices.

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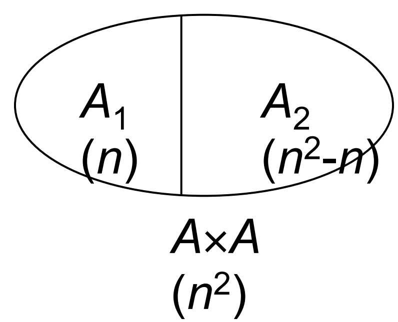


# Transitive

- Definition 7.4: A relation $\Re$ on a set $A$ is called **transitive if** $(x, y), (y, z) \in \Re \Rightarrow (x, z) \in \Re$ for all $x, y, z \in A$.

- **Ex 7.8** : Define the relation $\Re$ on the set $Z^+$ by $a \Re b$ if $a$ divides $b$. This is a transitive and reflexive relation but not symmetric.
- **Ex 7.9** : Define the relation $\Re$ on the set Z by $a \Re b$ if $a \times b \geq 0$. What properties do they have?
  - Reflexive, symmetric
  - Not transitive, e.g., $(3,0),(0,-7) \in \Re$, but $(3,-7)$ not

# Antisymmetric

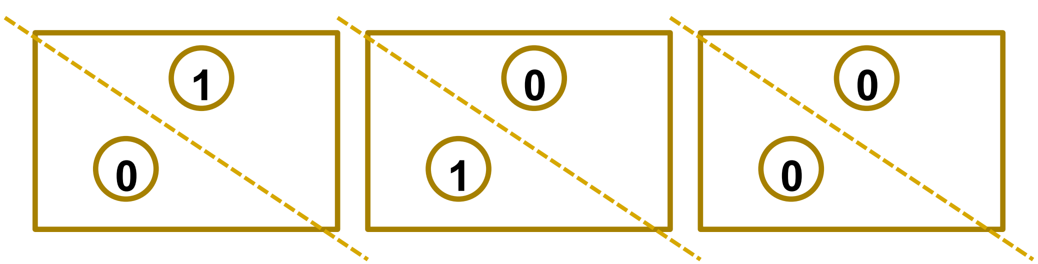
- Definition 7.5: A relation $\Re$ on a set $A$ is called **antisymmetric** if $(x, y)\in\Re$ and $(y, x)\in\Re \Rightarrow x = y$ for all x, $y\in A$.
  - Both **a** related to **b** and **b** related to **a,** if **a** and **b** are one and the same element from A

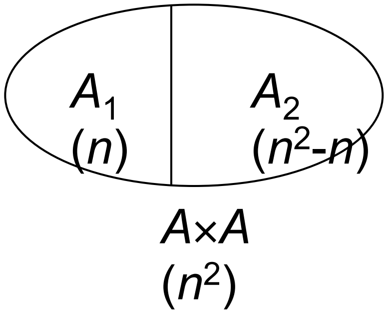
- **Ex 7.11** : Define the relation $(A, B)\in\Re$ if $A\subseteq B$. Then it is an anti-symmetric relation.

- Note that "*not symmetric*" is different from anti-symmetric.
- **Ex 7.12** : A = {1, 2, 3}, what properties do the following relations on $A$ have?
  - $\Re=\{(1, 2), (2, 1), (2, 3)\}$ (not symmetric, not antisymmetric)
  - $\Re=\{(1, 1), (2, 2)\}$ (symmetric and antisymmetric)

# Antisymmetric

- To count the antisymmetric relations on $A = \{a_1, a_2, \ldots, a_n\}$.
  - $A \times A = A_1 \cup A_2$, $A_1 = \{(a_i, a_i) | 1 \le i \le n\}$, $A_2 = \{(a_i, a_j) | i \ne j, 1 \le i, j \le n\}$
  - $A_1$ contains $n$ pairs, and $A_2$ contains $n^2$-$n$ pairs.
  - $A_2$ contains $(n^2$-$n)/2$ subsets $S_{i,j}$ of the form $\{(a_i, a_j), (a_j, a_i) \mid i < j\}$.
  - Each element in $A_1$ can be selected or not.
  - Each element in $S_{i,j}$ can be selected **in three alternatives**: ***either $(a_i, a_j)$, or $(a_j, a_i)$, or none*.**
  - So, we have totally $2^n \times 3^{(1/2)(n^2-n)}$ anti-symmetric relations on A.



# Antisymmetric

- **Ex 7.13** : Define the relation $\Re$ on the functions by $f \Re g$ if $f$ is dominated by $g$ (or $f \in O(g)$). What are their properties?
  - Reflexive
  - Transitive
  - not symmetric (e.g., $g$=n, $f$ =n$^{2,}$ g=O(f), but f $\neq$O(g))
  - not antisymmetric (e.g., $g$(n)= n, $f(n)$ = n+5, $f\Re g$ and $g\Re f$, *but* $f \neq g$)

# Example

- A={a1,a2,…an}, students in class discrete math.
- a relation R on a set A, if (ai, aj)$\in\Re$, the midterm score (ai) $\geqq$ midterm score (aj)
  - No equal score
- R is reflexive
- R is transitive
- R is not symmetric
- R is antisymmetric

# Partial Order

- Definition 7.6: A relation $\mathfrak{R}$ is called a partial order (partial ordering relation), if $\mathfrak{R}$ is *reflexive, anti-symmetric and transitive*.

- (A,R) is a **p**artially **o**rdered **set** / poset if R is a partial ordering on A. Typical notation: $(A,\leq)$;
  - "**no loops**"

- If $a\leq b$ or $b\leq a$, the elements a and b are comparable.
- If all pairs are comparable, $\leq$ is a total ordering or chain.

# Partial Order

- **Ex 7.15** : Let $A$ be the set of positive integers divisors of $n$, the relation $\Re$ on $A$ by $a\Re b$ if $a$ divides $b$, it defines a *partial order*. How many ordered pairs does it occur in $\Re$.
  - E.g. n=12, $A = \{1, 2, 3, 4, 6, 12\}$, $\Re = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (1, 12), (2, 2), (2, 4), (2, 6), (2, 12), (3, 3), (3, 6), (3, 12), (4, 4), (4, 12), (6, 6), (6, 12), (12, 12)\}$
  - If $(a, b) \in \Re$, then $a = 2^m \cdot 3^n$ and $b = 2^p \cdot 3^q$ with $\mathbf{0 \le m \le p \le 2, 0 \le n \le q \le 1}$.
  - Selection of size 2 from a set of size 3, with **repetition**.

$$\binom{3+2-1}{2} = \binom{4}{2} = 6 \text{ for } m, p; \quad \binom{2+2-1}{2} = \binom{3}{2} = 3 \text{ for } n, q$$

$$\therefore \text{total} = 6 \cdot 3 = 18 \text{ ordered pairs}$$

  - For $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} \Rightarrow$ the number of ordered pairs $= \prod_{i=1}^{k} \binom{(e_i+1)+2-1}{2} = \prod_{i=1}^{k} \binom{e_i+2}{2}$

*Maximal element*

# Equivalence relation

- Definition 7.7. A relation $\Re$ is called an **equivalence relation**, if $\Re$ is *reflexive, symmetric and transitive*.

- Given an equivalence relation R on A, for each a∈A the equivalence class [a] is defined by $\{x \mid (x,a) \in R\}$.
  - E.g., Modulo 3 equivalences on Z, such that $[0] = \{\ldots,-6,-3,0,3,6,\ldots\}$ and $[1] = \{\ldots,-5,-2,1,4,7,\ldots\}$

- **Ex 7.16** (b): If $A = \{1, 2, 3\}$, the following are all equivalence relations
  - $\Re_1 = \{(1, 1), (2, 2), (3, 3)\}$
  - $\Re_2 = \{(1, 1), (2, 2), (3, 3), (2, 3), (3,2)\}$
  - $\Re_3 = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$
  - $\Re_4 = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

# Examples

- Ex 7.16 (c): For a finite set $A$, $A \times A$ is the largest equivalence relation on $A$. If $A = \{a_1, a_2, \ldots, a_n\}$, then the equality relation $\Re = \{(a_i, a_i) | 1 \leq i \leq n\}$ is the smallest equivalence relation on $A$.

- Ex 7.16 (d): Let $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{x, y, z\}$, and $f : A \rightarrow B$ be the onto function. $f = \{(1, x), (2, z), (3, x), (4, y), (5, z), (6, y), (7, x)\}$.
Define the relation $\Re$ on $A$ by $a \Re b$ if $f(a) = f(b)$.
$\Re$ is reflexive, symmetric, and transitive, so it is an equivalence relation.
(e.g., $f$(a)=$f$(b), $f$(b)=$f$(c)=> $f$(a)=$f$(c))

- Ex 7.16 (e): If $\Re$ is a relation on $A$, then $\Re$ is both an equivalence relation and a partial order relation iff $\Re$ is the equality relation on $A$.
  - equality relation $\{(a_i, a_i) | a_i \in \mathrm{A}\}$

# 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

- Definition 7.8: Let relations $\Re_1 \subseteq A \times B$ and $\Re_2 \subseteq B \times C$. The <u>composite relation</u> $\Re_1 \circ \Re_2$ is a relation defined by $\Re_1 \circ \Re_2 = \{(x, z) \mid \exists y \in B$ such that $(x, y) \in \Re_1$ and $(y, z) \in \Re_2\}$.

  *(Note the different ordering with function composition.)*

  $f : A \rightarrow B, \quad g : B \rightarrow C, \quad g \circ f : A \rightarrow C$

- **<u>Ex 7.17</u>** : Consider $\Re_1 = \{(1, x), (2, x), (3, y), (3, z)\}$ and $\Re_2 = \{(w, 5), (x, 6)\}$, and $\Re_3 = \{(w, 5), (w, 6)\}$. $\Re_1 \circ \Re_2 = \{(1, 6), (2, 6)\}$, and $\Re_1 \circ \Re_3 = \emptyset$ ?
- **<u>Ex 7.18</u>** : Let $A$ be the set of employees {L. Alldredge,…} at a computer center, while $B$ denotes a set of programming language {C++, Java,…}, and $C$ is a set of projects $\{p_1, p_2, \ldots\}$, consider $\Re_1 \subseteq A \times B$, $\Re_2 \subseteq B \times C$. What is the means of $\Re_1 \circ \Re_2$ ?

join operation in database

# Composite Relation

- Theorem 7.1: $\Re_1 \subseteq A \times B$, $\Re_2 \subseteq B \times C$, and $\Re_3 \subseteq C \times D \Rightarrow \Re_1 \circ (\Re_2 \circ \Re_3) = (\Re_1 \circ \Re_2) \circ \Re_3$
- Definition 7.9. We define the powers of relation $\Re$ by (a) $\Re^1 = \Re$; (b) $\Re^{n+1} = \Re \circ \Re^n$.

