◎: 補充. $w(y_1, y_2)$.

一組函數集合 $\{u_1(x), u_2(x) \cdots u_n(x)\}$

考慮 $c_1u_1(x) + c_2u_2(x) + \cdots + c_nu_n(x) = 0$，其中 $c_1 \sim c_n \in$ const

(A) 若上式成立 $\overset{\text{iff}}{\Longleftrightarrow}$ $c_1 = c_2 = \cdots = c_n = 0$

則 $u_1(x), u_2(x) \cdots u_n(x)$ 為線性獨立 (linearly independent).

(B). 若 $\exists\, c_i \neq 0$ s.t 上式成立

則 $u_1(x), u_2(x) \cdots u_n(x)$ 為線性相依 (linearly dependent)

其中 $u_i(x)$ 可用 $u_1(x), u_2(x) \cdots u_{i-1}(x), u_{i+1}(x) \cdots u_n(x)$ 來表示.

pf: 令 $c_i \neq 0$.

$$c_1u_1 + c_2u_2 + \cdots + c_iu_i + \cdots + c_nu_n = 0$$

$$\Rightarrow u_i = \frac{-c_1}{c_i}u_1 + \frac{-c_2}{c_i}u_2 + \cdots + \frac{-c_{i-1}}{c_i}u_{i-1} + \frac{-c_{i+1}}{c_i}u_{i+1} + \cdots + \frac{-c_n}{c_i}u_n$$

ex. $u_1(x) = 3x$, $u_2(x) = -2x$.

$\Rightarrow c_1(3x) + c_2(-2x) = 0 \quad \Rightarrow (3c_1 - 2c_2)x = 0$

$\Rightarrow 3c_1 = 2c_2$ 　取 $c_1 = 1$, $c_2 = \frac{3}{2}$.

$\Rightarrow u_1(x) + u_2(x)$ 為線性相依

$\Rightarrow u_1(x) = 3x = -\frac{c_2}{c_1}(-2x)$.

ex. $u_1(x) = x^2$, $u_2(x) = x$

$\Rightarrow c_1x^2 + c_2x = 0 \Leftrightarrow c_1, c_2 = 0$

$\Rightarrow u_1(x) + u_2(x)$ 線性獨立

推廣：n 個函數.

$$\Rightarrow \begin{cases} c_1u_1 + c_2u_2 + \cdots + c_nu_n = 0 \\ c_1u_1' + c_2u_2' + \cdots + c_nu_n' = 0 \\ \vdots \end{cases}$$

聯立 $\left\{ \begin{array}{l} \vdots \\ c_1 u_1^{n-1} + c_2 u_2^{n-1} + \cdots + c_n u_n^{n-1} = 0 \end{array} \right.$

$$\begin{cases} ax+by=0 \\ cx+dy=0 \end{cases} \Rightarrow x = \frac{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

① 若 $x = y = 0 \iff \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$

② x, y 存在非零解 $\iff \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$

上述聯立方程式的行列式為

$$\begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \vdots & & & \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix} = \text{Wronski}(u_1, u_2 \cdots u_n)$$

(A). $W(u_1, u_2 \cdots u_n) = 0 \iff$ 至少有非零解在 $c_1 \sim c_n$ 之間
$\iff$ 線性相依.

(B). $W(u_1, u_2 \cdots u_n) \neq 0 \iff c_1, c_2 \cdots c_n = 0$
$\iff$ 線性独立

ex. $3x, -2x$.
$\Rightarrow W = \begin{vmatrix} 3x & -2x \\ 3 & -2 \end{vmatrix} = 0 \Rightarrow$ 線性相依.

ex. x, x^2.
$\Rightarrow W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2 \neq 0 \ \forall x \Rightarrow$ 線性独立

ex. e^{x}, e^{2x}, e^{3x}.

$$\Rightarrow W = \begin{vmatrix} e^{x} & e^{2x} & e^{3x} \\ e^{x} & 2e^{2x} & 3e^{3x} \\ e^{x} & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x} \Rightarrow \text{線性独立}$$

◎. 変係数微分方程式 (Eular-Cauchy Differential Equations)
(科西尤拉 / 尤拉科西, or 等維).

$$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_{n-1} x y' + a_n y = r(x)$$

$r(x) = 0 \longrightarrow$ homogeneous case.

想法：將原式変成常係数.

