# Master Title

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**Abstract**

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# Introduction

# Theory

## Spectral analysis

In 1807, the French mathematician and physicist Jean-Baptiste Joseph Fourier discovered that an arbitraty periodic function $x(t)$ can be written as a linear combination of sines and cosines [3]:

$$x(t) = \sum_{k=-\infty}^{k=\infty} c_k e^{ik\Omega_0 t}$$

With $c_k$ being the Fourier coefficients, and where the summation is periodic with period $T_0 = \frac{2\pi}{\Omega_0}$.

In other words, we can rewrite any signal as a sum of harmonic oscillators through a Fourier transform. This is also true for a discontinous time series $x[n]$:

$$x[n] = \sum_{k=0}^{N-1} c_k e^{i\frac{2\pi}{N}kn}$$

The Fourier coefficients $c_k$ form a Fourier series, defined by:

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-i\frac{2\pi}{N}kn}$$

Every Fourier coefficent is a complex number $a + ib$. The real component relating to cosines, while the complex component relates to sines as in Euler's formula:

$$Re^{ix} = a\cos(x) + ib\sin(x), \;\; R = |c_k|$$

When interpreting Fourier transforms, it is common to look at the Power Spectrum Density (PSD). This is done by looking at the absolute value of the Fourier coefficients $|c_k|$. Doing this lets us inspect what frequencies a signal is composed of, but truncates information on the phase.

The Fourier transform is a powerful tool. This is because it allows us to decompose a seemingly chaotic signal into its frequency components. Let us for example look at a plain sine wave with a frequency of 0.04 Hz, sampled over 100 seconds at a sampling frequency of 1 Hz:

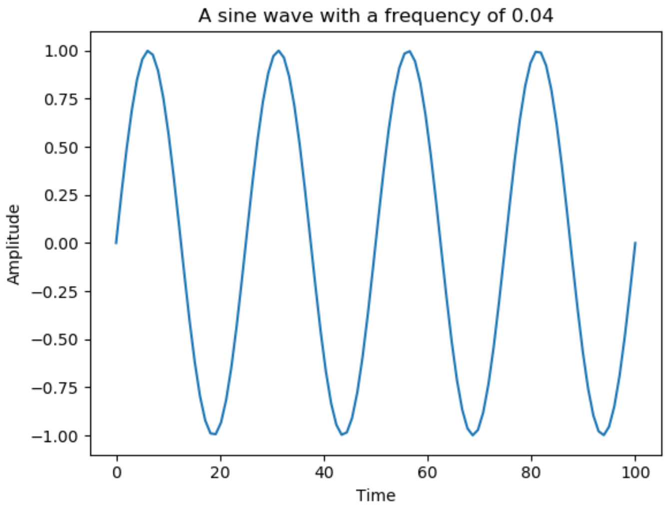


Figure 1: A plain sine signal with a frequency of 0.04 Hz, sampled over 100 seconds at a sampling frequency of 1 Hz.

Taking the Fourier transform of this lets us inspect what frequencies the signal is composed of, up to the Nyquist frequency $f_n = \dfrac{f_s}{2}$ which in this case is 0.5 Hz:

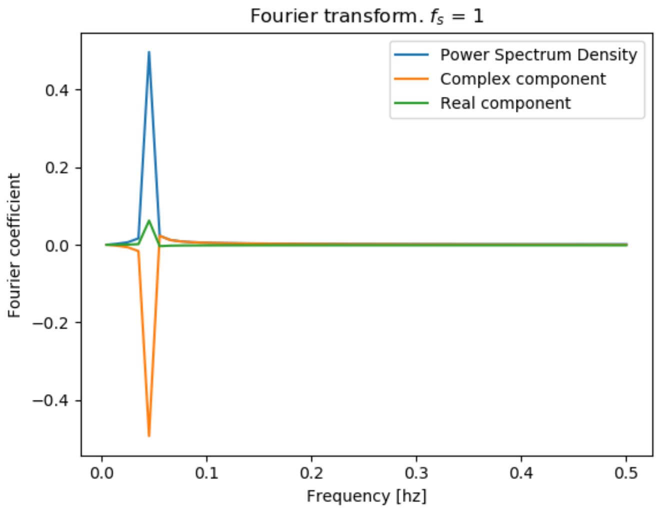


Figure 2: The Fourier transform of the signal in figure 1. We see a sharp peak in PSD around 0.04 Hz, as expected. We see that this is due to the complex, the sine, component. The negative sign is due to the phase of the signal.