- **Ex 7.19** : If $\Re = \{(1, 2), (1, 3), (2, 4), (3, 2)\}$, then $\Re^2 = \{(1, 4), (1, 2), (3, 4)\}$, $\Re^3 = ?$ and $\Re^4 = ?$

$\Re^3 = \{(1,4)\}$
and for n ≥ 4, $\Re^n = \emptyset$

# Relation Matrix

- Definition 7.10: An <u>$m \times n$ zero-one matrix</u> $E = (e_{ij})_{m \times n}$ is a rectangular array of numbers arranged in $m$ rows and $n$ columns, where each $e_{ij}$ denotes the <u>entry</u> in the $i$th row and $j$th column of $E$, and each such entry is 0 or 1.
- **Relation matrix**: A relation can be represented by an $m \times n$ zero-one matrix.

- **Ex 7.21** : Consider $\Re_1 = \{(1, x), (2, x), (3, y), (3, z)\}$, $\Re_2 = \{(w, 5), (x, 6)\}$, and $\Re_1 \circ \Re_2$ to be represented by relation matrices?

$$M(\mathcal{R}_1) = \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (4) \end{array} \begin{array}{c} \begin{array}{cccc} (w) & (x) & (y) & (z) \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \qquad M(\mathcal{R}_2) = \begin{array}{c} \\ (w) \\ (x) \\ (y) \\ (z) \end{array} \begin{array}{c} \begin{array}{ccc} (5) & (6) & (7) \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$M(\mathcal{R}_1) \cdot M(\mathcal{R}_2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (4) \end{array} \begin{array}{c} \begin{array}{ccc} (5) & (6) & (7) \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} = M(\mathcal{R}_1 \circ \mathcal{R}_2).$$

*Boolean addition' with 1+1=1*

# Relation Matrix

- <u>Ex 7.22</u>: If $\Re=\{(1, 2), (1, 3), (2, 4), (3, 2)\}$, then what are the relation matrices of $\Re^2$, $\Re^3$ and $\Re^4$?

$$M(\mathscr{R}) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(M(\mathscr{R}))^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(M(\mathscr{R}))^2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# Relation Matrix

- Let $A$ be a set with $|A| = n$ and $\Re$ be a relation on $A$. If $M(\Re)$ is the relation matrix for $\Re$, then
  - $M(\Re) = \mathbf{0}$ if and only if $\Re = \phi$.
  - $M(\Re) = \mathbf{1}$ if and only if $\Re = A \times A$.
  - $M(\Re^m) = [M(\Re)]^m$
- Definition 7.11: Let $E = (e_{ij})_{m \times n}$, $F = (f_{ij})_{m \times n}$ be two *mxn* ero-one matrices. We say that $E$ precedes, or is less than, $F$, written as $E \le F$, if $e_{ij} \le f_{ij}$ for all $i, j$.

- Ex 7.23 : $E \le F$. How many zero-one matrices $G$ do have the results of $E \le G$?

$$E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$2^3=8$

# Relation Matrix

- Definition 7.12: $I_n = (\delta_{ij})_{n\times n}$ is the $n\times n$ zero-one matrix, where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

- Definition 7.13: $A = (a_{ij})_{m\times n}$ is a zero-one matrix, the transpose of $A$, written $A^{\text{tr}}$, is the matrix $(a^*_{ji})_{n\times m}$ where $a^*_{ji} = a_{ij}$

- **Ex 7.24** :

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \qquad A^{\text{tr}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

- **Theorem 7.2**: If $M$ denote the relation matrix for $\Re$ on $A$, then
  (A) $\Re$ is reflexive if and only if $I_n \leq M$.
  (B) $\Re$ is symmetric if and only if $M = M^{\text{tr}}$.
  (C) $\Re$ is transitive if and only if $M^2 \leq M$.
  (D) $\Re$ is anti-symmetric if and only if $M \cap M^{\text{tr}} \leq I_n$.

# Directed Graph

- Definition 7.14. A directed graph can be denoted as $G = (V, E)$, where $V$ is the vertex set and $E$ is the edge set.
  - $(a, b)$: if $a, b \in V$ $(a, b) \in E$, then there is a edge from $a$ to $b$. Vertex $a$ is called source (origin) of the edge, and $b$ is terminating vertex.
  - $(a, a)$: is called a loop.
- $V = \{1, 2, 3, 4, 5\}$, $E = \{(1, 1), (1, 2), (1, 4), (3, 2)\}$
  - Isolated vertex: vertex 5 in Fig. 7.1.
- Single undirected edge $\{a, b\} = \{b, a\}$ in Fig. 7.2 (b) is used to represent the two directed edges shown in Fig. 7.2 (a).

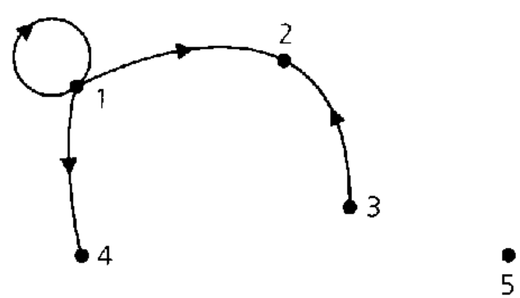


Figure 7.1

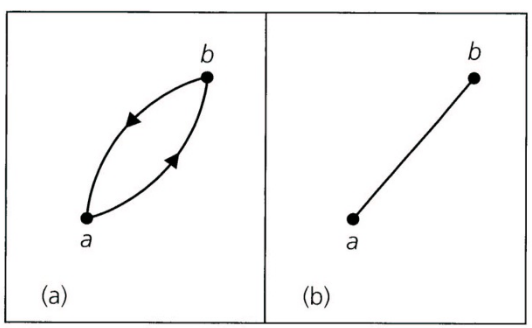


Figure 7.2

# Directed Graph

- Ex 7.26 precedence graph

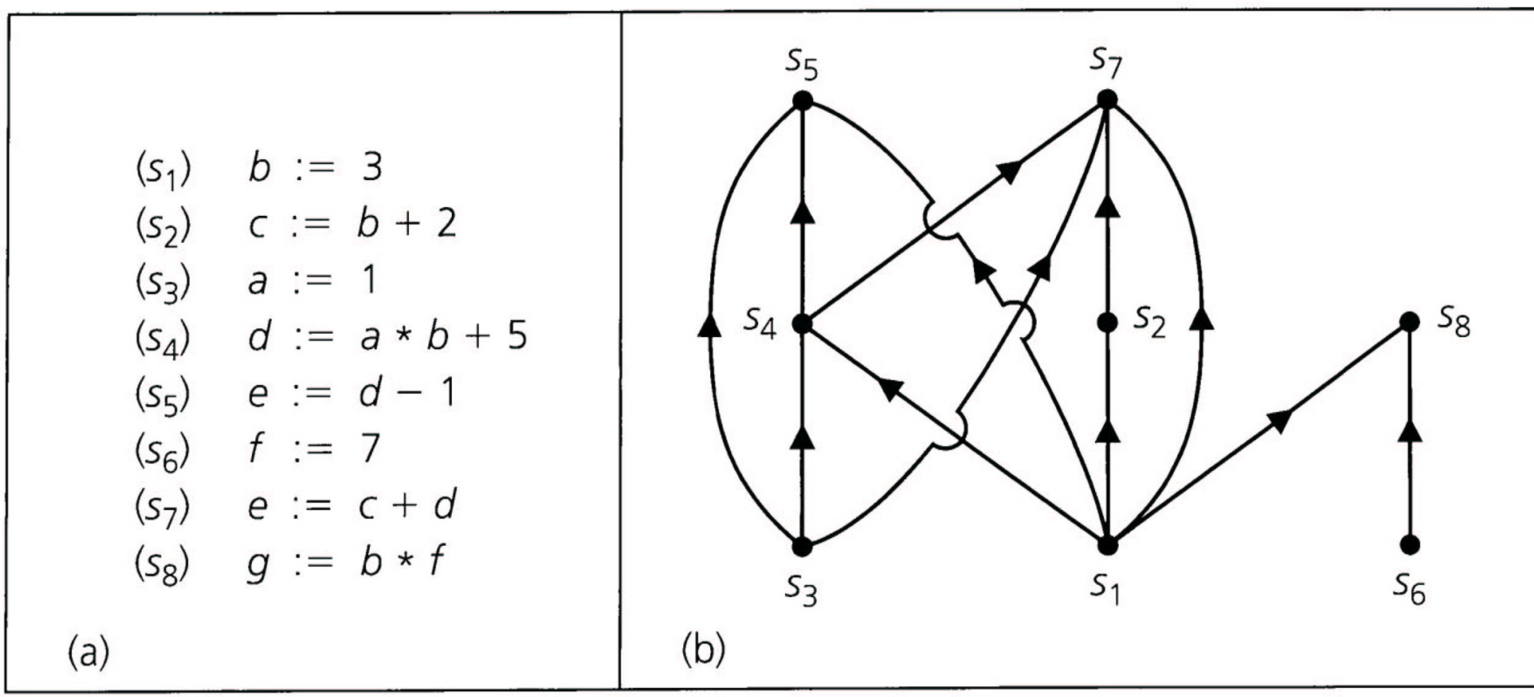



**Figure 7.3**

# Directed Graph

- Ex 7.27 : R = {(1,1),(1,2),(2,3),(3,2),(3,3),(3,4),(4,2)}
  - directed graph in Fig. 7.4 (a)
  - (associated) undirected graph in Fig. 7.4 (b)
  - path: In the connected graph, any two vertices $x$, $y$, with $x \neq y$, there is a path starting at $x$ and ending at $y$.
  - cycle: a closed path starts and terminates at the same vertex, containing at least three edges.
    - E.g.: {3, 4}, {4, 2}, and {2, 3}

*No repeated vertex*

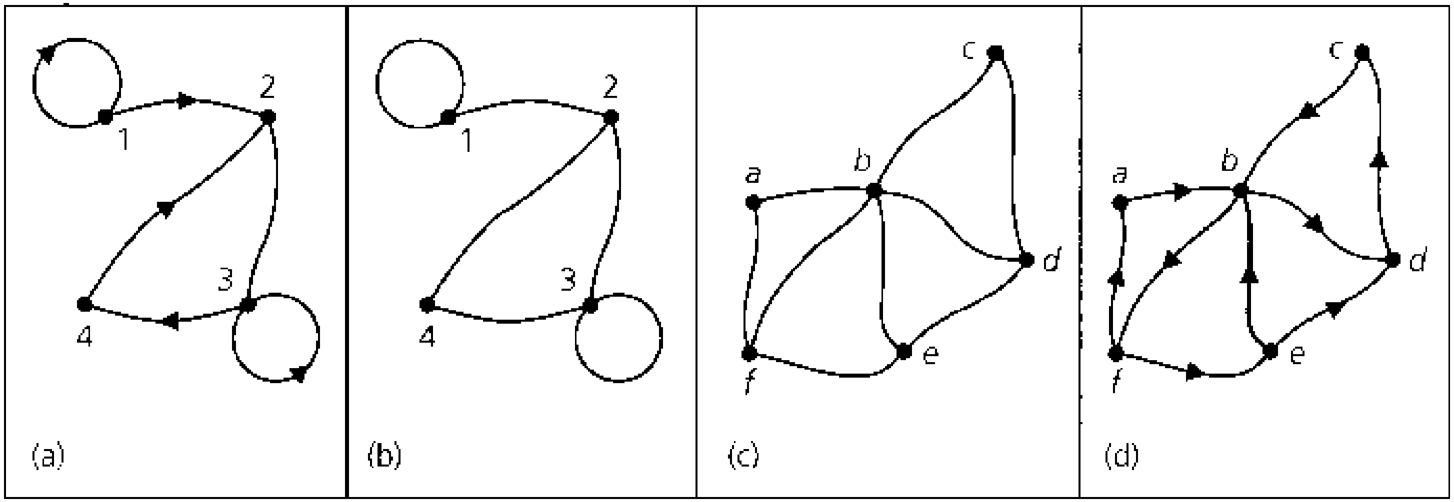



Figure 7.4

# Directed Graph

- Definition 7.15: A directed graph $G$ on $V$ is called **strongly connected**, if for all $x, y \in V$, where $x \neq y$, there is a path (in $G$) of directed edges from $x$ to $y$.
  - e.g., Fig. 7.5
- **Disconnected graph**: is the union of two connected pieces called the components of the graph.
  - e.g., Fig. 7.6

