ex. $F(s) = \dfrac{s}{(s+1)(s-2)^2} = \dfrac{k_1}{s+1} + \dfrac{k_2}{s-2} + \dfrac{k_3}{(s-2)^2}$

$\Rightarrow f(t) = k_1 e^{-t} + k_2 e^{2t} + k_3 t \cdot e^{2t}$

$k_1 = \dfrac{-1}{(-1-2)^2} = -\dfrac{1}{9}$

$k_3 = \dfrac{2}{2+1} = \dfrac{2}{3}$

$(k_2 = -k_1 = \dfrac{1}{9})$

↳ why? $\because F(s) = \dfrac{k_1(s-2)^2 + k_2(s+1)(s-2) + k_3(s+1)}{(s+1)(s-2)^2}$,

其中 s^2 項係數 $= 0 = k_1 + k_2$

ex. $F(s) = \dfrac{3s+2}{(s+3)(s-1)^2}$

$F(s) = \dfrac{3s+2}{(s+3)(s-1)^2} = \dfrac{k_1}{s+3} + \dfrac{k_2}{s-1} + \dfrac{k_3}{(s-1)^2}$

$k_1 = \dfrac{-9+2}{(-3-1)^2} = \dfrac{-7}{16}$

$k_2 = \dfrac{7}{16}$

$k_3 = \dfrac{3+2}{1+3} = \dfrac{5}{4}$

$f(t) = -\dfrac{7}{16} e^{-3t} + \dfrac{7}{16} e^{t} + \dfrac{5}{4} t e^{t}$ #.

ex. $F(s) = \dfrac{s^2+s+1}{(s-1)(s-3)^2}$

$F(s) = \dfrac{s^2+s+1}{(s-1)(s-3)^2} = \dfrac{k_1}{s-1} + \dfrac{k_2}{s-3} + \dfrac{k_3}{(s-3)^2}$

$k_1 = \dfrac{3}{4}$, $k_2 = \dfrac{1}{4}$, $k_3 = \dfrac{13}{2}$

$f(t) = \dfrac{3}{4} e^{t} + \dfrac{1}{4} e^{3t} + \dfrac{13}{2} t e^{3t}$

ex. $F(s) = \dfrac{s}{(s-1)(s^2+4s+13)}$

$F(s) = \dfrac{k_1}{s-1} + \dfrac{As+B}{s^2+4s+13}$

$\Rightarrow k_1 = \dfrac{1}{18} \Rightarrow F(s) = \dfrac{\frac{1}{18}(s^2+4s+13)+(As+B)(s-1)}{(s-1)(s^2+4s+13)}$

$\Rightarrow \begin{cases} \frac{1}{18} + A = 0 \\ \frac{13}{18} - B = 0 \end{cases} \Rightarrow \begin{matrix} A = -\frac{1}{18} \\ B = \frac{13}{18} \end{matrix} \Rightarrow F(s) = \dfrac{\frac{1}{18}}{s-1} + \dfrac{-\frac{1}{18}s + \frac{13}{18}}{s^2+4s+13}$

$= \dfrac{\frac{1}{18}}{s-1} + \dfrac{-\frac{1}{18}(s+2) + \frac{15}{18}}{(s+2)^2+3^2}$

$\Rightarrow f(t) = \frac{1}{18}e^{t} + \left(-\frac{1}{18}\cos 3t \cdot e^{-2t}\right) + \left(\frac{5}{18}\sin 3t \cdot e^{-2t}\right)$

ex. $y'' - 5y' + 6y = e^{-t}$, $y(0) = 0$, $y'(0) = 2$

$\Rightarrow L\{y'' - 5y' + 6y\} = L\{e^{-t}\}$

$\Rightarrow L\{y''\} + L\{-5y'\} + L\{6y\} = L\{e^{-t}\}$

$\Rightarrow s^2Y(s) - sy(0) - y'(0) - 5(sY(s) - y(0)) + 6Y(s) = \dfrac{1}{s+1}$

$\Rightarrow s^2Y(s) - 2 - 5sY(s) + 6Y(s) = \dfrac{1}{s+1}$

$\Rightarrow Y(s)(s^2 - 5s + 6) = 2 + \dfrac{1}{s+1} = \dfrac{2s+3}{s+1}$

$\Rightarrow Y(s) = \dfrac{2s+3}{(s^2-5s+6)(s+1)}$

$= \dfrac{k_1}{s-2} + \dfrac{k_2}{s-3} + \dfrac{k_3}{s+1}$

$\Rightarrow y(t) = -\frac{7}{3}e^{2t} + \frac{9}{4}e^{3t} + \frac{1}{12}e^{-t}$ #.