$\Rightarrow$ 令 $x = e^{t}$.

ex. $x^2 y'' - 2xy' + 2y = 0$.

令 $x = e^{t}$.

$$\Rightarrow \frac{dx}{dt} = e^{t}, \quad \frac{dy}{dt} = \frac{dy}{dx} \cdot e^{t} = \frac{dy}{dx} x$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\right) \cdot \frac{dx}{dt}$$

$$= \frac{d}{dx}\left(x \cdot \frac{dy}{dx}\right) \cdot x = \left(\frac{dy}{dx} + x \cdot \frac{d^2y}{dx^2}\right) \cdot x$$

$$= x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx}.$$

$$\left.\begin{aligned} &\Rightarrow xy' = x \cdot \frac{dy}{dx} = \frac{dy}{dt} \\ &x^2 y'' = x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - x\frac{dy}{dx} = \frac{d^2y}{dt^2} - \frac{dy}{dt} \end{aligned}\right\} \text{代回去.}$$

$$\Rightarrow \frac{d^2y}{dt^2} - \frac{dy}{dt} - 2 \cdot \frac{dy}{dt} + 2y = 0$$

$$\Rightarrow \frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0 \Rightarrow \text{変成常係数了}$$

$\Rightarrow \lambda^2 - 3\lambda + 2 = 0 \quad \Rightarrow \lambda = 1, 2.$

$\Rightarrow y(t) = c_1 e^{t} + c_2 e^{2t}.$

$\Rightarrow y(x) = c_1 x + c_2 x^2.$ #

note: ① $x \dfrac{dy}{dx} = \dfrac{dy}{dt}.$

② $x^2 \dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}$

③ $x^3 \dfrac{d^3y}{dx^3} = \dfrac{d^3y}{dt^3} - 3\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt}.$

其推導过於冗長

$\Rightarrow$ 引用 $D \equiv \dfrac{d}{dx}. \quad \mathcal{D} = \dfrac{d}{dt}.$

$\Rightarrow$ ①: $xDy = \mathcal{D}y$

②: $x^2D^2y = \mathcal{D}^2y - \mathcal{D}y = \mathcal{D}(\mathcal{D}-1)y$

③: $x^3D^3y = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y$

$\vdots$

$x^nD^ny = \mathcal{D}(\mathcal{D}-1)\cdots(\mathcal{D}-(n-1))y$

回頭來看 $a_0x^ny^{(n)} + a_1x^{n-1}y^{(n-1)} + \cdots + a_{n-1}xy^{(1)} + a_ny = 0$

$\Rightarrow a_0\mathcal{D}(\mathcal{D}-1)\cdots(\mathcal{D}-(n-1))y + a_1\mathcal{D}(\mathcal{D}-1)\cdots(\mathcal{D}-(n-2))y + \cdots + a_ny = 0$

意即 $L(\mathcal{D})y = 0$

ex. $x^2y'' + 4xy' + 2y = 0$

令 $x = e^t$

$\Rightarrow \mathcal{D}(\mathcal{D}-1)y + 4\mathcal{D}y + 2y = 0$

$\Rightarrow (\mathcal{D}^2 + 3\mathcal{D} + 2)y = 0$

$\Rightarrow \lambda^2+3\lambda+2=0 \quad \Rightarrow \lambda=-1,-2.$

$\Rightarrow y = c_1 e^{-t} + c_2 e^{-2t}$

$= c_1 x^{-1} + c_2 x^{-2}$ #.

ex. $x^3 y''' + 4x^2 y'' - 5xy' - 15y = 0$

令 $x = e^t$, $D = \frac{d}{dt}$.

$\Rightarrow D(D-1)(D-2)y + 4D(D-1)y - 5Dy - 15y = 0$

$\Rightarrow (D^3 + D^2 - 7D - 15)y = 0$

$\Rightarrow \lambda^3 + \lambda^2 - 7\lambda - 15 = 0$

$\Rightarrow \lambda = 3, \ -2 \pm i$

$\Rightarrow y(t) = c_1 e^{3t} + c_2 e^{-2t}\cos t + c_3 e^{-2t}\sin t$

$y(x) = c_1 x^3 + c_2 x^{-2}\cos(\ln x) + c_3 x^{-2}(\ln x)$.

ex. $x^2 y'' - xy' - 3y = 4x$.

y_h: $x^2 y_h'' - x y_h' - 3y_h = 0$

令 $x = e^t$, $D = \frac{d}{dt}$.

