Engineering Mathematics

Midterm Exam, Fall 2017/11/13

1. Let R be a rectangular region in xy-plan defined by a ≤ x ≤ b, c ≤ y ≤ d, that contains the point (x_0, y_0) in its interior. If f(x, y) and $\frac{\partial f}{\partial y}$ are continuous on R, then there exists some interval I_0: $(x_0 - h, x_0 + h)$, h > 0, contained in [a b], and a unique function y(x) defined on I_0 that is a solution of the initial-value problem.

2. Dividing the equation by $e^y \cos x$ gives $\frac{e^{2y}-y}{e^y}dy = \frac{\sin 2x}{\cos x}dx.$

   We know that $\sin 2x = 2\sin x \cos x$, then

   $\int(e^y - ye^{-y})dy = 2\int \sin x \, dx$. Integration by parts

   $e^y + ye^{-y} + e^{-y} = -2\cos x + c.$

   The initial condition y=0 when x=0 implies c=4. Thus a solution of the initial-value problem is $e^y + ye^{-y} + e^{-y} = 4 - 2\cos x.$

3. By writing the differential equation in the form

$$(\cos x \sin x - xy^2)dx + y(1 - x^2)dy = 0$$

   We recognize that the equation is exact because

$$\frac{\partial M}{\partial y} = -2xy = \frac{\partial N}{\partial x}$$

$$\frac{\partial f}{\partial y} = y(1 - x^2)$$

$$\mathrm{f(x,y)} = \frac{y^2}{2}(1 - x^2) + h(x)$$

$$\frac{\partial f}{\partial x} = -xy^2 + h'(x) = \cos x \sin x - xy^2$$

   The last equation implies that $\mathrm{h}'(\mathrm{x}) = \cos x \sin x$. Integrating gives

$$\mathrm{h(x)} = -\int (\cos x)(-\sin x \, dx) = -\frac{1}{2}cos^2 x$$

   Thus $\frac{y^2}{2}(1 - x^2) - \frac{1}{2}cos^2 x = c_1$ or $y^2(1 - x^2) - cos^2 x =$ c, by initial condition, the solution of the equation is $y^2(1 - x^2) - cos^2 x = 3.$

4. M=xy, N=$2x^2+3y^2-20$, $M_y = x$ and $N_x = 4x$.

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3x}{xy} = \frac{3}{y}$$

The integration factor is then $e^{\int 3/y dy} = e^{3\ln y} = y^3$.After multiplying the given DE by y^3 the resulting equation is $\mathrm{x}y^4dx + (2x^2y^3 + 3y^5 - 20y^3)dy = 0$.

The family of solution is $\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = c$.

5. (課本解)Let $\mathrm{u} = -2\mathrm{x} + \mathrm{y}$, then du/dx=-2+dy/dx, and so the differential equation is transformed into $\frac{du}{dx} = u^2 - 9$.

Using partial fractions, $\frac{1}{6}[\frac{1}{u-3} - \frac{1}{u+3}]du = dx$,

and integrating, the yields $\frac{1}{6}\ln|\frac{u-3}{u+3}| = x + c_1$ or $\frac{u-3}{u+3} = e^{6x+6c_1} = ce^{6x}$.

$$\mathrm{u} = \frac{3(1+ce^{6x})}{1-ce^{6x}} \quad or \quad y = 2x + \frac{3(1+ce^{6x})}{1-ce^{6x}}$$

Applying the initial condition y(0)=0 to the last equation gives c = -1. And the solution is $y = 2x + \frac{3(1-e^{6x})}{1+e^{6x}}$.