As we see in figure 2, the Fourier transform lets us see that the signal in figure 1 is composed of a sine wave with a frequency of 0.04 Hz. We could see this just from looking at the figure, so let us look at a more complicated signal, composed of multiple sine waves:

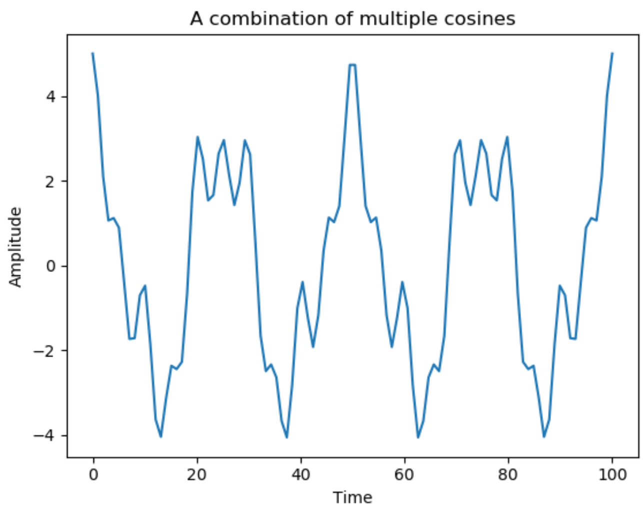


Figure 3: A more complicated signal, composed of multiple cosines. It is a lot more diffcult to see what frequencies it is composed of.

Once again, we take the Fourier transform of the signal and plot the PSD:

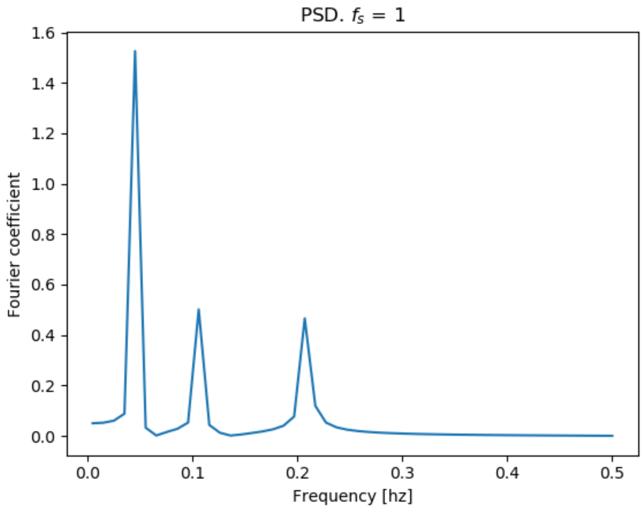


Figure 4: The Power Spectrum Density of the signal in figure 3. We see that the signal is composed of 3 frequencies; 0.04, 0.1 and 0.2 Hz. We also see that the amplitude of the 0.04 Hz component is 3 times larger than the other 2.

We end up with figure 4, where we see 3 distinct peaks in the PSD. Rather than the complicated mess we started with, we end up with a clear description of what is in the signal.

## Time difference between measurement devices

If we have some kind of positionally dependant structure, we can measure it by moving a measurement device with a velocity $v$ and sampling frequency $f_s$ through it. Doing so would result in a time series $x[t]$. If we measure the same structure again at the same positions with the same velocity and sampling frequency but at a later time, we would end up with a new time series $y[t]$. Given that the structure we are measuring is time independant, the two time series would be equal except for a time shift, so that $x[t] = y[t + \Delta t]$. If we know the positions of the measuring devices, we can rewrite our time series as functions of position instead: $x[t] \to x[\vec{r}]$ and $y[t] \to y[\vec{r}]$. We see, as $y$ was measured at the same positions as $x$, that these must be equivalent: $x[\vec{r}] = y[\vec{r}]$. As such, if we find the time difference $\Delta t$ between the measurements, we could go from a time dependant series to a spatially dependant series. Consequently, we could also find $\Delta t$ by finding the difference in position in real time between the measurement devices, and index-shifting the time series $y[t]$ until this distance becomes zero.

## Movement of plasma structures relative to 2 observation points.

Consider 2 satellites A and B following the same path, with B following A with a time delay $s$, through a plasma structure while measuring electron density. Looking at the electron density measurements over time for these satellites, you would get something akin to 5.