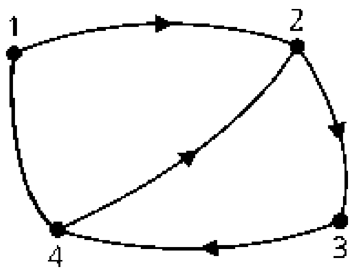



Figure 7.5

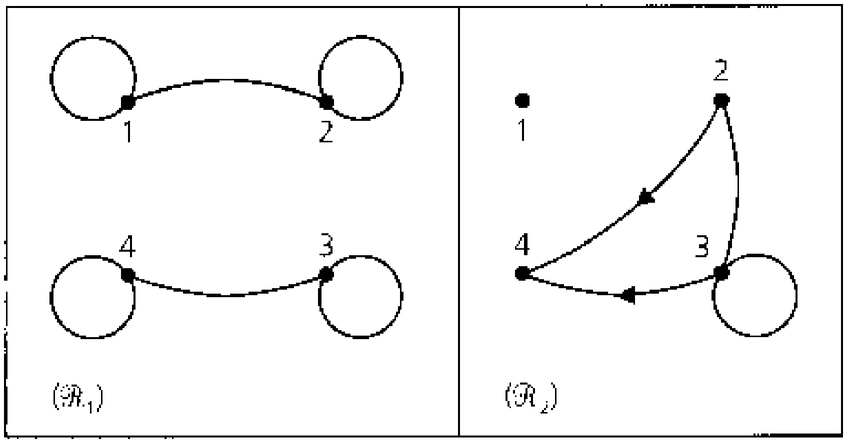



Figure 7.6

# Directed Graph

- **Complete graph**: the graphs of undirected graphs that are loop-free and have an edge for every pair of distinct vertices, which are denoted by $K_n$.
  - e.g., Fig. 7.7

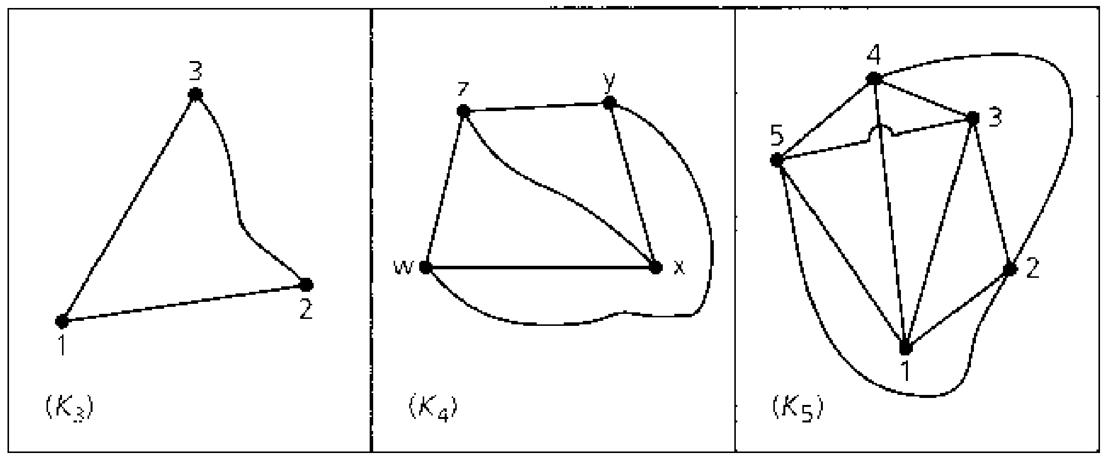
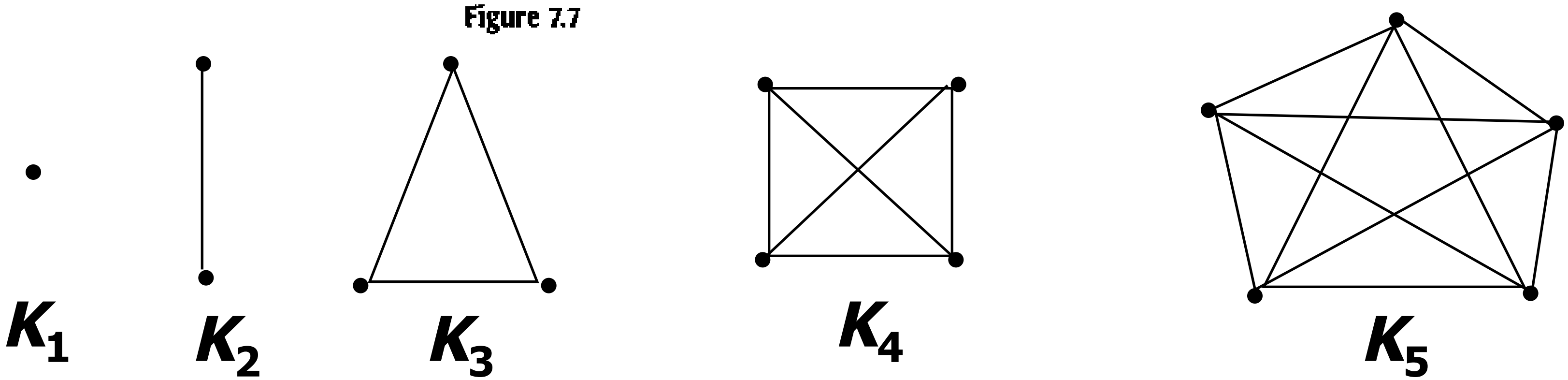



Figure 7.7



# Directed Graph

- Ex 7.30 : $\mathfrak{R}$ is reflexive if and only if its directed graph contains a loop at each vertex.
  - e.g., Fig 7.8, $A = \{1, 2, 3\}$ and $\mathfrak{R} = \{(1,1), (1, 2), (2, 2), (3, 3), (3, 1)\}$
- Ex 7.31 : $\mathfrak{R}$ is symmetric if and only if its directed graph may be drawn only by loops and undirected edges.
  - e.g., Fig 7.9, $A = \{1, 2, 3\}$ and $\mathfrak{R} = \{(1,1), (1, 2), (2, 1), (2, 3), (3, 2)\}$
- Ex 7.32 : $\mathfrak{R}$ is anti-symmetric if and only if for any $x \neq y$ the graph contains at most one of the edges $(x, y)$ or $(y, x)$
  - e.g., Fig 7.10, $A = \{1, 2, 3\}$ and $\mathfrak{R} = \{(1,1), (1, 2), (2, 3), (1, 3)\}$

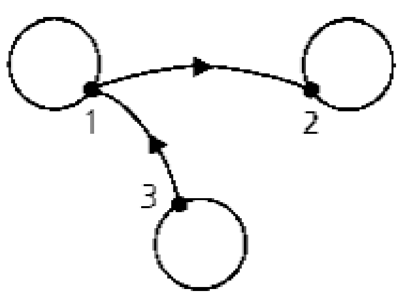


**Figure 7.8**

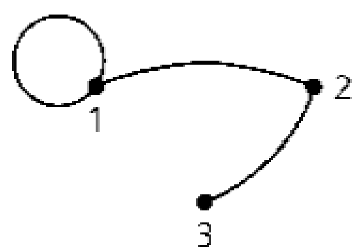


**Figure 7.9**

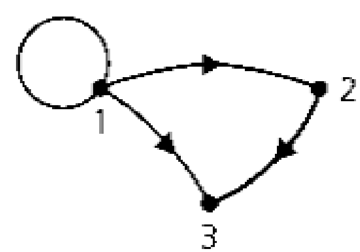


**Figure 7.10**

# Directed Graph

- **Ex 7.32** : $\Re$ is transitive if and only if for all $x, y \in A$, if there is a **path** from $x$ to $y$ in the associated graph, then there is an **edge** $(x, y)$ also.
  - e.g., Fig 7.10, $A = \{1, 2, 3\}$ and $\Re = \{(1, 1), (1, 2), (2, 3), (1, 3)\}$
- **Ex 7.33** : Fig 7.11, a relation is an equivalence relation if and only if its graph is one **complete graph** augmented by loops at every vertex or consists of disjoint union of complete graphs augmented by loops at each vertex.
  - e.g., Fig 7.11, $A = \{1, 2, 3, 4, 5\}$ and $\Re_1 = \{(1,1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5)\}$, $\Re_2 = \{(1,1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 4), (4, 5), (5, 4) (5, 5)\}$.

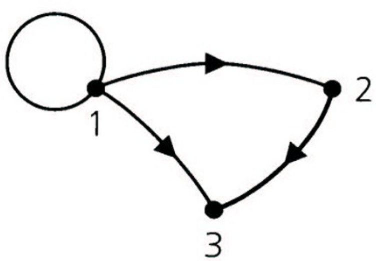



Figure 7.10

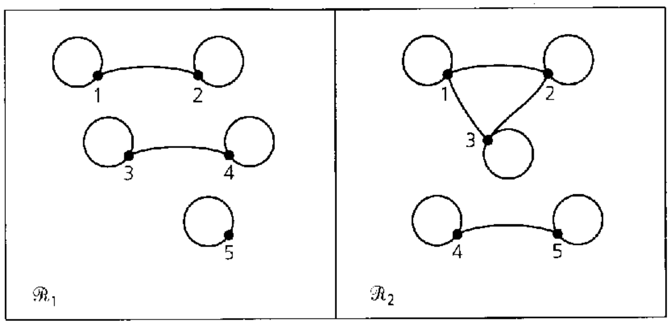



Figure 7.11

# Directed Graph

*reflexive: loop on each vertex*
*symmetric: undirected edge + loops*
*transitive: one path → one edge*
*equivalence: disjoint union of complete graphs + loops at every vertex*

$$\begin{array}{ccc} \text{directed graphs} & \longleftrightarrow & \text{relations} \\ \Downarrow & & \Downarrow \\ \text{adjacency matrices} & \longleftrightarrow & \text{relation matrices} \end{array}$$

# 7.3 Partial Orders: Hasse Diagrams

- Definition: Let $A$ be a set with $\mathfrak{R}$ a relation on $A$. The pair ($A$, $\mathfrak{R}$) is called a **partially ordered set**, or poset, if relation $\mathfrak{R}$ on $A$ is partially ordered.
  - If $A$ is called a poset, we understand that there is a partially order $\mathfrak{R}$ on $A$ that makes $A$ into this set.

natural counting: **N**

$x+5=2$ : **Z**

$2x+3=4$ : **Q**

$x^2-2=0$ : **R**

$x^2+1=0$ : **C**

Something was lost when we went from **R** to **C**. We have lost **the ability to "order"** the elements in **C**.