(正解)

Let $\mathrm{u} = -2\mathrm{x} + \mathrm{y}$, then du/dx=-2+dy/dx, and so the differential equation is transformed into $\frac{du}{dx} = u^2 + 5$.

$$\frac{1}{u^2+5}du = dx$$

$$\frac{\arctan\frac{u}{\sqrt{5}}}{\sqrt{5}} + c = x$$

$$\frac{\arctan\frac{-2x+y}{\sqrt{5}}}{\sqrt{5}} + c = x$$

$$\frac{\arctan 0}{\sqrt{5}} + c = 0, \mathrm{c} = 0$$

Ans: $\arctan\frac{-2x+y}{\sqrt{5}} = \sqrt{5}x$

6. To solve the equation we must solve the cubic polynomial auxiliary equation $3m^3+5m^2+10m-4=0$. With the identification $a_0=-4$ and $a_3=3$ then the factors of a_0 and a_3 are, respectively, p:$\pm1,\pm2,\pm4$ and q:$\pm1,\pm3$. So the possible rational roots of the cubic equation are $\frac{p}{q}:\pm1,\pm2,\pm4,\pm\frac{1}{3},\pm\frac{2}{3},\pm\frac{4}{3}$.

   We discover the root $m_1=\frac{1}{3}$ and the factorization

$$3m^3+5m^2+10m-4=(m-\frac{1}{3})(3m^2+6m+12)$$

   $m_2=-1-\sqrt{3}i$ and $m_3=-1+\sqrt{3}i$. The general solution of the given differential equation is $\mathrm{y}=c_1e^{x/3}+e^{-x}(c_2\cos\sqrt{3}\,x+c_3\sin\sqrt{3}\,x)$.

7. $y_h=c_1e^{-4x}+c_2xe^{-4x}$

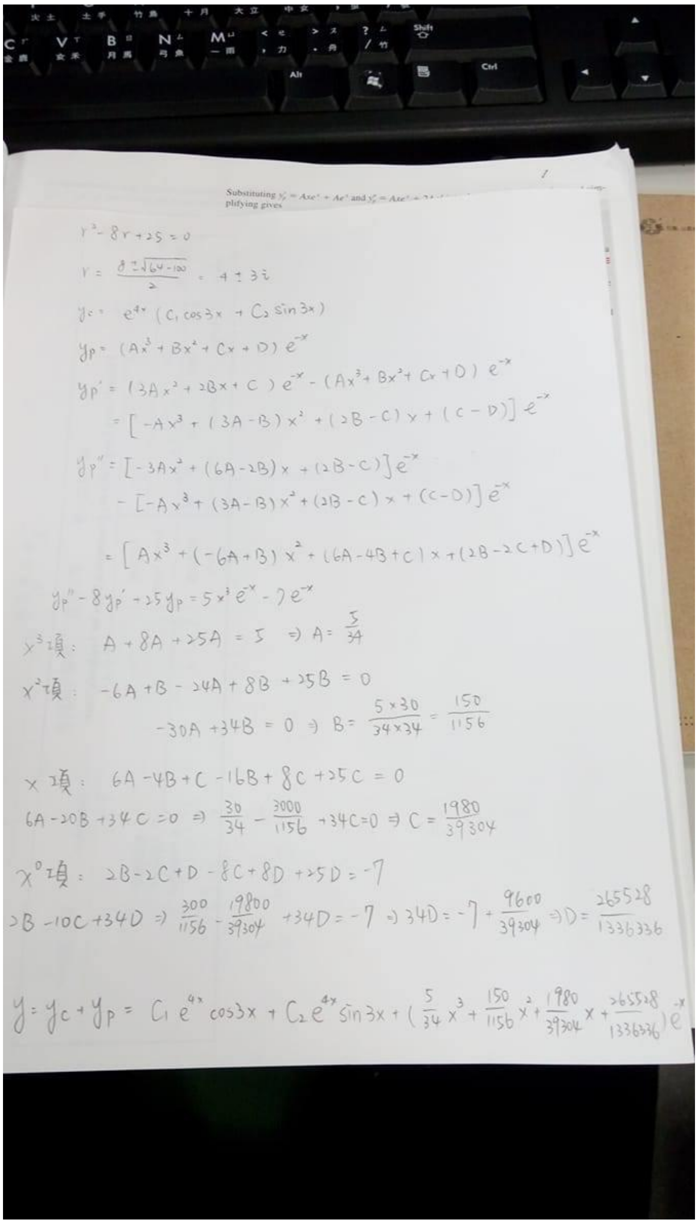
$$y_p=kx^2e^{-4x}$$
$$y_p'=kx^2(-4e^{-4x})+2kxe^{-4x}$$
$$=k\left(-4x^2e^{-4x}+2xe^{-4x}\right)$$