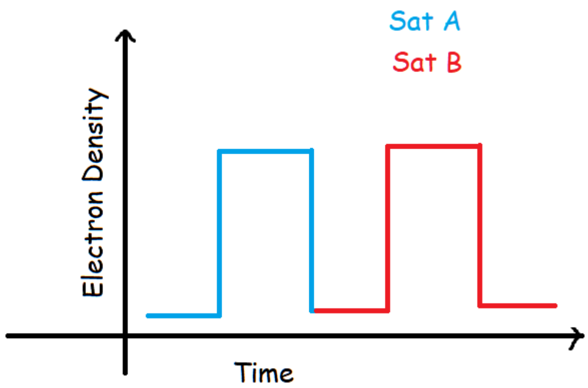


Figure 5: What you would expect the electron density measurements of 2 satellites following the same path with a time delay would look like. They measure approximately the same thing, but at different times.

Assuming we know the position of the satellites for every point of measurement, we can move from the time domain to the spatial domain by shifting the index of the time series measured by satellite B through minimizing the distance between each data point.
After shifting the time series of satellite B, there are 3 possible cases we can look at. The velocity of the plasma structure relative to the satellites, $V_B$, can either be 0, greater than 0, or less than 0 as seen in 6. From this data, we can find the time difference $n$ between the satellites finding the plasma structure.

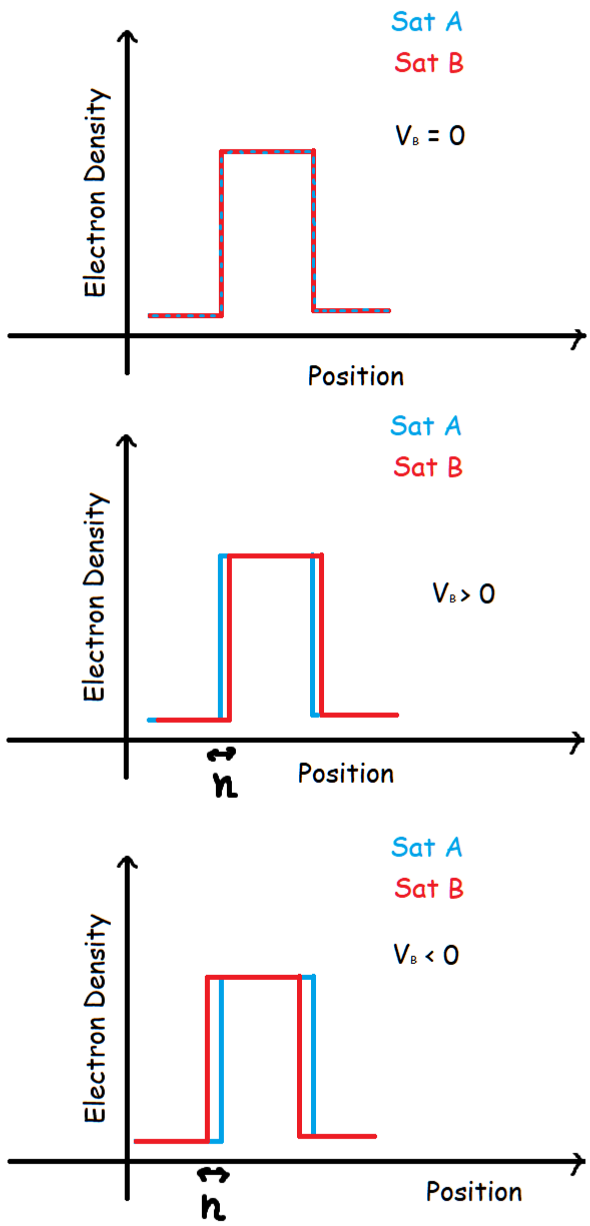


Figure 6: What the data would look like in the spatial domain, after minimizing distance between data points. We have 3 cases: $V_B = 0$, $V_B > 0$ and $V_B < 0$. For the first case, the data sets overlap perfectly. For the second case, the data from satellite B is slightly delayed. For the third case, the data from satellite B appears in front of the data from satellite A.

To find the velocity of the plasma structure relative to the satellites, we can imagine the satellites as stationary in space while the plasma structure passes through them with a velocity $\Delta V = V_{sat} \mp V_B$.

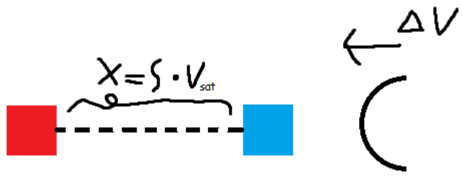


Figure 7: Keeping the satellites stationary, the plasma structure will move with a velocity $\Delta V = V_{sat} \mp V_B$ towards the satellites, with the direction dependant on the sign of $V_B$. The distance $x$ between the satellites is the time delay $s$ times the velocity of the satellites $V_{sat}$.