$\Rightarrow D(D-1)y_h - Dy_h - 3y_h = 0$

$\Rightarrow (D^2 - 2D - 3)y_h = 0$

$\Rightarrow \lambda^2 - 2\lambda - 3 = 0 \quad \Rightarrow \lambda = -1, 3.$

$\Rightarrow y_h(t) = c_1 e^{3t} + c_2 e^{-t}$

$y_h(x) = c_1 x^3 + c_2 x^{-1}$

y_p: $x^2 y_p'' - x y_p' - 3y_p = 4x$

(1) UC (未定係數法).

$\Rightarrow y_p'' - \frac{1}{x} y_p' - \frac{3}{x^2} y_p = \frac{4}{x} \quad \Rightarrow$ 不能猜

(2) RO (降階法).

令 $x = e^t$, $D = \frac{d}{dt}$

$\Rightarrow (D^2 - 2D - 3)y_p = 4e^t$

$\Rightarrow (D-3)(D+1)y_p = 4e^t$

$I_1 = e^t$, $I_2 = e^{-3t}$

$\Rightarrow y_p = I_2^{-1}\int I_2 \cdot I_1^{-1}\int I_1 r\,dt\,dt = -e^t$

$\Rightarrow y_p(x) = -x$ #.

(3). 線性微分運算子法.

$\Rightarrow y_p = \dfrac{1}{D^2-2D-3}4e^t = \dfrac{1}{1-2-3}4e^t = -e^t$

$\Rightarrow y_p(x) = -x$ #

(4) 變異係數法.

$y_1 = e^{3t}$, $y_2 = e^{-t}$

$w(y_1, y_2) = \begin{vmatrix} e^{3t} & e^{-t} \\ 3e^{3t} & -e^{-t} \end{vmatrix} = -4e^{2t}$

$\Rightarrow y_p = y_1\int \dfrac{-ry_2}{w}dx + y_2\int \dfrac{ry_1}{w}dx = -e^t$

$\Rightarrow y_p(x) = -x$ #.

* 注意, 若是用 $y_h = C_1 \underbrace{x^3}_{y_1} + C_2 \underbrace{x^{-1}}_{y_2}$

$\Rightarrow w = -4x$

$y_p = y_1\int \frac{-ry_2}{w}dx + y_2\int \frac{ry_1}{w}dx$

(此処の r 為 $\frac{4}{x}$, $\because 1y'' - \frac{1}{x}y' - \frac{3}{x^2}y = \frac{4}{x}$)

$= -x$ #.

ex. $(2x-3)^2y'' - 6(2x-3)y' + 12y = 0$

令 $u = 2x-3$.

$\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx} = \dfrac{dy}{du}\cdot 2$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\cdot\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(2\frac{dy}{du}\right) = 4\frac{d^2y}{du^2}$$

原式 : $u^2\cdot 4\frac{d^2y}{du^2} - 6u\cdot 2\frac{dy}{du} + 12y = 0$

令 $u = e^t$, $D = \frac{d}{dt}$.

$\Rightarrow [4D(D-1) - 12D + 12]y = 0$

$\Rightarrow (4D^2 - 16D + 12)y = 0$

$\Rightarrow 4\lambda^2 - 16\lambda + 12 = 0 \Rightarrow \lambda = 1, 3.$

$\Rightarrow y(t) = c_1 e^t + c_2 e^{3t}$

$y(u) = c_1 u^1 + c_2 u^3.$

$y(x) = c_1(2x-3) + c_2(2x-3)^3$ #

ex. $(3x+4)^2y'' - 6(3x+4)y' + 18y = 9\ln(3x+4).$

令 $u = 3x+4$.

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 3\frac{dy}{du}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\cdot 3\frac{dy}{du}\cdot\frac{du}{dx} = 9\frac{d^2y}{du^2}$$

$\Rightarrow$ 原式: $u^2\cdot 9\frac{d^2y}{du^2} - 6\cdot u\cdot\frac{dy}{du} + 18y = 9\ln u$

令 $u = e^t$, $D = \frac{d}{dt}$.