$$y_p''=k\left(16x^2-8x-8x+2\right)e^{-4x}$$
$$y_p''+8y_p'+16y_p$$
$$=k\left(16x^2-8x-8x+2\right)e^{-4x}+8k\left(-4x^2e^{-4x}+2xe^{-4x}\right)+16kx^2e^{-4x}$$
$$=2ke^{-4x}=3e^{-4x}$$
$$\Rightarrow k=\frac{3}{2}$$
$$y_p=\frac{3}{2}x^2e^{-4x}$$

   $\mathrm{y}=y_h+y_p=c_1e^{-4x}+c_2xe^{-4x}+\frac{3}{2}x^2e^{-4x}$.

8. (a)$y_p=(Ax^3+Bx^2+Cx+E)e^{-x}$

   $y_c=e^{4x}(c_1\cos3x+c_2\sin3x)$.



$r^2-8r+25=0$

$r=\frac{8\pm\sqrt{64-100}}{2}=4\pm3i$

$y_c=e^{4x}(C_1\cos3x+C_2\sin3x)$

$y_p=(Ax^3+Bx^2+Cx+D)e^{-x}$

$y_p'=(3Ax^2+2Bx+C)e^{-x}-(Ax^3+Bx^2+Cx+D)e^{-x}$

$\quad=[-Ax^3+(3A-B)x^2+(2B-C)x+(C-D)]e^{-x}$

$y_p''=[-3Ax^2+(6A-2B)x+(2B-C)]e^{-x}$

$\quad-[-Ax^3+(3A-B)x^2+(2B-C)x+(C-D)]e^{-x}$

$\quad=[Ax^3+(-6A+B)x^2+(6A-4B+C)x+(2B-2C+D)]e^{-x}$

$y_p''-8y_p'+25y_p=5x^3e^{-x}-7e^{-x}$

x^3項: $A+8A+25A=5 \Rightarrow A=\frac{5}{34}$

x^2項: $-6A+B-24A+8B+25B=0$

$-30A+34B=0 \Rightarrow B=\frac{5\times30}{34\times34}=\frac{150}{1156}$

x 項: $6A-4B+C-16B+8C+25C=0$

$6A-20B+34C=0 \Rightarrow \frac{30}{34}-\frac{3000}{1156}+34C=0 \Rightarrow C=\frac{1980}{39304}$

x^0項: $2B-2C+D-8C+8D+25D=-7$

$2B-10C+34D \Rightarrow \frac{300}{1156}-\frac{19800}{39304}+34D=-7 \Rightarrow 34D=-7+\frac{9600}{39304} \Rightarrow D=\frac{265528}{1336336}$

$y=y_c+y_p=C_1e^{4x}\cos3x+C_2e^{4x}\sin3x+(\frac{5}{34}x^3+\frac{150}{1156}x^2+\frac{1980}{39304}x+\frac{265528}{1336336})e^{-x}$

(b) $y_p=(Ax+B)\cos x+(Cx+D)\sin x.$

(b) $y' + 4y = x\cos x$

$r + 4 = 0$

$r = -4$

$y_c = C_1 e^{-4x}$

令 $y_p = (Ax + B)\cos x + (Cx + D)\sin x$

$y_p' = A\cos x - (Ax + B)\sin x + C\sin x + (Cx + D)\cos x$

$y_p' + 4y_p = x\cos x$

$(Cx + D + A)\cos x + (C - Ax - B)\sin x + (4Ax + 4B)\cos x$
$+ (4Cx + 4D)\sin x$

$= \big((4A + C)x + (A + D + 4B)\big)\cos x + \big((4C - A)x + (4D + C - B)\big)\sin x$