The time it takes the plasma structure to pass both satellites is:

$$t = \frac{x}{\Delta v} = \frac{s \cdot V_{sat}}{V_{sat} \mp V_B}$$

We can see that the time $t$ can be written as $t = s \mp n$, with $n$ being $t_A - t_B$ and therefore being negative when $V_B > 0$. This gives us:

$$s \mp n = \frac{s \cdot V_{sat}}{V_{sat} \mp V_B}$$

Finally, solving for $V_B$ gives us:

$$\begin{cases} V_B = V_{sat}(1 - \dfrac{s}{s-n}) & n < 0 \\ V_B = 0 & n = 0 \\ V_B = V_{sat}(\dfrac{s}{s+n} - 1) & n > 0 \end{cases} \tag{1}$$

# Instrumentation and Method

## The SWARM Mission

In this paper, we look at electron density and positional data from the European Space Agency(ESA)'s SWARM mission.
On 22 November 2013, ESA launched the SWARM mission. SWARM consists of 3 identical satellites (named A, B and C), carrying an array of instruments intended for creating a highly detailed survey of Earth's geomagnetic field.

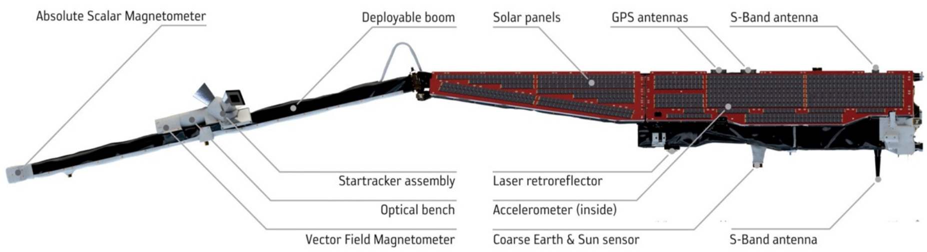


Figure 8: Side view of SWARM satellite. Source: ESA [2]

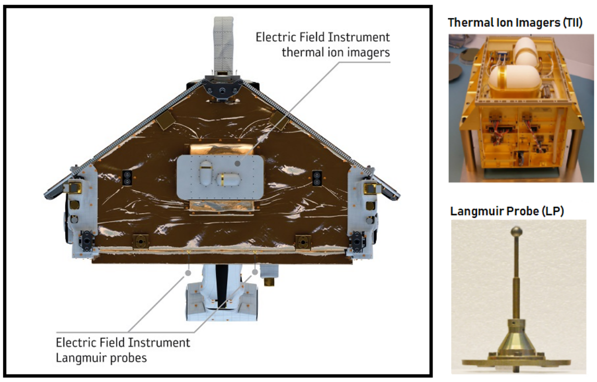


Figure 9: Rear view of SWARM satellite. Source: ESA [2]

Part of the instruments carried by the SWARM satellites is the Electric Field Instrument (EFI), as seen in figure 9. The EFI includes Langmuir probes which measure electron density, electron temperature and the spacecraft potential. In our case, we are interested in the electron density measurements from the Langmuir probes in conjuction with positional measurements from the GPS. The electron density is measured in electrons per cubed centimeter [$cm^{-3}$] while the positionals use spherical geocentric coordinates in the International Terrestrial Reference Frame (ITRF). Both are syncronized at a sampling frequency of 2 Hz.

During December 2013, the SWARM satellites orbited Earth in a cross polar pearls-on-a-string configuration, following the same line of longitude. On the 11th of December, the satellites were flying at an altitude of 507 km, with an orbital velocity of 7600 m/s. The satellites flew in a configuration with satellite B in the lead, followed 25.5 seconds later by satellite A, followed in turn 51 seconds later by satellite C. This is equivalent to a distance of 190 km and 385 km respectively.

As time passed, the satellites drifted apart. On the 30th of December, the time difference between the satellites was 59 seconds and 106.5 seconds.

### Largescale movement

## Results

## Discussion

## Conclusion

## Acknowledgements

## References

[1] Paschmann, G & Daly, P.W (2000) *Analysis Methods for Multi-Spacecraft Data*. Noordwijk: ESA Publications Division

[2] ESA (2021) *SWARM Mission* [internet]. Available from https://earth.esa.int/eogateway/missions/swarm

[3] Manolakis, D.G & Ingle, V.K (2011) *Applied digital signal processing*. New York: Cambridge university press

## Appendix