ex. $x'' + 4x' + 4x = 4$, $x(0) = 0$, $x'(0) = 0$

$$\Rightarrow s^2X(s) - sx(0) - x'(0) + 4(sX(s) - x(0)) + 4X(s) = \frac{4}{s}$$

$$\Rightarrow (s^2+4s+4)X(s) = \frac{4}{s}$$

$$\Rightarrow X(s) = \frac{4}{s(s+2)^2} = \frac{k_1}{s} + \frac{k_2}{s+2} + \frac{k_3}{(s+2)^2}$$

$$k_1 = \frac{4}{4} = 1, \quad k_2 = -1, \quad k_3 = -2.$$

$$\Rightarrow x(t) = 1 - e^{-2t} - 2e^{-2t}\cdot t$$ #.

ex. $y'' + 4y' + 5y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$

$$\Rightarrow s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 5Y(s) = \frac{1}{s+1}$$

$$\Rightarrow s^2Y(s) - 1 + 4sY(s) + 5Y(s) = \frac{1}{s+1}$$

$$\Rightarrow (s^2+4s+5)Y(s) = \frac{s+2}{s+1}$$

$$\Rightarrow Y(s) = \frac{k_1}{s+1} + \frac{As+B}{s^2+4s+5}$$

$$\Rightarrow Y(s) = \frac{\frac{1}{2}}{s+1} + \frac{-\frac{1}{2}s - \frac{1}{2}}{(s+2)^2+1} = \frac{\frac{1}{2}}{s+1} + \frac{-\frac{1}{2}(s+2) + \frac{1}{2}}{(s+2)^2+1}$$

$$\Rightarrow y(t) = \frac{1}{2}e^{-t} - \frac{1}{2}(\cos t)\cdot e^{-2t} + \frac{1}{2}(\sin t)\cdot e^{-2t}$$ #.

要記得驗算 =V= 看 $y(0) == 0$ & $y'(0) == 1$

ex. solve the following integral-differential equation

$$y' - 2y = \left(\int_0^t e^{2(t-\tau)}\cos 3\tau \, d\tau\right), \quad y(0) = 0$$ ⇒ Voltera 積分方程式.

$$\hookrightarrow f(t)*g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t g(\tau)f(t-\tau)\,d\tau$$

$$\Rightarrow y' - 2y = f(t)*g(t) = e^{2t} * \cos 3t$$

$$\mathcal{L}\{\quad\} = \mathcal{L}\{\quad\}$$

$$\Rightarrow sY(s) - y(0) - 2Y(s) = F(s)G(s) = \frac{1}{s-2}\cdot\frac{s}{s^2+9}$$

$$(s-2)Y(s) = \frac{s}{(s-2)(s^2+9)}$$

$$Y(s) = \frac{s}{(s-2)^2(s^2+9)} = \frac{k_1}{s-2} + \frac{k_2}{(s-2)^2} + \frac{As+B}{s^2+9}$$

$$k_2 = \frac{2}{2^2+9} = \frac{2}{13}$$

$$k_1 = \frac{d}{ds}\left(\frac{s}{s^2+9}\right)\Big|_{s=2} = \frac{5}{169}$$

$$\Rightarrow Y(s) = \frac{\frac{5}{169}(s-2)(s^2+9) + \frac{2}{13}(s^2+9) + (As+B)(s-2)^2}{(s-2)^2(s^2+9)}$$

$$\Rightarrow A = \frac{-5}{169}, \quad B = \frac{-36}{169}$$

$$\Rightarrow y(t) = \frac{5}{169}e^{2t} + \frac{2}{13}e^{2t}\cdot t - \frac{5}{169}\cos 3t - \frac{12}{169}\sin 3t \quad \#$$

ex. $f(t) = -1 + \int_0^t f(t-\tau)e^{3\tau}\,d\tau$, 求$f(t)$?

$$F(s) = -\frac{1}{s} + F(s)\cdot\frac{1}{s+3}$$

$$\Rightarrow F(s) = \frac{k_1}{s} + \frac{k_2}{s+2} \quad \Rightarrow f(t) = -\frac{3}{2} + \frac{1}{2}e^{-2t} \quad \#.$$

ex. $f(t) = e^{-2t} - 3e^{-3t}\int_0^t f(\tau)e^{3\tau}\,d\tau$, 求$f(t)$?