*$2+i < 1+2i$ ?*

# 7.3 Partial Orders: Hasse Diagrams

- **Ex 7.34** : Let $A$ be the set of courses offered at a college. Define the relation $\Re$ on $A$ by $x\Re y$ if $x$, $y$ are the same course or if $x$ is a prerequisite for $y$.

- **Ex 7.35** : Define $\Re$ on $A = \{1, 2, 3, 4\}$ by $x\Re y$ if $x$ divide $y$. Then $\Re = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (1, 4), (2, 4)\}$ is a partial order, and $(A, \Re)$ is a poset.

- **Ex 7.36** : PERT (Program Evaluation and Review Technique) network is first used by U.S. Navy in 1950.
  - E.g., Let $A$ be the set of tasks that must be performed to build a house. Define the relation $\Re$ on $A$ by $x\Re y$ if $x$, $y$ are the same task or if $x$ must be performed before $y$.

# Partial Orders: Hasse Diagrams

- **Ex 7.37** : Figure 7.17 (b) illustrates a simpler diagram for (a), called the **Hasse diagram**. The directions are assumed to go from the bottom to the top.

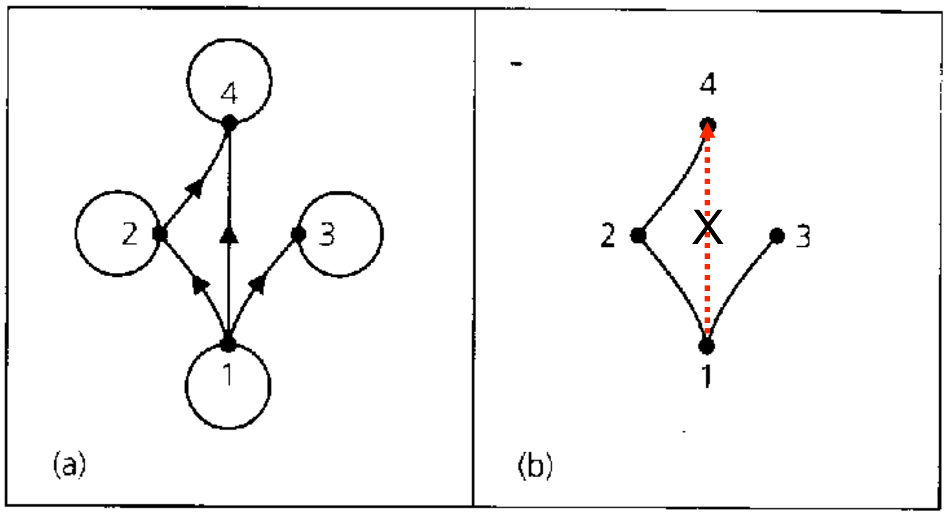
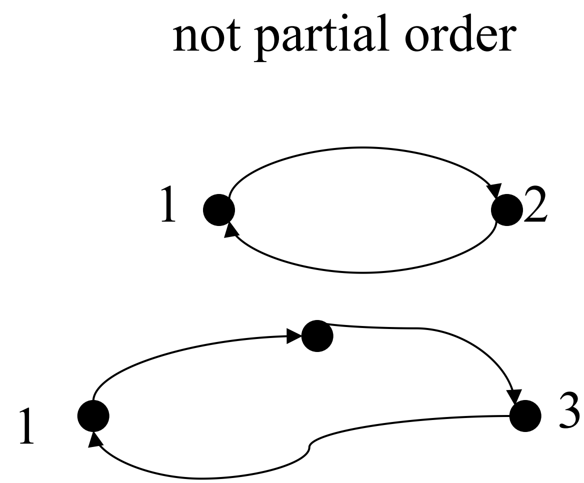



Figure 7.17



# Hasse Diagram

- If $(A, \Re)$ is a poset, we construct a Hasse diagram for $\Re$ on $A$ by drawing a line segment from x up to y, if
  - $x \Re y$
  - there is no other $z$ such that $x \Re z$ and $z \Re y$. *(in between x and y)*
- **Ex 7.38** : In Fig. 7.18 we have the Hasse diagrams for the following four posets.
  - (a) $\Re$ is the subset relation on $A$ is the power set of $\mathcal{U}$ with $\mathcal{U} = \{1, 2, 3\}$
  - (b), (c), and (d) are the divide relations.

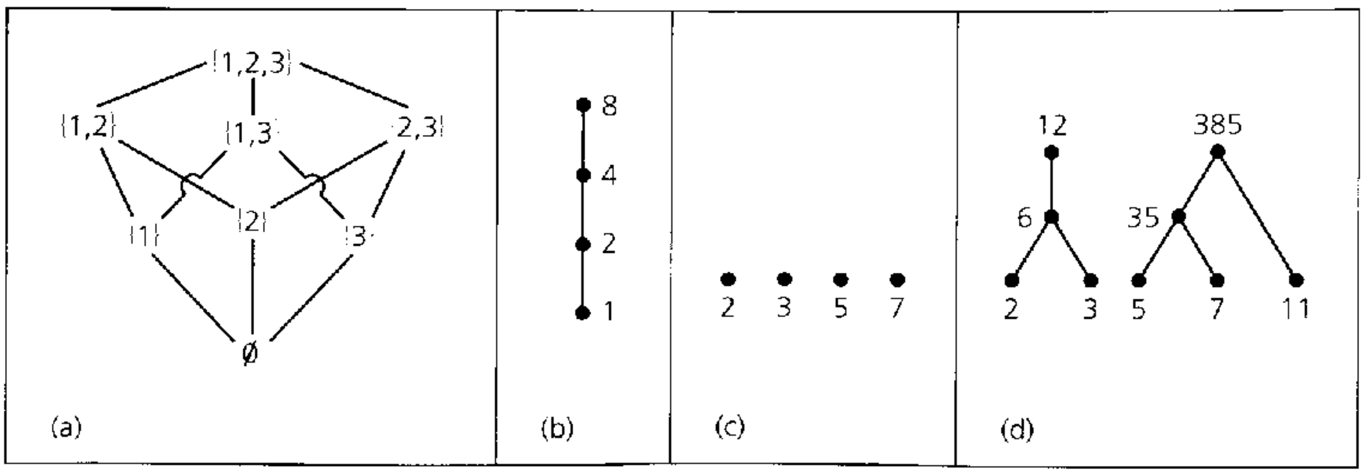



**Figure 7.18**

# Totally Ordered

- Definition 7.16. If $(A, \Re)$ is a poset, we say that A is **totally ordered** (linearly ordered) if for all $x, y \in A$ either $x\Re y$ or $y\Re x$. In this case, $\Re$ is called a total order.
- **Ex 7.40**

a) On the set **N**, the relation $\Re$ defined by $x\Re y$ if $x \leq y$ is a total order.

b) The subset relation is a partial order but not total order, e.g., $\{1, 2\}, \{1, 3\} \in A$, but $\{1, 2\} \not\subseteq \{1, 3\}$ or $\{1, 3\} \not\subseteq \{1, 2\}$.

c) Fig 7.19 is a total order.

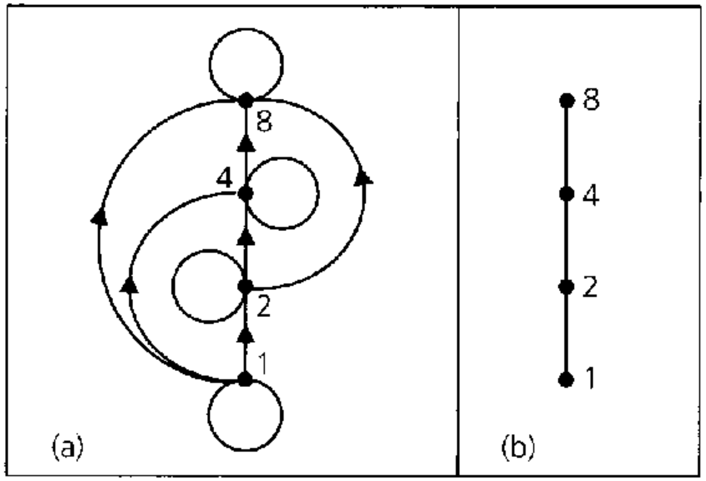



Figure 7.19

# Topological Sorting

- Given a Hasse diagram for a partial order relation $\Re$, how to find a total order $\Im$ for which $\Re \subseteq \Im$.