$= x\cos x$

$$\Rightarrow \begin{cases} 4A + C = 1 & \cdots ① \\ 4C - A = 0 & \cdots ② \\ A + D + 4B = 0 & \cdots ③ \\ 4D + C - B = 0 & \cdots ④ \end{cases}$$

① + 4 × ② $\Rightarrow 17C = 1 \Rightarrow C = \frac{1}{17},\ A = \frac{4}{17}$

③ $\Rightarrow 4B + D = -\frac{4}{17}$ ⋯ ⑤

④ $\Rightarrow -B + 4D = -\frac{1}{17}$ ⋯ ⑥

⑤ + 4 × ⑥ $\Rightarrow 17D = -\frac{8}{17} \Rightarrow D = -\frac{8}{289}$

$\Rightarrow B = -\frac{15}{289}$

Ans. $y = c_1 e^{-4x} + \left(\frac{4}{17}x - \frac{15}{289}\right)\cos x + \left(\frac{1}{17}x - \frac{8}{289}\right)\sin x$

9. $y_c = c_1\cos 3x + c_2\sin 3x$, using $y_1 = \cos 3x$, $y2 = \sin 3x$, and $\mathrm{f(x)} = \frac{1}{4}\csc 3x$

We obtain $\mathrm{W}(\cos 3x, \sin 3x) = \begin{vmatrix} \cos 3\mathrm{x} & \sin 3\mathrm{x} \\ -3\sin 3\mathrm{x} & 3\cos 3\mathrm{x} \end{vmatrix} = 3,$

$$W_1 = \begin{vmatrix} 0 & \sin 3\mathrm{x} \\ \frac{1}{4}\csc 3x & 3\cos 3\mathrm{x} \end{vmatrix} = -\frac{1}{4}$$

$$W_2 = \begin{vmatrix} \cos 3\mathrm{x} & 0 \\ -3\sin 3\mathrm{x} & \frac{1}{4}\csc 3x \end{vmatrix} = \frac{1}{4}\frac{\cos 3x}{\sin 3x}.$$

Integrating $u_1' = \frac{W_1}{W} = -\frac{1}{12}$ and $u_2' = \frac{W_2}{W} = \frac{1}{12}\frac{\cos 3x}{\sin 3s}$

gives $u_1 = -\frac{1}{12}x$ and $u_2 = \frac{1}{36}\ln|\sin 3x|$.

Thus $y_p = -\frac{1}{12} x \cos 3x + \frac{1}{36} (\sin 3x) \ln|\sin 3x|$.

$$the\ \ general\ \ solution\ \ is$$

$y = y_c + y_p = c_1 \cos 3x + c_2 \sin 3x - \frac{1}{12} x \cos 3x + \frac{1}{36} (\sin 3x) \ln|\sin 3x|$.

10. $y_h = ?$

$\lambda^2 + a^2 = 0 \quad \lambda = \pm ai$

$y_h = C_1 \cos ax + C_2 \sin ax$

$\left(D^2 + a^2\right) y_p = \cos ax$

$\Rightarrow y_p = \dfrac{1}{D^2 + a^2} \cos ax$

$= \dfrac{1}{-a^2 + a^2} \cos ax$

$=$？ (因此用極限)

$\Rightarrow y_p = \dfrac{1}{D^2 + a^2} \cos ax$

$= \lim\limits_{\Delta \to 0} \dfrac{1}{-(a+\Delta)^2 + a^2} \cos(a+\Delta)x$

$= \lim\limits_{\Delta \to 0} \dfrac{1}{-2a\Delta - \Delta^2} \cos(a+\Delta)x$

$f(t)$於 $t = \mathrm{a}$ 之 *Taylor* 級數展開式

$$f(t) = f(a) + f'(a)(t-a) + \frac{1}{2\,!} f''(a)(t-a)^2 + \cdots + \frac{1}{n\,!} f^{(n)}(a)(t-a)^n + \cdots$$

$\cos t$ 於 $t = ax$ 之 Taylor 展開

$$\cos t = \cos ax - \sin ax (t - ax) - \frac{1}{2\,!} \cos ax (t-ax)^2 + \frac{1}{3\,!} \sin ax (t-ax)^3 + \cdots$$