$$f(t) = e^{-2t} - 3\int_0^t f(\tau)e^{3\tau-3t}\,d\tau$$

$$= e^{-2t} - 3\int_0^t f(\tau)e^{3(\tau-t)}\,d\tau$$

$$= e^{-2t} - 3\left(\int_0^t f(\tau)e^{-3(t-\tau)}\,d\tau\right)$$

$\hookrightarrow f(t) * e^{-3t}$

$$\Rightarrow F(s) = \frac{1}{s+2} - 3F(s)\cdot\frac{1}{s+3}$$

$$F(s) = \frac{s+3}{(s+6)(s+2)} = \frac{k_1}{s+6} + \frac{k_2}{s+2}$$

$$k_1 = \frac{3}{4}, \quad k_2 = \frac{1}{4}$$

$$f(t) = \frac{3}{4}e^{-6t} + \frac{1}{4}e^{-2t} \quad \#$$

§ 聯立方程式.

$$\text{ex.}\begin{cases}\dfrac{dx_1(t)}{dt} = x_2(t) \\ \dfrac{dx_2(t)}{dt} = -2x_1(t) - 3x_2(t)\end{cases}, \quad \begin{matrix} x_1(0)=1 \\ x_2(0)=1\end{matrix}$$

偶合 (coupled)

$\downarrow \mathcal{L}\{\ \}$

$$\Rightarrow \begin{cases} sX_1(s) - x_1(0) = X_2(s) \\ sX_2(s) - x_2(0) = -2X_1(s) - 3X_2(s)\end{cases}$$

$\downarrow$ 整理

$$\Rightarrow \begin{cases} sX_1(s) - X_2(s) = 1 \\ 2X_1(s) + (s+3)X_2(s) = 1\end{cases}$$

$\downarrow$ 用矩陣處理.

$$\Rightarrow X_1(s) = \frac{\begin{vmatrix} 1 & -1 \\ 1 & s+3 \end{vmatrix}}{\begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix}} = \frac{s+4}{s^2+3s+2} = \frac{3}{s+1} + \frac{-2}{s+2}$$

$$\Rightarrow x_1(t) = 3e^{-t} - 2e^{-2t}$$

$$X_2(s) = \frac{\begin{vmatrix} s & 1 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix}} = \frac{s-2}{s^2+3s+2} = \frac{-3}{s+1} + \frac{4}{s+2}$$

$$\Rightarrow x_2(t) = -3e^{-t} + 4e^{2t}.$$

以線性代數來看

$$\Rightarrow \begin{bmatrix} s & -1 \\ 2 & s+1 \end{bmatrix}\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s(s+3)+2}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \dfrac{s+4}{s^2+3s+2} \\ \dfrac{s-2}{s^2+3s+2} \end{bmatrix}$$

$$\Rightarrow x_1(t) = \ldots, \quad x_2(t) = \ldots$$

difficult

ex. $y'' + 2ty' - 4y = 6$, $y(0) = 0$, $y'(0) = 0$

$\Rightarrow s^2Y(s) - sy(0) - y'(0) + 2\left(-\dfrac{d(sY(s))'}{ds}\right) - 4Y(s) = \dfrac{6}{s}$

↳ $L\{ty'\} = -\dfrac{dL\{y'\}}{ds}$

$\Rightarrow -2sY'(s) + (s^2-6)Y(s) = \dfrac{6}{s}$

↳ 其解 $Y(s) = cI^{-1} + I^{-1}\int I r\,dx$, $I = e^{\int p\,dx}$

$\Rightarrow Y'(s) + \dfrac{s^2-6}{-2s}Y(s) = \dfrac{6}{s(-2s)}$

$\Rightarrow I(s) = e^{\int P(s)ds} = e^{-\frac{1}{4}s^2}\cdot s^3$

$\Rightarrow Y(s) = cI^{-1} + I^{-1}\int I r\,ds$

$= c e^{\frac{1}{4}s^2}\cdot s^{-3} + e^{\frac{s^2}{4}}\cdot s^{-3}\left(\int e^{-\frac{s^2}{4}}\cdot s^3 \cdot \dfrac{6}{s(-2s)}\,ds\right)$