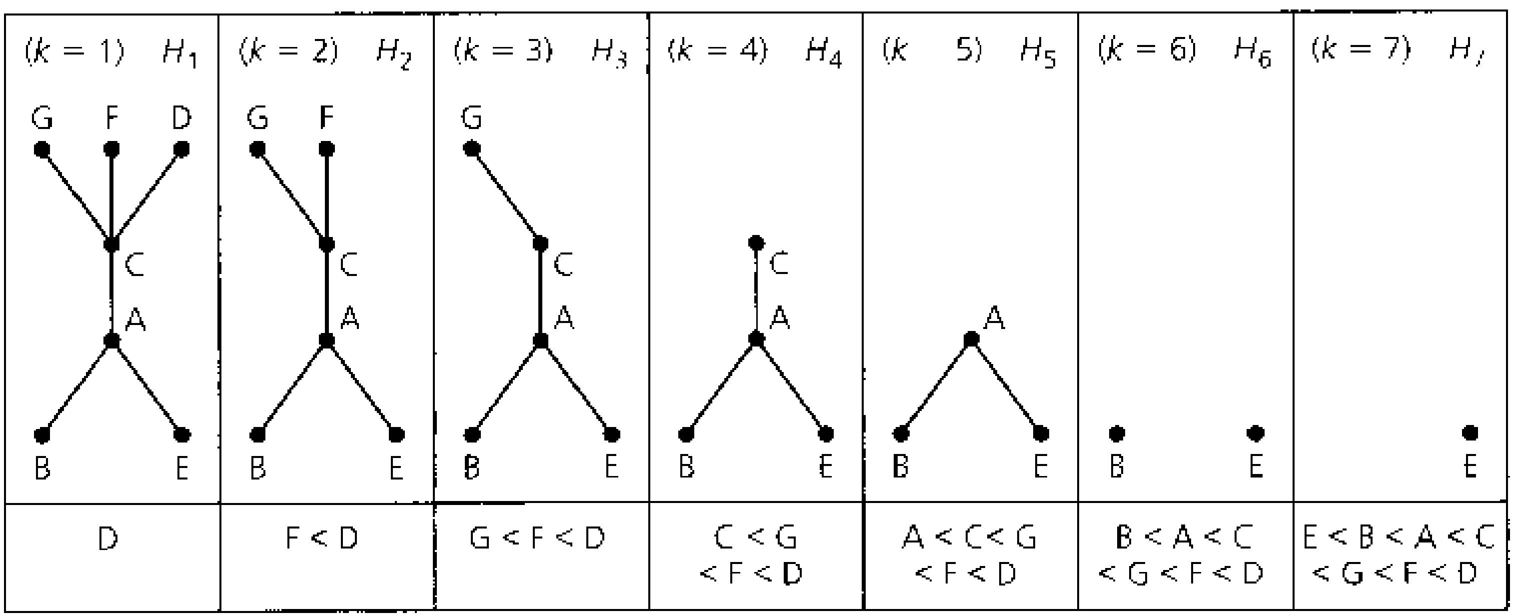



**Figure 7.21**

*Not unique, 12 answers*

# Topological Sorting

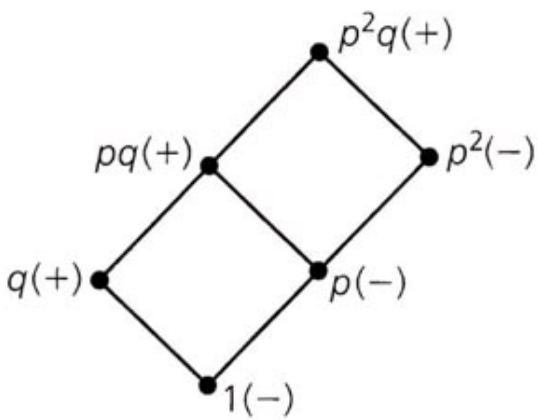
- For a partial order $\Re$ on a set $A$ with $|A| = n$
  - Step 1: Set $k = 1$. Let $H_1$ be the Hasse diagram of the partial order.
  - Step 2: Select a vertex $v_k$ in $H_k$ such that no edge in $H_k$ starts at $v_k$.
  - Step 3: If $k = n$, the process is completed and we have a total order

    $$\Im : v_n < v_{n-1} < \cdots < v_1$$

    that contains $\Re$.
  - If $k < n$, then remove from $H_k$ the vertex $v_k$ and all edges of $H_k$ that terminate at $v_k$. Call the result $H_{k+1}$. Increase $k$ by 1 and return to step (2).

# Topological Sorting

- **Ex 7.41**




(a)

$\mathcal{T}_1$: $p^2q > pq > q > p^2 > p > 1$
$+, +, +, -, -, -$

$\mathcal{T}_2$: $p^2q > pq > p^2 > p > q > 1$
$+, +, -, -, +, -$

$\mathcal{T}_3$: $p^2q > p^2 > pq > q > p > 1$
$+, -, +, +, -, -$

$\mathcal{T}_4$: $p^2q > pq > p^2 > q > p > 1$
$+, +, -, +, -, -$

$\mathcal{T}_5$: $p^2q > p^2 > pq > p > q > 1$
$+, -, +, -, +, -$

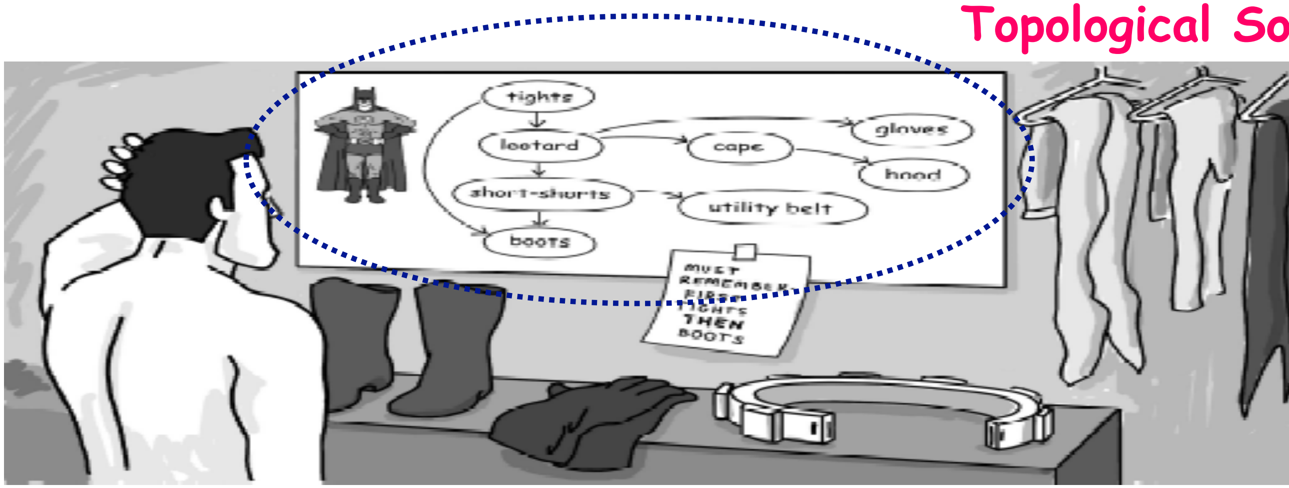
(b)