令 t=(a+Δ)x

$$\cos\left(a+\Delta\right)x=\cos ax-\sin ax\left(\left(a+\Delta\right)x-ax\right)-\frac{1}{2!}\cos ax\left(\left(a+\Delta\right)x-ax\right)^{2}+$$

$$\frac{1}{3!}\sin ax\left(\left(a+\Delta\right)x-ax\right)^{3}+\cdots$$

$$=\cos ax-\sin ax\Delta x-\frac{1}{2!}\cos ax\left(\Delta x\right)^{2}+\frac{1}{3!}\sin ax\left(\Delta x\right)^{3}+\cdots$$

$$\Rightarrow y_{p}=\lim_{\Delta\to 0}\frac{1}{-(2a\Delta+\Delta^{2})}\left[\cos ax-\Delta x\sin ax-\frac{1}{2!}(\Delta x)^{2}\cos ax+\frac{1}{3!}(\Delta x)^{3}\sin ax+\cdots\right]$$

$$\begin{pmatrix}\text{因解不下去,想一想}\cos(ax)\text{是否可以不考慮?}\\ \text{YES 因為 } y_{h} \text{ 已含}\cos(ax)\text{, 可以消去}\end{pmatrix}$$

$$=\lim_{\Delta\to 0}\frac{1}{-(2a+\Delta)}\left[-x\sin ax-\frac{1}{2!}\Delta x^{2}\cos ax+\frac{1}{3!}\Delta^{2}x^{3}\sin ax+\cdots\right]$$

$$=\frac{1}{-2a}-x\sin ax=\frac{1}{2a}x\sin ax$$

$$\Rightarrow y''+a^{2}y=\cos ax$$

$$y=C_{1}\cos ax+C_{2}\sin ax+\frac{1}{2a}x\sin ax$$

11. $y=y_h+y_p$

$$y_{h}=Ce^{-2x}$$

$$y_{p}=e^{-2x}\varphi(x)$$

$$y_{p}'=e^{-2x}\varphi'(x)-2e^{-2x}\varphi(x)$$

$$e^{-2x}\varphi'(x)-2e^{-2x}\varphi(x)+2e^{-2x}\varphi(x)=e^{x}$$

$$\varphi'(x)=e^{3x}$$

$$\varphi(x)=\frac{1}{3}e^{3x}+k\qquad(k\text{可略})$$

$$y=y_{h}+y_{p}=Ce^{-2x}+\frac{1}{3}e^{x}$$

12. From the auxiliary equation (m - 1)(m - 3) = 0 we find $y_c=c_1x+c_2x^3$. Put the differential equation into the standard form y''+P(x)y'+Q(x)y=f(x). Therefore we divide the given equation by x^2, and form

$$y''-\frac{3}{x}y'+\frac{3}{x^{2}}y=2x^{2}e^{x}$$

We make the identification f(x)=$2x^2e^x$. Now with $y_1=x, y_2=x^3$ and

$$W=\begin{vmatrix}x & x^{3}\\ 1 & 3x^{2}\end{vmatrix}=2x^{3},$$

$$W_1 = \begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix} = -2x^5e^x, \qquad W_2 = \begin{vmatrix} \mathrm{x} & 0 \\ 1 & 2x^2e^x \end{vmatrix} = 2x^3e^x$$

we find $u_1' = -\frac{2x^5e^x}{2x^3} = -x^2e^x$ and $u_2' = \frac{2x^3e^x}{2x^3} = e^x$. The results are

$u_1 = -x^2e^x + 2xe^x - 2e^x \;\; and \;\; u_2 = e^x$. Hence

$$y_p = u_1y_1 + u_2y_2 = (-x^2e^x + 2xe^x - 2e^x)x + e^xx^3 = 2x^2e^x - 2xe^x$$

$$\mathrm{y} = c_1x + c_2x^3 + 2x^2e^x - 2xe^x.$$