$\Rightarrow [D(D-1) - 2D + 2]y = \ln u = \ln e^t = t.$

$\Rightarrow (D^2 - 3D + 2)y = t$

$\Rightarrow \lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda = 1, 2$

$\Rightarrow y_h = c_1 e^t + c_2 e^{2t}$

$= c_1 u + c_2 u^2$

$= c_1(3x+4) + c_2(3x+4)^2$

用方法3 找 y_p

$$y_p = \frac{1}{D^2 - 3D + 2}\cdot t$$

$$= \frac{1}{2\left(1+\frac{D^2-3D}{2}\right)} \cdot t$$

$$= \frac{1}{2}\left(1 - \frac{D^2-3D}{2} + \left(\frac{D^2-3D}{2}\right)^2 - \cdots\right) t$$

$$= \frac{1}{2} t + \frac{3}{4}$$

$$= \frac{1}{2}\ln u + \frac{3}{4} = \frac{1}{2}\ln(3x+4) + \frac{3}{4}$$

$$\Rightarrow y = C_1(3x+4) + C_2(3x+4)^2 + \frac{1}{2}\ln(3x+4) + \frac{3}{4} \quad \#$$

§2.5 ex. Bernoulli equation [非線性]

$$y'(x) + p(x)y = r(x)\cdot y^n \quad (n \neq 0)$$

$$n=1 \Rightarrow y'(x) + (p-r)y = 0$$

$$n \neq 0 \neq 1 \Rightarrow y^{-n}y' + p\cdot y^{1-n} = r \quad \text{——— } (*)$$

令 $z = y^{1-n}$

$$\frac{dz}{dx} = (1-n)y^{1-n-1}\cdot\frac{dy}{dx} = (1-n)y^{-n}\cdot\frac{dy}{dx}$$

$$\Rightarrow y^{-n}\cdot\frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx}$$

原式 $(*)$

$$= \frac{1}{1-n}\frac{dz(x)}{dx} + p(x)\cdot z(x) = r(x)$$

$$\Rightarrow \frac{dz(x)}{dx} + \underbrace{\boxed{(1-n)p(x)}}_{p'} z(x) = \underbrace{\boxed{(1-n)}\,r(x)}_{r'}$$

$$\Rightarrow \frac{dz}{dx} + p'z = r'$$

$$\Rightarrow z = cI^{-1} + I^{-1}\int I r\,dx, \quad I = e^{\int p'dx}$$

$$\Rightarrow z = ce^{-\int(1-n)p\,dx} + e^{-\int(1-n)p\,dx}\int e^{\int(1-n)p\,dx}(1-n)r(x)\,dx = y^{1-n}$$

$$\Rightarrow y = z^{\frac{1}{1-n}}$$

ex. $x\frac{dy}{dx} + y = x^2y^2$, $n=2$

$\Rightarrow \frac{dy}{dx} + (\frac{1}{x})y = xy^2 = xy^n$.

令 $z = y^{1-n} = y^{-1}$

$\frac{dz}{dx} - \frac{1}{x}\cdot z(x) = -x \quad , \quad I = \frac{1}{x}$

$\Rightarrow z(x) = cx - x^2 = y^{-1} \Rightarrow y = \frac{1}{cx - x^2}$

若能把非線性転成線性是最好的，以下是幾种配的剛好的函式.

型①. $f'(y)\frac{dy}{dx} + p(x)f(y) = g(x)$.

令 $z = f(y)$

$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = \frac{df(y)}{dy} \cdot \frac{dy}{dx} = f'(y)\frac{dy}{dx}$.

原式: $\frac{dz}{dx} + p(x)\cdot z = g(x)$.

ex. $x^2\cos y\frac{dy}{dx} = 2x\sin y - 1$

令 $z = \sin y$，則 $\frac{dz}{dx} = \cos y \cdot \frac{dy}{dx}$

$x^2\frac{dz}{dx} - 2x\,dz = -1$

$\frac{dz}{dx} - \frac{2}{x}dz = \frac{-1}{x^2}$

$z = cI^{-1} + I^{-1}\int Ir\,dr$

$\Rightarrow f(y) = y^2, y^3 \ldots \sin(y) \ldots e^y$

型② 白努力

型③ Riccati

$\frac{dy}{dx} + p(x)\cdot y = g(x) + y^2\cdot r(x)$ 当 y_1 為上式之一特解時 則令 $y = y_1 + \frac{1}{z}$