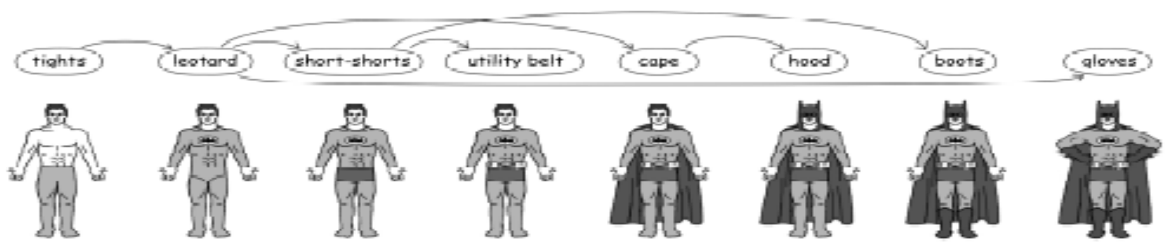
**Figure 7.22**

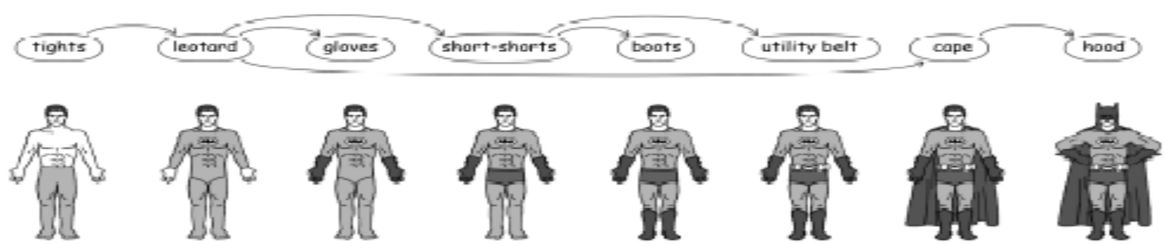
More practices, exercise7.27

# EX: Dressing in the morning

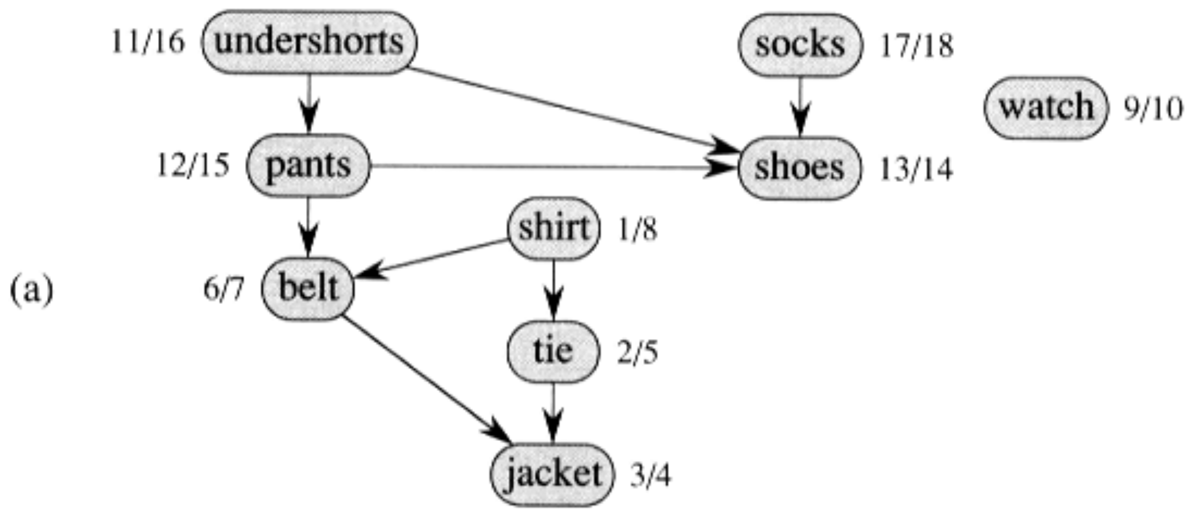
Topological Sort

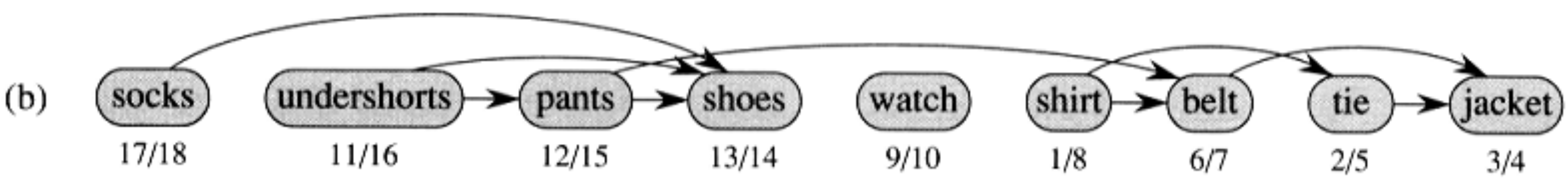


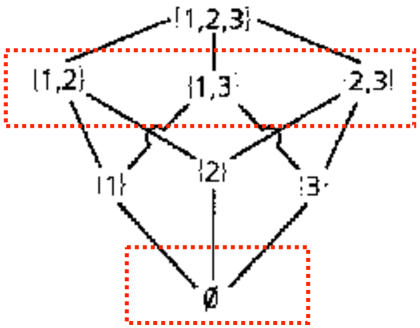
*DFS sequence*



# Maximal and Minimal

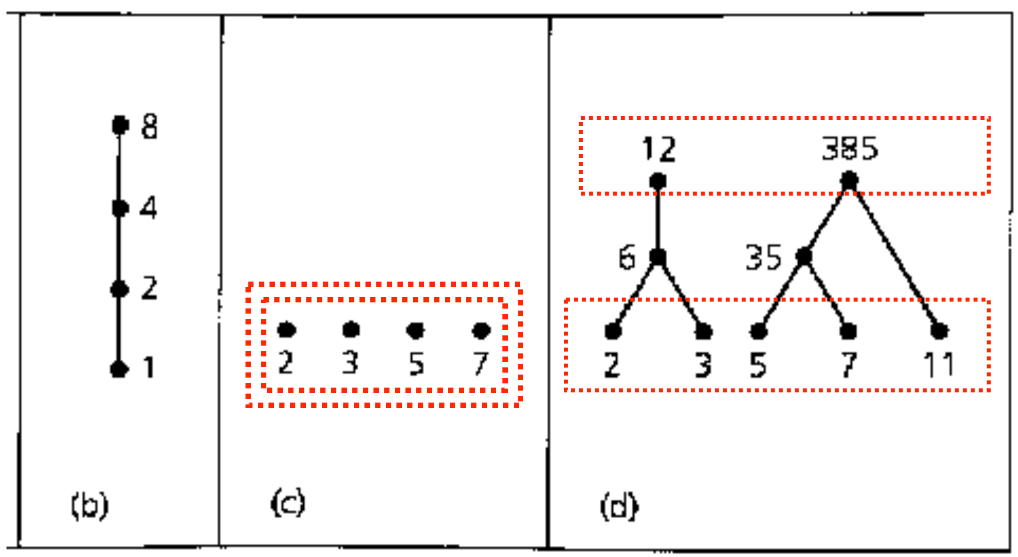
- Definition 7.17: If $(A, \Re)$ is a poset, then $x$ is a maximal element of $A$ if for all $a \in A$, $a \neq x \Rightarrow x \not\Re\, a$. Similarly, $y$ is a minimal element of $A$ if for all $b \in A$, $b \neq y \Rightarrow b \not\Re\, y$.
- Ex 7.42 : $\mathcal{U} = \{1, 2, 3\}$, $A = P(\mathcal{U})$.
  - For the poset $(A, \subseteq)$, $\mathcal{U}$ is the maximal and $\phi$ is the minimal.
  - Let $B$ be the *proper subsets* of $\{1, 2, 3\}$. Then we have multiple maximal elements $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$ for the poset $(B, \subseteq)$, and $\phi$ is still the only minimal element.




- Ex 7.43 : For the poset $(\mathbf{Z}, \leq)$, we have neither a maximal nor a minimal element. The poset $(\mathbf{N}, \leq)$, has no maximal element but a minimal element 0.

# Maximal and Minimal

- **Ex 7.44** : How about the poset in (b), (c), and (d) of Fig. 7.18? Do they have maximal or minimal elements?
- Theorem 7.3: If $(A, \mathfrak{R})$ is a poset and $\underline{A \text{ is finite}}$, then $A$ has both a maximal and a minimal element.

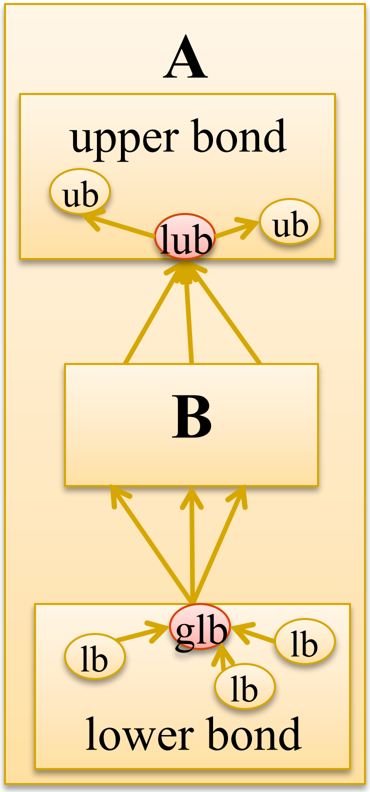


# Least and Greatest

- Definition 7.18: If $(A, \Re)$ is a poset, then $x$ is a **least** element of $A$ if for all $a \in A$, $x \Re a$. Similarly, $y$ is a **greatest** element of $A$ if for all $a \in A$, $a \Re y$.
- **Ex 7.45** : $\mathcal{U} = \{1, 2, 3\}$, $A = P(\mathcal{U})$.
  - For the poset $(A, \subseteq)$, $\mathcal{U}$ is the greatest and $\phi$ is the least.
  - Let $B$ be the nonempty subsets of $\mathcal{U}$. Then we have $\mathcal{U}$ as the greatest element and three minimal elements for the poset $(B, \subseteq)$, but no least element.
- Theorem 7.4: If poset $(A, \Re)$ has a greatest **or** a least element, then that element is unique.
  - Proof: Assume $x$ and $y$ are both greatest elements.
    Since x is a greatest element, $y \Re x$. Likewise, $x \Re y$ while $y$ is a greatest element. As $\Re$ is antisymmetric, it follows x = y.

# Lower and Upper Bound

- Definition 7.19: If $(A, \Re)$ is a poset with $B \subseteq A$, then
  - x∈A is called a **lower bound** of $B$ if $x \Re b$ **for all $b \in B$**
  - y∈A is called an **upper bound** of $B$ if $b \Re y$ **for all $b \in B$**
- An element $x' \in A$ is called a *greatest lower bound* (**glb**) of $B$ if for all other lower bounds $x''$ of $B$ we have $x'' \Re x'$. Similarly, an element $x' \in A$ is called a *least upper bound* (**lub**) of $B$ if for all other upper bounds $x''$ of $B$ we have $x' \Re x''$.
- Theorem 7.5: If $(A, \Re)$ is a poset and $B \subseteq A$, then B has **at most one** lub (glb).



# Lower and Upper Bound

- **Ex 7.47** : Let $\boldsymbol{U}$ ={1, 2, 3, 4} with $A$ = P($\boldsymbol{U}$ ) and let $\Re$ be the subset relation on $B$. If $B$ = {{1}, {2}, {1, 2}}, then what are the upper bounds of $B$, lower bounds of $B$, the greatest lower bound and the least upper bound?
  - Upper bounds: {1, 2}, {1, 2, 3}, {1, 2, 4}, and {1, 2, 3, 4}
  - lub: {1, 2}
  - glb = $\phi$

*{2, 3, 4} is not.*

- **Ex 7.48** : Let $\Re$ be the "≤" relation on A. What are the results for the following cases?
  - A = **R** and $B$ = [0, 1] => lub:1, glb:0
  - A = **R** and $B = \{q \in \mathbf{Q} \mid q^2 < 2\}$ => $\text{lub}: \sqrt{2}$, $\text{glb}: -\sqrt{2}$
  - A = **Q** and $B = \{q \in \mathbf{Q} \mid q^2 < 2\}$ => ? *No lub and glb*

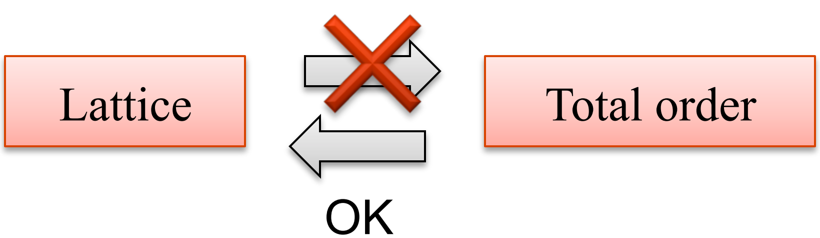
# Lattice

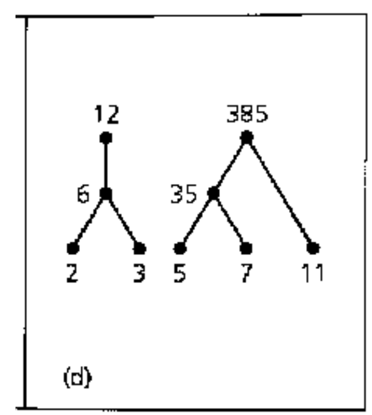
- Definition 7.20. The poset $(A, \Re)$ is called a **lattice** if for *all $x, y \in A$* the elements $\text{lub}\{x, y\}$ and $\text{glb}\{x, y\}$ both exist in $A$.

- **Ex 7.49** : For $A = \mathbf{N}$ and $x, y \in \mathbf{N}$, define $x \Re y$ by $x \le y$. Then $\text{lub}\{x, y\} = \max\{x, y\}$, $\text{glb}\{x, y\} = \min\{x, y\}$, and $(\mathbf{N}, \le)$ is a lattice.

- **Ex 7.50** : For the poset $(P(\boldsymbol{U}), \subseteq)$, if $S, T \subseteq \boldsymbol{U}$, we have $\text{lub}\{S, T\} = S \cup T$ and $\text{glb}\{S, T\} = S \cap T$ and it is a lattice.

- **Ex 7.51**: consider the poset in Example 7.38(d). Here we find that $\text{lub}\{2, 3\} = 6$ exists, but there is no glb for the elements 2 and 3.
  - This partial order is not a lattice.



# 7.4 Equivalence Relations and Partitions

- Equivalence relation: reflexive, symmetric, and transitive
- Examples:
  - For any set $A \neq \phi$, the relation of equality is an equivalence relation on $A$.
    - "sameness" among the elements of A
  - Let the relation on **Z** defined by $x \mathfrak{R} y$ if $x$-$y$ is a multiple of 2, then $\mathfrak{R}$ is an equivalence relation on **Z**, where all even integers are related, as are all odd integers.
    - The above relation **splits Z** into two subsets: $\{\ldots, -3, -1, 1, 3, \ldots\} \cup \{\ldots, -4, -2, 0, 2, 4, \ldots\}$

# Partition

- Definition 7.21. Given a set $A$ and index set $I$, let $\phi \neq A_i \subseteq A$ for $i \in I$.
  - Then $\{A_i\}_{i\in I}$ is a partition of $A$ if (a) $A = \cup_{i\in I} A_i$ and (b) $A_i \cap A_j = \phi$ for $i \neq j$.
  - Each subset $A_i$ is called a cell (block) of the partition.

- **Ex 7.52** : $A = \{1, 2,\ldots,10\}$
  - $A_1 = \{1, 2, 3, 4, 5\}$, $A_2 = \{6, 7, 8, 9, 10\}$.
  - $A_i = \{i, i+5\}$, $1 \leq i \leq 5$.
- **Ex 7.53** : Let $A = \mathbf{R}$, for each $i \in \mathbf{Z}$, let $A_i = [i, i+1)$. Then $\{A_i\}_{i\in Z}$ is a partition of $\mathbf{R}$.

# Equivalence Class

- Definition 7.22: Let $\Re$ be an equivalence relation on a set $A$. For each $x \in A$, the **equivalence class** of $x$, denoted $[x]$, is defined by $[x] = \{y \in A \mid y \Re x\}$

(1) $x \Re y$?

(2) $[x]=[y]$?