↳ 令 $u = \dfrac{-s^2}{4}$, $du = -\frac{1}{2}s\,ds$

$\Rightarrow 6\cdot e^{-\frac{s^2}{4}}$

$\Rightarrow Y(s) = c e^{\frac{s^2}{4}}\cdot s^{-3} + 6s^{-3}$

解 c ⇒ 初值定理

$\Rightarrow y(0) = \lim\limits_{s\to\infty} sY(s) = \lim\limits_{s\to\infty}\left(c\cdot e^{\frac{s^2}{4}}\cdot s^{-2} + \dfrac{6}{s^2}\right) = 0$

$\therefore c = 0$

$\therefore Y(s) = 6s^{-3}$, $y(t) = 3t^2$ #

ex. $y'' + 3y' + 2y = \delta(t-2)$, $y(0) = 0$, $y'(0) = 1$

$\Rightarrow (s^2Y - 0 - 1) + 3(sY(s) - 0) + 2Y(s) = e^{-2s}$

$\Rightarrow Y(s) = \dfrac{1 + e^{-2s}}{s^2+3s+2} = \dfrac{1}{(s+1)(s+2)} + \dfrac{1}{(s+1)(s+2)}e^{-2s}$

$= \dfrac{1}{s+1} + \dfrac{-1}{s+2} + \left[\dfrac{1}{s+1} + \dfrac{-1}{s+2}\right]e^{-2s}$

$y(t) = e^{-t} - e^{-2t} + \left[e^{-(t-2)} - e^{-2(t-2)}\right]H(t-2)$

ex. solve the following problems

(i) $y'' + 4y' + 3y = 3\delta(t-2) + H(t-1)$, $y(0) = y'(0) = 0$

(ii) $y'' + y = f(t)$, $f(t) = \begin{cases} 0, & 0 \le t \le \pi \\ 1, & \pi \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$, $y(0) = 0$, $y'(0) = 1$

(iii). $f(t) = \int_0^t f(t-\tau) e^{-\tau} d\tau + 3t^5$

(i). $s^2 Y(s) - s y(0) - y'(0) + 4(sY(s) - y(0)) + 3Y(s) = 3e^{-2s} + \frac{1}{s}e^{-s}$

$(s^2+4s+3) Y(s) = 3e^{-2s} + \frac{1}{s}e^{-s}$

$Y(s) = \frac{3}{s^2+4s+3} e^{-2s} + \frac{1}{s(s^2+4s+3)} e^{-s}$

$= \left(\frac{\frac{3}{2}}{s+1} + \frac{-\frac{3}{2}}{s+3}\right)e^{-2s} + \left(\frac{\frac{1}{3}}{s} + \frac{-\frac{1}{2}}{s+1} + \frac{\frac{1}{6}}{s+3}\right)e^{-s}$

$\Rightarrow y(t) = \left[\frac{3}{2}e^{-(t-2)} - \frac{3}{2}e^{-3(t-2)}\right] \cdot H(t-2)$

$+ \left(\frac{1}{3} - \frac{1}{2}e^{-(t-1)} + \frac{1}{6}e^{-3(t-1)}\right) H(t-1)$ #

(ii). 非週期函数 ⇒ 乖² Laplace.

$(s^2 Y(s) - 0 - 1) + Y(s) = \left(\frac{1}{s}e^{-\pi s} - \frac{1}{s}e^{-2\pi s}\right)$

是由 $L\{f(t)\}$来

$\Rightarrow \int_\pi^{2\pi} 1 \cdot e^{-st} dt = -\frac{1}{s} e^{-st}\Big|_\pi^{2\pi}$

$\Rightarrow Y(s) = \frac{1}{s^2+1} + \frac{1}{s(s^2+1)}(e^{-\pi s} - e^{-2\pi s})$

$= \frac{1}{s^2+1} + \boxed{\left(\frac{1}{s} + \frac{-s}{s^2+1}\right)(e^{-\pi s}} - e^{-2\pi s})$

$\Rightarrow y(t) = \sin t + \boxed{[1 - \cos(t-\pi)] H(t-\pi)} - (1 - \cos(t-2\pi)) H(t-2\pi)$

(iii). $f(t) = \left(\int_0^t f(t-\tau)e^{-\tau}\,d\tau\right) + 3t^5$

↳ $f(t)*e^{-t}$

$\Rightarrow F(s) = F(s)\frac{1}{s+1} + 3\cdot\frac{5!}{s^{5+1}}$

$\Rightarrow F(s) = \frac{3\cdot 5!\,(s+1)}{s^7} = \frac{3\cdot 5!}{s^6} + \frac{3\cdot 5!\cdot\frac{6!}{6!}}{s^7}$

$\Rightarrow f(t) = 3t^5 + \frac{1}{2}t^6$ #.