- **Ex 7.54** : Define the relation $\Re$ on $\mathbf{Z}$ by $x \Re y$ if $4 \mid (x-y)$.
  - $[0] = \{\ldots, -8, -4, 0, 4, \ldots\} = \{4k \mid k \in \mathbf{Z}\}$
  - $[1] = \{\ldots, -7, -3, 1, 5, \ldots\} = \{4k+1 \mid k \in \mathbf{Z}\}$
  - $[2] = \{\ldots, -6, -2, 2, 6, \ldots\} = \{4k+2 \mid k \in \mathbf{Z}\}$
  - $[3] = \{\ldots, -5, -1, 3, 7, \ldots\} = \{4k+3 \mid k \in \mathbf{Z}\}$
- **Ex 7.55** : Define the relation $\Re$ on $\mathbf{Z}$ by $a \Re b$ if $a^2=b^2$, $\Re$ is an equivalence relation.
  - $[n] = [-n] = \{-n, n\}$

$$Z = \{0\} \cup \left( \bigcup_{n \in Z^+} \{-n, n\} \right)$$

# Equivalence Class

- **Theorem 7.6**: If $\Re$ is an equivalence relation on a set $A$ and $x, y \in A$, then
  - (a) $x \in [x]$
  - (b) $x \Re y$ if and only if $[x] = [y]$
  - (c) $[x] = [y]$ or $[x] \cap [y] = \phi$. *(identical or disjoint)*

- **Ex 7.56** :
  - Let $A = \{1, 2, 3, 4, 5\}$, $\Re = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}$. $[1] = \{1\}$, $[2] = \{2, 3\} = [3]$, $[4] = \{4, 5\} = [5]$. Then, we have $A = [1] \cup [2] \cup [4]$.
  - Consider an onto function $f: A \to B$. $f(\{1, 3, 7\}) = x$; $f(\{4, 6\}) = y$; $f(\{2, 5\}) = z$. The relation $\Re$ defined on $A$ by $a \Re b$ if $f(a) = f(b)$.
    - $A = [1] \cup [4] \cup [2] = f^{-1}(x) \cup f^{-1}(y) \cup f^{-1}(z)$.

- **Ex 7.58** : If an equivalence relation $\Re$ on $A = \{1, 2, 3, 4, 5, 6, 7\}$ induces the partition $A = \{1, 2\} \cup \{3\} \cup \{4, 5, 7\} \cup \{6\}$, what is $\Re$? [2007台大資工]
  - $[1] = \{1, 2\} = [2] = \{(1, 1), (2, 2), (1, 2), (2, 1)\}$
  - $\Re = (\{1,2\} \times \{1,2\}) \cup (\{3\} \times \{3\}) \cup (\{4, 5, 7\} \times \{4, 5, 7\}) \cup (\{6\} \times \{6\})$
  - $|\Re| = 2^2 + 1^2 + 3^2 + 1^2 = 15$

{(1,1), (2,2)} v.s. {1,2}x{1,2}

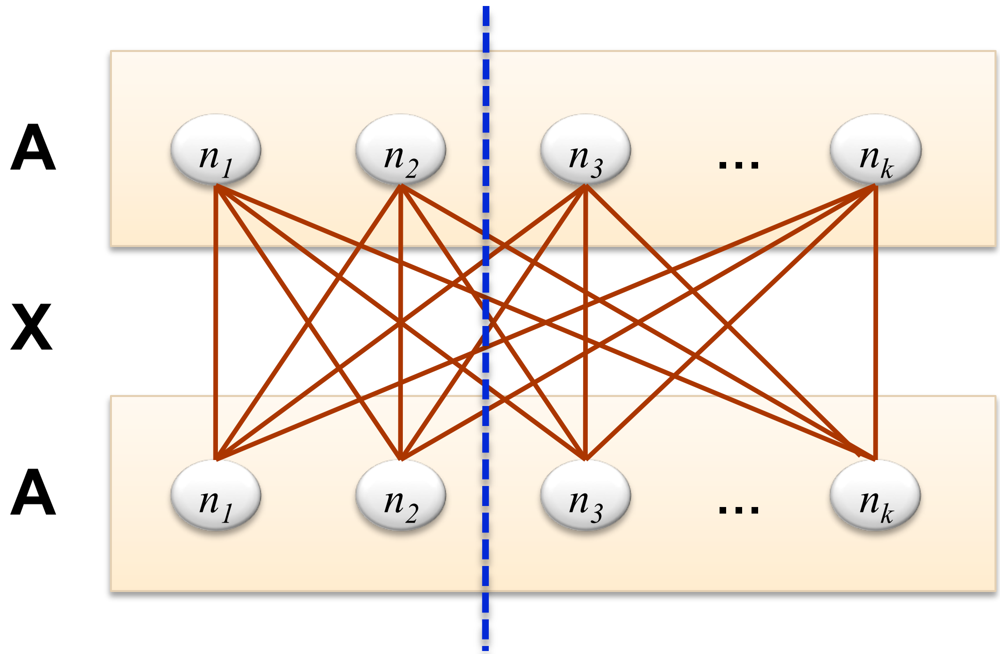
# Equivalence and Partition

- **Theorem 7.7**: If $A$ is a set, then
  - (a) any equivalence relation $\Re$ on $A$ induces a partition of $A$; and
  - (b) any partition of $A$ gives rise to an equivalence relation $\Re$ on $A$.
- **Theorem 7.8**: For any set $A$, there is one-to-one correspondence between the set of equivalence relations on $A$ and the set of partitions of $A$.

- Ex 7.59 :
  - (a) If A= {1, 2, 3, 4, 5, 6}, how many relations on A are equivalence relations? (identical containers)
    - *a partition of A: a distribution of the (distinct) elements of A into identical containers with no container left empty* $\sum_{i=1}^{6} S(6,i) = 203$
  - (b) How many of the equivalence relations in part (a) satisfy 1, 2 $\in$ [4]?

$$\sum_{i=1}^{4} S(4,i) = 15$$

# Equivalence and Partition




*k=10*

$k^2$ *100*

$2^2+(k-2)^2$ *4+64=68*

***r partitions?***

# 7.5 Finite State Machines: The Minimization Process

- Two finite state machines of the same function may have different number of internal states.
  - Some of these states are redundant.
- A process of transforming a given machine into one that has *no redundant internal states* is called the minimization process.
  - Rely on the concepts of *equivalence relation* and *partition*.

# Finite State Machines: The Minimization Process

- 1-Equivlence: Given the finite state machine $M = \{S, I, O, v, w\}$, we define the ***relation $E_1$*** on $S$ by $s_1 E_1 s_2$ if $w(s_1, x) = w(s_2, x)$ for all $x \in I$.
- The relation $E_1$ is an equivalence relation on $S$, and it partitions $S$ into subsets such that two states are in the same subset if they produce the same output for each $x \in I$.

- Here $s_1$ and $s_2$ are called 1-equivlent.

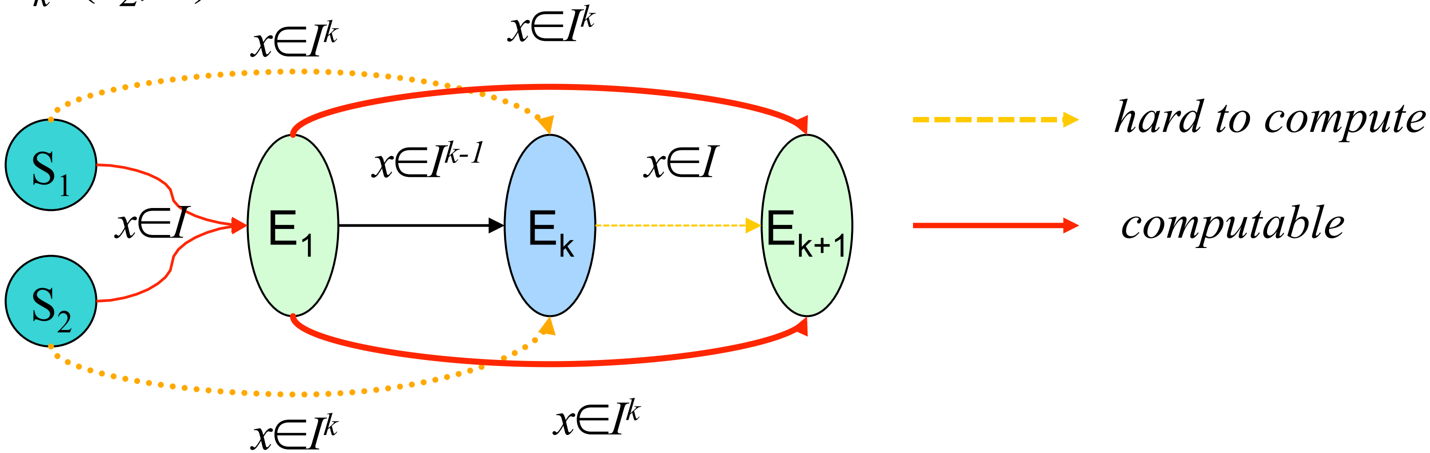
# Finite State Machines: The Minimization Process

- For the states $S$, we define the <u>*k-equivalence*</u> relation $E_k$ on $S$ by $s_1 E_k s_2$ if $w(s_1, x) = w(s_2, x)$ for all $x \in I^k$.
- The relation $E_k$ is an equivalence relation on $S$, and it partitions $S$ into subsets such that two states are in the same subset if they produce the same output for each $x \in I^k$.

- We call two states $s_1$ and $s_2$ equivalent if they are $k$-equivalent for all $k \geq 1$.

# Finite State Machines: The Minimization Process

- Goal: *Determine the partition of S induced by E and select one state for each equivalent class.*
- Observations:
  - If two states are not 2-equivalent, they can not be 3-equivalent.
  - For $s_1, s_2 \in S$, where $s_1 E_k s_2$, we find that $s_1 E_{k+1} s_2$ if and only if $v(s_1, x) E_k v(s_2, x)$ for all $\boldsymbol{x \in I}$.



# An Algorithm for the Minimization of a Finite State Machine

1. Set $k = 1$. $s_1 \mathrm{E}_1 s_2$ when $s_1$ and $s_2$ have the same output rows. ($P_i$ be the partitions of S induced by $E_i$)
2. Having determined $P_k$, we want to obtain $P_{k+1}$. Determine the states that are ($k$+1)-equivalent. Note that if $s_1 \mathrm{E}_k s_2$, then $s_1 \mathrm{E}_{k+1} s_2$ if and only if $v(s_1, x)\mathrm{E}_k v(s_2, x)$ for all $x \in I$.
3. If $P_{k+1} = P_k$, the process is completed.
   If $P_{k+1} \neq P_k$, $k = k+1$, goto step 2.

# Minimization of a Finite State Machine

- **Ex 7.60** : $M$ is given by the state table shown in Table 7.1.
  - Looking at the output rows: which states are 1-equivalent?
  - $P_1$ partitions $S$ as $P_1$: $\{s_1\}$, $\{s_2, s_5, s_6\}$, $\{s_3, s_4\}$
  - $P_2$: $\{s_1\}$, $\{s_2, s_5\}$, $\{s_6\}$, $\{s_3, s_4\}$ (Table 7.2), $P_2 \neq P_1$, continue
  - $P_3$: $\{s_1\}$, $\{s_2, s_5\}$, $\{s_6\}$, $\{s_3, s_4\}$, $P_3 = P_2$, stop

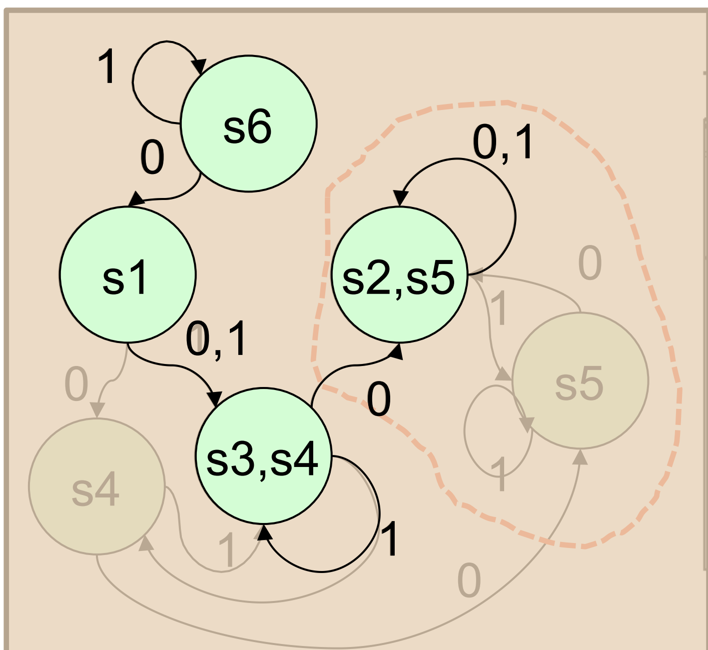



**Table 7.1**

| | $\nu$ | | $\omega$ | |
|---|---|---|---|---|
| | 0 | 1 | 0 | 1 |
| $s_1$ | $s_4$ | $s_3$ | 0 | 1 |
| $s_2$ | $s_5$ | $s_2$ | 1 | 0 |
| $s_3$ | $s_2$ | $s_4$ | 0 | 0 |
| $s_4$ | $s_5$ | $s_3$ | 0 | 0 |
| $s_5$ | $s_2$ | $s_5$ | 1 | 0 |
| $s_6$ | $s_1$ | $s_6$ | 1 | 0 |

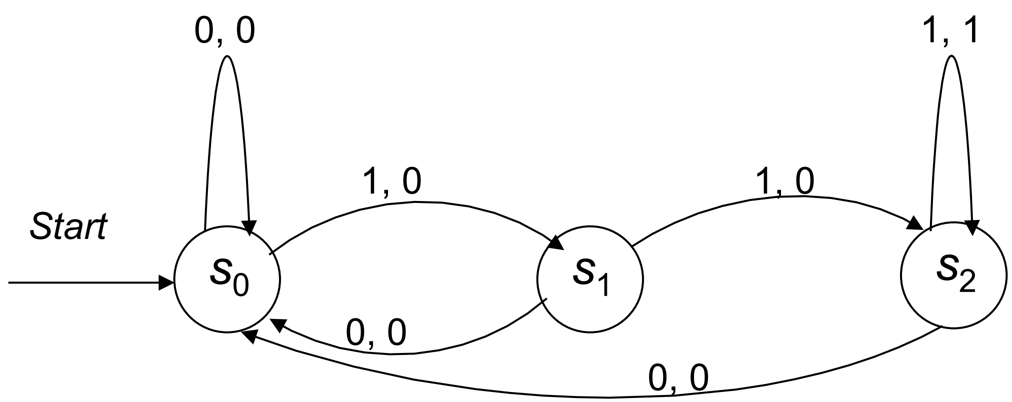
**Table 7.2**

| | $\nu$ | | $\omega$ | |
|---|---|---|---|---|
| | 0 | 1 | 0 | 1 |
| $s_1$ | $s_3$ | $s_3$ | 0 | 1 |
| $s_2$ | $s_2$ | $s_2$ | 1 | 0 |
| $s_3$ | $s_2$ | $s_3$ | 0 | 0 |
| $s_6$ | $s_1$ | $s_6$ | 1 | 0 |

*See also 15.2 Karnaugh Maps, a similar idea*

# More Minimization example

- <u>In Ex 6.20</u> : Construct a machine that recognizes each occurrence of the sequence 111.



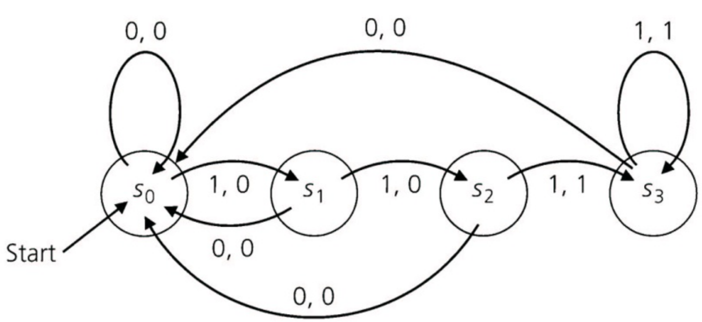




Figure 6.10

| | $v$ | | $w$ | |
|---|---|---|---|---|
| | 0 | 1 | 0 | 1 |
| $s_0$ | $s_0$ | $s_1$ | 0 | 0 |
| $s_1$ | $s_0$ | $s_2$ | 0 | 0 |
| $s_2$ | $s_0$ | $s_3$ | 0 | 1 |
| $s_3$ | $s_0$ | $s_3$ | 0 | 1 |

$P_1$: $\{s_0, s_1\}, \{s_2, s_3\}$

$P_2$: $\{s_0\}, \{s_1\}, \{s_2, s_3\}$

$P_3$: $\{s_0\}, \{s_1\}, \{s_2, s_3\}$

# Refinement

- Definition 7.23: If $P_1$ and $P_2$ are partitions of set $A$, then $P_2$ is called a refinement of $P_1$, denoted as $P_2 \leq P_1$, if every cell of $P_2$ is contained in a cell of $P_1$.
- When $P_2 \leq P_1$ and $P_2 \neq P_1$, we write $P_2 < P_1$.
  - In Example 7.60, $P_3 = P_2 < P_1$
- Theorem 7.9: In the minimization process, if $P_{k+1} = P_k$, then $P_{r+1} = P_r$ for all $r \geq k+1$.

# Distinguishing String

- If $s_1 E_k s_2$ but $s_1 \not{E}_{k+1} s_2$, then we have a string $x = x_1 x_2 \ldots x_k x_{k+1} \in I^{k+1}$ such that $w(s_1, x) \neq w(s_2, x)$ but $w(s_1, x_1 x_2 \ldots x_k) = w(s_2, x_1 x_2 \ldots x_k)$. We call this string $x$ as **distinguishing string**.
- $s_1 \not{E}_{k+1} s_2 \Rightarrow \exists x_1 \in I,\ [v(s_1, x_1) \not{E}_k\ \mathrm{v}(s_2, x_1)]$

# Distinguishing String

- **Ex 7.61** : From Example 7.60, $s_2 E_1 s_6$ but $s_2 \not{E}_2 s_6$, so we seek a distinguishing string of length 2.
- $x$ = 00 is the minimal distinguishing string for $s_2$ and $s_6$
  - $w(s_2, 00) = 11 \neq 10 = w(s_6, 00)$

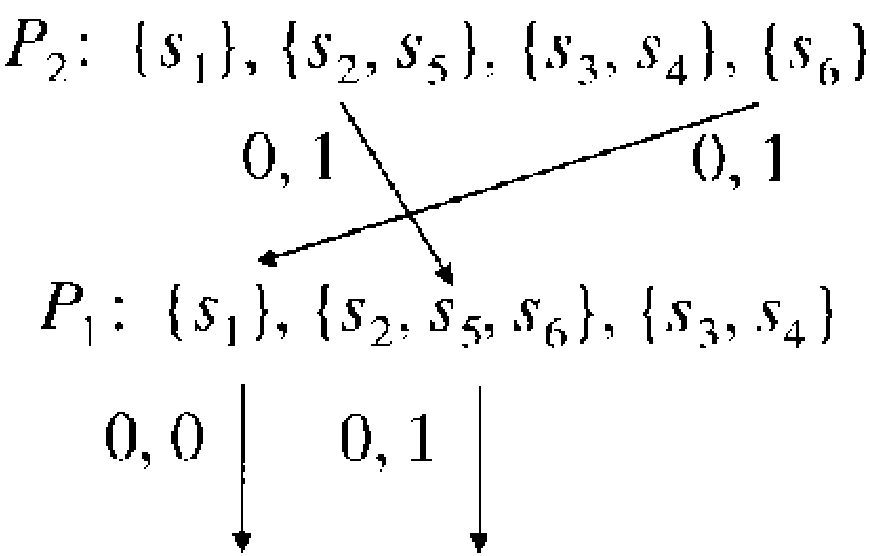



**Table 7.1**

| | $\nu$ | | $\omega$ | |
|---|---|---|---|---|
| | 0 | 1 | 0 | 1 |
| $s_1$ | $s_4$ | $s_3$ | 0 | 1 |
| $s_2$ | $s_5$ | $s_2$ | 1 | 0 |
| $s_3$ | $s_2$ | $s_4$ | 0 | 0 |
| $s_4$ | $s_5$ | $s_3$ | 0 | 0 |
| $s_5$ | $s_2$ | $s_5$ | 1 | 0 |
| $s_6$ | $s_1$ | $s_6$ | 1 | 0 |

# Distinguishing String

- **Ex 7.62** : $s_1$ and $s_4$ are 2-equivalent but are not 3-equivalent.
- $x = 111$ is the minimal distinguishing string for $s_1$ and $s_4$
  - $w(s_1, 111) = 100 \neq 101 = w(s_4, 111)$

**Table 7.3**

| | $\nu$ | | $\omega$ | |
|---|---|---|---|---|
| | 0 | 1 | 0 | 1 |
| $s_1$ | $s_4$ | $s_2$ | 0 | 1 |
| $s_2$ | $s_5$ | $s_2$ | 0 | 0 |
| $s_3$ | $s_4$ | $s_2$ | 0 | 1 |
| $s_4$ | $s_3$ | $s_5$ | 0 | 1 |
| $s_5$ | $s_2$ | $s_3$ | 0 | 0 |

(a)

$P_3$: $\{s_1, s_3\}, \{s_2\}, \{s_4\}, \{s_5\}$

1, 1     1, 1

$P_2$: $\{s_1, s_3, s_4\}, \{s_2\}, \{s_5\}$

1, 0     1, 0

$P_1$: $\{s_1, s_3, s_4\}, \{s_2, s_5\}$

1, 1     1, 0

(b)

can't choose '0'

X=11 to distinguish s2 and s5

typo

2) Then $\nu(s_1, 1) \not E_2 \nu(s_4, 1) \Rightarrow \exists x_2 \in \mathcal{I}$ (here $x_2 = 1$) with $\nu(\nu(s_1, 1), 1) \not E_1 \nu(\nu(s_4, 1), 1)$, or $\nu(s_1, 11) \not E_1 \nu(s_4, 11)$. We used the partitions $P_2$ and $P_1$ to obtain $x_2 = 1$.

3) Now we use the partition $P_1$ where we find that for $x_3 = 1 \in \mathcal{I}$,